MAT454 Notes

Jad Elkhaleq Ghalayini

March 11 2020

Theorems of Montel and Picard

Picard's Big Theorem

Picard's big theorem says that in the neighborhood of an essential singularity a holomorphic function omits at most one complex value. That is,

Theorem 1 (Picard's Big Theorem)**.** *If z_0 is an isolated essential singularity of a holomorphic function $f(z)$, then f takes every complex value with one possible exception in any neighborhood Ω of z_0, i.e. $\#(\mathbb{C}\backslash f(\Omega)) \leqslant 1$.*

So, is this *really* the best possible statement of this kind: that is, is it really possible that a complex function can omit one possible value? Yes: for example, $e^{1/z} \neq 0$, so it omits value 0 even though it has an essential singularity at the origin.

So, there's also Picard's Little Theorem:

Theorem 2 (Picard's Little Theorem)**.** *A non-constant entire function f omits at most a point, i.e. $\#(\mathbb{C}\backslash f(\mathbb{C})) \leqslant 1$*

Proof. So why does Picard's Little Theorem follow from Picard's Big Theorem (of course, Picard proved the little theorem first)? Well, either ∞ is a pole or it is an essential singularity.

- If ∞ is a pole, by the fundamental theorem of algebra we have that f is a polynomial (since there are no poles at any finite points). So in this case, it does take *every* value.
- If ∞ is an essential singularity, then we can apply Picard's Big Theorem to a neighborhood of ∞.

□

So, we're going to prove Picard's big theorem using the theory of normal families, and in fact we're going to deduce it as well as a closely related, very strong theorem by Montel, from a strange lemma. This is going to be a necessary and sufficient condition for normality, but we'll express it as a necessary and sufficient condition for *failure* of normality:

Lemma 1 (Zalcman's Lemma)**.** *Let $\mathcal{S}$ be a family of meromorphic functions on a domain Ω. f is <u>not</u> normal in the chordal metric if and only if there is a convergent sequence $\{a_n\} \to a_\infty \in \Omega$, a convergent sequence of positive numbers $\{\rho_n\} \to 0$ and a sequence of functions $\{f_n\} \subset \mathcal{S}$ such that the sequence*

$$g_n(z) = f_n(a_n + \rho_n z)$$

converges uniformly to g in the chordal metric on compact subsets of $\mathbb{C}$ where g is nonconstant and meromorphic on all of $\mathbb{C}$.