

# MAT454 Notes

Jad Elkhaleq Ghalayini

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Let's, as we usually do, consider a holomorphic function  $f(z)$  in an open set  $\Omega \subseteq \mathbb{C}$ . Last time, we showed that  $f$  has a convergent power series expansion in any open disc in  $\Omega$  (centered at the center of the disc). For example, around  $a = 0 \in \Omega$ , we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Writing  $z = re^{i\theta}$ , we get

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

We can write out the following formula for these Fourier coefficients:

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta$$

Today we're going to be looking at the consequences of this formula. First of all, this formula gives a simple but useful upper bound on  $a_n$ : if we take the maximum absolute value of  $f$  along the circle of radius  $r$ , written

$$M(r) = \sup_{\theta} |f(re^{i\theta})|$$

we get

$$|a_n| \leq \frac{M(r)}{r^n}$$

These are called **Cauchy's inequalities**. These have some important consequences, like **Liouville's theorem**: a bounded holomorphic function on all of  $\mathbb{C}$  is a constant. How does this follow? Well, if  $c$  is the upper bound of  $f$  on  $\mathbb{C}$ , we have each

$$\forall r \in \mathbb{R}^+, M(r) \leq c \implies |a_n| \leq \frac{M(r)}{r^n} \leq \frac{c}{r^n}$$

Hence, for  $n > 0$ ,  $0 \leq a_n \leq \epsilon$  for all  $\epsilon > 0$ , implying  $a_n = 0$ . It follows that  $f = a_0 = c$ , a constant. Another consequence is that we can write, for any  $r$

$$f(0) = a_0 = a_0 r^0 = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

This generalizes readily to stating that holomorphic functions satisfy the **Mean Value Property (MVP)**:

$$f(\text{center of disk}) = \text{mean value on boundary}$$

Another property which we won't prove is the **Maximum Modulus Principle (MMP)**: if  $f$  is a continuous complex-valued function on an open  $\Omega \subseteq \mathbb{C}$  with the MVP, then it satisfies the MMP, that is, if  $|f|$  has a local maximum at a point  $a$  of  $\Omega$ , then  $f$  is constant in a neighborhood of  $a$ .

We can use this to prove **Schwarz's Lemma**:

**Theorem 1** (Schwarz's Lemma). *Suppose  $f(z)$  is holomorphic in  $|z| < 1$ ,  $f(0) = 0$  and  $|f(z)| < 1$ . Then*

1.  $|f(z)| \leq |z|$  if  $|z| < 1$

2. If  $|f(z_0)| = |z_0|$  at some  $z_0 \neq 0$ , then  $f(z) = \lambda z$  for some  $|\lambda| = 1$ .

We recall a sketch of the proof

*Proof.* (Sketch) By the convergent power series expansion,  $g(z)/z$  is holomorphic, and can hence have the maximum modulus principle applied to it.  $\square$