MAT454 Notes

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Zeros and Poles

Definition 1 (Zero)**.** *If f is holomorphic in a neighborhood of $z_0 \in \mathbb{C}$ and $f(z_0) = 0$, we can write, for some $k \in \mathbb{N}$,*

$$f(z) = (z - z_0)^k f_1(z)$$

*where$f_1(z)$ is nonvanishing near z_0. In this case k is called the **order** or **multiplicity** of the **zero** z_0*

Zeros of holomorphic functions form a discrete set. We want to study, however, not only holomorphic functions, but also quotients of holomorphic functions

Definition 2 (Meromorphic)**.** *A function f is **meromorphic** on an open $\Omega \subseteq \mathbb{C}$ if it is defined and holomorphic in the complement of a discrete set such that in some neighborhood of every point of Ω we can write $f(z) = g(z)/h(z)$ where g, h are holomorphic and h is not identically zero.*

Why is it interesting to work with meromorphic and not just holomorphic functions? Essentially, it's because meromorphic functions in a domain Ω form a field (whereas holomorphic functions only form a ring). Note that, in this course, when we say "domain", what we mean is a connected open set. If $f(z), g(z)$ are holomorphic near z_0, like before, we can write

$$f(z) = (z - z_0)^k f_1(z), \qquad g(z) = (z - z_0)^\ell g_1(z)$$

where $f_1(z_0), g_1(z_0) \neq 0$. Near z_0, then, the quotient looks like

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-\ell} \frac{f_1(z)}{g_1(z)}$$

So what are the different possibilities? If $k \geqslant \ell$, then this function extends to be holomorphic at z_0. On the other hand, if $k < \ell$, then, of course,

$$\lim_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| = \infty$$

Note: *not* undefined, but ∞. In this case, we say that z_0 is a pole of order $\ell - k$. Holomorphic functions in an annulus $r < |z| < R$ have a convergent Laurent expansion in an annulus

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n<0} a_n z^n + \sum_{n \geqslant 0} a_n z^n$$

Note that the LHS converges when $r < |z|$, whereas the RHS converges when $|z| < R$. This actually comes from Cauchy's theorem, just in the case of a convergent power series expansion of a holomorphic function. So this is from Cauchy's integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=-\infty}^{\infty} a_n z^n$$

for z between γ_1, γ_2. So

$$a_n = \frac{1}{2\pi i} \int_{\gamma_?} \frac{f(\xi)}{\xi^{n+1}} d\xi$$

So of course, this should be just like before for the positive part. Note that this is the integral over γ_1 if $n \geqslant 0$ and over γ_2 if $n < 0$.

Previously, we talked about holomorphic functions on the Riemann sphere, noting there were very few of them: namely, constants. Now, there are a few more meromorphic functions on the Riemann sphere, but not much. Specifically,

Theorem 1. *Every meromorphic function f on S^2 is rational.*

Proof. This theorem uses what is probably the only thing in first year calculus you don't prove: the partial fraction decomposition. So we prove it now. Say $f(z)$ has poles $b_1, ..., b_k$ (finite) and maybe ∞. So what can we say about the Laurent expansion at a pole? There's only finitely many negative terms, specifically, the order of the pole.

So these negative parts of the Laurent expansions around each b_j are like polynomials $P_j(\frac{1}{z-b_j})$. We'll call these principal parts. What about the principal part at ∞? It's a polynomial in z, as it's a polynomial in $\frac{1}{z'}$, where z' is the coordinate at ∞, which is $1/z$. Call this $P_\infty(z)$. So we can write

$$f(z) - P_\infty(z) - \sum_{j=1}^{k} P_j\left(\frac{1}{z-b_j}\right)$$

which is holomorphic on S^2, and hence must be a constant a. So we can write

$$f(z) = a + P_\infty(z) + \sum_{j=1}^{k} P_j\left(\frac{1}{z-b_j}\right)$$

And that's rational. $\square$