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The friend of my enemy is my enemy, the enemy of my enemy is my friend: Axioms for structural balance and bi-polarity

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ABSTRACT

Structural balance is a simple equilibrium model of positive and negative relationships, such as friendship and enmity. Some relational patterns (e.g. friends sharing an enemy) are balanced; others (e.g. enemies sharing a friend) are not. The model has tested well in a variety of applications, from the social psychology of small groups to the politics of international conflict. Several versions are at least implicit in the literature but had not previously been identified. Here I frame each as an axiom system and prove its equivalence to a version of *bi-polarity*, the idea that interactive subjects can always be partitioned into two opposing sides.

Those lovely old aphorisms in the title capture the idea of *structural balance*, an equilibrium model of "positive" and "negative" relationships, such as friendship and enmity. Certain relational patterns are balanced, others not. Only the balanced ones are expected to emerge and endure. For example, two friends might share a friend or share an enemy, but it would be hard for one friend to befriend the other's enemy: only the first two patterns are balanced. In each of several versions, structural balance has the notable consequence that any set of interactive subjects can be partitioned into two opposing sides.

The theory of structural balance has migrated from cognitive and social psychology (Heider, 1946, 1958; Taylor, 1970), where it has fared remarkably well in experimental tests (Davis, 1967; Morrissette, 1958), to international relations (Harary, 1961; Guetzkow, 1957; McDonald and Rosecrance, 1985), where it has also tested well (Kim, 2007), beating rivals as a predictive formulation of the balance of power (Healy and Stein, 1973). Heider (1946) first presented the model as an exhaustive list of balanced and unbalanced triads. Cartwright and Harary (1956) then generalized that list to a condition on signed graphs: they are balanced when no cycles therein have odd numbers of negative relationships. Later mathematical work has used those two versions of the model to study degrees of imbalance and the dynamics of balancing, in both cases emphasizing computational problems (Abell and Ludwig, 2009; Hummon and Doreian, 2003; Kulakowski, 2007; Notsu et al., 2006). Implicit in the literature, however, are several other versions of structural balance, but they have not previously been identified as such, much less distinguished from one another.

Here I frame all these versions as axiom systems, bring out their logical relationships, and show how each amounts to a version of two-sidedness, or bi-polarity. For example, as Cartwright and Harary (1956) show, their formulation of structural balance partitions the universe in two so that friends are always on the same side, enemies on opposite sides. This is a weak form of two-sidedness, for the partition does not have to be unique: there can be more than one way to draw the line between sides. By contrast, and without mentioning structural balance, Lave and March (1975, p. 67) proffer an axiom system consisting of my two titular aphorisms and two more, all unattributed by them but customarily credited to Arab folklore, plus an axiom of Completeness: every pair of subjects are directly related. They deduce that the universe is strictly bi-polar, uniquely partitioned in two so that any two subjects on the same side are friends, on opposite sides enemies. Again, if we drop the highly restrictive axiom of Completeness, that bi-polarity consequence no longer follows, but this weaker one does: the universe can be partitioned into one or more subuniverses so that no two are related at all but internally each of them is strictly bi-polar. And there are further, subtler, in some ways more realistic variations. Altogether we shall examine seven axiom systems,

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forming four equivalence groups, along with five versions of bi-polarity.

1. The Cartwright–Harary system

With one exception the axiom systems below have two primitive binary relations, F and E, on a nonempty set S, its members denoted x, y, z, etc. Cartwright and Harary speak of a signed graph rather than a pair of relations, but the difference is stylistic. They also assume finitude, but needlessly so. To motivate and interpret axioms and consequences, think of F (for friendship) as positive, E (for enmity) as negative, and S (for subjects) as comprising organisms or organizations of any sort (though Heider also allowed some inanimate objects of people's attitudes).

Plausible on their face and implicit in most graph formulations are five *background assumptions*:

Reflexivity of F in S (*F-rflx*). xFx.

Symmetry of F (*F-sym*). $xFy \Rightarrow yFx$.

Symmetry of E (*E-sym*). $xEy \Rightarrow yEx$.

Irreflexivity of E (*E-irrflx*). Not xEx.

Incompatibility of F and E (*FincE*). $xFy \Rightarrow$ not xEy.

These I include in every axiom set unless they follow from other axioms. *FincE* always does. *F-rflx* and *F&E-sym* (as I shall refer to the two symmetry assumptions) rarely do.

The Cartwright and Harary condition of structural balance says that no cycle of positive and negative relationships has an odd number of negative ones. Define the *sign*, $\sigma(x_1, \ldots, x_n)$, of any finite sequence of elements of S:

$$\sigma(x) = 1$$

$$\sigma(x_1, \ldots, x_k, x_{k+1}) = \begin{cases} \sigma(x_1, \ldots, x_k) & \text{if } x_k F x_{k+1} \\ -\sigma(x_1, \ldots, x_k) & \text{if } x_k E x_{k+1} \\ 0 & \text{if neither.} \end{cases}$$

Obviously a sequence represents a path of positive and/or negative relationships if its sign is 1 or -1, a cyclic path if its first element is the same as its last, and one with an odd number of negative (E) relationships if its sign is -1. So the Cartwright–Harary condition amounts to:

Structural Balance (*SB*). $\sigma(x_1, \ldots, x_n) = -1 \Rightarrow x_1 \neq x_n$.

And their implicit axiom set is

{*SB*, *F-rflx*, *F&E-sym*}.

E-irrflx follows, for $xEx \Rightarrow \sigma(x, x) = -1$, contrary to *SB*. Likewise *FincE*, for xFy & $xEy \Rightarrow xFyEx$ (by *E-sym*) $\Rightarrow \sigma(x, y, x) = -1$, again contrary to *SB*. But *F-rflx*, *F-sym*, and *E-sym* are each independent of the other axioms. Appendix B shows that every member of every axiom set is thus independent.

One might challenge *F-rflx*, *F-sym*, or *E-sym* by contending that self-friendship flouts conventional usage, or that friendship and enmity are not always reciprocated (a possibility entertained by Harary, 1957). However, suppose we drop those assumptions but define:

$$xF^*y \Leftrightarrow x = y \text{ or } xFy \text{ or } yFx,$$

$$xE^*y \Leftrightarrow xEy \text{ or } yEx,$$

and replace F and E by F^* and E^* in the definition of σ. Then *SB* is still plausible, and F^* is reflexive and F^* and E^* symmetric by definition. Those who object to *F-rflx* or *F&E-sym* can simply substitute F^* for F and E^* for E and find nothing wrong in what follows. Meanwhile the rest of us are spared a proliferation of systems that accommodate variations in minor assumptions.

2. Weak bi-polarity equivalents

Cartwright and Harary's chief finding is that structural balance is tantamount to a version of two-sidedness, unnamed by them, which I shall christen:

> *Weak Dichotomy* (*WkDi*). S has a partition into two subsets such that, for all x and y, xFy only if x and y belong to the same subset whereas xEy only if they belong to different subsets.

Possibly one of those subsets is empty; then xEy never holds, and *WkDi* is vacuously true.

All wars are two-sided, says Trager's Law (after geopolitical theorist Robert Trager, personal communication). One might extend it to systems of alliances. Maybe there are exceptions, but they are few enough to make it worth recasting the "law" in more exact language. *WkDi* falls short because the partition need not be unique: we cannot always tell from F and E who belongs with whom on which side. Say S has five members, x, y, z, w, and v, and *F-rflx* aside the only F relationship is between x and y (xFy and, therefore, yFx) and the only E relationship between z and w. Then one partition fulfilling *WkDi* is $\{\{x, y, w\}, \{z, v\}\}$. But another is $\{\{x, y, z, v\}, \{w\}\}$, and there are two more.

By contrast, look at the two world wars. In each, some countries were neutral, and neutrals in one conflict might have their own conflicts and alliances. But among the combatants there was a unique partition into two opposing sides, this despite the fact that some pairs of combatants were not directly related as friends or enemies. In WWI the US and Ottoman Turkey were neither friends nor enemies, neither allied nor at war. Yet they were unambiguously on opposite sides. In WWII Peru and Finland were clearly on opposite sides but never fought each other. Added to *SB*, a transitivity axiom considered in Section 3 implies that the US and Turkey (or Peru and Finland) were enemies of a sort, for the US was allied to Turkey's enemy, Britain. But *SB* alone implies only that they were not friends.

To capture the two-sidedness of the world wars, let A be any nonempty subset of S, and define:

> A *path* from x to y is any sequence of members of S that begins with x and ends with y and has a sign of 1 or -1.
>
> A is *path connected* if and only if, for all x, y in A, there exists a path from x to y that consists solely of members of A.
>
> A is *bi-polar* just in case it is path connected and has a unique partition into two subsets such that, for all x, y in A, xFy only if x and y belong to the same subset whereas xEy only if they belong to different subsets.

To capture the world-war structure we can attribute bi-polarity to any set of combatants while allowing for subjects that are neutrals in that conflict:

> *Local Bi-Polarity* (*Local BiPol*). S has a unique partition into one or more bi-polar subsets such that, for all x and y, xFy or xEy only if x and y belong to the same subset.

Any subject that has no enemies and no friends but itself occupies a unit set in this partition.

Local BiPol may look stronger than Cartwright and Harary's simpler *WkDi*. But assuming *F-rflx* and *F&E-sym*, the two versions of two-sidedness are equivalent. For each is equivalent to *SB*:

Theorem 1. *These three axiom sets are equivalent: {SB, F-rflx, F&E-sym}, {WkDi, F-rflx, F&E-sym}, {Local BiPol, F-rflx, F&E-sym}.*

Appendix A has proofs of theorems.

3. Axioms of connectedness

Cartwright and Harary call attention to another potential axiom:

Completeness. *xFy* or *xEy*.

They deduce nothing from it and find it less compelling than *SB* but see a "tendency" to satisfy it in the long run—a tendency of subjects to close any relational gaps between themselves. Others occasionally assume it, and Heider's triads all satisfy it.

It is worth adding *Completeness* to our initial axiom set to see what follows, but we should also consider two weaker axioms of *connectedness*:

Path Connexity (*PthCnx*). *S* is path connected (from every *x* to every *y* there exists a path).
Graph Transitivity (*GrTrans*). $F \cup E$ is transitive ($xF \cup EyF \cup Ez \Rightarrow xF \cup Ez$).

Because $F \cup E$ is the union of *F* and *E*, $xF \cup Ey$ holds if *xFy* or *xEy*, hence if *x* and *y* are directly related in the Cartwright–Harary signed graph. Note that *GrTrans* does not imply the transitivity of *F* or *G*. For example, if *xEyEz* then *GrTrans* implies only that *xEz or xFz* (and *SB* then rules out *xEz*, blocking transitivity of *E*). Where *Completeness* says that any pair of subjects are directly related, *PthCnx* says they are at least indirectly related, and *GrTrans* that any indirectly related pair are also directly related. So *PthCnx* holds for restricted domains of interactive subjects, such as the combatants in either world war. One might defend *GrTrans* by contending, possibly on semantic grounds, that friendship and enmity are never merely indirect: by fighting Germany and supporting Britain in WWI, the US opposed Turkey willy-nilly.

The obvious logical connection among the connectedness axioms is this:

Theorem 2. *Completeness is equivalent to PthCnx plus GrTrans.*

We now have three stronger variants of the original Cartwright–Harary axiom set:

{*SB*, *Completeness*},

{*SB*, *PthCnx*, *F-rflx*, *F&E-sym*},

{*SB*, *GrTrans*, *F-rflx*, *F&E-sym*}.

I omitted the three background assumptions from the first set because they follow. For example, *SB* implies not *xEx*, whence *xFx* (*F-rflx*) by *Completeness*. Again, if *xFy* but *not yFx* (contrary to *F-sym*), then *xFyEx* by *Completeness*, contrary to *SB*.

4. Equivalence of new axiom sets with stronger versions of bi-polarity

Our three new axiom sets, each stronger than Cartwright and Harary's, are tantamount to three new versions of bi-polarity, each stronger, than *WkDi* or *Local BiPol*.

By making *S* as a whole path connected, *PthCnx* reduces the mutually unrelated subsets allowed by *Local BiPol* to one, *S* itself:

Global Bi-Polarity (*Global BiPol*). *S* is bi-polar.

By making all relationships direct, *GrTrans* ensures that, not only are friends always on the same side and enemies on opposite sides of any bi-polar partition, but subjects on the same side are always friends and those on opposite sides enemies:

Local Strict Bi-Polarity (*Local Strict BiPol*). *S* has a partition into one or more strictly bi-polar subsets such that, for all *x*, *y*, if *xFy* or *xEy* then *x* and *y* belong to the same subset,

where:

A is *strictly bi-polar* just in case it has a unique partition into two subsets such that, for all *x*, *y* in *A*, *xFy* if *x* and *y* belong to the same subset whereas *xEy* if they belong to different subsets.

Finally, by combining *PthCnx* and *GrTrans*, *Completeness* ensures the strongest possible version of bi-polarity:

Global Strict Bi-Polarity (*Global Strict BiPol*). *S* is strictly bi-polar.

These, then, are the equivalences to be proved:

Theorem 3. *{SB, PthCnx, F-rflx, F&E- sym} is equivalent to Global BiPol plus F-rflx and F&E-sym.*

Theorem 4. *{SB, GrTrans, F-rflx, F&E- sym} is equivalent to Local Strict BiPol.*

Theorem 5. *{SB, Completeness} is equivalent to Global Strict BiPol.*

Note that *Strict BiPol*, *Local* and *Global*, is strong enough to incorporate *F-rflx* and *F&E-sym*.

5. Lave and March's aphoristic axioms

In their classic textbook, Lave and March (1975, p. 67) assume the two aphorisms of my title and two more: The friend of my friend is my friend, and The enemy of my friend is my enemy. To these they add *Completeness* and assert (without proof) that *Global Strict Bi-Polarity* follows. For the moment, however, set *Completeness* aside and look at those aphoristic axioms:

FFF .$xFyFz \Rightarrow xFz$,

FEE .$xFyEz \Rightarrow xEz$,

EFE .$xEyFz \Rightarrow xEz$,

EEF .$xEyEz \Rightarrow xFz$.

We cannot deduce *F-rflx*, *F-sym*, *E-sym*, or even *E-irrflx* and so must add them. But we can drop *EFE*, for $xEyFz \Rightarrow zFyEx$ (by *F&E-sym*) $\Rightarrow zEx$ (by *FEE*) $\Rightarrow xEz$ (by *E-sym*). (Yes, we could have dropped *FEE* instead.) Here is the relaxed Lave–March axiom set:

{*FFF*, *FEE*, *EEF*, *F-rflx*, *F&E-sym*, *E-irrflx*}.

Obviously the new axioms capture at least part of *SB* and *GrTrans*. In fact:

Theorem 6. *{FFF, FEE, EEF, F-rflx, F&E-sym, E-irrflx} is equivalent to {SB, GrTrans, F-rflx, F&E-sym}.*

That is, the Lave–March aphoristic system, with four background assumptions added but without the very strong *Completeness* or the redundant *EFE*, is equivalent to the original Cartwright–Harary system plus *GrTrans*. It follows by Theorem 4 that a third equivalent is *Local Strict BiPol*.

Now add *Completeness*. Then we can drop *F-rflx*, *F-sym*, and *E-sym*. We can even drop *FFF*. For if *xFyFz* but *not xFz* (contrary to *FFF*), then *xEz* by *Completeness*. So $zFx \Rightarrow zFxEz \Rightarrow zEz$ (by *FEE*), contrary to *E-irrflx*. But *not*-$zFx \Rightarrow zEx$ (by *Completeness*) $\Rightarrow yFzEx \Rightarrow yEx$ (by *FEE*) $\Rightarrow xFyEx \Rightarrow xEx$ (by *FEE*), contrary again to *E-irrflx*. Our new axiom set is:

{*FEE*, *EEF*, *Completeness*, *E-irrflx*}.

It amounts to the strongest of the *SB*-based systems:

Theorem 7. *{FEE, EEF, Completeness, E-irrflx} is equivalent to {SB, Completeness}.*

Thanks to Theorem 5, a third equivalent is *Global Strict BiPol*.

6. Simplification of the strongest system

The beauty of the last two axiom sets, the Lave–March aphoristic ones, is that they are *elementary*—couched in terms of F and E and first-order logic. Because *SB* refers to numbers and sequences, our previous axiom sets were not. So the two aphoristic systems are simplifications of a sort.

The second and stronger of the two – the one that has *Completeness* – allowed a second simplification: we dropped *FFF*. It also allows a third. For it obviously implies:

Definability of E (*DefE*). $xEy \Leftrightarrow$ not xFy.

This lets us drop E as a primitive relation, then define it as the complement of F (we can just as well drop F). That simplification opens the door to a fourth. For we can now drop *E-irrflx* and *Completeness*, so long as we replace *FEE* by *FFF* (a tad simpler in primitive notation) and add the more trifling *F-rflx* and *F-sym*. The result:

{*DefE*, *FFF*, *EEF*, *F-rflx*, *F-sym*}.

I put *DefE* in this axiom set because it is essential to the following equivalence:

Theorem 8. *{DefE, FFF, EEF, F-rflx, F-sym} is equivalent to {FEE, EEF, Completeness, E-irrflx}.*

That is, to prove sufficiency we must assume *DefE*, and to prove necessity we must deduce *DefE*. But *DefE* is a definition, not an axiom. As such it does not raise but reduces the complexity of the new system by eliminating a primitive notion. The obvious corollary to Theorem 8 is that this system is tantamount to *Global Strict BiPol*, the strongest version of two-sidedness.

That corollary adds a nice wrinkle to a well known fact about *equivalence relations*. *FFF*, *F-rflx*, and *F-sym* make F an equivalence relation on S (transitive, reflexive in S, symmetric). Unless it is empty, the complement of such a relation, in this case E, cannot be transitive. It does not have to be *in*transitive, but it might be, and *EEF* says it is. The well known fact is that an equivalence relation (F) partitions its field (S) into equivalence classes; there is no limit on how many. The wrinkle is that the intransitivity of the complement (E) turns that partition into a dichotomy: there are but two equivalence classes.

We have come full circle. Heider began with an exhaustive list of balanced and unbalanced triads, all satisfying *Completeness*. Dropping *Completeness*, Cartwright and Harary generalized beyond triads to structures of unlimited complexity. Restoring *Completeness*, Lave and March got the same effect with four elementary axioms. But each of those is tantamount to a ban on one of Heider's unbalanced triads. For example, *FFF* bans cases in which *xFyFzEx*. The chief difference with Heider's formulation is that couching a few triad bans as axioms allowed us to deduce all the others along with Cartwright and Harary's *SB* and much more. Finally, all we needed were two of those axioms, *FFF* and *EEF*, hence two triad bans—on cases in which *xFyFzEx* and *xEyEzEx*.

But Davis (1967) and Kulakowski (2007) have suggested that we generalize Heider by dropping that second ban, to accommodate more than two mutually hostile alliances. See how smoothly that irons out our wrinkle on equivalence relations. It amounts to dropping *EEF*, leaving only the bare assumption that F is an equivalence relation. Recalling that $E =$ not-F, our axioms now say in effect that there is a partition of S into mutually hostile alliances, of which there may be any number—any number of poles.

7. Conclusion

The idea of structural balance is beautiful in its sweep, simplicity, and power: certain patterns of bi-lateral relationship, "positive" and "negative" but otherwise unconstrained in content, are balanced, others not, and a general balance effects a bi-polar partition among interactive subjects. Potential applications extend beyond the small-group intra-actions of social psychology and the grand conflicts and alliances of international relations to the interactions and stratifications of domestic politics and economics. Elections typically have allies and opponents, friends and enemies of a sort, along with a partition into competing parties, of which the modal number is two. Sometimes voting rules encourage a finer division of parties, but even then those parties form two opposing legislative blocs after election. Ethnic differences often become political divisions, of course, too often murderous civil wars. Less often remarked is how often those divisions line up and congeal as two opposing sides. In the field of industrial organization, when fully contested markets give way, by merger and acquisition and cartelization, to restricted competition, as often as not we observe markets that are monopolistic or duopolistic (either way, two-fold partitions of erstwhile competitors, possibly with one subset empty).

Structural balance owes this breadth, in part, to its compatibility with more than one underlying mechanism. Heider (1946, 1958) attributed it to the avoidance of cognitive dissonance, or a feeling of discomfiture. But it can also be attributed to the rational reckoning of benefits and costs. Maybe enemies of my enemies are my friends because they help me keep my enemies at bay. Maybe friends of my enemies are my enemies too because they help my enemies harm me.

When we attempt to axiomatize structural balance and deduce bi-polarity, we encounter interesting variations. The weakest axiomatization implies Cartwright and Harary's *Weak Dichotomy*, but also the more interesting *Local Bi-Polarity*. Stronger axiomatizations, all based on added connectedness axioms, give us stronger versions of bi-polarity. Two have elementary equivalents, based on Lave and March's *axioms*. Besides deducing a version of bi-polarity from each axiom set, we can prove equivalence of the two. Our axiom sets and bi-polarity equivalents then form four equivalence groups, distinguished by connectedness axioms, as follows:-

Group I (no connectedness axiom)

Ia. *Structural Balance* (*SB*), *F-rflx*, *F&E-sym*
Ib. *Weak Dichotomy* (*WkDi*), *F-rflx*, *F&E-sym*
Ic. *Local Bi-Polarity* (*Local BiPol*), *F-rflx*, *F&E-sym*.

Group II (*Path Connexity*)

IIa. *SB*, *Path Connexity* (*PthCnx*), *F-rflx*, *F&E-sym*
IIb. *Global Bi-Polarity* (*Global BiPol*), *F-rflx*, *F&E-sym*.

Group III (*Graph Transitivity*)

IIIa. *SB*, *Graph Transitivity* (*GrTrans*), *F-rflx*, *F&E-sym*
IIIb. *Local Strict Bi-Polarity* (*Local Strict BiPol*)
IIIc. *FFF*, *FEE*, *EEF*, *F-rflx*, *F&E-sym*, *E-irrflx*.

Group IV (*Completeness*, or *PthCnx* plus *GrTrans*)

IVa. *SB*, *Completeness*
IVb. *Global Strict Bi-Polarity* (*Global Strict BiPol*)
IVc. *FEE*, *EEF*, *Completeness*, *E-irrflx*
IVd. *FFF*, *EEF*, *F-rflx*, *F-sym*, with E defined as not-F.

Some implications between groups are obvious:

$$\text{IV} \Leftrightarrow \text{II} + \text{III}, \qquad \text{II} \Rightarrow \text{I}, \qquad \text{III} \Rightarrow \text{I}.$$

That is all there are: IV is independent of II and of III, II and III are mutually independent, and both are independent of I. These independence results are demonstrated in Appendix B, along with the independence of each axiom within each set.

To grasp the axiomatic story in a fell swoop, assume *F-rflx*, *F&E-sym*, and *E-irrflx*. Then the weakest axiom system has *Structural*

Balance (*SB*): no cycle of bi-lateral relationships has an odd number of negative ones. This amounts to postulating a partition of the universe into one or more unrelated subuniverses, each internally bi-polar—uniquely partitioned in two so that friends are on the same side, enemies on opposite sides. Add *Path Connexity*, and any plurality of subuniverses disappears. Instead add *Graph Transitivity*, and each bi-polar partition becomes strict: subjects on the same side are always friends, on opposite sides enemies. *Completeness* combines those two additions, making the universe as a whole strictly bi-polar. *Graph Transitivity* allows an equivalent but elementary axiomatization based on three folkloric aphorisms, or two under the stronger *Completeness*. The effect of the latter can then be got more simply by defining enmity as nonfriendship. So a strictly bi-polar partition of the universe is tantamount to the transitivity of friendship and the intransitivity of its complement; drop the latter and the universe remains polar but no longer "bi".

Appendix A. Proofs of theorems

To prove Theorem 1, the hardest of the lot, we shall need two definitions and five lemmata. An *A-path* is a path whose components all belong to A. A path is *positive* if its sign is 1, *negative* if it is -1. Note that every path is positive or negative: a sequence of sign 0 is not a path.

Lemma 1. *Assume F-rflx, F&E-sym, and SB. Then for no x, y does there exist both a positive and a negative path from x to y.*

Proof. Suppose on the contrary that $(z_1, \ldots, z_n)$ is a positive path and $(w_1, \ldots, w_k)$ a negative path from $z_1 = w_1$ to $z_n = w_k$. Then

$$1 = \sigma(z_1, \ldots, z_n)$$

and

$$-1 = \sigma(w_1, \ldots, w_k) = r_1 \cdot r_2 \cdots r_k$$

where $r_1 = 1$

$$r_{i+1} = \begin{cases} 1 & \text{if } w_i F w_{i+1} \\ -1 & \text{if } w_i E w_{i+1}, \quad i = 1, 2, \ldots, k-1. \end{cases}$$

So *F&E-sym* implies:

$$\sigma(w_k, \ldots, w_1) = r_k \cdot r_{k-1} \cdots r_1 = -1,$$

the same product in reverse. But $z_n = w_k$, so $z_n F w_k$ by *F-rflx*, and thus

$$\sigma(z_1, \ldots, z_n, w_k, \ldots, w_1) = \sigma(z_1, \ldots, z_n) \cdot r_k \cdots r_1 = 1 \cdot -1 = -1.$$

But that violates *SB* because $z_1 = w_1$. □

Lemma 2. *Assume SB, F-rflx, and F&E-sym. Then A is path connected ⇒ A is bi-polar.*

Proof. Take any x in A, and let

$P = \{y \in A \mid$ there exists a positive A-path from x to $y\}$

$N = \{y \in A \mid$ there exists a negative A-path from x to $y\}$.

By Lemma 1, P and N are disjoint. And because A is path connected, $A = P \cup N$. So $\{P, N\}$ is a partition of A.

Let $y, z \in A$ and yFz. To show that y and z belong to the same subset in that partition, suppose not; say $y \in P$ and $z \in N$. Then there exists a positive A-path $(x_1, \ldots, x_n)$ from $x = x_1$ to $y = x_n$. But because yFz, $(x_1, \ldots, x_n, z)$ is a positive A-path too, so we cannot have $z \in N$ after all.

Next let $y, z \in A$ and yEz. To show that y and z belong to different subsets, suppose not. Then if $y, z \in P$, there exists a positive A-path $(x_1, \ldots, x_n)$ from $x = x_1$ to $y = x_n$, making $(x_1, \ldots, x_n, z)$ a negative A-path from x to z because yEz, so that $z \in N$ after all. Or if $y, z \in N$, there exists a negative A-path $(x_1, \ldots, x_n)$ from $x = x_1$ to $y = x_n$, making $(x_1, \ldots, x_n, z)$ a positive A-path from x to z and putting z in P after all.

Finally, to show that $\{P, N\}$ is unique, suppose $\{B, C\}$ is a cross-cutting partition of A for which xFy only if x, y belong to the same subset whereas xEy only if they belong to different subsets. I shall deduce that $\{B, C\} = \{P, N\}$. So far we have $xFy \Rightarrow$ either $x, y \in P$ or $x, y \in N$, but likewise $xFy \Rightarrow$ either $x, y \in B$ or $x, y \in C$. Hence,

$xFy \Rightarrow x, y$ both belong to one of these intersections: $P \cap B$, $P \cap C$, $N \cap B$, or $N \cap C$.

Similarly,

$xEy \Rightarrow$ one of x, y belongs to $P \cap B$ and the other to $N \cap C$ or else one of them belongs to $N \cap B$ and the other to $P \cap C$.

Consequently,

$$xF \cup Ey \Rightarrow \text{ either } x, y \in (P \cap B) \cup (N \cap C) \quad \text{or} \quad x, y \in (N \cap B) \cup (P \cap C).$$

Therefore, every A-path lies wholly within $(P \cap B) \cup (N \cap C)$ or wholly within $(N \cap B) \cup (P \cap C)$. But A is path connected. So one of those two unions must exhaust A, making the other one empty. But that is possible only if the partitions $\{P, N\}$ and $\{B, C\}$ are the same. □

Lemma 3. *Assuming F-rflx and F&E-sym, SB ⇒ Local BiPol.*

Proof. Define: $xQy \Leftrightarrow$ there exists a path from x to y. By hypothesis Q is reflexive in S and symmetric, and by construction it is transitive. So it is an equivalence relation on S. Hence it partitions S in a unique way into equivalence classes. But each such class is path connected by construction, hence bi-polar by Lemma 2. And obviously Q never holds between members of different equivalence classes. So xFy or xEy only if x, y belong to the same equivalence class, ensuring that *Local BiPol* is satisfied. □

Lemma 4. *Local BiPol ⇒ WkDi.*

Proof. By hypothesis there exists a partition of S into subsets $A_1, \ldots, A_n$ for which (1) $xF \cup Ey$ only when x, y belong to the same subset, and (2) each A_i has a further partition into two subsets B_i, C_i such that, for all x, y in A_i, xFy only if x, y belong to the same subset and xEy only if they belong to different subsets. Then $\{U_i B_i, U_i C_i\}$ is a partition of S itself that meets the requirements for *WkDi*: xFy only if x, y belong to the same subset in that partition, and xEy only if they belong to different subsets. □

Lemma 5. *WkDi ⇒ SB.*

Proof. By hypothesis S has a partition into two subsets such that xFy only if x, y belong to the same subset whereas xEy only if they belong to different subsets. To deduce *SB*, take any path $(x_1, \ldots, x_n)$. We must show that it is negative only if $x_1 \neq x_n$, i.e., that $x_1 = x_n$ only if it is positive. I shall prove, by induction on n, the stronger proposition that x_1 and x_n belong to the same subset in the partition only if $\sigma(x_1, \ldots, x_n) = 1$ and to different subsets only if $\sigma(x_1, \ldots, x_n) = -1$. Trivial if $n = 1$. Otherwise there are two cases.

Case 1. x_1, x_{n-1} are in the same subset. Then $\sigma(x_1, \ldots, x_{n-1}) = 1$ by inductive hypothesis. But if $x_{n-1} F x_n$ then x_{n-1} and x_n are in the same subset, whence so are x_1 and x_n, and $\sigma(x_1, \ldots, x_{n-1}, x_n) = 1$. And if $x_{n-1} E x_n$ then x_{n-1} and x_n are in different subsets, whence so are x_1 and x_n, and $\sigma(x_1, \ldots, x_{n-1}, x_n) = -1$.

Case 2. x_1, x_{n-1} are in different subsets. Then $\sigma(x_1, \ldots, x_{n-1}) = -1$ by inductive hypothesis. But if $x_{n-1} F x_n$ then x_{n-1} and x_n are in the same subset, whence x_1 and x_n are in difference subsets, and $\sigma(x_1, \ldots, x_{n-1}, x_n) = -1$. And if $x_{n-1} E x_n$ then x_{n-1} and x_n are in different subsets, whence x_1 and x_n are in the same subset, and $\sigma(x_1, \ldots, x_{n-1}, x_n) = 1$. □

Theorem 1. *These three axiom sets are equivalent: {SB, F-rflx, F&E-sym}, {WkDi, F-rflx, F&E-sym}, and {Local BiPol, F-rflx, F&E-sym}.*

Proof. Lemmata 3–5 show:

{*SB*, *F-rflx*, *F&E-sym*} $\Rightarrow$ *LocalBiPol* $\Rightarrow$ *WkDi* $\Rightarrow$ *SB*,

whence the theorem follows immediately. □

The next four theorems are about our three connectedness assumptions and their effects on bi-polarity.

Theorem 2. *Completeness is equivalent to PthCnx plus GrTrans.*

Proof. Sufficiency is trivial. To prove necessity, take any x, y. By *PthCnx* there exists a path $(z_1, \ldots, z_n)$ from $x = z_1$ to $y = z_n$. It follows by $n - 2$ applications of *GrTrans* that $xF \cup Ey$: *Completeness* holds. □

Theorem 3. *{SB, PthCnx, F-rflx, F&E-sym} is equivalent to Global BiPol plus F-rflx and F&E-sym.*

Proof. By Theorem 1 those four axioms imply *Local BiPol*, and thanks to *PthCnx* the partition of S into bi-polar subsets can have but one member, S itself, so that *Global BiPol* is satisfied too. Conversely, *Global BiPol* immediately implies *Local BiPol* and *PthCnx*, and together with *F-rflx* and *F&E-sym*, *Local BiPol* in turn implies *SB* by Theorem 1. □

Theorem 4. *{SB, GrTrans, F-rflx, and F&E-sym} is equivalent to Local Strict BiPol.*

Proof. For sufficiency: *Local BiPol* follows by Theorem 1. So each subset A in the partition of S is bi-polar, hence path connected. Therefore, as in the proof of Theorem 2, we have $xF \cup Ey$ for all x, y in A, and thus, because A is bi-polar, xFy if x, y belong to the same subset in the partition of A, and xEy if they belong to different subsets: the bi-polarity of A is strict.

To prove necessity, assume *Local Strict BiPol*. So every x belongs to one of the two subsets B, C in the strictly bi-polar partition of some A in the partition of S. Then xFx: *F-rflx* holds. And if xFy then x, y both belong to some such B or C, whence yFx as well, whereas if xEy then x, y belong to different such subsets of some such A, so that yEx: *F&E-sym* is satisfied. But if $xF \cup EyF \cup Ez$ then x, y, z all belong to some such A, and thus xFy if x, y belong to the same subset in the partition of A whereas xEy if not: *GrTrans* holds. And thanks to *Local BiPol*, *F-rflx*, and *F&E-sym*, *SB* holds by Theorem 1. □

Theorem 5. *{SB, Completeness} is equivalent to Global Strict BiPol.*

Proof. Sufficiency: *Completeness* implies *GrTrans* by Theorem 2, whence *Local Strict BiPol* follows by Theorem 4. And *Completeness* obviously implies that there exists but one strictly bi-polar subset in the partition of S: S itself. So *Global Strict BiPol* is satisfied as well.

Now necessity: *Global Strict BiPol* implies *Local Strict BiPol*, which implies *SB* by Theorem 4, and it suffices to deduce *Completeness*. Take any x, y. In the strictly bi-polar partition of S, x and y belong to the same or different subsets, so xFy or xEy. □

The remaining three theorems assert the equivalence of some of the Cartwright–Harary *SB*-based axiom sets, not with versions of bi-polarity, but with sets based on Lave and March's aphoristic axioms.

Theorem 6. *{FFF, FEE, EEF, F-rflx, F&E-sym, E-irrflx} is equivalent to {SB, GrTrans, F-rflx, F&E-sym}.*

Proof. For sufficiency I shall deduce *GrTrans*, then *SB*. So suppose $xF \cup EyF \cup Ez$; to deduce xFz or xEz. There are four possibilities. If $xFyFz$ then xFz by *FFF*. If $xFyEz$ then xEz by *FEE*. If $xEyFz$ then $zFyEx$ by *F&E-sym*, whence zEy by *FEE*, so xEz by *E-sym*. Finally, if $xEyEz$ then xFz by *EEF*.

For *SB* we must show that $\sigma(x_1, \ldots, x_n) = -1 \Rightarrow x_1 \neq x_n$. But by *E-irrflx*, $x_1Ex_n \Rightarrow x_1 \neq x_n$. So it suffices to show, by induction on n, that $\sigma(x_1, \ldots, x_n) = -1 \Rightarrow x_1Ex_n$ whereas $\sigma(x_1, \ldots, x_n) = 1 \Rightarrow x_1Fx_n$. Trivial if $n = 1$ or $n = 2$. Otherwise there are two cases.

Case 1. $\sigma(x_1, \ldots, x_{n-1}) = 1$. Then x_1Fx_{n-1} by inductive hypothesis. So if $x_{n-1}Fx_n$ then $\sigma(x_1, \ldots, x_{n-1}, x_n) = 1$, and x_1Fx_n by *FFF*. Or if $x_{n-1}Ex_n$ then $\sigma(x_1, \ldots, x_{n-1}, x_n) = -1$, and x_1Ex_n by *FEE*.

Case 2. $\sigma(x_1, \ldots, x_{n-1}) = -1$. Then x_1Ex_{n-1} by inductive hypothesis. So if $x_{n-1}Fx_n$ then $\sigma(x_1, \ldots, x_{n-1}, x_n) = -1$, and $x_nFx_{n-1}Ex_1$ by *F&E-sym*, whence x_nEx_1 by *FEE*, and thus x_1Ex_n by *E-sym*. Or if $x_{n-1}Ex_n$ then $\sigma(x_1, \ldots, x_{n-1}, x_n) = 1$, and x_1Fx_n by *EFF*.

As for necessity, we saw in Section 1 that *SB* $\Rightarrow$ *E-irrflx*, and it suffices to deduce *FFF*, *FEE*, and *EEF*. Suppose $xFyEz$. Then xFz or xEz by *GrTrans*. But $xEz \Rightarrow zEx$ (by *E-sym*) $\Rightarrow \sigma(x, y, z, x) = -1$, contrary to *SB*. Proofs of *FEE* and *EEF* are similar. □

Theorem 7. *{FEE, EEF, Completeness, E-irrflx} is equivalent to {SB, Completeness}.*

Proof. For sufficiency we have to deduce *SB*, and for that it is enough, by Theorem 6, to deduce *F-rflx* and *F&E-sym*. For any x we have xFx (*F-rflx*) by *Completeness* and *E-irrflx*. To deduce *F-sym*, suppose xFy. Then if *not* yFx, we have yEx by *Completeness*, so $xFyEx$ and thus xEx by *FEE*, contrary to *E-irrflx*. For *E-sym*, suppose xEy. Then if *not* yEx, we have yFx by *Completeness*, so $yFxEy$, whence yEy by *FEE*, contrary once more to *E-irrflx*.

To prove necessity, recall that *SB* implies *E-irrflx*, and it and *Completeness* imply *F-rflx*. But *Completeness* implies *GrTrans* by Theorem 2, and {*SB*, *GrTrans*, *F-rflx*, *F&E-sym*} implies *FFF*, *FEE*, and *EEF* by Theorem 6. So it suffices to deduce *F-sym* and *E-sym*. For the former, suppose xFy. Then if *not* yFx, we have yEx by *Completeness*, so $yExFy$, contrary to *SB*. For *E-sym*, suppose xEy. Then if *not* yEx, we have yFx by *Completeness*, so $yFxEy$, again contrary to *SB*. □

Theorem 8. *{DefE, FFF, EEF, F-rflx, F-sym} is equivalent to {FEE, EEF, Completeness, E-irrflx}.*

Proof. To prove sufficiency, we first deduce *FEE*, then *Completeness*, then *E-irrflx*. Assume $xFyEz$ but *not* xEz. Then xFz by *DefE*, whence zFx by *F-sym*, so $zFxFy$, and thus zFy by *FFF*. Therefore yFz by *F-sym*. It follows by *DefE* that not yEz, contrary to our assumption. *Completeness* follows immediately from *DefE*. And to deduce *E-irrflx*, note that $xEx \Rightarrow$ *not* xFx, contrary to *F-rflx*.

For necessity, we have already deduced *FFF* in Section 5. Now we must deduce *F-rflx*, *F-sym*, and *DefE*. *F-rflx* holds because *not* $xFx \Rightarrow xEx$ by Completeness, contrary to *E-irrflx*. To deduce *F-sym*, suppose the contrary: xFy but *not* yFx. Then yEx by *Completeness* and thus $xFyEx$. It follows by *FEE* that xEx, contrary again to *E-irrflx*. To deduce *DefE* we first show that $xEy \Rightarrow$ not xFy. Suppose instead that xEy but xFy too. Then yFx by *F-sym*, so $yFxEy$, and thus yEy by *FEE*, contrary yet again to *E-irrflx*. Hence $xEy \Rightarrow$ not xFy after all. The converse follows immediately from *Completeness*. □

Appendix B. Independence of axioms

To prove an axiom independent within a set of axioms, we need an *independence example*, an interpretation of S, F, and E (except for IVd, where E is not primitive) that makes the given axiom false but all the others in the set true. The following independence examples cover all the axioms and axiom sets listed in Section 7, Conclusion. *Global Strict BiPol* needs no independence example because it is never combined with other assumptions. "Every" means every set containing the given axiom. Only *F-sym* and *E-sym* need different independence examples for different axiom sets (or different equivalence groups) containing them. Sometimes one independence example works for two or more axioms (which never belong to the same axiom set). In each example but the first, I omit mention of S because it is simply the field of S (e.g. $S = \{1, 2, 3\}$ in example 2). I also omit mention of E when $E = \emptyset$ (as in 1–3 but not 4).

	Axiom	Axiom set	Independence example
1.	*F-rflx*	Every	$S = \{1\}, F = \phi$
2.	*F-sym*	I, III, IV	$F = \{(1, 1), (2, 2), (1, 2)\}$
3.	*F-sym*	II	$F = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (3, 1)\}$
4.	*E-sym*	I, III	$F = \{(1, 1), (2, 2)\}, E = \{(1, 2)\}$
5.	*E-sym*	II	$F = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$, $E = \{(1, 2), (2, 3)\}$
6.	*E-irrflx*	Every	$F = E = \{(1, 1)\}$
7.	*SB* *WkDi* *Local BiPol* *EEF*	Every	$F = \{(1, 1), (2, 2), (3, 3)\}$, $E = \{(1, 2), (2, 1), (2, 3), (3, 2), (3, 1), (1, 3)\}$
8.	*GrTrans* *Local Strict BiPol* *FFF*	Every	$F = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$
9.	*FEE*	Every	$F = \{(1, 1), (2, 2), (1, 2), (2, 1)\}$, $E = \{(1, 2), (2, 1)\}$
10.	*PthCnx* *Completeness* *Global BiPol*	Every	$F = \{(1, 1), (2, 2)\}$

As for the independence of whole equivalence groups, example 10 proves Groups II and IV independent of I and III, and 8 proves III and IV independent of I and II.

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