Riemann surfaces   March 23, 2020

Example ③   Riemann surface to make the mult. valued function

$$y = (1-x^3)^{1/3}$$

single valued:

$$\begin{tikzcd} X \subset \mathbb{C}\times\mathbb{C} \arrow[d, "\varphi"'] & (x,y) \arrow[d, mapsto] \\ \mathbb{C} & x \end{tikzcd} \qquad : \quad x^3 + y^3 = 1$$

$$\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[dr, "{(x,y)\mapsto y}"] & \\ \mathbb{C} \arrow[r, dashed, "{y=(1-x^3)^{1/3}}"'] & \mathbb{C} \end{tikzcd}$$

X is a manifold:

Local coords at a point $(x_0, y_0) \in X$:

$y_0 \neq 0$: x $x \longmapsto (x, y)$, where $y =$ branch of $(1-x^3)^{1/3}$ which $= y_0$, when $x = x_0$.

$y_0 = 0$: y
(then $x_0 \neq 0$)

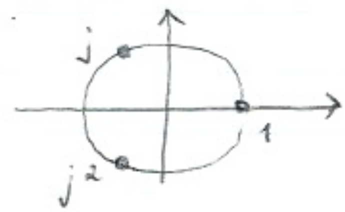
Compatibility of coord charts:
If $x_0 \neq 0$, $y_0 \neq 0$, then
$(1-x^3)^{1/3}$ has holom branch $= y_0$ when $x = x_0$;
$(1-y^3)^{1/3}$ " " " $= x_0$ " $y = y_0$

Note x, y are both holom fns on X.
E.g., x:
at (x_0, y_0), $y_0 \neq 0$, x is local coord
" " , $y_0 = 0$, $x = (1-y^3)^{1/3}$

local coord

On X, $y = (1 - x^3)^{1/3}$ single-valued holom. fn.

X has 3 points over each pt. of $\mathbb{C}$ (3 sheets)
coincide if $x = 1, j, j^2$, $j = e^{2\pi i/3}$



We can extend X to a Riemann surface over $S^2 = P^1(\mathbb{C})$:

Recall $P^2(\mathbb{C})$: $[x, y, z]$ homog. coords.
$\mathbb{C}^2 = \{ [x, y, 1] \}$
Complex curve at ∞ : $\{ z = 0 \} = P^1(\mathbb{C})$

Complex curve $x^3 + y^3 = z^3$ in $P^2(\mathbb{C})$
obtained by homogenizing the affine curve $x^3 + y^3 = 1$:

$$\left(\frac{x}{z}\right)^3 + \left(\frac{y}{z}\right)^3 = 1$$

This defines Hausdorff space $X' \subset P^2(\mathbb{C})$
Manifold structure : Problems 5, # 4.

X can be identified with subspace of X' :
$(x, y) \longmapsto [x, y, 1]$

X' consists of X together with 3 points "at infinity":
$[1, -1, 0]$, $[j, -1, 0]$, $[j^2, -1, 0]$.

<u>Remark</u> $X' : \quad x^3 + y^3 = z^3$
is a smooth cubic curve, so it can be parametrized by the Weierstrass $\wp$-function!

X' can be written in <u>Weierstrass normal form</u>
$$y^2 = 4x^3 - 20a_2 x - 28 a_4$$
after a <u>homogeneous linear transformation</u>:

Substitute
$$x = \xi + \eta$$
$$y = \xi - \eta$$
$$z = \zeta$$

We have $(\xi + \eta)^3 + (\xi - \eta)^3 = \zeta^3$
i.e. $2\xi^3 + 6\xi\eta^2 = \zeta^3$

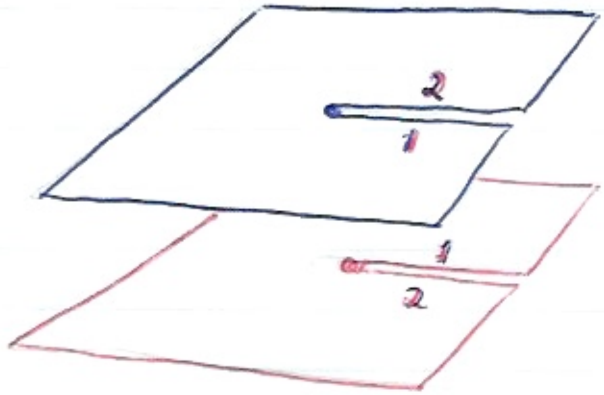
Dehomogenize wrt. ξ: (i.e. look in appropriate coord chart)
$$2 + 6\eta^2 = \zeta^3$$
or
$$6\eta^2 = \zeta^3 - 2$$

After homogeneous linear change of variable:
$$y^2 = \underbrace{x^3 - 1}_{3 \text{ distinct roots}}$$

<u>Exercise</u> Explicitly find the lattice such that the cubic curve $x^3 + y^3 = 1$ is parametrized by the corresponding Weierstrass $\wp$-function.

Construction of Riemann surfaces by cutting and pasting.

$y = x^{1/2}$



$y = (1 - x^3)^{1/3}$