3. Normal families

$\Omega \subset \mathbb{C}$ open

$\mathcal{C}(\Omega)$ metrizable

$\mathcal{H}(\Omega)$ closed subsp, ind metric

$\Omega = \bigcup E_i$ closed disks

$\frac{a+b}{1+(a+b)} \le \frac{a}{1+a} + \frac{b}{1+b}$

$$d(f) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{z \in E_i} \frac{|f(z)|}{1+|f(z)|}$$

$d(f,g) = d(f-g)$

Metric space compact iff every infinite sequence has cgt (inf.) subseq.

$\mathcal{S} \subset \mathcal{C}(\Omega)$ <u>normal family</u> if every sequence in $\mathcal{S}$ has subseq that converges in $\mathcal{C}(\Omega)$
(i.e. converges unif on comp. sets

e.g. $\mathcal{S} = \{z^n\}$ normal on D but limit not in $\mathcal{S}$

$\{z^n\}$ cges unif on comp subsets of D, but not on D

$\{g_n = \begin{smallmatrix} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{smallmatrix}\}$ normal but doesn't converge

$\mathcal{S}$ compact iff normal and limit fns themselves in $\mathcal{S}$

$\mathcal{S}$ normal $\Rightarrow$ $\bar{\mathcal{S}}$ normal (exercise)

So:

<u>Lemma</u> $\mathcal{S} \subset \mathcal{C}(\Omega)$ normal iff $\bar{\mathcal{S}}$ compact

<u>Lemma</u> $\Omega = \bigcup_i E_i$ union of closed disks

$\mathcal{S} \subset \mathcal{C}(\Omega)$ normal iff, for every i, every seq in $\mathcal{S}$ contains subseq that cges unif on E_i.

<u>Proof</u>

"Only if" by defn

"If" Suppose $\{f_n\} \subset \mathcal{S}$.

$\exists$ subseq $\{f_n^{(1)}\} \subset \{f_n\}$ that cges unif on E_1

$\{f_n^{(2)}\} \subset \{f_n^{(1)}\}$ " " " " E_2

. . .

$\{f_n^{(n)}\} \subset \{f_n^{(n-1)}\}$ " " " " E_n

Diagonal seq. $\{f_n^{(n)}\}$ cges unif on each E_n

$\therefore$ unif on any compact $K \subset \Omega$. $\square$

$X \subset \mathbb{C}$, $\quad \mathcal{S} \subset \mathcal{C}(X)$

$\mathcal{S}$ <u>equicontin at</u> $a \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$
s.t. if $z \in X$, $|z-a| < \delta$, then
$|f(z) - f(a)| < \epsilon$, $\forall f \in \mathcal{S}$

$\mathcal{S}$ <u>equicontin</u> (on X) if equicontin at each $a \in X$
<u>unif equicont</u> (on X) if $\forall \epsilon > 0$, $\exists \delta > 0$
s.t. if $z, w \in X$, $|z-w| < \delta$, then
$|f(z) - f(w)| < \epsilon$, $\forall f \in \mathcal{S}$

e.g. $\mathcal{S} \subset \mathcal{C}(D)$ holom fns f on open disk D

s.t. $|f'| \leq M < \infty$ on D
$|f(z) - f(w)| \leq M|z-w|$, $z, w \in D$
given ϵ, take $\delta = \epsilon/M$ unif equicontin on D

Family of fns that is equicontin on compact set is unif. equicontin.

<u>Arzela-Ascoli Thm</u>. $\Omega \subset \mathbb{C}$ <u>domain</u>
$\mathcal{S} \subset \mathcal{C}(\Omega)$ normal iff
(1) $\mathcal{S}$ equicontin on Ω
(2) $\exists$ $z_0 \in \Omega$ s.t. $\{f(z_0) : f \in \mathcal{S}\}$ bounded subset of $\mathbb{C}$
(In proof we show $\mathcal{S}$ normal $\Rightarrow$ (2) holds at every pt.)
Here we cons families of contin fns with vals in $\mathbb{C}$
$|f(z) - f(w)|$ dist bet. 2 values in $\mathbb{C}$
Arzela-Ascoli thm holds for families of contin fns with values in complete metric space
e.g. Riemann sphere S^2 with chordal metric!

For families of holom fns, Arzela-Ascoli and Cauchy ings give criterion for normality

$\mathcal{S} \subset C(\Omega)$ locally bounded on Ω
if, $\forall z_0 \in \Omega$, $\exists \delta > 0$, $M < \infty$ s.t.
$|f(z)| \le M$, $|z - z_0| < \delta$, $f \in \mathcal{S}$ $\quad M = M(z_0)$

$\iff$ unif bounded on compact subsets of Ω:
$\forall K \subset \Omega$ compact, $\exists M = M(K) < \infty$ s.t.
$|f(z)| \le M$, $z \in K$, $f \in \mathcal{S}$.

Theorem. $\mathcal{S} \subset \mathcal{H}(\Omega)$, $\Omega \subset \mathbb{C}$ domain. (Montel)
Foll equiv:
(1) $\mathcal{S}$ normal
(2) $\mathcal{S}$ locally bounded
(3) $\mathcal{S}' = \{f' : f \in \mathcal{S}\}$ locally bounded
and $\exists z_0 \in \Omega$ s.t. $\{f(z_0) : f \in \mathcal{S}\}$ bd'd in $\mathbb{C}$

Cor. $\mathcal{S} \subset \mathcal{H}(\Omega)$ compact iff closed and loc bd'd

~~Proof~~.
(1) $\Rightarrow$ (2) Suppose $\mathcal{S}$ normal
$\forall w \in \Omega$ $\{f(w) : f \in \mathcal{S}\}$ bd'd (by Arz-Asc)
Hence $\mathcal{S}$ equicontin, then $\mathcal{S}$ loc bd'd (")
$|f(z) - f(w)| < 1$ on disc centre w, $\forall f \in \mathcal{S}$
(2) $\Rightarrow$ (3) Suppose $\mathcal{S}$ loc. bd'd.
Given $z_0 \in \Omega$, $\exists r > 0$, $M < \infty$
s.t. $|f(z)| \le M$ on closed disk centre z_0, radius r in Ω
Then $|f'(z)| \le \frac{2M}{r}$ ~~$\frac{4M}{r^2}$~~ on $B(z_0, r/2)$;
i.e. $\mathcal{S}'$ loc. bd'd.
(3) $\Rightarrow$ (1)
Given $z_0 \in \Omega$, $|f'(z)| \le M < \infty$ in disk E centre z_0.
By integration along line seg,
$|f(z) - f(z_0)| \le M|z - z_0|$, $\forall f \in \mathcal{S}$
$\therefore$ $\mathcal{S}$ equicontin at z_0; $\therefore$ normal by Arz-Asc $\square$

<u>Proof of Arzela-Ascoli</u>

Suppose $\mathcal{F} \subset C(\Omega)$ normal. Then:

(2) holds for all $z_0 \in \Omega$:

Otherwise, $\exists$ $z_0 \in \Omega$, $\{f_n\} \subset \mathcal{F}$ s.t. $|f_n(z_0)| \to \infty$.

But normality implies there is subseq of $\{f_n\}$ which converges at z_0; contra.

(1) holds:

Otherwise, $\exists$ $z_0 \in \Omega$ s.t. $\mathcal{F}$ not equicont at z_0 i.e. $\exists$ $\epsilon > 0$, $z_n \in \Omega$, $f_n \in \mathcal{F}$ s.t.

(*) $|z_n - z_0| < 1/n$, $|f_n(z_n) - f_n(z_0)| \geq \epsilon$, for $n \geq n_0$.

By normality, $\{f_n\}$ contains subseq which converges unif to contin fn. f on $|z - z_0| \leq 1/n_0$.

By passing to this subseq and relabelling pts z_n and seq f_n, we can suppose (*) holds and f_n goes unif to f on $|z - z_0| \leq 1/n_0$

$$\epsilon \leq |f_n(z_n) - f_n(z_0)|$$
$$\leq |f_n(z_n) - f(z_n)| + |f(z_n) - f(z_0)| + |f(z_0) - f_n(z_0)|$$

For n large enough

1st, 3rd terms $< \epsilon/3$ by unif gce.

2nd term $< \epsilon/3$ by contin of f; contra.

Now suppose (1), (2) hold

Then (2) holds at every $z \in \Omega$:
By equicontin, each $w \in \Omega$ lies in open disk $D_w \subset \Omega$ s.t.
$|f(z) - f(w)| < 1, \quad z \in D_w, \quad f \in \mathcal{F}$

Let $U = \{z_0 \in \Omega : (2) \text{ holds}\}$
By above if $w \in U$, then there is disk $D_w \subset U$, so U open
Also by above, if $w \in \Omega \setminus U$, then $D_w \subset \Omega \setminus U$, so U closed.
$U \neq \emptyset$, by (2).
Since Ω conn, $U = \Omega$.

Suppose $\{f_n\} \subset \mathcal{F}$
Let $T = \{z_k\}$ be countable dense subset of Ω

Since $\{f(z_1) : f \in \mathcal{F}\}$ bd'd,
$\exists$ subsq $\{f_n^{(1)}\}$ of $\{f_n\}$ s.t. $\{f_n^{(1)}(z_1)\}$ cges
Likewise,
" " $\{f_n^{(2)}\}$ of $\{f_n^{(1)}\}$ " $\{f_n^{(2)}(z_2)\}$ "
...
" " $\{f_n^{(k)}\}$ of $\{f_n^{(k-1)}\}$ " $\{f_n^{(k)}(z_k)\}$ "
$\therefore$ $\underbrace{\{f_n^{(n)}\}}_{\text{relabel as } \{f_n\}}$ converge at z_k, $\forall k$

Enough to show: For any closed disk $E \subset \Omega$.
$\{f_n\}$ ~~has subsq that~~ cges unif on E.

Let $\epsilon > 0$. We'll find M s.t. if $p, q > M$, then $|f_p(z) - f_q(z)| < \epsilon$ on E.

By (1), $\mathcal{S}$ unif equicontin on ~~...~~ E.
$\heartsuit$ So $\exists\ \delta > 0$ s.t. if $z, w \in E$, $|z - w| < \delta$,
then $|f(z) - f(w)| < \epsilon/3$, $f \in \mathcal{S}$

Choose finite subset $z_{k_1}, \dots, z_{k_N}$ of $T \cap E$
s.t. every $z \in E$ lies in $B(z_{k_j}, \delta)$, some j

$\diamondsuit$ Take $M < \infty$ s.t., $\forall\ p, q \geq M$, $1 \leq j \leq N$,
$|f_p(z_{k_j}) - f_q(z_{k_j})| < \epsilon/3$

Given $z \in E$, choose z_{k_j} s.t. $z \in B(z_{k_j}, \delta)$
Then, if $p, q \geq M$,

$$|f_p(z) - f_q(z)| \leq |f_p(z) - f_p(z_{k_j})| + |f_p(z_{k_j}) - f_q(z_{k_j})| + |f_q(z_{k_j}) - f_q(z)|$$

1st & 3 terms $< \epsilon/3$ by $\heartsuit$
2nd " $< \epsilon/3$ by $\diamondsuit$
as req'd.

Since $\epsilon > 0$ arb., $\{f_n\}$ Cauchy sequence (in unif top)
Since $\mathbb{C}$ complete, $\{f_n\}$ cges (unif) on E.
By Lemma p. 3-1, $\mathcal{S}$ normal $\square$

Arzela-Ascoli Thm holds for families of contin fns with values in complete metric space
e.g. contin fns with values in Riem sphere S^2 (or extended plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$) with (induced) chordal metric (Probs 1, # 2)

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}} \qquad d(z, w) = d\left(\frac{1}{z}, \frac{1}{w}\right)$$

f contin at z_0 with $f(z_0) = \infty$
means : $\forall\ R < \infty\ \ \exists\ \delta > 0$
s.t., if $|z - z_0| < \delta$, then $|f(z)| > R$

(Topology induced on $\mathbb{C}$ by chordal metric is usual Euclid. top)

Family $\mathcal{F}$ of contin fns on Ω is normal in chordal metric iff equicontin in chord. metric
(Condn (ii) of Arz-Asc. thm not needed because $\mathbb{C}^*$ or S^2 compact in this top)

E.g. We are interested in family $\mathcal{F}$ of merom fns on $\Omega \subset \mathbb{C}$ (or $\Omega \subset S^2$) (holom fns w values in S^2)

<u>Lemma</u> $\{f_n\}$ seq of merom fns which converge unif on comp subsets of domain $\Omega \subset \mathbb{C}$ (or S^2) (in chordal metric)
Then limit fn (either) merom (or id $= \infty$)

<u>Proof</u>
Chordal metric $d(z, w) = d(1/z, 1/w)$ ♤
Limit fn. f contin as map to S^2

If $|f(z_0)| < \infty$, then f bd'd in nbd of z_0, $f_n \to f$ unif (in Euc metric) in nbd of z_0
so f holom in nbd of z_0

If $f(z_0) = \infty$, $1/f_n$ bd'd in nbd of z_0 n large enough ; $1/f$ anal in nbd of z_0

If $1/f$ not id zero in nbd of z_0
then zero at z_0 isolated ;
so f has pole at z_0 (isol)

$\therefore$ set of non isolated zeros of $1/f$
open & closed in Ω
Since Ω conn, f either id. $= \infty$,
or f anal except for poles $\square$

Does thm (p. 3-3) on characterization of normal families have analogues for merom fns in chordal metric ?
(1) $\Leftarrow$ (2) No : all contin fns bounded by 2 in chordal metric
(1) $\Leftarrow$ (3) has analogue using <u>spherical deriv</u>

<u>Defn.</u> <u>Spherical deriv</u>
f merom on domain $\Omega \subset \mathbb{C}$ (or S^2)

$$f^{\#}(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{|z - w|}$$

Would have to be more careful in this case: are $d(z,w)$ in denom too! (+)

Same using Leacless metric!

If z not a pole

$$f^{\#}(z) = \lim_{w \to z} \frac{2|f(z) - f(w)|}{|z-w|\sqrt{(1+|f(z)|^2)(1+|f(w)|^2)}}$$

$$= \frac{2|f'(z)|}{1 + |f(z)|^2}$$

By ♤, $(1/f)^{\#} = f^{\#}$ (+), so that $f^{\#}(z)$
is finite and contin. at all $z \in \Omega$
(and > 0 at z iff f 1-1 near z)

Marty's thm

$\mathcal{F}$ family of merom fns on domain Ω

$\mathcal{F}$ normal in chordal metric

iff $\mathcal{F}^\# = \{ f^\# : f \in \mathcal{F} \}$ loc. bdd

Proof

Suppose $\mathcal{F}$ normal in chordal metric.

Suppose spherical derivs not bdd in any nbd of pt z_0

$\exists\ f_n \in \mathcal{F}$, $z_n \to z_0$ s.t. $f_n^\#(z_n) \to \infty$

By normality, can assume f_n converges unif on comp subsets of Ω, in chordal metric

Limit f either merom or ∞ (by Lemma)

If $f(z_0) \neq \infty$:

f bdd in Euc. metric in nbd U of z_0

Since $f_n \to f$ in chordal metric,

f_n also bdd in U (n large enough)

so $f_n \to f$ unif on comp sub in Euc metric on U

$\therefore$ $f_n' \to f'$ unif on comp subsets of U

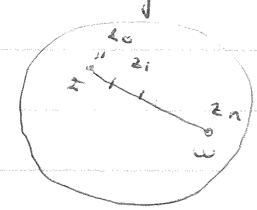
$f_n^\# \to f^\#$ " " " " "

Contra to $f_n^\#(z_n)$ unbdd.

If $f(z_0) = \infty$: Apply same argument to $1/f$, $1/f_n$.

Converse: Suppose spherical derivs bdd by M in disk $D \subset \Omega$

If $z, w \in D$, set $z_j = z + \frac{j}{n}(w - z)$, $0 \le j \le n$



n large:

$d(f(z), f(w)) \le \sum_{j=1}^{n} d(f(z_{j-1}), f(z_j))$

$f \in \mathcal{F}$ $\quad \approx \sum_{j=1}^{n} f^\#(z_j)\,|z_j - z_{j-1}| \le M|z - w|$

$\therefore$ $\mathcal{S}$ equicontin wrt chordal metric on D

By Arzela-Ascoli for chordal metric,
$\mathcal{S}$ normal on D

So by (chordal version of) Lemma, p. 3-1,
$\mathcal{S}$ normal on Ω. $\square$