Lemmas for Abel's theorem March 30, 2020

Abel's theorem Given a_2, a_4 such that

$$P(x) = 4x^3 - 20a_2 x - 28a_4$$

has 3 distinct roots, there is a discrete subgroup Γ of $\mathbb{C}$ such that

$$a_2 = 3\sum_{\substack{\omega\in\Gamma\\ \omega\neq 0}} \frac{1}{\omega^4}, \quad a_4 = 5\sum \frac{1}{\omega^6}$$

Proof

$\omega = \frac{dx}{y} = dz$ defines many-val'd function z on ellipti curve $X' \subset P^2(\mathbb{C})$.

$$X' : \quad y^2 t = 4x^3 - 20a_2 x t^2 - 28 a_4 t^3$$

Lemma 1. The different branches of z are obtained from each other by adding constants that ~~form~~ form discrete subgroup Γ of $\mathbb{C}$, and Γ is generated by 2 elements $e_1, e_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$

Proof.

$z = \int_{P_0}^{P} \omega$, $P_0 = [0, 1, 0]$ Fix base point as pt. at ∞ (so $z = 0$ at ∞)

integral over curve from P_0 to P in X'

ω has primitive locally, so that, locally, $z(p)$ is single-val'd holomorphic fn. of p.

Globally, $z(p)$ depends on choice of curve; well-defined up to addition of a <u>period</u> of ω,

$$\pi(\gamma) := \int_\gamma \omega$$

γ 1-<u>cycle</u> in X'
(essentially piecewise C^1 closed curve)

Formally: C^1 1-<u>chain</u> with <u>boundary</u> 0

formal $\mathbb{Z}$-linear combination of singular C^1 1-<u>simplices</u>

If γ is a boundary itself; i.e.,
$\gamma = \partial\sigma$, σ 2-chain,
then

$$\pi(\gamma) = \int_{\partial\sigma} \omega = \int_\sigma d\omega = 0$$

Stokes' thm $\qquad$ $\omega = f(z)\,dz$ in local coords

$$d\omega = df \wedge dz$$
$$= \left(\frac{\partial f}{\partial z}\,dz + \underbrace{\frac{\partial f}{\partial \bar z}}_{0}\,d\bar z\right) \wedge dz$$
$$= \frac{\partial f}{\partial z}\,dz \wedge dz = 0.$$

Also,

$$\pi(\gamma_1 + \gamma_2) = \pi(\gamma_1) + \pi(\gamma_2)$$

Thus π induces homomorphism from 1st homology group $H_1(X'; \mathbb{Z})$ of X' to $\mathbb{C}$.

i.e. $\quad z : X' \longrightarrow \mathbb{C}/\Gamma$

where $\Gamma = \text{Image } \pi$

$$= \underbrace{\{\pi(\gamma) : \gamma \in H_1(X'; \mathbb{Z})\}}_{\text{group of periods}}$$

$H_1(X'; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$

by Riemann - Hurwitz formula :

X' $\overset{n}{2}$-to-1 covering
$\downarrow$ with 4^{k} ramification points
S^2 each of ramification index (order) 2

Euler characteristic of compact, connected orientable surface S

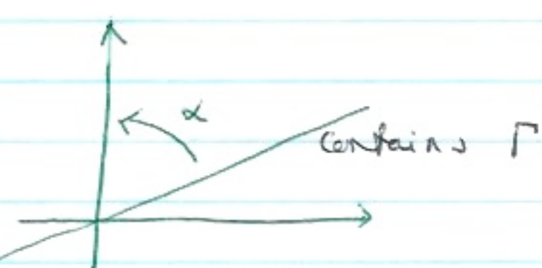
$$\chi(S) = 2 - 2g \qquad g = \text{genus}$$

$$\chi(X') = \underbrace{n}_{2} \underbrace{\chi(S')}_{2} - \underbrace{\sum_{\text{ramif pts}}}_{(4 \text{ points})} \left(\underbrace{\text{ramif index}}_{2} - 1 \right)$$

$$= 0$$

so genus of X' is 1 : X' topologically a torus

Γ is a lattice in $\mathbb{C}$:
If not, then Γ contained in 1-dim'l real subspace of $\mathbb{C}$;
i.e. $\exists$ $\alpha \in \mathbb{C}$, $\alpha \neq 0$, s.t.
$\mathrm{Re}(\alpha \pi(\gamma)) = 0$, $\forall \gamma$



Then $\mathrm{Re}(\alpha z)$ has no periods:
single - valued harmonic function on X'

Since X' compact, it has maximum :
impossible unless $\mathrm{Re}(\alpha z)$ constant ;
therefore αz constant : contradiction $\square$

(X'', φ'') Riemann surface over S^2 corresponding to elliptic curve

$$y^2 t = 4x^3 - 20 b_2 x t^2 - 28 b_4 t^3$$

where

$$b_2 = 3 \sum \frac{1}{\omega^4}, \quad b_4 = 5 \sum \frac{1}{\omega^6}$$

Then we have

$$X' \xrightarrow{z} \mathbb{C}/\Gamma \underset{[\wp, \wp', 1]}{\xrightarrow{\cong}} X''$$

<u>Lemma 2.</u> $z = \int_{P_0} \omega$ defines biholomorphism $X' \to \mathbb{C}/\Gamma$

<u>Proof.</u>

$X' \xrightarrow{z} \mathbb{C}/\Gamma$ finite sheeted covering.

(z everywhere local coord on X',
so z local homeo $X' \to \mathbb{C}/\Gamma$.
z onto since image open and closed (∵ X' compact))

Such covering corresponds to sublattice Γ' of Γ; i.e. z factors as

$$z : X' \underset{\text{homeomorphism}}{\xrightarrow{\sim}} \mathbb{C}/\Gamma' \longrightarrow \mathbb{C}/\Gamma$$

Lifting to universal covering spaces

$$\begin{array}{ccccc} \tilde{X}' & \xrightarrow{\tilde{z}} & \mathbb{C} & \xrightarrow{=} & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{z} & \mathbb{C}/\Gamma' & \longrightarrow & \mathbb{C}/\Gamma \end{array}$$

$\tilde{z}(q_1) - \tilde{z}(q_2) \in \Gamma'$, for all $q_1, q_2 \in \tilde{X}'$ over same point of X'.

Therefore
$\{\tilde{z}(q_1) - \tilde{z}(q_2) : q_1, q_2$ in fibre over $p \in X'\}$
is precisely group of periods $\pi(\gamma)$;
i.e., $\underline{\underline{\Gamma}}$

So $\Gamma = \Gamma'$ $\square$