# MAT454 Notes

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**Definition 1** ($n$-dimensional complex projective space)**.** *We define*

$$\mathrm{P}^n(\mathbb{C}) = \mathbb{C}^{n+1}\backslash\{0\}/\sim$$

*where*

$$(x_0, ..., x_n) \sim (x'_0, ..., x'_n) \iff \exists \lambda \in \mathbb{C}, (x'_0, ..., x'_n) = (\lambda x_0, ..., \lambda x_n)$$

*We denote the equivalence class of $(x_0, ..., x_n)$ by $[x_0, ..., x_n]$.*

**Definition 2** (Homogeneous coordinates)**.** *We define coordinate charts $U_i = \{[x_0, ..., x_n] \in \mathrm{P}^n(\mathbb{C}) : x_i \neq 0\}$ with affine coordinates $U_i \to \mathbb{C}^n$,*

$$[x_0, ..., x_n] \mapsto \left( \frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i} \right)$$

*with inverse*

$$(g_1, ..., g_n) \mapsto [g_1, ..., g_{i-1}, 1, g_{i+1}, .., g_n]$$

Using these coordinates, we have that $\mathrm{P}^n(\mathbb{C})$ has the structure of an $n$-dimensional complex manifold, as the transition mappings are rational. Let's take one of the charts here, say $U_0$, to be $\mathbb{C}^n$. So

$$\mathrm{P}^n(\mathbb{C}) = U_0 \cup \text{ everything else}$$

But what's everything else? So $U_0$ is all the points where $x_0 \neq 0$, so everything else is the set of points

$$\{x_0 = 0\} = \{[0, x_1, ..., x_n]\} \simeq \mathrm{P}^{n-1}(\mathbb{C}) \implies \mathrm{P}^n(C) = U_0 \cup \mathrm{P}^{n-1}(\mathbb{C})$$

We call this copy of $\mathrm{P}^{n-1}(\mathbb{C}) \simeq \{x_0 = 0\}$ the **hyperplane at infinity**. This is like a generation of the Riemann sphere which we saw before, which we saw was given by $S^2 = \mathrm{P}^1(\mathbb{C})$. So when we talk about $\mathrm{P}^2(\mathbb{C})$, that's like having 2-complex coordinates with a line at infinity. Specifically, we can write it as

$$\mathrm{P}^2(\mathbb{C}) = \{[x, y, t]\} = \mathbb{C}^2_{(x,y)} \cup \{t = 0\}$$

the **projective line at infinity**. Now assume we have a curve $X \subset \mathbb{C}^2$ generated by the equation

$$y^2 = 4x^3 - 20a_2 x - 28a_4$$

where the RHS has three distinct roots. We want to compute the **compactification of $X$ in $\mathrm{P}^2(\mathbb{C})$**. We can write this down in homogeneous coordinates

$$y^2 t = 4x^3 - 20a^2 x t^2 - 28a_4 t^3$$

taking $X'$ to be the solution set of this. Why is this the right thing? When you look at $\mathrm{P}^2(\mathbb{C})$, and look in here at the set of points

$$\{[x, y, t] : t \neq 0\} \simeq \mathbb{C}^2_{(x,y)}$$

we see that it is has homomorphism

$$[x, y, t] \mapsto \left( \frac{x}{t}, \frac{y}{t} \right)$$

Hence, we rewrite our eqaution in our new coordinates for $\mathbb{C}^2$,

$$\frac{y^2}{t^2} = 4\frac{x^3}{t^3} - 20a_2\frac{x}{t} - 28a_4$$

Now we can just multiply both sides by $t^3$. So if you haven't seen this before, this takes a little bit of familiarity, but the actual operations involved are very simple operations. Of course, our *original* $X$ is a subspace of $X'$. But how much have we added to $X$? Well, if we set $t = 0$, we get $x = 0$. So, how many points are we adding? One point, at $\infty$:

$$X' = X \cup \{[0, 1, 0]\}$$

Now, in the neighborhood of any finite point, $X'$ just looks like $X$. What about in a neighborhood of the point at $\infty$, $[0, 1, 0]$? What does it look like? So this point $[0, 1, 0]$ doesn't actually lie in the coordinate chart $U_0 = \{x \neq 0\}$, it lies in $U_1 = \{y \neq 0\}$. This chart has affine coordinates given by $(x', t') = (x/y, t/y)$. So what's the equation of $X'$ in *this* coordinate chart? It's

$$t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3$$

In some neighborhood of $(x', t') = (0, 0)$ (the point at infinity), the implicit function theorem tells us that $t'$ is a holomorphic function of $x'$:

$$t' = 4x'^3 - 320a_2x'^7 + ...$$

As an exercise, we can take the Taylor series

$$t' = b_0 + b_1x + b_2x^2 + ...$$

plug it into the equation and solve successively for the coefficients. If you haven't done that before, do the exercise.

This tells us, in general, though, that $t'$ is a function of $x'$. So in a neighborhood of the point at infinity, $X'$ looks like the graph of this function.