MAT454 Notes

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The main objects of study in this course are holomorphic functions.

Definition 1 (Holomorphic function)**.** *$f(t)$ is called* ***holomorphic at*** *$z \in \mathbb{C}$ if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, i.e. there is $c \in \mathbb{C}$ such that

$$f(z+h) = f(z) + c \cdot h + \varphi(h) \cdot h, \lim_{h \to 0} \varphi(h) = 0$$

Now, from this perspective, this looks no different from the usual case of a differentiable function. But it is different, because the variables are complex, and hence we can write

$$z = x + iy, \qquad f(z) = u(z) + iv(z)$$

Hence, this function mapping $z \mapsto f(z)$ is, from the real perspective, a function from $\mathbb{R}^2 \to \mathbb{R}^2$, taking

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Naturally, in the above definition, we can also write $a + ib$ and $h = \xi + i\eta$. Hence the derivative $h \mapsto c \cdot h$ can be written as

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

In other words, this says that

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \iff \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are what is called the Cauchy-Riemann equations. So the moral of the story is that holomorphic is *not* the same as differentiable as a function of two real variables. It' the same as differentiable as a function of two real variables *plus* satisfying the Cauchy-Riemann equations.

It's going to convenient throughout this course to think about derivatives in terms of differential forms. Let's suppose, to begin a bit more generally, that we're considering a complex-valued *differentiable* (not necessarily holomorphic) function $f(x, y)$.

Definition 2. *The* ***differential*** *of f is given by*

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy$$

But, we're thinking about x and y as parts of a complex number, with $z = x + iy$ and $\bar{z} = x - iy$. So we can solve for x and y in terms of z and $\bar{z}$. We can also compute the differentials

$$dz = dx + idy, \qquad d\bar{z} = dx - idy$$

So we can solve for dz and $d\bar{z}$ in terms of dz and $d\bar{z}$, getting

$$dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z})$$

In particular, we can take df and rewrite it in terms of dz and $d\bar{z}$ by substituting in these expressions. So if we do that we get

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar{z}$$

So, if we would like to define partial derivatives with respect to z and $\bar{z}$, how should we define them? Well... the coefficients above seem to be natural choices, giving

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \implies df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

In terms of *this* expression, what's a third way of writing the Cauchy-Riemann equations? It's simply

$$\frac{\partial f}{\partial \bar{z}} = 0$$

And of course, this basically captures your "feeling" of what a holomorphic function should be: it's supposed to be a function of z, and not $\bar{z}$. Ok, so this is the basic definition of holomorphic.