MAT454 Academic Offense Sheet

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A quick collection of useful facts, theorems, and definitions for complex analysis. May be incorrect, and is certainly incomplete. Use at your own risk!

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1 Basic Definitions and Theorems

For $f = u + iv$ holomorphic, we have

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

Definition 1. *The **differential** of f is given by*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{2}$$

$$dz = dx + idy, \qquad d\bar{z} = dx - idy \iff dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z}) \tag{3}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{4}$$

Definition 2 (Harmonic)**.** *We say a real or complex valued function $f(x, y)$ is **harmonic** if f is $\mathcal{C}^2$ and*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \tag{5}$$

Proposition 1. *Every real-valued harmonic function is, not necessarily everywhere but at least locally, the real part of a holomorphic function.*

Theorem 1. *ω has a primitive in Ω if and only if, for any piecewise differentiable closed curve $\gamma : [a, b] \to \Omega$ (i.e. with $\gamma(a) = \gamma(b)$), or equivalently any piecewise differentiable $\gamma : S^1 \to \Omega$, we have*

$$\int_\gamma \omega = 0 \tag{6}$$

Definition 3. *We say a differential form ω on a domain Ω is **closed** if every point in Ω has a neighborhood in which ω has a primitive.*

Theorem 2. *Any closed differential form ω in a simply-connected open set Ω has a primitive.*

Theorem 3 (Cauchy's Theorem)**.** *Let Ω be a domain and let $f(z)$ be continuous in Ω and holomorphic except on a set of discrete lines and points. Then the differentiable form $f(z)dz$ is closed.*

Corollary 1. *A holomorphic function $f(z)$ locally has a primitive, which is holomorphic (i.e. a function F such that $dF = f(z)dz$)*

Corollary 2 (Morera's Theorem)**.** *If $f(z)$ is continuous in Ω and $df = f(z)dz$ is closed, then $f(z)$ is holomorphic.*

Definition 4. *Let $\gamma : S^1 \to \Omega$ be a closed curve and $a \notin \gamma(S^1)$ be a point not in the image of γ. Then the **winding number of γ with respect to** a is given by the integral*

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - a} \tag{7}$$

This integral is an integer as it is the difference between two branches of log.

Theorem 4 (Cauchy's Integral Formula)**.** *If $f(z)$ is holomorphic in Ω, $a \in \Omega$ and $\gamma : S^1 \to \Omega$ is a closed curve with $a \notin \gamma$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - a} = w(\gamma, a)f(a) \tag{8}$$

Theorem 5 (Liouville's Theorem)**.** *A bounded holomorphic function on all of $\mathbb{C}$ is a constant.*

Definition 5 (Zero). *If f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, we can write, for some $k \in \mathbb{N}$,*

$$f(z) = (z - z_0)^k f_1(z) \tag{9}$$

*where$f_1(z)$ is nonvanishing near z_0. In this case k is called the **order** or **multiplicity** of the **zero** z_0*

Definition 6 (Meromorphic). *A function f is **meromorphic** on an open $\Omega \subseteq \mathbb{C}$ if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of Ω we can write $f(z) = g(z)/h(z)$ where g, h are holomorphic and h is not identically zero.*

Definition 7 (Laurent expansion). *Holomorphic functions in an annulus $r < |z| < R$ have a convergent **Laurent expansion** in an annulus*

$$\sum_{n=-\infty}^{\infty} a_n z^n = P(z) + R(z), \qquad P(z) = \sum_{n<0} a_n z^n, \qquad R(z) = \sum_{n\geqslant 0} a_n z^n \tag{10}$$

Theorem 6. *Every meromorphic function f on S^2 is rational.*

Definition 8 (Isolated singularity). *A holomorphic function in a punctured disk $0 < |z| < R$ has an **isolated singularity** at 0 if $f(z)$ cannot be extended to be holomorphic at 0.*

Theorem 7 (Weierstrass Theorem). *If 0 is an essential singularity, then for all $\epsilon > 0$, $f(\{0 \leqslant |z| \leqslant \epsilon\})$ is dense in $\mathbb{C}$.*

2 Useful Tools

- Projection from the Riemann Sphere:

$$\pi : S^2 \backslash \{N\} \to \mathbb{C}, \pi(x, y, t) = \frac{x + iy}{1 - t} \tag{11}$$

- Green's Formula:

  Theorem 8 (Green's formula).

$$\int_\gamma P dx + Q dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \tag{12}$$

- Schwarz Reflection Principle:

  Theorem 9 (Schwarz Reflection Principle). *If $f : H \to \mathbb{C}$ is continuous on the closed upper half-plane H, holomorphic on the open upper half-plane and takes real values on the real axis (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$) then it can be extended to an entire function by $f(\overline{z}) = \overline{f(z)}$. More generally, this can be applied to reflecting any half-domain over any line.*

- Fourier coefficients and Cauchy inequalities:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z} \implies f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \tag{13}$$

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{i\pi\theta}, \qquad a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta \tag{14}$$

$$M(r) = \sup_\theta |f(re^{i\theta})| \implies |a_n| \leqslant \frac{M(r)}{r^n} \tag{15}$$

- The Mean Value Property (MVP): *harmonic* functions satisfy

$$f(\text{center of disk}) = \text{ mean value on boundary} \tag{16}$$

- The Maximum Modulus Principle (MMP): if f is a continuous complex-valued function on an open $\Omega \subseteq \mathbb{C}$ with the MVP, then it satisfies the MMP, that is, if $|f|$ has a local maximum at a point a of Ω, then f is constant in a neighborhood of a.

- Schwarz's Lemma:

  Theorem 10 (Schwarz's Lemma). *Suppose $f(z)$ is holomorphic in $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$. Then*

  1. $|f(z)| \leqslant |z|$ *if* $|z| < 1$
  2. *If* $|f(z_0)| = |z_0|$ *at some* $z_0 \neq 0$, *then* $f(z) = \lambda z$ *for some* $|\lambda| = 1$.

3 Residues and Integrals

Definition 9 (Residue). *Let $f(z)$ be a holomorphic function in a punctured disc centered around a, and let γ be a closed curve lying entirely in the punctured disc (in particular, never touching a) with winding number $w(\gamma, a) = 1$. We define the **residue** of the differential form $f(z)dz$ (or "of f") at a to be*

$$\mathrm{Res}_a(f) = \frac{1}{2\pi i} \int_\gamma f(z)dz = a_{-1} \tag{17}$$

where a_n are the coefficients in the Laurent expansion of f at a. Note that this is independent of the choice of curve γ.

Definition 10 (Residue at ∞). *Writing $z = \frac{1}{z'}$, we have in coordinates at ∞*

$$f(z)dz = -\frac{1}{z'^2} f(1/z')dz' = g(z')dz' \tag{18}$$

We define

$$\mathrm{Res}_\infty(f) = \mathrm{Res}_0(g) = -a_{-1} \tag{19}$$

where a_n are the terms of the Laurent expansion in $|z| > R$.

Theorem 11 (Residue Theorem). *Let $\Omega \subset S^2$ be open and let $f(z)$ be holomorphic in Ω except perhaps on a discrete set of isolated points. Let Γ be the oriented (piecewise $\mathcal{C}^1$) boundary of a compact set $K \subset \Omega$ not containing any singularity (either essential singularities or poles). Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_a \mathrm{Res}_a(f) \tag{20}$$

where a ranges over the singularities contained in K, perhaps including ∞.

4 Elliptic Curves