2. TOPOLOGY OF A SPACE OF HOLOMORPHIC FUNCTIONS

Ω open in $\mathbb{C}$
$\mathcal{C}(\Omega)$ ring of contin $\mathbb{C}$-val'd fns on Ω
$\mathcal{H}(\Omega)$ subring of holom fns

Topology of $\mathcal{C}(\Omega)$:
Seq $\{f_n\} \subset \mathcal{C}(\Omega)$ <u>cges unif on compact subsets</u> if $\forall\ K \subset \Omega$ comp., $\{f_n|K\}$ cges unif

Unif cce on compact subsets det's <u>topology</u> on $\mathcal{C}(\Omega)$ ("compact-open" topology)

Fund system of open nbhds of 0:
$V(K, \epsilon) = \{f : |f(z)| < \epsilon,\ z \in K\}$
$K \subset$ comp, $\epsilon > 0$
($f_n \to f$ unif on compact subsets iff $\forall\ K, \epsilon$, $f - f_n \in V(K, \epsilon)$, n large enough)

Nbhds of any f can be obt'd by translating nbhds of 0 by f.

$\mathcal{C}(\Omega)$ is metrizable (in fact, top can be def'd by a translation inv't. metric):

$\Omega = \bigcup K_i$, $K_1 \subset K_2 \subset \dots$ exhaustion by comp set, s.t. each comp K in some K_i

$$d(f) := \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, M_i(f)\}$$

Cons cover of Ω by cl. disks of rational centre and radius
$K_j = \bigcup_{i \le j} D_i$

where $M_i(f) = \max_{K_i} |f(z)|$

$d(f) = 0$ iff $f = 0$
$d(f+g) \le d(f) + d(g)$
Hence $d(f-g)$ is a metric (sat. Δ ineq), transl. inv.

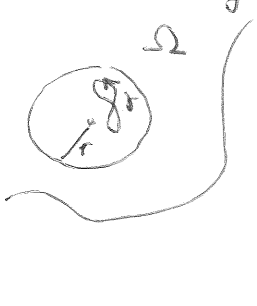
$\mathcal{C}(\Omega)$ is <u>complete</u>. Limit of a seq of fns that converges unif on comp sets is contin.

We give $\mathcal{H}(\Omega)$ the <u>subspace topology</u>

THM (Weierstrass)
(1) $\mathcal{H}(\Omega)$ is a closed subsp of $\mathcal{C}(\Omega)$
(2) The mapping $\mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $f \mapsto f'$ contin.

(1) means:
If $\{f_n\} \in \mathcal{H}(\Omega)$ cges unif on comp sets, then $f = \lim f_n \in \mathcal{H}(\Omega)$
(2) means:
If $\{f_n\} \in \mathcal{H}(\Omega)$ cges unif to f on comp sets then $\{f_n'\}$ cges to f' unif on comp sets

<u>Proof</u>
(1) Enough to show $f(z)\,dz$ closed form (by Morera)



γ closed curve in $|z-a| < r$

$\int_\gamma f_n(z)\,dz = 0$

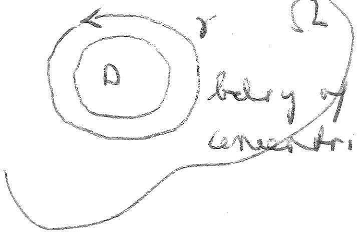
unif limit of f_n on γ

$\int_\gamma f(z)\,dz = \lim \int_\gamma f_n(z)\,dz = 0$

so $f(z)\,dz$ closed

(2) Suppose $f_n \to f$ unif on comp subsets

Enough to show $f_n' \to f'$ unif on closed disk $D \subset \Omega$



bdry of larger concentric disk

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi, \qquad z \in D$$

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)^2}\, d\xi$$

$$f'(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n(\xi)}{(\xi - z)^2}\, d\xi = \lim_{n\to\infty} f_n'(z)$$

and limit approached unif for $z \in D$ $\square$

<u>Applicn to series of holom fns</u>

COR. If series of holom fns Σf_n cges unif on comp subset of Ω, then sum $f = \Sigma f_n$ holom on Ω, and series can be differentiated term-by-term.

PROP. (Hurwitz).

Ω <u>domain</u>

If $\{f_n\} \subset \mathcal{H}(\Omega)$ cges unif on comp. sets and each f_n vanishes nowhere in Ω, then $f = \lim f_n$ either never zero or ident. zero

<u>Proof</u>

Suppose f not id zero

Zeros of f isolated (since Ω conn)

Suppose $f(z_0) = 0$

Mult. of this zero is

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{f_n'(z)}{f_n(z)}\, dz = 0$$

γ small circle centre z_0

since f_n never 0 : contra $\square$

COR. Ω domain
If $\{f_n\} \subset \mathcal{H}(\Omega)$ cges unif on comp sets and each f_n 1-1, then $f = \lim f_n$ either 1-1 or const.

Proof.
Assume f not const and $f(z_1) = f(z_2) = a$, say, $z_1 \neq z_2$ in Ω

U (z_1), V (z_2) disj open nbds in Ω

$f(z) - a$ vanishes at pt of U so there is subseq $\{f_{n_i}\}$ ~~which~~ s.t. $f_{n_i}(z) - a$ van. at pt of U

Then there is subseq $\{f_{n_{i_j}}\}$ s.t. $f_{n_{i_j}}(z) - a$ van at pt of V; contra. $\square$

Series of meromorphic fns

$\{f_n\}$ seq of merom fns on open $\Omega \subset \mathbb{C}$

$\sum_{n=1}^{\infty} f_n$ converges unif (resp. unif & abs) on $X \subset \Omega$ if all but fin many f_n have no pole in X and form unif (resp. unif & abs) cgt ~~seq~~ series on X

We cons. series of merom fns that cge unif on comp. subsets of Ω. Can define sum on relatively compact open $U \subset \Omega$ as

$$\underbrace{\sum_{n \leq n_0} f_n}_{\text{merom}} + \underbrace{\sum_{n > n_0} f_n}_{\substack{f_n \text{ no pole in } \bar{U}: \\ \text{sum unif cgt on } \bar{U}}}$$

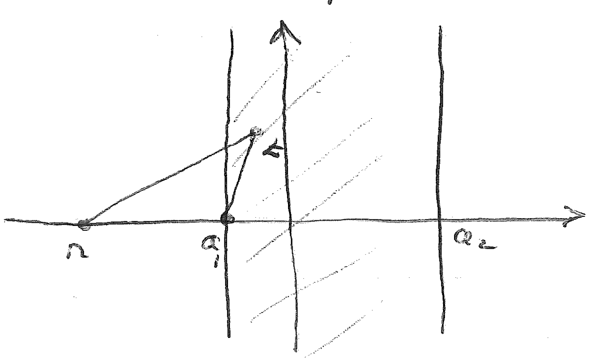
Merom fn thus def'd on U indep of choice of n_0

THM. Σf_n series of merom fns on Ω
If series unif cgt on compact subsets of Ω.
then sum f is merom fn on Ω. and
$\Sigma f_n'$ cges unif on comp subset of Ω to f'

<u>Rmk</u> Pole sets $P(f) \subset \cup P(f_n)$
(= if $P(f_n)$ pairwise disjoint; in this case, pole of order k of f_n is pole of order k of f)

<u>Examples</u> ① $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$

Unif & abs cgt on compact subset of $\mathbb{C}$:
Enough to show unif & abs cgt on any vertical strip
$a_1 \le x \le a_2$;
thus, after removing fin many terms, have series of holom fns unif & abs. cgt in strip



$\displaystyle\sum_{n < a_1} \frac{1}{(z-n)^2}$ unif & abs cgt in strip $\because$ each |term| bd'd above by
$$\sum \frac{1}{(a_1-n)^2}$$

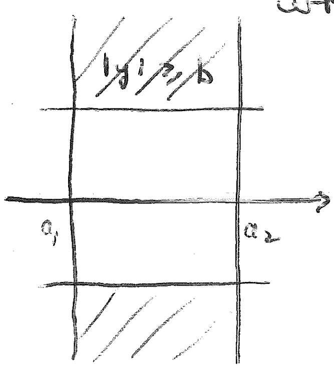
$\displaystyle\sum_{n > a_2}$ likewise

Let $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ merom on $\mathbb{C}$

<u>period</u> 1 $\quad (f(z+1) = f(z))$
poles : $z = n$, all double poles,
princ part $\dfrac{1}{(z-n)^2}$ $\quad (\therefore \text{res} = 0)$

$f(z) = \underbrace{\left(\frac{\pi}{\sin \pi z}\right)^2}_{g(z)}$ Enough to show $g(z)$ merom w. same pole and princ parts as f $\therefore$ and $f-g$ bdd

<u>Note</u> $f(z) \to 0$ as $|y| \to \infty$, unif wrt x:
i.e., $\forall \epsilon > 0$ $\exists$ b s.t. $|f(z)| < \epsilon$ when $|y| \geq b$



In shaded region, ~~[illegible]~~ series of holom fns unif & abs. cgt.

When $|y| \to \infty$, each term $\to 0$ unif. wrt x in strip: $\therefore$ sum of series $\to 0$ as $|y| \to \infty$, unif wrt x in strip (or unif wrt x, by periodicity)

$g(z)$ has foll same properties as $f(z)$:

(1) merom in $\mathbb{C}$, periodic of period 1
(2) poles $z = n$, each pole double, princ part $1/(z-n)^2$ (cons ...)

(3) $g(z) \to 0$ as $|y| \to \infty$, unif wrt x
(because
$$|\sin \pi z|^2 = \sin^2 \pi x + \sinh^2 \pi y \to \infty \text{ as } |y| \to \infty \text{ (unif wrt } x\text{)})$$

$f - g$ holom in $\mathbb{C}$ $\because$ f, g same poles & princ parts

$f-g$ bdd : In strip $a_1 \le x \le a_2$, bdd
for $|y| \le b$ (contin on comp set),
& " $|y| \ge b$ ($\to 0$ as $|y| \to \infty$, unif wrt x)
$\therefore$ bdd in strip, so in $\mathbb{C}$ by periodicity

$f-g$ const by Liouville's thm
const $= 0$ $\because$ $f-g \to 0$ as $|y| \to \infty$

<u>Applicn</u>. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler) :

$$\underbrace{\left(\frac{\pi}{\sin \pi z}\right)^2 - \frac{1}{z^2}}_{\lim_{z\to 0} = \frac{\pi^2}{3}} = \sum_{n \ne 0} \frac{1}{(z-n)^2}$$ holom fn $h(z)$ in nbd of 0

$h(0) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{(\pi z - \frac{1}{6}\pi^3 z^3 + \dots)^2} = \frac{1}{z^2}\left(1 - \frac{1}{6}\pi^2 z^2 + \dots\right)^{-2} = \frac{1}{z^2} + \frac{\pi^2}{3} + z^2(\dots)$$

Ex (2)

$$\frac{1}{z} + \sum_{n \ne 0} \underbrace{\left(\frac{1}{z-n} + \frac{1}{n}\right)}_{\frac{z}{n(z-n)}}$$

$\frac{1}{n^2} \sim \frac{z}{n(z-n)}$ Series unif & abs cgt on comp subsets of $\mathbb{C}$

$\therefore$ Sum $f(z)$ merom in $\mathbb{C}$, poles $z = n$,
simple poles, of residue 1

Series can be diff'd term-by-term :

$$f'(z) = -\frac{1}{z^2} - \sum_{n \ne 0} \frac{1}{(z-n)^2} = -\left(\frac{\pi}{\sin \pi z}\right)^2 = \frac{d}{dz}(\pi \cot \pi z)$$

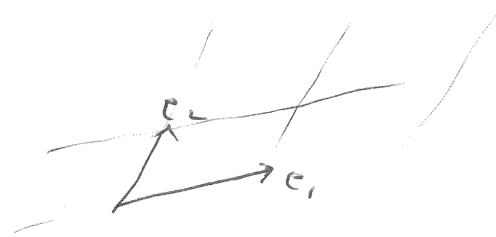
$\therefore$ $f(z) - \pi \cot \pi z$ const.
odd fn. since $f(z)$ odd, ~~do~~ so const $= 0$

Get

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi \cot \pi z$$

Weierstrass $\wp$-function

Doubly periodic fns

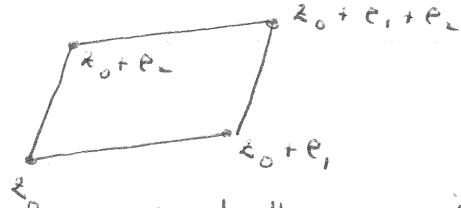


$e_1, e_2 \in \mathbb{C}$, lin indep over $\mathbb{R}$

$\{n_1 e_1 + n_2 e_2 : n_1, n_2 \in \mathbb{Z}\}$ discrete subgp Γ of $\mathbb{C}$

$f(z)$ has Γ ~~as gp of periods~~
if $f(z + n_1 e_1 + n_2 e_2) = f(z)$, $z \in \mathbb{C}$, $n_1, n_2 \in \mathbb{Z}$

$(\Longleftrightarrow \quad f(z+e_1) = f(z), \quad f(z+e_2) = f(z))$



period $\|$gm with 1st vertex z_0

Γ discrete subgp of $\mathbb{C}$
with basis $\{e_1, e_2\}$

$\{e_1', e_2'\}$ basis of Γ
iff e_1', e_2' lin comb of e_1, e_2 w. integer coeffs
and det of matrix of coeffs $= \pm 1$
(Proof. e_1', e_2' basis $\Rightarrow$ det of mat of coeffs
is integer which is invertible in ring of integers : ± 1
converse from formula for inverse)

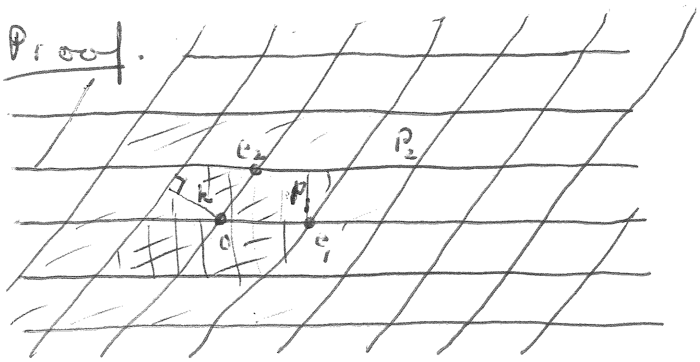
$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}^{\heartsuit}$$

We'll see unif & abs cgt on comp subsets of $\mathbb{C}$

Weierstrass $\wp$-fn. is even (depends on Γ)

LEMMA. $\displaystyle\sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \frac{1}{|\omega|^3}$ converges

Proof.



$P_n := \{ t_1 e_1 + t_2 e_2 : \max\{|t_1|, |t_2|\} = n \}$

There are $8n$ pts of Γ on P_n, each of dist $\geq kn$ from 0 (where $k > 0$ const)

$$\sum_{\omega \in P_n} \frac{1}{|\omega|^3} \leq \frac{8n}{k^3 n^3} = \frac{8}{k^3 n^2}$$

$$\sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \leq \sum_{n=1}^{\infty} \frac{8}{k^3 n^2} < \infty \qquad \square$$

Series $\wp$ cges unif & abs in disk $|z| \leq r$:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega z - z^2}{\omega^2 (z-\omega)^2} \right|$$

$$= \frac{|z(2 - z/\omega)|}{|\omega|^3 \left|1 - \frac{z}{\omega}\right|^2}$$

$$\leq \frac{r \cdot 5/2}{|\omega|^3 \cdot 1/4}$$ for $|z| \leq r$, $|\omega| \geq 2r$ (latter true for all but fin many ω, $\therefore$ for all but fin many terms of series)

$$= \frac{10r}{|\omega|^3}$$

Poles of $\wp$:
$\omega \in \Gamma$, double poles, princ part $\frac{1}{(z-\omega)^2}$

$\wp$ even fn.

$$\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$$ unif & abs cgt on comp subsets of $\mathbb{C}$

$\wp'$ periodic: $\wp'(z+\omega) = \wp'(z)$

$\wp'$ odd

$\wp$ itself has Γ as gp of period

Enough to show $\wp(z+e_i) = \wp(z)$, $i=1,2$

$\wp(z+e_i) - \wp(z) = \text{const}$ ($\because$ deriv 0)

Put $z = -e_i/2$ (not poles of $\wp$):

const $= \wp(e_i/2) - \wp(-e_i/2)$

$= 0$ since $\wp$ even

<u>Summary</u> $\wp$ merom fn w Γ as gp of periods, poles precisely at pts of Γ, each with prime part $\frac{1}{(z-\omega)^2}$

<u>Laurent expansion of $\wp(z)$</u>

$$\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \dots \qquad \diamond$$

in some nbd of 0

because $\wp$ even

$\wp(z) - \frac{1}{z^2} = g(z)$ vanishes at 0

Can find coeffs by diff $g(z)$ term-by-term:

	deriv of $1/(z-\omega)^2$
1	$-2/(z-\omega)^3$
2	$6/(z-\omega)^4 \div 2!$
3	$-24/(z-\omega)^5 \div 3!$
4	$120/(z-\omega)^6 \div 4!$
$2k$	$\frac{(2k+1)!}{(2k)!\,(z-\omega)^{2k+2}}$

$$a_2 = 3 \sum_{\omega \neq 0} \frac{1}{\omega^4}$$

$$a_4 = 5 \sum_{\omega \neq 0} \frac{1}{\omega^6}$$

$\dots$

$$a_{2k} = (2k+1) \sum_{\omega \neq 0} \frac{1}{\omega^{2(k+1)}}$$

Diff'l eqn satis'd by $\wp$

Differentiate $\diamond$ term-by-term:

$$\wp'(z) = -\frac{2}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots$$

and square both sides:

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + \cdots$$

Cube both sides of $\diamond$:

$$\wp(z)^3 = \frac{1}{z^6} + \frac{3a_2}{z^2} + 3a_4 + \cdots$$

Therefore,

$$\wp'(z)^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots)$$

So $(\wp')^2 - 4\wp^3 + 20a_2\wp + 28a_4$

is holom near 0, 0 at 0

Since it has Γ as gp of periods, it is holom near each $\omega \in \Gamma$, 0 at $\omega \in \Gamma$.
In fact, holom in $\mathbb{C}$ $\because$ no poles outside Γ, bounded in $\mathbb{C}$ by periodicity, zero at 0, so id zero by Liouville

$$\wp'^2 = 4\wp^3 - 20a_2\wp - 28a_4$$

Formulas $x = \wp(z)$, $y = \wp'(z)$
give param repn of alg. curve

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

We'll show any pt. (x, y) of this curve is image of a pt. $z \in \mathbb{C}$ which is uniquely det'd up to addn of el of Γ.

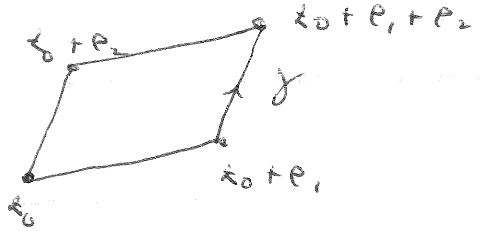
~~Doubly periodic fns~~

Γ gen by e_1, e_2 lin indep over $\mathbb{R}$

PROP 1. f nonconst merom fn in $\mathbb{C}$, Γ as gp of periods

Then # zeros of f in a period $\parallel$gm = # poles in same $\parallel$gm (provided no zeros or poles on bdry)

Proof



Choose z_0 so $f(z)$ has no zeros, poles on bdry γ

Argument princ:

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \text{\# zeros} - \text{\# poles}$$

(each counted w. mult)

vanishes by periodicity

□

COR. Holom fn in $\mathbb{C}$ w. Γ as gp of periods is const

Anyway clear by Liouville

Otherwise, # zeros of $f(z) - a$ = # poles $(0!)$ $\forall a$

PROP 2. $f(z)$ nonconst merom fn in $\mathbb{C}$, Γ gp of periods, $a \in \mathbb{C}$

Let α_i be roots of $f(z) = a$ (counted w. mult)

β_i poles of $f(z)$ (" " ")

contained in a period ‖ gm

Then $\sum \alpha_i \equiv \sum \beta_i \mod \Gamma$

(In part., $\sum \alpha_i \mod \Gamma$ indep of a)

$z = \alpha_i + (z - \alpha_i)$

$f(z) - a = c(z-\alpha_i)^k + \text{h.o.t.}$

$f'(z) = kc(z-\alpha_i)^{k-1} + \text{h.o.t.}$

$$\frac{z f'(z)}{f(z) - a} = \frac{k\alpha_i}{z - \alpha_i} + \text{hol}$$

Proof

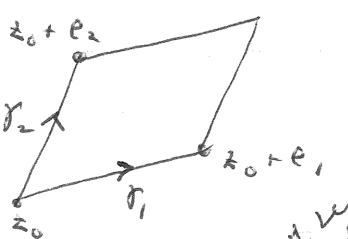
not periodic

$$\frac{1}{2\pi i}\int_\gamma \frac{z f'(z)}{f(z)-a}\,dz = \text{sum of residues of } \frac{z f'(z)}{f(z)-a}$$

At zeros α_i of $f(z) - a$, res $= k\alpha_i$, (k = mult)

At poles β_i of $f(z)$, res $= -k\beta_i$ (k = mult)

$$\text{LHS} = -\frac{e_2}{2\pi i}\int_{\gamma_1} \frac{f'(z)}{f(z)-a}\,dz + \frac{e_1}{2\pi i}\int_{\gamma_2}\frac{f'(z)}{f(z)-a}\,dz$$



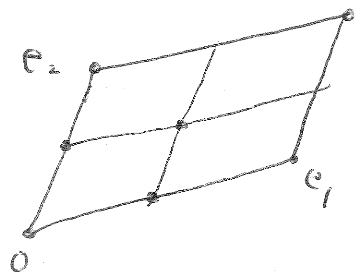
select zero or of pole cancel and avoid mod!

integers $\times 2\pi i$ (differences between 2 determinations of $\log(f(z_0) - a)$ ~~$\div 2\pi i$~~) $\square$

THM Given discrete gp Γ, the eqn
$$4x^3 - 20a_2 x - 28a_4 = 0,$$
where $a_2 = 3\sum_{\omega \neq 0} \frac{1}{\omega^4}$, $a_4 = 5\sum_{\omega\neq 0}\frac{1}{\omega^6}$

has 3 distinct roots. Moreover, $\forall (x, y) \in \mathbb{C}^2$ on alg curve $y^2 = 4x^3 - 20a_2 x - 28 a_4$, there is unique $z \in \mathbb{C} \mod \Gamma$ s.t.
$$x = \wp(z), \quad y = \wp'(z).$$

<u>Later</u>. Conversely, given eqn
$$y^2 = 4x^3 - 20a_2x - 28a_4,$$
where R H S has 3 distinct roots (any non zero cubic!) there exists Γ s.t. a_2, a_4 given as above.
(So if $\wp$ is Weier $\wp$-fn assoc to Γ, then $x = \wp(z)$, $y = \wp'(z)$ gives param of curve, as in Thm).



~~Proof of thm.~~

Cons pts $z \in \mathbb{C}$ s.t. $2z \in \Gamma$, $z \notin \Gamma$

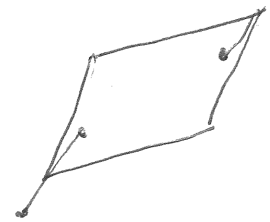
e.g., $e_1/2$, $e_2/2$, $(e_1+e_2)/2$
Clear that any z s.t. $2z \in \Gamma$, $z \notin \Gamma$ congruent mod Γ to one of these 3 pts, and that classes of these pts mod Γ distinct

At such pt z: $\wp'(z) = \wp'(-z)$ by periodicity
$\wp'(z) = -\wp'(-z)$ $\because$ $\wp'$ odd
$\therefore$ $\wp'(z) = 0$

But $\wp'$ has ! triple pole in each period ll gm
By Prop. 1, $\wp'$ has at most 3 dist. zeros in period ll gm; $\therefore$ they are above 3 pts or their conjugates mod Γ.

Moreover, each value $\wp(e_1/2)$, $\wp(e_2/2)$, $\wp((e_1+e_2)/2)$ taken only once in ll gm, and these 3 values distinct ∎

By Prop. 1, $\wp$ doesn't take given value more than twice in period ll'gm ;
also $\wp(z_0) = \wp(-z_0)$, $z_0 \in \mathbb{C}$



$\therefore$ if $2z_0 \notin \Gamma$, $\wp$ takes value $\wp(z_0)$ exactly twice.
if $2z_0 \in \Gamma$, $z_0 \notin \Gamma$, then $\wp'(z_0) = 0$,
so $\wp(z) = \wp(z_0)$ has z_0 as double root,
and $\wp$ takes val $\wp(z_0)$ only once in ll'gm.

Since $\wp'^2 = 4\wp^3 - 20a_2\wp - 28a_4$,
these val's are the 3 roots of RHS
(i.e. RHS has 3 distinct roots)

2nd statement in thm:
If $(x,y) \in$ curve, $y \neq 0$,
then x is value of $\wp$ that occurs twice in each period ll'gm : $x = \wp(z) = \wp(-z)$.
For this x,
$y = \wp'(z)$, $-y = -\wp'(z) = \wp'(-z)$
are the 2 values that work. $\square$

$X \subset \mathbb{C}^2 : \quad y^2 = 4x^3 - 20a_2 x - 28 a_4$

smooth curve : (i.e. locally graph of holom fn.
$y = f(x)$ or $x = g(y)$: 1-diml cx-submanif of $\mathbb{C}^2$)

$(x_0, y_0) \in X, \quad y_0 \neq 0$:
$y = f(x)$ near (x_0, y_0), by impl fn. thm ;
i.e. x local coord

$(x_0, y_0) \in X, \quad y_0 = 0$:
$p'(x_0) \neq 0$, so $x = g(y)$ near $(x_0, 0)$;
i.e. y local coord

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "y"', "\text{onto}"] \arrow[dr, "(x,y)"] & \mathbb{C}^2 \arrow[dl] \\ \mathbb{C} & x \end{tikzcd}$$

<u>Riem. surface</u> over $\mathbb{C}$

We can compactify X to get Riem surface over $S^2 = P^1(\mathbb{C})$

<u>n-diml cx proj. space</u> $\quad P^n(\mathbb{C}) = \mathbb{C}^{n+1} / \{0\} / \sim$
where $(x_0', \dots, x_n') \sim (x_0, \dots, x_n)$
if $\exists \ \lambda \in \mathbb{C} \setminus \{0\}$ s.t. $(x_0', \dots, x_n') = (\lambda x_0, \dots, \lambda x_n)$

$[x_0, \dots, x_n]$ equiv. class of $(x_0, \dots, x_n)$
"homog coords"

<u>Coord charts</u>

$U_i = \{ [x_0, \dots, x_n] \in P^1(\mathbb{C}) : x_i \neq 0 \}, \quad i = 0, \dots, n$

Affine coords on U_i :
$$[x_0, \dots, x_n] \longmapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots \frac{x_n}{x_i} \right)$$

Complex (alg) manif. structure

$P^n(\mathbb{C}) = U_0 \cong \mathbb{C}^n$ together with complex
"hyperplane at ∞" $\quad \{x_0 = 0\} \cong P^{n-1}(\mathbb{C})$

e.g. $P^1(\mathbb{C}) \cong S^2$ Riem sphere
$P^2(\mathbb{C})$: $[x, y, t]$

Complex curve
X' : $y^2 t = 4x^3 - 20 a_2 x t^2 - 28 a_4 t^3$ ◊
in $P^2(\mathbb{C})$, obt'd by homogenizing
affine curve
X : $y^2 = 4x^3 - 20 a_2 x - 28 a_4$

i.e. $\mathbb{C}^2 \hookrightarrow U_0 \subset P^2(\mathbb{C})$
(x, y) $[x, y, t]$, $t \neq 0$
$(x, y) \longmapsto [x, y, 1]$
$\left(\frac{x}{t}, \frac{y}{t}\right) \longleftarrow [x, y, t]$ $\left(\frac{y}{t}\right)^2 = 4\left(\frac{x}{t}\right)^3 - 20 a_2 \left(\frac{x}{t}\right) - 28 a_4$
def's Haus. sp. $X' \subset P^2(\mathbb{C})$

X can be id'd w. ~~op~~ subspace of X'
X' consists of X together w. single pt at infinity
$t = 0$: $[0, 1, 0]$

$[0, 1, 0]$ in chart $\{[x, y, t] \in P^2(\mathbb{C}) : y \neq 0\}$
$(x', t') = \left(\frac{x}{y}, \frac{t}{y}\right)$ affine coords in this chart

In this chart, ◊ equiv to
$t' = 4x'^3 - 20 a_2 x' t'^2 - 28 a_4 t'^3$

In some nbd of $(x', t') = (0, 0)$ (pt at ∞), implicit fn thm.
gives t' as holom fn of x'
$t' = 4x'^3 - 320 a_2 x'^7 + \dots$
i.e., in nbd of pt at ∞ in X', can take x'
as local coord.
Get complex manifold structure on X'

$X \subset \mathbb{C}^2$ $y^2 = 4x^3 - 20a_2x - 28a_4$

$\downarrow$ $(x,y) \mapsto x$

$\mathbb{C}$

$X' \hookrightarrow P^2(\mathbb{C})$ $y^2t = 4x^3 - 20a_2xt^2 - 28a_4t^3$

$$\begin{array}{l} X' \hookrightarrow P^2(\mathbb{C}) \\ \downarrow \varphi' \\ S^2 \end{array}$$

$\varphi' = \varphi$ on X

$\infty \mapsto$ pt at ∞ of S^2

$t' = 4x'^3 - 20a_2x't'^2 - 28a_4t'^3$

In chart $y \neq 0$, curve consists of pts

$[x', 1, t'] = [x', 1, 4x'^3 - 320a_2x'^7 + \dots]$

φ takes this to $[x', t']$ in $P^1(\mathbb{C}) = S^2$;

i.e. to $\frac{x'}{t'} \in \mathbb{C}$, or $\frac{t'}{x'}$ in coords at ∞

$$\frac{t'}{x'} = \frac{4x'^3 - \dots}{x'} \longrightarrow 0 \text{ as } x' \to 0$$

$[\wp(z), \wp'(z), 1] = [\frac{\wp}{\wp'}, 1, \frac{1}{\wp'}] \to [0,1,0]$ $\frac{z^3}{z^2}$ $[\wp(z), \wp'(z), 1]$ $[\frac{\wp}{\wp'}, 1, \frac{1}{\wp'}]$ z^3

The merom transf $x = \wp(z)$, $y = \wp'(z)$

def's isom $\mathbb{C}/\Gamma \xrightarrow{\simeq} X'$

$[\wp(z), \wp'(z), 1]$
$[\frac{\wp(z)}{\wp'(z)}, 0, \frac{1}{\wp'(z)}]$
$[z, 0, z^3]$

quot top: $U \subset \mathbb{C}/\Gamma$ open iff inv. image open in $\mathbb{C}$

homeo since contin, bij

In fact, isom as cx manifs

Inverse isom def's z as holom many-valued fn on X', whose branches differ in value by consts $\in \Gamma$.

$dx = y\,dz$ $y^2 = P(x) = 4x^3 - 20a_2x - 28a_4$

$2y\,dy = P'(x)\,dx$

$$dz = \frac{dx}{y} = \frac{2\,dy}{P'(x)}$$

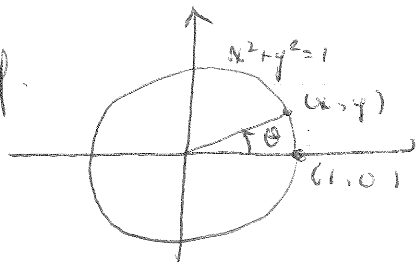
dz extension to X' of holom diff'l form

$$\frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}$$

$$z = \wp^{-1}(x) = \int \frac{dx}{\sqrt{4x^3 - 20a_2x - 28a_4}}$$

elliptic integral

Weier $\wp$-fn. obt'd by "inversion" of elliptic integral cf.



$$x\,dx + y\,dy = 0$$

$$\frac{dy}{x} = -\frac{dx}{y}$$

$$\shortparallel$$

$$d\theta = \frac{dy}{\sqrt{1-y^2}}$$

$x = \cos\theta = \sin'\theta$

$y = \sin\theta$

$dy = x\,d\theta$

We invert $\int \frac{dy}{x} = \int \frac{dy}{\sqrt{1-y^2}}$ in nbd of $(1,0)$

by defining trig fns by reln

$$\theta = \int_{(1,0)}^{(\cos\theta,\, \sin\theta)} \frac{dy}{x} = \int_0^{\sin\theta} \frac{dy}{\sqrt{1-y^2}}$$

θ near 0

Remark on holom fns in several variables

$f(z_1, \dots, z_n)$ in open $\Omega \subset \mathbb{C}^n$

holom if f C^1 and $df = \sum \frac{\partial f}{\partial z_j} dz_j$

Not hard to show holom equiv to foll.

(1) f contin & separately holom in each var.

(2) f analytic (use Cauchy integral)

In fact, separately holom $\Rightarrow$ holom (hard)

Implicit fn thm (2 variable)

$f(x,y) = 0$, $(x,y) \in \mathbb{C}^2$

$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ Then can solve locally for y as holom fn of x.

Proof $\quad z = f(x,y)$

$$x = x_1 + i x_2$$
$$y = y_1 + i y_2$$
$$z = z_1 + i z_2$$
$$f = f_1 + i f_2$$

For x fixed.

$$dz = \frac{\partial f}{\partial y}\, dy$$
$$d\bar z = \frac{\partial \bar f}{\partial \bar y}\, d\bar y$$
$$dz \wedge d\bar z = \left|\frac{\partial f}{\partial y}\right|^2 dy \wedge d\bar y$$

$dy \wedge d\bar y = -2i\, dy_1 \wedge dy_2$

write in terms of real & im part

$$\frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} = \left|\frac{\partial f}{\partial y}\right|^2 \neq 0$$

By implicit fn. thm, can solve for $y = y_1 + i y_2$ as C^1 fn. of $x = x_1 + i x_2$

$$f(x,y) = 0$$

But $\quad \dfrac{\partial f}{\partial x}\, dx + \dfrac{\partial f}{\partial y}\, dy = 0$ $\quad$ ~~$f(x, y(x)) = 0$ $\quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$~~ $\quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dz$

i.e. $\dfrac{dy}{dx}$ holom

$f(x, y(x)) = 0$

$\dfrac{\partial f}{\partial x}\, dx + \dfrac{\partial f}{\partial y}\left(\dfrac{\partial y}{\partial x}\, dx + \dfrac{\partial y}{\partial \bar x}\, d\bar x\right)$ $\quad y$ holom fn of x by Morera's thm.

By implicit fn thm applied to eqns $\begin{cases} z = f(x,y) \\ \bar z = \bar f(x,y) \end{cases}$ can write y_1, y_2 locally as C^1 fns of x_1, x_2, z_1, z_2

$dz = \dfrac{\partial f}{\partial x}\, dx + \dfrac{\partial f}{\partial y}\, dy$ $\quad$ shows dy lin comb of dx, dz: i.e. y holom in x, z

<u>Proof</u> $\quad z = f(x,y)$

$$x = x_1 + i x_2$$
$$y = y_1 + i y_2$$
$$z = z_1 + i z_2$$

$$dx \wedge d\bar{x} = -2i\, dx_1 \wedge dx_2$$

$$dx \wedge d\bar{x} = (dx_1 + i\,dx_2) \wedge (dx_1 - i\,dx_2) = -2i\, dx_1 \wedge dx_2$$

$$z = f(x,y)$$
$$\bar{z} = \bar{f}(x,y)$$

For x fixed:

$$dz = \frac{\partial f}{\partial y}\, dy$$
$$d\bar{z} = \frac{\partial \bar{f}}{\partial \bar{y}}\, d\bar{y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2}\right)(z_1 + i z_2)$$

$$\frac{\partial \bar{f}}{\partial \bar{y}} = \frac{1}{2}\left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right)(z_1 - i z_2) = \overline{\frac{\partial f}{\partial y}}$$

$$dz \wedge d\bar{z} = \left|\frac{\partial f}{\partial y}\right|^2 dy \wedge d\bar{y} \qquad (x \text{ fixed})$$

so $\quad dz_1 \wedge dz_2 = \left|\frac{\partial f}{\partial y}\right|^2 dy_1 \wedge dy_2$

$$\frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} = \left|\frac{\partial f}{\partial y}\right|^2 \neq 0$$

Show directly using CR eqns?

$$dz_1 = \frac{\partial z_1}{\partial y_1} dy_1 + \frac{\partial z_1}{\partial y_2} dy_2$$
$$dz_2 = \frac{\partial z_2}{\partial y_1} dy_1 + \frac{\partial z_2}{\partial y_2} dy_2$$
$$dz_1 \wedge dz_2 = \frac{\partial(z_1, z_2)}{\partial(y_1, y_2)}\, dy_1 \wedge dy_2$$

~~By impl. fn. thm, can solve $f(x,y) = 0$ locally as $y_1, y_2 = C^1$ fns of x_1, x_2~~

By impl. fn. thm applied to eqns
$$z = f(x,y)$$
$$\bar{z} = \bar{f}(x,y)$$
(vars: x, y)
can write y_1, y_2 locally as C^1 fns of x_1, x_2, z_1, z_2

~~$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$~~

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

shows dy lin comb of dx, dz, i.e. y holom in x, z. $\square$

Theorem of Mittag-Leffler (1877)

Recall Merom fn in S^2 is rational
(partial fraction decomp)

(f has pole at ∞ if $g(z) = f(1/z)$ has pole at 0;
prin part of f at ∞ is poly.)

$\tan z$, $\sec z$ merom in $\mathbb{C}$,
but pt at ∞ is limit of pole

e.g. $\sum \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}$

THM $b_k \in \mathbb{C}$, $\lim_{k \to \infty} b_k = \infty$
$P_k(z)$ polys without const term
Then $\exists$ merom fn in $\mathbb{C}$ with poles b_k,
corresp prin parts $P_k\left(\frac{1}{z-b_k}\right)$

The most general merom fn. with these poles and prin parts can be written

$$f(z) = \sum_k \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right) + g(z)$$

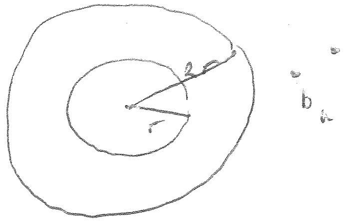
where $g(z)$ holom in $\mathbb{C}$
$p_k(z)$ polys s.t. series unif & abs cgt on comp subsets of $\mathbb{C}$

Proof
We can assume that no $b_k = 0$

$P_k\left(\frac{1}{z-b_k}\right)$ holom in $|z| < |b_k|$
so can expand in Taylor series at 0.
Let $p_k(z)$ = sum of terms of degree $\leq n_k$
where n_k chosen so that

$$\left| P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right| \leq \frac{1}{2^k} \quad \text{if} \quad |z| \leq \frac{|b_k|}{2}$$

We show series cges unif & ab. in $|z| \le r$, any r:



Choose m s.t. $|b_k| > 2r$, $k > m$

$$\sum_{k=m+1}^{\infty} \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right)$$

unif & abs cgt in $|z| \le r$

by comp w. $\Sigma\, 1/2^k$ (Weier M-test)

$\therefore$ $\displaystyle\sum_{k=1}^{\infty} \left(P_k\left(\frac{1}{z-b_k}\right) - p_k(z) \right)$ merom in $\mathbb{C}$, with given poles & prin parts

Any merom fn. with same poles and prin parts differs from preceding by holom fn. $\square$

<u>Infinite products of holom fns</u>

$\displaystyle\prod_{n=1}^{\infty} b_n$, $b_n \in \mathbb{C}$ $\qquad p_n = b_1 \cdots b_n$

<u>cges</u> to $p = \lim_{n\to\infty} p_n$ if lim exists, $\ne 0$

Too restrictive. Say <u>converges</u> if all but fin. many factors ~~nonzero~~ nonzero, and partial prods formed by nonzero factors have nonzero limit.

$\prod b_n$ cges $\Rightarrow$ $\underset{p_n/p_{n-1}}{\overset{\shortparallel}{b_n}} \to 1$ (omitting 0 factors)

$\therefore$ write inf prods

$\heartsuit$ $\quad \prod (1 + a_n)$ $\qquad a_n \to 0$ nec for cge.

cf.

$\diamondsuit$ $\quad \Sigma \log(1 + a_n)$ $\qquad$ prin branch of log (def'd for n large enough)

S_n = n'th partial sum

$p_n = e^{s_n}$

So s_n cges $\Rightarrow$ p_n cges

p_n cges $\Rightarrow$ s_n cges

$p_n \to p$

Fix $\log p = \log|p| + i \arg p$

Only problem is convergence of im part of s_n

Set $\log p_n = \log|p_n| + i \arg p_n$

where $\arg p_n \in (\arg p - \pi, \arg p + \pi]$

Then $s_n = \log p_n + 2\pi i k_n$, $k_n \in \mathbb{Z}$

$\log(1 + a_{n+1}) = s_{n+1} - s_n = \log p_{n+1} - \log p_n + 2\pi i (k_{n+1} - k_n)$

n large enough: $|\arg(1 + a_{n+1})|$, $|\arg p_n - \arg p|$, $|\arg p_{n+1} - \arg p|$ all $< 2\pi/3$

$\therefore$ $|k_{n+1} - k_n| < 1$, so $k_{n+1} = k_n$, n large enough

i.e. k_n eventually const, say k

$s_n \longrightarrow \log p + 2\pi i k$

Say $\prod(1 + a_n)$ <u>cges abs</u> if $\sum \log(1 + a_n)$ cge abs

This is equiv to abs cge of $\sum a_n$:

$\lim_{z \to 0} \frac{\log(1+z)}{z} = 1$ $\qquad \left| \frac{|\log(1+a_n)|}{|a_n|} - 1 \right| < \epsilon$

given ϵ, $(1-\epsilon)|a_n| < |\log(1 + a_n)| < (1+\epsilon)|a_n|$

for n large enough

$\prod_{n=1}^{\infty} f_n(z)$ $\quad f_n$ contin, $\mathbb{C}$-val'd on open $\Omega \subset \mathbb{C}$

<u>Converges unif & abs</u> on $K \subset \Omega$ $\iff$

(1) $f_n(z) \to 1$ unif on K $\quad\longleftarrow\quad$ $\log f_n$ part. prime branch of $\log f_n$ def'd, n large

(2) $\sum \log f_n$ unif & abs cgt on K

Set $f_n = 1 + u_n$

Then $\prod (1+u_n)$ cges unif & abs on K

$\iff$ $\sum u_n$ " " " " "

If $\prod f_n$ cges unif & abs on comp subsets of Ω then partial prods cge unif on comp subsets to a limit $f(z)$, which is therefore contin

THM. f_n holom in Ω

$\prod f_n$ cge unif & abs on comp subsets of Ω

Then

(1) $f = \prod f_n$ holom on Ω (unif limit of holom fns)

$f = f_1 \cdots f_p \underbrace{\prod_{n>p} f_n}_{\text{contin nonvan on } U}$ (obvious on rel comp open U)

(2) Set of zeros of f = union of zero sets of f_n

Mult of zero of f = sum of mults for each f_n

(f_n has no zeros in rel comp U, n large)

(3) Series of merom fns $\sum_{n=1}^{\infty} f_n'/f_n$ cges unif & abs on comp subsets of Ω; sum is f'/f

<u>Proof of last statement</u>

Cons. $U \subset \Omega$ rel. comp. $f = f_1 \cdots f_p \cdot g_p$

$$\frac{f'}{f} = \sum_{n \le p} \frac{f_n'}{f_n} + \frac{g_p'}{g_p} \quad \text{where } g_p = \exp\left(\sum_{n>p} \log f_n\right)$$

(def'd, holom in U, p large enough)

by thm$^{(i)}$ above & prod rule

$$\frac{g_p'}{g_p} = \sum_{n>p} \frac{f_n'}{f_n}$$

since $\sum_{n>p} \log f_n$ ges unif on comp sets in U to branch of $\log g_p$ so Σ deriv. ge unif on comp subset to g_p'/g_p

$\therefore$ On U, $\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n}$ ges unif & abs on comp subsets of U, $\therefore$ on Ω $\square$

<u>Example</u> $\sin \pi z$ as infinite prod

$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ ges unif & abs on comp subsets of $\mathbb{C}$ ($\because \sum \frac{z^2}{n^2}$ does)

$\therefore$ $f(z)$ holom in $\mathbb{C}$, zeros $\mathbb{Z}$, all simple

Can diff logarithmically term-by-term $\qquad \frac{n^2 - z^2}{n^2}$

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

$$= \pi \cot \pi z$$

$$= \frac{g'(z)}{g(z)} \quad \text{where } g(z) = \sin \pi z$$

$\therefore$ $f(z) = c\, g(z)$, c const $\left(\frac{d}{dz}\left(\frac{f}{g}\right) = 0\right)$

$$\left.\begin{array}{l} \frac{f(z)}{z} \longrightarrow 1 \quad \text{as } z \to 0 \\ \frac{\sin \pi z}{z} \longrightarrow \pi \quad \text{as } z \to 0 \end{array}\right\} \quad c = 1/\pi \qquad \frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

what about fn w +ve, -ve nbs as zeros?

Prob #5. f_1, f_2 entire, zeros disjoint 2-28

$\Rightarrow \exists\ g_1, g_2$ entire, $f_1 g_1 + f_2 g_2 = 1$

Proof $g(z)$ entire fn. w. zeros of order n_k+1 at a_k
$h(z)$ merom., poles a_k,

principal part = principal part of $\dfrac{b_k}{g(z)}$ at a_k $+ \dfrac{1}{z-a_k}$

$f(z) = g(z)\,h(z)$ holom outside a_k

Near a_k $\quad f(z) = g(z)\left\{\dfrac{b_k}{g(z)} + h(z) - \dfrac{b_k}{g(z)}\right\}$

$$= b_k + \underbrace{g(z)}_{\substack{\text{zero of} \\ \text{order } n_k+1}} \underbrace{\left\{ h(z) - \frac{b_k}{g(z)} \right\}}_{\frac{1}{z-a_k} + \text{holom}}$$

$\square$

Normal families and compact subsets of $\mathcal{H}(\Omega)$

$C(\Omega)$ metrizable
$\mathcal{H}(\Omega)$ closed subspace, induced metric

Metric space compact iff every infinite sequence has cgt (inf) subsequence

$\mathcal{S} \subset \mathcal{H}(\Omega)$ normal family if every seq in $\mathcal{S}$ has subseq that converges unif on comp subsets of Ω (i.e., converges in $\mathcal{H}(\Omega)$)

Thus:

~~Whenever~~ $\mathcal{S}$ compact iff normal and limit fns themselves in $\mathcal{S}$.

$\mathcal{S}$ normal $\Rightarrow \bar{\mathcal{S}}$ normal (exercise)

So

PROP. $\mathcal{S} \subset \mathcal{H}(\Omega)$ normal iff $\bar{\mathcal{S}}$ compact