MAT454 Notes

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March 2 2020

The Conformal Mapping Problem

Let f be a holomorphism, and assume $f'(z_0) = 0$. Then f^{-1} exists in a neighborhood of $f(z)$, and f is **conformal** at z_0 (preserves angles and their orientations). A nonconstant holomorphic mapping $f : \Omega \to \mathbb{C}$ is **open**, if it is one to one, then f is a homeomorphism onto its image $f(\Omega)$, and f^{-1} is a holomorphism.

Definition 1. *A* ***conformal*** *or* ***biholomorphic*** *mapping $f : \Omega \to \Omega'$ is a holomorphic mapping with aholomorphic inverse.*

We are now faced with the **conformal mapping problem**:

- Given domains $\Omega, \Omega' \subset \mathbb{C}$, are they biholomorphic?
- If so, can we find all biholomorphisms?

We note that, for $f, g : \Omega \to \Omega'$, f, g are biholomorphisms if and only if $g^{-1} \circ f \in \operatorname{Aut} \Omega$, the group of biholomorphisms of Ω with itself. Furthermore, f induces a conjugation map

$$\operatorname{Aut} \Omega \to \operatorname{Aut} \Omega', \quad S \mapsto f \circ S \circ f^{-1}$$

Now let's consider some examples, starting with the complex plane itself. We have that

$$\operatorname{Aut} \mathbb{C} = \{\text{linear transformations } w = az + b, \quad a \neq 0\}$$

Suppose $w = f(z) \in \operatorname{Aut} \mathbb{C}$. At ∞, f has either an essential singularity or a pole. But we can show we don't have an essential singularity. On the other hand, what about when f is a polynomial, say of degree n, then that must mean it is not one-to-one, because $f(z) = w$ has n distinct roots for almost every value of w, except at roots of the derivative $w = f(z), f'(z) = 0$. So $n = 1$.

What about the Riemann sphere, $\operatorname{Aut} S^2$. What should this group of biholomorphisms look like? Wec consider fractional linear transformations

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

These coefficient, of course, are not uniquely determined, being only determined up to a constant. The inverse of a fractional linear transformation is that given by the inverse matrix, namely

$$\frac{dz - b}{-cz + a}$$

(uniquely determined up to a constant, so we don't have to write the $\frac{1}{ad-bc}$).

Lemma 1. *Suppose G is a subgroup of $\operatorname{Aut} \Omega$ such that G is transitive on Ω and for some z_0 the subgroup of automorphisms which fix this point lies inside of G. Then $G = \operatorname{Aut} \Omega$.*

Proof. Let $S \in \operatorname{Aut} \Omega$ be arbitrary. We have to show that $S \in G$. To do so, we note that since G acts transitively, we can take $T \in G$ such that $T(z_0) = S(z_0)$. There exists such a T because it acts transitively. Then of course we can write

$$S = T \circ (T^{-1} \circ S), \quad (T^{-1} \circ S)(z_0) = z_0 \implies T^{-1} \circ S \in G \implies S \in G$$

being the composition of elements of G. ∎

So that's what we've shown here: the subgroup of automorphisms fixing ∞ lies in this subgroup of fractional linear transformations, and the subgroup is transitive, and hence it composes the entire automorphism group $\operatorname{Aut} S^2$.