2a

The covariance matrix of $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ (where $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are rows of the matrix **X**) is:

$$cov(\boldsymbol{X}) = \begin{bmatrix} cov(\boldsymbol{x_1 x_1}) & cov(\boldsymbol{x_1 x_2}) \\ cov(\boldsymbol{x_2 x_1}) & cov(\boldsymbol{x_2 x_2}) \end{bmatrix}$$

Be definition

$$cov(\boldsymbol{x_1 x_2}) = E\{\boldsymbol{x_1 x_2}\} - E\{\boldsymbol{x_1}\}E\{\boldsymbol{x_2}\} \tag{1}$$

As $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are independent vectors of gaussian distributed random variables ($E\{\boldsymbol{x_1 x_2}\} = E\{\boldsymbol{x_1}\}E\{\boldsymbol{x_2}\}$, $E\{\boldsymbol{x_1}\} = E\{\boldsymbol{x_2}\} = 0$), we get:

$$\begin{aligned} cov(\boldsymbol{x_1 x_2}) = E\{\boldsymbol{x_1 x_2}\} - E\{\boldsymbol{x_1}\}E\{\boldsymbol{x_2}\} = cov(\boldsymbol{x_2 x_1}) = 0 \\ cov(\boldsymbol{x_1 x_1}) = cov(\boldsymbol{x_2 x_2}) = E\left\{\boldsymbol{x_1^2}\right\} = E\left\{\boldsymbol{x_2^2}\right\} = 1 \end{aligned} \tag{2}$$

Finally, the theoretical value of $cov(\boldsymbol{X})$ is equal to:

$$cov(\boldsymbol{X}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Important! There is no need to transpose $\boldsymbol{X}$ before calculation the covariance matrix. In contrast to Matlab implementation, *numpy.cov()* applies along rows.

2b

Multiply $\boldsymbol{A}$ by $\boldsymbol{X}$ and rename variables for convenience:

$$\boldsymbol{AX} = \begin{bmatrix} A_{11}\boldsymbol{x_1} + A_{12}\boldsymbol{x_2} \\ A_{21}\boldsymbol{x_1} + A_{22}\boldsymbol{x_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix}$$

The covariance matrix of $\boldsymbol{AX}$ is:

$$cov(\boldsymbol{AX}) = \begin{bmatrix} cov(\boldsymbol{x_3 x_3}) & cov(\boldsymbol{x_3 x_4}) \\ cov(\boldsymbol{x_4 x_3}) & cov(\boldsymbol{x_4 x_4}) \end{bmatrix}$$

$$\begin{aligned} cov(\boldsymbol{x_3 x_4}) &= E\{(A_{11}\boldsymbol{x_1} + A_{12}\boldsymbol{x_2})(A_{21}\boldsymbol{x_1} + A_{22}\boldsymbol{x_2})\} \\ &= E\left\{A_{11}A_{21}\boldsymbol{x_1}^2\right\} + E\{A_{11}A_{22}\boldsymbol{x_1 x_2}\} + \\ &\quad + E\{A_{12}A_{21}\boldsymbol{x_1 x_2}\} + E\left\{A_{12}A_{22}\boldsymbol{x_2}^2\right\} \end{aligned} \tag{3}$$

Collect the terms and using the fact that $\boldsymbol{x_3}$ and $\boldsymbol{x_4}$ are independent variables, finally get:

$$\begin{aligned} cov(\boldsymbol{x_3 x_4}) &= A_{11}A_{21}E\left\{\boldsymbol{x_1}^2\right\} + A_{11}A_{22}E\{\boldsymbol{x_1}\}E\{\boldsymbol{x_2}\} + \\ &\quad + A_{12}A_{21}E\{\boldsymbol{x_1}\}E\{\boldsymbol{x_2}\} + A_{12}A_{22}E\left\{\boldsymbol{x_2}^2\right\} = \\ &= A_{11}A_{21} + A_{12}A_{22} = 2 * 0 + 3 * 1 = 3 \end{aligned} \tag{4}$$

Using the same notation, calculate:

$$\begin{aligned} cov(\boldsymbol{x_3 x_3}) = E\left\{(A_{11}\boldsymbol{x_1} + A_{12}\boldsymbol{x_2})^2\right\} = A_{11}^2 + A_{21}^2 = 4 + 9 = 13 \\ cov(\boldsymbol{x_4 x_4}) = E\left\{(A_{21}\boldsymbol{x_1} + A_{22}\boldsymbol{x_2})^2\right\} = A_{21}^2 + A_{22}^2 = 0 + 1 = 1 \end{aligned} \tag{5}$$

Therefore, the covariance matrix of AX is

$$cov(\boldsymbol{AX}) = \begin{bmatrix} cov(\boldsymbol{x_3 x_3}) & cov(\boldsymbol{x_3 x_4}) \\ cov(\boldsymbol{x_4 x_3}) & cov(\boldsymbol{x_4 x_4}) \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ 3 & 1 \end{bmatrix}$$

Calculate $cov(\boldsymbol{A}\boldsymbol{A}^T)$ theoretically:

$$\boldsymbol{K} = \boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 13 & 3 \\ 3 & 1 \end{bmatrix}$$

Denote $\boldsymbol{K_1} = \boldsymbol{K}(1,:)$ and $\boldsymbol{K_2} = \boldsymbol{K}(2,:)$. Hence, the covariance of $\boldsymbol{K}$ is defined:

$$cov(\boldsymbol{K}) = \begin{bmatrix} cov(\boldsymbol{K_1}\boldsymbol{K_1}) & cov(\boldsymbol{K_1}\boldsymbol{K_2}) \\ cov(\boldsymbol{K_2}\boldsymbol{K_1}) & cov(\boldsymbol{K_2}\boldsymbol{K_2}) \end{bmatrix}$$

where

$$cov(\boldsymbol{AB}) = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{A_i} - \mu_A)(\boldsymbol{B_i} - \mu_B) \tag{6}$$

More specifically, $\mu_{K_1} = (13+3)/2 = 8$, $\mu_{K_2} = (3+1)/2 = 2$, $N = 2$. Therefore,

$$\begin{aligned} cov(\boldsymbol{K_1}\boldsymbol{K_1}) &= \frac{1}{2-1} \sum_{i=1}^{2} (\boldsymbol{K_1}[i] - \mu_{K_1})(\boldsymbol{K_1}[i] - \mu_{K_1}) = \\ &= (13-8)(13-8) + (3-8)(3-8) = 50 \\ cov(\boldsymbol{K_1}\boldsymbol{K_2}) = cov(\boldsymbol{K_2}\boldsymbol{K_1}) &= (13-8)(3-2) + (3-8)(1-2) = 10 \\ cov(\boldsymbol{K_2}\boldsymbol{K_2}) &= (3-2)(3-2) + (1-2)(1-2) = 2 \end{aligned} \tag{7}$$

$$cov(\boldsymbol{K}) = \begin{bmatrix} 50 & 10 \\ 10 & 2 \end{bmatrix}$$