

# MAT454 Notes

Jad Elkhaleq Ghalayini

March 11 2020

## Theorems of Montel and Picard

### Picard's Big Theorem

Picard's big theorem says that in the neighborhood of an essential singularity a holomorphic function omits at most one complex value. That is,

**Theorem 1** (Picard's Big Theorem). *If  $z_0$  is an isolated essential singularity of a holomorphic function  $f(z)$ , then  $f$  takes every complex value with one possible exception in any neighborhood  $\Omega$  of  $z_0$ , i.e.  $\#(\mathbb{C} \setminus f(\Omega)) \leq 1$ .*

So, is this *really* the best possible statement of this kind: that is, is it really possible that a complex function can omit one possible value? Yes: for example,  $e^{1/z} \neq 0$ , so it omits value 0 even though it has an essential singularity at the origin.

So, there's also Picard's Little Theorem:

**Theorem 2** (Picard's Little Theorem). *A non-constant entire function  $f$  omits at most a point, i.e.  $\#(\mathbb{C} \setminus f(\mathbb{C})) \leq 1$*

*Proof.* So why does Picard's Little Theorem follow from Picard's Big Theorem (of course, Picard proved the little theorem first)? Well, either  $\infty$  is a pole or it is an essential singularity.

- If  $\infty$  is a pole, by the fundamental theorem of algebra we have that  $f$  is a polynomial (since there are no poles at any finite points). So in this case, it does take *every* value.
- If  $\infty$  is an essential singularity, then we can apply Picard's Big Theorem to a neighborhood of  $\infty$ .

□

So, we're going to prove Picard's big theorem using the theory of normal families, and in fact we're going to deduce it as well as a closely related, very strong theorem by Montel, from a strange lemma.