

MAT454 Academic Offense Sheet

Jad Elkhaleq Ghalayini

April 22, 2020

A quick collection of useful facts, theorems, and definitions for complex analysis. May be incorrect, and is certainly incomplete. Use at your own risk!

Contents

1 Basic Definitions and Theorems	2
2 Useful Results and Formulas	5
3 Residues and Integrals	6
4 Elliptic Curves	7

1 Basic Definitions and Theorems

For $f = u + iv$ holomorphic, we have

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (1)$$

Definition 1. The **differential** of f is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \quad (2)$$

$$dz = dx + idy, \quad d\bar{z} = dx - idy \iff dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = \frac{1}{2i}(dz - d\bar{z}) \quad (3)$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \quad (4)$$

Definition 2 (Harmonic). We say a real or complex valued function $f(x, y)$ is **harmonic** if f is \mathcal{C}^2 and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \quad (5)$$

Proposition 1. Every real-valued harmonic function is, not necessarily everywhere but at least locally, the real part of a holomorphic function.

Theorem 1. ω has a primitive in Ω if and only if, for any piecewise differentiable closed curve $\gamma : [a, b] \rightarrow \Omega$ (i.e. with $\gamma(a) = \gamma(b)$), or equivalently any piecewise differentiable $\gamma : S^1 \rightarrow \Omega$, we have

$$\int_{\gamma} \omega = 0 \quad (6)$$

Definition 3. We say a differential form ω on a domain Ω is **closed** if every point in Ω has a neighborhood in which ω has a primitive.

Theorem 2. Any closed differential form ω in a simply-connected open set Ω has a primitive.

Theorem 3 (Cauchy's Theorem). Let Ω be a domain and let $f(z)$ be continuous in Ω and holomorphic except on a set of discrete lines and points. Then the differentiable form $f(z)dz$ is closed.

Corollary 1. A holomorphic function $f(z)$ locally has a primitive, which is holomorphic (i.e. a function F such that $dF = f(z)dz$)

Corollary 2 (Morera's Theorem). If $f(z)$ is continuous in Ω and $df = f(z)dz$ is closed, then $f(z)$ is holomorphic.

Definition 4. Let $\gamma : S^1 \rightarrow \Omega$ be a closed curve and $a \notin \gamma(S^1)$ be a point not in the image of γ . Then the **winding number of γ with respect to a** is given by the integral

$$w(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} \quad (7)$$

This integral is an integer as it is the difference between two branches of \log .

Theorem 4 (Cauchy's Integral Formula). If $f(z)$ is holomorphic in Ω , $a \in \Omega$ and $\gamma : S^1 \rightarrow \Omega$ is a closed curve with $a \notin \gamma$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z - a} = w(\gamma, a)f(a) \quad (8)$$

Theorem 5 (Liouville's Theorem). A bounded holomorphic function on all of \mathbb{C} is a constant.

Definition 5 (Zero). If f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, we can write, for some $k \in \mathbb{N}$,

$$f(z) = (z - z_0)^k f_1(z) \quad (9)$$

where $f_1(z)$ is nonvanishing near z_0 . In this case k is called the **order** or **multiplicity** of the **zero** z_0

Definition 6 (Meromorphic). A function f is **meromorphic** on an open $\Omega \subseteq \mathbb{C}$ if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of Ω we can write $f(z) = g(z)/h(z)$ where g, h are holomorphic and h is not identically zero.

Definition 7 (Laurent expansion). Holomorphic functions in an annulus $r < |z| < R$ have a convergent **Laurent expansion** in an annulus

$$\sum_{n=-\infty}^{\infty} a_n z^n = P(z) + R(z), \quad P(z) = \sum_{n < 0} a_n z^n, \quad R(z) = \sum_{n \geq 0} a_n z^n \quad (10)$$

Theorem 6. Every meromorphic function f on S^2 is rational.

Definition 8 (Isolated singularity). A holomorphic function in a punctured disk $0 < |z| < R$ has an **isolated singularity** at 0 if $f(z)$ cannot be extended to be holomorphic at 0.

Theorem 7 (Weierstrass Theorem). If 0 is an essential singularity, then for all $\epsilon > 0$, $f(\{0 \leq |z| \leq \epsilon\})$ is dense in \mathbb{C} .

Definition 9. Let $\Omega \subset \mathbb{C}$ be open. We define $\mathcal{C}(\Omega)$ to be the **ring of continuous, complex-valued functions on Ω** and $\mathcal{H}(\Omega)$ to be the **subring of holomorphic functions on Ω**

Definition 10 (Uniform convergence on compact subsets). We say that a sequence of functions $\{f_n\} \subset \mathcal{C}(\Omega)$ **converges uniformly on compact subsets** if for all compact subsets $K \subset \Omega$, $\{f_n|_K\}$ converges uniformly, i.e.

$$\forall \text{ compact } K \subset \Omega, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N, \forall z \in K, |f_m(z) - f_n(z)| < \epsilon \quad (11)$$

Theorem 8 (Weierstrass). 1. $\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$, i.e. if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly to f on compact sets then $f = \lim_{n \rightarrow \infty} f_n \in \mathcal{H}(\Omega)$ is holomorphic.

2. The mapping $\mathcal{H}(\Omega) \rightarrow \mathcal{H}(\Omega) f \mapsto f'$ is continuous, i.e. if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly to f on compact sets then $\{f'_n\}$ converges uniformly to f' on compact sets.

Corollary 3. Let $\{f_n\}$ be a series of holomorphic functions. If $\{g_n = \sum_{k=0}^n f_k\}$ converges uniformly on compact subsets of Ω , then the sum

$$f = \sum f_n \quad (12)$$

is holomorphic on Ω and the series can be differentiated term by term.

Proposition 2. Let Ω be a domain. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets and each f_n vanishes nowhere in Ω then $f = \lim_{n \rightarrow \infty} f_n$ is either never zero or identically zero.

Corollary 4. Let Ω be a domain. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets and each f_n is one-to-one, then $\lim_{n \rightarrow \infty} f_n$ is either one-to-one or constant.

Definition 11. We say that $\sum_{n=1}^{\infty} f_n$ **converges uniformly** (respectively **converges uniformly absolutely**) on $X \subset \mathbb{C}$ if all but finitely many f_n have no pole in X and form a uniformly convergent (respectively uniformly absolutely convergent) series on X .

Definition 12. Let $X \subset \mathbb{C}$ and $\mathcal{S} \subset \mathcal{C}(X)$. We say that \mathcal{S} is **equicontinuous** at $a \in X$ if

$$\forall \epsilon > 0, \exists \delta > 0, \forall z \in X, |z - a| < \delta \implies \forall f \in \mathcal{S}, |f(z) - f(a)| < \epsilon \quad (13)$$

\mathcal{S} is **equicontinuous on X** if it is equicontinuous at each $a \in X$. It is **uniformly equicontinuous on X** if

$$\forall \epsilon > 0, \exists \delta > 0, \forall z, w \in X, |z - w| < \delta \implies \forall f \in \mathcal{S}, |f(z) - f(w)| < \epsilon \quad (14)$$

Theorem 9 (Arzela-Ascoli). *Let $\Omega \subset \mathbb{C}$ be a **domain**. Then $\mathcal{S} \subset \mathcal{C}(\Omega)$ is normal if and only if*

1. \mathcal{S} is equicontinuous on Ω
2. There exists $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}\}$ is a bounded subset of \mathbb{C}

Definition 13. $\mathcal{S} \subset \mathcal{C}(\Omega)$ is **locally bounded** on Ω if

$$\forall z_0 \in \Omega, \exists \delta > 0, M < \infty, \forall z \in \Omega, f \in \mathcal{S}, |z - z_0| < \delta \implies |f(z)| \leq M \quad (15)$$

This is true if and only if \mathcal{S} is **uniformly bounded on compact subsets of Ω** , i.e. for all $K \subset \Omega$ compact,

$$\exists M = M(K), \forall z \in K, \forall f \in \mathcal{S}, |f(z)| \leq M \quad (16)$$

Theorem 10 (Montel). *Let $\mathcal{S} \subset \mathcal{H}(\Omega)$ where $\Omega \subset \mathbb{C}$ is a domain. Then the following are equivalent:*

1. \mathcal{S} is normal
2. \mathcal{S} is locally bounded
3. $\mathcal{S}' = \{f' : f \in \mathcal{S}\}$ is locally bounded and there exists $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}\}$ is bounded in \mathbb{C} .

The Arzela-Ascoli theorem holds for families of continuous functions with values in a complete metric space, e.g. continuous functions with values in the Riemann sphere S^2 (or the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$) with the (induced) **chordal metric**

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|)^2(1 + |w|)^2}} \quad (17)$$

(we note that the topology induced by \mathbb{C} by the chordal metric is the usual Euclidean topology).

Definition 14 (Normal in the chordal metric). *A family \mathcal{S} of continuous functions on Ω is **normal in the chordal metric** if and only if it is equicontinuous in the chordal metric: condition (2) of the Arzela-Ascoli theorem is not needed because \mathbb{C}^* or S^2 is compact in this topology.*

We can use this definition to analyze, e.g., a family \mathcal{S} of meromorphic functions on $\Omega \subset \mathbb{C}$ (or $\Omega \subset S^2$), since these can be considered holomorphic functions with values in S^2 .

Lemma 1. *Let $\{f_n\}$ be a sequence of meromorphic functions which converges uniformly on compact subsets of the domain $\Omega \subset \mathbb{C}$ (on S^2 , in the chordal metric). Then the limit function is either meromorphic or identically ∞ .*

Definition 15 (Spherical derivative). *If f is meromorphic on a domain $\Omega \subset \mathbb{C}$ (or S^2), we define the **spherical derivative of f** at $z \in \Omega$ by*

$$f^\sharp(z) = \lim_{w \rightarrow z} \frac{d(f(z), f(w))}{|z - w|} \quad (18)$$

If z is not a pole, we have

$$f^\sharp(z) = \lim_{w \rightarrow z} \frac{2|f(z) - f(w)|}{|z - w|\sqrt{(1 + |f(z)|^2)(1 + |f(w)|^2)}} = \frac{2|f'(z)|}{1 + |f(z)|^2} \quad (19)$$

We have that

$$\left(\frac{1}{f}\right)^\sharp = f^\sharp \quad (20)$$

implying that $f^\sharp(z)$ is finite and continuous at all $z \in \Omega$, and greater than zero at z if and only if f is one-to-one near z .

Theorem 11 (Marty's Theorem). *Let \mathcal{S} be a family of meromorphic functions on a domain Ω . Then \mathcal{S} is normal in the chordal metric if and only if*

$$\mathcal{S}^\# = \{f^\# : f \in \mathcal{S}\} \quad (21)$$

is bounded.

Theorem 12 (Riemann mapping theorem). *Any simply connected open $\Omega \subset \mathbb{C}$ except \mathbb{C} itself has a biholomorphic mapping onto the open unit disc D*

Theorem 13 (Picard's Little Theorem). *A non-constant entire function f omits at most a point, i.e. $\#(\mathbb{C} \setminus f(\mathbb{C})) \leq 1$*

Theorem 14 (Picard's Big Theorem). *If z_0 is an isolated essential singularity of a holomorphic function $f(z)$, then f takes every complex value with one possible exception in any neighborhood Ω of z_0 , i.e. $\#(\mathbb{C} \setminus f(\Omega)) \leq 1$.*

2 Useful Results and Formulas

- Projection from the Riemann Sphere:

$$\pi : S^2 \setminus \{N\} \rightarrow \mathbb{C}, \pi(x, y, t) = \frac{x + iy}{1 - t} \quad (22)$$

- Green's Formula:

Theorem 15 (Green's formula).

$$\int_{\gamma} Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \quad (23)$$

- Schwarz Reflection Principle:

Theorem 16 (Schwarz Reflection Principle). *If $f : H \rightarrow \mathbb{C}$ is continuous on the closed upper half-plane H , holomorphic on the open upper half-plane and takes real values on the real axis (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$) then it can be extended to an entire function by $f(\bar{z}) = \overline{f(z)}$. More generally, this can be applied to reflecting any half-domain over any line.*

- Fourier coefficients and Cauchy inequalities:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} \implies f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \quad (24)$$

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \quad a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta \quad (25)$$

$$M(r) = \sup_{\theta} |f(re^{i\theta})| \implies |a_n| \leq \frac{M(r)}{r^n} \quad (26)$$

- The Mean Value Property (MVP): *harmonic* functions satisfy

$$f(\text{center of disk}) = \text{mean value on boundary} \quad (27)$$

- The Maximum Modulus Principle (MMP): if f is a continuous complex-valued function on an open $\Omega \subseteq \mathbb{C}$ with the MVP, then it satisfies the MMP, that is, if $|f|$ has a local maximum at a point a of Ω , then f is constant in a neighborhood of a .

- Schwarz's Lemma:

Theorem 17 (Schwarz's Lemma). *Suppose $f(z)$ is holomorphic in $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$. Then*

1. $|f(z)| \leq |z|$ if $|z| < 1$
2. If $|f(z_0)| = |z_0|$ at some $z_0 \neq 0$, then $f(z) = \lambda z$ for some $|\lambda| = 1$.

- Automorphisms of the complex plane:

$$\text{Aut } \mathbb{C} = \{\text{linear transformations } w = az + b, \quad a \neq 0\} \quad (28)$$

- Automorphisms of the Riemann Sphere:

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \quad (29)$$

Inverse:

$$\frac{dz - b}{-cz + a} \quad (30)$$

- Automorphisms of the upper half-plane:

$$w = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R} \quad (31)$$

- Holomorphic logarithm theorem: If f is nonvanishing on a simply connected set, then there exists a holomorphic function H on this set such that $f = e^H$.

3 Residues and Integrals

Definition 16 (Residue). *Let $f(z)$ be a holomorphic function in a punctured disc centered around a , and let γ be a closed curve lying entirely in the punctured disc (in particular, never touching a) with winding number $w(\gamma, a) = 1$. We define the **residue** of the differential form $f(z)dz$ (or “of f ”) at a to be*

$$\text{Res}_a(f) = \frac{1}{2\pi i} \int_{\gamma} f(z)dz = a_{-1} \quad (32)$$

where a_n are the coefficients in the Laurent expansion of f at a . Note that this is independent of the choice of curve γ .

Definition 17 (Residue at ∞). *Writing $z = \frac{1}{z'}$, we have in coordinates at ∞*

$$f(z)dz = -\frac{1}{z'^2} f(1/z')dz' = g(z')dz' \quad (33)$$

We define

$$\text{Res}_{\infty}(f) = \text{Res}_0(g) = -a_{-1} \quad (34)$$

where a_n are the terms of the Laurent expansion in $|z| > R$.

Theorem 18 (Residue Theorem). *Let $\Omega \subset S^2$ be open and let $f(z)$ be holomorphic in Ω except perhaps on a discrete set of isolated points. Let Γ be the oriented (piecewise C^1) boundary of a compact set $K \subset \Omega$ not containing any singularity (either essential singularities or poles). Then*

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_a \text{Res}_a(f) \quad (35)$$

where a ranges over the singularities contained in K , perhaps including ∞ .

At a simple pole a , we have

$$\text{Res}_a(f) = \lim_{z \rightarrow a} (z - a)f(z) \quad (36)$$

More generally, for a pole of degree n , we have

$$\text{Res}_a(f) = \frac{1}{(n-1)!} \lim_{z \rightarrow a} \frac{d^{n-1}}{dz^{n-1}} ((z - c)^n f(z)) \quad (37)$$

4 Elliptic Curves

Definition 18. Let $e_1, e_2 \in \mathbb{C}$ be linearly independent over \mathbb{R} . We can define a discrete subgroup of \mathbb{C} with *basis* e_1, e_2

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\} \quad (38)$$

We say that f has Γ as **group of periods** if

$$\forall z \in \mathbb{C}, f(z) = f(z + e_1) = f(z + e_2) \quad (39)$$

Definition 19 (Weierstrass \wp -function). We define the Weierstrass \wp -function by the infinite sum

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w \in \Gamma \\ w \neq 0}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right] \quad (40)$$

Claim. 1. \wp has a double pole at each $w \in \Gamma$ with prime part $\frac{1}{(z-w)^2}$

2. \wp is an even function

3. $\wp' = -z \sum_{w \in \Gamma} (z-w)^{-2}$ converges absolutely uniformly on compact subsets of \mathbb{C}

4. \wp' is doubly periodic: $\forall w \in \Gamma, \wp'(z+w) = \wp'(z)$

5. \wp' is odd

6. \wp itself has Γ as group of periods

We have

$$(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4 \quad (41)$$

Proposition 3. If f is a non-constant meromorphic function on \mathbb{C} with Γ as group of periods, then the number of zeros of f in a period parallelogram is equal to the number of poles in the same parallelogram