Algorithm Design: Chapter 1 Exercise Solutions

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Inle Bush's solutions to all odd exercises of Algorithm Design: Chapter 1, by Jon Kleinberg and Eva Tardos.
I use the the syntax $A \overset{C}{>} B$ to denote C prefers A to B and $A \overset{C}{=} B$ to denote C is indifferent between A and B.

1. The statement "In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m." is false. As a counterexample, take the scenario with individuals M1, M2, W1, W2 and rankings $W1 \overset{M1}{>} W2$, $W2 \overset{M2}{>} W1$, $M2 \overset{W1}{>} M1$, $M1 \overset{W2}{>} M2$. Neither of the 2 possible matchings for this instance contain a pair in which each partner is ranked first on their partner's list.

3. There is not always a stable pair of schedules for this problem. As a counterexample take the instance where n=2, A has shows ranking 1 and 3, and B has shows ranking 2 and 4. Disregarding the ordering of the time slots, there are two possible schedules: 1,2,3,4 and 1,42,3 where numbers represent the show with the respective rating and two numbers being contained in the same element indicates being in the same slot. Clearly, A and B prefer the first and second schedule respectively. Thus in either schedule, one of the networks would opt to change so there is no stable schedule.

5. (a) There always exists a perfect matching with no strong instability. To generate such a perfect matching for a given problem instance, we generate artificial total ordering preference lists then use these preference lists and the Gale-Shapley algorithm to compute a perfect matching. For each individual, we construct an artificial preference list by keeping preferences from their preference list but substituting indifferences for randomly assigned preferences. As the artificial preference lists denote total orderings, with them one may produce a matching S using the Gale-Shapley algorithm.

   Previous results on termination of the algorithm hold. We may show by contradiction that S cannot have a strong instability and thus show the correctness of the algorithm. If there were a strong instability in S, then there would be a man m and a woman w such that each of m, w prefer the other, according to their original preference lists, to their partner in S. Since preferences are conserved in generating the artificial ranking, this would mean there would also be an instability (according to the original definition) according to their artificial preference lists. However, the book has shown that Gale-Shapley generates matchings without instabilities and thus we know that S cannot have a strong instability.

   (b) There does not always exist a perfect matching with no weak instability. As a counterexample take the scenario with individuals M1, M2, W1, W2 with rankings $W1 \overset{M1}{>} W2$, $W1 \overset{M2}{>} W2$, $M2 \overset{W1}{=} M1$, $M2 \overset{W2}{=} M1$. There are two possible matchings, in one, there are the pairs $(W1, M1)$ and $(W2, M2)$ which has a weak instability between W1 and M2. The other has the pairs $(W1, M2)$ and $(W2, M1)$ which has a weak instability between W1 and M1.

7. By translating the problem of finding a valid switching into an analogous stable matching problem, we may apply the Gale-Shapley algorithm to the analogous problem and translate the solution back to find a valid switching. Here is the algorithm to generate a valid switching for a given wiring: Input and output streams are represented as men and women respectively and a switching is represented as a marriage. Output streams are ranked in the preference list of input streams in source to sink order along that input stream. Input streams are ranked in the preference list of output streams in sink to source order along that output stream. With these sets of individuals and preference lists, we may use Gale-Shapley algorithm to generate a stable matching.

   In this way we generate a pairing of input and output wires, P, that has no instabilities in terms of our generated preference list. We now show that a switching scheme, S, on the original wires with the same pairs as P is valid.

Assume towards contradiction that the corresponding switching schemes S is invalid. Then there exists a junction box through which two data streams pass following a switching operation. More formally, this means that there exist a pair of inputs i_1 and i_2 and outputs o_1 and o_2 where i_1 switches onto o_1, i_2 switches onto o_2, i_1 is upstream of i_2 on o_1, and o_1 is upstream of o_2 on i_2. This would imply that in the corresponding pairing scheme P, there would exist men m_1 and m_2 and women w_1 and w_2 corresponding to i_1, i_2, o_1, and o_2 respectively such that i_1 were married to o_1, i_2 were married to o_2, $i_2 \overset{o_1}{>} i_1$, and $o_1 \overset{i_2}{>} o_2$. By definition, this would constitute an instability. However, this is a contradiction as P is a stable matching. Thus, S must be a valid switching scheme.

Since P has no instabilities the switching with the same pairs, S, is valid. Since Gale-Shapley always terminates with a stable-matching, this algorithm will always produce a valid switching.