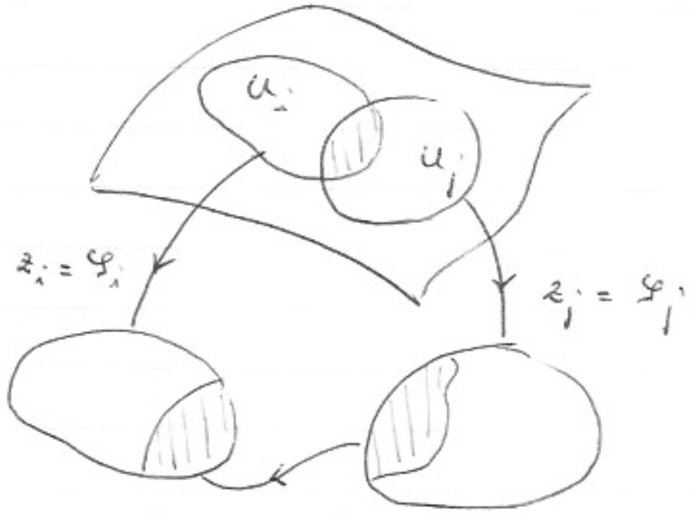
March 20, 2020

Integration of holomorphic differential forms



ω holomorphic differential form on M

$$z_i = g_{ij}(z_j) = (\varphi_i \circ \varphi_j^{-1})(z_j) \qquad dz_i = g_{ij}'(z_j)\, dz_j$$

$$\omega_i = f_i(z_i)\, dz_i$$
$$\omega_j = f_j(z_j)\, dz_j = f_i(g_{ij}(z_j))\, g_{ij}'(z_j)\, dz_j \quad \text{in overlap}$$

e.g. S^2 $\qquad z = 1/z'$ $\qquad z \in \mathbb{C}$, z' coordinate at ∞

$$\omega = f(z)\, dz = f\left(\frac{1}{z'}\right)\left(-\frac{1}{z'^2}\right) dz'$$

In some neighbourhood of each point of M, ω has <u>primitive</u> g (i.e. holomorphic function g such that $dg = \omega$); g uniquely det'd up to addn. of const.

If M simply-connected, ω has a global primitive

In general $\int_{\substack{\text{closed}\\ \text{curve}}} \omega$ needn't be zero

period of the integral $\int \omega$

Residue of a holomorphic differential form

ω holomorphic differential form in complement of a discrete set $E \subset M$.

Consider $a \in E$, z local coordinate at a $(z(a) = 0)$

$$\omega = f(z)\, dz$$
$$= \underset{\substack{\text{holom. in}\\ \text{nbhd. of } a}}{\omega_1} + \left(\frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \right) dz$$

using Laurent expansion of f at a

γ closed path in small nbhd of a s.t. winding no. wrt. a is $+1$.

residue of ω at a

$$\frac{1}{2\pi i} \int_\gamma \omega = c_1$$

(so c_1 indep. of choice of local coord)

Residue theorem

ω holomorphic diff'l form in complement of discrete set E

Γ oriented bdry of compact K s.t. Γ contains no point of E.

Complex manifold has natural orientation so this makes sense

Then

$$\int_\Gamma \omega = 2\pi i \times \text{sum of residues of } \omega \text{ at points of } E \text{ in } K$$

Riemann surfaces

Y complex curve e.g. $Y = \mathbb{C}$, S^2

Riemann surface over Y

$\underset{\text{connected complex curve}}{\varphi : X} \to Y$ non-constant holomorphic mapping

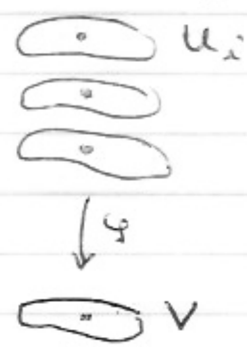
Ramification point of φ :
point where multiplicity > 1

φ is open mapping
Ramification points isolated
Inverse image of point of Y is discrete

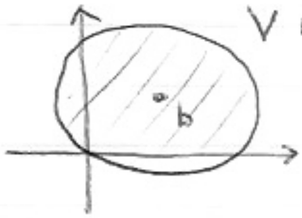
φ not necessarily injective,
even if unramified.

Example ①. $Y = \mathbb{C}^*$ (not simply-conn.)
$X = \mathbb{C}$, $\varphi : z = e^t$

In this example, $\varphi : X \to Y$ is a covering space of Y ; i.e. unramified Riemann surface such that, every $b \in Y$ has a nbhd V s.t. $\varphi^{-1}(V)$ = disjoint union of open $U_i \subset X$, each mapped isomorphically onto V by φ



An example:



$V: |z-b| < |b|$

Each branch of $\log z$ defines isomorphism $V \xrightarrow{\sim}$ open set in $\mathbb{C}$. These open sets disjoint,
$\bigcup_i U_i = \varphi^{-1}(V)$,
$\varphi | U_i : U_i \xrightarrow{\sim} V$

(Theorem. Any connected open set in $\mathbb{C}$ (or any connected complex manifold Y) has a simply-connected covering space.)

An example:
$z = e^t$ is a local coord in a nbhd of any point of X (i.e., holomorphic function on X can be expressed locally as holom function of z; not in general globally).

e.g. t is a holom. function on X; in a nbhd of each point, it is a branch of $\log z$

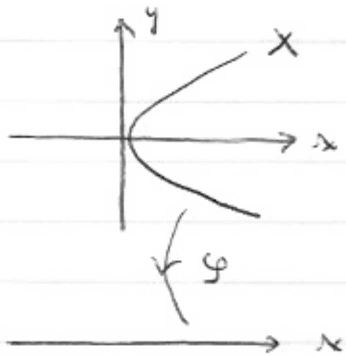
"We make $\log z$ single-valued by lifting it to the Riemann surface"

$$\begin{tikzcd} \mathbb{C} = X \arrow[d, "e^t = \varphi"'] \arrow[dr, "t"] & \\ \mathbb{C}^* = Y \arrow[r, "\log z"'] & \mathbb{C} \end{tikzcd}$$

multivalued

2 $y = x^{1/2}$ We make this multivalued function single-valued by introducing Riemann surface

$$\begin{array}{ccc} X & \subset & \mathbb{C} \times \mathbb{C} \quad : \quad x - y^2 = 0 \\ \downarrow \varphi & : & (x, y) \mapsto x \\ \mathbb{C} & & \end{array}$$



$$\begin{tikzcd} X \arrow[d, "\varphi"'] \arrow[dr, "(x,y) \mapsto y"] & \\ \mathbb{C} \arrow[r, "x^{1/2}"'] & \mathbb{C} \end{tikzcd}$$

x is a local coordinate on X <u>outside</u> of 0