$\therefore$ $\mathcal{S}$ equicontin wrt chordal metric on D

By Arzela - Ascoli for chordal metric,
$\mathcal{S}$ normal on D
So by (chordal version of) Lemma, p. 3-1,
$\mathcal{S}$ normal on Ω. $\square$

THEOREMS OF MONTEL AND PICARD

Picard's Big Thm If $f(z)$ holom fn. with ess. sing. at z_0, then $\exists\ \lambda \in \mathbb{C}$ s.t. in any nbd of z_0, f assumes every value in $\mathbb{C} \setminus \{\lambda\}$ infinitely many times

i.e. holom fn omits at most one cx. value in any nbd of ess sing.

It may omit one; e.g. $e^{1/z}$ omits value 0 in every nbd of ess sing 0.

Picard's little thm Nonconst entire fn. omits at most one cx. value

Much stronger than Liouville's thm

Follows from Big Picard:
Either pole at ∞: poly, so takes every val
or ess sing at ∞ ...

Zalcman's lemma.

$\mathcal{S}$ family of merom fns on domain Ω

$\mathcal{S}$ is <u>not</u> normal in chordal metric

iff $\exists$ sequence $a_n \to a_\infty$ in Ω

seq of pos. nos. $\rho_n \to 0$

seq $\{f_n\} \subset \mathcal{S}$

s.t. $g_n(z) := f_n(a_n + \rho_n z)$ converges unif in chordal metric on comp. subsets of $\cancel{\Omega}$ $\mathbb{C}$ to nonconst fn. $g(z)$ merom on all of $\mathbb{C}$

Moreover, if $\mathcal{S}$ not normal, then we can choose data above s.t.

$$g^{\#}(z) \leq g^{\#}(0) = 1, \qquad \forall z \in \mathbb{C}$$

<u>Rmk.</u> Remarkable that we can describe non-normality in terms of a cgt. sequence

If $\{f_n\}$ were itself cgt., then $\{g_n\}$ would converge to const on compact subsets of $\mathbb{C}$, since radius $\rho_n \to 0$

Example

$\mathcal{S} = \{\underset{f_n}{\underset{\shortparallel}{z^n}}\}$ normal on D and on $\mathbb{C} \setminus \bar{D}$ (chordal metric)

(by Marty's thm or char of normality in Euc. metric)

but not on $\{z : |z| < 2\}$:

unif cge fails at every pt of S^1

Set $a_n = 1$, $\rho_n = 1/n$

$\forall z \in \mathbb{C}$, $f_n(a_n + \rho_n z) = \left(1 + \frac{z}{n}\right)^n \to e^z$ as $n \to \infty$

i.e. $\log(1 + z/n)^n \to z$

$$\frac{n \log(1 + z/n)}{z} = \frac{\log(1 + z/n) - 0}{z/n} \to 1$$

diff quot for deriv of $\log(1+z)$ at 0

$g(z) = e^z \qquad g^\#(z) = \dfrac{2|g'(z)|}{1+|g(z)|^2} = \dfrac{2|e^z|}{1+|e^z|^2}$

$g^\#(0) = 1 \qquad g^\#(z) \le 1 \qquad \dfrac{2t}{1+t^2} \le 1 \qquad 2t \le 1+t^2$ ✓

Proof of Z's lemma.

Suppose $\mathcal{F}$ normal

Any sq $\{f_n\} \subset \mathcal{F}$ has cgt subseq which we can relabel $\{f_n\}$:

$f_n \to f$, say

For any $a_n \to a_\infty \in \Omega$ and $\rho_n \to 0$,

$g_n(z) := f_n(a_n + \rho_n z) \to f(a_\infty)$ as $n \to \infty$

$\forall z \in \mathbb{C}$, since $\{f_n\}$ unif equicontin in nbd of a_∞ (Arzela-Ascoli)

$\therefore\ g = \lim g_n$ const.

Suppose $\mathcal{F}$ not normal — i.e. $\mathcal{F}^\# = \{f^\# : f \in \mathcal{F}\}$ not loc. bd'd (Marty)

so $\exists\ b_n \to b_\infty \in \Omega$

$f_n \in \mathcal{F}$

s.t. $f_n^\#(b_n) \to \infty$

Can assume $b_\infty = 0$, $\{|\zeta| \le r\} \subset \Omega$

$M_n := \max\limits_{|\zeta| \le r} (r - |\zeta|) f_n^\#(\zeta) = (r - |a_n|) f_n^\#(a_n)$

for some a_n, $|a_n| < r$,

since $f_n^\#$ contin

$M_n \to \infty$ since $b_n \to 0$

Then $g_n(z) := f_n\left(a_n + \dfrac{z}{f_n^\#(a_n)}\right)$ def'd on $|z| \le M_n$

(On $|z| \le M_n$, $\left|a_n + \dfrac{z}{f_n^\#(a_n)}\right| \le |a_n| + \dfrac{M_n}{f_n^\#(a_n)} = |a_n| + r - |a_n| = r$)

Fix $R < \infty$

If $|z| \le R < M_n$, then

$$g_n^\#(z) = \frac{f_n^\#(a_n + z/f_n^\#(a_n))}{f_n^\#(a_n)}$$

$$\le \frac{M_n}{r - |a_n + z/f_n^\#(a_n)|} \cdot \frac{r - |a_n|}{M_n}$$

$$\le \frac{r - |a_n|}{r - |a_n| - |z|/f_n^\#(a_n)} \qquad |a_n + z/f_n^\#(a_n)| \le |a_n| + |z|/f_n^\#(a_n)$$

$$= \frac{1}{1 - |z|/M_n} \longrightarrow 1 \quad \text{as } n \to \infty$$

By Marty's thm. $\{g_n\}$ contains cgt subseq in chordal metric ; relabelling subseq, we have

$$g_n(z) = f_n(a_n + \rho_n z), \qquad \rho_n = \frac{1}{f_n^\#(a_n)}$$

By above, $g^\#(z) \le g^\#(0) = 1$

g is merom by Lemma, p. 3-7

not const since $g^\#(0) = 1$ — (also not ∞)

Since $\{a_n\} \subset$ compact subset of Ω

we can assume $a_n \longrightarrow a_\infty \in \Omega$ $\square$

Montel's theorem — Improvement of loc. bdd. cond'n of normality

Family $\mathcal{F}$ of merom fns on domain Ω which omits 3 distinct values $a, b, c \in \mathbb{C}^*$ is normal in chordal metric

Proof.

We can assume $\Omega = D$ (∵ normality local cond'n: can be checked on closed disks)

Can assume $a = 0,\ b = 1,\ c = \infty$ by composing with fract. lin transf (which is unif contin in chordal metric)

Therefore we can assume $\mathcal{F}$ is family of all holom fns on D which omit vals $0, 1$.

Set $\mathcal{F}_m = \{ f \in \mathcal{H}(D) :$ f omits values 0 and $e^{2\pi i k/2^m}$ $k = 1, \dots, 2^m \}$

$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \dots$

If $f \in \mathcal{F}_m$, then f doesn't vanish, so has holom square root $f^{1/2}$; $f^{1/2} \in \mathcal{F}_{m+1}$ In part $\mathcal{F}_m \neq \emptyset$

Suppose $\mathcal{F}$ not normal
$\exists$ seq $\{f_n\} \subset \mathcal{F}$ with no cgt subseq
Then $\{f_n^{1/2}\} \subset \mathcal{F}_1$ " " " "
. . .

By induction, each $\mathcal{F}_m$ not normal

For each m, we can const. limit fn. $\underline{h_m}$ " g " as in Zalcman's lemma non const
Moreover, each h_m entire (by exercise below)
~~and nonconst. ... $h_m^\#$ ...~~
/ (hence $h_m^\#(z) \le h_m^\#(0) = 1$)
By Marty's thm, $\{h_m\}$ normal family

If h is limit of a subseq (unif on comp subsets of $\mathbb{C}$ in chordal metric) then h entire by exercise, non const $\because$ $h^\#(0) = 1$

By Hurwitz, h omits 2^m-roots of unity for each m : all these are dense in S^1.

Since $h(\mathbb{C})$ connected and open,
either $h(\mathbb{C}) \subset D$: h bdd
or $h(\mathbb{C}) \subset \mathbb{C} \setminus D$: $1/h$ bdd
By Liouville, h = const ;
contra to $h^{\#}(0) = 1$.
$\therefore$ $\mathcal{S}$ is normal $\square$

<u>Exercise</u> $\{f_n\}$ seq of holom fns on domain $\Omega \subset \mathbb{C}$,
which converges unif in chordal metric
on compact subsets of Ω
Then limit either holom or $\equiv \infty$ (Hint: Hurwitz)
Moreover, if limit holom, then
convergence is unif in Euc metric on
compact subsets of Ω.

<u>Remark</u>
Families $\mathcal{S}_m$ in proof above are not closed.
(Const. fns 0, 1, ∞ are in closure)
e.g. $\left(\frac{1+z}{2}\right)^n$ omits 0, 1 in D,
$\longrightarrow 0$ in D as $n \to \infty$
However, in pf of Montel, Zalc's lemma
yields limit fns that are not const,
$\therefore$ in family $\mathcal{S}_m$ by Hurwitz.

<u>Picard's big theorem</u>

If f merom in punctured disk $\{0 < |z - z_0| < \delta\}$ and f omits 3 distinct values in $\mathbb{C}^*$ then f extends to merom fn in $|z - z_0| < \delta$.

Equiv statement: Holom fn omits at most one cx val in nbd of ess. sing.

Thm $\Rightarrow$ Equiv statement:

Have f merom in punctured disk, omits ∞ so can't omit 2 values $\neq \infty$ since doesn't extend to merom at z_0

Equiv statement $\Rightarrow$ Thm:

By fract lin transf, can ass f omits ∞ and 2 other vals, so z_0 not ess sing.

<u>Proof</u>

Can assume $z_0 = 0$, f omits vals $0, 1, \infty$.

Take $\epsilon_n \to 0$

~~$\mathcal{S} = \{f(\epsilon_n z)\}$ normal family, on Ω in chordal metric [illegible] on compact subsets of [illegible]~~

$\mathcal{S} = \{f(\epsilon_n z)\}$ normal family on Ω in chordal metric by Montel. $\Omega = \{\frac{1}{2} < |z| < 2\}$

Relabelling a subseq, can assume $f(\epsilon_n z)$ converges unif on comp subset of Ω to holom fn. g on Ω (or to $g = \infty$) (~~limit merom~~ by Exercise, or id $= \infty$)

If g holom:

$|g(z)| \leq M < \infty$ on $|z| = 1$

$\therefore |f(z)| \leq M + 1$ on $|z| = \epsilon_n$,

for $n \geq n_0$.

By max prine,

$|f| \leq M+1$ on $\epsilon_{n+1} < |z| < \epsilon_n$, $n \geq n_0$

$|f| \leq M+1$ on $0 < |z| < \epsilon_{n_0}$

so f extends to be holom at 0.

If $g \equiv \infty$, apply similar argument to $1/f(\epsilon_n z)$ to conclude $1/f$ extends to be holom at 0;

i.e. f merom in nbd of 0. $\square$