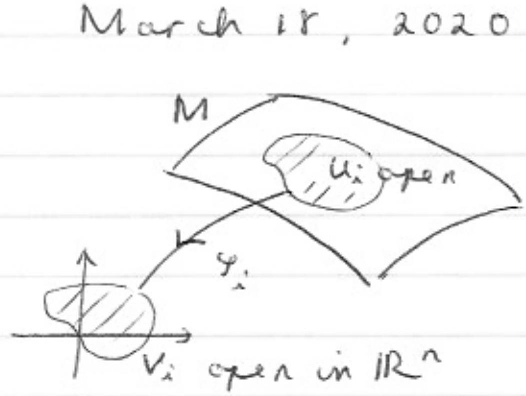
RIEMANN SURFACES

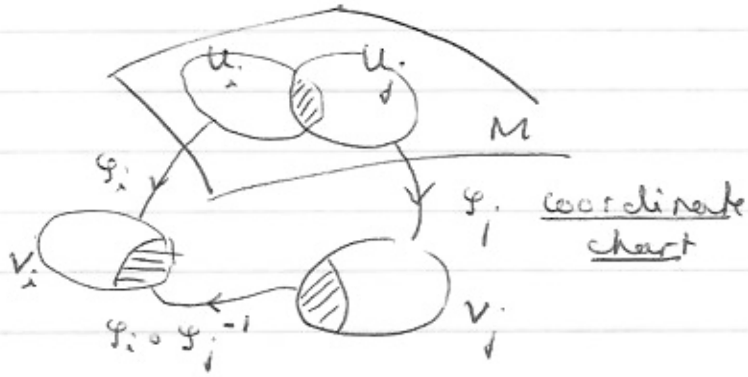
<u>Complex manifolds</u> March 18, 2020

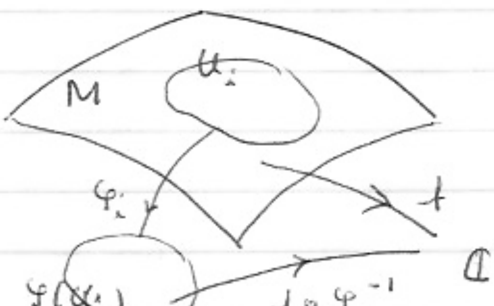
<u>n-diml topological manifold</u>:
Hausdorff, paracompact topological space, locally homeomorphic to $\mathbb{R}^n$



Complex manifold: manif with a complex structure:

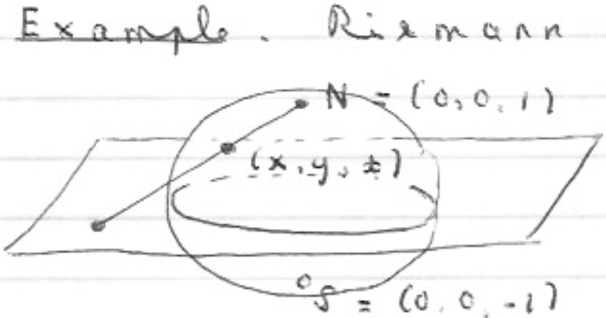
<u>Complex structure on M</u>:
Replace $\mathbb{R}^n$ by $\mathbb{C}^n$
$\varphi_i \circ \varphi_j^{-1}$ holomorphic, for all i, j





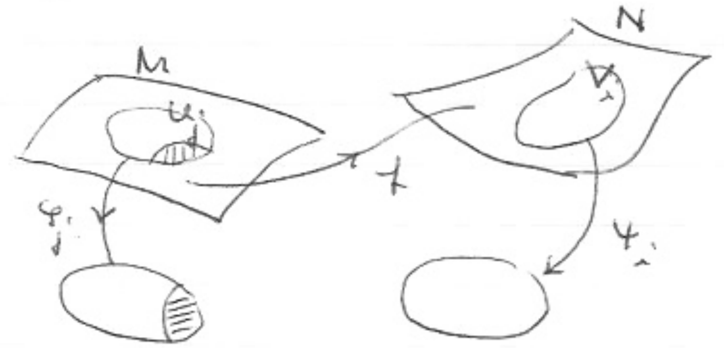
Continuous $\mathbb{C}$-valued function f on M is <u>holomorphic</u> if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ holomorphic, for all i
i.e., f can be expressed as holom. function with respect to each system of local coordinates $(z_1, \dots, z_n) = (\varphi_{i1}, \dots, \varphi_{in})$ (where $\varphi_i = (\varphi_{i1}, \dots, \varphi_{in})$)

<u>Example</u>. Riemann sphere S^2



$$\varphi_U : \underset{S^2 \setminus \{N\}}{U} \xrightarrow{\sim} \mathbb{C}$$
$$(x, y, t) \longmapsto \frac{x + iy}{1 - t}$$
$$\varphi_V : V \xrightarrow{\sim} \mathbb{C} \qquad V = S^2 \setminus \{S\}$$
$$(x, y, t) \longmapsto \frac{x - iy}{1 + t}$$

$$\varphi_u \circ \varphi_v^{-1} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$z \longmapsto 1/z$$



$f : M \to N$ <u>holomorphic mapping</u> if $\psi_i \circ f \circ \varphi_j^{-1}$ holom, for all i, j
in $\varphi_j(U_j \cap f^{-1}(V_i))$

<u>Isomorphism</u> (or <u>biholomorphism</u>) of M onto N:
Homeomorph $f : M \xrightarrow{\sim} N$ s.t. f, f^{-1} holomorphic

Say that 2 complex structures on same topological manifold M are <u>equivalent</u> if id map isomorphism

<u>Complex manifold</u>: topological manifold M together with equivalence class of complex structures

<u>Examples</u>

(1) Open subsets of $\mathbb{C}$ (or of any complex manif, with induced cx. manif. structure)

(2) Riemann sphere $S^2 = P^1(\mathbb{C})$ (<u>compact</u>)

(3) $\mathbb{C}/\mathbb{Z}$ — additive subgroup of $\mathbb{C}$
(equivalence classes: pts whose differences $\in \mathbb{Z}$)

Hausdorff space, with quotient topology
$p : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$
Complex manifold structure: Take $V \subset \mathbb{C}$ small enough that $p|V$ injective (e.g. diam $V = 1$)
$U = p(V)$; local coord $z \circ p^{-1}$ (def. d in U)

(4) $\mathbb{C}/\Gamma$ ← discrete group as before (generators e_1, e_2)
$p : \mathbb{C} \to \mathbb{C}/\Gamma$ complex manifold structure as in (3)

$M = \mathbb{C}/\Gamma$ compact
($M = p$ (closed period llgm))
topologically, a torus

<u>Complex curve</u> or <u>abstract Riemann surface</u> :
1-diml complex manifold M

Local properties of holomorphic functions extend to complex curves; e.g.,

principal of analytic continuation (2 holom maps coincide if coincide on set with limit pt)

maximum modulus principal

<u>Meromorphic function on M</u> :
holomorphic mapping $M \to S^2$

e.g. (1) $\mathbb{C} \xrightarrow{p} \mathbb{C}/\Gamma$
$f \circ p \longleftarrow f$ bijection between meromorphic functions on $\mathbb{C}/\Gamma$ and meromorphic functions on $\mathbb{C}$ with Γ as group of periods

(2) $\mathbb{C} \longrightarrow \mathbb{C}^*$ ← nonzero cx. nos.
$z \mapsto e^{2\pi i z}$
induces holomorphic mapping $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ (1-1, onto ; ∴ biholom^{+}; in fact isomorphism of topological sps)

+ Bijective holomorphic mapping of complex curves is biholomorphism
(inverse clearly locally holomorphic)