MAT454 Notes

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We're interested in a biholomorphism $w = f(z)$ from a polygonal region enclosed by$z_1, ..., z_n$, with $w_k = f(z_k)$ to the unit disc, where

- $0 < \alpha_k < 2$
- $-1 < \beta_k < 1, \sum \beta_k = 2$
- $\alpha_k + \beta_k = 1$,
- The intersection of the line between z_{k-1} and z_k and the line between z_k and z_{k+1} has angle $\alpha_k \pi$ inside the polygon and $\beta_k \pi$ outside the polygon

We want to find a formula for the inverse function $z = F(w)$. The statement of the theorem (though last time we wrote it as an integral) is that, for some constant c,

$$F'(w) = c \prod (w - w_k)^{\beta_k}$$

We have that $\zeta = (z - z_k)^{1/\alpha_k}$ is invertible and maps the "angle" α_k to the half-disc. Writing

$$w = f(z_k + \zeta^{\alpha_k}) = g(\zeta), \zeta = (w - w_k)g(w) \implies F(w) = z_k + (w - w_k)_k^{\alpha} G_k(w)$$
$$\implies F'(w) = (w - w_k)^{\alpha_k - 1} G_k(w)$$

So

$$F'(w)(w - w_k)^{\beta_k}$$

is holomorphic and nonzero near w_h. So

$$H(w) = F'(w) \prod (w - w_k)_k^{\beta}$$

is holomorphic and nonzero in a neighborhood of the closed unit disk. To show that $H(w)$ is constant, it is enough to show that $\arg H(w) = \Im \log H(w)$ is constant on S^1 (this is well defined as zero is not included so there is a branch of log). This works because H is a harmonic function, and therefore we can use the Mean Value Property and the Maximum Modulus Principle.

So we just have to compute the argument. Let's look at what happens at a point $e^{i\theta}$ on the arc between w_{k-1} and w_k. We compute

$$\frac{d}{d\theta} F(e^{i\theta}) = F'(e^{i\theta}) i e^{i\theta}$$

We have that, since $F(e^{i\theta})$ is a parametrization of a straight line,

$$\arg \frac{d}{d\theta} F(e^{i\theta}) = 0 \implies \arg F'(e^{i\theta}) = \mathit{const} - (\theta + \pi/2)$$

We have that

$$\arg(e^{i\theta} - w_k) = \theta/2 + \mathit{const} \implies \arg F'(e^{i\theta}) \prod (e^{i\theta} - w_k)^{\beta_k} = \mathit{const} - \theta + (\sum \beta_k) \frac{\theta}{2} = \mathit{const}$$

This shows $\arg H(w)$ is constant on the open arc from w_k to w_{k+1} for all k, but it's continuous because $\log H(w)$ is well-defined. Therefore, H is constant on S^1, completing the proof.