Monitoring Refinement via Symbolic Reasoning

Abstract

Efficient implementations of concurrent objects such as semaphores, locks, and atomic collections are essential to modern computing. Programming such objects is error prone: in minimizing the synchronization overhead between concurrent object invocations, one risks the conformance to reference implementations — or in formal terms, one risks violating *observational refinement*. Precisely testing this refinement even within a single execution is intractable, limiting existing approaches to executions with very few object invocations.

We develop scalable and effective algorithms for detecting refinement violations. Our algorithms are founded on incremental, symbolic reasoning, and exploit foundational insights into the refinement-checking problem. Our approach is *sound*, in that we detect only actual violations, and scales far beyond existing violation-detection algorithms. Empirically, we find that our approach is *practically* complete, in that we detect nearly all violations arising in actual executions.

1. Introduction

Efficient implementations of concurrent objects such as semaphores, locks, and atomic collections including stacks and queues are vital to modern computer systems. Programming them is however error prone. To minimize synchronization overhead between concurrent object-method invocations, implementors avoid blocking operations like lock acquisition, allowing methods to execute concurrently. However, concurrency risks unintended inter-operation interference, and risks conformance to reference implementations. Conformance is formally captured by *observational refinement*: given two libraries L_1 and L_2 implementing the methods of some concurrent object, we say L_1 *refines* L_2 if and only if every computation of every program using L_1 would also be possible were L_2 used instead.

Verifying observational refinement is intrinsically hard: it is undecidable even for finite-state implementations whose methods can be called concurrently by arbitrarily-many threads [Bouajjani et al. 2013]. In fact, even checking the conformance of a single program execution using one library with respect to another is intractable [Gibbons and Korach 1997]. In practice, fully-automated techniques for checking observational refinement have been limited to detecting violations in executions with very few library-method invocations.

In this work we develop a fully-automated and highly-scalable means of detecting violations to observational refinement by monitoring program executions. The key challenge we address is how to achieve scalability while maintaining precision/completeness. Detecting refinement violations may require observing large executions, with many *operations*, i.e., object-method invocations. However, the complexity of precise violation-checking is exponential in the number of operations. Essentially, this check amounts to considering every possible *linearization* of an execution's operations, which are only partially-ordered by their happens-before relation. Only if none of the linearizations represent a valid sequence of method invocations does the execution witness a violation.

Our approach is based on sound yet possibly-incomplete means for avoiding the practical scalability pitfalls, the most immediate pitfall being the explicit enumeration of possible linearizations. We discover that naturally-occurring concurrent objects, including atomic collections, locks, and semaphores, can be expressed symbolically, in a first-order language over their method names, argument/return values, and invocation-ordering constraints. Ultimately this allows us to reduce violation detection for single executions to satisfiability in propositional logic. Practically speaking, this allows us to exploit the highly-developed algorithms of modern symbolic reasoning engines in place of the explicit enumeration of linearizations. Furthermore, symbolic reasoning lends itself to *incrementality*: as the symbolic representation of each successive execution step differs monotonically, only by the addition of a new operation or return value, we can reuse all logical implications of previous steps. Conceptually, this avoids recomputing the set of possible linearizations from scratch after each execution step.

While exploiting symbolic reasoning engines makes sense practically, and is likely to be more efficient, the link to symbolic reasoning also reveals insights leading to more-drastic optimizations. In particular, we notice that in proving satisfiability, the solver must essentially build a model of some linearization of the given execution which represents a valid object method-invocation sequence. In doing so, the solver may need to make branching decisions in addition to logical deductions, or *unit propagation*, possibly backtracking later, about (a) how pending operations should be completed/-dropped, and (b) which operations should be linearized before

others. However, from the perspective of detecting violations, it is always sound to forgo such costly branching/backtracking and, for instance, wait until the given pending operations are actually completed later in the execution to determine their return values. Though it is unclear whether such strategies might actually be complete in theory, we hypothesize that they are effective in practice, uncovering most violations.

Though limiting symbolic reasoning to saturation, or unit propagation, does avoid the exponential cost in the number of operations, a truly useful runtime monitor for observational refinement ought to be *linear* in the number of operations, and incur only a constant space overhead; otherwise it will progressively retard program execution and/or eventually exhaust program memory. Achieving this complexity goal implies that the monitor cannot store all previously-executed operations. However, simply forgetting arbitrary operations is unsound, in the sense that we may conclude a violation when none actually occurred. For instance, if a monitor for an atomic queue object observes ordered enqueue(a) and enqueue(b) operations, and dequeue(b), having dropped dequeue(a), the monitor may detect a violation, thinking dequeue(b) should not precede dequeue(a). To resolve this issue, we develop a sound theory for the removal of *matched* operations, e.g., the enqueue operations together with the dequeue operations removing the same added elements. We hypothesize that such strategies are, too, effective in practice, in that they do not cause many violations to be missed.

Empirically we demonstrate that our violation-detection strategies are profoundly-more scalable than existing techniques. We also validate our aforementioned completeness hypotheses: that in practice, most violations are caught despite the possibility for incompleteness.

To summarize, we make the following contributions:

- First-order logical characterizations of naturally-occurring concurrent objects, allowing for symbolic reasoning about observational refinement (§3).
- A reduction from refinement-violation detection for single executions to propositional logic satisfiability (§4).
- A sound theory of matched-operation removal, allowing the scalability required for runtime monitoring (§5).
- Empirical validation that our violation-detection optimizations are scalable and effectively-complete (§6).

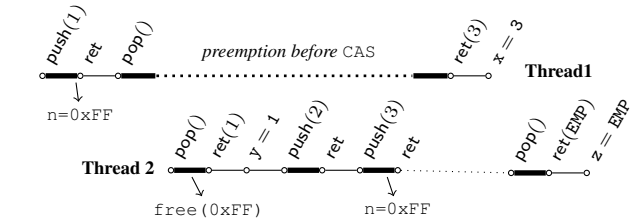
We begin by formalizing observational refinement (§2), and conclude by discussing related work (§7).

2. Observational Refinement

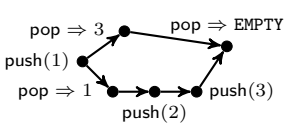
Figure 1 lists a non-blocking stack [Treiber 1986] which stores its elements in a singly-linked list rooted at `Top`, and avoids blocking lock acquisitions in favor of non-blocking compare-and-swap (`CAS`) instructions in order to maximize parallelism. In one atomic step, the `CAS` instruction assigns `Top = n` only if `Top == t`.

```
struct node *Top;

void push(int v):
  struct node *n,*t;
  n = malloc(sizeof( *n));
  n->data = v;
  do {
    struct node *t = Top;
    n->next = t;
  } while (! CAS (&Top, t, n
      ));

int pop():
  struct node *n,*t;
  do {
    *t = Top;
    if (t==NULL) return
        EMPTY;
    n = t->next;
  } while (! CAS (&Top, t,
      n))
  int result = t->data;
  free(t);
  return result;

struct node {
  int data;
  struct node *next;
}

void Thread1():
  push(1);
  int x = pop();

void Thread2():
  int y = pop();
  push(2);
  push(3);
  int z = pop();
```

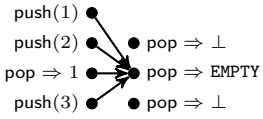
Figure 1. An implementation of Treiber's stack. The `pop` operation returns the value `EMPTY` when the stack is empty.



(a) An execution e of the program; it depicts calls, returns, and assignments, and time progresses from left to right.



(b) The history of the above execution.



(c) A weaker history.

Figure 2. An execution and its history.

Unfortunately this implementation suffers from a subtle concurrency bug [Michael 2004] exposed by the two-thread program of Figure 1 via the execution depicted in Figure 2(a). Essentially, Thread 1 wrongfully assumes the absence of interference from other threads on the successful `CAS` operation. Thread 1 is preempted right before executing its `CAS` in the `pop` method; at that moment, its `t` variable points to the first element in the list at address `0xFF` added by `push(1)`, and `n == NULL`. While Thread 2 updates the list with two additional elements, added by `push(2)` and `push(3)`, the `t` variable of Thread 1 still points to the list's first element at address `0xFF`, which was freed by Thread 2's call to `pop`, and reallocated in the call to `push(3)`. When Thread 1 resumes, its `CAS` succeeds, effectively removing two elements from the list instead of one. The final `pop` of Thread 2 thus erroneously returns `EMPTY`. Intuitively, this is a problem because the `EMPTY` value should not have been returned since more elements have been pushed than popped

prior to Thread 2's final pop operation. This bug exposes the fact that our CAS-based implementation does not conform to programmers' expectations of a stack object whose operations execute atomically. In particular, the assignment z = EMPTY should never have occurred.

Formally this conformance is *observational refinement*. Essentially, a library implementation L_1 *refines* L_2 if every observable behavior of programs using L_1 is also observable using L_2. This is not the case between the CAS-based implementation L_1 of Figure 1 and an atomic implementation L_2, since z = EMPTY is observable with L_1 yet not with L_2.

We capture the interaction between programs and libraries by their *histories*, representing the partial happens-before orders of library method invocations. Fixing arbitrary sets $\mathbb{O}$, $\mathbb{M}$, and $\mathbb{V}$ of operation identifiers, method names, and parameter/return values, respectively, we define the set of *operation labels* as $\mathbb{L} = (\mathbb{M} \times \mathbb{V} \times (\mathbb{V} \cup \{\bot\}))$. We write $m(u) \Rightarrow v$ to denote the label $\ell = \langle m, u, v\rangle$. When $v = \bot$ we say ℓ is *pending*, and otherwise *completed*. A *history* $h = \langle O, <, f\rangle$ is a partial order $<$ on a set $O \subseteq \mathbb{O}$ of operations labeled by $f : O \to \mathbb{L}$ such that operations with pending labels are maximal. An operation o with label ℓ is an *ℓ-operation*, and o is *pending/completed* when ℓ is. We say h is *sequential* when $<$ is a total order on O, and *(in)complete* when (not) all operations are complete. A history set is *sequential* when it contains only sequential histories.

Example 2.1. *The history of Figure 2(b) captures the execution of Figure 2(a), where arrows depict the transitive reduction of the order relation. Essentially, operations o_1 and o_2 are ordered if o_1 happens before o_2. For example, although the* push(1)*-operation precedes* pop $\Rightarrow 3$*, since the* push(1)*-operation returns before the* (pop $\Rightarrow 3$)*-operation is called, the* (pop $\Rightarrow 1$)*-operation is incomparable to* (pop $\Rightarrow 3$)*, since it returns after the* (pop $\Rightarrow 3$)*-operation is called.*

This notion of histories gives rise to a natural *weaker-than* relation $\preceq$ relating any two histories h_1 and h_2 such that h_2 includes all completed operations of h_1, and preserves the order between the operations common with h_1. Pending operations in h_1 can be either omitted or completed in h_2. Formally, $\langle O_1, <_1, f_1\rangle \preceq \langle O_2, <_2, f_2\rangle$ iff there exists an injection $g : O_2 \to O_1$ such that

- $o \in \text{range}(g)$ when $f_1(o) = m(u) \Rightarrow v$ and $v \neq \bot$,
- $g(o_1) <_1 g(o_2)$ implies $o_1 <_2 o_2$ for each $o_1, o_2 \in O_2$,
- $f_1(g(o)) \ll f_2(o)$ for each $o \in O_2$.

where $(m_1(u_1) \Rightarrow v_1) \ll (m_2(u_2) \Rightarrow v_2)$ iff $m_1 = m_2$, $u_1 = u_2$, and $v_1 \in \{v_2, \bot\}$. When the injection g need be fixed, we write $h_1 \preceq_g h_2$. We say h_1 and h_2 are *equivalent* when $h_1 \preceq h_2$ and $h_2 \preceq h_1$. We do not distinguish between equivalent histories, and we assume every set H of histories is closed under inclusion of equivalent histories, i.e., if h_1 and h_2 are equivalent and $h_1 \in H$, then $h_2 \in H$ as well. Finally, $\overline{H}$ denotes the closure $\{h : \exists h' \in H.\ h \preceq h'\}$ of a history set H under weakening.

Example 2.2. *The history of Figure 2(c) is weaker than that of Figure 2(b). While one of the two pending* pop*-operations is mapped to the completed* (pop $\Rightarrow 3$)*-operation, the other is dropped.*

We model libraries as sets of histories. Since libraries only dictate methods' executions between their respective calls and returns, for any execution they admit, they must also admit executions with weaker inter-operation ordering, in which calls may happen earlier, and/or returns later. Thus any weakening of a history admitted by a library must also be admitted. Formally, a *library* L is a set of histories closed under weakening, i.e. $\overline{L} = L$, and a *kernel of* L is any set H such that $\overline{H} = L$. A library L is called *atomic* if it has a sequential kernel. Atomic libraries are often considered as specifications for concurrent objects. In practice, libraries can be made atomic by guarding their methods bodies with global lock acquisitions.

Example 2.3. *The* atomic stack *is the library whose unique kernel is the set of all sequential histories for which the return value of each* pop *operation is either the argument value v to the last unmatched* push *operation, or* EMPTY *if there are no unmatched* push *operations.*

We define *observational refinement* between two libraries as history-set inclusion, saying L_1 *refines* L_2 iff $L_1 \subseteq L_2$. Although this refinement is typically defined with respect to the admissibility of program executions, recent work shows these definitions are equivalent [Bouajjani et al. 2015].

3. Refinement via Symbolic Reasoning

In this section we represent the kernels of typical concurrent objects, including atomic collections and locks, in a simple first-order language. Besides function and predicate symbols describing the operation labels and the order relation of a history, this language includes a predicate match describing a *matching function* M that maps operations o which "remove" or "test the presence" of values to the operations $M(o)$ which added them. For instance, a matching function M for a history of the atomic stack object is injective, and maps each (pop $\Rightarrow v_1$)-operation o to a push(v_2)-operation $M(o)$ such that $v_1 = v_2$ when $M(o)$ is defined. Similarly, the matching function M for an atomic lock object is also injective, and maps each unlock-operation o to a lock-operation $M(o)$ if $M(o)$ is defined.

Figures 3–7 list the properties characterizing typical concurrent objects in a first-order language whose variables range over operation identifiers, and whose functions and predicates are interpreted over the operation labels and order relation of a history, and a given matching function. We use the function symbols meth-od, arg-ument, and ret-urn, as well as the predicate symbols match and b-efore for this purpose. Note that we interpret the predicate match(x_1, x_2) by $o_1 = M(o_2)$

ATOMIC

$\forall x.\ \neg \mathsf{b}(x,x) \wedge \forall x_1,x_2,x_3.\ (\mathsf{b}(x_1,x_2) \wedge \mathsf{b}(x_2,x_3)) \Rightarrow \mathsf{b}(x_1,x_3)$

$\forall x_1,x_2.\ \mathsf{b}(x_1,x_2) \oplus \mathsf{b}(x_2,x_1) \wedge \forall x.\ \mathsf{ret}(x) \neq \bot$

COMPLETED

$\forall x.\ \neg \mathsf{b}(x,x) \wedge \forall x_1,x_2,x_3.\ (\mathsf{b}(x_1,x_2) \wedge \mathsf{b}(x_2,x_3)) \Rightarrow \mathsf{b}(x_1,x_3)$

$\forall x.\ \mathsf{ret}(x) \neq \bot$

INJECTIVE

$\forall x,y_1,y_2.\ \mathsf{match}(x,y_1) \wedge \mathsf{match}(x,y_2) \Rightarrow y_1 = y_2$

INJECTIVE(X)

$\forall x,y_1,y_2.\ \mathsf{match}(x,y_1) \wedge \mathsf{match}(x,y_2) \wedge \mathsf{meth}(y_1) = \mathsf{meth}(y_2) = X$
$\Rightarrow x_1 = x_2$

SYMMETRIC

$\forall x_1,x_2.\ \mathsf{match}(x_1,x_2) \Leftrightarrow \mathsf{match}(x_2,x_1)$

TOTAL(Y)

$\forall y.\ \mathsf{meth}(y) = Y \Rightarrow \exists x.\ \mathsf{match}(x,y) \wedge \mathsf{b}(x,y)$

MATCH1(X,Y)

$\forall x,y.\ \mathsf{match}(x,y) \Rightarrow \mathsf{meth}(x) = X \wedge \mathsf{meth}(y) = Y \wedge \mathsf{arg}(x) = \mathsf{ret}(y)$

MATCH2(X,Y_1,Y_2)

$\forall x,y.\ \mathsf{match}(x,y) \Rightarrow \mathsf{meth}(x) = X \wedge (\mathsf{meth}(y) = Y_1 \vee \mathsf{meth}(y) = Y_2)$
$\wedge\ \mathsf{arg}(x) = \mathsf{arg}(y)$

MATCH3(X,Y)

$\forall x,y.\ \mathsf{match}(x,y) \Rightarrow \mathsf{meth}(x) = X \wedge \mathsf{meth}(y) = Y$

Figure 3. Generic formulæ used across many objects ($\oplus$ denotes exclusive or).

when each x_i binds to o_i. We represent the kernel H of each of the following objects by a first-order formula THEORY(H) such that $h \in H$ iff $h, M \models$ THEORY(H), for some M, where the satisfaction relation $\_,\_ \models \_$ is defined as usual, using the aforementioned interpretations.

Example 3.1 (Atomic collections)**.** *The kernel of atomic queue objects is represented by the conjunction of properties stating:*

- *values are added before they are removed (*ADDREM*),*
- *values are removed in the order they are added (*FIFO*), and*
- *remove operations returning empty are not surrounded by matching adds and removes (*EMPTY*).*

We thus represent the kernel H_{q} of atomic queues by

ATOMIC $\wedge$ ADDREM $\wedge$ FIFO $\wedge$ EMPTY

Similarly, we represent the kernel H_{pq} of priority queues by

ATOMIC $\wedge$ ADDREM $\wedge$ MAX $\wedge$ EMPTY

and the kernel H_{st} of atomic stacks by

ATOMIC $\wedge$ ADDREM $\wedge$ LIFO $\wedge$ EMPTY

ADDREM

$\forall r.\ \mathsf{meth}(r) = \mathsf{rem} \wedge \mathsf{ret}(r) \neq \mathsf{empty} \Rightarrow \exists a.\ \mathsf{match}(a,r) \wedge \mathsf{b}(a,r)$

EMPTY

$\forall e,a.\ \mathsf{meth}(e) = \mathsf{rem} \wedge \mathsf{ret}(e) = \mathsf{empty} \wedge \mathsf{meth}(a) = \mathsf{add}$
$\wedge\ \mathsf{b}(a,e) \Rightarrow \exists r.\ \mathsf{match}(a,r) \wedge \mathsf{b}(r,e)$

FIFO

$\forall a_1,a_2,r_2.\ \mathsf{meth}(a_1) = \mathsf{add} \wedge \mathsf{match}(a_2,r_2)$
$\wedge\ \mathsf{b}(a_1,a_2) \Rightarrow \exists r_1.\ \mathsf{match}(a_1,r_1) \wedge \mathsf{b}(r_1,r_2)$

LIFO

$\forall a_1,a_2,r_1.\ \mathsf{meth}(a_2) = \mathsf{add} \wedge \mathsf{match}(a_1,r_1)$
$\wedge\ \mathsf{b}(a_1,a_2) \wedge \mathsf{b}(a_2,r_1) \Rightarrow \exists r_2.\ \mathsf{match}(a_2,r_2) \wedge \mathsf{b}(r_2,r_1)$

MAX

$\forall a_1,a_2,r_1.\ \mathsf{meth}(a_2) = \mathsf{add} \wedge \mathsf{match}(a_1,r_1) \wedge \mathsf{b}(a_2,r_1)$
$\wedge\ \mathsf{arg}(a_1) < \mathsf{arg}(a_2) \Rightarrow \exists r_2.\ \mathsf{match}(a_2,r_2) \wedge \mathsf{b}(r_2,r_1)$

Figure 4. Formulæ for collection objects.

Additionally, we enforce the sanity of the underlying matching function by adding the formulæ MATCH1(add, rem) *and* INJECTIVE*, ensuring removes are matched to adds adding the removed value and that every two removes are matched to different adds.*

Example 3.2 (Atomic sets)**.** *Unlike the atomic queues and stacks which behave as* multisets *and return removed values, the atomic set's* remove *method takes as an argument a value to be removed, and succeeds whether or not the value is present. The formulæ*INCLUDE *and* EXCLUDE *specify when a* contains*-operation may return* true*. We represent the kernel H_{s} of atomic sets by*

ATOMIC $\wedge$ INCLUDE $\wedge$ EXCLUDE

Additionally, we enforce the sanity of the underlying matching function by adding the formulæ: MATCH2(add, remove, contains) *ensuring removes and contains returning true are matched to adds of the same value,* INJECTIVE(remove) *ensuring that every two removes are matched to different adds,* ADDREM *ensuring that the matching function maps removes to preceding adds, and* MATCHINGADD *(resp.,* MATCHINGREM*) ensuring that every add matched to a remove (resp., remove matched to an add) is the first in a sequence of adds adding (resp., removes removing) the same value.*

Example 3.3 (Atomic register)**.** *Atomic registers with* read *and* write *methods essentially ensure that each value read is written by the most recent* WRITE*-operation. We represent the kernel H_{r} of atomic registers by*

ATOMIC $\wedge$ READWRITE $\wedge$ READFROM

and enforce the sanity of the underlying matching function by adding the formulæ INJECTIVE *and* MATCH1(write, read)*, ensuring reads are matched to writes writing the read value.*

INCLUDE

$\forall c.\ \mathsf{meth}(c) = \mathsf{contains} \wedge \mathsf{ret}(c) = \mathsf{true}$
$\quad \Rightarrow \exists a.\ \mathsf{match}(c, a) \wedge \mathsf{b}(a, c)$

$\forall c, a, r.\ \mathsf{meth}(c) = \mathsf{contains} \wedge \mathsf{ret}(c) = \mathsf{true}$
$\quad \wedge\ \mathsf{meth}(r) = \mathsf{remove} \wedge \mathsf{meth}(a) = \mathsf{add} \wedge \mathsf{match}(c, a) \wedge \mathsf{match}(r, a)$
$\quad \Rightarrow \mathsf{b}(a, c) \wedge \mathsf{b}(c, r)$

EXCLUDE

$\forall c, a.\ \mathsf{meth}(c) = \mathsf{contains} \wedge \mathsf{ret}(c) = \mathsf{false}$
$\quad \wedge\ \mathsf{meth}(a) = \mathsf{add} \wedge \mathsf{arg}(c) = \mathsf{arg}(a) \wedge \mathsf{b}(a, c)$
$\quad \Rightarrow \exists r.\ \mathsf{meth}(r) = \mathsf{remove} \wedge \mathsf{match}(r, a) \wedge \mathsf{b}(r, c)$

ADDREM

$\forall a, r.\ \mathsf{match}(a, r) \wedge \mathsf{meth}(r) = \mathsf{remove} \Rightarrow \mathsf{meth}(a) = \mathsf{add} \wedge \mathsf{b}(a, r)$

MATCHINGADD

$\forall a, x, p.\ \mathsf{match}(a, x) \wedge \mathsf{arg}(p) = \mathsf{arg}(a)$
$\quad \wedge\ \forall q.\ \mathsf{b}(q, p) \vee \mathsf{b}(a, q) \vee \mathsf{arg}(q) \neq \mathsf{arg}(a)$
$\quad \Rightarrow \mathsf{meth}(p) = \mathsf{remove} \vee (\mathsf{meth}(p) = \mathsf{contains} \wedge \mathsf{ret}(p) = \mathsf{false})$

MATCHINGREM

$\forall a, r, p.\ \mathsf{match}(r, a) \wedge \mathsf{arg}(p) = \mathsf{arg}(r) \wedge \mathsf{meth}(r) = \mathsf{remove}$
$\quad \wedge\ \forall q.\ \mathsf{b}(q, p) \vee \mathsf{b}(r, q) \vee \mathsf{arg}(q) \neq \mathsf{arg}(r)$
$\quad \Rightarrow \mathsf{meth}(p) = \mathsf{add} \vee (\mathsf{meth}(p) = \mathsf{contains} \wedge \mathsf{ret}(p) = \mathsf{true})$

Figure 5. Formulæ for set objects.

READWRITE

$\forall r.\ \mathsf{meth}(r) = \mathsf{read} \Rightarrow \exists w.\ \mathsf{match}(w, r) \wedge \mathsf{b}(w, r)$

READFROM

$\forall w_1, r.\ \mathsf{meth}(w_1) = \mathsf{write} \wedge \mathsf{meth}(r) = \mathsf{read} \wedge \neg\mathsf{match}(w_1, r)$
$\quad \wedge\ \mathsf{b}(w_1, r) \Rightarrow \exists w_2.\ \mathsf{match}(w_2, r) \wedge \mathsf{b}(w_1, w_2) \wedge \mathsf{b}(w_2, r)$

Figure 6. Formulæ for register objects.

Example 3.4 (Synchronization objects)**.** *Atomic lock objects with* lock *and* unlock *methods ensure that at most one thread holds a lock at any moment. We represent the kernel of atomic locks by*

$$\text{ATOMIC} \wedge \text{TOTAL}(\mathsf{unlock}) \wedge \text{MUTEX}$$

and enforce coherent matching by adding the formulæ MATCH3(lock, unlock) *and* INJECTIVE*. Atomic semaphore objects with* acquire *and* release *methods generalize atomic locks, ensuring that at most n copies of a resource are held at any moment, for some fixed $n \in \mathbb{N}$. We represent the kernel of atomic semaphores by*

$$\text{ATOMIC} \wedge \text{TOTAL}(\mathsf{release}) \wedge \text{LIMIT}$$

and enforce coherent matching by adding the fomrulæ MATCH3(acquire, release) *and* INJECTIVE*.*

Exchanger objects are used to pair up threads so they can atomically swap values. The only method of this object is exchange *which receives as input a value v that a thread it offers to swap and returns a value $v' \neq \mathsf{null}$ if it has paired up with an* exchange(v') *operation. The latter operation will return the value v. We represent the kernel of exchanger objects by*

$$\text{COMPLETED} \wedge \text{EXCHANGE}$$

The kernel of this object contains non-sequential histories because the time spans of exchange *operations that pair up overlap. Additionally, we enforce the sanity of the underlying matching function by adding the formulæ:* MATCH1(exchange, exchange)*,* INJECTIVE*, and* SYMMETRIC *ensuring that the matching function is injective and symmetric and that it associates operations returning a value v to operations that receive v as input, and* MATCHOVERLAP *ensuring that matched operations overlap in time.*

MUTEX

$\forall \ell_1, \ell_2.\ \mathsf{meth}(\ell_1) = \mathsf{meth}(\ell_2) = \mathsf{lock} \wedge \mathsf{b}(\ell_1, \ell_2)$
$\quad \Rightarrow \exists u.\ \mathsf{meth}(u) = \mathsf{unlock} \wedge \mathsf{b}(\ell_1, u) \wedge \mathsf{b}(u, \ell_2)$

LIMIT

$$\forall x_0, \ldots, x_n.\ \bigwedge_{0 \leq i < n} \mathsf{b}(x_i, x_n) \wedge \bigwedge_{0 \leq i \leq n} \mathsf{meth}(x_i) = \mathsf{acquire}$$
$$\Rightarrow \exists r.\ \mathsf{b}(r, x_n) \wedge \bigvee_{0 \leq i \leq n} \mathsf{match}(y, x_i)$$

EXCHANGE

$\forall x.\ \mathsf{ret}(x) \neq \mathsf{null} \Rightarrow \exists y.\ \mathsf{match}(x, y)$

MATCHOVERLAP

$\forall x_1, x_2.\ \mathsf{match}(x_1, x_2) \Rightarrow \neg\mathsf{b}(x_1, x_2) \wedge \neg\mathsf{b}(x_2, x_1)$

Figure 7. Formulæ for synchronization objects.

4. Refinement via Propositional Reasoning

In this section we demonstrate that the history membership problem $h \in \overline{H}$ reduces to propositional satisfiability, given a formula THEORY(H) characterizing the histories of the library kernel H. Note that $h \in \overline{H}$ iff h is weaker than some history $h' \in H$, or equivalently weaker than some history h' such that $h', M \models \text{THEORY}(H)$, for some matching function M. When h is complete, the fact that any stronger history contains exactly the same set of operations enables the construction of a formula STRONGER(h) characterizing the histories stronger than h. Together with THEORY(H), this formula describes all stronger histories satisfying THEORY(H), and is therefore equivalent to $h \in H(L)$. We show how to construct these formulæ in Section 4.1.

When h contains pending operations, stronger histories h' may contain fewer operations, since some pending operations of h may be omitted in h', and others completed. In this case the joint satisfiability of THEORY(H) and STRONGER(h) must be enhanced with additional constraints to ensure that the operations of h' include at least the completed operations

$$\textsc{Domain}(h) \quad \bigwedge_{o_1,o_2\in O} o_1 \neq o_2 \wedge \forall x.\ \bigvee_{o\in O} x = o$$

$$\textsc{Labels}(h) \quad \bigwedge_{f(o)=(m(u)\Rightarrow v)} \mathsf{meth}(o) = m \wedge \mathsf{arg}(o) = u \wedge [\mathsf{ret}(o) = v]_{v\neq\perp}$$

$$\textsc{Order}(h) \quad \bigwedge_{o_1<o_2} \mathsf{b}(o_1, o_2)$$

$$\textsc{Used} \quad \forall x.\ \mathsf{ret}(x) \neq \perp \Rightarrow \mathsf{used}(x)$$

Figure 8. Formulæ characterizing histories $h = \langle O, <, f\rangle$ ($\mathsf{ret}(o) = v$ is present in $\textsc{Labels}(h)$ only if $v \neq \perp$).

of h, and possibly some pending operations of h. We tackle this problem in Section 4.2.

4.1 Complete Histories

The following lemma characterizes the weaker than relation between histories. It states that a history h' stronger than a complete history h can only differ in having more order constraints between the operations, the operation labels being the same in h and h'.

Lemma 4.1. *A complete history $h = \langle O, <, f\rangle$ is weaker than another history $h' = \langle O', <', f'\rangle$ iff there exists a bijection $g : O' \to O$ such that:*

- *operations related by g have the same label, i.e., for each $o \in O$, $f(o) = f'(g^{-1}(o))$, and*
- *order constraints are preserved from h to h', i.e., for each $o, o' \in O$, $o < o'$ implies $g^{-1}(o) <' g^{-1}(o')$.*

We characterize histories stronger than h by the formula $\textsc{Stronger}(h)$, defined as the conjunction

$$\textsc{Domain}(h) \wedge \textsc{Labels}(h) \wedge \textsc{Order}(h)$$

using the formulæ of Figure 8 characterizing the identifiers, labels, and order constraints of h. Note that the $\textsc{Domain}(h)$ formula restricts the interpretation domain of each variable to the operations of h. As a consequence of Lemma 4.1, the formula $\textsc{Stronger}(h)$ does indeed characterize all histories at least as strong as h.

Lemma 4.2. *Let h and h' be histories. Then $h \preceq h'$ iff $h' \models \textsc{Stronger}(h)$.*

It follows that the library membership test $h \in \overline{H}$ for complete histories h reduces to first-order satisfiability.

Theorem 1. *Let h be a complete history, and H a history set. Then $h \in \overline{H}$ iff $\textsc{Stronger}(h) \wedge \textsc{Theory}(H)$ is satisfiable.*

As long as the formula $\textsc{Theory}(H)$ contains only a fixed set of predicates e.g., $=$ and $\leq$, as is the case for all formulæ of Figures 3–7, satisfiability of $\textsc{Stronger}(h) \wedge \textsc{Theory}(H)$ reduces to propositional satisfiability. Intuitively this holds since the domain of (quantified) variables is restricted to operations appearing in h. Thus for a given h and H, we construct the propositional formula $[\![\textsc{Stronger}(h) \wedge \textsc{Theory}(H)]\!]$ by replacing each universally-quantified subformula $\forall x.\varphi$ by $\bigwedge_{o\in O} \varphi[x \mapsto o]$, and each existentially-quantified subformula $\exists x.\varphi$ by $\bigvee_{o\in O} \varphi[x \mapsto o]$. It follows that this propositional formula is equisatisfiable to the original first-order formula, and is constructed in polynomial time.

Corollary 1. *Let h be a complete history, and H a history set. Then $h \in \overline{H}$ iff the propositional formula*

$$[\![\textsc{Stronger}(h) \wedge \textsc{Theory}(H)]\!]$$

is satisfiable.

4.2 Incomplete Histories

The pending operations of a history h may be omitted in a stronger history or they may be completed with arbitrary return values. Therefore, the set of histories stronger than a history h can be characterized by a formula obtained from $\textsc{Stronger}(h)$ by adding a domain predicate used that is constrained to contain all the completed operations of h and by omitting the constraints on the return values of pending operations. Moreover, every operation of h whose return value is different from $\perp$ in a model of this formula (this may be an operation which is pending in h) should satisfy used. It can be proved that every history stronger than h corresponds to a model of this formula, projected on the set of operations satisfying used.

Thus, we override the $\textsc{Stronger}(h)$ formula for incomplete histories h as the conjunction

$$\textsc{Domain}(h) \wedge \textsc{Labels}(h) \wedge \textsc{Order}(h) \wedge \textsc{Used},$$

where $\textsc{Used}$ is defined in Figure 8.

For arbitrary, not necessarily complete, histories h, the models of $\textsc{Stronger}(h)$ are histories paired with an interpretation $U : O \to \mathbb{B}$ for the domain predicate used, mapping the operations O of h to $\{\mathsf{true}, \mathsf{false}\}$. Given such a model $\langle h, U\rangle$, let $U(h)$ be the history obtained from h by deleting operations o such that $\neg U(o)$.

Lemma 4.3. *Let h and h' be histories, and $U : O' \to \mathbb{B}$. Then $h', U \models \textsc{Stronger}(h)$ iff $h \preceq U(h')$.*

We leverage the predicate used to guard the domain of quantifiers in $\textsc{Theory}(H)$. For simplicity, we assume that $\textsc{Theory}(H)$ is given in prenex normal form, whose quantifier prefix is of the form $\forall^*\exists^*$. All of the formulæ of Figures 3–7 can be written in this form. We thus define the guarded formula $G(\varphi)$ of the formula $\varphi = \forall\vec{x}.\exists\vec{y}.\psi$ as

$$G(\varphi) = \forall\vec{x}.\exists\vec{y}.\ \bigwedge_{x\in\vec{x}} \mathsf{used}(x) \Rightarrow \bigvee_{y\in\vec{y}} \mathsf{used}(y) \wedge \psi$$

As usual, universal quantifiers are guarded using implication and existential quantifiers using conjunction. It follows from Lemma 4.3 that history membership reduces to first-order satisfiability.

Theorem 2. *Let h be a history, and H a history set. Then $h \in \overline{H}$ iff the first-order formula*

$$\textsc{Stronger}(h) \wedge G(\textsc{Theory}(H))$$

is satisfiable.

We again reduce this first-order satisfiability problem to propositional satisfiability by limiting the domain of quantifiers to the operations of h via the function $[\![\cdot]\!]$.

Corollary 2. *Let h be a history, and H a history set. Then $h \in \overline{H}$ iff the propositional formula*

$$[\![\textsc{Stronger}(h) \wedge G(\textsc{Theory}(H))]\!]$$

is satisfiable.

5. Removing Matched Operations

While the reduction to symbolic reasoning enabled by the previous sections already offers practical advantages over the explicit enumeration of history linearizations, this reduction does nothing to avoid the increasing cost of refinement checking as execution-length increases. A truly useful runtime monitor must be *linear* in the number of operations in order to avoid a progressive slowdown of the monitored implementation. Achieving this complexity goal implies forgetting increasingly-many previously-executed operations. However, forgetting arbitrary operations from a valid history can result in a false violation. For example, removing the $\mathsf{write}(1)$ operation from the history of the atomic register in Figure 10 results in a history which is not anymore admitted by this object. To identify the removable operations that preserve history membership, we rely on the matching functions used to characterize library kernels.

5.1 Unique matching functions

Recall that a history $h \in \overline{H}$ iff there exists a history h' stronger than h and a matching function $M : O \rightharpoonup O$ such that $h', M \models \textsc{Theory}(H)$. We assume in the following that for every history h, the matching function $M : O \rightharpoonup O$ is unique and that it is defined by the operation labels in h, i.e., there exists a function $\mathsf{M} : \mathbb{L} \rightharpoonup \mathbb{L}$ such that $M(o) = o'$ iff $\mathsf{M}(f(o)) = f(o')$. The matching function associated to some history h is denoted by M_h. We give some examples of how to construct histories of standard libraries with unique matching functions.

Example 5.1 (Collections)**.** *For usual implementations of collections such as stacks, queues, and sets, each operation adding a value to the collection is going to receive as input a value which is uniquely identified by a tag. When a method removing an element from the collection succeeds it is also going to return the unique tag associated with that element, thus defining a unique matching function from remove operations to add operations. The same strategy can be adapted to implementations of a register: the inputs to write operations are tagged and the read operations return tagged values.*

Example 5.2 (Locks)**.** *The implementations of a lock object usually have two abstract states, one where the object is unlocked, and one where it is locked. The* lock *operations can be modified to receive as input a value which is unique for every* lock *operation in an execution and every successful execution of a* lock *operation results in an object state that stores that input value. When an* unlock *operation succeeds, it returns the value stored in the object state. Therefore, the matching function maps* $\mathsf{unlock} \Rightarrow v$ *operations to* $\mathsf{lock}(v)$ *operations.*

Example 5.3 (Semaphores)**.** *Semaphore objects are usually implemented using a counter, which counts the number of* acquire *operations which successfully entered the semaphore and which are not yet released. We can instrument the implementation by keeping a map which maps each* slot *(from* 1 *to the capacity* c*) to a unique tag which was received as input by the* acquire *operation which has that slot, if any. When a* release *succeeds in decrementing the counter, it returns the unique tag of the* acquire *which had the slot.*

5.2 Closure under removing matched operations

A *match* of a history h is an operation o together with the maximal set of operations mapped by M_h to o. Moreover, a match consists only of completed operations and at least one operation mapped to o. Formally, a match of a history $h = \langle O, <, f\rangle$ is a set of operations $m = o \cup M_h^{-1}(o)$ such that $o \in O$, $M_h^{-1}(o) \neq \emptyset$, and all the operations in m are completed. The operation o of a match $m = o \cup M_h^{-1}(o)$ is denoted by $+(m)$.

Example 5.4. *Let h be the history in Figure 9 such that the matching function M_h maps every* $\mathsf{pop} \Rightarrow i$ *operation to the* $\mathsf{push}(i)$ *operation. The matches of h are* $m_1 = \{\mathsf{add}(1), \mathsf{rem} \Rightarrow 1\}$, $m_2 = \{\mathsf{add}(2), \mathsf{rem} \Rightarrow 2\}$, *and* $m_3 = \{\mathsf{add}(3), \mathsf{rem} \Rightarrow 3\}$. *Since every operation has a different label, we abuse the notation and write matches as sets of operation labels. Then,* $+(m_i) = \mathsf{add}(i)$*, for each i.*

A set of histories H is *match-removal closed* iff for every history $h \in H$ and every match m of h, H contains the history obtained from h by deleting the operations in m. The history obtained from another history h by deleting a set of operations O is denoted by $h \setminus O$.

The kernels of all the reference implementations described in Section 3 are match-removal closed. For instance, consider a sequential history h in the basis of an atomic queue and a match $m = \{\mathsf{add}(1), \mathsf{rem} \Rightarrow 1\}$. The history obtained from h by removing the match m is also a valid sequential queue history because essentially, the remaining values are still removed in the order in which they are added.

The match-removal closure property extends from a kernel H to the entire library $\overline{H}$. Therefore, if by deleting matches

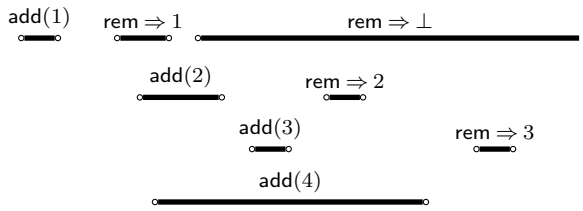


Figure 9. A history that is not admitted by the atomic stack. Each operation is represented by an horizontal line segment. The line segment of an operation ending before the line segment of another operation means that the two operations are ordered (reading from left to right).

from a history we get a history which is not admitted by a library $\overline{H}$ then the initial history is also not admitted by $\overline{H}$.

Theorem 3. *$\overline{H}$ is match-removal closed if H is.*

Proof. Let $h \in \overline{H}$ and m a match of h. By definition, there exists a history $h' \in H$ such that $h \preceq h'$. By the assumptions on the matching functions, m is also a match of h'. By hypothesis, H is match-removal closed which implies that the history h'' obtained from h' by deleting the operations in m is also in H. From the definition of $\preceq$ it follows that the history $h \setminus m$ is weaker than h'' which implies $h \setminus m \in \overline{H}$. □

While the statement of Theorem 3 implies that a history $h \setminus m$, where m is a match of h, belongs to $\overline{H}$ whenever $h \in \overline{H}$, Example 5.5 shows that the reverse is not true.

Example 5.5. *Figure 9 pictures a history which is not admitted by the atomic stack: since the element* 3 *was pushed after* 2, $\mathsf{pop} \Rightarrow 3$ *should not have started after* $\mathsf{pop} \Rightarrow 2$ *has finished. The matching function M_h maps every* $\mathsf{pop} \Rightarrow i$ *operation to the* $\mathsf{push}(i)$ *operation.. The history obtained by removing the match* $\{\mathtt{push}(2), \mathtt{pop} \Rightarrow 2\}$ *is however admitted by the atomic stack.*

5.3 Forgetting matched operations

When continuously checking membership to a reference implementation on a history that keeps extending with new operations, one must ensure that forgetting matches doesn't lead to spurious violations at later times. In the continuation of Theorem 3, one should also prove that a match of a history h is a match of every extension of h (a match of h could be strictly included in a match of an extension and therefore, not a match of the extension). Otherwise, removing an arbitrary match before extending a history may be the same as removing a set of operations which is not a match from the extended history (therefore, Theorem 3 doesn't apply). This result holds when the matching function of all histories in the library is injective because every match has exactly two operations. All the reference implementations in Section 3 fall into this case except for the atomic sets and registers.

A history h_2 *extends* a history h_1 if intuitively it is the history of an execution that extends the execution represented by

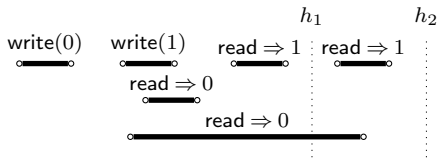


Figure 10. Two histories h_1 and h_2 of the atomic register, where h_2 is an extension of h_1.

h_1. Formally, $h_2 = \langle O_2, <_2, f_2\rangle$ *extends* $h_1 = \langle O_1, <_1, f_1\rangle$, written $h_1 \rhd h_2$, iff

- $O_1 \subseteq O_2$,
- $f_1(o) \ll f_2(o)$ for each $o \in O_1$,
- $o_1 <_1 o_2$ iff $o_1 <_2 o_2$ for each $o_1, o_2 \in O_1$, and
- $o_1 <_2 o_2$ for each $o_1 \in O_1$ and $o_2 \in O_2 \setminus O_1$.

Lemma 5.1. *Let $\overline{H}$ be a library s.t. M_h is injective for all $h \in \overline{H}$. For every two histories $h_1, h_2 \in \overline{H}$ s.t. $h_1 \rhd h_2$, if m is a match of h_1 then m is also a match of h_2.*

Example 5.6 shows that this result doesn't hold when the matching function is not injective.

Example 5.6. *Figure 10 pictures two histories h_1 and h_2 of the atomic register, the latter being an extension of the former. We assume that the matching function maps every* $\mathsf{read} \Rightarrow i$ *operation to the* $\mathsf{write}(i)$ *operation. The match* $\{\mathsf{write}(1), \mathsf{read} \Rightarrow 1\}$ *of h_1 is strictly included in the match* $\{\mathsf{write}(1), \mathsf{read} \Rightarrow 1, \mathsf{read} \Rightarrow 1\}$ *of h_2, and therefore not a match of h_2. Removing the match of h_1 and extending it with the* $\mathsf{read} \Rightarrow 1$ *operation results in a spurious violation.*

For libraries with non-injective matching functions, we impose additional conditions on their kernels, and insist that a match m of h is a match of every extension of h.

Let R be a relation on operation labels. We say that a history h is *R-ordered* iff for any two matches m_1 and m_2 of h such that $R(+(m_1), +(m_2))$, each operation of m_1 is ordered before each operation of m_2.

Consider the atomic register and the relation R_{reg} which holds for every two labels of two write operations. Then, any history from its kernel (which by definition is sequential) is R-ordered because every read operation is mapped by the matching function to the closest preceding write operation. Similarly, one can show that any history from the kernel of the atomic set is R_{set}-ordered, where R_{set} holds for every two labels $\mathsf{add}(x)$ and $\mathsf{add}(y)$ with $x = y$.

A set of histories H is R-ordered iff every history in H is R-ordered. Let $\overline{H}$ be a library such that H is R-ordered. A match m of a history $h \in \overline{H}$ is *overwritten* by another match m' of h iff the labels of $+(m)$ and $+(m')$ are related by R, $+(m)$ finishes before $+(m')$, and every operation overlapping with $+(m')$ is completed.

Example 5.7. *Given the histories h_1 and h_2 in Figure 10, the match* $\{\mathsf{write}(0), \mathsf{read} \Rightarrow 0, \mathsf{read} \Rightarrow 0\}$ *is overwritten by*

the match $\{\mathsf{write}(1), \mathsf{read} \Rightarrow 1, \mathsf{read} \Rightarrow 1\}$ *in* h_2 *but not in* h_1 *since a* $\mathsf{read} \Rightarrow 0$ *operation is pending in* h_1.

Given a match m overwritten by another match m' in $h \in \overline{H}$, every new completed operation o from an extension h' of h cannot be matched to $+(m)$, therefore m is also a match of h'. Otherwise, since all the operations overlapping with $+(m')$ are completed in h, o starts after $+(m')$ and H would contain a history where the operations $+(m)$, $+(m')$, o occur in this order. Therefore, H is not R-ordered.

Lemma 5.2. *Let* H *be a* R*-ordered, and* $h_1, h_2 \in \overline{H}$*. If* $h_1 \rhd h_2$ *and* m *is a match of* h_1 *overwritten by another match* m' *of* h_1*, then* m *is also a match of* h_2.

A match m of a history $h \in \overline{H}$ is called *obsolete* if (1) it is simply a match when the matching function of every history $h \in \overline{H}$ is injective or (2) it is overwritten by another match of h, when H is R-ordered. Lemmas 5.1 and 5.2, and Theorem 3 imply the following.

Corollary 3. *Let* H *be a match-removal closed history set s.t.* M_h *is injective for all* $h \in \overline{H}$ *or* H *is* R*-ordered, for some* R*. For every two histories* $h_1, h_2 \in \overline{H}$ *s.t.* $h_1 \rhd h_2$ *and* m *an obsolete match of* h_1*,* $h_2 \setminus m$ *is a history of* $\overline{H}$.

The following illustrates a possible source of incompleteness for monitoring algorithms which remove operations.

Example 5.8. *Figure 11 pictures two histories* h_1 *and* h_2*, the latter being an extension of the former. The* $(\mathsf{rem} \Rightarrow 2)$*-operation is pending in* h_1 *and only completes in* h_2*. However, removing the match* $\{\mathsf{add}(1), \mathsf{rem} \Rightarrow 1\}$ *from* h_1 *before the pending* rem *completes results in a history admitted by the atomic stack. This highlights a risk of incompleteness which we must account for in our development of refinement-monitoring algorithms. Should we simply forget about the match* $\{\mathsf{add}(1), \mathsf{rem} \Rightarrow 1\}$ *while monitoring, we may not detect the violation obviated when the pending* rem *completes. In order to avoid such incompleteness, at the very least, our monitoring algorithm must remember that the* $\mathsf{add}(2)$ *operation should be ordered before* $(\mathsf{rem} \Rightarrow \mathtt{empty})$ *(since* $\mathsf{add}(2)$ *must be ordered before* $\mathsf{add}(1)$*, which is ordered before* $(\mathtt{rem} \Rightarrow \mathtt{empty})$*), even after removing the match. Then when* $(\mathsf{rem} \Rightarrow 2)$ *completes, we could deduce that* $(\mathsf{rem} \Rightarrow \mathit{empty})$ *must be ordered after* $\mathsf{add}(2)$ *and before* $(\mathsf{rem} \Rightarrow 2)$*, a contradiction with the atomic stack theory* $\textsc{Theory}(H_{\mathsf{st}})$ *of Section 3.*

6. Empirical Validation

To demonstrate the practical value of the theory developed in the previous sections, we argue that our symbolic refinement-checking algorithms

- scale far beyond existing algorithms, and
- are often complete in practice.

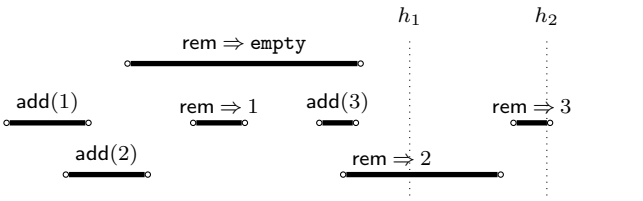


Figure 11. Two histories which are not admitted by the atomic stack, h_2 being an extension of h_1.

To argue these points we have implemented several variations of three basic refinement-checking algorithms.

ENUMERATE checks each history h by enumerating the linearizations h' of h's completions. We check whether each h' is included in the kernel H by asking[1] whether $h' \models \textsc{Theory}(H)$. As soon as this check succeeds, we conclude that $h \in \overline{H}$. Otherwise if this check fails for all linearizations h', we conclude $h \notin \overline{H}$. We also implement a variation which skips the enumeration of h's completions, allowing the solver to search the space of completions symbolically. The `-c` flag specifies explicit enumeration of completions.

SYMBOLIC checks each history h by reduction to the satisfiability of $\textsc{Stronger}(h) \wedge \textsc{Theory}(H)$, as described in Section 4, delegating the enumeration of both completions and linearizations to an underlying solver. If the satisfiability check succeeds, or is inconclusive, we conclude that $h \in \overline{H}$. Otherwise if unsatisfiability is found, we conclude that $h \notin \overline{H}$. We also implement variations where the completions of h' are enumerated explicitly (`-c`), where the formula $\textsc{Stronger}(h)$ is asserted to the solver incrementally as operations begin and end (`-i`), where the match-removal optimization of Section 5 is performed (`-r`), and any combination of the three.

SATURATE avoids the expensive propositional backtracking inherent to the aforementioned SYMBOLIC checker by limiting the satisfiability check to Boolean constraint propagation. Essentially, we implement a customized incremental solver which only saturates with unit propagation, avoiding any propositional branching. If a contradiction is found, we conclude that $h \notin \overline{H}$. Otherwise if saturation fails to reveal a contradiction, we conclude $h \in \overline{H}$. We also implement a variation where the match-removal optimization of Section 5 is performed (`-r`).

We have studied actual concurrent data structure implementations, including the Scal[2] High-Performance Multicore-Scalable Computing suite. Some of these implementations,

[1] Classically this check is performed by set inclusion, as the length of h is assumed to be bounded by some number $n \in \mathbb{N}$ of operations, and the subset $H_n \subseteq H$ of n-operation histories of H is computable in finite time. As we assume no such bound on the number of operations, we perform this check via theorem-prover query instead.

[2] http://scal.cs.uni-salzburg.at

such as the Michael-Scott Queue [Michael and Scott 1996], are meant to preserve observational refinement[3], while others, such as the non-blocking bounded-reordering queue [Kirsch et al. 2013], are meant to preserve weaker properties.

The input to our checking algorithms are histories given as text files consisting of line-separated call and return actions. For the selected set of concurrent object implementations, we generated the histories of several executions under pseudo-random scheduling by logging calls and returns in the order in which they occurred. While scanning an input history, the selected algorithm performs a membership test at each prefix at which an operation completes — i.e., at return actions.

Our first set of experiments (§6.1) demonstrates that our symbolic algorithms are drastically more scalable than existing algorithms, in that they are able to process vastly more history operations in much shorter time. Our second set of experiments (§6.2) demonstrates that while our more efficient algorithms can be incomplete on certain hand-crafted cases, failing to identify particularly intricate refinement violations, the violations surfacing in the logs of actual executions are consistently discovered. We made all measurements on similar MacBook Pro 2.XGHz Intel Core i5/i7 machines, and discharged theorem-prover queries with an in-process instance of Z3 compiled from revision `cee7dd3`.

Our implementation of the algorithms, and all (generated) histories used in these experiments, are available on Github[4].

6.1 Scalability of Symbolic Checking

Our first experiment measures the number of steps each algorithm is able to process for varying time limits, averaged over 10 histories of an atomic stack implementation with an average of 1020.8 steps. We used five per-history time limits of 5s, 25s, 50s, 75s, and 100s. The graph of Figure 12 shows the results with time plotted on the y-axis, and number of steps plotted on the x-axis on a logarithmic scale. The ENUMERATE algorithm performs worst, progressing only from 16 steps in 5s to 18 steps in 100s. The SYMBOLIC algorithm is a significant improvement, progressing from 31.6 steps in 5s to 52.7 steps in 100s. While adding match removal helps, achieving roughly an order-of-magnitude improvement over ENUMERATE, the cost of checking remains exponential in the number of steps.

Even without match removal, the SATURATE checker achieves roughly a two-orders-of-magnitude improvement over ENUMERATE, progressing from 307.7 steps in 5s to 744.6 steps in 100s. Most impressively, adding match removal to the SATURATE checker allows it to process *all* 1020.8 steps in under 5s.

Our second experiment measures the average number of steps each algorithm is able to process in a fixed time limit of 5s per history over 100 histories of Scal's Michael-Scott Queue and 100 histories of Scal's bounded-reordering queue. Each history contains 20 enqueue-operations and 30 dequeue operations (total 100 steps) obtained from highly-concurrent executions in which most operations are concurrent. Although most sequential executions of the bounded-reordering queue exhibit violations with respect to an atomic queue, the extreme concurrency in this set of histories allows great freedom in the linearization of operations, rendering all but 4 histories violation free. Table 1 shows the results, including the average number of steps (out of 100) processed, the number of violations (out of 4) discovered, within the 5s time limit. While the ENUMERATE algorithm is never able to process enough steps in 5s to detect any of the violations, the SYMBOLIC checker, at least without computing history completions, does detect 3 out of 4 violations. Once again, the SATURATE checker swiftly processes all histories to their entirety well under the time limit, averaging 0.1s, correctly detecting all 4 violations.

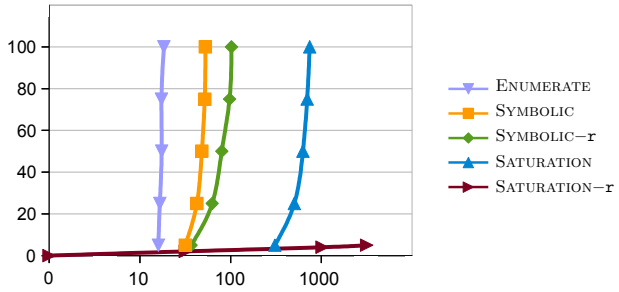


Figure 12. The number of steps each algorithm is able to process in given time limits of 5s, 25s, 50s, 75s, and 100s, per history averaged over 10 histories of an atomic stack implementation. Further right is better. The x-axis plots steps on a logarithmic scale, and the y-axis plots time.

Algorithm	Avg. steps	Avg. time to viol.	Nb. violations
ENUMERATE	35	–	0/4
ENUMERATE `-c`	38	–	0/4
SYMBOLIC	52	2.350s	3/4
SYMBOLIC `-r`	52	2.322s	3/4
SYMBOLIC `-i`	51	0.984s	3/4
SYMBOLIC `-c`	38	–	0/4
SYMBOLIC `-c -i`	38	–	0/4
SATURATE	100	0.040s	4/4
SATURATE `-r`	100	0.050s	4/4

Table 1. The average number of steps each algorithm is able to process in a fixed time limit of 5s per history, averaged over 200 very-concurrent histories.

6.2 Completeness in Practice

While the second experiment of Table 1 already demonstrates that "needle-in-a-haystack" violations are swiftly discovered by the SATURATE checker, our third experiment elaborates, measuring the percentage of violations each algorithm is

[3] More precisely, they are designed to be linearizable.

[4] https://github.com/pldi15-anonymous/refinement-monitor-experiments

Algorithm	% violations	Avg. step detected
ENUMERATE	55%	23.8
ENUMERATE -c	88%	23.8
SYMBOLIC	75%	23.8
SYMBOLIC -r	78%	23.8
SYMBOLIC -i	90%	23.8
SYMBOLIC -c	100%	23.8
SYMBOLIC -c -i	100%	23.8
SATURATE	100%	24.2
SATURATE -r	100%	24.2

Table 2. The percentage of violations each algorithm is able to discover in a fixed time limit of 5s per history, over 100 barely-concurrent histories.

able discover in a fixed time limit of 5s per history over 100 histories of Scal's bounded-reordering queue. Each history contains 20 enqueue-operations and 30 dequeue operations (total 100 steps) obtained from highly-sequential executions in which most operations are sequential. Consequently, the extreme sequentiality in this set of histories greatly limits freedom in the linearization of operations of the reordering queue, rendering all 100 histories violations. Table 2 shows the results, including the percentage of histories in which, and average step at which, the violation was discovered within the 5s time limit. Since most operations are already ordered, the ENUMERATE checker has very few completions and linearizations to perform, and succeeds in exposing most, but not all violations. The SYMBOLIC checker is more efficient, particularly when interacting with the solver incrementally, and computing completions explicitly. Still, the SATURATE checker is far more efficient, and succeeds in discovering *every* violation. Notice that in some cases the SATURATE checker requires more steps to detect a violation. This is of course by design, as we wish to avoid branching and backtracking on the possible completions of operations.

Finally, we address the possible sources of incompleteness due to match removal discussed in Section 5. Recall the history h_1 and its extension h_2 of Figure 11 from Example 5.8. Since the SATURATE algorithm does not speculate on whether the pending rem-operation might match the add of 2 or 3 (or both!), it will not detect the violation in h_1 until the pending rem-operation completes. Simply removing the add-rem match of value 1 from h_1 before the pending rem-operation completes would result in a non-violating history. However, by applying the stack-theory axioms $\text{THEORY}(H_{\mathsf{st}})$ to completed operations before removing this match, the SATURATE algorithm infers the constraints

$$\mathsf{add}(2) < \mathsf{add}(1) < (\mathsf{rem} \Rightarrow 1) < (\mathsf{rem} \Rightarrow \mathtt{empty})$$

of which $\mathsf{add}(2) < (\mathsf{rem} \Rightarrow \mathtt{empty})$ persists after the match removal. Finally, when the pending rem-operation does complete, returning 2, the SATURATE algorithm derives a contradiction, since

$$\mathsf{add}(2) < (\mathsf{rem} \Rightarrow \mathtt{empty}) < (\mathsf{rem} \Rightarrow 2).$$

The incremental nature of the SATURATE algorithm thus avoids the practically-occurring sources of possible incompleteness due to match removal of which we are aware.

7. Related Work

Previous work on automated refinement verification focuses on *linearizability* [Herlihy and Wing 1990], which is now known to be equivalent when considering atomic reference implementations [Filipovic et al. 2010; Bouajjani et al. 2015]. The theoretical limits of linearizability checking are well studied. While checking a single execution is NP-complete [Gibbons and Korach 1997], checking all executions of a finite-state implementation is in EXPSPACE when the number of program threads is bounded [Alur et al. 2000], and undecidable otherwise [Bouajjani et al. 2013].

Several semi-automated verification approaches rely on annotating method bodies with *linearization points* [Abdulla et al. 2013; Amit et al. 2007; Dragoi et al. 2013; Liu et al.; O'Hearn et al.; Vafeiadis; Zhang] to reduce the otherwise-exponential number of possible linearizations to one single linearization. These methods typically rely on programmer annotation, and do not admit conclusive evidence of a violation in the case of a failed proof. "Aspect-oriented proofs" reduce verification for certain atomic objects to a small set of simpler properties. Specifically, checking executions of atomic queue implementations against the theory $\text{THEORY}(H_q)$ of the atomic queue[5] *kernel* from Section 3 is sound and complete for complete histories [Henzinger et al. 2013].

Most automated approaches for detecting linearizability violations [Burckhardt et al.; Burnim et al. 2011; Zhang et al. 2013; Wing and Gong 1993] enumerate the exponentially-many linearizations of each execution, limiting them to executions with few operations, as observed in Section 6. Colt [Shacham et al. 2011]'s approach mitigates this cost with programmer-annotated linearization points, as mentioned above, and ultimately suffers from the same problem: a failed proof only indicates incorrect annotation.

One closely-related work aims to reduce the complexity of refinement checking by approximating executions with weaker, bounded histories [Bouajjani et al. 2015]. While this approach is also sound, in the sense that only actual violations are flagged, its completeness in practice is reported to rely on observing *many* executions with varying thread schedules, and its applicability is limited to executions with a bounded number of argument/return values. On the contrary, the requirements for long-term execution monitoring which we address demand completeness for each individual execution without bounding the number of data values. Technically, our approximations differ as well: while theirs forgets the order between some operations, our match removal optimization forgets operations completely. Note however that by removing operations only *after* saturation of their logical implications, the effect of forgotten operations can persist.

[5] Technically, the axioms of this theory without our totality axiom.

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