§6 Kronecker 乘积

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1　基本概念和基本性质

定义 1 设 $\boldsymbol{A}=(a_{ij})\in\mathbf{P}^{m\times n},\boldsymbol{B}=(b_{ij})\in\mathbf{P}^{p\times q}$, 则

$$\boldsymbol{A}\otimes\boldsymbol{B}=\begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B}\\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \cdots & a_{2n}\boldsymbol{B}\\ \cdots & \cdots & \cdots & \cdots\\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B}\end{bmatrix}$$

称为矩阵 $\boldsymbol{A}$ 与 $\boldsymbol{B}$ 的 Kronecker 乘积 (或直积, 张量积).

例 1 若 $\boldsymbol{A}=\begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}2&1\\2&3\end{bmatrix}$, 则

$$\boldsymbol{A}\otimes\boldsymbol{B}=\begin{bmatrix}\boldsymbol{B}&2\boldsymbol{B}&3\boldsymbol{B}\\3\boldsymbol{B}&2\boldsymbol{B}&\boldsymbol{B}\end{bmatrix}=\begin{bmatrix}2&1&4&2&6&3\\2&3&4&6&6&9\\6&3&4&2&2&1\\6&9&4&6&2&3\end{bmatrix}.$$

一般情况下，$\boldsymbol{A}\otimes\boldsymbol{B}\neq\boldsymbol{B}\otimes\boldsymbol{A}$.

定理 1 设 $\boldsymbol{A}\in\mathbf{P}^{m\times n},\boldsymbol{B}\in\mathbf{P}^{p\times q},\boldsymbol{C}\in\mathbf{P}^{r\times s},\boldsymbol{D}\in\mathbf{P}^{k\times h}$, 则

(1) 单位矩阵之积: $\boldsymbol{E}_m\otimes\boldsymbol{E}_n=\boldsymbol{E}_{mn}$;

(2) 纯量积: $\lambda(\boldsymbol{A}\otimes\boldsymbol{B})=(\lambda\boldsymbol{A})\otimes\boldsymbol{B}=\boldsymbol{A}\otimes(\lambda\boldsymbol{B}),\forall\lambda\in\mathbf{P}$;

(3) 分配律: 当 $m=p,n=q$ 时, $(\boldsymbol{A}+\boldsymbol{B})\otimes\boldsymbol{C}=(\boldsymbol{A}\otimes\boldsymbol{C})+(\boldsymbol{B}\otimes\boldsymbol{C})$, $\boldsymbol{C}\otimes(\boldsymbol{A}+\boldsymbol{B})=(\boldsymbol{C}\otimes\boldsymbol{A})+(\boldsymbol{C}\otimes\boldsymbol{B})$;

(4) 结合律: $(\boldsymbol{A}\otimes\boldsymbol{B})\otimes\boldsymbol{C}=\boldsymbol{A}\otimes(\boldsymbol{B}\otimes\boldsymbol{C})$;

(5) 转置及共轭: $(\boldsymbol{A}\otimes\boldsymbol{B})^{\boldsymbol{T}}=\boldsymbol{A}^{\boldsymbol{T}}\otimes\boldsymbol{B}^{\boldsymbol{T}}$, $\overline{(\boldsymbol{A}\otimes\boldsymbol{B})}=\bar{\boldsymbol{A}}\otimes\bar{\boldsymbol{B}}$;

(6) 混合积: 当 $\boldsymbol{n}=\boldsymbol{r},\boldsymbol{q}=\boldsymbol{k}$ 时, 有 $(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D})=(\boldsymbol{AC})\otimes(\boldsymbol{BD})$;

(7) 逆: 若 $\boldsymbol{A}^{-1},\boldsymbol{B}^{-1}$ 存在, 则

$$(\boldsymbol{A}\otimes\boldsymbol{B})^{-1}=\boldsymbol{A}^{-1}\otimes\boldsymbol{B}^{-1};$$

(8) 迹: 当 $\boldsymbol{m}=\boldsymbol{n},\boldsymbol{p}=\boldsymbol{q}$ 时, 有 $tr(\boldsymbol{A}\otimes\boldsymbol{B})=tr\boldsymbol{A}\cdot tr\boldsymbol{B}$;

(9) 秩: $rank(\boldsymbol{A}\otimes\boldsymbol{B})=rank\boldsymbol{A}\cdot rank\boldsymbol{B}$;

(10) 行列式: 当 $\boldsymbol{m}=\boldsymbol{n},\boldsymbol{p}=\boldsymbol{q}$ 时, 有 $\det(\boldsymbol{A}\otimes\boldsymbol{B})=(\det\boldsymbol{A})^{\boldsymbol{p}}(\det\boldsymbol{B})^{\boldsymbol{m}}$;

(11) 当 $\boldsymbol{A},\boldsymbol{B}$ 为对称矩阵时, $\boldsymbol{A}\otimes\boldsymbol{B}$ 也是对称矩阵; 当 $\boldsymbol{A},\boldsymbol{B}$ 为 *Herimtian* 矩阵时, $\boldsymbol{A}\otimes\boldsymbol{B}$ 也是 *Hermitian* 矩阵;

(12) $\boldsymbol{U}\in\mathrm{P}^{\boldsymbol{n}\times\boldsymbol{n}},\boldsymbol{V}\in\boldsymbol{P}^{\boldsymbol{m}\times\boldsymbol{m}}$ 均为酉矩阵, $\boldsymbol{U}\otimes\boldsymbol{V}$ 也是酉矩阵;

(13) 若令 $\boldsymbol{A}^{[0]}=\mathbf{1}$, $\boldsymbol{A}^{[1]}=\boldsymbol{A},\boldsymbol{A}^{[\boldsymbol{k}]}=\boldsymbol{A}\otimes\boldsymbol{A}\otimes\cdots\otimes\boldsymbol{A}$, 则 $(\boldsymbol{AB})^{[\boldsymbol{k}]}=\boldsymbol{A}^{[\boldsymbol{k}]}\boldsymbol{B}^{[\boldsymbol{k}]}$.

证明：(6)

$$\begin{aligned}(\boldsymbol{A}\otimes\boldsymbol{B})(\boldsymbol{C}\otimes\boldsymbol{D})&=\begin{bmatrix}a_{11}B&\cdots&a_{1n}B\\\vdots&\ddots&\vdots\\a_{m1}B&\cdots&a_{mn}B\end{bmatrix}\begin{bmatrix}c_{11}D&\cdots&c_{1s}D\\\vdots&\ddots&\vdots\\c_{n1}D&\cdots&c_{ns}D\end{bmatrix}\\&=\begin{bmatrix}\sum\limits_{k=1}^{n}a_{1k}c_{k1}BD&\cdots&\sum\limits_{k=1}^{n}a_{1k}c_{ks}BD\\\vdots&\ddots&\vdots\\\sum\limits_{k=1}^{n}a_{mk}c_{k1}BD&\cdots&\sum\limits_{k=1}^{n}a_{mk}c_{ks}BD\end{bmatrix}\\&=AC\otimes BD.\end{aligned}$$

(10) 由 Jordan 标准型分解知，存在可逆矩阵 $\boldsymbol{P},\boldsymbol{Q}$，使得

$$\boldsymbol{A}=\boldsymbol{P}^{-1}\begin{bmatrix}\lambda_1&&&\star\\&\lambda_2&&\\&&\ddots&\\0&&&\lambda_m\end{bmatrix}\boldsymbol{P}=\boldsymbol{P}^{-1}\boldsymbol{J_A}\boldsymbol{P}$$

和

$$\boldsymbol{B}=\boldsymbol{Q}^{-1}\begin{bmatrix}\mu_1&&&\star\\&\mu_2&&\\&&\ddots&\\0&&&\mu_m\end{bmatrix}\boldsymbol{Q}=\boldsymbol{Q}^{-1}\boldsymbol{J_B}\boldsymbol{Q},$$

则利用第 (6) 条性质有

$$\boldsymbol{A\otimes B=(P^{-1}J_AP)\otimes(Q^{-1}J_BQ)=(P\otimes Q)^{-1}(J_A\otimes J_B)(P\otimes Q).}$$

于是有

$$\begin{aligned}\det(A\otimes B)&=\det(J_A\otimes J_B)\\&=(\prod_{j=1}^{p}\lambda_1\mu_j)(\prod_{j=1}^{p}\lambda_2\mu_j)\cdots(\prod_{j=1}^{p}\lambda_m\mu_j)\\&=(\prod_{i=1}^{m}\lambda_i)^p(\prod_{j=1}^{p}\mu_j)^m\\&=(\det(A))^p(\det(B))^m.\end{aligned}$$

2 Kronecker 积的特征值

定理 2 设 $\lambda_i(i=1,2,\dots,m)$ 是 $A\in C^{m\times m}$ 的特征值, x_i 是相应的特征向量; $\mu_j(j=1,2,\dots,n)$ 是 $B\in C^{n\times n}$ 的特征值, y_j 是相应的特征向量, 则 $A\otimes B$ 的 mn 个特征值是 $\lambda_i\mu_j$, 对应的特征向量是 $x_i\otimes y_j$, $1\le i\le m, 1\le j\le n$.

证明: 因为 $Ax_i=\lambda_ix_i$, $By_j=\mu_jy_j$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, 则

$$\begin{aligned}(A\otimes B)(x_i\otimes y_j)&=Ax_i\otimes By_j\\&=\lambda_ix_i\otimes\mu_jy_j\\&=\lambda_i\mu_j(x_i\otimes y_j),\end{aligned}$$

得证.

定义 2 m 阶矩阵 A 与 n 阶矩阵 B 的 *Kronecker* 和定义为

$$A\oplus_k B=A\otimes E_n+E_m\otimes B.$$

定理 3 设 $\lambda_i(i=1,2,\dots,m)$ 是 $A\in C^{m\times m}$ 的特征值, x_i 是相应的特征向量; $\mu_j(j=1,2,\dots,n)$ 是 $B\in C^{n\times n}$ 的特征值, y_j 是相应的特征向量, 则 $A\oplus_k B$ 的 mn 个特征值是 $\lambda_i+\mu_j$, 对应的特征向量是 $x_i\otimes y_j$, $1\le i\le m, 1\le j\le n$.

证明: 因为 $Ax_i=\lambda_ix_i$, $By_j=\mu_jy_j$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, 则

$$\begin{aligned}(A\oplus_k B)(x_i\otimes y_j)&=(A\otimes E_n)(x_i\otimes y_j)+(E_m\otimes B)(x_i\otimes y_j)\\&=(Ax_i)\otimes y_j+x_i\otimes(By_j)\\&=(\lambda_i+\mu_j)(x_i\otimes y_j),\end{aligned}$$

得证.

3 向量化算符

设矩阵

$$\boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

记 $\boldsymbol{A}$ 的列为 $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$, 即 $\boldsymbol{A}=(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n)$. 向量化算符: $\mathbf{Vec}(\boldsymbol{A})=\begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{A}_n \end{bmatrix}$

定理 4 设 $\boldsymbol{A}\in \mathbf{C}^{m\times n}$, $\boldsymbol{X}\in \mathbf{C}^{n\times r}$, $\boldsymbol{B}\in \mathbf{C}^{r\times s}$, 则

$$\mathbf{Vec}(\boldsymbol{AXB})=(\boldsymbol{B}^T\otimes \boldsymbol{A})\mathbf{Vec}(\boldsymbol{X}).$$

证明: 令 $\boldsymbol{B}_k$ 为矩阵 $\boldsymbol{B}$ 的第 $\boldsymbol{k}$ 列, 则

$$\begin{aligned} (\boldsymbol{AXB})_k &= \boldsymbol{A}(\boldsymbol{XB})_k \\ &= \boldsymbol{AXB}_k \\ &= \boldsymbol{A}(\boldsymbol{X}_1 b_{1k}+\ldots+\boldsymbol{X}_r b_{rk}) \\ &= b_{1k}\boldsymbol{AX}_1+\ldots+b_{rk}\boldsymbol{AX}_r \\ &= \begin{bmatrix} b_{1k}\boldsymbol{A}, \ldots, b_{rk}\boldsymbol{A} \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_1 \\ \vdots \\ \boldsymbol{X}_r \end{bmatrix} \\ &= (\boldsymbol{B}_k^T\otimes \boldsymbol{A})\mathbf{Vec}(\boldsymbol{X}), \quad k=1,\ldots,s. \end{aligned}$$

于是有

$$\mathbf{Vec}(\boldsymbol{AXB})=\begin{bmatrix} \boldsymbol{B}_1^T\otimes \boldsymbol{A} \\ \vdots \\ \boldsymbol{B}_s^T\otimes \boldsymbol{A} \end{bmatrix}\mathbf{Vec}(\boldsymbol{X})=(\boldsymbol{B}^T\otimes \boldsymbol{A})\mathbf{Vec}(\boldsymbol{X}).$$

推论 1 设 $\boldsymbol{A}\in \mathbf{C}^{m\times m}$, $\boldsymbol{X}\in \mathbf{C}^{m\times n}$, $\boldsymbol{B}\in \mathbf{C}^{n\times n}$, 则

(1) $\mathbf{Vec}(\boldsymbol{AX})=(\boldsymbol{E}_n\otimes \boldsymbol{A})\mathbf{Vec}(\boldsymbol{X})$;

(2) $\mathbf{Vec}(\boldsymbol{XB})=(\boldsymbol{B}^T\otimes \boldsymbol{E}_m)\mathbf{Vec}(\boldsymbol{X})$;

(3) $\mathbf{Vec}(\boldsymbol{AX}+\boldsymbol{XB})=(\boldsymbol{E}_n\otimes \boldsymbol{A}+\boldsymbol{B}^T\otimes \boldsymbol{E}_m)\mathbf{Vec}(\boldsymbol{X})$.