MAT454 Notes

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Definition 1 (n-dimensional complex projective space)**.** *We define*

$$\mathrm{P}^n(\mathbb{C}) = \mathbb{C}^{n+1} \backslash \{0\} / \sim$$

where

$$(x_0, ..., x_n) \sim (x_0', ..., x_n') \iff \exists \lambda \in \mathbb{C}, (x_0', ..., x_n') = (\lambda x_0, ..., \lambda x_n)$$

We denote the equivalence class of $(x_0, ..., x_n)$ *by* $[x_0, ..., x_n]$*.*

Definition 2 (Homogeneous coordinates)**.** *We define coordinate charts* $U_i = \{[x_0, ..., x_n] \in \mathrm{P}^n(\mathbb{C}) : x_i \neq 0\}$ *with affine coordinates* $U_i \to \mathbb{C}^n$*,*

$$[x_0, ..., x_n] \mapsto \left(\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right)$$

with inverse

$$(g_1, ..., g_n) \mapsto [g_1, ..., g_{i-1}, 1, g_{i+1}, .., g_n]$$

Using these coordinates, we have that $\mathrm{P}^n(\mathbb{C})$ has the structure of an n-dimensional complex manifold, as the transition mappings are rational. Let's take one of the charts here, say U_0, to be $\mathbb{C}^n$. So

$$\mathrm{P}^n(\mathbb{C}) = U_0 \cup \text{ everything else}$$

But what's everything else? So U_0 is all the points where $x_0 \neq 0$, so everything else is the set of points

$$\{x_0 = 0\} = \{[0, x_1, ..., x_n]\} \simeq \mathrm{P}^{n-1}(\mathbb{C}) \implies \mathrm{P}^n(C) = U_0 \cup \mathrm{P}^{n-1}(\mathbb{C})$$

We call this copy of $\mathrm{P}^{n-1}(\mathbb{C}) \simeq \{x_0 = 0\}$ the **hyperplane at infinity**. This is like a generation of the Riemann sphere which we saw before, which we saw was given by $S^2 = \mathrm{P}^1(\mathbb{C})$. So when we talk about $\mathrm{P}^2(\mathbb{C})$, that's like having 2-complex coordinates with a line at infinity. Specifically, we can write it as

$$\mathrm{P}^2(\mathbb{C}) = \{[x, y, t]\} = \mathbb{C}^2_{(x,y)} \cup \{t = 0\}$$

the **projective line at infinity**. Now assume we have a curve $X \subset \mathbb{C}^2$ generated by the equation

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

where the RHS has three distinct roots. We want to compute the **compactification of** X **in** $\mathrm{P}^2(\mathbb{C})$. We can write this down in homogeneous coordinates

$$y^2t = 4x^3 - 20a^2xt^2 - 28a_4t^3$$

taking X' to be the solution set of this. Why is this the right thing? When you look at $\mathrm{P}^2(\mathbb{C})$, and look in here at the set of points

$$\{[x, y, t] : t \neq 0\} \simeq \mathbb{C}^2_{(x,y)}$$

we see that it is has homomorphism

$$[x, y, t] \mapsto \left(\frac{x}{t}, \frac{y}{t}\right)$$

Hence, we rewrite our eqaution in our new coordinates for $\mathbb{C}^2$,

$$\frac{y^2}{t^2} = 4\frac{x^3}{t^3} - 20a_2\frac{x}{t} - 28a_4$$

Now we can just multiply both sides by t^3. So if you haven't seen this before, this takes a little bit of familiarity, but the actual operations involved are very simple operations. Of course, our *original* X is a subspace of X'. But how much have we added to X? Well, if we set $t = 0$, we get $x = 0$. So, how many points are we adding? One point, at ∞:

$$X' = X \cup \{[0, 1, 0]\}$$

Now, in the neighborhood of any finite point, X' just looks like X. What about in a neighborhood of the point at ∞, $[0, 1, 0]$? What does it look like? So this point $[0, 1, 0]$ doesn't actually lie in the coordinate chart $U_0 = \{x \neq 0\}$, it lies in $U_1 = \{y \neq 0\}$. This chart has affine coordinates given by $(x', t') = (x/y, t/y)$. So what's the equation of X' in *this* coordinate chart? It's

$$t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3$$

In some neighborhood of $(x', t') = (0, 0)$ (the point at infinity), the implicit function theorem tells us that t' is a holomorphic function of x':

$$t' = 4x'^3 - 320a_2x'^7 + ...$$

As an extercise, we can take the Taylor series

$$t' = b_0 + b_1x + b_2x^2 + ...$$

plug it into the equation and solve successively for the coefficients. If you haven't done that before, do the exercise.

This tells us, in general, though, that t' is a function of x'. So in a neighborhood of the point at infinity, X' looks like the graph of this function. Therefore, in the chart where $y \neq 0$, X' consists of the points given by the formula

$$[x', 1, t' = 4x'^3 - 320a_2x'^7 + ...]$$

Let's now consider a map $\varphi' = \varphi$ on X, $\infty \mapsto \infty$ in the Riemann sphere. φ sends the above point to $\frac{x'}{t'} \in \mathbb{C}$ or $\frac{t'}{x'}$ in coordinates at ∞, i.e. to $[x', t'] \in \mathrm{P}^1(\mathbb{C}) = S^2$. So this is a well-defined holomorphic function.

Now suppose

$$a_2 = 3\sum_{\substack{w\in\Gamma \\ w\neq 0}} \frac{1}{w^4}, \qquad a_4 = 5\sum_{\substack{w\in\Gamma \\ w\neq 0}} frac1w^6$$

and let's look at the meromorphic mapping $(x, y) = (\wp(z), \wp'(z))$, i.e.

$$z \mapsto [\wp(z), \wp'(z), 1]$$

We claim that this defines a homeomorphism from $\mathbb{C}/\Gamma$ to X'.

If $\wp'(z) \neq 0$, this is the same thing in homogeneous coordinates as

$$\left[\frac{\wp(z)}{\wp'(z)}, 1, \frac{1}{\wp'(z)}\right]$$

Now, what does this look like? $\wp$ begins with a/z^2, whereas $\wp'$ begins with b/z^3, so $\frac{\wp}{\wp'}$ begins with cz, whereas $\frac{1}{\wp'}$ begins with dz^3. On the other hand, $0 \mapsto \infty$, and in fact $\Gamma \mapsto \infty$.