MAT454 Notes

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Let's, as we usually do, consider a holomorphic function $f(z)$ in an open set $\Omega \subseteq \mathbb{C}$. Last time, we showed that f has a convergent power series expansion in any open disc in Ω (centered at the center of the disc). For example, around $a = 0 \in \Omega$, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Writing $z = re^{i\theta}$, we get

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

We can write out the following formula for these Fourier coefficients:

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) d\theta$$

Today we're going to be looking at the consequences of this formula. First of all, this formula gives a simple but useful upper bound on a_n: if we take the maximum absolute value of f along the circle of radius r, written

$$M(r) = \sup_{\theta} |f(re^{i\theta})|$$

we get

$$|a_n| \leqslant \frac{M(r)}{r^n}$$

These are called **Cauchy's inequalities**. These have some important consequences, like **Liouville's theorem**: a bounded holomorphic function on all of $\mathbb{C}$ is a constant. How does this follow? Well, if c is the upper bound of f on $\mathbb{C}$, we have each

$$\forall r \in \mathbb{R}^+, M(r) \leqslant c \implies |a_n| \leqslant \frac{M(r)}{r^n} \leqslant \frac{c}{r^n}$$

Hence, for $n \leqslant 0$, $0 \leqslant c \leqslant \epsilon$ for all $\epsilon > 0$, implying $c = 0$.