

## 一、初等矩阵

**定义 1** 设  $u, v \in C^n, \sigma \in C$ , 则称

$$E(u, v, \sigma) = E - \sigma uv^H \quad \text{为初等矩阵.}$$

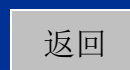
1. 初等矩阵的特征向量 ( $u, v \neq 0, \sigma \neq 0$ ).

(1)  $u \in v^\perp$ , 设  $u_1, \dots, u_{n-1}$  是  $v^\perp$  的一组基, 它们也是  $E(u, v, \sigma)$  的  $n-1$  个线性无关的特征向量.

(2)  $u \notin v^\perp$ , 设  $u_1, \dots, u_{n-1}$  是  $v^\perp$  的一组基, 则  $u, u_1, \dots, u_{n-1}$  是  $E(u, v, \sigma)$  的  $n$  个线性无关的特征向量.

2. 初等矩阵的特征值

$$\lambda(E(u, v, \sigma)) = \{1, 1, \dots, 1, 1 - \sigma v^H u\}$$



$$3. \det(E(u, v, \sigma)) = 1 - \sigma v^H u$$

$$4. E(u, v, \sigma)^{-1} = E(u, v, \frac{\sigma}{\sigma v^H u - 1}), (1 - \sigma v^H u \neq 0)$$

5. 非零向量  $a, b \in C^n$ , 存在  $u, v, \sigma$ , 使得

$$E(u, v, \sigma)a = b, (\sigma u = \frac{a - b}{v^H a}).$$

6. 初等变换矩阵

$$E_{ij} = E - (e_i - e_j)(e_i - e_j)^T = E(e_i - e_j, e_i - e_j, 1)$$

$$E_{ij}(k) = E + k e_j e_i^T = E(e_j, e_i, -k)$$

$$E_i(k) = E - (1 - k)e_i e_i^T = E(e_i, e_i, 1 - k)$$



## 7. 初等酉阵 (*Householder*变换)

$$H(u) = E(u, u; 2) = E - 2uu^H, (u^H u = 1)$$

$$(1) H(u)^H = H(u) = H(u)^{-1}$$

$$(2) H(u)(a + ru) = a - ru, \forall a \in u^\perp, r \in \mathbb{C} \text{ (镜象变换)}$$

### (3) *Householder*变化的特性

$$\begin{aligned} (H(u)x, H(u)y) &= (x, y) \Rightarrow \|H(u)x\|^2 = (H(u)x, H(u)x) = (x, x) = \|x\|^2 \\ &\Rightarrow \|H(u)x\| = \|x\| \end{aligned}$$



## 二. 投影变换与矩阵

(1) **斜投影**  $P_{LM}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)A \Rightarrow A = A^2$

(2) **正交投影**  $P_{LM}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)A \Rightarrow \begin{cases} A^2 = A \\ A^H = A \end{cases}$

(3)  $A \in C^{m \times n}$ , 则

$R(A) = \{y \mid y = Ax, \forall x \in C^n\}$  ———  $A$  的值域;

$N(A) = \{x \mid Ax = 0, \forall x \in C^n\}$  ———  $A$  的核.

$$\text{rank}(A) = \text{rank}(AB) \Rightarrow R(AB) = R(A)$$

(4)  $A \in C^{n \times m}, B \in C^{n \times s}, R(A) \perp R(B) \Leftrightarrow A^H B = 0$



$$(5) A \in C^{m \times n} \Rightarrow \begin{cases} \dim R(A) + \dim N(A^H) = m \\ \dim R(A^H) + \dim N(A) = n \\ C^m = R(A) \oplus N(A^H) \\ C^n = R(A^H) \oplus N(A) \end{cases}$$

$$(6) A \in C^{n \times n}, A = A^2 \Rightarrow \begin{cases} (1) & A^H = (A^H)^2, E - A = (E - A)^2 \\ (2) & \sigma(A) = \{\lambda \mid Ax = \lambda x, x \neq 0\} = \{0, 1\} \\ (3) & rank(A) = tr(A) \\ (4) & A(E - A) = (E - A)A = 0 \\ (5) & A\alpha = \alpha \Leftrightarrow \alpha \in R(A) \\ (6) & N(A) = R(E - A), R(A) = N(E - A) \end{cases}$$

$$(7) A \in C^{n \times n}, A = A^2 \Rightarrow A = A^H \Leftrightarrow C^n = R(A) \oplus N(A), R^\perp(A) = N(A)$$



### 三、Kronecker积(和)

$$A = (a_{ij}) \in P^{m \times n}, B = (b_{ij}) \in P^{p \times q}$$

$$1. \text{Kronecker积} \Leftrightarrow A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in P^{mp \times nq}$$

2.  $m$ 阶矩阵 $A$ 与 $n$ 阶矩阵 $B$ 的Kronecker 和

$$A \oplus_k B = A \otimes E_n + E_m \otimes B$$

$$(A \oplus_k B = E_n \otimes A + B \otimes E_m)$$



### 3. Kronecker积的性质:

设  $A \in P^{m \times n}$ ,  $B \in P^{p \times q}$ ,  $C \in P^{r \times s}$ ,  $D \in P^{k \times h}$ , 则

$$(1) \quad E_m \otimes E_n = E_{mn}$$

$$(2) \quad \lambda(A \otimes B) = (\lambda A) \otimes B = A \otimes (\lambda B)$$

$$(3) \quad (A + B) \otimes C = (A \otimes C) + (B \otimes C)$$

$$(4) \quad (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$(5) \quad (A \otimes B)^T = A^T \otimes B^T, \quad \overline{(A \otimes B)} = \bar{A} \otimes \bar{B}, \quad (A \otimes B)^H = A^H \otimes B^H$$



(6)  $A \in P^{m \times n}, B \in P^{p \times q}, C \in P^{n \times s}, D \in P^{q \times h}$ , 则

$$\underline{(A \otimes B)(C \otimes D) = AC \otimes BD}$$

(7)  $A \in P^{m \times m}, B \in P^{p \times p}$ , 且  $A, B$  可逆, 则

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

(8)  $A \in P^{m \times m}, B \in P^{p \times p}$ , 则  $tr(A \otimes B) = trA \bullet trB$

(9)  $rank(A \otimes B) = rankA \bullet rankB$





(10)  $A \in P^{m \times m}, B \in P^{p \times p}$ , 则

$$\det(A \otimes B) = (\det A)^p \cdot (\det B)^m$$

(11) 当  $A^T = A, B^T = B$  时,  $A \otimes B$  也是对称矩阵;

当  $A^H = A, B^H = B$  时,  $A \otimes B$  也是 Hermite 矩阵;

(12) 当  $U, V$  均为酉矩阵时,  $U \otimes V$  也是酉矩阵;



返回

## 4、Kronecker积(和)的特征值

**定理1:** 设 $\lambda_i (i = 1, 2, \dots, m)$ 为 $A \in C^{m \times m}$ 的特征值,  $x_i (i = 1, 2, \dots, m)$ 为相应的特征向量;  $\mu_j (j = 1, 2, \dots, n)$ 为 $B \in C^{n \times n}$ ,  $y_j (j = 1, 2, \dots, n)$ 为相应的特征向量, 则 $A \otimes B$ 有 $mn$ 个特征值为 $\lambda_i \mu_j$ , 对应的特征向量为 $x_i \otimes y_j$ .

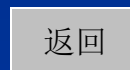
**定理2:** 设 $\lambda_i (i = 1, 2, \dots, m)$ 为 $A \in C^{m \times m}$ 的特征值,  $x_i (i = 1, 2, \dots, m)$ 为相应的特征向量;  $\mu_j (j = 1, 2, \dots, n)$ 为 $B \in C^{n \times n}$ ,  $y_j (j = 1, 2, \dots, n)$ 为相应的特征向量, 则 $\lambda_i + \mu_j$ 是 $A \oplus_k B$ 的特征值,  $x_i \otimes y_j$ 为对应的特征向量.



## 5、向量化算符

$$\text{设 } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = (A_{c1}, A_{c2}, \dots, A_{cn})$$

(1). 向量化算符:  $Vec A = \begin{pmatrix} A_{c1} \\ A_{c2} \\ \vdots \\ A_{cn} \end{pmatrix}$  ---- 矩阵A的拉直.



**(2).性质:**  $Vec (kA + lB) = kVec A + lVec B$

$$A \in P^{m \times n}, \text{ 且 } A = \alpha\beta^T, Vec (\alpha\beta^T) = \beta \otimes \alpha$$

**定理3:** 设  $A \in C^{m \times n}, X \in C^{n \times r}, B \in C^{r \times s}$ , 则

$$Vec (AXB) = (B^T \otimes A)Vec X$$

**推论1:** 设  $A \in C^{m \times m}, B \in C^{n \times n}, X \in C^{m \times n}$ , 则

$$(1) Vec (AX) = (E_n \otimes A)Vec X;$$

$$(2) Vec (XB) = (B^T \otimes E_m)Vec X.$$

$$(3) vec(AX + XB) = [(E_n \otimes A) + (B^T \otimes E_m)]vec(X)$$



## 6.Kronecker 乘积的应用 -----矩阵方程的求解

### (1): *Sylvester* 方程

$$AX + XB = D$$

$$\Leftrightarrow \text{vec}(AX + XB) = [(E \otimes A) + (B^T \otimes E)]\text{vec}(X) = \text{vec}(D)$$

特别  $B = A^T$ , 即  $AX + XA^T = D$  -----*Lyapunov* 方程

### (2): $AXB = D$

$$\Leftrightarrow \text{vec}(AXB) = (B^T \otimes A)\text{vec}(X) = \text{vec}(D)$$

### (3): $A_1XB_1 + A_2XB_2 = D$

$$\Leftrightarrow [(B_1^T \otimes A_1) + (B_2^T \otimes A_2)]\text{vec}(X) = \text{vec}(D)$$

