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Bounds for the Least Laplacian Eigenvalue of a Signed Graph

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Abstract A signed graph is a graph with a sign attached to each edge. This paper extends some fundamental concepts of the Laplacian matrices from graphs to signed graphs. In particular, the relationships between the least Laplacian eigenvalue and the unbalancedness of a signed graph are investigated.

Keywords Signed graph, Laplacian matrix, The least eigenvalue, Balanced signed graph
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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G) = \{e_1, e_2, \ldots, e_m\}$. The Laplacian matrix of the graph G is $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(d_{v_1}, d_{v_2}, \ldots, d_{v_n})$ is the diagonal matrix of vertex degrees and $A(G) = (a_{ij})$ is the $(0, 1)$-adjacency matrix of the graph G, that is, $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. There is a long history of results related to the Laplacian matrix of a graph. The first of these is the celebrated 1847 result of Kirchoff referred to as the Matrix-Tree Theorem. More recent investigations were stimulated by the results of Fiedler [1], and there has been a lot of activities in this area since then. See, for example, [2, 3], or book [4] for a survey of some of the recent work.

A *signed graph* $\Gamma = (G, \sigma)$ consists of an unsigned graph $G = (V, E)$ and a mapping σ: $E \to \{+, -\}$, the edge labelling. We may write $V(\Gamma)$ for the vertex set and $E(\Gamma)$ for the edge set if necessary. The degree sequence of a signed graph $\Gamma = (G, \sigma)$ and the degree sequence of its underlying graph G are the same. Signed graphs were introduced by Harary in [5] in connection with the study of the theory of social balance in social psychology (see [6]), and the matroids of graphs were extended to matroids of signed graphs by Zaslavsky in [7]. The Matrix-Tree Theorem for a signed graph was obtained by Zaslavsky in [7] and by Chaiken in [8]. More recent results on signed graphs can be found in [9].

Let $\Gamma = (G, \sigma)$ be a signed graph. The *Laplacian matrix* of Γ, denoted by $L(\Gamma)$ or $L(G, \sigma)$, is defined as $D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix of vertex degrees of signed graph Γ and $A(\Gamma)$ is the *signed adjacency matrix* of the signed graph Γ, $A(\Gamma) = (a^{\sigma}_{ij})$ is defined in a natural way: $a^{\sigma}_{ij} = \sigma(v_i v_j) a_{ij}$, where $a_{ij} = 1$ if v_i and v_j is adjacent, and $a_{ij} = 0$ otherwise.

From the above definitions, it follows that $L(G) = L(G, +)$, and $D(G) + A(G) = L(G, -)$, where $\sigma = +$ and $\sigma = -$ are the all-positive and all-negative edge labelling, respectively. Thus $L(G, \sigma)$ may be viewed as a common generalization of the Laplacian matrix $L(G)$ and $D(G) + A(G)$ of a graph G. The aim of this paper is to extend the concept of Laplacian matrix of a graph to a signed graph.

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This paper is organized as follows. In Section 2 we recall some results on signed graphs and obtain some elementary properties on the Laplacian matrices of signed graphs. In Section 3, we give some bounds on the least Laplacian eigenvalue of a signed graph, and investigate the relation between the least Laplacian eigenvalue and the unbalancedness of a signed graph.

2 Preliminary

Let $\Gamma = (G, \sigma)$ be a signed graph and C be a cycle of Γ, the sign of C being denoted by $\mathrm{sign}(C)=\prod_{e\in C}\sigma(e)$. A cycle whose sign is $+$ (respectively, $-$) is said to be *positive* (respectively, *negative*). A signed graph is said to be *balanced* if all its cycles are positive.

Suppose $\Gamma = (G, \sigma)$ is a signed graph and $\theta:\ V \to \{+,-\}$ is any sign function. *Switching* Γ by θ means forming a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ whose underlying graph is the same as G, but whose sign function is defined on an edge $e = v_iv_j$ by $\sigma^\theta(e) = \theta(v_i)\sigma(e)\theta(v_j)$.

Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs with the same underlying graph. Γ_1 and Γ_2 are said to be *switching equivalent,* written $\Gamma_1 \sim \Gamma_2$, if there exists a *switching function* θ such that $\Gamma_2 = \Gamma_1^\theta$. Switching leaves the many signed-graphic characteristics invariant, such as the set of positive cycles. Switching was first introduced by Seidel and it plays an important role in the discussions on signed graphs (see for example [9]).

Two matrices M_1 and M_2, of order n, are said to be *signature similar* if there exists a signature matrix, that is, a diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$ with diagonal entries $s_i = \pm 1$ such that $M_2 = SM_1S$. By the definitions of switching equivalent and signature similar, we obtain:

Lemma 2.1 *Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be signed graphs on the same underlying graph. Then $\Gamma_1 \sim \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.*

The following result essentially comes from [7, 9]:

Theorem 2.2 *Let $\Gamma = (G, \sigma)$ be a signed graph. Then the following conditions are equivalent*:

(1) Γ *is balanced*;

(2) $\Gamma = (G, \sigma) \sim (G, +)$;

(3) *There exists a signature matrix S such that $SL(\Gamma)S$ has all off-diagonal entries of $SL(\Gamma)S$ 0 or -1*;

(4) *There exists a partition $V(\Gamma) = V_1 \cup V_2$ such that every edge between V_1 and V_2 is negative and every edge within V_1 or V_2 is positive.*

Similarly to an unsigned graph, we may define the Laplacian matrix of a signed graph $\Gamma = (G, \sigma)$ by means of the *incidence matrix* of Γ. For each edge $e_k = v_iv_j$ of G, we choose one of v_i or v_j to be the head of e_k and the other to be the tail. We call this an *orientation* of Γ. The vertex-edge incidence matrix $C = C(\Gamma)$ afforded by a fixed orientation of Γ is the n-by-m matrix $C = (c_{ij})$ given by

$$c_{ij} = \begin{cases} +1, & \text{if } v_i \text{ is the head of } e_j; \\ -1, & \text{if } v_i \text{ is the tail of } e_j, \text{ and } \ \sigma(e_j) = +; \\ +1, & \text{if } v_i \text{ is the tail end of } e_j, \text{ and } \ \sigma(e_j) = -; \\ 0, & \text{otherwise.} \end{cases}$$

The general rule behind this is that for each edge $e = uv$ of Γ, $c_{ue} = -\sigma(e)c_{ve}$. While C depends on the orientation of Γ, CC^t does not, and it is easy to verify that $L(\Gamma) = CC^t$. Hence $L(\Gamma)$ is a positive semidefinite symmetric matrix.

We call a subgraph H of a connected signed graph $\Gamma = (G, \sigma)$ an *essential spanning subgraph* of Γ if either Γ is balanced and H is a spanning tree of G, or Γ is not balanced, $V(H) = V(\Gamma)$, and every component of H is a unicyclic graph whose unique cycle is negative. The following is the Matrix-Tree Theorem for signed graphs:

Theorem 2.3 (Matrix-Tree Theorem for signed graphs [7, 8]) *Let Γ be a connected signed graph with n vertices and let b_l be the number of essential spanning subgraphs which contain l cylces. Then* $\det L(\Gamma) = \sum_{l=0}^{n} 4^l b_l$.

We can also describe $L(\Gamma)$ by means of its quadratic form:

$$x^t L(\Gamma) x = x^t C C^t x = \sum_{v_i v_j \in E(\Gamma)} (x_i - \sigma(v_i v_j) x_j)^2,$$

where $x = (x_1, x_2, \ldots, x_n)^t \in \mathbf{R}^V$. By the Matrix-Tree Theorem for signed graphs we obtain:

Corollary 2.4 *Let $\Gamma = (G, \sigma)$ be a connected signed graph and $L(\Gamma)$ be its Laplacian matrix. Then* $\det L(\Gamma) = 0$ *if and only if Γ is balanced.*

It follows from Corollary 2.4 that 0 is a Laplacian eigenvalue of Γ if and only if Γ is balanced.

3 The Least Laplacian Eigenvalue and the Unbalanceness of a Signed Graph

Let $\Gamma = (G, \sigma)$ be a signed graph and $L(\Gamma)$ be its Laplacian matrix. Since $L(\Gamma)$ is positive semidefinite, all eigenvalues of $L(\Gamma)$ are nonnegative, we call the eigenvalues of $L(\Gamma)$ Laplacian eigenvalues of the signed graph Γ. Let $\lambda_1(\Gamma)$ or $\lambda_1(\sigma)$ denote the least Laplacian eigenvalue of the signed graph $\Gamma = (G, \sigma)$. Among the Laplacian eigenvalues of Γ, the most important ones are the least eigenvalues $\lambda_1(\Gamma)$ and the largest eigenvalue $\lambda_{\max}(\Gamma)$. In [10] the largest eigenvalue $\lambda_{\max}(\Gamma)$ was investigated and in this section we focus our attention on the least eigenvalue of a signed graph. By means of the Rayleigh quotient, we have

$$\lambda_1(\sigma) = \min \left\{ \sum_{uv \in E(\Gamma)} (f(u) - \sigma(uv) f(v))^2 \,\middle|\, \sum_{v \in V} f^2(v) = 1, f \in \mathbf{R}^V \right\}. \tag{3.1}$$

Expression (3.1) can be used to get combinatorial upper bounds on $\lambda_1(\Gamma)$. For example, we have

Lemma 3.1 *Let s, t be adjacent vertices of a signed graph Γ. Then*

$$\lambda_1(\Gamma) \le \frac{1}{2}(d_s + d_t - 2). \tag{3.2}$$

Proof Let $f \in \mathbf{R}^V$ be defined by

$$f(v) = \begin{cases} \dfrac{1}{\sqrt{2}}, & v = s, \\ \sigma(st)\frac{1}{\sqrt{2}}, & v = t, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\sum_{v \in V(\Gamma)} f^2(v) = 1$, (3.1) yields $\lambda_1(\Gamma) \le \sum_{uv \in E(\Gamma)} (f(u) - \sigma(uv) f(v))^2 = \dfrac{1}{2}(d_s + d_t - 2)$.

Let $\Gamma = (G, \sigma)$ be a connected signed graph. By Corollary 2.4, then the least Laplacian eigenvalue $\lambda_1(\Gamma)$ is zero if and only Γ is balanced, that is, all cycles in Γ are positive. Thus the least Laplacian eigenvalue of a signed graph Γ and its balancedness are closely related. In this section we will establish a relation between the least eigenvalue $\lambda_1(\Gamma)$ and a graph-theoretic parameter $\omega(\Gamma)$ of a signed graph Γ, which is somewhat similar to isoperimetric number inequality in [3] and non-bipartiteness in [11].

Let S be a subset of $V(\Gamma)$. We denote by ∂S the set of edges which connect the vertices in S to the vertices outside S. Then we define a parameter $\omega(\Gamma)$ of a signed graph Γ as

$$\omega(\Gamma) = \min_{\emptyset \ne S \subseteq V} \frac{e_{\min}(S) + |\partial(S)|}{|S|}, \tag{3.3}$$

where $e_{min}(S)$ is the minimum number of edges that need to be removed from the signed subgraph induced by S to make it balanced. It follows from Theorem 2.2 that if $\omega(\Gamma) = \frac{e_{\min}(S) + |\partial(S)|}{|S|}$ then $S = S_1 \cup S_2$ for subsets S_1, S_2 of S and $e_{\min}(S) = |E^-(S_1)| + |E^-(S_2)| + |E^+(S_1, S_2)|$, where $E^-(S_i)$ is the set of negative edges whose vertices are in S_i, $i = 1, 2$, and $E^+(S_1, S_2)$ is the set of positive edges with one vertex in S_1 and the other in S_2. Note that $\Gamma = (G, -)$ is balanced if and only if G is bipartite. Thus the parameter $\omega(\Gamma)$ here is a generalization of the parameter defined in [11].

Proposition 3.2 *Let Γ be a signed graph. Then $\omega(\Gamma) = 0$ if and only if $L(\Gamma)$ is singular.*

Proof Note that $\omega(\Gamma) = 0$ if and only if there exists an $S \subseteq V$ such that $e_{\min}(S) = 0$ and $|\partial(S)| = 0$. That is, $\omega(\Gamma) = 0$ if and only if Γ has a balanced component. Hence Proposition 3.2 follows from Corollary 2.4.

The next result gives an upper bound of $\lambda_1(\Gamma)$ in terms of the parameter $\omega(\Gamma)$.

Proposition 3.3 *Let Γ ba a signed graph. Then $\lambda_1(\Gamma) \leq 4\omega(\Gamma)$.*

Proof Let $\omega(\Gamma) = \frac{e_{\min}(S)+|\partial(S)|}{|S|}$. Then $S = S_1 \cup S_2$ for subsets S_1, S_2 of S and $e_{\min}(S) = |E^-(S_1)| + |E^-(S_2)| + |E^+(S_1, S_2)|$. Set

$$f(u) = \begin{cases} 1, & u \in S_1; \\ -1, & u \in S_2; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lambda_1(\Gamma) \leq \frac{f^t L(\Gamma) f}{f^t f} = \frac{4|E^-(S_1)| + 4|E^-(S_2)| + 4|E^+(S_1, S_2)| + |\partial(S)|}{|S|} \leq 4\omega(\Gamma).$$

In the main result of this paper, Theorem 3.4, we will give a lower bound of $\lambda_1(\Gamma)$ in terms of the parameter $\omega(\Gamma)$ and the largest degree Δ of a signed graph. It is somewhat similar to a result of Mohar [3, Theorem 3.10] about the relationship between $\lambda_1(\Gamma)$ and the isoperimetric number of a graph G. Before stating it, we need an auxiliary function $h(S)$ defined on a nonempty subset S of $V(\Gamma)$ as follows:

$$h_\Gamma(S) = \min_{T \subseteq S} \frac{|\partial(T)|}{|T|}, \tag{3.4}$$

where the minimization is carried out over all nonempty subsets T of S.

Theorem 3.4 *Let $\Gamma = (G, \sigma)$ be a connected signed graph with the largest degree Δ. $L(\Gamma)$ and $\omega(\Gamma)$ are as above. Then the smallest Laplacian eigenvalue $\lambda_1(\Gamma)$ is bounded below as*

$$\lambda_1(\Gamma) \geq \Delta - \sqrt{\Delta^2 - \omega^2(\Gamma)}. \tag{3.5}$$

Proof Let f be a unit eigenvector corresponding to $\lambda_1(\Gamma)$. Then

$$\lambda_1(\Gamma) = f^t L(\Gamma) f = \sum_{uv \in E(\Gamma)} (f(u) - \sigma(uv) f(v))^2. \tag{3.6}$$

Let the vertex set of G be $V = \{1, 2, \ldots, n\}$, $S_1 = \{i \in V : f(i) > 0\}$, and $S_2 = \{i \in V : f(i) < 0\}$. Then at least one of S_1 and S_2 is non-empty. We will construct a new signed graph $\Gamma' = (G', \sigma')$ as follows: Create a set S_1', which consists of copies of vertices in S_1 i.e., $S_1' = \{i' : i \in S_1\}$. In the same way, we create a set S_2' of copies of vertices in S_2. Now we define the graph $\Gamma' = (G', \sigma')$ with a vertex set $V' = V \cup S_1' \cup S_2'$ and a signed edge set E' defined as follows. If $(i, j) \in E(S_k)$, $k = 1, 2$, and $\sigma(ij) = -$ then we introduce two negative edges (i, j') and (j, i') in E'. If $(i, j) \in E(S_1, S_2)$, and $\sigma(ij) = +$ then we introduce two positive edges (i, j') and (j, i') in E'. For any other signed edge $(i, j) \in E$, we just introduce the single signed edge (i, j) in E', whose sign is the same as $\sigma(ij)$. Clearly, Γ' has the same maximum degree Δ as Γ. Define the function $g : V' \mapsto \mathbf{R}$ by $g(i) = f(i)$ for $i \in S_1 \cup S_2$, and $g(i) = 0$ otherwise. It is not difficult to check that

$$\sum_{ij \in E} (f(i) - \sigma(ij) f(j))^2 \geq \sum_{ij \in E'} (g(i) - \sigma'(ij) g(j))^2 = g^t L(\Gamma') g \tag{3.7}$$

and $\sum_{i \in V'} g^2(i) = \sum_{i \in V} f^2(i) = 1$. In what follows we show that the parameter $h_{\Gamma'}(S_1 \cup S_2)$ defined for the graph Γ' as in (3.4) is at least $\omega(\Gamma)$. Let W be an arbitrary subset of $S_1 \cup S_2$.

Suppose $T_1 = W \cap S_1$ and $T_2 = W \cap S_2$. Then, in Γ, we have

$$\omega(\Gamma)|T_1 \cup T_2| \leq |E^-(T_1)| + |E^-(T_2)| + |E^+(T_1, T_2)| + |\partial(T_1 \cup T_2)|. \tag{3.8}$$

But for each edge ij in $E^-(T_k)$, $k = 1, 2$, and each edge ij in $E^+(T_1, T_2)$, there correspond two edges ij' and ji' in $\partial(T_1 \cup T_2)$ in Γ' and for each edge in $\partial(T_1 \cup T_2)$ in Γ, there corresponds a uniquely defined edge in $\partial(T_1 \cup T_2)$ in Γ'. Let $\partial'(T_1 \cup T_2)$ denote $\partial(T_1 \cup T_2)$ in Γ'. Thus, from (3.8), we conclude that, in Γ',

$$\omega(\Gamma)|T_1 \cup T_2| \leq |\partial'(T_1 \cup T_2)|, \quad \text{for } \ T_1 \cup T_2 = W \subseteq S_1 \cup S_2. \tag{3.9}$$

From (3.4) and (3.9), $h_{\Gamma'}(S_1 \cup S_2) \geq \omega(\Gamma)$ in Γ'.

Let we introduce

$$F := \sum_{(i,j)\in E'} |g^2(i) - g^2(j)|, \tag{3.10}$$

and also let $0 = t_0 < t_1 < \cdots < t_m$, be all distinct values of $|g(i)|(i \in V(G'))$. For $k = 0, 1, \ldots, m$, we let $V'_k = \{i \in V(\Gamma') : |g(i)| \geq t_k\}$. Then

$$\begin{aligned}
\sum_{(i,j)\in E'} |g^2(i) - g^2(j)| &= \sum_{\substack{(i,j)\in E' \\ |g(i)|>|g(j)|}} (g^2(i) - g^2(j)) = \sum_{k=1}^{m} \sum_{(i,j)\in\partial(V'_k)} (t_k^2 - t_{k-1}^2) \\
&= \sum_{k=1}^{m} |\partial(V'_k)|(t_k^2 - t_{k-1}^2) \geq h_{\Gamma'}(S_1 \cup S_2) \sum_{k=1}^{m} |V'_k|(t_k^2 - t_{k-1}^2) \\
&= h_{\Gamma'}(S_1 \cup S_2) \sum_{k=0}^{m} t_k^2(|V'_k| - |V'_{k+1}|) = h_{\Gamma'}(S_1 \cup S_2) \sum_{i\in V'} g(i)^2 \\
&= h_{\Gamma'}(S_1 \cup S_2).
\end{aligned} \tag{3.11}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
F^2 &= \left(\sum_{(i,j)\in E'} |g^2(i) - g^2(j)| \right)^2 \leq \sum_{(i,j)\in E'} (g(i) - g(j))^2 \sum_{(i,j)\in E'} (g(i) + g(j))^2 \\
&= (g^t L(\Gamma')g)\{g^t[2\,\mathrm{diag}(d_1, d_2, \ldots, d_{|V(\Gamma')|}) - L(\Gamma')]g\} \\
&\leq \{f^t(L(\Gamma))f\}\{f^t[2\,\mathrm{diag}(d_1, d_2, \ldots, d_{|V(\Gamma)|})f - f^t L(\Gamma)]f\} \\
&\leq \{f^t(L(\Gamma))f\}\{2\Delta f^t f - f^t L(\Gamma)f\} = \lambda_1(\Gamma)[2\Delta - \lambda_1(\Gamma)].
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) we obtain $\lambda_1(\Gamma)[\lambda_1(\Gamma) - 2\Delta] + h^2_{\Gamma'}(S_1 \cup S_2) \leq 0$. Thus, from the above inequality and $h_{\Gamma'}(S_1 \cup S_2) \geq \omega(\Gamma)$, we obtain the following lower bound of $\lambda_1(\Gamma)$:

$$\lambda_1(\Gamma) \quad \geq \Delta - \sqrt{\Delta^2 - h^2_{\Gamma'}(S_1 \cup S_2)} \geq \Delta - \sqrt{\Delta^2 - \omega^2(G)}.$$

In general, computing $\lambda_1(\Gamma)$ is easier than computing $\omega(\Gamma)$. The other form of Theorem 3.4 provides an upper bound for $\omega(\Gamma)$.

Corollary 3.5 *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph. Then* $\omega(\Gamma) \leq \sqrt{\lambda_1(\Gamma)[2\Delta - \lambda_1(\Gamma)]}$.

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