

# MAT454 Notes

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## Zeros and Poles

**Definition 1** (Zero). If  $f$  is holomorphic in a neighborhood of  $z_0 \in \mathbb{C}$  and  $f(z_0) = 0$ , we can write, for some  $k \in \mathbb{N}$ ,

$$f(z) = (z - z_0)^k f_1(z)$$

where  $f_1(z)$  is nonvanishing near  $z_0$ . In this case  $k$  is called the **order** or **multiplicity** of the **zero**  $z_0$

Zeros of holomorphic functions form a discrete set. We want to study, however, not only holomorphic functions, but also quotients of holomorphic functions

**Definition 2** (Meromorphic). A function  $f$  is **meromorphic** on an open  $\Omega \subseteq \mathbb{C}$  if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of  $\Omega$  we can write  $f(z) = g(z)/h(z)$  where  $g, h$  are holomorphic and  $h$  is not identically zero.

Why is it interesting to work with meromorphic and not just holomorphic functions? Essentially, it's because meromorphic functions in a domain  $\Omega$  form a field (whereas holomorphic functions only form a ring). Note that, in this course, when we say "domain", what we mean is a connected open set. If  $f(z), g(z)$  are holomorphic near  $z_0$ , like before, we can write

$$f(z) = (z - z_0)^k f_1(z), \quad g(z) = (z - z_0)^\ell g_1(z)$$

where  $f_1(z_0), g_1(z_0) \neq 0$ . Near  $z_0$ , then, the quotient looks like

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-\ell} \frac{f_1(z)}{g_1(z)}$$

So what are the different possibilities? If  $k \geq \ell$ , then this function extends to be holomorphic at  $z_0$ . On the other hand, if  $k < \ell$ , then, of course,

$$\lim_{z \rightarrow z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$$

Note: *not* undefined, but  $\infty$ . In this case, we say that  $z_0$  is a pole of order  $\ell - k$ . Holomorphic functions in an annulus  $r < |z| < R$  have a convergent Laurent expansion in an annulus

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n<0} a_n z^n + \sum_{n \geq 0} a_n z^n$$

Note that the LHS converges when  $r < |z|$ , whereas the RHS converges when  $|z| < R$ . This actually comes from Cauchy's theorem, just in the case of a convergent power series expansion of a holomorphic function. So this is from Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=-\infty}^{\infty} a_n z^n$$

for  $z$  between  $\gamma_1, \gamma_2$ . So

$$a_n = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\xi)}{\xi^{n+1}} d\xi$$

So of course, this should be just like before for the positive part. Note that this is the integral over  $\gamma_1$  if  $n \geq 0$  and over  $\gamma_2$  if  $n < 0$ .

Previously, we talked about holomorphic functions on the Riemann sphere, noting there were very few of them: namely, constants. Now, there are a few more meromorphic functions on the Riemann sphere, but not much. Specifically,

**Theorem 1.** *Every meromorphic function  $f$  on  $S^2$  is rational.*

*Proof.* This theorem uses what is probably the only thing in first year calculus you don't prove: the partial fraction decomposition. So we prove it now. Say  $f(z)$  has poles  $b_1, \dots, b_k$  (finite) and maybe  $\infty$ . So what can we say about the Laurent expansion at a pole? There's only finitely many negative terms, specifically, the order of the pole.

So these negative parts of the Laurent expansions around each  $b_j$  are like polynomials  $P_j(\frac{1}{z-b_j})$ . We'll call these principal parts. What about the principal part at  $\infty$ ? It's a polynomial in  $z$ , as it's a polynomial in  $\frac{1}{z'}$ , where  $z'$  is the coordinate at  $\infty$ , which is  $1/z$ . Call this  $P_\infty(z)$ . So we can write

$$f(z) - P_\infty(z) - \sum_{j=1}^k P_j\left(\frac{1}{z-b_j}\right)$$

which is holomorphic on  $S^2$ , and hence must be a constant  $a$ . So we can write

$$f(z) = a + P_\infty(z) + \sum_{j=1}^k P_j\left(\frac{1}{z-b_j}\right)$$

And that's rational. □

**Definition 3** (Isolated singularity). *A holomorphic function in a punctured disk  $0 < |z| < R$  has an **isolated singularity** at 0 if  $f(z)$  cannot be extended to be holomorphic at 0.*

Extension is possible if and only if  $f$  is bounded in a neighborhood of 0 in this punctured disk.