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## Riemann surface associated to an elliptic curve

$X \subset \mathbb{C}^2 : \quad y^2 = 4x^3 - 20a_2x - 28a_4$
where RHS has 3 distinct roots

$X' \subset P^2(\mathbb{C}) : \quad y^2t = 4x^3 - 20a_2xt^2 - 28a_4t^3$

$$\begin{tikzcd}
\mathbb{C}^2 \supset X \arrow[r, hook, "{(x,y) \mapsto [x,y,1]}"] \arrow[d, "\varphi"] \arrow[d, bend right, "{(x,y) \mapsto x}"'] & X' \subset P^2(\mathbb{C}) \arrow[d, "\varphi'"] \\
\mathbb{C} \arrow[r, hook] & S^2 = P^1(\mathbb{C})
\end{tikzcd}$$

THM.
Suppose $a_2, a_4$ obt'd from discrete subgp. $\Gamma$ of $\mathbb{C}$:
$$a_2 = 3\sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{1}{\omega^4}, \quad a_4 = 5\sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{1}{\omega^6}$$

Then the meromorphic transf.
$$x = \wp(z), \quad y = \wp'(z) \qquad dx = y\,dz$$

defines biholomorphism
$$\begin{tikzcd}
\mathbb{C}/\Gamma \arrow[r, "\cong"] & X' \arrow[l, bend left, "z = \int \omega \ *"]
\end{tikzcd}$$

Inverse defines $z$ as holomorphic many-val'd function on $X'$, whose branches differ by consts. belonging to $\Gamma$.

$*$ $dx = y\,dz$
so $dz$ is the form $\omega$ described last time.

Conversely:

ABEL'S THEOREM.

Given $a_2, a_4$ such that
$$P(x) = 4x^3 - 20a_2 x - 28a_4$$

has 3 distinct roots, there is discrete subgp $\Gamma$ of $\mathbb{C}$ s.t. $a_2, a_4$ as above.

(Moreover, the elliptic curve
$$X' : \quad y^2 t = 4x^3 - 20a_2 x t^2 - 28a_4$$

has parametrization given by
$$[\wp(z), \wp'(z), 1] \quad .)$$

<u>Sketch of proof.</u>

$\omega = dz$ defines many-val'd fn. $z$ on $X'$.

<u>Lemma 1.</u> The different branches of $z$ are obtained from each other by adding constants that form discrete subgp $\Gamma$ of $\mathbb{C}$, and $\Gamma$ is generated by two elements $e_1, e_2$ linearly independent over $\mathbb{R}$.

Then we can introduce elliptic curve
$$y^2 = 4x^3 - 20b_2 x - 28b_4$$

where
$$b_2 = 3\sum_{\omega \neq 0} \frac{1}{\omega^4}, \qquad b_4 = 5\sum_{\omega \neq 0} \frac{1}{\omega^6}$$

Let $(X'', \wp'')$ be corresponding Riem. surface over $S^2$.

The many-valued fn. $z$ def's holom mapping

$$X' \xrightarrow{z} \mathbb{C}/\Gamma \xrightarrow{\cong} X''$$
$$[\wp, \wp', 1]$$

Lemma 2.

$$z = \int_{[0,1,0]} \omega \quad \text{def's biholomorphism } X' \to \mathbb{C}/\Gamma$$

Therefore the (non homogeneous) coords $(x, y)$ of a point of $X'$ are merom fns. of $z$ with $\Gamma$ as group of periods.

We can then show that $X' \xrightarrow{\cong} \mathbb{C}/\Gamma \xrightarrow{\cong} X''$ is the identity:

Recall:

$$X' \text{ at } \infty : \quad \underset{\underset{[x, y, 1]}{\parallel}}{[x', 1, t']} = [x', 1, 4x'^3 - 320a_2x'^7 + \cdots]$$

$x'$ and $z$ are both coords at $\infty$

$$\omega = \frac{dx}{y} = t' d\left(\frac{x'}{t'}\right) = dx' - x'\frac{dt'}{t'}$$

$$= dx' - \frac{12x'^2 + \cdots}{4x'^2 + \cdots} dx'$$

$$= -2dx'(1 + g(x'))$$

holom near $x' = 0$, $g(0) = 0$

$$x = \frac{x'}{t'} = \frac{x'}{4x'^3 - 320\, a_2 x'^7 + \cdots}$$

$$= \frac{1}{4x'^2} + \cdots = \frac{1}{z^2} + \cdots$$

since $dz = \omega = -2\,dx'\,(1 + g(x'))$

<u>Claim</u>. $x$ is a merom fn. of $z$ with double pole at $z = 0$, princ. part $1/z^2$, and no poles other than points of $\Gamma$:

Pole of order 2 by calculation above (or because each value $x(z)$ taken twice near pole: $(x(z), \pm y(z))$ )

$$x(z) = \frac{1}{z^2} + \frac{c}{z} + d + \cdots$$

$$x'(z) = -\frac{2}{z^3} - \frac{c}{z^2} + e + \cdots$$

$dz = \dfrac{dx}{y}$ : $\quad x'(z)^2 = y^2 = 4x^3 - 20\, a_2 x - 28\, a_4$

So

$$\frac{4}{z^6} + \frac{4c}{z^5} + \frac{c^2}{z^4}$$

$$= \frac{4}{z^6} + \frac{12c}{z^5} + \frac{12d + 12c^2}{z^4} + \cdots$$

$4c = 12c \implies c = 0, \quad d = 0$

Hence $x = \wp(z)$, the Weierstrass $\wp$-function associated with $\Gamma$

(since $x - \wp(z)$ holom, doubly-periodic, const term $= 0$)

$$y = \frac{dx}{dz} = \wp'(z)$$

Therefore $X' \longrightarrow X''$ is the identity; i.e.

$$b_2 = a_2, \quad b_4 = a_4 \qquad \square$$