

MAT454 Academic Offense Sheet

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A quick collection of useful facts, theorems, and definitions for complex analysis. May be incorrect, and is certainly incomplete. Use at your own risk!

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1 Basic Definitions and Theorems

For $f = u + iv$ holomorphic, we have

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (1)$$

Definition 1. The **differential** of f is given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \quad (2)$$

$$dz = dx + idy, \quad d\bar{z} = dx - idy \iff dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = \frac{1}{2i}(dz - d\bar{z}) \quad (3)$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \quad (4)$$

Definition 2 (Harmonic). We say a real or complex valued function $f(x, y)$ is **harmonic** if f is \mathcal{C}^2 and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \quad (5)$$

Proposition 1. Every real-valued harmonic function is, not necessarily everywhere but at least locally, the real part of a holomorphic function.

Theorem 1. ω has a primitive in Ω if and only if, for any piecewise differentiable closed curve $\gamma : [a, b] \rightarrow \Omega$ (i.e. with $\gamma(a) = \gamma(b)$), or equivalently any piecewise differentiable $\gamma : S^1 \rightarrow \Omega$, we have

$$\int_{\gamma} \omega = 0 \quad (6)$$

Definition 3. We say a differential form ω on a domain Ω is **closed** if every point in Ω has a neighborhood in which ω has a primitive.

Theorem 2. Any closed differential form ω in a simply-connected open set Ω has a primitive.

Theorem 3 (Cauchy's Theorem). Let Ω be a domain and let $f(z)$ be continuous in Ω and holomorphic except on a set of discrete lines and points. Then the differentiable form $f(z)dz$ is closed.

Corollary 1. A holomorphic function $f(z)$ locally has a primitive, which is holomorphic (i.e. a function F such that $dF = f(z)dz$)

Corollary 2 (Morera's Theorem). If $f(z)$ is continuous in Ω and $df = f(z)dz$ is closed, then $f(z)$ is holomorphic.

Definition 4. Let $\gamma : S^1 \rightarrow \Omega$ be a closed curve and $a \notin \gamma(S^1)$ be a point not in the image of γ . Then the **winding number of γ with respect to a** is given by the integral

$$w(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} \quad (7)$$

This integral is an integer as it is the difference between two branches of \log .

Theorem 4 (Cauchy's Integral Formula). If $f(z)$ is holomorphic in Ω , $a \in \Omega$ and $\gamma : S^1 \rightarrow \Omega$ is a closed curve with $a \notin \gamma$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)dz}{z - a} = w(\gamma, a)f(a) \quad (8)$$

Theorem 5 (Liouville's Theorem). A bounded holomorphic function on all of \mathbb{C} is a constant.

2 Useful Tools

- Projection from the Riemann Sphere:

$$\pi : S^2 \setminus \{N\} \rightarrow \mathbb{C}, \pi(x, y, t) = \frac{x + iy}{1 - t} \quad (9)$$

- Green's Formula:

Theorem 6 (Green's formula).

$$\int_{\gamma} Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \quad (10)$$

- Schwarz Reflection Principle:

Theorem 7 (Schwarz Reflection Principle). *If $f : H \rightarrow \mathbb{C}$ is continuous on the closed upper half-plane H , holomorphic on the open upper half-plane and takes real values on the real axis (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$) then it can be extended to an entire function by $f(\bar{z}) = \overline{f(z)}$. More generally, this can be applied to reflecting any half-domain over any line.*

- Fourier coefficients and Cauchy inequalities:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z} \implies f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \quad (11)$$

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \quad a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta \quad (12)$$

$$M(r) = \sup_{\theta} |f(re^{i\theta})| \implies |a_n| \leq \frac{M(r)}{r^n} \quad (13)$$

- The Mean Value Property (MVP): *harmonic* functions satisfy

$$f(\text{center of disk}) = \text{mean value on boundary} \quad (14)$$

- The Maximum Modulus Principle (MMP): if f is a continuous complex-valued function on an open $\Omega \subseteq \mathbb{C}$ with the MVP, then it satisfies the MMP, that is, if $|f|$ has a local maximum at a point a of Ω , then f is constant in a neighborhood of a .

- Schwarz's Lemma:

Theorem 8 (Schwarz's Lemma). *Suppose $f(z)$ is holomorphic in $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$. Then*

1. $|f(z)| \leq |z|$ if $|z| < 1$
2. If $|f(z_0)| = |z_0|$ at some $z_0 \neq 0$, then $f(z) = \lambda z$ for some $|\lambda| = 1$.

3 Residues and Integrals

4 Elliptic Curves