

# MAT454 Notes

Jad Elkhaleq Ghalayini

March 11 2020

## Theorems of Montel and Picard

### Picard's Big Theorem

Picard's big theorem says that in the neighborhood of an essential singularity a holomorphic function omits at most one complex value. That is,

**Theorem 1** (Picard's Big Theorem). *If  $z_0$  is an isolated essential singularity of a holomorphic function  $f(z)$ , then  $f$  takes every complex value with one possible exception in any neighborhood  $\Omega$  of  $z_0$ , i.e.  $\#(\mathbb{C} \setminus f(\Omega)) \leq 1$ .*

So, is this *really* the best possible statement of this kind: that is, is it really possible that a complex function can omit one possible value? Yes: for example,  $e^{1/z} \neq 0$ , so it omits value 0 even though it has an essential singularity at the origin.

So, there's also Picard's Little Theorem:

**Theorem 2** (Picard's Little Theorem). *A non-constant entire function  $f$  omits at most a point, i.e.  $\#(\mathbb{C} \setminus f(\mathbb{C})) \leq 1$*

*Proof.* So why does Picard's Little Theorem follow from Picard's Big Theorem (of course, Picard proved the little theorem first)? Well, either  $\infty$  is a pole or it is an essential singularity.

- If  $\infty$  is a pole, by the fundamental theorem of algebra we have that  $f$  is a polynomial (since there are no poles at any finite points). So in this case, it does take *every* value.
- If  $\infty$  is an essential singularity, then we can apply Picard's Big Theorem to a neighborhood of  $\infty$ .

□

So, we're going to prove Picard's big theorem using the theory of normal families, and in fact we're going to deduce it as well as a closely related, very strong theorem by Montel, from a strange lemma. This is going to be a necessary and sufficient condition for normality, but we'll express it as a necessary and sufficient condition for *failure* of normality:

**Lemma 1** (Zalcman's Lemma). *Let  $\mathcal{S}$  be a family of meromorphic functions on a domain  $\Omega$ .  $f$  is not normal in the chordal metric if and only if there is a convergent sequence  $\{a_n\} \rightarrow a_\infty \in \Omega$ , a convergent sequence of positive numbers  $\{\rho_n\} \rightarrow 0$  and a sequence of functions  $\{f_n\} \subset \mathcal{S}$  such that the sequence*

$$g_n(z) = f_n(a_n + \rho_n z)$$

*converges uniformly to  $g$  in the chordal metric on compact subsets of  $\mathbb{C}$  where  $g$  is nonconstant and meromorphic on all of  $\mathbb{C}$ . Moreover, in the case where  $\mathcal{S}$  is not normal, then we can choose the data above such that*

$$\forall z \in \mathbb{C}, g^\sharp(z) \leq g^\sharp(0) = 1$$

So, what's strange about this lemma? Of course, it's strange because normality is about the existence of convergent sequences: non-normal should be that the stuff doesn't converge, and yet here we're saying that we can give a criterion for non-normality in terms of a convergent sequence. So, why should that be? I mean, like, what about the case where  $\mathcal{S}$  were normal and we did this kind of construction: why would we get something worse? Or *would* we get something worse? That's sort of a quandry, and the point is in the case that  $\mathcal{S}$  is normal, i.e. every sequence in  $\mathcal{S}$  contains a convergent subsequence, it's not that we get something worse, it's that we get something better: we can still perform the above construction, but  $g$  would be constant!

So, what's an example? Consider  $\mathcal{S} = \{f_n(z) = z^n\}$ . Of course, this is uniformly convergent on compact subsets of the unit disc, and actually, is also uniformly convergent in the chordal metric on compact subsets of the complement of the closed disc. But, it's not uniformly convergent on compact subsets of a *bigger* open disc, e.g.  $|z| < 2$ . The issue is that it's not uniformly continuous on the unit circle, as the limit would not be continuous there. So let's look at this situation, and take  $a_n = 1$ ,  $\rho_n = \frac{1}{n}$ . That means that

$$g(z) = \lim_{n \rightarrow \infty} g_n(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

which is obviously nonconstant and entire. We know this from a proof from first year calculus, coming from the fact that

$$n \log(1 + z/n) \rightarrow z \iff \frac{\log(1 + z/n)}{z/n} \rightarrow 1$$

So, how do you compute the spherical derivative of  $g$ ? So, it's just from the formula,

$$g^\sharp(z) = \frac{2|g'(z)|}{1 + |g(z)|^2} = \frac{2|e^z|}{1 + |e^z|^2} \implies g^\sharp(0) = \frac{2}{2} = 1, \quad g^\sharp(z) \leq 1 \iff \forall t \in \mathbb{R}^+, 1 + t^2 \leq 2t$$

So that's sort of a phenomenon. Now let's prove it:

*Proof.* Suppose  $\mathcal{S}$  is normal. Then, by definition, any sequence  $\{f_n\} \subset \mathcal{S}$  has a convergent subsequence: let's relabel  $\{f_n\}$  to denote this subsequence, in other words, we can assume that  $\{f_n\} \rightarrow f$ . Let's look at any choice of data  $\{a_n\} \rightarrow a_\infty \in \omega$ ,  $\{\rho_n\} \rightarrow 0$  like in the theorem. We have

$$g_n(z) = f_n(a_n + \rho_n z)$$

By the Arzelà–Ascoli theorem, namely that normality is equivalent to equicontinuity, we have that the sequence  $\{f_n\}$  is equicontinuous, and in fact, if we want to restrict our attention to a relatively compact neighborhood of  $a_\infty$ , it will tell us that  $\{f_n\}$  is uniformly equicontinuous in a neighborhood of  $a_\infty$ . That means that

$$g(z) = \lim_{n \rightarrow \infty} g_n(z) = \lim_{n \rightarrow \infty} f_n(a_n + \rho_n z) = f(a_\infty)$$

i.e.  $g$  is a constant. □

As an exercise, let's look at a sequence  $\{f_n\}$  of holomorphic functions on a domain  $\Omega$  which converges uniformly on compact subsets of  $\Omega$  in the chordal metric. Then the limit is either holomorphic or identically equal to  $\infty$ , and moreover, in the case that it is holomorphic, the convergence is uniform not only in the chordal metric but also in the usual Euclidean metric. Try to do this for next time, and if it's not clear to you, tell me so I'll put it on the problem set. So I think I'm going to stop here today rather than getting into the middle.