MAT454 Notes

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Definition 1 (n-dimensional complex projective space)**.** *We define*

$$\mathrm{P}^n(\mathbb{C}) = \mathbb{C}^{n+1}\backslash\{0\}/\sim$$

where

$$(x_0, ..., x_n) \sim (x_0', ..., x_n') \iff \exists\lambda \in \mathbb{C}, (x_0', ..., x_n') = (\lambda x_0, ..., \lambda x_n)$$

We denote the equivalence class of $(x_0, ..., x_n)$ *by* $[x_0, ..., x_n]$.

Definition 2 (Homogeneous coordinates)**.** *We define coordinate charts* $U_i = \{[x_0, ..., x_n] \in \mathrm{P}^n(\mathbb{C}) : x_i \neq 0\}$ *with affine coordinates* $U_i \to \mathbb{C}^n$*,*

$$[x_0, ..., x_n] \mapsto \left(\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right)$$

with inverse

$$(g_1, ..., g_n) \mapsto [g_1, ..., g_{i-1}, 1, g_{i+1}, .., g_n]$$

Using these coordinates, we have that $\mathrm{P}^n(\mathbb{C})$ has the structure of an n-dimensional complex manifold, as the transition mappings are rational. Let's take one of the charts here, say U_0, to be $\mathbb{C}^n$. So

$$\mathrm{P}^n(\mathbb{C}) = U_0 \cup \text{ everything else}$$

But what's everything else? So U_0 is all the points where $x_0 \neq 0$, so everything else is the set of points

$$\{x_0 = 0\} = \{[0, x_1, ..., x_n]\} \simeq \mathrm{P}^{n-1}(\mathbb{C}) \implies \mathrm{P}^n(C) = U_0 \cup \mathrm{P}^{n-1}(\mathbb{C})$$

We call this copy of $\mathrm{P}^{n-1}(\mathbb{C}) \simeq \{x_0 = 0\}$ the **hyperplane at infinity**. This is like a generation of the Riemann sphere which we saw before, which we saw was given by $S^2 = \mathrm{P}^1(\mathbb{C})$. So when we talk about $\mathrm{P}^2(\mathbb{C})$, that's like having 2-complex coordinates with a line at infinity. Specifically, we can write it as

$$\mathrm{P}^2(\mathbb{C}) = \{[x, y, t]\} = \mathbb{C}^2_{(x,y)} \cup \{t = 0\}$$

the **projective line at infinity**. Now assume we have a curve $X \subset \mathbb{C}^2$ generated by the equation

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

where the RHS has three distinct roots. We want to compute the **compactification of X in** $\mathrm{P}^2(\mathbb{C})$. We can write this down in homogeneous coordinates

$$y^2t = 4x^3 - 20a^2xt^2 - 28a_4t^3$$

taking X' to be the solution set of this. Why is this the right thing? When you look at $\mathrm{P}^2(\mathbb{C})$, and look in here at the set of points

$$\{[x, y, t] : t \neq 0\} \simeq \mathbb{C}^2_{(x,y)}$$

we see that it is has homomorphism

$$[x, y, t] \mapsto \left(\frac{x}{t}, \frac{y}{t}\right)$$

Hence, we rewrite our eqaution in our new coordinates for $\mathbb{C}^2$,

$$\frac{y^2}{t^2} = 4\frac{x^3}{t^3} - 20a_2\frac{x}{t} - 28a_4$$

Now we can just multiply both sides by t^3. So if you haven't seen this before, this takes a little bit of familiarity, but the actual operations involved are very simple operations. Of course, our *original* X is a subspace of X'. But how much have we added to X? Well, if we set $t = 0$, we get $x = 0$. So, how many points are we adding? One point, at ∞:

$$X' = X \cup \{[0, 1, 0]\}$$