Evaluation of integrals by residues on Riemann surface — March 25, 2020

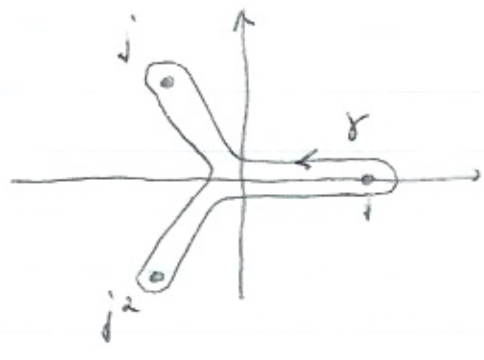
Example $\displaystyle\int_0^1 \frac{dx}{(1-x^3)^{1/3}}$

Consider Riemann surface of $y = (1-x^3)^{1/3}$ :

$X : \quad x^3 + y^3 = 1$ ;

branch points $1, j, j^2$ , $j = e^{2\pi i/3}$

There are 3 closed curves on $X$ with the image $\gamma$ since $(1-x^3)^{1/3}$ returns to the same branch as $x$ describes $\gamma$ :




Variation of argument of $(1-x^3)^{1/3}$ as $x$ describes $\gamma$ $= 2\pi \times$

$$\frac{1}{2\pi i}\int_\gamma \frac{d(1-x^3)^{1/3}}{(1-x^3)^{1/3}}$$

$x^3 + y^3 = 1$

$x^2\,dx + y^2\,dy = 0$

$\dfrac{dy}{y} = -\dfrac{x^2}{y^3}\,dx$

$$= -\frac{1}{2\pi i}\int_\gamma \frac{x^2\,dx}{1-x^3}$$

$$= -\left(\text{sum of residues of } \frac{x^2}{1-x^3} \text{ inside } \gamma\right)$$

$$= -3 \times \left(-1/3\right) = 1 \;;$$

e.g., residue at $x = 1$ is coeff of $1/t$ in

$$\frac{(1+t)^2}{1-(1+t)^3} = -\frac{(1+t)^2}{3t + \text{higher order}}$$

$$= -\frac{1}{3}$$

Let $I = \int_0^1 \frac{dx}{(1-x^3)^{1/3}}$

Consider following holom diff'l form $\omega$ on $X$:
Near $(x_0, y_0)$, $y_0 \neq 0$ : $\omega = \frac{dx}{y}$

Near $(x_0, y_0)$, $y_0 = 0$ : $\omega = -\frac{y\,dy}{x^2}$

i.e. $\omega$ is diff'l form $\frac{dx}{(1-x^3)^{1/3}}$ on $\mathbb{C}$,
made holomorphic by introducing Riem surface

Integral of $\omega$ around one of lifted curves:

$$\int_0^1 \frac{dx}{(1-x^3)^{1/3}} + j^2 \int_1^0$$
$$+ \int_0^1 + j^2 \int_1^0 + \int_0^1 + j^2 \int_1^0 = 3(1-j^2)I$$

(Factor $j^2$ in 2nd term is from argument or residue calculation above: argument changes by $2\pi/3$ going around the point 1. The third term is deduced from second by substitution, etc.)

$\omega$ has poles at $\infty$ ; we calculate the residues in coordinate $u$ at infinity:

$$x = \frac{1}{u} \qquad \frac{dx}{(1-x^3)^{1/3}} = -\frac{du}{u^2\left(1-\frac{1}{u^3}\right)^{1/3}}$$

$$= -\frac{du}{u(u^3-1)^{1/3}}$$

from one of the branches

$$= -\frac{du}{(-1)^{1/3}\,u(1-u^3)^{1/3}}$$

Above gives residue $(-1)^{2/3} = j$.
Residues of the other poles at $\infty$
are $1$, $j^2$.

Since $\gamma$ is negatively oriented with
respect to $\infty$,

$$3(1-j^2)\, I = -2\pi i \text{ (one of residues above)}$$

Let's try $j$:

$$3(1-j^2)\, I = -2\pi i\, j$$

Multiply both sides by $j^2$:

$$3\underbrace{(j^2-j)}_{-\sqrt{3}\, i}\, I = -2\pi i$$

$$I = \frac{2\pi}{3\sqrt{3}}$$

(This must be correct choice of residue
because $I$ is real, and other residues
would give preceding answer multiplied
by $j$ or $j^2$, both nonreal.)

## Riemann surface associated with an elliptic curve

Recall:

$$y^2 = 4x^3 - 20a_2 x - 28 a_4$$

Assume RHS $P(x)$ has 3 distinct roots, so equation defines nonsingular curve.

(Every nonsingular cubic has an equation of this form in suitable affine coordinates ~ Weierstrass normal form.)

$X: \ y^2 = P(x)$, Riemann surface over $\mathbb{C}$:

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[dr, "(x,y)\mapsto x"'] & \supset X \arrow[r, hook, "(x,y)\mapsto[x,y,1]"] \arrow[d, "\varphi"] & X' \subset \mathbb{P}^2(\mathbb{C}) \arrow[d, "\varphi'"] \\
& \mathbb{C} \arrow[r, hook] & S^2 = \mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

$$X': \quad y^2 t = 4x^3 - 20 a_2 x t^2 - 28 a_4 t^3$$

Single point at $\infty$: $[0, 1, 0]$

$\varphi' = \varphi$ on $X$

$\infty \mapsto$ point at $\infty$ of $S^2$

$dx/\sqrt{P(x)}$ lifts to holom diff'l form $\omega$ on $X$:

$\omega = \dfrac{dx}{y}$ near pt $(x_0, y_0)$ where $y_0 \neq 0$

$\omega = \dfrac{dy}{6x^2 - 10a_2}$ near $(x_0, 0)$

$$2y\,dy = (12x^2 - 20a_2)\,dx$$

$\omega$ has primitive in nbhd of each pt. of $X$.
Globally, primitive is a many-valued function
$z = z(x,y)$, holom in nbhd of each pt of $X$

$dz = \omega$ cf. $\omega = d\theta$, $\theta = \theta(x,y)$
$dx = y\, dz$ on $x^2 + y^2 = 1$

Each branch of $z$ in nbhd of each point $(x_0, y_0) \in X$ is a local coord:

$y_0 \neq 0$ : $x$ local coord
$y_0 = 0$ : $y$ " "

$\omega$ extends to holom diffl. form on compact curve $X'$

Review this for next time!