MAT454 Academic Offense Sheet

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A quick collection of useful facts, theorems, and definitions for complex analysis. May be incorrect, and is certainly incomplete. Use at your own risk!

Contents

1 Basic Definitions and Theorems

For $f = u + iv$ holomorphic, we have

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1}$$

Definition 1. *The **differential** of f is given by*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{2}$$

$$dz = dx + idy, \qquad d\bar{z} = dx - idy \iff dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z}) \tag{3}$$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} \tag{4}$$

Definition 2 (Harmonic)**.** *We say a real or complex valued function $f(x, y)$ is **harmonic** if f is $\mathcal{C}^2$ and*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = 0 \tag{5}$$

Proposition 1. *Every real-valued harmonic function is, not necessarily everywhere but at least locally, the real part of a holomorphic function.*

Theorem 1. *ω has a primitive in Ω if and only if, for any piecewise differentiable closed curve $\gamma : [a, b] \to \Omega$ (i.e. with $\gamma(a) = \gamma(b)$), or equivalently any piecewise differentiable $\gamma : S^1 \to \Omega$, we have*

$$\int_\gamma \omega = 0 \tag{6}$$

Definition 3. *We say a differential form ω on a domain Ω is **closed** if every point in Ω has a neighborhood in which ω has a primitive.*

Theorem 2. *Any closed differential form ω in a simply-connected open set Ω has a primitive.*

Theorem 3 (Cauchy's Theorem)**.** *Let Ω be a domain and let $f(z)$ be continuous in Ω and holomorphic except on a set of discrete lines and points. Then the differentiable form $f(z)dz$ is closed.*

Corollary 1. *A holomorphic function $f(z)$ locally has a primitive, which is holomorphic (i.e. a function F such that $dF = f(z)dz$)*

Corollary 2 (Morera's Theorem)**.** *If $f(z)$ is continuous in Ω and $df = f(z)dz$ is closed, then $f(z)$ is holomorphic.*

2 Useful Tools

- Projection from the Riemann Sphere:

$$\pi : S^2 \backslash \{N\} \to \mathbb{C}, \pi(x, y, t) = \frac{x + iy}{1 - t} \tag{7}$$

- Green's Formula:

  Theorem 4 (Green's formula)**.**

$$\int_\gamma Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{8}$$

- Schwarz Reflection Principle:

  Theorem 5 (Schwarz Reflection Principle)**.** *If $f : H \to \mathbb{C}$ is continuous on the closed upper half-plane H, holomorphic on the open upper half-plane and takes <u>real</u> values on the real axis (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$) then it can be extended to an entire function by $f(\bar{z}) = \overline{f(z)}$. More generally, this can be applied to reflecting any half-domain over any line.*

3 Residues and Integrals

4 Elliptic Curves