MAT454 Notes

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Let's, as we usually do, consider a holomorphic function $f(z)$ in an open set $\Omega \subseteq \mathbb{C}$. Last time, we showed that f has a convergent power series expansion in any open disc in Ω (centered at the center of the disc). For example, around $a = 0 \in \Omega$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Writing $z = re^{i\theta}$, we get

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

We can write out the following formula for these Fourier coefficients:

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta$$

Today we're going to be looking at the consequences of this formula. First of all, this formula gives a simple but useful upper bound on a_n: if we take the maximum absolute value of f along the circle of radius r, written

$$M(r) = \sup_{\theta} |f(re^{i\theta})|$$

we get

$$|a_n| \leqslant \frac{M(r)}{r^n}$$

These are called **Cauchy's inequalities**. These have some important consequences, like **Liouville's theorem**: a bounded holomorphic function on all of $\mathbb{C}$ is a constant. How does this follow? Well, if c is the upper bound of f on $\mathbb{C}$, we have each

$$\forall r \in \mathbb{R}^+, M(r) \leqslant c \implies |a_n| \leqslant \frac{M(r)}{r^n} \leqslant \frac{c}{r^n}$$

Hence, for $n > 0$, $0 \leqslant a_n \leqslant \epsilon$ for all $\epsilon > 0$, implying $a_n = 0$. It follows that $f = a_0 = c$, a constant. Another consequence is that we can write, for any r

$$f(0) = a_0 = a_0 r^0 = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$$

This generalizes readily to stating that holomorphic functions satisfy the **Mean Value Property (MVP)**:

$$f(\text{center of disk}) = \text{ mean value on boundary}$$

Another property which we won't prove is the **Maximum Modulus Principle (MMP)**: if f is a continuous complex-valued function on an open $\Omega \subseteq \mathbb{C}$ with the MVP, then it satisfies the MMP, that is, if $|f|$ has a local maximum at a point a of Ω, then f is constant in a neighborhood of a.

We can use this to prove **Schwarz's Lemma**:

Theorem 1 (Schwarz's Lemma)**.** *Suppose $f(z)$ is holomorphic in $|z| < 1$, $f(0) = 0$ and $|f(z)| < 1$. Then*

1. $|f(z)| \leqslant |z|$ *if* $|z| < 1$
2. *If* $|f(z_0)| = |z_0|$ *at some* $z_0 \neq 0$, *then* $f(z) = \lambda z$ *for some* $|\lambda| = 1$.

We recall a sketch of the proof

Proof. (Sketch) By the convergent power series expansion, $g(z)/z$ is holomorphic, and can hence have the maximum modulus principle applied to it. □

So let's spend a little time looking at functions with the MVP in general. Continuous functions with the MVP are precisely the <u>harmonic functions</u>. The real and imaginary parts of a complex valued function with the MVP also satisfy the MVP. A real valued harmonic function g is locally the real part of a holomorphic function, uniquely determined up to addition of a constant:

$$\frac{\partial^2 g}{\partial z \partial \bar{z}} = 0 \implies \frac{\partial g}{\partial z} \text{ holomorphic}$$

Therefore, $\frac{\partial g}{\partial z}$ locally has primitive f, defined up to a constant. Since g is real valued, we can write

$$df = \frac{\partial g}{\partial z} dz, \quad d\bar{f} = \frac{\partial g}{\partial \bar{z}} d\bar{z}$$

Hence,

$$d(f + \bar{f}) = dg \implies g = 2\Re f + \text{ const}$$

So harmonic functions satisfy the MVP and MMP, and conversely, a continuous function in an open set $\Omega \subseteq \mathbb{C}$ satisfying the MVP is harmonic. Just a couple of words as to why this is true, as this is really something that you should review: this comes from the solution to what's called the Dirichlet problem for a disk. What is this problem? It says that, given any continuous function on the boundary of a disk, you can extend it to a continous function on the whole disk which is harmonic on the interior.