1. REVIEW

$f(z)$ holom fn at $z \in \mathbb{C}$:

$z = x + iy$

$f = u + iv$

$z \mapsto f(z)$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$

$\lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ exists

$f(z+h) - f(z) = c \cdot h + \varphi(h) \cdot h$, $\lim_{h \to 0} \varphi(h) = 0$

$c = f'(z)$

Deriv $h \mapsto c \cdot h$

$c = a + ib$

$h = \xi + i\eta$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}$$

i.e., $\dfrac{\partial f}{\partial x} + i \dfrac{\partial f}{\partial y} = 0$ or $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

f holom at z iff diff'ble at z, as fn. of x, y and sat's CR eq'ns

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$f(x,y)$ diff'ble, $\mathbb{C}$-valued

Differential $df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$

$z = x + iy \qquad dz = dx + i\,dy \qquad dx = \frac{1}{2}(dz + d\bar{z})$

$\bar{z} = x - iy \qquad d\bar{z} = dx - i\,dy \qquad dy = \frac{1}{2i}(dz - d\bar{z})$

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) d\bar{z}$$

Thus write

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \qquad (*)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

so we have

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

CR eq'ns : $\dfrac{\partial f}{\partial \bar{z}} = 0$

$f(x,y)$ <u>harmonic</u> if C^2 and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

<u>Laplace's eqn</u>

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ <u>Laplacian</u>

By $(*)$, Laplace's eqn becomes

$$\frac{\partial^2 f}{\partial z\, \partial \bar{z}} = 0$$

$\mathbb{C}$-val'd fn f harmonic iff $\mathrm{Re}\, f$, $\mathrm{Im}\, f$ harmonic

Holom fn harmonic, $\therefore$ real and im parts of holom fn harmonic

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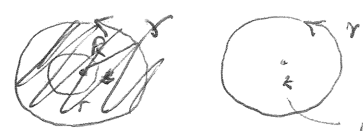
Holom $\Longleftrightarrow$ <u>analytic</u>; i.e. rep by cgt power series in nbd of every pt

By Cauchy's thm:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$



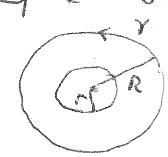
not nec centre

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right)$$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} z^n \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$$

if $|z| \le r$, unif & abs cgt on $|\zeta| = R$

so can integrate term-by-term



$$= \sum a_n z^n. \qquad a_n = \frac{1}{2\pi i} \int_{|\zeta| = R} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$$

$$\frac{\partial^2 g}{\partial z\, \partial \bar{z}} = 0 \qquad \frac{\partial}{\partial \bar{z}}\left(\frac{\partial g}{\partial z}\right) = 0$$

holo

<u>Rmk.</u> Real-val'd harmonic fn. $g(x,y)$ is <u>locally</u> real part of holom fn f, ! det'd up to addn of const

(from Cauchy's thm)

(or above: holom fn has prim)

But not nec <u>globally</u>

e.g. $\log|z|$ in $\mathbb{C} \setminus \{0\}$

not real part of holom fn.

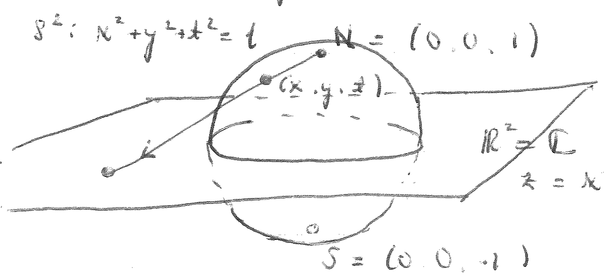
because $\log z$ has no single-val'd branch in Ω

Extended complex plane, or <u>Riemann sphere</u> S^2

$f(z)$ <u>holom at ∞</u> iff $f(1/z)$ holom at 0

Holom fn on Riem sphere is const
by max mod prin (f local max at a
$\Rightarrow$ const in nbhd of a)

~~Riemann sphere~~



Stereographic proj. from N.

$$z = \frac{x+iy}{1-t}$$

(Check $(0,0,1)$, (x,y,t) and $\left(\frac{x}{1-t}, \frac{y}{1-t}, 0\right)$ collinear)

Homeo of $S^2 \setminus \{N\}$ onto $\mathbb{C}$ (coord. chart)

Complex conj of stereo proj from S: $\quad z' = \dfrac{x-iy}{1+t}$

Metrics on S^2: geodesic, chordal

Homeo of $S^2 \setminus \{S\}$ onto $\mathbb{C}$ (coord chart)

For any pt $(x,y,t) \in S^2$ other than S, N,
$zz' = 1 \qquad z' = 1/z$

<u>1-diml cx. proj. space</u> $\quad P^1(\mathbb{C}) = \mathbb{C}^2 \setminus \{0\} / \sim$

$(x_0, x_1) \sim (x_0', x_1')$
if $\exists \lambda \in \mathbb{C}$ s.t. $(x_0', x_1') = (\lambda x_0, \lambda x_1)$

$[x_0, x_1]$ equiv class of (x_0, x_1) "homog coords"

Coord chart:

$$U_i = \{ [x_0, x_1] \in P^1(\mathbb{C}) : x_i \neq 0 \}, \quad i = 0, 1$$

$U_0 \longrightarrow \mathbb{C}$ $\qquad\qquad$ $U_1 \longrightarrow \mathbb{C}$

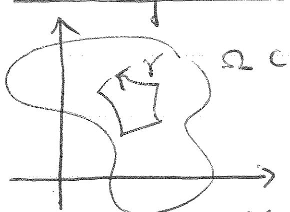
$[x_0, x_1] \longmapsto \dfrac{x_1}{x_0} = z$ $\qquad\qquad$ $[x_0, x_1] \longmapsto \dfrac{x_0}{x_1} = z'$

$\overset{\shortparallel}{[1, x_1/x_0]}$

$z z' = 1$ $\qquad$ $P^1(\mathbb{C})$ obt'd by gluing together 2 copies of $\mathbb{C}$ along complements of $\{0\}$, by formula $z' = 1/z$.

i.e., $P^1(\mathbb{C}) \cong S^2$.

Cauchy's thm



$\Omega \subset \mathbb{R}^2$

Differential form

$\omega = P\,dx + Q\,dy$ $\qquad$ P, Q contin ($\mathbb{R}$ or $\mathbb{C}$-val'd)

γ piecewise C^1 curve in Ω

$\gamma : [a, b] \longrightarrow \Omega$ $\qquad$ $\gamma(t) = (x(t), y(t))$

$$\int_\gamma \omega := \int_a^b f(t)\,dt, \qquad f(t) = P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)$$

$f(t)\,dt = \gamma^*(\omega)$ $\qquad$ $\gamma^*(P) = P \circ \gamma$

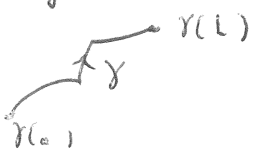
$\gamma^*(dx) = d(x \circ \gamma) = x'(t)\,dt$

Indep of param: $t : [c, d] \longrightarrow [a, b]$,

$t(c) = a, \quad t(d) = b, \quad t'(s) > 0$

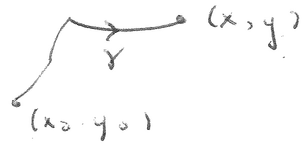
$\delta(s) = \gamma(t(s))$

$\int_\gamma \omega = \int_\delta \omega$ $\qquad$ by integn by subst'n e.g. $\omega = \underset{C^1}{dF} = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$ F <u>primitive</u> of ω

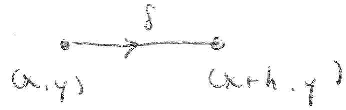


$$\int_\gamma dF = \int_a^b (F\circ\gamma)'(t)\,dt = F(\gamma(b)) - F(\gamma(a))$$

ω has a primitive in Ω
iff $\int_\gamma \omega = 0$ for every piecewise C^1 closed curve γ in Ω



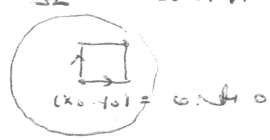
$$F(x,y) := \int_\gamma \omega$$



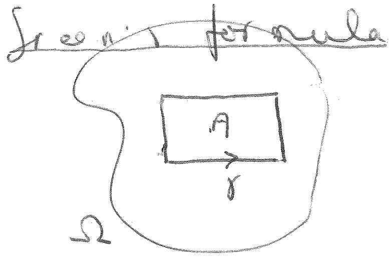
$$F(x+h,y) - F(x,y) = \int_\delta \omega = \int_x^{x+h} P(\xi,y)\,d\xi$$

$$\lim_{h\to 0} \frac{1}{h}(\quad) = P(x,y)$$

In case Ω open disk,
ω has prim in Ω iff $\int_\gamma \omega = 0$ whenever γ
bdry of rect in Ω with sides $\parallel$ axes
(suff condn :



<u>Green's formula</u>



$$\int_\gamma P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

assuming these exist, contin

$$\omega = P\,dx + Q\,dy$$
$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

ω <u>closed</u> if any pt has open nbhd in which ω has primitive

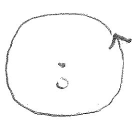
ω closed iff $\int_\gamma \omega = 0$ whenever γ bdry of small rect in Ω sides $\parallel$ axes.

So any closed form in open disk has prim

Assuming ω C^1:
ω closed iff $d\omega = 0$: i.e. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$
(by Green's formula)

e.g. $\Omega = \mathbb{C} \setminus \{0\}$
$\omega = \frac{dz}{z}$ closed in Ω (prim exist. locally: branch of log)

but has no primitive:



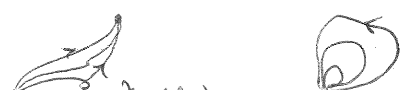
$\gamma(\theta) = e^{i\theta}$

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} = 2\pi i \neq 0$$

(cf. $d\theta$)

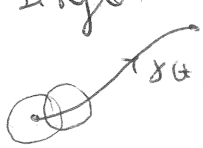
THM. Any closed diff. form ω in a simply-conn open set Ω has a prim.

simply-conn: connected and any closed (contin) curve in Ω is homot to pt in Ω



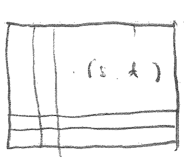
$\gamma_s(t)$

Closed form ω in domain Ω doesn't nec have single valid prim, but has prim along curve



$\gamma(t)$ $f(t)$ contin; $f(t) = F(\gamma(t))$ ← local prim $\quad \int_\gamma \omega = f(1) - f(0)$

(2 different local prims differ by const)

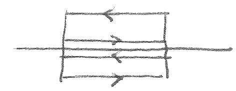


primitive over homotopy $\gamma(s,t) = \gamma_s(t)$

$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ if γ_0, γ_1 homotopic
(with fixed endpts)
(or closed curves homot as cl. curves)

Cauchy's thm $\Omega \subset \mathbb{C}$, $f(z)$ holom in Ω
Then diff'l form $f(z)\,dz$ closed

or contin & holom except on line (or some lines & pts)



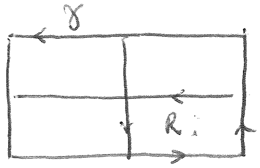
Proof. Follows from Green's thm and CR eqns assuming $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ contin

$$f(z)\,dz = \underbrace{f(z)}_{P}\,dx + i\underbrace{f(z)}_{Q}\,dy$$

By Green, enough to show $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$: CR!

$\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$ $\qquad$ $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$

Enough to show $\underbrace{\int_\gamma f(z)\,dz}_{\mu(R)} = 0$, γ bdry of rect $R \subset \Omega$



γ_i, $1 \le i \le 4$

$$\mu(R) = \int_\gamma f(z)\,dz = \sum \int_{\gamma_i} f(z)\,dz = \sum \mu(R_i)$$

$\therefore$ $|\mu(R_i)| \geq \frac{1}{4}|\mu(R)|$ for some i; ~~arg~~. say $R^{(1)} = R_i$, $\gamma^{(1)} = \gamma_i$

$$R \supset R^{(1)} \supset R^{(2)} \supset \dots$$

$$\left| \int_{\gamma^{(k)}} f(z)\,dz \right| \geq \frac{1}{4^k}\mu(R)$$

$$z_0 \in \bigcap_k R^{(k)}$$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z)(z - z_0)$$

$$\lim_{z \to z_0} \varphi(z) = 0$$

$$\int_{\gamma^{(k)}} f(z)\,dz = f(z_0)\underbrace{\int_{\gamma^{(k)}} dz}_{0} + f'(z_0)\underbrace{\int_{\gamma^{(k)}} (z - z_0)\,dz}_{0} + \underbrace{\int_{\gamma^{(k)}} \varphi(z)(z - z_0)\,dz}$$

Given $\epsilon > 0$, if k large enough, $| \quad \cdot \quad | \le \epsilon \operatorname{diag} R^{(k)} \operatorname{perim} R^{(k)}$
$= \frac{\epsilon}{4^k} \operatorname{diag} R \operatorname{perim} R$

$\therefore$ $|\mu(R)| \le \epsilon \operatorname{diag} R \operatorname{perim} R$, $\forall \epsilon > 0$

$\therefore = 0$ $\qquad \square$

COR. Holom fn. $f(z)$ in Ω locally has a primitive, ~~which~~ which is holom

$$\underline{dF} = f(z)\,dz$$

$$\frac{\partial F}{\partial z}dz + \frac{\partial F}{\partial \bar z}d\bar z$$

COR. (Morera's thm: Converse of Cauchy's thm)
$f(z)$ contin in Ω
If $f(z)dz$ closed, then $f(z)$ holom

Pf. Loc., $f(z)$ has prim. $g(z)$

$$f(z)\,dz = dg = \frac{\partial g}{\partial z}dz + \underbrace{\frac{\partial g}{\partial \bar z}}_{\therefore\ =0}d\bar z$$

$f(z) = \frac{\partial g}{\partial z}$ cont $\therefore g$ holom

Cauchy's integral formula

γ closed curve in Ω

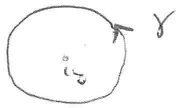
a not in image γ

Winding no of γ wrt a

$$w(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

integer (diff bet 2 branches of log)

Knot under homot of γ not passing thro' a.
As fn of a. Const on conn. comps of compl of γ



γ circle closed in pos. sense
$w(\gamma, c_i) = 1$

$$w(\gamma, a) = \begin{cases} 1 & a \text{ inside circle} \\ 0 & a \text{ outside} \end{cases}$$

(Orientn of γ; e.g., simple closed curve +vely oriented wrt 0 is −vely oriented wrt ∞)

THM.

$f(z)$ holom in Ω
$a \in \Omega$
γ closed curve in Ω $a \notin$ image γ,
γ homot to pt in Ω

Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-a} = w(\gamma, a)\, f(a)$$

Proof

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

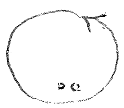
contin, holom in $\Omega \setminus \{a\}$

$$\therefore \int_\gamma \frac{f(z) - f(a)}{z-a}\,dz = 0$$

(since $g(z)dz$ closed, by Cauchy)

$$\int_\gamma \frac{f(z)\,dz}{z-a} = \int_\gamma \frac{f(a)\,dz}{z-a} = 2\pi i\, w(\gamma, a)\, f(a)$$

$\Box$

e.g.  γ bdry of disk. f holom in nbd.

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-a} = \begin{cases} f(a) & a \text{ inside disk} \\ 0 & a \text{ outside} \end{cases}$$

<u>Summary</u> $f(z)$ contin in Ω.

Foll equiv:

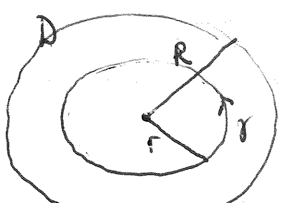
(1) $f(z)$ holom

(2) $f(z)\,dz$ closed

(3) $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi$ $\quad z$ in open disk, bdy γ

e.g. $f(z)$ contin, holom except on line $\Rightarrow$ holom

<u>Integral formulas for Taylor coeffs</u>



$f(z)$ holom in D: $|z| < R$

$\gamma(\theta) = re^{i\theta}$, $r < R$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - z} \quad *$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{(\xi - z)^{n+1}}$$

If $|z| \le r'$, $r' < r$ then unif and abs cgt

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi^{n+1}}$$

$\xi = re^{i\theta}$

$d\xi = ire^{i\theta}\,d\theta$

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

periodic in θ

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\,d\theta$$

Fourier coeffs

$$* \quad \frac{1}{\xi - z} = \frac{1}{\xi}\left(1 - \frac{z}{\xi}\right)^{-1} = \frac{1}{\xi}\left(1 + \frac{z}{\xi} + \frac{z^2}{\xi^2} + \dots\right)$$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \underbrace{\sum_{n=0}^{\infty} z^n \frac{f(\xi)}{\xi^{n+1}}}\,d\xi$$

If $|z| < r$, then unif & abs cgt and $|\xi| = r$, so can integrate term by term

Integral formula gives upper bd for a_n:
Let $M(r) = \sup_\theta |f(re^{i\theta})|$ upper bd for $|f|$ on circle rad. r

$|a_n r^n| \leq M(r)$

$|a_n| \leq \frac{M(r)}{r^n}$ $\forall n$ Cauchy inequalities.

Liouville's thm

Bounded holom fn on $\mathbb{C}$ is const

Proof

$M(r) \leq M$

$|a_n| \leq \frac{M}{r^n}$ $\forall r \Rightarrow a_n = 0,\ n \geq 1$ $\square$

Fund thm of alg in Cor. $\frac{1}{p}$ bd'd if p never 0

Mean value property

$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\,d\theta$ $D \subset \Omega$ comp disk $f(\text{centre}) =$ mean value on bdry

Max. mod. principle

f contin cx. val'd fn in $\Omega \subset \mathbb{C}$ with MVP.
If $|f|$ has (local) max at $a \in \Omega$
then f const in nbd of a. $\square$

Schwarz's lemma

Harmonic functions

Real and im parts of fn w. MVP also have MVP
Fns w. MVP are precisely harmonic fns

Real-val'd harmonic fn $g(x,y)$ loc. real part of holom fn, ! det'd up to addn of const

$\frac{\partial^2 g}{\partial z \partial \bar z} = 0 \Rightarrow \frac{\partial g}{\partial z}$ ~~holom : loc has prim~~ $d\bar{}$ holom

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \qquad \overline{df} = d\bar{f} \qquad \overline{\frac{\partial f}{\partial \bar z}} = \frac{\partial \bar f}{\partial z}$$

$\frac{\partial^2 g}{\partial z\,\partial \bar z} = 0 \Rightarrow \frac{\partial g}{\partial z}$ holom, so loc. has prim f
(! det'd up to const)

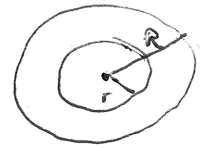
$$df = \frac{\partial g}{\partial z}\,dz$$
$$d\bar f = \frac{\partial g}{\partial \bar z}\,d\bar z$$

$\frac{\partial g}{\partial \bar z}$ conj of $\frac{\partial g}{\partial z}$ since g real val'd

$d(f+\bar f) = dg \qquad g = 2\,\mathrm{Re}\, f$ + real const $\square$

Harmonic fns therefore have MVP
Thus max mod princ applies to harmonic fns

real-val'd harmonic $g(x,y)$ = real part of holom fn $f(z) = \sum a_n z^n$



Can ass $a_0 \in \mathbb{R}$

$$g(r\cos\theta, r\sin\theta) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n\left(a_n e^{in\theta} + \bar a_n e^{-in\theta}\right)$$

unif & abs cgt wrt $\theta \in [0, 2\pi]$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\,d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{g(r\cos\theta, r\sin\theta)}{(re^{i\theta})^n}\,d\theta$$

$$\int_0^{2\pi} g(\quad)\,e^{-in\theta}\,d\theta = \frac{1}{2}\,2\pi r^n a_n$$

For $|z| < r$, get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\underbrace{\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right\}}_{\frac{re^{i\theta}+z}{re^{i\theta}-z}}\,d\theta$$

expresses holom fn $f(z)$
in $|z| < r$ in terms
of real part on bdry

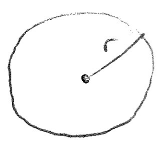
Equate real parts:

$$g(x,y) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\underbrace{\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}}_{\text{Poisson kernel}}\,d\theta$$

formula valid in $x^2+y^2<r$ for any $\mathbb{R}$-valued harmonic fn in $x^2+y^2<R$, where $r<R$

also true for $\mathbb{C}$-valued harmonic fn

<u>Dirichlet problem for a disk</u>



Given contin fn $f(\theta)$ on circle centre 0, radius r, ($f(\theta)$ periodic, period 2π), Can we find seek fn. $F(z)$ contin in $|z|\le r$, harm in $|z|<r$, s.t. $F(re^{i\theta}) = f(\theta)$?

THM Dirichlet problem for disk has ! soln.

<u>Pf.</u>

Can restrict to case f, F $\mathbb{R}$ valued

Uniqueness from max mod princ

Soln: $|z|<r$:

Def. $$F(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$F(z)$ harmonic since real part of

$$\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\frac{re^{i\theta}+z}{re^{i\theta}-z}\,d\theta$$

(holom by diff under)

Need only show

$$f(\theta_0) = \lim_{\substack{z\to re^{i\theta_0}\\ |z|<r}} F(z) \quad ✓$$

COR. Any contin fn in open Ω with MVP is harmonic

On disk in Ω:

$f|$ bdry extend to F harmc

$f-F$ zero on bdry $\therefore 0$ by max mod prin

domain = open & conn
Prerequisites = 1st 4 Chapters of Ahlfors

Zeros and poles

$f(z)$ holom

$f(z_0) = 0$: $f(z) = (z - z_0)^k f_1(z)$, $f_1(z_0) \neq 0$ (are f_1 (see below))

k = order or mult of zero z_0

Zeros of an fn. that doesn't vanish id form discrete set

Meromorphic fn. in open set Ω

def'd and anal. in compl of discrete set; in nbd of any pt of Ω, expressible as quotient of an fns $f(z)/g(z)$) not id zero

Merom fns in domain Ω form field

$$f(z) = (z - z_0)^k f_1(z) \qquad f_1(z_0),\ g_1(z_0) \neq 0$$
$$g(z) = (z - z_0)^l g_1(z)$$

For $z \neq z_0$ near z_0, $\dfrac{f(z)}{g(z)} = (z - z_0)^{k-l} \dfrac{f_1(z)}{g_1(z)}$

$k \geq l$ extends to be holom at z_0

$k < l$:

z_0 pole of $\frac{f}{g}$ of order (mult) $l - k$

$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$ (val ∞ in Riem sphere)

A merom fn on Ω is a holom fn $\Omega \longrightarrow S^2$.

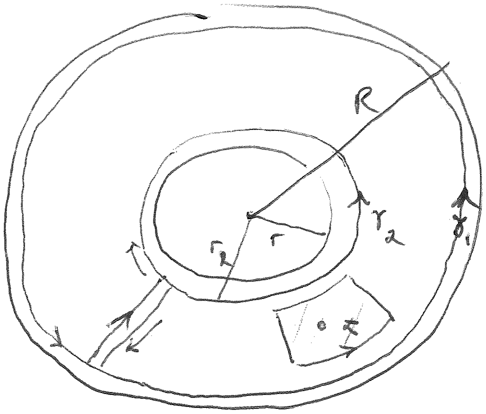
Laurent expansion

Holom fn in annulus $r < |z| < R$

has Laurent exp in annulus

$$\sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n<0} a_n z^n}_{\text{holom } |z|>r} + \underbrace{\sum_{n\geq 0} a_n z^n}_{\text{holom in } |z|<R}$$

i.e. $\sum_{n<0} a_n z^{-n} = \sum_{n=1}^{\infty} a_{-n} z^n$ holom in $|z| < \frac{1}{r}$



By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\xi)\,d\xi}{\xi - z} - \underbrace{\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)\,d\xi}{\xi - z}}$$

$$\frac{1}{\xi - z} = -\frac{1}{z}\,\frac{1}{1-\frac{\xi}{z}} = -\sum_{n<0} \frac{z^n}{\xi^{n+1}}$$

$$-\sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}$$

unif & abs cgt on $|\xi| = r_2$
for $|z| \geq r_2' > r_2$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)}{\xi^{n+1}}\,d\xi, \qquad n < 0$$

unif & abs cgt $\quad r_2 \leq |z| \leq r_1 \qquad 1 + \frac{\xi}{z} +$

THM. Any m. merom on S^2 is rational

<u>Proof</u>

$f(z)$ poles $b_1, \dots, b_n, \infty$

prine parts $P_k\left(\frac{1}{z - b_k}\right)$, $P_\infty(z)$

poly

f has pole at ∞
if $g(z) = f(1/z)$ pole at 0;
prine part of f at ∞
is poly

$$f(z) - \sum_{k=1}^{n} P_k\left(\frac{1}{z-b_k}\right) - P_\infty(z)$$

holom on S^2; $\therefore$ const

$$f(z) = \underbrace{a + P_\infty(z)} + \sum_{k=1}^{n} P_k\left(\frac{1}{z - b_k}\right)$$

expan by partial fraction

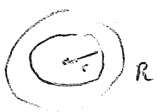
If we write $f = p/q$, then this part present only if $\deg p \geq \deg q$: quotient after divn. $\square$

Holom fn. $f(z)$ in punctured disk; e.g. $0<|z|<R$
isol. sing at $z=0$ if $f(z)$ can't be extended to holom fn on $|z|<R$;
pole or essential singularity

Extension possible iff $f(z)$ bd'd in nbd of 0

$$f(z) = \sum_{n=-\infty} a_n z^n \qquad \rho < |z| < R$$

$$f(re^{i\theta}) = \sum a_n r^n e^{in\theta} \qquad \rho < r < R$$



$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \qquad n \in \mathbb{Z}$$

(integrate term-by-term)

$$|a_n| \le \frac{M(r)}{r^n} \qquad M(r) \text{ upper bd for } |f(z)|,\ |z| = r$$

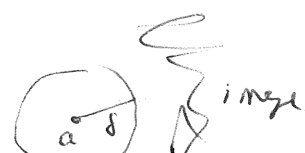
If f bd'd in punctured disk,

$$|a_n| \le \frac{M}{r^n} \quad \forall \text{ small } r \implies a_n = 0, \quad n < 0$$

$\cancel{M r^n}$

Weierstrass thm If 0 ess. sing, then $\forall \epsilon > 0$ $f(\{0<|z|<\epsilon\})$ dense in $\mathbb{C}$

Proof



Otherwise, $\exists\ a \in \mathbb{C}$ s.t.

$$|f(z) - a| \ge \delta, \qquad 0 < |z| < \epsilon$$

$$g(z) = \frac{1}{f(z)-a} \quad \text{holom and bd'd in } 0<|z|<\epsilon$$

$\therefore$ holom in $|z|<\epsilon$

$$\frac{1}{g(z)} \quad \text{merom in } |z| < \epsilon$$

$$f(z) = a + \frac{1}{g(z)} \quad \text{merom; contra.} \qquad \square$$

Residue

$f(z)$ holom in punctured disk,

$w(\gamma, a) = 1$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz \qquad \underline{\text{residue}} \text{ of } f(z)\,dz \text{ (or "of } f(z)\text{") at } a$$

$$= a_{-1}$$

(coeff in Laurent exp of a)

Residue at ∞

$$z = 1/z' \qquad f(z)\,dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz' \quad \text{in coord}^{\underline{s}} \text{ at } \infty$$

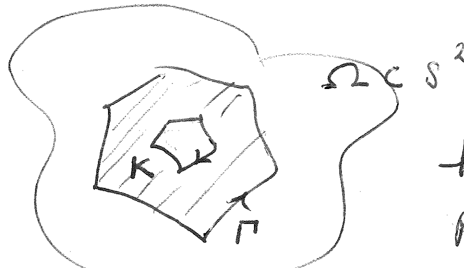
residue at ∞ is residue of

$$-\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz' = -a_{-1}$$

$$\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + \cdots$$

Laurent exp in $|z| > R$

Residue thm

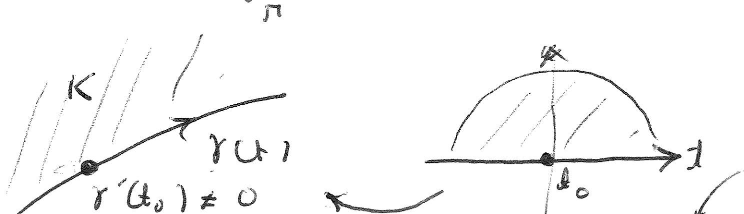


$f(z)$ holom in Ω except perhaps at isol. pts

Γ oriented bdry of compact $K \subset \Omega$, not containing any sing or ∞ (piecewise C^1)

Then $\displaystyle\int_\Gamma f(z)\,dz = 2\pi i \sum \operatorname{res}(f, z_n)$

sings in K (perhaps incl. ∞)



Jac det > 0

orientn pres, restricts to γ on real axis
then upper disk goes to interior of K