MAT454 Notes

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Zeros and Poles

Definition 1 (Zero). *If f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, we can write, for some $k \in \mathbb{N}$,*

$$f(z) = (z - z_0)^k f_1(z)$$

*where$f_1(z)$ is nonvanishing near z_0. In this case k is called the **order** or **multiplicity** of the **zero** z_0*

Zeros of holomorphic functions form a discrete set. We want to study, however, not only holomorphic functions, but also quotients of holomorphic functions

Definition 2 (Meromorphic). *A function f is **meromorphic** on an open $\Omega \subseteq \mathbb{C}$ if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of Ω we can write $f(z) = g(z)/h(z)$ where g, h are holomorphic and h is not identically zero.*

Why is it interesting to work with meromorphic and not just holomorphic functions? Essentially, it's because meromorphic functions in a domain Ω form a field (whereas holomorphic functions only form a ring). Note that, in this course, when we say "domain", what we mean is a connected open set. If $f(z), g(z)$ are holomorphic near z_0, like before, we can write

$$f(z) = (z - z_0)^k f_1(z), \qquad g(z) = (z - z_0)^\ell g_1(z)$$

where $f_1(z_0), g_1(z_0) \neq 0$. Near z_0, then, the quotient looks like

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-\ell} \frac{f_1(z)}{g_1(z)}$$

So what are the different possibilities? If $k \geqslant \ell$, then this function extends to be holomorphic at z_0. On the other hand, if $k < \ell$, then, of course,

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$$

Note: *not* undefined, but ∞. In this case, we say that z_0 is a pole of order $\ell - k$.