JOURNAL OF MATHEMATICAL PSYCHOLOGY 9, 66–91 (1972)

A Derivation of a Measure of Relative Balance for Social Structures and a Characterization of Extensive Ratio Systems

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As an extension of the Heider and Cartwright–Harary theory of balance in social structures (signed digraphs), a measure of relative balance which is derived from some plausible axioms is needed. Such a measure should reflect the balance ordering, i.e., the relation 'one social structure is more balanced than another'.

Axioms for such a balance ordering are stated below in terms of the signed digraph representing the social structure as in the Cartwright–Harary theory. The axioms give rise to a measure which determines the balance ordering uniquely, subject to choice of a parameter $f(m)$ which represents the relative importance of a semicycle of length m in the signed digraph. Special values of the parameter $f(m)$ give rise to measures previously suggested as plausible in the literature. Techniques for estimating the values $f(m)$ are described.

A second axiomatization for the balance ordering is obtained by translating the requirements on balance into a typical problem of measurement theory in the Suppes–Zinnes sense. The resulting axioms define a so-called extensive ratio system.

1. HISTORICAL BACKGROUND

Many empirical relationships in the social sciences seem to have positive and negative evaluations. Thus, for example, the relations "likes," "associates with," "tells truth to," etc., all seem to have opposites: "hates," "avoids," "lies to," etc.

A good model for a social structure with such positive and negative relationships, say a small group, is a *signed digraph*. This is a directed graph (digraph)[1] $G = (V, E)$ together with a function $f\colon E \to \{+, -\}$ associating with each line (element of E)

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[1] That is, V is finite and $E \subseteq V \times V$. The reader is referred to Harary, Norman and Cartwright (1965) for all digraph definitions and for a discussion of signed digraphs.

a sign. The points (members of V) are thought of as elements of the structure, e.g., members of the small group. If a line from x to y is positive (negative), this indicates that the relationship between x and y is positive (negative). The absence of a line between x and y is interpreted as either indifference or absence of the relationship in question. Much of the literature relevant to our topic has been restricted to the case where the digraph is symmetric, and where pairs of symmetric lines have the same sign, thus dealing with *signed graphs*. But the results all apply without much modification to the case of signed digraphs.

Sociologists have been concerned with the notion of "balance" of a social structure. Early work in the theory of structural balance is surveyed in Berger, Cohen, Snell and Zelditch (1962) and the theory has recently been surveyed in detail by Taylor (1970). This theory goes back to Heider (1946), who made an attempt to describe what he meant by balance by giving a series of examples and asserted that unbalanced social structures exhibit a certain tension resulting in a tendency to change in the direction of balance. Search for a precise definition of balance led to the formulation of such a notion for signed graphs and digraphs. The concept which turned out to be relevant was that of a *semicycle*, namely, a collection of distinct points $x_1, x_2, \ldots, x_n$ together with n lines, one from each pair x_1x_2 or x_2x_1 ; x_2x_3 or x_3x_2 ;...; $x_{n-1}x_n$ or x_nx_{n-1} ; x_nx_1 or x_1x_n . A semicycle is determined simply by giving its lines. Its *length* is n. Thus the signed digraph of Fig. 1 has the following semicycles: $e_1e_2e_3$, $e_1e_2e_4$,

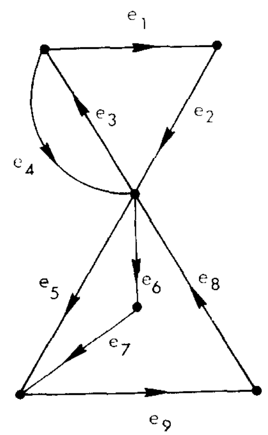

FIG. 1. Signed digraph, with semicycles $e_1e_2e_3$, $e_1e_2e_4$, $e_5e_6e_7$, $e_5e_8e_9$, and $e_6e_7e_9e_8$.

$e_5e_6e_7$, $e_5e_8e_9$, $e_6e_7e_9e_8$. There is also a semicyle e_3e_4 , but we shall disregard semicycles of length 2.

Heider (1946) observed that of the four structures in Fig. 2, the first and third are balanced while the second and fourth are unbalanced. [For a more detailed discussion

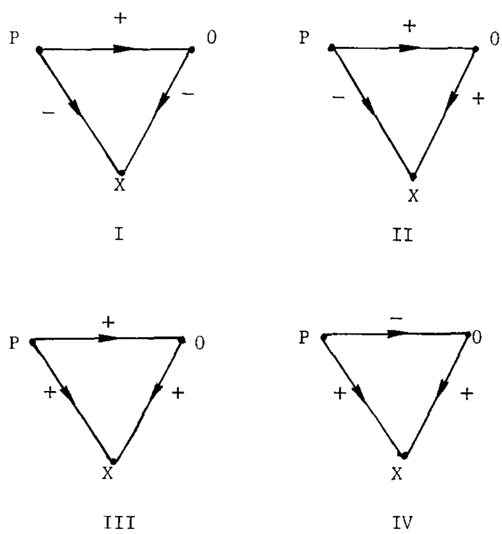

FIG. 2. Four social structures. P is interpreted to be a person, 0 another person, and X an impersonal entity. The relationships shown are P's evaluation of 0, P's evaluation of X, and 0's evaluation of X. According to Heider (1946), structures I and III are balanced, II and IV are not.

of this example, the reader is referred to Harary, Norman, and Cartwright (1965)]. In particular, the first and third semicycles in Fig. 2 have an even number of negative lines while the remaining two semicycles have an odd number of negative lines.

This observation led Harary (1954) and Cartwright and Harary (1956) to generalize to longer semicycles and to suggest that an arbitrary semicycle should be called *balanced*, or *positively balanced*, or simply *positive*, precisely if it has an even number of negative lines; and *negatively balanced*, or simply *negative*, otherwise. These authors suggested calling a signed digraph *balanced* if each semicycle of it is balanced, and established several interesting criteria for balance. There is some empirical justification for this definition.

This work on balance was intriguing as far as it went, both mathematically and from the point of view of applications. But it seemed quite clear that very few social structures are balanced and that of the rest it makes sense to say that some are more balanced than others. Indeed, Heider's observation that a structure tends to change in the direction of balance calls for a measure of the extent to which one imbalanced structure is more balanced than another. Morissette (1958) speaks of "degrees" of balance and reformulates Heider's basic propositions about balance as follows:

PROPOSITION 1. *The magnitude of forces toward balance is inversely related to the degree of balance of the system.*

PROPOSITION 2. *The magnitude of tension created by a system is inversely related to the degree of balance of that system.*

From a practical point of view, then, much more useful than a definition of the absolute concept "balance" would be a measure of relative balance—a way of deciding whether one structure is more balanced than another.

Aware of the need for such a measure, Cartwright and Harary (1956) suggested as a first approximation to relative balance of a signed digraph the ratio of the number c^+ of balanced semicycles to the total number c of semicycles. Using this definition, Morissette (1958) performed an experiment which tended to confirm Heider's Propositions 1 and 2 as formulated above. Taylor (1970) gives a survey of empirical work which tests these propositions.

Harary (1955) also introduced the notion of limited balance (of order N). Namely, a signed digraph is said to be *N-balanced* if each semicycle of length at most N is balanced. It seems natural to speak of *relative N-balance* as well, and use as a measure the ratio of the number of balanced semicycles of length at most N to the number of semicycles of length at most N. [For a discussion of the psychological and sociological significance of limited balance, see Cartwright and Harary (1956).]

We first became interested in the problem of measuring relative balance when we became dissatisfied with the rather naive measure c^+/c. Our first informal idea was that semicycles of different length contribute differently to balance, with longer semicycles being relatively less important, an idea also mentioned by Flament (1963), by Cartwright and Harary (1956), and by Taylor (1970). Indeed, according to Taylor (p. 77), "the empirical utility of formalization (of the theory of balance) seems to rest to a great degree on a solution to" the problem of how to incorporate or measure differing effects of semicycles of differing length. In Norman–Roberts (1972) we give an example which indicates why it is not only convenient but necessary to allow different weights for different length semicycles.

Our first idea was to use a measure of the form $\Sigma f(m)\, c_m^+/\Sigma f(m)\, c_m$, where c_m^+ is the number of balanced semicycles of length m and c_m is the number of semicycles of length m, and $f(m)$ measures the importance of semicycles of length m.[2] In particular, a monotone decreasing function f, such as $1/m$ or $1/m^2$ or $1/2^m$, seemed appropriate. (According to Taylor (1970, p. 76, pp. 206 ff), some evidence accumulated by Zajonc and Burnstein (1965) about the difficulty of learning and recalling longer semicycles suggests that longer semicycles will have less effect upon a person's tension than shorter ones.) The Cartwright–Harary measure uses $f(m) \equiv 1$ and the relative N-balance measure above uses

$$f(m) = \begin{cases} 1 & \text{if } m \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

[2] Flament (1963) has a slightly different, but closely related, suggestion.

(Considerably different balance measures have been suggested, in particular, the so-called line-index. We refer the reader to Harary–Norman–Cartwright (1965), Harary (1959), and Flament (1963) for a discussion, and to Taylor (1970, pp. 60 ff, pp. 188 ff) for a survey of such measures.)

While research was continuing on a measure of balance, a group of sociologists at the Stanford University Laboratory for Social Research was applying balance theory to the study of small groups and was interested in a balance measure to apply to their studies. [Cf. Berger, Zelditch, and Anderson (1966), Zelditch, Berger and Cohen, (1965), etc.] *In particular, what was desired was a balance measure which was not ad hoc, but which could rather be derived from a simple set of assumptions or axioms.* The task of formulating such a set of axioms and deriving from them a balance measure by "following our mathematical noses" led to the present research. It turned out to be fruitful to axiomatize only the order of balance, rather than the measure itself, and regard as an acceptable *measure* any one which gave rise to the given balance ordering. Much to our surprise, the axioms we listed led us back to our original idea of obtaining a measure of relative balance by differentially weighting different length semicycles, summing, and then taking ratios.

In the next section we present a set of axioms for balance and a theorem which shows how the axioms determine the balance ordering, subject to the specification of a certain parameter. The list of axioms we present is of course somewhat different from the list we originally started with. Most of the modifications have been suggested for reasons of mathematical elegance.

In Section 3 we discuss the interpretation of the parameter and make some brief remarks on how the parameter can be chosen. In Section 4 we discuss the problem of axiomatizing the balance ordering directly, rather than using the indirect approach applied in Section 2. We present a second axiomatization for the balance ordering which results in the same representation theorem, and in the process solve the problem of axiomatizing what we call extensive ratio systems. Section 4 contains the major mathematical results of the paper. The earlier axiomatization for balance may be considered motivation for the interest in the theorems of Section 4. In the final section, we indicate how the results of this paper can be applied to measure local balance rather than global balance.

It should be noted here that our balance ordering applies only to social structures which can be translated into signed digraphs. The significance of this translation procedure should not be overlooked. It is not too hard to give an example which shows how adding new information to several signed digraphs representing alternative social situations can change the relative balance of these situations. In Norman-Roberts (1972), we discuss the translation procedure by giving a detailed formalization of the theory of distributive justice [cf. Berger, Zelditch, Anderson and Cohen (1972)] and derive a relative balance ordering of the 16 cases of distributive justice.

2. The Balance Ordering

2.1. *Distance Axioms*

Our approach to obtaining the balance ordering is axiomatic. The approach is strongly motivated by that of Kemeny and Snell (1962) in their attempt to obtain a consensus from the rankings among alternatives obtained from a group of experts. They first axiomatize a measure of distance between two rankings and then define a consensus ranking in terms of the distance measure. Similarly, instead of directly axiomatizing an order of balance, we first axiomatize a measure of distance between two so-called "semicycle sequences" and then relate this to the relative balance of the corresponding signed digraphs. It will turn out that the axioms determine the distance function and the balance ordering uniquely subject to choice of a parameter f.

To proceed with the development, suppose G is a signed digraph. By its *positive semicycle sequence* we shall mean the infinite sequence $A = (a_3, a_4, \ldots, a_m, \ldots)$, where a_m gives the number of (positively) balanced semicycles of length m in G. The entries of A are of course 0 from some point on. Similarly, if b_m gives the number of negative semicycles of length m, then $B = (b_3, b_4, \ldots, b_m, \ldots)$ is the *negative semicycle sequence.* The pair (A, B) is the *semicycle sequence pair,* or the *semicycle pair* for short. In the following we shall develop a technique for determining the relative balance of two signed digraphs. The order of balance will be based entirely on the semicycle pair. (Some people allow the possibility of semicycles of length 2. The following discussion is easily modified to include these semicycles, adding components a_2 and b_2 to the appropriate semicycle sequences.)

Let $\mathscr{S}$ be the class of all infinite sequences of nonnegative integers with only finitely many nonzero terms. (It should be noted that every member of $\mathscr{S}$ is the positive (negative) semicycle sequence of some signed digraph.) Sequences in $\mathscr{S}$ will be denoted by capital letters A, B, C,... and A will always be the sequence $(a_3, a_4, \ldots, a_m, \ldots)$, B the sequence $(b_3, b_4, \ldots, b_m, \ldots)$, etc.

Our first objective is to find a distance measure d on $\mathscr{S}$, i.e., a function d: $\mathscr{S} \times \mathscr{S} \to \mathscr{R}$, satisfying certain axioms motivated by the Kemeny–Snell (1962) axioms for distance between preference rankings. To introduce the axioms, we need to define the notion of one sequence being between two others. We use the very simple Kemeny–Snell definition here, and say that if A, B, and C are sequences in $\mathscr{S}$, then C is *between* A and B if for all m, $a_m \leqslant c_m \leqslant b_m$ or $a_m \geqslant c_m \geqslant b_m$. We denote this $[A, C, B]$.

We shall assume that the measure d satisfies the following four axioms.

Distance Axioms. Let A, B, C be sequences in $\mathscr{S}$.

Axiom 1 (nonnegativity). $d(A, B) \geqslant 0$ and $d(A, B) > 0$ if $A \neq B$.

Axiom 2 (symmetry). $d(A, B) = d(B, A)$.

AXIOM 3 (betweenness). If $[A, B, C]$, then $d(A, C) = d(A, B) + d(B, C)$.

AXIOM 4 (translation invariance). $d(A + C, B + C) = d(A, B)$.

There are properties of distance to which we are accustomed which are not stated in these axioms, in particular the assumption $d(A, A) = 0$ and the triangle inequality. It will follow from Theorem 1 below that our axioms in fact imply all the usual metric properties.

2.2. *Balance Axioms*

Our study of balance will be restricted to the class $\mathscr{G}$ of all signed digraphs having at least one semicycle. The notion of balance is ambiguous for signed digraphs with no semicycles.[3] So we omit these signed digraphs from our discussion.

Our axioms for balance will be stated in terms of a primitive $\preccurlyeq$, a binary relation defined on $\mathscr{G}$, where $G \preccurlyeq H$ is interpreted as meaning that G is no more balanced than H. The first balance axiom asserts that the order of balance depends only on the semicycle pairs. Formally, it follows from Axiom 6.

AXIOM 5. If G and G' in $\mathscr{G}$ have the same semicycle pairs, then for all H in $\mathscr{G}$, $G \preccurlyeq H$ if and only if $G' \preccurlyeq H$ and $H \preccurlyeq G$ if and only if $H \preccurlyeq G'$.

If we denote by 0 the sequence consisting of all zeroes, then we can define a quaternary relation L on $\mathscr{S}$ as follows.

DEFINITION: $L(A, B, C, D)$ if and only if $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$ and whenever G and H are signed digraphs in $\mathscr{G}$ with semicycle pairs (A, B) and (C, D), then $G \preccurlyeq H$.

The restrictions $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$ are of course necessary because by assumption there is no signed digraph in $\mathscr{G}$ with semicycle pair $(0, 0)$.

The remaining balance axiom is introduced in terms of L, relating L to the distance measure d discussed in the previous section.

AXIOM 6. Suppose A, B, C, D are in $\mathscr{S}$ and $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$. Then

$$L(A, B, C, D) \leftrightarrow [d(A, 0)/d(0, B)] \leqslant [d(C, 0)/d(0, D)].$$

Axiom 6 is unambiguous because by Axiom 1, $d(A, 0) \neq 0$ or $d(0, B) \neq 0$ and similarly $d(C, 0) \neq 0$ or $d(0, D) \neq 0$.[4] This axiom allows us to determine the balance ordering $\preccurlyeq$ from the distance measure. It says that relative balance is dependent

[3] These signed digraphs have been called *vacuously balanced* by Cartwright and Harary (1956), who discuss some experimental work of Jordan (1953) which indicates that the vacuously balanced social structures turn out to have a degree of balance (more precisely, a degree of "pleasantness") essentially neutrally located between the balanced and the unbalanced ones.

[4] Note that the quotients are allowed to be infinite.

on how far the respective positive and negative semicycle sequences are from those of signed digraphs having no positive semicycles and having no negative semicycles. Specifically, it says that the relative balance is dependent on the ratio between these distances. To partly motivate the use of the ratio here, rather than, say, the difference, we note that we are assuming the following principle: if the distances from 0 of both the positive and negative semicycle sequences are doubled, then we would expect the balance to remain unchanged.

Formally, it would be nice to derive a balance ordering strictly from axioms on L, and not bring in the extra concept of distance. This we do in Section 4. We used the present approach here because we were motivated by the Kemeny–Snell approach to preference rankings and, more important, because we wanted to list some quite simple axioms and see where they led us. The problem of listing a more formally satisfying set of axioms was secondary. Thus, Section 4 contains the major mathematical results of this paper, and those of this section may be considered as motivation for interest in the theorems of Section 4.

2.3. *Uniqueness of Distance*

In the following, I_m is the sequence $(a_3, a_4, ..., a_i, ...)$ in $\mathscr{S}$ with $a_i = 0$ for $i \neq m$ and $a_m = 1$. The signed digraphs G and H of Fig. 3 have semicycle pairs (I_3, I_3) and $(I_4, 5I_3)$, respectively.

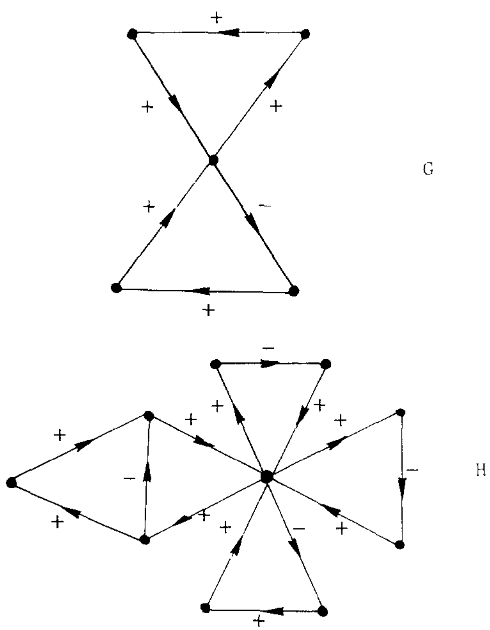

FIG. 3. G has semicycle pair (I_3, I_3). H has semicycle pair $(I_4, 5I_3)$.

THEOREM 1. *For any given positive real numbers* $f(3), f(4),..., f(m),...$, *there is a unique distance measure d on* $\mathscr{S}$ *satisfying Axioms* 1–4 *and having the property that for each integer* $m \geqslant 3$, $d(0, I_m) = f(m)$. *Specifically, if* $A = (a_3, a_4,..., a_m,...)$ *and* $B = (b_3, b_4,..., b_m,...)$, *then*

$$d(A, B) = \sum_{m=3}^{\infty} f(m) \mid b_m - a_m \mid. \tag{1}$$

Proof. We begin by showing that if there is such a measure d, it must satisfy (1). First, we prove that $d(0, sI_m) = sf(m)$ for any nonnegative integer s. This is true if $s = 0$, for we have $[0, 0, 0]$, and thus by Axiom 3, we conclude $d(0, 0) + d(0, 0) = d(0, 0)$; so $d(0, 0) = 0$. We complete the proof by induction on s. If $s > 0$, then $[0, (s-1)I_m, sI_m]$, and so by Axiom 3, $d(0, sI_m) = d(0, (s-1)I_m) + d((s-1)I_m, sI_m)$. By Axiom 4 and the definition of $f(m)$, the latter term is $d(0, I_m) = f(m)$, and so, using the induction hypothesis, $d(0, sI_m) = (s-1)f(m) + f(m) = sf(m)$.

To continue the proof, we define S_k to be that subset of $\mathscr{S}$ consisting of sequences $(a_3, a_4,..., a_m,...)$ in which only components 3 through k may differ from 0. We establish (1) by induction on k. If $k = 3$, and $A, B \in S_k$, then $A = a_3I_3$ and $B = b_3I_3$. If $b_3 \geqslant a_3$, then $B - A \in \mathscr{S}$ and so by Axiom 4, $d(A, B) = d(0, B - A) = d(0, (b_3 - a_3) I_3) = f(3) \mid b_3 - a_3 \mid$, establishing (1). The proof is similar if $b_3 \leqslant a_3$, and so (1) follows for $A, B \in S_3$.

Continuing with induction, we assume (1) holds for sequences in S_{k-1} and show it for sequences in S_k. If $A, B \in S_k$, define $C = (b_3, b_4,..., b_{k-1}, a_k, 0,...)$. Trivially, $[A, C, B]$, and so by Axiom 3, $d(A, B) = d(A, C) + d(C, B)$. But $d(C, B) = f(k) \mid b_k - a_k \mid$, since either $B - C$ or $C - B$ is in $\mathscr{S}$ and $d(C, B) = d(0, B - C) = d(0, (b_k - a_k) I_k)$ in the former case and $d(C, B) = d(C - B, 0) = d((a_k - b_k) I_k, 0)$ in the latter case. Finally, $d(A, C) = \sum_{m=3}^{k-1} f(m) \mid b_m - a_m \mid$. This follows by inductive assumption and Axiom 4, since $A = A' + a_kI_k$ and $C = B' + a_kI_k$, where $A' = (a_3, a_4,..., a_{k-1}, 0, 0,...)$ and $B' = (b_3, b_4,..., b_{k-1}, 0, 0,...)$.

We have now shown that if there is a distance measure satisfying our axioms, then it is the measure (1). The proof is completed by verifying that the measure (1) satisfies the axioms. We omit the details. Q.E.D.

Remark. We remarked earlier that our Axioms 1–4 for distance imply the triangle inequality, i.e., $d(A, C) \leqslant d(A, B) + d(B, C)$. This is clear from the representation for d given in Theorem 1. Moreover, it is clear from this representation that Axioms 1–4 also imply a stronger version of Axiom 4, namely that $d(A, C) = d(A, B) + d(B, C)$ if and only if $[A, B, C]$. Finally, analysis of the proof of Theorem 1 shows that the representation (1) may be obtained with arbitrary $f(m)$ if only Axioms 2–4 are assumed. Then Axiom 1 is equivalent to the positivity of f.

2.4. *Uniqueness of the Balance Ordering*

THEOREM 2. *Suppose L is a quaternary relation on $\mathscr{S}$ and suppose there is a distance measure d on $\mathscr{S}$ satisfying Axioms* 1–4 *and related to L by Axiom* 6. *Then*:

(a) *There are positive numbers $f(3), f(4),..., f(m),...$ so that for all A, B, C, D in $\mathscr{S}$, if $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$ then*

$$L(A, B, C, D) \leftrightarrow \frac{\sum_{m=3}^{\infty} f(m)\, a_m}{\sum_{m=3}^{\infty} f(m)\, b_m} \leqslant \frac{\sum_{m=3}^{\infty} f(m)\, c_m}{\sum_{m=3}^{\infty} f(m)\, d_m}, \tag{2}$$

where $A = (a_3, a_4, ..., a_m, ...)$, $B = (b_3, b_4, ..., b_m, ...)$, $C = (c_3, c_4, ..., c_m, ...)$ *and* $D = (d_3, d_4, ..., d_m, ...)$.

(b) *If $f(m)$ and $f'(m)$ are two positive functions satisfying* (2), *then there is a real number q so that $f(m) = qf'(m)$, all m.*

(c) *If d' is another distance measure on $\mathscr{S}$ satisfying Axioms* 1–4 *and related to L by Axiom* 6, *then there is a real number q so that for all $A, B \in \mathscr{S}$, $d(A, B) = qd'(A, B)$.*

(d) *L is determined uniquely by* (2), *given the numbers*

$$f(m) = \inf\{l/k: L(kI_m, lI_3, I_3, I_3)\}. \tag{3}$$

Proof. (a) Theorem 1.

(b) Using (2), given m, we have for all k, l,

$$f(m)/f(3) \leqslant l/k \leftrightarrow L(kI_m, lI_3, I_3, I_3) \leftrightarrow f'(m)/f'(3) \leqslant l/k.$$

Thus, $f(m)/f(3) = f'(m)/f'(3)$, $f(m) = [f(3)/f'(3)]\, f'(m)$.

(c) (b) and Theorem 1.

(d) By (a) there is a positive function $f'(m)$ so that

$$L(A, B, C, D) \leftrightarrow \frac{\sum_{m=3}^{\infty} f'(m)\, a_m}{\sum_{m=3}^{\infty} f'(m)\, b_m} \leqslant \frac{\sum_{m=3}^{\infty} f'(m)\, c_m}{\sum_{m=3}^{\infty} f'(m)\, d_m}. \tag{4}$$

It follows that $L(I_m, lI_3, I_3, I_3)$ for some $l > 0$, so $f(m)$ is well-defined.

In particular, we have $f'(m)/f'(3) \leqslant l/k$ iff $L(kI_m, lI_3, I_3, I_3)$. It is now sufficient to prove that $L(kI_m, lI_3, I_3, I_3)$ iff $f(m) \leqslant l/k$. For then it follows that $f(m) = f'(m)/f'(3)$, and so (2) follows from (4).

To complete the proof, we note first that if $L(kI_m, lI_3, I_3, I_3)$, then $f(m) \leqslant l/k$ follows by definition of f. Suppose now that $f(m) \leqslant l/k$. If $f(m) < l/k$, then by definition of f, there are l', k' so that $l'/k' < l/k$ and $L(k'I_m, l'I_3, I_3, I_3)$. Therefore $k'f'(m)/l'f'(3) \leqslant 1$, whence $kf'(m)/lf'(3) \leqslant 1$, and so $L(kI_m, lI_3, I_3, I_3)$. If $f(m) = l/k$, suppose not $L(kI_m, lI_3, I_3, I_3)$. Thus, $kf'(m)/lf'(3) > 1$. There are k', l' so that $l'/k' > l/k$ and $k'f'(m)/l'f'(3) > 1$. By definition of $f(m) = l/k$, we can find k'', l'' so that $l'/k' > l''/k'' \geqslant l/k$ and so that $L(k''I_m, l''I_3, I_3, I_3)$. But $k''f'(m)/l''f'(3) > k'f'(m)/l'f'(3) > 1$, contradicting $L(k''I_m, l''I_3, I_3, I_3)$. Q.E.D.

Remark. The function f satisfying (2) is monotone decreasing if and only if whenever $m \geqslant n$, we have $L(I_m, I_n, I_3, I_3)$.

In conclusion, suppose there is a distance measure d on $\mathscr{S}$ satisfying Axioms 1–4 and related to the balance ordering $\preccurlyeq$ by Axiom 6. Then $\preccurlyeq$ is completely determined. It is obtained by calculating the numbers

$$B(G) = \sum_{m=3}^{\infty} f(m)\, a_m \Big/ \sum_{m=3}^{\infty} f(m)\, b_m, \tag{5}$$

and using the relationship (2), where (A, B) is the semicycle pair and $f(m)$ is interpreted as a measure of the relative importance of a semicycle of length m. We shall have more to say on the choice of the parameter f in practice. We defer this discussion to Section 3.

2.5. *A Measure of Balance*

According to Theorem 2, the balance ordering is completely determined by the numbers $B(G)$ given by (5). It is now simple to define a measure of balance $b(G)$, which takes values between 0 and 1 and which gives rise to the balance ordering determined by (5). Any two measures giving rise to the same balance ordering are regarded as equivalent from our point of view. In particular, we simply take

$$b(G) = \sum_{m=3}^{\infty} f(m)\, a_m \Big/ \sum_{m=3}^{\infty} f(m)(a_m + b_m), \tag{6}$$

where G has semicycle pair (A, B). It is interesting to note that this is exactly the measure $\sum f(m)\, c_m^+ / \sum f(m)\, c_m$ discussed in Section 1. In particular, choice of $f \equiv 1$ leads to the measure c^+/c suggested by Cartwright and Harary (1956). Only the relative N-balance measure discussed in Section 1 must be redefined, since by Theorem 2, f is required to be positive. We simply take $f(m)$ to be 1 if $m \leqslant N$ and $f(m)$ arbitrarily small for $m > N$, where how small we take $f(m)$ depends on the length of the longest semicycle appearing in any of the signed digraphs under study.

2.6. *A Remark on Counting Semicycles*

It is necessary to make a remark about the counting of positive and negative semicycles. The signed digraph of Fig. 4a has two semicycles: e_1, e_2, e_3 and e_1, e_2, e_4.[5] If there are lines going in both directions (as from x_1 to x_3 and x_3 to x_1), and both

[5] It also has a semicycle e_3, e_4, but we are disregarding semicycles of length 2.

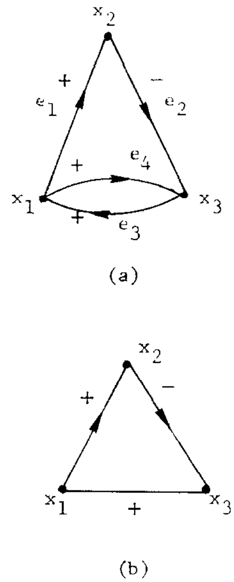

FIG. 4. Fig. 4(a) shows a signed digraph which is conveniently represented in Fig. 4(b) by drawing a nondirected positive line between points x_1 and x_3 in place of the two directed positive lines.

lines have the same sign, then the signed digraph can be conveniently represented by drawing nondirected lines between such points, as shown in Fig. 4b. In practice, in applications of balance theory, a signed digraph like this is sometimes considered to have only one (negative) semicycle rather than two. In general, if all the 2-way relations can be represented by nondirected lines with signs, rather than dealing with the number of positive and negative semicycles, users of balance theory frequently count the number of positive and negative *cycles* in the signed *graph* obtained by omitting all arrows.

Our balance ordering is based entirely on the semicycle pair. The previous discussion applies equally well to the situation where instead of the semicycle pair one is dealing with the sequence pair obtained by this modified method of counting semicycles. But it should be pointed out that the relative balance ordering obtained can vary considerably depending on the semicycle counting procedure used. It is up to the user to decide on one or the other.

Fortunately, there are two important special cases where we can use either semicycle counting procedure and get the same results. One is the case where the digraph is asymmetric. Here of course the issue does not arise. The second important case occurs when all the relationships are symmetric and symmetric pairs of lines have the same sign. Here we are dealing with a signed graph. (This situation might arise, for example, in the study of small groups, where each point of the group represents a person and each line a relationship such as "associates with.") Here, for any balance ordering obtained by a given choice of parameter f and with the strict semicycle counting procedure, there is a corresponding choice of parameter f' which gives the same balance ordering under the modified semicycle counting procedure, i.e., the cycle counting procedure. Specifically, since in a given cycle of length m there are 2^m semicycles, it is clear that $f'(m) = 2^m f(m)$ will do the job.

3. The Parameter f

In using the balance ordering derived in Section 2, it is first necessary to specify the function f, which is a measure of the effect of the length of a semicycle. In this section, we discuss very briefly the selection of an appropriate f.

The positivity of f alone is frequently sufficient to determine relative balance. That is, in computing balance, it often will follow that $\sum f(m)\, a_m / \sum f(m)(a_m + b_m)$ is greater than $\sum f(m)\, c_m / \sum f(m)(c_m + d_m)$ simply because $f(m) > 0$. In this case, the user of the measure need not select a particular f.

More often, it is possible to avoid specifying f exactly by just using several of its qualitative properties. For example, as we argued earlier, it is natural to assume that longer semicycles are not as important for balance (or imbalance) as shorter ones. Thus, we would expect that for all $m \geqslant n$, $L(I_m, I_n, I_3, I_3)$. It follows that f is monotone decreasing. Frequently this monotone decreasing property of f is sufficient to determine, on the basis of the formulas $\sum f(m)\, a_m / \sum f(m)(a_m + b_m)$, which of two signed digraphs is more balanced.

In general, we would hope to be able to provide the user of such a measure with a series of specific tests on f, a series of specific decisions as to which of two signed digraphs is more balanced, on the basis of which he will be able to choose a particular f. This is easy enough to do if we limit ourselves to a short list of plausible f's, such as $1/m$, $1/m^2$, $1/2^m$, $3/m$, $3/m^2$, $3/2^m$, $1/(m-2)$, $1/(m-2)^2$, $1/2^{m-2}$, etc. Then the representation (2) of Theorem 2 gives us obvious and simple tests to choose between any two of these which are not constant multiples of one another. For example, to choose between $1/(m-2)$ and $1/(m-2)^2$, we simply have to decide whether or not $L(3I_4, I_3, I_3, I_3)$. For in both cases $f(3) = 1$. And $L(3I_4, I_3, I_3, I_3)$ implies $f(4) \leqslant 1/3 < 1/2$, while $\sim L(3I_4, I_3, I_3, I_3)$ implies by (2) that $f(4) \geqslant 1/3 > 1/2^2$. On the other hand, there is no way of choosing between $1/m$ and $3/m$, for they both give rise to the same L via (2). The choice here should be based simply on ease of calculation. More details on choice of f using Theorem 2 are given in Norman–Roberts (1972), where an example is given showing that in certain circumstances taking $f(m) \equiv 1$ is unsatisfactory.

4. A Different Axiomatization of the Balance Ordering and a Characterization of Extensive Ratio Systems

From a formal point of view, it is a little unsatisfying to axiomatize the balance ordering by introducing the concept of distance. It would be nicer to state axioms purely in terms of the relation $\preccurlyeq$, or the relation L. Our reasons for not employing this approach were given in Section2.2. We now know where the axioms of Section 2 have led us, namely, to the representation (2) of Theorem 2. Now we want to state conditions strictly in terms of L or $\preccurlyeq$ which are necessary and sufficient for this representation.

4.1. *A Paradox Explained*

Before turning to this general problem, we note one plausible axiom on the balance ordering which is violated if f is monotone. If H and K are signed digraphs, let us denote by $H \oplus K$ the signed digraph which is the disjoint union of H and K. If $\preccurlyeq$

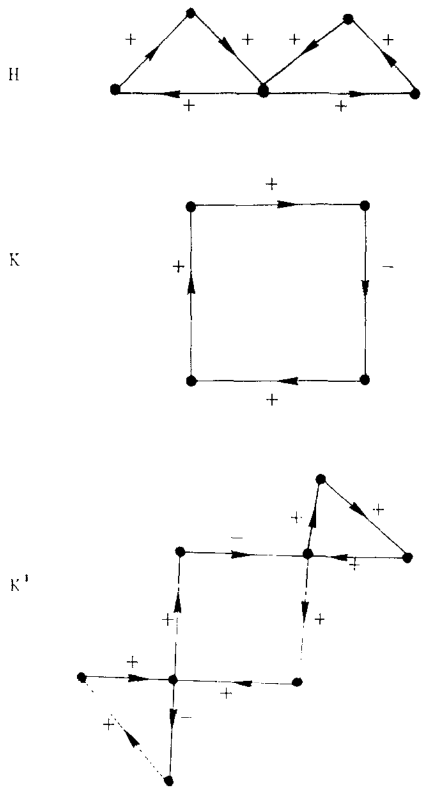

Fig. 5. Three signed digraphs H, K and K' such that $K \prec K'$ and $H \oplus K \succ H \oplus K'$.

denotes the relative balance ordering among signed digraphs in $\mathscr{G}$, it seems natural to assume that $H \oplus K \preccurlyeq H \oplus K'$ iff $K \preccurlyeq K'$. That is, if we augment a given social structure in two different ways, making sure none of the new elements has any relationship to any of the original ones, then we obtain more balance in that augmented structure in which we have added the more balanced structure. Unfortunately, this axiom is false. To give an example, let H, K, and K' be the signed digraphs shown in Fig. 5. Then using (5) of Section 2.4, we find that $K \prec K'$ and $H \oplus K \succ H \oplus K'$, where $\prec$ means $\preccurlyeq$ and not $\succcurlyeq$. For, $B(K) = 0 < f(3)/(f(3) + f(4)) = B(K')$, and $B(H \oplus K) = 2f(3)/f(4) = 6f(3)/3f(4)$, $B(H \oplus K') = 3f(3)/(f(3) + f(4)) = 6f(3)/(2f(3) + 2f(4))$, and $3f(4) < 4f(4) \leqslant 2f(3) + 2f(4)$, assuming that f is monotone.

This seeming paradox can be explained in the following way. The signed digraph K' is larger than, i.e., it has more semicycles than, the signed digraph K. Thus, although K' is less imbalanced than K, it plays a much more important role in determining the overall balance of $H \oplus K'$ than K does in $H \oplus K$. Thus, since K' is less balanced than H, addition of K' results in lower overall balance than does addition of K.

The following theorem shows that the desired axiom is true, if certain additional assumptions are made about K and K'. The proof is straightforward and is omitted.

THEOREM 3. *Under the Axioms of Section 2, suppose K and K' have the same number of semicycles of each length. Then for all H, $H \oplus K \preccurlyeq H \oplus K'$ iff $K \preccurlyeq K'$.*

4.2. *Axioms for Extensive Ratio Systems*

Our aim now is to characterize the representation (2) of Theorem 2 strictly in terms of the relation L. The characterization we obtain is a corollary of a more general result, which relates to a large body of work in the theory of measurement.

Suppose we start with an abstract set S (to be thought of as the set $\mathscr{S}$ above less the sequence all of whose terms are 0). And suppose we assume that on S there is given an operation o (to be thought of as addition of sequences) and a quaternary relation L. Let $\mathscr{R}^+$ denote the positive reals. To characterize the representation (2), we search for conditions on L and o necessary and sufficient for the existence of a function F: $S \to \mathscr{R}^+$ so that

$$L(A, B, C, D) \leftrightarrow F(A)/F(B) \leqslant F(C)/F(D), \tag{7}$$

and

$$F(AoB) = F(A) + F(B). \tag{8}$$

This problem is closely related to some problems studied in the literature of measurement theory. Finding a positive function F satisfying (7) is of course equivalent to finding any real-valued function G satisfying

$$L(A, B, C, D) \leftrightarrow G(A) - G(B) \leqslant G(C) - G(D). \tag{9}$$

This problem is important from the point of view of finding an additive utility function and is frequently called the problem of *utility difference measurement*. See Suppes and Zinnes (1963, p. 34) for a discussion of this interpretation. Conditions on (S, L) necessary and sufficient for the representation (9) alone have been given when S is finite by Scott (1964). Interesting sufficient conditions have been given for (9) if S is of arbitrary cardinality. Two sets of such conditions appear in Suppes and Zinnes (1963). To our knowledge, no axioms on (S, L) which are both necessary and sufficient for the existence of the representation (9) have been given without the restriction that S be finite.

Suppose we define from L the binary relation R on S by

$$ARB \leftrightarrow L(A, B, A, A). \tag{10}$$

Then it is easy to see that if F satisfies (7), it also preserves R, i.e., it satisfies

$$ARB \leftrightarrow F(A) \leqslant F(B). \tag{11}$$

Necessary and sufficient conditions on the system (S, R, o) for the existence of a (positive) function F satisfying (8) and (11) have been given in Roberts and Luce (1968) and in Holman (1969). An interesting set of sufficient conditions was given by Suppes (1951), and of course a very old set of sufficient conditions goes back to Hölder (1901). The representation {(8), (11)} is referred to as *extensive measurement*.

We shall show that in the presence of a very simple assumption, Axiom 8 below, there is a positive function F satisfying (7) and (8) if and only if there are functions F and G, F positive, so that G satisfies (9) and F satisfies (8) and (11). Thus, combining conditions necessary and sufficient for the existence of utility difference measurement and extensive measurement would give us conditions for the representation we desire. But unfortunately, no nice necessary and sufficient conditions are known for utility difference measurement without the restriction of finiteness, and so we have had to tackle our representation problem independently.

We state the needed axioms in the following definition, in which E denotes the quaternary relation on S given by

$$E(A, B, C, D) \leftrightarrow L(A, B, C, D) \,\&\, L(C, D, A, B).$$

If $n > 0$, nA is $Ao(Ao(Ao(\ldots A)))\ldots$, with n iterations.

DEFINITION. Suppose S is a set, L is a quaternary relation on S, and o is a binary operation on S. Then (S, L, o) is an *extensive ratio system* if the following axioms are satisfied for all $A, B, C, D, E, F, A', C' \in S$.

AXIOM 1 (connectedness). $L(A, B, C, D)$ or $L(C, D, A, B)$.

AXIOM 2 (transitivity). $L(A, B, C, D) \,\&\, L(C, D, E, F) \rightarrow L(A, B, E, F)$.

AXIOM 3 (reversal condition). $L(A, B, C, D) \rightarrow L(D, C, B, A)$.

AXIOM 4 (quadruple condition). $L(A, B, C, D) \rightarrow L(A, C, B, D)$.

AXIOM 5 (weak commutativity). $E(AoB, BoA, C, C)$.

AXIOM 6 (weak associativity). $E(Ao(BoC), (AoB)\,oC, D, D)$.

AXIOM 7 (positivity). $\sim L(C, C, A, AoB)$.

AXIOM 8 (additivity). $L(A, B, C, D) \,\&\, L(A', B, C', D) \rightarrow L(AoA', B, CoC', D)$.

AXIOM 9 (density of rationals). $\sim L(A, B, C, D) \rightarrow (\exists k, l > 0)(\sim L(A, B, kE, lE)$ & $\sim L(kE, lE, C, D))$.

Discussion of the Axioms. Axioms 1–4 are standard axioms for utility difference measurement. [Cf. Suppes and Zinnes (1963).] Axioms 5 and 6 say that the operation o is "weakly" commutative and "weakly" associative, in the sense that AoB and BoA, for example, though not equal, are equivalent for all our purposes. Axiom 7 reflects the fact that F is positive and Axiom 8 that F is additive. Axiom 9 says that between any two reals there is a rational.

Our aim is to prove the following theorem.

THEOREM 4. *If S is a set, L is a quaternary relation on S and o is a binary operation on S, then (S, L, o) is an extensive ratio system if and only if there is a function $F\colon S \rightarrow \mathscr{R}^+$ so that for all $A, B, C, D \in S$:*

$$L(A, B, C, D) \leftrightarrow F(A)/F(B) \leqslant F(C)/F(D) \tag{12}$$

and

$$F(AoB) = F(A) + F(B). \tag{13}$$

Before giving a proof of Theorem 4, we note that an alternative set of axioms which is sufficient (but not necessary) for the representation $\{(12), (13)\}$ can be obtained as a direct corollary of Theorem 2 of Marley (1968), which improves on a result of Luce (1965).[6] Marley's theorem combines the axioms of Luce and Tukey (1964) for conjoint measurement and those of Suppes (1951) for extensive measurement, and adds an additional axiom called Axiom C–E'.

To explain the relation of Marley's result to the desired representation, we use the notation of Luce (1965) and Marley (1968). We assume that their two sets A_1 and A_2 are identical, and both equal our set S, and we assume $o_1 = o_2 = o$. The binary relation R used by Luce and Marley corresponds to the converse of the relation $\preccurlyeq$ defined by $(A, B) \preccurlyeq (C, D) \leftrightarrow L(A, B, C, D)$. Luce and Marley's binary

[6] We are indebted to the referee for pointing out this result.

relation I corresponds to the quaternary relation E defined above. And their binary relations R_1 and R_2 are the relations R' and R respectively, where R is defined by (10) and R' is the converse of R. We take $R_1{}^* = R_1$ and $R_2{}^* = R_2{}'$.

The conjoint measurement Axioms C_1–C_4 and the extensive measurement Axioms $E1$–$E6$ used by Luce and Marley are easily restated in terms of L. Part (ii) of Axiom C–E' is needed in the special form where $m = n = 1$, from which it of course follows. We restate this special form as Axiom C–E''.

AXIOM C–E'': For all $A, B \in S$ and $i, j > 0$, $E(iA, iB, jA, jB)$.

It follows from the discussion in Luce (1965) and Marley (1968) that, if (S, L, o) satisfies Axioms $C1$–$C4$, $E1$–$E6$, and C–E'', there are positive extensive measures ν_ρ on A_ρ, i.e., positive additive measures preserving $R_\rho{}^*$. Since $R_2{}^* = R_1{}^*$, we may take $\nu_2 = \nu_1$. Also, by the discussion in Marley (1968), there are positive conjoint measures μ_ρ preserving R_ρ and so that

$$L(A, B, C, D) \leftrightarrow \mu_1(A)\,\mu_2(B) \leqslant \mu_1(C)\,\mu_2(D). \tag{14}$$

By Theorem 2 of Marley, it now follows that for some constants β_ρ and λ_ρ, we have

$$\mu_1 = \beta_1 \nu_1^{\lambda_1}, \qquad \mu_2 = \beta_2 \nu_2^{\lambda_2} \qquad \text{and} \qquad |\lambda_1| = |\lambda_2|.$$

We note that $\lambda_1 \geqslant 0$. For, by Axiom $E6$, $AoAP_1{}^*A$. Thus $AoAR_1A$ or $(AoA, X)\,R(A, X)$, some X, or $L(A, X, AoA, X)$. This implies $\beta_1\beta_2\nu_1(A)^{\lambda_1}\,\nu_2(X)^{\lambda_2} \leqslant \beta_1\beta_2[2\nu_1(A)]^{\lambda_1}\,\nu_2(X)^{\lambda_2}$, whence, since $\nu_1, \nu_2 > 0$, $1 \leqslant 2^{\lambda_1}$, $\lambda_1 \geqslant 0$.

Next, note that $\lambda_1 = -\lambda_2$. For, suppose $\lambda_1 = \lambda_2$. Then $\lambda_2 \geqslant 0$, and so

$$\beta_1\beta_2\nu_1(X)^{\lambda_1}\,\nu_2(A)^{\lambda_2} \leqslant \beta_1\beta_2\nu_1(X)^{\lambda_1}[2\nu_2(A)]^{\lambda_2},$$

whence $L(X, A, X, AoA)$, or $(X, AoA)\,R(X, A)$, or $AoAR_2A$. This implies AR_1AoA. Since also $AoAR_1A$, we have $\sim AoAP_1{}^*A$, in violation of Axiom $E6$.

Finally, defining $F(A) = \nu_1(A)$, we obtain a positive function satisfying (12) and (13). For, ν_1 is additive and

$$L(A, B, C, D) \leftrightarrow \beta_1\beta_2 \frac{\nu_1(A)^{\lambda_1}}{\nu_1(B)^{\lambda_1}} \leqslant \beta_1\beta_2 \frac{\nu_1(C)^{\lambda_1}}{\nu_1(D)^{\lambda_1}} \leftrightarrow \frac{F(A)}{F(B)} \leqslant \frac{F(C)}{F(D)}.$$

Thus we have shown

THEOREM 5. *If (S, L, o) satisfies Axioms $C1$–$C4$ and $E1$–$E6$ of Luce and Marley and Axiom C–E'', then there is a function F: $S \to \mathscr{R}^+$ satisfying* (12) *and* (13).

Turning to the proof of Theorem 4, we note that all of the axioms for an extensive ratio system are clearly necessary for the existence of a positive F satisfying (12) and (13). We shall not be able to include all the details of a proof of their sufficiency, but instead, occasionally give only an outline. We begin by summarizing in the following lemma some of the basic properties of L used often in the proof. Part (c) especially will usually be tacitly assumed, allowing us for example to operate as if o were commutative and associative, etc.

LEMMA 1. (a) $E(A, A, B, B)$.

(b) $L(A, B, X, Y) \& L(B, C, Y, Z) \rightarrow L(A, C, X, Z)$.

(c) *If* $E(A, B, C, C)$, *then*

$$L(A, X, Y, Z) \leftrightarrow L(B, X, Y, Z)$$

and

$$L(X, A, Y, Z) \leftrightarrow L(X, B, Y, Z).$$

(d) $E(A, B, kA, kB)$ *for all positive integers* k.

(e) $L(kA, lA, mA, nA) \leftrightarrow k/l \leqslant m/n$.

Proof. (a) Note either $L(A, A, B, B)$ or $L(B, B, A, A)$ by Axiom 1, and apply Axiom 3.

(b) Axiom 4 implies $L(A, X, B, Y)$ and $L(B, Y, C, Z)$. Thus $L(A, X, C, Z)$ by Axiom 2 and $L(A, C, X, Z)$ by Axiom 4.

(c) $E(B, A, Y, Y)$ follows easily from $E(A, B, C, C)$. Then $L(B, A, Y, Y)$ $\& L(A, X, Y, Z)$ imply $L(B, X, Y, Z)$, by (b). The rest of the proof is analogous.

(d) One establishes $L(kA, A, kB, B)$ by induction on k, since by Axiom 8, $L(A, A, B, B)$ and $L((k-1)A, A, (k-1)B, B)$ imply $L(kA, A, kB, B)$. $L(kB, B, kA, A)$ follows similarly.

(e) Suppose $L(kA, lA, mA, nA)$. By (d), $L(knA, lnA, lmA, lnA)$, so $L(lnA, lnA, lmA, knA)$ by Axioms 4 and 3. If $k/l > m/n$, then Axiom 7 is violated. So $k/l \leqslant m/n$. Conversely, if $\sim L(kA, lA, mA, nA)$, then $L(mA, nA, kA, lA)$, so $m/n \leqslant k/l$. If $m/n = k/l$, then $kn = lm$. It follows by (a) that $L(knA, lmA, lnA, lnA)$, whence $L(knA, lnA, lmA, lnA)$ by Axiom 4. Now part (d) implies $L(kA, lA, mA, nA)$. Q.E.D.

Remark. It should be noted that Lemma 1(d) is essentially equivalent to Axiom C–E'', the special axiom added to those for conjoint measurement and extensive measurement to obtain the representation {(12), (13)}.[7] The proof shows that under standard Axioms 1–6, Axiom C–E'' follows from Axiom 8 (additivity). It is interesting to note that Axiom 8 is exactly the axiom which must be added to those of utility difference measurement and extensive measurement to obtain the representation {(12), (13)}, as will be shown in Theorem 6(d) below.

[7] We are indebted to the referee for making this observation.

LEMMA 2. *The binary relation R defined by*

$$ARB \leftrightarrow L(A, B, A, A)$$

is a weak order.

Proof. Note that by Lemma 1(a), $ARB \leftrightarrow (\forall X \in S)[L(A, B, X, X)]$. Reflexivity and connectedness of R are trivial. Transitivity follows by Lemma 1(b), since $L(A, B, X, X) \,\&\, L(B, C, X, X) \rightarrow L(A, C, X, X)$. Q.E.D.

LEMMA 3 (Archimedean condition). *For all A, B, C, D in S, there is $n > 0$ such that $\sim L(A, B, C, nD)$.*

Proof. By Axiom 7 (positivity), $\sim L(A, B, A, BoB)$. By Axiom 9, there are $k, l > 0$ so that $\sim L(A, B, kC, lC)$. Choose $m > 0$ so that $k/l > 1/m$. It follows by Lemma 1(e) and Axiom 2 that $\sim L(A, B, C, mC)$. We have thus verified the sublemma that for all A, B, C there is $m > 0$ such that $\sim L(A, B, C, mC)$.

Applying this result to D, C, C, we find that for some $m > 0$, $\sim L(D, C, C, mC)$. Thus, by Axioms 1 and 3, $L(C, D, mC, C)$. By Lemma 1(d), we conclude $L(mC, mD, mC, C)$, or, using Axioms 3 and 4, $L(C, mD, mC, mC)$.

Applying the sublemma to A, B, D, we find there is $m' > 0$ so that $\sim L(A, B, D, m'D)$. We conclude that $\sim L(A, B, C, mm'D)$. For suppose $L(A, B, C, mm'D)$. We also have $L(C, mm'D, mD, mm'D)$, since $L(C, mD, mC, mC)$ and $E(mC, mC, mm'D, mm'D)$. And we have $L(mD, mm'D, D, m'D)$, by Lemma 1(d). By Transitivity (Axiom 2), we conclude $L(A, B, D, m'D)$, which is a contradiction. Q.E.D.

LEMMA 4. *There is a function $F\colon S \rightarrow \mathscr{R}^+$ which is additive and preserves R, i.e., which satisfies* (13) *and* (11).

Proof. By Roberts–Luce (1968, Theorem 3), the following conditions on (S, R, o) are sufficient for the existence of such a function.

(i) R is a weak order.
(ii) $AoB\ R\ BoA$ and $BoA\ R\ AoB$; $Ao(BoC)\ R\ (AoB)oC$ and $(AoB)oC\ R\ Ao(BoC)$.
(iii) $ARB \leftrightarrow AoC\ R\ BoC$.
(iv) $\sim BRA \rightarrow (\forall C, D \in S)(\exists n > 0)(nAoC\ R\ nBoD)$.
(v) $\sim (AoB)\ R\ A$.

These conditions all follow from the axioms for extensive ratio systems. Condition (i) is Lemma 2. Condition (ii) is essentially Weak Commutativity and Weak Associativity (Axioms 5 and 6). Condition (v) follows from Positivity (Axiom 7). The implication $ARB \rightarrow AoC\ R\ BoC$ of Condition (iii) follows by Axioms 4 and 8, using $L(A, B, X, X) \,\&\, L(C, C, X, X)$. The reverse implication follows from the other conditions. For, suppose $AoC\ R\ BoC$ and $\sim ARB$. By Condition (iv), there is $n' > 0$ so that $n'Bo(BoC)\ R\ n'AoA$. Letting $n = n' + 1$, we have $nBoC\ R\ nA$. By the already proved direction of Condition (iii), $nBonCoC\, R\, nAonC$. Now, $AoC\, R\, BoC$ implies $nAonC\, R\, nBonC$, as can be seen by applying Lemma 1(d) to $L(AoC, BoC, E, E)$. We conclude from Condition (i) that $nBonCoC\, R\, nBonC$, violating Condition (v). It remains to establish Condition (iv). If $\sim BRA$, then for any E, $\sim L(B, A, E, E)$, and so by Axiom 9 and Lemma 1(e), there are $k, l > 0$ so that $\sim L(B, A, kE, lE)$ and $k/l > 1$. Thus, by Axioms 1 and 3, $L(A, B, lE, kE)$. By Lemma 3, there is $n > 0$ so that $L(C, nB, E, kE)$. By Lemma 1(d), $L(nA, nB, lE, kE)$. Applying Axiom 8, we find $L(nAoC, nB, (l + 1)\, E, kE)$. By Axiom 7, $L(nAoC, nBoD, nAoC, nB)$, and by Lemma 1(e), $L((l + 1)\, E, kE, E, E)$. Applying Transitivity (Axiom 2), we conclude $L(nAoC, nBoD, E, E)$, as desired. Q.E.D.

Remark. A system (S, R, o) satisfying Conditions (i)–(v) is called by Roberts-Luce (1968) a *positive extensive system.*

LEMMA 5. *Suppose there is a function G: $S \to \mathscr{R}^+$ satisfying*

$$L(A, B, C, D) \leftrightarrow G(A)/G(B) \leqslant G(C)/G(D). \tag{15}$$

Then any function F: $S \to \mathscr{R}^+$ which is additive and preserves R, i.e., satisfies (13) *and* (11), *also satisfies* (12).

Proof. Fix $I \in S$. Since any positive multiple of G also satisfies (15), we may assume $G(I) = 1$. Now by Lemma 1(d), $E(kA, A, kB, B)$ holds for all A, B. Thus for all A, B, $G(kA)/G(A) = G(kB)/G(B)$. We shall denote this common ratio by $\lambda(k)$. Note first that λ is multiplicative, i.e., $\lambda(kl) = \lambda(k)\,\lambda(l)$. For, $G(klA)/G(A) = G(klA)/G(lA) \cdot G(lA)/G(A)$. λ is also strictly increasing, for we have $\lambda(k) \leqslant \lambda(l) \leftrightarrow G(kI) \leqslant G(lI) \leftrightarrow L(kI, lI, I, I) \leftrightarrow kI\, R\, lI \leftrightarrow F(kI) \leqslant F(lI) \leftrightarrow k \leqslant l$. Since λ is multiplicative, we may extend λ to the rationals by defining $\lambda(k/l) = \lambda(k)/\lambda(l)$. It follows by Lemma 4 of Luce (1965) that there is a positive constant c so that $\lambda(k) = k^c$, all k.

Now, we have for all k, l: $F(A)/F(I) \leqslant l/k \leftrightarrow kA\, R\, lI \leftrightarrow L(kA, lI, I, I) \leftrightarrow G(A)/G(I) \leqslant \lambda(l)/\lambda(k) \leftrightarrow [G(A)]^{1/c} \leqslant l/k$. It follows that $F(A)/F(I) = [G(A)]^{1/c}$. Finally, we have $F(A)/F(B) \leqslant F(C)/F(D) \leftrightarrow [G(A)]^{1/c}/[G(B)]^{1/c} \leqslant [G(C)]^{1/c}/[G(D)]^{1/c} \leftrightarrow G(A)/G(B) \leqslant G(C)/G(D) \leftrightarrow L(A, B, C, D)$. Thus, F satisfies (12). Q.E.D.

By Lemmas 4 and 5, the proof of Theorem 4 will be complete if we can find a function G: $S \to \mathscr{R}^+$ satisfying (15). We can find such a G if there is a function b: $S \times S \to \mathscr{R}^+$ so that for all $A, B, C, D \in S$:

$$L(A, B, C, D) \leftrightarrow b(A, B) \leqslant b(C, D) \tag{16}$$

and

$$b(A, B) = 1/b(B, A) \tag{17}$$

and

$$b(A, C) = b(A, B)\, b(B, C). \tag{18}$$

For, fix I in S and simply define $G(A) = b(A, I)$. G is positive because b is. G satisfies (15) because b satisfies (16) and by (17) and (18), $G(A)/G(B) = b(A, I)/b(B, I) = b(A, I)\, b(I, B) = b(A, B)$.

We shall now define a positive function b satisfying (16)–(18), thus completing the proof of Theorem 4. Fix I in S. We define $b(A, B)$ by

$$b(A, B) = \inf\{k/l : L(A, B, kI, lI)\}.$$

This inf exists because by the Archimedean Condition (Lemma 3), $L(A, B, nI, I)$ holds for some n.

LEMMA 6. (a) $L(A, B, kI, lI) \leftrightarrow b(A, B) \leqslant k/l$.

(b) $L(kI, lI, A, B) \leftrightarrow b(A, B) \geqslant k/l$.

(c) $b(A, B) > 0$.

(d) $b(A, B) = 1/b(B, A)$.

Proof. (a) One direction follows by definition. To complete the proof, suppose $b(A, B) \leqslant k/l$. Then there are k', l' so that $k'/l' \leqslant k/l$ and $L(A, B, k'I, l'I)$. Now by Lemma 1(e), $L(k'I, l'I, kI, lI)$, so we have $L(A, B, kI, lI)$ by Axiom 2.

(b) Suppose $L(kI, lI, A, B)$. By Lemma 1(e), if $m/n < k/l$, $\sim L(kI, lI, mI, nI)$, so by Axiom 2, $\sim L(A, B, mI, nI)$. This implies $b(A, B) \geqslant k/l$. Suppose next $\sim L(kI, lI, A, B)$. By Axiom 9, there are k', l' so that $\sim L(k'I, l'I, A, B)$ and $\sim L(kI, lI, k'I, l'I)$. Thus, by Lemma 1(e), $k'/l' < k/l$. Since $L(A, B, k'I, l'I)$, this shows $b(A, B) < k/l$.

(c) By the Archimedean Condition (Lemma (3)) there is an n so that $\sim L(A, B, I, nI)$.

(d) $b(A, B) \leqslant k/l \leftrightarrow L(A, B, kI, lI) \leftrightarrow L(lI, kI, B, A) \leftrightarrow b(B, A) \geqslant l/k \leftrightarrow 1/b(B, A) \leqslant k/l$. Q.E.D.

Lemma 6 shows property (17) holds for b. To verify property (18) for b, we note that if $L(A, B, kI, lI)$ and $L(B, C, mI, nI)$, then by Lemma 1(d) and Axiom 2, $L(A, B, kmI, lmI)$ and $L(B, C, lmI, lnI)$. By Lemma 1(b), $L(A, C, kmI, lnI)$. Picking k/l and m/n arbitrarily close to $b(A, B)$ and $b(B, C)$, we find $b(A, C) \leqslant b(A, B)\, b(B, C)$. A similar argument establishes $b(A, C) \geqslant b(A, B)\, b(B, C)$.

We are now ready to prove (16), i.e.,

$$L(A, B, C, D) \leftrightarrow b(A, B) \leqslant b(C, D).$$

One direction of the equivalence is trivial, since if $L(A, B, C, D)$, then $L(C, D, kI, lI) \rightarrow L(A, B, kI, lI)$. To prove the converse, suppose $\sim L(A, B, C, D)$.

Choose k, l as in Axiom 9, with I replacing E. By Lemma 6, we conclude $b(A, B) > k/l > b(C, D)$. This completes the proof of Theorem 4.

There are some interesting corollaries of the proof which we summarize in the following theorem, along with a uniqueness result.

THEOREM 6. *Suppose S is a set, L is a quaternary relation on S, o is a binary operation on S, and R is a binary relation on S defined from L by* (10). *Then the following hold:*

(a) *If F and F' are two positive functions on S satisfying* (12) *and* (13), *then there is a positive number q so that for all A, $F(A) = qF'(A)$.*

(b) *If there is a positive function F on S satisfying* (12) *and* (13), *then one such function is given by*

$$F(A) = \inf\{k/l : L(A, I, kI, lI)\} = \inf\{k/l : L(lA, kI, I, I)\}.$$

(c) *Suppose there is a positive function G on S satisfying* (15) *and a positive function F on S which is additive and preserves R, i.e., satisfies* (13) *and* (11). *Suppose also that Axiom* 8 (*Additivity*) *holds. Then F also satisfies* (12).

(d) *The following conditions on (S, L, o) are necessary and sufficient for the existence of a positive F on S satisfying* (12) *and* (13).

(i) *There is a positive function G on S satisfying* (15).

(ii) *(S, R, o) is a positive extensive system, i.e., satisfies Conditions* (i)–(v) *in Lemma* 4.

(iii) *(S, L, o) satisfies Additivity* (*Axiom* 8).

Proof. (a) F and F' both satisfy (11). The result now follows from Theorem 3 of Roberts–Luce (1968).

(b) Since F exists, Axioms 1–9 hold. Choose such an F and without loss of generality take $F(I) = 1$. Define

$$G(A) = \inf\{k/l : L(A, I, kI, lI)\}.$$

It is a direct corollary of the proof of Theorem 4 that G satisfies (15). By the proof of Lemma 5, since F satisfies (11) and (13), $F(A) = F(I) \cdot [G(A)]^{1/c} = [G(A)]^{1/c}$. Now $G(mI) = m$, since by Lemma 1(e), $L(mI, I, kI, lI) \leftrightarrow m \leqslant k/l$. Thus, to use the notation of Lemma 5, $\lambda(m) = G(mI) = m$, and $c = 1$. Thus, $F(A) = G(A)$ satisfies (12) and (13).

(c) Analysis of the proof of Lemma 5 shows that (12) for F is proved using (11) and (13) for F and (15) for G and Axioms 1–8. Axioms 1–7 all follow from (11) and (13) for F and (15) for G, and positivity of F and G.

(d) Use (c) plus Theorem 3 of Roberts–Luce (1968), which says that (S, R, o) is a positive extensive system if and only if there is a positive F satisfying (11) and (13).

Q.E.D.

4.3. *Axioms for the Balance Ordering*

We now state for completeness the specialization of Theorem 4 to the case of the balance ordering. Here o is addition of sequences, a commutative and associative operation, so Axioms 5 and 6 hold in the presence of the other axioms, by Lemma 1(a). We have:

THEOREM 7. *Suppose $\mathscr{S}$ is the set of all infinite sequences of nonnegative integers with only finitely many nonzero terms, suppose L is a quaternary relation on $\mathscr{S}$, and suppose o is the operation of termwise addition. Let S be $\mathscr{S}$ less the sequence all of whose terms are zero. Then the following conditions are necessary and sufficient for there to be positive numbers $f(m)$ so that if $A, B, C, D \in \mathscr{S}$ and $(A, B) \neq (0, 0)$ and $(C, D) \neq (0, 0)$, then*

$$L(A, B, C, D) \leftrightarrow \frac{\sum_{m=3}^{\infty} f(m)\, a_m}{\sum_{m=3}^{\infty} f(m)\, b_m} \leqslant \frac{\sum_{m=3}^{\infty} f(m)\, c_m}{\sum_{m=3}^{\infty} f(m)\, d_m},$$

where $A = (a_3, a_4, ..., a_m, ...)$, $B = (b_3, b_4, ..., b_m, ...)$, $C = (c_3, c_4, ..., c_m, ...)$ *and* $D = (d_3, d_4, ..., d_m, ...)$.

(i) *(S, L, o) satisfies Axioms 1–4 and 7–9.*

(ii) *If $A, B, C \in S$, then $E(A, 0, B, 0)$ and $E(0, A, 0, B)$ and $\sim L(C, 0, A, B)$ and $\sim L(A, B, 0, C)$.*

Moreover, the function $f(m)$ is monotone decreasing if and only if whenever $m \geqslant n$, we have $L(I_m, I_n, I_3, I_3)$.

Remark. Theorem 2 now follows as a corollary of Theorems 1, 6, and 7.

5. LOCAL BALANCE

In practice we frequently have in the signed digraph a distinguished point P. In that case, a possibly more useful measure is the so-called "local balance" at P, introduced by Harary (1955). Here, the procedure is the same as described above, except that every time the word "semicycle" is mentioned, it should be replaced by the words "semicycle through P." Thus, let $a_{m,P}$ = the number of balanced semicycles of length m through P, $b_{m,P}$ = the number of unbalanced semicycles of length m through P. Then, following Section 2.5, the local balance at P, b_P, is given by

$$\sum_{m=3}^{\infty} f(m)\, a_{m,P} \Big/ \sum_{m=3}^{\infty} f(m)[a_{m,P} + b_{m,P}].$$

It also seems tempting to speak of different degrees of local balance. Thus, a second degree local balance measure at P might be obtained by some sort of (weighted) average of the local balance at points of distance at most one from P. Higher degree local balance measures might be obtained by taking points of distance at most N, for some N. These suggestions are of course ad hoc, and the whole concept of different degrees of local balance, if it turns out to be valuable, should probably be put on a firmer axiomatic foundation.

A much more difficult situation arises if we have no distinguished point P. It still may be useful to measure local balance. If the computations are not too involved, it may be useful to compute not only global balance, but also local balance at all the points (and possibly several degrees of local balance at each point). Investigation of all this data might reveal that a particular point is fairly crucial.

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RECEIVED: September 25, 1970