MAT454 Academic Offense Sheet

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A quick collection of useful facts, theorems, and definitions for complex analysis. May be incorrect, and is certainly incomplete. Use at your own risk!

Contents

1 Basic Definitions and Theorems

For $f = u + iv$ holomorphic, we have

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

Definition 1. *The **differential** of f is given by*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{2}$$

$$dz = dx + idy, \qquad d\bar{z} = dx - idy \iff dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z}) \tag{3}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{4}$$

Definition 2 (Harmonic)**.** *We say a real or complex valued function $f(x, y)$ is **harmonic** if f is $\mathcal{C}^2$ and*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \tag{5}$$

Proposition 1. *Every real-valued harmonic function is, not necessarily everywhere but at least locally, the real part of a holomorphic function.*

Theorem 1. *ω has a primitive in Ω if and only if, for any piecewise differentiable closed curve $\gamma : [a, b] \to \Omega$ (i.e. with $\gamma(a) = \gamma(b)$), or equivalently any piecewise differentiable $\gamma : S^1 \to \Omega$, we have*

$$\int_\gamma \omega = 0 \tag{6}$$

Definition 3. *We say a differential form ω on a domain Ω is **closed** if every point in Ω has a neighborhood in which ω has a primitive.*

Theorem 2. *Any closed differential form ω in a simply-connected open set Ω has a primitive.*

Theorem 3 (Cauchy's Theorem)**.** *Let Ω be a domain and let $f(z)$ be continuous in Ω and holomorphic except on a set of discrete lines and points. Then the differentiable form $f(z)dz$ is closed.*

Corollary 1. *A holomorphic function $f(z)$ locally has a primitive, which is holomorphic (i.e. a function F such that $dF = f(z)dz$)*

Corollary 2 (Morera's Theorem)**.** *If $f(z)$ is continuous in Ω and $df = f(z)dz$ is closed, then $f(z)$ is holomorphic.*

Definition 4. *Let $\gamma : S^1 \to \Omega$ be a closed curve and $a \notin \gamma(S^1)$ be a point not in the image of γ. Then the **winding number of γ with respect to** a is given by the integral*

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - a} \tag{7}$$

This integral is an integer as it is the difference between two branches of log.

Theorem 4 (Cauchy's Integral Formula)**.** *If $f(z)$ is holomorphic in Ω, $a \in \Omega$ and $\gamma : S^1 \to \Omega$ is a closed curve with $a \notin \gamma$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z - a} = w(\gamma, a)f(a) \tag{8}$$

Theorem 5 (Liouville's Theorem)**.** *A bounded holomorphic function on all of $\mathbb{C}$ is a constant.*

Definition 5 (Zero). *If f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, we can write, for some $k \in \mathbb{N}$,*

$$f(z) = (z - z_0)^k f_1(z) \tag{9}$$

*where$f_1(z)$ is nonvanishing near z_0. In this case k is called the **order** or **multiplicity** of the **zero** z_0*

Definition 6 (Meromorphic). *A function f is **meromorphic** on an open $\Omega \subseteq \mathbb{C}$ if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of Ω we can write $f(z) = g(z)/h(z)$ where g, h are holomorphic and h is not identically zero.*

Definition 7 (Laurent expansion). *Holomorphic functions in an annulus $r < |z| < R$ have a convergent **Laurent expansion** in an annulus*

$$\sum_{n=-\infty}^{\infty} a_n z^n = P(z) + R(z), \qquad P(z) = \sum_{n<0} a_n z^n, \qquad R(z) = \sum_{n\geqslant 0} a_n z^n \tag{10}$$

Theorem 6. *Every meromorphic function f on S^2 is rational.*

Definition 8 (Isolated singularity). *A holomorphic function in a punctured disk $0 < |z| < R$ has an **isolated singularity** at 0 if $f(z)$ cannot be extended to be holomorphic at 0.*

Theorem 7 (Weierstrass Theorem). *If 0 is an essential singularity, then for all $\epsilon > 0$, $f(\{0 \leqslant |z| \leqslant \epsilon\})$ is dense in $\mathbb{C}$.*

Definition 9. *Let $\Omega \subset \mathbb{C}$ be open. We define $\mathbb{C}(\Omega)$ to be the **ring of continuous, complex-valued functions on** Ω and $\mathcal{H}(\Omega)$ to be the **subring of holomorphic functions on** Ω*

Definition 10 (Uniform convergence on compact subsets). *We say that a sequence of functions $\{f_n\} \subset \mathcal{C}(\Omega)$ **converges uniformly on compact subsets** if for all compact subsets $K \subset \Omega$, $\{f_n|K\}$ converges uniformly, i.e.*

$$\forall \text{ compact } K \subset \Omega, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geqslant N, \forall z \in K, |f_m(x) - f_n(x)| < \epsilon \tag{11}$$

Theorem 8 (Weierstrass).

1. *$\mathcal{H}(\Omega)$ is a closed subspace of $\mathcal{C}(\Omega)$, i.e. if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly to f on compact sets then $f = \lim_{n\to\infty} f_n \in \mathcal{H}(\Omega)$ is holomorphic.*
2. *The mapping $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ $f \mapsto f'$ is continous, i.e. if $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly to f on compact sets then $\{f_n'\}$ converges uniformly to f' on compact sets.*

Corollary 3. *Let $\{f_n\}$ be a series of holomorphic functions. If $\{g_n = \sum_{k=0}^{n} f_k\}$ converges uniformly on compact subsets of Ω, then the sum*

$$f = \sum f_n \tag{12}$$

is holomorphic on Ω and the series can be differentiated term by term.

Proposition 2. *Let Ω be a domain. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets and each f_n vanishes nowhere in Ω then $f = \lim_{n\to\infty} f_n$ is either never zero or identically zero.*

Corollary 4. *Let Ω be a domain. If $\{f_n\} \subset \mathcal{H}(\Omega)$ converges uniformly on compact sets and each f_n is one-to-one, then $\lim_{n\to\infty} f_n$ is either one-to-one or constant.*

Definition 11. *We say that $\sum_{n=1}^{\infty} f_n$ **converges uniformly** (respectively **converges uniformly absolutely**) on $X \subset \Omega$ if all but finitely many f_n have no pole in X and form a uniformly convergent (respectively uniformly absolutely convergent) series on X.*

Definition 12. *Let $X \subset \mathbb{C}$ and $\mathcal{S} \subset \mathcal{C}(X)$. We say that $\mathcal{S}$ is **equicontinuous** at $a \in X$ if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall z \in X, |z - a| < \delta \implies \forall f \in \mathcal{S}, |f(z) - f(a)| < \epsilon \tag{13}$$

*$\mathcal{S}$ is **equicontinuous on** X if it is equicontinous at each $a \in X$. It is **uniformly equicontinuous on** X if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall z, w \in X, |z - w| < \delta \implies \forall f \in \mathcal{S}, |f(z) - f(w)| < \epsilon \tag{14}$$

Theorem 9 (Arzela-Ascoli). *Let $\Omega \subset \mathbb{C}$ be a* ***domain****. Then $\mathcal{S} \subset \mathcal{C}(\Omega)$ is normal if and only if*

1. *$\mathcal{S}$ is equicontinuous on Ω*
2. *There exists $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}$ is a bounded subset of $\mathbb{C}$*

Definition 13. *$\mathcal{S} \subset \mathcal{C}(\Omega)$ is* ***locally bounded*** *on Ω if*

$$\forall z_0 \in \Omega, \exists \delta > 0, M < \infty, \forall z \in \Omega, f \in \mathcal{S}, |z - z_0| < \delta \implies |f(z)| \leqslant M \tag{15}$$

This is true if and only if $\mathcal{S}$ is ***uniformly bounded on compact subsets of*** *Ω, i.e. for all $K \subset \Omega$ compact,*

$$\exists M = M(K), \forall z \in K, \forall f \in \mathcal{S}, |f(z)| \leqslant M \tag{16}$$

Theorem 10 (Montel). *Let $\mathcal{S} \subset \mathcal{H}(\Omega)$ where $\Omega \subset \mathbb{C}$ is a domain. Then the following are equivalent:*

1. *$\mathcal{S}$ is normal*
2. *$\mathcal{S}$ is locally bounded*
3. *$\mathcal{S}' = \{f' : f \in \mathcal{S}\}$ is locally bounded and there exists $z_0 \in \Omega$ such that $\{f(z_0) : f \in \mathcal{S}\}$ is bounded in $\mathbb{C}$.*

The Arzela-Ascoili theorem holds for families of continuous functions with values in a complete metric space, e.g. continuos functions with values in the Riemann sphere S^2 (or the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$) with the (induced) **chordal metric**

$$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|)^2(1 + |w|)^2}} \tag{17}$$

(we note that the topology induced by $\mathbb{C}$ by the chordal metric is the usual Euclidean topology).

Definition 14 (Normal in the chordal metric). *A family $\mathcal{S}$ of continuous functions on Ω is* ***normal in the chordal metric*** *if and only if it is equicontinuous in the chordal metric: condition (2) of the Arzela-Ascoli theorem is not needed because $\mathbb{C}^*$ or S^2 is compact in this topology.*

We can use this definition to analyze, e.g., a family $\mathcal{S}$ of meromorphic functions on $\Omega \subset \mathbb{C}$ (or $\Omega \subset S^2$), since these can be considered holomorphic functions with values in S^2.

Lemma 1. *Let $\{f_n\}$ be a sequence of meromorphic functions which converges uniformly on compact subsets of the domain $\Omega \subset \mathbb{C}$ (on S^2, in the chordal metric). Then the limit function is either meromorphic or identically ∞.*

Definition 15 (Spherical derivative). *If f is meromorphic on a domain $\Omega \subset \mathbb{C}$ (or S^2), we define the* ***spherical derivative of*** *f at $z \in \Omega$ by*

$$f^\sharp(z) = \lim_{w \to z} \frac{d(f(z), f(w))}{|z - w|} \tag{18}$$

If z is not a pole, we have

$$f^\sharp(z) = \lim_{w \to z} \frac{2|f(z) - f(w)|}{|z - w|\sqrt{(1 + |f(z)|^2)(1 + |f(w)|^2)}} = \frac{2|f'(z)|}{1 + |f(z)|^2} \tag{19}$$

We have that

$$\left(\frac{1}{f}\right)^\sharp = f^\sharp \tag{20}$$

implying that $f^\sharp(z)$ is finite and continuous at all $z \in \Omega$, and greater than zero at z if and only if f is one-to-one near z.

Theorem 11 (Marty's Theorem). *Let $\mathcal{S}$ be a family of meromorphic functions on a domain Ω. Then $\mathcal{S}$ is normal in the chordal metric if and only if*

$$\mathcal{S}^\sharp = \{f^\sharp : f \in \mathcal{S}\} \tag{21}$$

is bounded.

Theorem 12 (Riemann mapping theorem). *Any simply connected open $\Omega \subset \mathbb{C}$ except $\mathbb{C}$ itself has a biholomorphic mapping onto the open unit disc D*

Theorem 13 (Picard's Little Theorem). *A non-constant entire function f omits at most a point, i.e. $\#(\mathbb{C}\backslash f(\mathbb{C})) \leqslant 1$*

Theorem 14 (Picard's Big Theorem). *If z_0 is an isolated essential singularity of a holomorphic function $f(z)$, then f takes every complex value with one possible exception in any neighborhood Ω of z_0, i.e. $\#(\mathbb{C}\backslash f(\Omega)) \leqslant 1$.*

2 Useful Results and Formulas

- Projection from the Riemann Sphere:

$$\pi : S^2\backslash\{N\} \to \mathbb{C}, \pi(x,y,t) = \frac{x+iy}{1-t} \tag{22}$$

- Green's Formula:

  Theorem 15 (Green's formula).

$$\int_\gamma Pdx + Qdy = \iint\limits_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{23}$$

- Schwarz Reflection Principle:

  Theorem 16 (Schwarz Reflection Principle). *If $f : H \to \mathbb{C}$ is continuous on the closed upper half-plane H, holomorphic on the open upper half-plane and takes real values on the real axis (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$) then it can be extended to an entire function by $f(\overline{z}) = \overline{f(z)}$. More generally, this can be applied to reflecting any half-domain over any line.*

- Fourier coefficients and Cauchy inequalities:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z} \implies f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \tag{24}$$

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{i\pi\theta}, \qquad a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})d\theta \tag{25}$$

$$M(r) = \sup_\theta |f(re^{i\theta})| \implies |a_n| \leqslant \frac{M(r)}{r^n} \tag{26}$$

- The Mean Value Property (MVP): *harmonic* functions satisfy

$$f(\text{center of disk}) = \text{ mean value on boundary} \tag{27}$$

- The Maximum Modulus Principle (MMP): if f is a continuous complex-valued function on an open $\Omega \subseteq \mathbb{C}$ with the MVP, then it satisfies the MMP, that is, if $|f|$ has a local maximum at a point a of Ω, then f is constant in a neighborhood of a.

- Schwarz's Lemma:

  Theorem 17 (Schwarz's Lemma)**.** *Suppose $f(z)$ is holomorphic in $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$. Then*

  1. $|f(z)| \leqslant |z|$ *if* $|z| < 1$
  2. *If* $|f(z_0)| = |z_0|$ *at some* $z_0 \neq 0$*, then* $f(z) = \lambda z$ *for some* $|\lambda| = 1$.

- Automorphisms of the complex plane:

$$\operatorname{Aut}\mathbb{C} = \{\text{linear transformations } w = az + b, \quad a \neq 0\} \tag{28}$$

- Automorphisms of the Riemann Sphere:

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0 \tag{29}$$

  Inverse:

$$\frac{dz-b}{-cz+a} \tag{30}$$

- Automorphisms of the upper half-plane:

$$w = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R} \tag{31}$$

3 Residues and Integrals

Definition 16 (Residue)**.** *Let $f(z)$ be a holomorphic function in a punctured disc centered around a, and let γ be a closed curve lying entirely in the punctured disc (in particular, never touching a) with winding number $w(\gamma, a) = 1$. We define the **residue** of the differential form $f(z)dz$ (or "of f") at a to be*

$$\operatorname{Res}_a(f) = \frac{1}{2\pi i}\int_\gamma f(z)dz = a_{-1} \tag{32}$$

where a_n are the coefficients in the Laurent expansion of f at a. Note that this is independent of the choice of curve γ.

Definition 17 (Residue at ∞)**.** *Writing $z = \frac{1}{z'}$, we have in coordinates at ∞*

$$f(z)dz = -\frac{1}{z'^2}f(1/z')dz' = g(z')dz' \tag{33}$$

We define

$$\operatorname{Res}_\infty(f) = \operatorname{Res}_0(g) = -a_{-1} \tag{34}$$

where a_n are the terms of the Laurent expansion in $|z| > R$.

Theorem 18 (Residue Theorem)**.** *Let $\Omega \subset S^2$ be open and let $f(z)$ be holomorphic in Ω except perhaps on a discrete set of isolated points. Let Γ be the oriented (piecewise $\mathcal{C}^1$) boundary of a compact set $K \subset \Omega$ not containing any singularity (either essential singularities or poles). Then*

$$\int_\Gamma f(z)dz = 2\pi i \sum_a \operatorname{Res}_a(f) \tag{35}$$

where a ranges over the singularities contained in K, perhaps including ∞.

4 Elliptic Curves

Definition 18. *Let $e_1, e_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. We can define a discrete subgroup of $\mathbb{C}$ with* ***basis*** *e_1, e_2*

$$\Gamma = \{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\} \tag{36}$$

We say that f has Γ as ***group of periods*** *if*

$$\forall z \in \mathbb{C}, f(z) = f(z + e_1) = f(z + e_2) \tag{37}$$

Definition 19 (Weierstrass $\wp$-function)**.** *We define the Weierstrass $\wp$-function by the infinite sum*

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{w \in \Gamma \\ w \neq 0}} \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right] \tag{38}$$

Claim.

1. *$\wp$ has a double pole at each $w \in \Gamma$ with prime part $\frac{1}{(z-w)^2}$*
2. *$\wp$ is an even function*
3. *$\wp' = -z \sum_{w \in \Gamma} (z-w)^{-2}$ converges absolutely uniformly on compact subsets of $\mathbb{C}$*
4. *$\wp'$ is doubly periodic: $\forall w \in \Gamma, \wp'(z+w) = \wp'(z)$*
5. *$\wp'$ is odd*
6. *$\wp$ itself has Γ as group of periods*

We have

$$(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4 \tag{39}$$

Proposition 3. *If f is a non-constant meromorphic function on $\mathbb{C}$ with Γ as group of periods, then the number of zeros of f in a period parallelogram is equal to the number of poles in the same parallelogram*