MAT454 Notes

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Proof of Zalcman's Lemma

Let $\mathcal{S}$ be a family of meromorphic functions on a domain Ω which is not normal in the chordal metric. We wish to find $a_n \to a_\infty \in \Omega$, $\rho_n \to 0$ such that

$$g_n(z) = f_n(a_n + \rho_n z)$$

converges to a nonconstant meromorphic function g in the chordal metric on compact subsets of $\mathbb{C}$ such that

$$g^\sharp(z) \leqslant g^\sharp(0) = 1$$

So, not normal means that

$$S^\sharp = \{f^\sharp : f \in \mathcal{S}\}$$

is not locally bounded (as per the theorem by Marty). This means, in particular, that there is a sequence of points $b_n \to b_\infty \in \Omega$, $f_n \in \mathcal{S}$ such that $f_n^\sharp(b_n) \to \infty$. Of course, we can assume $b_\infty = 0$, and hence in particular that there issomedisc $\{|z| \leqslant r\} \subset \Omega$.

Let

$$M_n = \max(r - |\zeta|) f_n^\sharp(\zeta)$$

This is a continous function on a compact set, and so it takes on its maximum at some point a_n, at which we have that this is equal to

$$(r - |a_n|) f_n^\sharp(a_n)$$

In particular, $a_n \to \infty$ since $b_n \to 0$. So the sequence we're going to construct is

$$g_n(z) = f_n(a_n + z/f_n^\sharp(a_n))$$

As n goes to infinity, this function is defined on bigger and bigger compact sets, as

$$\left| a_n + \frac{z}{f_n^\sharp(a_n)} \right| \leqslant |a_n| + \frac{|z|}{|f_n^\sharp(a_n)|} \leqslant |a_n| + r - |a_n|$$

That means that this is defined on $|z| \leqslant M_n$. Fix $R \geqslant 0$. If $|z| \leqslant R \leqslant M_n$, then by the chain rule for spherical derivatives

$$|g_n^\sharp(z)| \leqslant \frac{|f_n^\sharp(a_n + z/f_n^\sharp(a_n))|}{|f_n^\sharp(a_n)|} \leqslant \frac{\cancel{M_n}}{r - |a_n + z/f_n^\sharp(a_n)|} \frac{r - |a_n|}{\cancel{M_n}} \leqslant \frac{r - |a_n|}{r - |a_n| - |z|/f_n^\sharp(a_n)} = \frac{1}{1 - |z|/M_n} \to 1$$

as $n \to \infty$. This tells us by Marty's theorem that $\{g_n\}$ contains a convergent subsequence in the chordal metric. Of course, taking that convergent subsequence and re-labeliling, we can just assume it is given by $\{g_n\}$. g is meromorphic by this "lemma", non-constant since $g^\sharp(0) = 1$ and the endnote shows $g^\sharp(z) \leqslant 1$.

Montel's Theorem

We've already seen a result of Montel, that was the consequence of the Arzela-Ascoli theorem characterising normality in terms of uniformly bounded. This is a "much souped up" version of that theorem, which says that if you have a family $\mathcal{S}$ of meromorphic functions on a domain Ω of $\mathbb{C}$ which omits three distinct values (*including* ∞) in $\mathbb{C}^*$l is normal in the chordal metric.

We'll do this on Monday, and we're going to see how the Big Picard Theorem follows from this. So, Big Picard's Theorem says that if f is meromorphic in a punctured disk $0 < |z - z_0| < \rho$ and omits three distinct values in $\mathbb{C}^*$ then it extends to be meromorphic in $|z - z_0| < \rho$.