Discrete time Kalman filter Formula

Predict : $\hat{X}_p[K] = A\hat{X}[K] + B\vec{U}[K] + G\vec{\omega}[K]$;State prediction

$\hat{P}_p[K] = A\hat{P}[K]A^T + \mathrm{GQG}^T$;Covariance prediction

Update: $\hat{Y}_p[K] = C\hat{X}_p[K] + D\vec{U}[K] + \vec{V}[K]$;Measurement prediction

$\tilde{Y}[K] = \vec{Y}[K] - \vec{Y}_P[K]$;Difference value between sensor and prediction value

$S[K] = C\hat{P}_p C^T + R$;Covariance sensor

$K[K] = \hat{P}_p[K]\, C^T\, S[K]^{-1}$;Kalman gain

$\hat{X}[K+1] = \hat{X}_p[K] + K[K]\,\tilde{Y}[K]$;State prediction next iteration

$\hat{P}[K+1] = (\,\mathrm{II} + K[K]\,C\,)\,\hat{P}_p[K]$;Covariance prediction next iteration

ps. K represent iteration

$\vec{X}$ is State $\left(\vec{X} \in \mathrm{IR}^{n_x}\right)$

$\vec{U}$ is Know input $\left(\vec{U} \in \mathrm{IR}^{n_u}\right)$

$\vec{\omega}$ is Process noise $\left(\vec{\omega} \in \mathrm{IR}^{n_\omega}\right)$

$\vec{V}$ is Measurement noise $\left(\vec{V} \in \mathrm{IR}^{n_v}\right)$

According to equation : $\dfrac{d\vec{\omega}}{\mathrm{dt}} = \vec{\alpha}$

$$\int_{\tau=0}^{\tau=t} \frac{d\vec{\omega}}{\mathrm{dt}}\, d\tau = \int_{\tau=0}^{\tau=t} \vec{\alpha}\, d\tau$$

$\vec{\omega}(t) - \vec{\omega}(0) = \vec{\alpha}(t) - \vec{\alpha}(0)$;$\vec{\alpha}(0)$ is 0

$\vec{\omega}(t) = \vec{\omega}(0) + \vec{\alpha}(t)$------------------- (eq 1)

According to equation : $\dfrac{d\vec{\theta}}{\mathrm{dt}} = \vec{\omega} = \vec{\omega}(0) + \vec{\alpha}(t)$;show in eq1

$$\int_{\tau=0}^{\tau=t} \frac{d\overrightarrow{\theta}}{\mathrm{dt}}\, d\tau = \int_{\tau=0}^{\tau=t} \left(\overrightarrow{\omega}(0) + \overrightarrow{\alpha}(t) \right) d\tau$$

$$\overrightarrow{\theta}(t) - \overrightarrow{\theta}(0) = \overrightarrow{\omega}(0)t + \frac{1}{2}\overrightarrow{\alpha}(t)^2 - \overrightarrow{\omega}(0)0 - \frac{1}{2}\overrightarrow{\alpha}(0)^2$$

$\overrightarrow{\theta}(t) = \overrightarrow{\theta}(0) + \overrightarrow{\omega}(0)t + \frac{1}{2}\overrightarrow{\alpha}(t)^2$------------------- (eq 2)

Give $t = t_i$

$\overrightarrow{\theta}(t_i) = \overrightarrow{\theta_0} + \overrightarrow{\omega_0} t_i + \frac{1}{2}\overrightarrow{\alpha}(t_i)^2$ ------------------- (eq 3)

$\overrightarrow{\omega}(t_i) = \overrightarrow{\omega_0} + \overrightarrow{\alpha}(t_i)$ ------------------- (eq 4)

Δt is timestep

Give $t = t_i + \Delta t$

$$\overrightarrow{\theta}(t_i + \Delta t) = \overrightarrow{\theta_0} + \overrightarrow{\omega_0}(t_i + \Delta t) + \frac{1}{2}\overrightarrow{\alpha}(t_i + \Delta t)^2$$

$$\overrightarrow{\theta}(t_i + \Delta t) = \overrightarrow{\theta_0} + \overrightarrow{\omega_0} t_i + \overrightarrow{\omega_0}\Delta t + \frac{1}{2}\overrightarrow{\alpha}(t_i)^2 + \overrightarrow{\alpha}\, t_i \Delta t + \frac{1}{2}\overrightarrow{\alpha}(\Delta t)^2$$

$\overrightarrow{\theta}(t_i + \Delta t) = \overrightarrow{\theta}(t_i) + \overrightarrow{\omega}(t_i)\Delta t + \frac{1}{2}\overrightarrow{\alpha}(\Delta t)^2$ ------------------- (eq 5) ;show in eq 3,eq 4

$$\overrightarrow{\omega}(t_i + \Delta t) = \overrightarrow{\omega_0} + \overrightarrow{\alpha}(t_i + \Delta t)$$

$$\overrightarrow{\omega}(t_i + \Delta t) = \overrightarrow{\omega_0} + \overrightarrow{\alpha}\, t_i + \overrightarrow{\alpha}\, \Delta t$$

$\overrightarrow{\omega}(t_i + \Delta t) = \overrightarrow{\omega}(t_i) + \overrightarrow{\alpha}\, \Delta t$ ------------------- (eq 6) ;show in eq4

According to equation 5 : $\overrightarrow{\theta}(t_i + \Delta t) = \overrightarrow{\theta}(t_i) + \overrightarrow{\omega}(t_i)\Delta t + \frac{1}{2}\overrightarrow{\alpha}(\Delta t)^2$ give to discrete time

$\overrightarrow{\theta}[K+1] = \overrightarrow{\theta}[K] + \overrightarrow{\omega}[K]\Delta t + \frac{1}{2}\overrightarrow{\alpha}[K](\Delta t)^2$ ------------------- (eq 7)

According to equation 6 : $\overrightarrow{\omega}(t_i + \Delta t) = \overrightarrow{\omega}(t_i) + \overrightarrow{\alpha}\Delta t$ give to discrete time

$\overrightarrow{\omega}[K+1] = \overrightarrow{\omega}[K] + \overrightarrow{\alpha}[K]\Delta t$ ------------------- (eq 8)

Assumption : Acceleration is 0 mean gausion $\alpha \sim N(0, \sigma_\alpha^2)$

: α is constant interm of kinematic derivations

According to equation : $\hat{X}_p[K] = A\hat{X}[K] + B\overrightarrow{U}[K] + G\overrightarrow{\omega}[K]$

$\hat{X}_p[K] = A\hat{X}[K] + G\overrightarrow{\omega}[K]$; $\overrightarrow{U}$ =0

$\hat{X}_p[K] = A\hat{X}[K] + G\alpha$; from assumption and eq 7, eq 8 $\overrightarrow{\omega}[K] = \alpha$

$$\begin{bmatrix}\theta[k+1]\\ \omega[k+1]\end{bmatrix} = \begin{bmatrix}1 & \Delta t\\ 0 & 1\end{bmatrix}\begin{bmatrix}\theta[k]\\ \omega[k]\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\Delta t^2\\ \Delta t\end{bmatrix}\alpha$$

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X =
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$$\begin{pmatrix}\text{Thetapre}\\ \text{Omegapre}\end{pmatrix}$$

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A =
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$$\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

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G =
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$$\begin{pmatrix}\frac{t^2}{2}\\ t\end{pmatrix}$$

```
stateprediction =
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$$\begin{pmatrix}\text{Thetapre} + \text{Omegapre}\ t\\ \text{Omegapre}\end{pmatrix}$$

According to equation : $\hat{Y}_p[K] = C\hat{X}_p[K] + D\overrightarrow{U}[K] + \overrightarrow{V}[K]$

$\hat{Y}_p[K] = C\hat{X}_p[K] + D\overrightarrow{U}[K]$; from assumption $\overrightarrow{V} = 0$

$\hat{Y}_p[K] = C\hat{X}_p[K]$; $\overrightarrow{U}$ =0

$$\theta[k] = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\theta[k]\\ \omega[k]\end{bmatrix}$$

```
C = 1×2
     1     0
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According to equation : $\hat{P}_p[K] = A\hat{P}[K]A^T + \mathrm{GQG}^T$

Q is Covariance process noise

Ppreloop is $\hat{P}_p$ Pupdate iteration k-1

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Ppreloop =
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$$\begin{pmatrix} \mathrm{Ppre}_{11} & \mathrm{Ppre}_{12} \\ \mathrm{Ppre}_{21} & \mathrm{Ppre}_{22} \end{pmatrix}$$

`Q =` Gll

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Ppredict =
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$$\begin{pmatrix} \mathrm{Ppre}_{11} + \bar{t}\ (\mathrm{Ppre}_{12} + \mathrm{Ppre}_{22}\,t) + \mathrm{Ppre}_{21}\,t + \frac{\mathrm{Gll}\,t^2\ (\bar{t})^2}{4} & \mathrm{Ppre}_{12} + \mathrm{Ppre}_{22}\,t + \frac{\mathrm{Gll}\,t^2\,\bar{t}}{2} \\ \frac{\mathrm{Gll}\,t\ (\bar{t})^2}{2} + \mathrm{Ppre}_{22}\,\bar{t} + \mathrm{Ppre}_{21} & \mathrm{Ppre}_{22} + \mathrm{Gll}\,t\,\bar{t} \end{pmatrix}$$

According to equation : $S[K] = C\hat{P}_p C^T + R$

R is Covariance sensor

`S =` is $CP_{11} + R$

$$\boxed{\mathrm{Ppre}_{11}} \boxed{+ R +} \boxed{\bar{t}\ (\mathrm{Ppre}_{12} + \mathrm{Ppre}_{22}\,t) + \mathrm{Ppre}_{21}\,t + \frac{\mathrm{Gll}\,t^2\ (\bar{t})^2}{4}}$$

According to equation : $K[K] = \hat{P}_p[K]\,C^T\,S[K]^{-1}$

`K =`

$$\begin{pmatrix} \frac{\mathrm{Ppre}_{11} + \bar{t}\ (\mathrm{Ppre}_{12} + \mathrm{Ppre}_{22}\,t) + \mathrm{Ppre}_{21}\,t + \frac{\mathrm{Gll}\,t^2\ (\bar{t})^2}{4}}{\sigma_1} \\ \frac{\frac{\mathrm{Gll}\,t\ (\bar{t})^2}{2} + \mathrm{Ppre}_{22}\,\bar{t} + \mathrm{Ppre}_{21}}{\sigma_1} \end{pmatrix}$$

is CP_{11}

is CP_{21}

$k_{11} = CP_{11} / CP_{11} + R$

$k_{21} = CP_{21} / CP_{11} + R$

where

$$\sigma_1 = \boxed{\mathrm{Ppre}_{11}} \boxed{+ R +} \boxed{\bar{t}\ (\mathrm{Ppre}_{12} + \mathrm{Ppre}_{22}\,t) + \mathrm{Ppre}_{21}\,t + \frac{\mathrm{Gll}\,t^2\ (\bar{t})^2}{4}}$$

is $CP_{11} + R$

According to equation : $\hat{P}[K+1] = (\,\mathrm{II} + K[K]\,C\,)\,\hat{P}_p[K]$

$\overset{\Lambda}{P}[K+1]$=PPreloop

`PPreloop =`

$$\begin{pmatrix} -\left(\frac{\sigma_2}{\sigma_1}-1\right)\sigma_2 & -\left(\frac{\sigma_2}{\sigma_1}-1\right)\sigma_4 \\ \mathrm{Ppre}_{21}+\mathrm{Ppre}_{22}\,\overline{t}+\frac{\mathrm{Gll}\,t\,\left(\overline{t}\right)^2}{2}-\frac{\sigma_3\,\sigma_2}{\sigma_1} & \mathrm{Ppre}_{22}-\frac{\sigma_4\,\sigma_3}{\sigma_1}+\mathrm{Gll}\,t\,\overline{t} \end{pmatrix}$$

where

$$\sigma_1 = \mathrm{Ppre}_{11}+R+\overline{t}\ \left(\mathrm{Ppre}_{12}+\mathrm{Ppre}_{22}\,t\right)+\mathrm{Ppre}_{21}\,t+\frac{\mathrm{Gll}\,t^2\,\left(\overline{t}\right)^2}{4}$$

$$\sigma_2 = \mathrm{Ppre}_{11}+\overline{t}\ \left(\mathrm{Ppre}_{12}+\mathrm{Ppre}_{22}\,t\right)+\mathrm{Ppre}_{21}\,t+\frac{\mathrm{Gll}\,t^2\,\left(\overline{t}\right)^2}{4}$$

$$\sigma_3 = \frac{\mathrm{Gll}\,t\,\left(\overline{t}\right)^2}{2}+\mathrm{Ppre}_{22}\,\overline{t}+\mathrm{Ppre}_{21}$$

$$\sigma_4 = \mathrm{Ppre}_{12}+\mathrm{Ppre}_{22}\,t+\frac{\mathrm{Gll}\,t^2\,\overline{t}}{2}$$

According to equation : $\widetilde{Y}[K] = \overrightarrow{Y}[K] - \overrightarrow{Y_P}[K]$

$\overrightarrow{Y}[K]$ is Sensor read θ

$\overrightarrow{Y_P}[K]$ is θ predict iteration K-1

`Ytelda =` $\mathrm{ReadThetafromSensor} - \mathrm{Thetapre}$

According to equation : $\overset{\Lambda}{X}[K+1] = \overset{\Lambda}{X}_p[K] + K[K]\,\widetilde{Y}[K]$

`StateEstmate =`

$$\left(\begin{array}{c} \text{Thetapre} + \text{Omegapre}\, t + \dfrac{(\text{ReadThetafromSensor} - \text{Thetapre})\left(\text{Ppre}_{11} + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\right.}{\sigma_1} \\ \text{Omegapre} + \dfrac{(\text{ReadThetafromSensor} - \text{Thetapre})\left(\dfrac{\text{Gll}\, t\, (\bar{t})^2}{2} + \text{Ppre}_{22}\, \bar{t} + \text{Ppre}_{21}\right)}{\sigma_1} \end{array} \right.$$

where

$$\sigma_1 = \boxed{\text{Ppre}_{11}}\,\boxed{+\,R\,+}\,\bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\, t + \frac{\text{Gll}\, t^2\, (\bar{t})^2}{4}$$

is CP11 + R

double t = time step

In microcontroller loop sequence program

```
stateprediction =
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$$\begin{pmatrix} \text{Thetapre} + \text{Omegapre}\, t \\ \text{Omegapre} \end{pmatrix}$$

StateTheta = ThetaPre + OmegaPre(t)

StateOmega = Omegapre

`Ytelda =` $\text{ReadThetafromSensor} - \text{Thetapre}$

YTheta_telda = GetRealread( ) - Thetapre

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Ppredict =
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$$\begin{pmatrix} \text{Ppre}_{11} + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\, t + \dfrac{\text{Gll}\, t^2\, (\bar{t})^2}{4} & \text{Ppre}_{12} + \text{Ppre}_{22}\, t + \dfrac{\text{Gll}\, t^2\, \bar{t}}{2} \\ \dfrac{\text{Gll}\, t\, (\bar{t})^2}{2} + \text{Ppre}_{22}\, \bar{t} + \text{Ppre}_{21} & \text{Ppre}_{22} + \text{Gll}\, t\, \bar{t} \end{pmatrix}$$

CPpre12 t + CPpre22 t^2

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StateEstmate =
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CP11 = CPpre11 + t (CPpre12 + CPpre22(t)) + CPpre21(t) + dt^4/4

CP12 = CPpre12 + CPpre22(t) + dt^3/2

CP21 = dt^3/2 + CPpre22(t) + CPpre21

CP22 = CPpre22 + dt^2

k11 = CP11 / CP11 + R

k21 = CP21 / CP11 + R

$$\left(\begin{array}{c} \overbrace{\text{Thetapre} + \text{Omegapre}\, t}^{\text{is StateTheta}} + \dfrac{\overbrace{(\text{ReadThetafromSensor} - \text{Thetapre})}^{\text{Y Theta\_telda}}\overbrace{\left(\text{Ppre}_{11} + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\right)}^{\text{is k11}}}{\underbrace{\sigma_1}_{\text{is CP11 + R}}} \\ \overbrace{\text{Omegapre}}^{\text{is StateOmega}} + \dfrac{\underbrace{(\text{ReadThetafromSensor} - \text{Thetapre})}_{\text{Y Theta\_telda}}\underbrace{\left(\dfrac{\text{Gll}\, t\, (\bar{t})^2}{2} + \text{Ppre}_{22}\, \bar{t} + \text{Ppre}_{21}\right)}_{\text{is CP21}}}{\underbrace{\sigma_1}_{\text{is CP11 + R}}}\ \text{is k21} \end{array}\right)$$

where

$$\sigma_1 = \boxed{\text{Ppre}_{11} + R + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\, t + \frac{\text{Gll}\, t^2\, (\bar{t})^2}{4}} \quad \text{is CP11 + R}$$

UpdateValue Thetapre Omegapre to next iteration

$$\begin{bmatrix} \text{Thetapre} \\ \text{Omegapre} \end{bmatrix} = \text{StateEstmate}$$

Thetapredict = StateTheta + (YTheta_telda)(k11)

Omegapredict = StateOmega + (YTheta_telda)(k21)

Update value

```
PPreloop =
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$$\begin{pmatrix} -\left(\dfrac{\sigma_2}{\sigma_1} - 1\right)\sigma_2 & -\left(\dfrac{\sigma_2}{\sigma_1} - 1\right)\sigma_4 \\ \underbrace{\text{Ppre}_{21} + \text{Ppre}_{22}\, \bar{t} + \dfrac{\text{Gll}\, t\, (\bar{t})^2}{2}}_{\text{is } \sigma_3 = \text{CP21}} - \dfrac{\sigma_3\, \sigma_2}{\sigma_1} & \underbrace{\text{Ppre}_{22} - \dfrac{\sigma_4\, \sigma_3}{\sigma_1} + \text{Gll}\, t\, \bar{t}}_{\text{is CP22}} \end{pmatrix}$$

CPpre11 = $-\left(\left[\text{CP11}/\text{CP11}+R\right] - 1\right)\text{CP11}$

CPpre12 = $-\left(\left[\text{CP11}/\text{CP11}+R\right] - 1\right)(\text{CP12})$

CPre21 = $\text{CP21} - \left(\left[\text{CP21} * \text{CP11}\right]/\text{CP11}+R\right)$

CPre22 = $\text{CP22} - \left(\text{CP12} * \text{CP21}\right)/\text{CP11}+R$

where

$$\sigma_1 = \boxed{\text{Ppre}_{11} + R + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\, t + \frac{\text{Gll}\, t^2\, (\bar{t})^2}{4}} \quad \text{is CP11 + R}$$

$$\sigma_2 = \boxed{\text{Ppre}_{11} + \bar{t}\,(\text{Ppre}_{12} + \text{Ppre}_{22}\, t) + \text{Ppre}_{21}\, t + \frac{\text{Gll}\, t^2\, (\bar{t})^2}{4}} \quad \text{is CP11}$$

$$\sigma_3 = \boxed{\frac{\text{Gll}\, t\, (\bar{t})^2}{2} + \text{Ppre}_{22}\, \bar{t} + \text{Ppre}_{21}} \quad \text{is CP21}$$

$$\sigma_4 = \boxed{\text{Ppre}_{12} + \text{Ppre}_{22}\, t + \frac{\text{Gll}\, t^2\, \bar{t}}{2}} \quad \text{is CP12}$$

UpdateValue Ppreloop to next iteration

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PPreloop =
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$$\begin{pmatrix} \text{Ppre}_{11} & \text{Ppre}_{12} \\ \text{Ppre}_{21} & \text{Ppre}_{22} \end{pmatrix}$$