MAT454 Notes

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Today, our goal is to prove the Riemann mapping theorem:

Theorem 1 (Riemann mapping theorem)**.** *Any simply connected open $\Omega \subset \mathbb{C}$ except $\mathbb{C}$ itself has a biholomorphic mapping onto the open unit disc D*

We will begin by proving a series of lemmas.

Lemma 1. *There is a biholomorphism of Ω onto a bounded open subset of $\mathbb{C}$*

Proof. Let $a \notin \Omega$ be a point, which exists as $\Omega \neq \mathbb{C}$. Then $\frac{1}{z-a}$ is a nonvanishing holomorphic function on the simply connected open set Ω, and so it has a primitive, some holomorphic function $g(z)$.

Now, a primitive of $\frac{1}{z-a}$ is like a branch of $\log(z-a)$, which means that

$$z - a = e^{g(z)}$$

One thing this tells us right away is that $g(z)$ is one to one, because $z-a$ is one to one and if the composition of a function with something else is one to one, then that function must be one to one.

Take a point $z_0 \in \Omega$. Since Ω is open and g is one to one, implying it is nonconstant, there is an open disc centered at $g(z_0)$ inside $g(\Omega)$. Now, I claim that if you look at this disc translated by $2\pi i$ it's outside of $g(\Omega)$, i.e. $E + 2\pi i \cap g(\Omega) = \varnothing$. Intuitively, this is the case because it's on a different branch of the log function. A cleaner way of saying this is that this is because $\exp \circ g$ is one-to-one, but exp will take two translated points to the same point, yielding a contradiction.

So then, how do we get a biholomorphism of Ω onto a bounded open subset of $\mathbb{C}$?

$\square$