Linear Algebra and its Applications 446 (2014) 133–147

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Balancedness and the least eigenvalue of Laplacian of signed graphs

Francesco Belardo

Department of Mathematics and Computer Science, University of Messina, 98 166 Sant'Agata, Messina, Italy

ARTICLE INFO

Article history:
Received 30 May 2013
Accepted 1 January 2014
Available online 22 January 2014
Submitted by R. Brualdi

MSC:
05C50
05C22
15A18

Keywords:
Signed graph
Least eigenvalue
Laplacian matrix
Balanced graph
Frustration

ABSTRACT

Let $\Gamma = (G, \sigma)$ be a connected signed graph, where G is the underlying simple graph and $\sigma : E(G) \to \{1, -1\}$ is the sign function on the edges of G. Let $L(\Gamma) = D(G) - A(\Gamma)$, be the Laplacian of Γ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$ be its eigenvalues. It is well-known that a signed graph Γ is balanced if and only if $\lambda_n(\Gamma) = 0$. Here we show that, if Γ is unbalanced, then λ_n estimates how much Γ is far from being balanced. In particular, let $\nu(\Gamma)$ (resp. $\epsilon(\Gamma)$) be the frustration number (resp. frustration index), that is the minimum number of vertices (resp. edges) to be deleted such that the signed graph is balanced. Then we prove that

$$\lambda_n(\Gamma) \leqslant \nu(\Gamma) \leqslant \epsilon(\Gamma).$$

Further we analyze the case when $\lambda_n(\Gamma) = \nu(\Gamma)$. In the latter setting, we identify the structure of the underlying graph G and we give an algebraic condition for $L(\Gamma)$ which leads to the above equality.

1. Introduction

Let $G = (V(G), E(G))$ be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where its order and size are $|V(G)| = |G| = n(G) = n$

E-mail address: fbelardo@gmail.com.

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2014.01.001

and $|E(G)| = \|G\|$, respectively. Let $\deg(v)$ denote the degree of the vertex v, and let $\Delta(G)$ be the maximum degree in G. If M is a $n \times n$ symmetric matrix, then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of M ordered in non increasing fashion. By $|M|$ and $\mathrm{Spec}(M)$ we denote the determinant of M and the multiset of the eigenvalues of M, respectively. In the literature, graphs are studied by means of the eigenvalues of several matrices associated to graphs. The *adjacency matrix* $A(G) = (a_{ij})$, where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise, is one of the most studied together with the *Laplacian matrix* $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is the diagonal matrix of vertex degrees.

In this paper we focus our attention to (connected) *signed graphs*. A signed graph (sometimes abbreviated to sigraph) Γ is a pair (G, σ), where $G = (V(G), E(G)) = G(\Gamma)$ is a connected graph and $\sigma : E(G) \to \{+1, -1\}$ is a sign function (or *signature*) on the edges of G. Due to the latter definition, sometimes signed graphs are treated as weighted graphs with (edge) weights equal to $+1$ and -1. The order of Γ is the order of G and it is denoted as $|\Gamma|$. By $E^+(G)$ and $E^-(G)$ we denote the subset of $E(G)$ for which the edges have positive and negative sign, respectively. $E(U, W)$, with $U, W \subset V(G)$, is the set of edges having one endpoint in U and the other in W. If $\sigma(e) = 1$ (resp. $\sigma(e) = -1$) for all edges in $E(G)$ then we write $(G, +)$ (resp. $(G, -)$). Also for signed graphs we consider matrices spectra. The adjacency matrix $A(\Gamma)$ is the matrix whose entries are $(a_{ij}^{\sigma}) = \sigma(ij)a_{ij}$, where a_{ij} is the ij-th entry of $A(G)$. Similarly we define $L(\Gamma) = D(G) - A(\Gamma)$ as the Laplacian matrix of the signed graph Γ. In this paper we focus on the Laplacian eigenvalues of Γ, so we will omit "Laplacian", if clear from the context.

As usual, K_n and C_n denote the complete graph and the cycle on n vertices, respectively. Given two graphs G_1 and G_2, the graphs $G_1 + G_2$ and $G_1 \vee G_2$ denote the disjoint union and the complete join of G_1 and G_2, respectively. Similarly, for the signed graphs $\Gamma_1 = (G_1, \sigma_1)$ and $\Gamma_2 = (G_2, \sigma_2)$, we can naturally define $\Gamma_1 + \Gamma_2$ as the signed graph $(G_1 + G_2, \sigma_1 \cup \sigma_2)$. Let I_n, J_n and O_n be the $n \times n$ identity matrix, the all-1 matrix, and the all-0 matrix, respectively. If A and B are two square matrices, then $A \oplus B$ is a diagonal block matrix whose diagonal blocks are A and B. For two n-order Hermitian matrices A and B, we say that $A \succcurlyeq B$ if $A - B$ is positive semidefinite. Finally, let A be a symmetric matrix, then by $A^{\#}$ we denote the Moore-Penrose pseudo-inverse. Recall that $A^{\#}$ is the unique symmetric matrix such that

$$\text{(i)} \quad AA^{\#}A = A, \qquad \text{(ii)} \quad A^{\#}AA^{\#} = A^{\#}, \qquad \text{(iii)} \quad A^{\#}A = AA^{\#}.$$

For all other notations not given here, we refer the reader to [7] and [2].

Signed graphs appear in the context of social (signed) networks describing the relation of being friend (positive edge) or enemy (negative edge) between people (vertices). Nowadays signed graphs appear in the literature of social network analysis and logical programming (see [1], for example), and spectral techniques are used in order to detect communities (clusters) in the signed network [10]. For a possibly complete bibliography on signed graphs, the reader is referred to [13]. In this paper, inspired by the work [6], we study the relation between the least eigenvalue of $L(\Gamma)$ and the property of signed graphs of being balanced.

Here is the remainder of the paper. In Section 2 we give some basic facts and notions on signed graphs. In Section 3 we prove our main results and give some examples.

2. Preliminaries

In this section we present some standard results and give some basic notions which will be useful later.

Let C be a cycle in $\Gamma = (G, \sigma)$, the *sign* of C is the product of edge signs in the cycle, namely $\operatorname{sign}(C) = \prod_{e \in C} \sigma(e)$. A cycle whose sign is 1 (resp. -1) is said to be *positive* (resp. *negative*). A signed graph is said to be *balanced* if all cycles are positive. Harary in [8] gave the following characterization for a balanced signed graph: $\Gamma = (G, \sigma)$ is balanced if and only if $V(G)$ can be divided in to two the color classes V_1 and V_2, such that $\sigma(uv) = 1$ for all $u, v \in V_i$ $(i = 1, 2)$ and $\sigma(uv) = -1$ for all $u \in V_1$ and $v \in V_2$. Further, the following characterization was given by Zaslavsky in [12].

Theorem 2.1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph and $L(\Gamma) = D(G) - A(\Gamma)$ be its Laplacian matrix. Then Γ is balanced if and only if* $\det(L(\Gamma)) = 0$.

If a signed graph Γ is unbalanced, then it properly contains a balanced signed subgraph B (for instance, (K_2, σ) is balanced). The smallest number of vertices (resp. edges) whose deletion leads to a balanced graph is known as the *frustration number* (resp. *frustration index*). We denote by $\nu(\Gamma)$ and $\epsilon(\Gamma)$ the frustration number and frustration index, respectively. It is worth to mention that Harary originally defined the frustration index as the *line index of balance*, so it is common to find the frustration index and the frustration number denoted by $\ell(\Gamma)$ and $\ell_0(\Gamma)$. However, in this paper we adopt the former notation.

We next introduce the notion of *sign switching*. Let $\Gamma = (G, \sigma)$ be a signed graph and $\theta : V(G) \to \{+1, -1\}$ be a function assigning signs on the vertices of G. The signed graph $\Gamma^\theta = (G, \sigma^\theta)$ where $\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)$ is said to be *switching equivalent* to Γ, or for short $\Gamma^\theta \sim \Gamma$. Switching preserves many signed graph invariants as, for instance, the set of positive cycles (see, for example, [9,12]). In particular, switching of balanced signed graphs leads to balanced signed graphs. In the literature it is well-known the notion of *signature similarity*: two matrices M_1 and M_2 of order n are signature similar if there exists a diagonal matrix $S = \operatorname{diag}(s_1, s_2, \ldots, s_n)$, where $s_i = \pm 1$, such that $M_2 = S^\top M_1 S = S M_1 S$. Clearly, signature similarity and switching in the context of signed graphs are equivalent, and we have the following simple lemma.

Lemma 2.2. *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on the same graph G. Then Γ_1 is switching equivalent to Γ_2 if and only if $L(\Gamma_1)$ is signature similar to $L(\Gamma_2)$.*

The following fact is well-known. If $\Gamma = (G, \sigma)$ is balanced, then we can consider the signature function θ such that $\theta(v) = 1$ if v belongs to V_1 and $\theta(v) = -1$ if v belongs to V_2, where V_1, V_2 are the two color classes.

Theorem 2.3. *Let $\Gamma = (G, \sigma)$ be a balanced graph. Then $\Gamma \sim (G, +)$, or equivalently $L(\Gamma)$ is signature similar to $L(G)$.*

If G is a bipartite graph then $(G, +)$ is switching equivalent to $(G, -)$, that is the well known fact $L(G) = D(G) - A(G)$ being signature similar to $Q(G) = D(G) + A(G)$. The matrix Q has been well studied recently (see [3–5] for basic results on $Q(G)$), and in [6] the least eigenvalue $\lambda_n(Q)$ has been regarded as a measure for the bipartiteness (or biparticity) of the graph G: the larger it is $\lambda_n(Q(G))$, the farther it is G from being bipartite. Clearly, $\lambda_n(Q(G)) = 0$ if and only if G is bipartite. On the other hand, $Q(G) = L(G, -)$, so the signless Laplacian can be seen as a signed graph in which all edges have negative sign. Hence, it is natural to think to $\lambda_n(L(\Gamma))$ as a distance of the signed graph Γ from being balanced. The following lemma is straightforward.

Lemma 2.4. *Let $\Gamma = (G, -)$ be a signed graph with just negative edges. Then Γ is balanced if and only if G is bipartite.*

The matrix $L(\Gamma) = D(G) - A(\Gamma)$ is symmetric and diagonally dominant, hence it is positive semidefinite. The following characterization for $L(\Gamma)$ is given in [9]:

$$X^T L(\Gamma) X = \sum_{v_i v_j \in E(G)} \big(x_i - \sigma(v_i v_j) x_j\big)^2, \tag{1}$$

where X is a vector of size $V(G)$, and x_i is the component of X related to the vertex $v_i \in G$. The eigenvalue equation related to eigenvalue λ at vertex v reads

$$\big(\deg(v) - \lambda\big) x_v = \sum_{u \sim v} \sigma(uv) x_u. \tag{2}$$

We conclude this section by giving some well-known results which will be useful in the sequel.

Lemma 2.5. *Let M be the 2×2 block matrix whose blocks are the matrices A, B, C and D.*

If A is a non-singular square matrix, then

$$|M| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \big| D - CA^{-1}B \big|. \tag{3}$$

If $D = A$ and $B = C$ are blocks of the same size, then

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A - B||A + B|. \tag{4}$$

Lemma 2.6. *Let $A = tI_k + sJ_k$. Then*

$$|A| = (t)^{k-1}(ks + t).$$

The following theorem is widely known as *Courant–Weyl inequalities.*

Theorem 2.7. *Let A and B be two $n \times n$ Hermitian matrices. Then the following inequalities hold*:

i) $\lambda_i(A + B) \leqslant \lambda_j(A) + \lambda_{i-j+1}(B)$, $n \geqslant i \geqslant j \geqslant 1$;
ii) $\lambda_i(A + B) \geqslant \lambda_j(A) + \lambda_{i-j+n}(B)$, $1 \leqslant i \leqslant j \leqslant n$.

3. Algebraic frustration

In [9], the author finds a correlation between the least eigenvalue of Γ and the property of being balanced. Here we continue that investigation by showing that $\lambda_n(\Gamma) \leqslant \nu(\Gamma) \leqslant \epsilon(\Gamma)$. The proof of the main result follows Theorem 2.1 from [6] extended to the case of signed graphs.

The following lemma is well-known and proved in [11], here we report it for the sake of completeness. Note also that it can be deduced from Theorem 3.2 as well.

Lemma 3.1. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n. Then $\lambda_n(\Gamma) \leqslant \epsilon(\Gamma)$.*

Proof. If Γ is balanced then $\lambda_n = \epsilon(\Gamma) = 0$ and the assertion holds. Hence, we assume in the remainder of the proof that Γ is unbalanced. Assume first that $|G| \geqslant 4$. Let $F \subseteq E(G) = E$, with $|F| = \epsilon(\Gamma) > 0$, be a minimum set of edges for which $\Gamma - F$ is balanced, and let V_1, V_2 the two color classes of $\Gamma - F$. Let us define the following vector X on the vertices of G, $X = (x_1, x_2, \dots, x_n)$ where $x_v = 1$ if $v \in V_1$, $x_v = -1$ if $v \in V_2$; so $X^\top X = |G|$. Observe that $(x_u - \sigma(uv)x_v)^2 = 0$ whenever $uv \in E \setminus F$. On the other hand, $F = E^-(V_1) \cup E^-(V_2) \cup E^+(V_1, V_2)$, so $(x_u - \sigma(uv)x_v)^2 = 4$ for each $uv \in F$. Hence, we have

$$\begin{aligned}
\lambda_n(\Gamma) &\leqslant \frac{\sum_E (x_u - \sigma(uv)x_v)^2}{X^\top X} \\
&= \frac{\sum_{E \setminus F} (x_u - \sigma(uv)x_v)^2}{n} + \frac{\sum_F (x_u - \sigma(uv)x_v)^2}{n} \\
&= \frac{4\epsilon(\Gamma)}{n} \leqslant \epsilon(\Gamma).
\end{aligned}$$

Finally let $|G| \leqslant 3$. Since Γ is unbalanced, then $|G| = 3$ and Γ is a triangle with either one negative edge, or three negative edges. In both cases (note the two signed graphs are signature similar), we have $\lambda_n(\Gamma) = \epsilon(\Gamma) = 1$. □

Note that $\nu(\Gamma) \leqslant \epsilon(\Gamma)$ is trivial and well-known. In fact, assume that we have to remove m edges in order to make the signed graph balanced. Alternatively, we can remove one of the end vertices for each edge and obtain the same result. So, $\nu(\Gamma) \leqslant \epsilon(\Gamma) = m$. The latter equality is achieved when the m edges form a matching.

In the next theorem, we prove that $\lambda_n(\Gamma) \leqslant \nu(\Gamma)$. However, before proving it we need some additional notation. If Γ is unbalanced, then there exists a signed subgraph $S \subset \Gamma$, with $|S| = \nu(\Gamma)$, such that $\Gamma - S$ is balanced. In particular, $\Gamma - S$ might be not connected, so $\Gamma - S = B_1 + B_2 + \cdots + B_m$ where B_i $(i = 1, 2, \ldots, m)$ is a connected balanced signed subgraph of Γ. Observe also that in the worst case $m = 1$ and $B_1 = (K_2, \sigma)$, so $|B_1| \geqslant 2$ and, consequently, $|S| \leqslant |\Gamma| - 2$. Finally, since B_i is balanced, then we denote by B_i^1 the first color class, and by B_i^2 the (eventually empty) second color class.

Theorem 3.2. *Let $\Gamma = (G, \sigma)$ be a signed graph. Then $\lambda_n(\Gamma) \leqslant \nu(\Gamma)$.*

Proof. Let $\nu(\Gamma) = k$ and $|\Gamma| = n$. If so, there exists $S \subset \Gamma$ with $|S| = k \leqslant n - 2$ such that $\Gamma - S = B_1 + B_2 + \cdots + B_m$ is balanced with each B_i being connected and balanced and $|B_i| = n_i$ $(i = 1, 2, \ldots, m)$. Recall that for each balanced component B_i, B_i^1 is the first color class, and B_i^2 is the second one. We define the following vector $X_i = (x_1, x_2, \ldots, x_n)$ on the vertices of Γ such that $x_v = 1$ if $v \in B_i^1$, $x_v = -1$ if $v \in B_i^2$, and $x_v = 0$ if $v \in \Gamma - B_i$, for some fixed $1 \leqslant i \leqslant m$. Let $\bar{\Gamma} = (\bar{G}, \bar{\sigma})$ be the signed graph obtained from Γ such that the underlying graph $\bar{G}$ is obtained from G by joining each vertex of $G(S)$ to each of $G(\Gamma - S)$, and $\bar{\sigma}$ is obtained from σ by randomly giving any sign on the (eventually) added edges.

Since, $E(G) \subseteq E(\bar{G})$, and by observing that $(x_u - \bar{\sigma}(uv)x_v)^2 = 1$ for any edge (in $\bar{G}$) of $E(S, B_i)$, and $(x_u - \bar{\sigma}(uv)x_v)^2 = 0$ otherwise, we have

$$\lambda_n(\Gamma) \leqslant \frac{X^\top L(\Gamma)X}{X^\top X} \leqslant \frac{X^\top L(\bar{\Gamma})X}{X^\top X} = \frac{\sum_{uv \in E(\bar{G})}(x_u - \bar{\sigma}(uv)x_v)^2}{|B_i|}$$
$$= \frac{\sum_{u \in S, v \in B_i}(x_u - \bar{\sigma}(uv)x_v)^2}{|B_i|} = \frac{|S||B_i|}{|B_i|} = k. \tag{5}$$

This completes the proof. □

From Theorem 3.2, we have that λ_n is a lower bound for both the frustration number $\nu(\Gamma)$ and the frustration index $\epsilon(\Gamma)$, so we call the eigenvalue λ_n the *algebraic frustration*.

In the next theorem we study which conditions on the signed graph Γ lead to the equality $\lambda_n(\Gamma) = \nu(\Gamma) = k$.

Theorem 3.3. *Let $\Gamma = (G, \sigma)$ be an unbalanced signed graph of order n with $\nu(\Gamma) = k$. Let $S \subset \Gamma$, with $|S| = k$, such that $\Gamma - S = B_1 + B_2 + \cdots + B_m$ is balanced and each B_i is connected for $i = 1, 2, \ldots, m$.*

Then, $\lambda_n(\Gamma) = \nu(\Gamma) = k$ if and only if

i) $G = G(S) \vee G(B_1 + B_2 + \cdots + B_m)$;
ii) *for any $s \in S$, $|E^+(s, B_i^1)| + |E^-(s, B_i^2)| = |E^-(s, B_i^1)| + |E^+(s, B_i^2)|$, and $|B_i|$ is even, for each $i = 1, 2, \ldots, m$;*
iii) $L(S) + (n - 2k)I_k - \sum_{i=1}^{m} S_i^\top L(B_i)^{\#} S_i \succcurlyeq O$;

where B_i^1 and B_i^2 are the color classes of B_i, S_i is the $|B_i| \times k$ matrix whose pq-th entry is $-\sigma(v_p s_q)$, $v_p \in B_i$ and $s_q \in S$, and $L(B_i)^{\#}$ denotes the Moore-Penrose pseudo-inverse of $L(B_i)$.

Proof. Assume first that $\lambda_n = \nu(\Gamma) = k$. Then in (5) all inequalities are equalities, in particular $G = G(S) \vee G(B_1 + B_2 + \cdots + B_m)$, where $|S| = k$ and B_i is connected and balanced for $i = 1, 2, \ldots, m$. In addition, for each $i = 1, 2, \ldots, m$, the vector X_i defined as in the proof of Theorem 3.2, is an eigenvector related to k. Let $V(S) = \{s_1, s_2, \ldots, s_k\}$, for each $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, m$ the eigenvalue equation (2) for $\lambda_n = k$ w.r.t. $X_i = (x_1, x_2, \ldots, x_n)$ computed at s_j reads

$$k x_{s_j} = \deg(s_j) x_{s_j} - \sum_{v \sim s_j} \sigma(v s_j) x_v.$$

Recall that $x_v = 1$ if $v \in B_i^1$, $x_v = -1$ if $v \in B_i^2$, and $x_v = 0$ otherwise. Hence the above relation yields to

$$\left|E^+\left(s_j, B_i^1\right)\right| + \left|E^-\left(s_j, B_i^2\right)\right| = \left|E^-\left(s_j, B_i^1\right)\right| + \left|E^+\left(s_j, B_i^2\right)\right|, \tag{6}$$

where $E^*(s_j, B_i)$ is the set of edges with sign $* \in \{+, -\}$ between the vertex s_j and B_i.

The relation (6) deserves a couple of comments.

Firstly, each B_i consists of an even number of vertices, otherwise the above relation cannot be satisfied. In fact, assume without loss of generality that $|B_i^1| = p$ and $|B_i^2| = d$, where p is an even number and d is odd. If so, $|E^+(s_j, B_1)| + |E^-(s_j, B_1)| = p$ and $|E^+(s_j, B_2)| + |E^-(s_j, B_2)| = d$ and by simple computation we get $2(|E^+(s_j, B_i^1)| + |E^-(s_j, B_i^2)|) = p + d$, that is a contradiction.

Secondly, the vector $S_{ij} = (-\sigma(v_1 s_j), -\sigma(v_2 s_j), \ldots, -\sigma(v_{n_i} s_j))^\top$, with $v_t \in B_i$, is orthogonal to $\tilde{X}_i$, the eigenvector for $\lambda_n(B_i)$. In fact, $\tilde{X}_i$, extended with zero's, is the eigenvector X_i for $\lambda_n(\Gamma) = k$, and due to (6) the orthogonality follows.

So far, k is an eigenvalue of Γ of multiplicity at least m. We now make use of the fact that k is the least eigenvalue. Let us consider the matrix $L' = L(\Gamma) - kI_n$. Hence, if λ is an eigenvalue of L then $\lambda - k$ is an eigenvalue of L'. In particular, $\lambda_n(L') = 0$ and

L' is positive semidefinite. There exists a labeling of vertices such that L' can be block described as follows:

$$L' = \left(\begin{array}{c|ccc} L(S) + (n-2k)I_k & S_1^\top & \cdots & S_m^\top \\ \hline S_1 & L(B_1) & & \\ \vdots & & \ddots & \\ S_m & & & L(B_m) \end{array}\right),$$

where S_i is the $n_i \times k$ matrix whose j-th column has the opposite adjacency between s_j and the vertices of B_i, namely $S_i = (S_{i1}|S_{i2}|\dots|S_{ik})$, with $S_{ij} = -(\sigma(v_1 s_j), \sigma(v_2 s_j), \dots, \sigma(v_{n_i} s_j))$, $v_r \in B_i$ and $s_j \in S$. We now look for a regular matrix C such that $C^\top L' C$ is block diagonal. For this purpose, we have to find the matrices Y_i such that $S_i - L(B_i)Y_i = O$. Indeed, as mentioned above, S_{ij} is orthogonal to $\tilde{X}_i$ the eigenvector for $\lambda_n(B_i) = 0$. Hence S_{ij} belongs to the column space of $L(B_i)$ and consequently there exists a non-zero vector Y_{ij} such that $S_{ij} = L(B_i)Y_{ij}$. Therefore, if $Y_i = (Y_{i1}|Y_{i2}|\dots|Y_{ik})$ then we get $S_i - L(B_i)Y_i = O$, as wanted. However we can give an expression to $L(B_i)Y_i$ that is independent from the choice of Y_i but depends on the matrices S_i's. Recall that the entries of $\tilde{X}_i$ are 1's for each vertex in B_1^1 and -1's for each vertex in B_1^2. Let $L(B_i)^\#$ be the Moore-Penrose inverse of $L(B_i)$ and $\mu_j = \lambda_j(B_i)$. Hence, if $L(B_i) = U \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_{n-1}, 0)U^\top$, where U is the orthogonal matrix whose columns consist of the normalized eigenvector of μ_i, including $\frac{1}{\sqrt{n_i}}\tilde{X}_i$, then $L(B_i)^\# = U \operatorname{diag}(\mu_1^{-1}, \mu_2^{-1}, \dots, \mu_{n-1}^{-1}, 0)U^\top$. So,

$$L(B_i)^\# L(B_i) = U \operatorname{diag}(1, \dots, 1, 0)U^\top = U\big(I_{n_i} - \operatorname{diag}(0, \dots, 0, 1)\big)U^\top = I_{n_i} - \frac{1}{n_i}\tilde{X}_i\tilde{X}_i^{\,\top}.$$

Now,

$$Y_{ip}^\top S_{iq} = S_{iq}^\top Y_{ip} = S_{iq}^\top\left(I_{n_i} - \frac{1}{n_i}\tilde{X}_i\tilde{X}_i^{\,\top}\right)Y_{ip} = S_{iq}^\top L(B_i)^\# L(B_i)Y_{ip} = S_{iq}^\top L(B_i)^\# S_{ip}. \tag{7}$$

Recall that Y_i is the matrix whose j-th column is Y_{ij}. Let C be the following block defined matrix

$$C = \left(\begin{array}{c|ccc} I_k & O & \cdots & O \\ \hline -Y_1 & I_{n_1} & & \\ \vdots & & \ddots & \\ -Y_m & & & I_{n_m} \end{array}\right).$$

Then it is easy to check that $C^\top L'C$ is a block diagonal matrix given by

$$C^\top L'C = \left[L(S) + (n-2k)I - \sum_{i=1}^{m} Y_i^\top S_i\right] \oplus L(B_1) \oplus \cdots \oplus L(B_m).$$

Since L' is positive semidefinite, then $C^\top L'C$ is positive semidefinite as well. In particular, we get

$$L(S) + (n-2k)I - \sum_{i=1}^{m} Y_i^\top S_i \succcurlyeq O.$$

Note that in view of (7) it is $Y_i^\top S_i = S_i^\top L(B_i)^{\#} S_i$. Hence, we finally have

$$L(S) + (n-2k)I - \sum_{i=1}^{m} Y_i^\top S_i = L(S) + (n-2k)I - \sum_{i=1}^{m} S_i^\top L(B_i)^{\#} S_i \succcurlyeq O.$$

Now assume that Γ is a signed graph satisfying i)–iii). If so, it is easy to see that k is the least eigenvalue of Γ. It remains to prove that $\nu(\Gamma) = k$ and we are done. According to Theorem 3.2, we have that $k \leqslant \nu(\Gamma)$. However, by deleting the vertices of S we have that Γ is balanced, hence $\nu(\Gamma) \leqslant k$, and equality follows.

This completes the proof. □

We now examine some corollaries which, under some constraint, simplify Condition iii) of Theorem 3.3.

Corollary 3.4. *Let Γ be a signed graph satisfying Theorem 3.3 i) and ii). If $S_{ij} = Z_i$ for all $s_j \in S$ and $i = 1, 2, \ldots, m$, then Condition iii) reduces to*

$$\text{iii}') \quad L(S) + (n-2k)I_k - \left(\sum_{i=1}^{m} Z_i^\top L(B_i)^{\#} Z_i\right) J_k \succcurlyeq O.$$

Corollary 3.5. *If Γ has only negative edges, then $L(\Gamma) = D(G) + A(G) = Q(G)$ and algebraic frustration reduces to algebraic bipartiteness. Furthermore, the conditions of Theorem 3.3 simplify to Conditions 1. and 3. of Theorem 2.1 in [6].*

It is worth to mention that the examples (negative and positive ones) given in [6] for the signless Laplacian matrix, are valid also in our context due to the fact that $Q(G) = L(\Gamma)$ with $\Gamma = (G, -)$. From the same examples, se can also deduce more complex settings by making use of signature similarity. For instance, assume that all B_i's are $\Delta(B_i)$-regular and bipartite, S consists of a single vertex s, and from s starts a positive edge if it goes to a vertex of a color class (in the sense of bipartiteness) of B_i and a negative edge if it goes to a vertex of the second color class of B_i. Then the

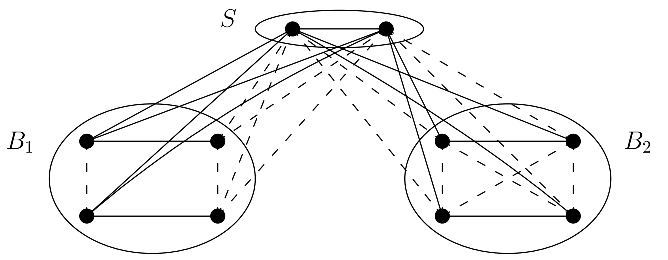

Fig. 1. A signed graph Γ with $\lambda_n(\Gamma) = \nu(\Gamma) = 2$.

vector $X_i = (-\sigma(v_1 s), \dots, -\sigma(v_{n_i} s))$ is an eigenvector for $2\Delta(B_i)$. Note also that X_i is an eigenvector of the eigenvalue $(2\Delta(B_i))^{-1}$ of $L(B_i)^{\#}$. We now check Condition iii) and we find that

$$(n-2) - \sum_{i=1}^{m} X_i^\top L(B_i)^{\#} X_i = (n-2) - \sum_{i=1}^{m} \frac{n_i}{2\Delta(B_i)} \geqslant (n-2) - \frac{n-1}{2} \geqslant 0.$$

We now give a numerical example.

Example 3.6. Consider signed graph Γ depicted in Fig. 1. The continuous lines are positive edges and the dashed ones are negative edges.

We have that

$$L(\Gamma) = \begin{pmatrix} L(S) + 8I_2 & S_1^\top & S_2^\top \\ S_1 & L(B_1) + 2I_4 & O \\ S_2 & O & L(B_2) + 2I_4 \end{pmatrix}.$$

Since Conditions i) and ii) of Theorem 3.3 hold, then 2 is an eigenvalue with multiplicity at least 2. Next, 2 is the least eigenvalue of Γ if and only if Condition iii) of Theorem 3.3 is satisfied as well. After some computations we find that

$$L(B_1)^{\#} = \begin{pmatrix} 1 & 3/4 & -1/2 & -1/4 \\ 1 & 3/2 & -1 & -1/2 \\ -1 & -5/4 & 3/2 & 3/4 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

$$L(B_2)^{\#} = \begin{pmatrix} 3/16 & -1/16 & 1/16 & 1/16 \\ -1/16 & 3/16 & 1/16 & 1/16 \\ 1/16 & 1/16 & 3/16 & -1/16 \\ 1/16 & 1/16 & -1/16 & 3/16 \end{pmatrix}.$$

Hence,

$$S_1^\top L(B_1)^{\#} S_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad S_2^\top L(B_2)^{\#} S_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Finally, we compute $L(S) + 6I_2 - \sum_{i=1}^{2} S_i^\top L(B_i)^\# S_i$, and we get

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix},$$

that is positive definite. Hence, Condition iii) of Theorem 3.3 is satisfied and 2 is indeed the least eigenvalue of Γ. $\square$

Theorem 3.3 gives a necessary and sufficient condition for a signed graph Γ to have $\lambda_n(\Gamma) = \nu(\Gamma)$. However, it is difficult to give a combinatorial interpretation of the Condition iii). In the following theorem we give a structural condition that leads to the equality between the algebraic frustration and the frustration number.

Theorem 3.7. *Let $\Gamma = (G, \sigma)$ be a signed graph of order n, and S the smallest signed subgraph such that $\Gamma - S$ is balanced. If Γ satisfies the following conditions*:

- $G = G(S) \vee K_{2t}$*, with* $|S| = k$ *and* $2t = n - k$;
- $\sigma(e) = 1$ *for* $e \in E(K_{2t})$, $\sigma(sv_i) = 1$ *for* $s \in S$ *and* $v_i \in K_{2t}$, $i = 1, 2, \dots, t$, *and* $\sigma(e) = -1$ *otherwise*;
- $n - 3k \geqslant 0$;

then $\lambda_n(\Gamma) = \nu(\Gamma) = k$.

Proof. Assume first that S is totally disconnected, namely $G(S) = kK_1 = \{s_1, s_2, \dots, s_k\}$. It is not difficult to see that the first condition comes from Theorem 3.3 i). The second condition satisfies Theorem 3.3 ii). In fact, take $B_1 = (K_{2t}, +)$ with $K_{2t} = \{v_1, v_2, \dots, v_{2t}\}$. Since B_1^2 is empty, the signs of the edges between kK_1 and K_{2t} are constrained by the condition (6), which becomes $|E^+(s_i, B_1^1)| = |E^-(s_i, B_1^1)|$. Without loss of generality, we can assume that $\sigma(s_j v_i) = 1$ for $j = 1, 2, \dots, k$ and $i = 1, 2, \dots, t$ and $\sigma(s_j v_i) = -1$ for $j = 1, 2, \dots, k$ and $i = t+1, t+2, \dots, 2t$. Note that the assumption $\Gamma - S = (K_{2t}, +)$ makes the computations easier, however via signature similarity we can pick more complex situations.

We now check the Condition iii) from Theorem 3.3. From any vertex of S to $(K_{2t}, +)$ we have the same sign pattern, so we can use Corollary 3.4 instead. In this case we have $L(S) = O_k$. So we just need to compute $Z^T L(K_{2t}, +)^\# Z$. It is easy to see that $Z = (1, \dots, 1, -1, \dots, -1)^T$ where the number of 1's is equals the number of -1's that is equal to t. Next, we need to compute $L(K_{2t}, +)^\#$. After some computation we get that the latter matrix has the following form:

$$L(K_{2t}, +)^\# = (l_{ij}) = \begin{cases} \frac{2t-1}{(2t)^2}, & \text{if } i = j; \\ -\frac{1}{(2t)^2}, & \text{if } i \neq j. \end{cases}$$

Hence we obtain

$$Z^T L(K_{2t},+)^{\#} Z = 1.$$

We now verify Condition iii)′ from Corollary 3.4:

$$L(S) + (n-2k)I_k - \big(Z^T L(K_{2t},+)^{\#} Z\big) J_k = (n-2k)I_k - J_k. \tag{8}$$

The above matrix has all diagonal entries equal to $n-2k-1$ and each non-diagonal entry equal to -1, which are $k-1$ for each row. Hence, if $n-2k-1 \geqslant k-1$, that is $n-3k \geqslant 0$, we have that $(n-2k)I_k - J_k$ is a symmetric diagonally dominant with diagonal positive entries. So, $(n-2k)I_k - J_k$ is a positive semidefinite matrix, with 0 as an eigenvalue if and only if $n-3k=0$.

Finally, if S is any signed graph of order k, it is easy to see that in (8), from $L(S)$ being a symmetric diagonally dominant with diagonal positive entries, we get that the matrix $L(S)+(n-2k)I_k - J_k$ is symmetric diagonally dominant with diagonal positive entries for any $n \geqslant 3k$. This completes the proof. □

In the next theorem we compute the spectrum of some signed graphs which satisfy the hypothesis of Theorem 3.7. It is worth to observe that some of them have integral spectrum. The power denotes the multiplicity.

Theorem 3.8. *Let Γ be a signed graph satisfying the hypothesis of Theorem 3.7.*

- *If $S = (kK_1, \emptyset)$ then* $\operatorname{Spec}(\Gamma) = \{k^{(2)}, 2k^{(k-1)}, 3k^{(2k-2)}, 4k\}$;
- *if $S = (K_k, +)$ then* $\operatorname{Spec}(\Gamma) = \{k^{(2)}, 3k^{(3k-3)}, 4k\}$;
- *if $S = (K_k, -)$ then* $\operatorname{Spec}(\Gamma) = \{k, (3k-2)^{(k-1)}, 3k^{(2k-2)}, 1/2(7k-2 \pm \sqrt{9k^2-4k+4})\}$.

Proof. We just prove the result when $S = (kK_1, \emptyset)$. The cases $S = (K_k, +)$ and $S = (K_k, -)$ can be proved similarly but they are a little harder to compute.

Recall that $L(K_{2k},+) = 2kI_{2k} - J_{2k}$, and S_1 is the 2×1 block matrix $(J_k, -J_k)^T$. From $G(S) = kK_1$ and $L(S) = O_k$, we have that $L(\Gamma)$ admits the following block form:

$$L(\Gamma) = \begin{pmatrix} L(S)+2kI_k & S_1^T \\ S_1 & L(K_{2k},+)+kI_{2k} \end{pmatrix} = \begin{pmatrix} 2kI_k & S_1^T \\ S_1 & 3kI_{2k} - J_{2k} \end{pmatrix}.$$

Hence the characteristic polynomial of Γ is the below determinant

$$\begin{vmatrix} (2k-x)I_k & S_1^T \\ S_1 & 3kI_{2k} - J_{2k} - xI_{2k} \end{vmatrix}$$

which in view of (3) equals

$$(2k-x)^k \left| 3kI_{2k} - J_{2k} - S_1\left(\frac{1}{2x-k} I_k\right) S_1^T - xI_{2k}\right|.$$

We now compute the latter determinant.

Observe that

$$S_1\left(\frac{1}{2x-k} I_k\right) S_1^T = \frac{1}{2x-k} S_1 S_1^T = \frac{k}{2k-x}\begin{pmatrix} J_k & -J_k \\ -J_k & J_k \end{pmatrix}.$$

So we get

$$\left| 3kI_{2k} - J_{2k} - \frac{k}{2k-x}\begin{pmatrix} J_k & -J_k \\ -J_k & J_k \end{pmatrix} - xI_{2k}\right| = \frac{1}{(2k-x)^{2k}}\begin{vmatrix} A & B \\ B & A \end{vmatrix},$$

where

$$A = (a_{ij}) = \begin{cases} x^2 + x - 5kx + 6k^2 - 3k, & \text{if } i = j; \\ x - 3k, & \text{if } i \neq j, \end{cases}$$

and $B = (x-k)J_k$. In view of (4) we have

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B||A-B|.$$

It is a matter of computation to check that

$$A + B = \left(x^2 - 5kx + 6k^2 - 2k\right) I_k + (2x - 4k) J_k, \quad \text{and}$$
$$A - B = \left(x^2 - 5kx + 6k^2\right) I_k - 2kJ_k.$$

The above determinants can be easily computed via Lemma 2.6, and we get that

$$|A - B| = -(x-2k)^{(k-1)}(x-3k)^{(k-1)}(x-k)(x-4k);$$
$$|A + B| = -(x-2k)^k(x-3k)^{(k-1)}(x-k).$$

By multiplying all contributions we obtain that $\mathrm{Spec}(\Gamma) = \{k^{(2)}, 2k^{(k-1)}, 3k^{(2k-2)}, 4k\}$, as claimed. This completes the proof. □

We conclude this paper by giving a sufficient (but somewhat difficult to achieve) condition for Γ such that $\lambda_n < k$.

Theorem 3.9. *Let $\Gamma = (G, \sigma)$ be a signed graph satisfying Conditions* i) *and* ii) *of Theorem 3.3. If $\lambda_n(S) < 2k - n$ then $\lambda_n(\Gamma) < k$.*

Proof. Since $G = G(S) \vee G(B_1 + B_2 + \cdots + B_m)$, we have that $L(\Gamma)$ has, after a proper labeling, the following block form

$$L(\Gamma) = \left(\begin{array}{c|ccc} L(S) + (n-k)I_k & S_1^\top & \cdots & S_m^\top \\ \hline S_1 & L(B_1) + kI_k & & \\ \vdots & & \ddots & \\ S_m & & & L(B_m) + kI_k \end{array}\right).$$

Consider the following signature $\theta : G \to \{1, -1\}$, such that $\theta(v) = 1$ if $v \in S$ and $\theta(v) = -1$ if $v \in G \setminus S$. After applying θ to Γ, we obtain a new signed graph Γ' cospectral to Γ. It is easy to check that Γ' has the following Laplacian matrix,

$$L(\Gamma') = \left(\begin{array}{c|ccc} L(S) + (n-k)I_k & -S_1^\top & \cdots & -S_m^\top \\ \hline -S_1 & L(B_1) + kI_k & & \\ \vdots & & \ddots & \\ -S_m & & & L(B_m) + kI_k \end{array}\right).$$

Consider now the matrix $\bar{L} = L(\Gamma) + L(\Gamma')$, then we get

$$\bar{L} = 2\big[L(S) + (n-k)I_k \oplus L(B_1) + kI_k \oplus \cdots \oplus L(B_m) + kI_k\big].$$

In view of Theorem 2.7 ii) by putting $i = j = n$ we get that

$$\lambda_n(\bar{L}) \geqslant \lambda_n(\Gamma) + \lambda_n\big(\Gamma'\big) = 2\lambda_n(\Gamma). \tag{9}$$

Note that

$$\lambda_n(\bar{L}) = \min\big\{2k, 2\lambda_n\big(L(S) + (n-k)I_k\big)\big\}.$$

If $\lambda_n(\bar{L}) = 2\lambda_n(L(S) + (n-k)I_k) < 2k$ then by (9) it is $\lambda_n(\Gamma) < k$. Since, $2\lambda_n(L(S) + (n-k)I_k) < 2k$ implies $\lambda_n(S) < 2k - n$, we deduce the assertion. □

We give an example of a signed graph that satisfies hypothesis of the above theorem.

Example 3.10. Consider the signed graph $\Gamma = (G, \sigma)$ below described. $G = C_5 \times K_4$, with $K_4 = \{v_1, v_2, v_3, v_4\}$. $\sigma(e) = -1$ for $e \in E(C_5)$, $\sigma(e) = 1$ for $e \in E(K_4)$, $\sigma(c, v_1) = \sigma(c, v_2) = 1$ and $\sigma(c, v_3) = \sigma(c, v_4) = -1$, for any $c \in C_5$. It is routine to check that Γ satisfies Theorem 3.3 i) and ii) by setting $S = (C_5, -) =$ and $B_1 = B_1^1 = (K_4, +)$ (here $B_2 = \emptyset$, but we can make it non-empty with a suitable switching). In addition we have $2k = 10 > 9 = n$ and $\lambda_n(C_5) < 0.39 < 1 = 2k - n$, hence the additional conditions of Theorem 3.9 are fulfilled, as well. After making a few computations we find that $\lambda_n(\Gamma) = 4 < 5 = k$.

Other examples can be obtained by taking $G = C_{2m+1} \times K_{2m}$ ($m \geqslant 2$) and σ having an analogous pattern. □

In conclusion, in this paper we discussed the relationship between the least eigenvalue and the property of being balanced for a signed graph, as it was partially detected in [9]. Also, by following the ideas and the proofs from [6] (especially Theorem 2.1 and the consequent examples) we have extended to signed graphs the algebraic bipartiteness defined for the signless Laplacian of graphs. Hence, we call $\lambda_n(\Gamma)$ the algebraic frustration. It is worth to mention that, as in [6], we are not able to give a combinatorial interpretation of Theorem 3.3 iii). Hence, it would be interesting to find a sign pattern, if it exists, for the adjacencies in $E(S, B_1 + B_2 + \cdots + B_m)$ which satisfies the algebraic condition of Theorem 3.3. A simpler but interesting variant of the latter problem is to identify families of signed graphs for which the algebraic frustration equals the frustration number.

Acknowledgements

The author is grateful to the unknown referees for their careful read of the paper and comments which have considerably improved the presentation of this paper. The author would like to thank Prof. Alessandro Provetti for introducing to the author the notion of signed graph and its applications in Social Networks and Logical Programming. The author is grateful to Prof. Enzo Maria Li Marzi and Slobodan K. Simić for the fruitful discussion on the arguments of this paper.

References

[1] S. Costanza, A. Provetti, Graph representations of logic programs: properties and comparison, in: Proceedings of the Sixth Latin American Workshop on Non-monotonic Reasoning, 2010, pp. 1–14.
[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Applications, III revised and enlarged edition, Johan Ambrosius Bart. Verlag, Heidelberg/Leipzig, 1995.
[3] D. Cvetković, S.K. Simić, Towards a spectral theory of graphs based on signless Laplacian, I, Publ. Inst. Math. (Beograd) (N.S.) 85 (99) (2009) 19–33.
[4] D. Cvetković, S.K. Simić, Towards a spectral theory of Graphs based on signless Laplacian, II, Linear Algebra Appl. 432 (2010) 2257–2272.
[5] D. Cvetković, S.K. Simić, Towards a spectral theory of Graphs based on signless Laplacian, III, Appl. Anal. Discrete Math. 4 (2010) 156–166.
[6] S. Fallat, Y.-Z. Fan, Bipartiteness and the least eigenvalue of signless Laplacian of graphs, Linear Algebra Appl. 436 (2012) 3254–3267.
[7] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[8] F. Harary, On the notion of balanced of a signed graph, Michigan Math. J. 2 (2) (1953) 143–146.
[9] Y.P. Hou, Bounds for the least Laplacian eigenvalue of a signed graph, Acta Math. Sin. (Engl. Ser.) 21 (4) (2005) 955–960.
[10] J. Kunegis, et al., Spectral analysis of signed graphs for clustering, prediction and visualization, in: Proceedings of the 2010 SIAM International Conference on Data Mining, 2010, pp. 559–570.
[11] Y.-Y. Tan, Y.-Z. Fan, On edge singularity and eigenvectors of mixed graphs, Acta Math. Sin. Engl. Ser. B 24 (1) (2008) 139–146.
[12] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1981) 47–74.
[13] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, Electron. J. Combin. (Dynamic Surveys) DS8 (2012), URL: http://www3.combinatorics.org/Surveys/ds8.pdf.