分段插值

分段插值函数

三次样条插值的概念

用一阶导数表示的样条

三次样条的极性

二元函数插值简介

分段线性插值

插值节点满足：$x_0<x_1<\cdots\cdots<x_n$ 已知

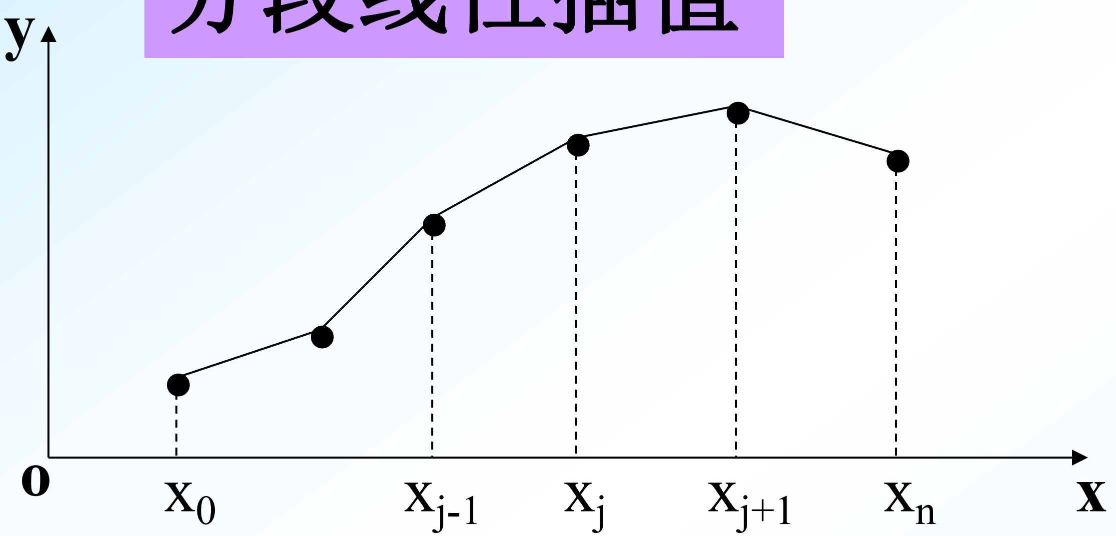
$$y_j=f(x_j) \qquad (j=0,1,2,\cdots,n)$$

$x\in[x_j,\ x_{j+1}]$时，线性插值函数

$$L_h(x)=\frac{x_{j+1}-x}{x_{j+1}-x_j}y_j+\frac{x-x_j}{x_{j+1}-x_j}y_{j+1}$$

$$(j=0,1,\cdots,n-1)$$

分段线性插值



$$L_n(x)=\sum_{j=0}^{n} y_j l_j(x)$$

$$l_j(x)=\begin{cases} \dfrac{x-x_{j-1}}{x_j-x_{j-1}}, & x_{j-1}\le x\le x_j \\ \dfrac{x-x_{j+1}}{x_j-x_{j+1}}, & x_j\le x\le x_{j+1} \\ 0, & \text{其它} \end{cases}$$

计算量与n无关；

n越大，误差越小.

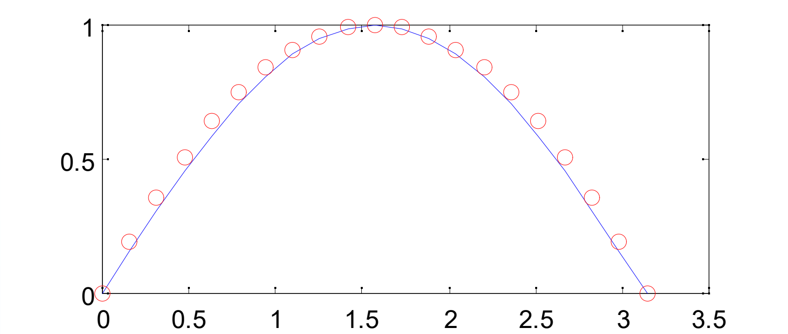
$$\lim_{n\to\infty} L_n(x)=g(x),\ x_0\le x\le x_n$$

三次样条插值的概念

例1. $\sin x$ 在区间$[0，\pi]$上的插值逼近

1. **二次插值**

x	0	$\pi/2$	π
$\sin x$	0	1	0



$$L_2(x) = \frac{4}{\pi^2} x(\pi - x)$$

$$|R_2(x)| = \frac{1}{6}|x(x-\pi/2)(x-\pi)|$$

定义 5.4 给定区间$[a, b]$上的一个分划:

$$a = x_0 < x_1 < \ldots < x_n = b$$

已知$f(x_j) = y_j \quad (j = 0,1,\cdots,n)$, 如果

$$S(x) = \begin{cases} S_1(x), x \in [x_0, x_1] \\ S_2(x), x \in [x_1, x_2] \\ \cdots\cdots\cdots\cdots \\ S_n(x), x \in [x_{n-1}, x_n] \end{cases}$$

满足: (1) $S(x)$在 $[x_j, \ x_{j+1}]$上为三次多项式;

(2) $S''(x)$在区间$[a, \ b]$上连续;

(3) $S(x_j) = y_j \ (j = 0,1,\cdots,n)$.

则称 $S(x)$为三次样条插值函数.

n个三次多项式, 待定系数共$4n$个!

当$x \in [x_j, x_{j+1}]$ $(j = 0,1,\ldots n-1)$时

$$S_j(x) = a_j + b_j x + c_j x^2 + d_j x^3$$

插值条件: $S(x_j) = y_j \quad (j = 0,1,\cdots,n)$

连续性条件: $S(x_j+0) = S(x_j-0) \quad (j = 1,\cdots,n-1)$

$$S'(x_j+0) = S'(x_j-0) \quad (j = 1,\cdots,n-1)$$

$$S''(x_j+0) = S''(x_j-0) \quad (j = 1,\cdots,n-1)$$

由样条定义,可建立方程$(4n-2)$个!

方程数少于未知数个数??

(1)自然边界条件: $S''(x_0)=0,\ S''(x_n)=0$

(2)周期边界条件: $S'(x_0)=S'(x_n),\ S''(x_0)=S''(x_n)$

(3)固定边界条件: $S'(x_0)=f'(x_0),\ S'(x_n)=f'(x_n)$

例2 5.7 已知$f(-1)=1, f(0)=0, f(1)=1$.求$[-1，1]$上的三次自然样条(满足自然边界条件).

解 设

$$S(x)=\begin{cases} a_1x^3+b_1x^2+c_1x+d_1, & x\in[-1,0] \\ a_2x^3+b_2x^2+c_2x+d_2, & x\in[0,1] \end{cases}$$

则有:

$$-a_1+b_1-c_1+d_1=1, \qquad d_1=d_2,$$

$$d_1=0, \qquad c_1=c_2,$$

$$a_2+b_2+c_2+d_2=1 \qquad b_1=b_2$$

由自然边界条件:

$$-6a_1+2b_1=0,\ 6a_2+2b_2=0$$

解方程组,得

$$a_1=-a_2=1/2,\quad b_1=b_2=3/2,$$

$$c_1=c_2=d_1=d_2=0.$$

问题的解

$$S(x)=\begin{cases}\dfrac{1}{2}x^3+\dfrac{3}{2}x^2, & x\in[-1,0]\\ -\dfrac{1}{2}x^3+\dfrac{3}{2}x^2, & x\in[0,1]\end{cases}$$

三次Hermite插值

$$H(x) = y_0\alpha_0(x) + y_1\alpha_1(x) + m_0\beta_0(x) + m_1\beta_1(x)$$

$$\alpha_0(x) = (1 + 2\frac{x - x_0}{x_1 - x_0})(\frac{x_1 - x}{x_1 - x_0})^2 \quad \beta_0(x) = (x - x_0)(\frac{x_1 - x}{x_1 - x_0})^2$$

$$\alpha_1(x) = (1 + 2\frac{x_1 - x}{x_1 - x_0})(\frac{x - x_0}{x_1 - x_0})^2 \quad \beta_1(x) = (x - x_1)(\frac{x - x_0}{x_1 - x_0})^2$$

x	x_0	x_1
$\alpha_0(x)$	1	0
$\alpha_0'(x)$	0	0
$\alpha_1(x)$	0	1
$\alpha_1'(x)$	0	0

x	x_0	x_1
$\beta_0(x)$	0	0
$\beta_0'(x)$	1	0
$\beta_1(x)$	0	0
$\beta_1'(x)$	0	1

用一阶导数表示的样条

已知函数表

x	x_0	x_1	……	x_n
$f(x)$	y_0	y_1	……	y_n

设 $f(x)$ 在各插值节点 x_j 处的一阶导数为 m_j

取 $x_{j+1}-x_j=h$，$(j=0,1,2,\cdots,n)$. 当 $x\in[x_j, x_{j+1}]$时,

分段Hermite插值

$$S(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2 y_j+(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2 y_{j+1}$$

$$+(x-x_j)(\frac{x_{j+1}-x}{h})^2 m_j+(x-x_{j+1})(\frac{x-x_j}{h})^2 m_{j+1}$$

由$S''(x)$连续,有等式： $S''(x_j+0)=S''(x_j-0)$

考虑 $S''(x)$ 在区间$[x_j, x_{j+1}]$和$[x_{j-1}, x_j]$上表达式.

当 $x\in[x_j, x_{j+1}]$时, $S(x)$ 由基函数组合而成

$$\alpha_j(x)=(1+2\frac{x-x_j}{h})(\frac{x_{j+1}-x}{h})^2$$

$$\alpha_{j+1}(x)=(1+2\frac{x_{j+1}-x}{h})(\frac{x-x_j}{h})^2$$

$$\beta_j(x)=(x-x_j)(\frac{x_{j+1}-x}{h})^2$$

$$\beta_{j+1}(x)=(x-x_{j+1})(\frac{x-x_j}{h})^2$$

$$\begin{cases} \alpha_j''(x_j) = [\frac{-8}{h^3}(x_{j+1} - x) + (1 + 2\frac{x - x_j}{h})\frac{2}{h^2}]_{x=x_j} = -\frac{6}{h^2} \\ \alpha_{j+1}''(x_j) = [-\frac{8}{h^3}(x - x_j) + (1 + 2\frac{x_{j+1} - x}{h})\frac{2}{h^2}]_{x=x_j} = \frac{6}{h^2} \end{cases}$$

$$\begin{cases} \beta_j''(x_j) = [\frac{4}{h^2}(x - x_{j+1}) + (x - x_j)\frac{2}{h^2}]_{x=x_j} = -\frac{4}{h} \\ \beta_{j+1}''(x_j) = [\frac{4}{h^2}(x - x_j) + (x - x_{j+1})\frac{2}{h^2}]_{x=x_j} = -\frac{2}{h} \end{cases}$$

$$\Rightarrow \quad S''(x_j + 0) = \alpha_j''(x_j)y_j + \alpha_{j+1}''(x_j)y_{j+1} + \beta_j''(x_j)m_j + \beta_{j+1}''(x_j)m_{j+1}$$

$$S''(x_j+0)=-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}$$

同理, 有

$$S''(x_j-0)=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

联立, 得

$$-\frac{6}{h^2}y_j+\frac{6}{h^2}y_{j+1}-\frac{4}{h}m_j-\frac{2}{h}m_{j+1}$$
$$=\frac{6}{h^2}y_{j-1}-\frac{6}{h^2}y_j+\frac{2}{h}m_{j-1}+\frac{4}{h}m_j$$

➔ $$m_{j-1}+4m_j+m_{j+1}=\frac{3}{h}(y_{j+1}-y_{j-1})$$

$(\ j=1, 2, \cdots\cdots, n\text{-}1\ )$

设自然边界条件成立, 即

$$S''(x_0+0) = -\frac{6}{h^2}y_0 + \frac{6}{h^2}y_1 - \frac{4}{h}m_0 - \frac{2}{h}m_1 = 0$$

$$S''(x_n-0) = \frac{6}{h^2}y_{n-1} - \frac{6}{h^2}y_n + \frac{2}{h}m_{n-1} + \frac{4}{h}m_n = 0$$

自然样条的导数值满足:

$$2m_0 + m_1 = \frac{3}{h}[y_1 - y_0] \triangleq g_0$$

$$m_{j-1} + 4m_j + m_{j+1} = \frac{3}{h}(y_{j+1} - y_{j-1}) \triangleq g_j$$

(j=1, 2, ⋯⋯, n-1)

$$m_{n-1} + 2m_n = \frac{3}{h}[y_n - y_{n-1}] \triangleq g_n$$

$$
\begin{bmatrix}
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1 & 4 & 1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & 1 & 4 & 1 \\
 & & & 1 & 2
\end{bmatrix}
\begin{bmatrix}
m_0 \\ m_1 \\ \vdots \\ m_{n-1} \\ m_n
\end{bmatrix}
=
\begin{bmatrix}
g_0 \\ g_1 \\ \vdots \\ g_{n-1} \\ g_n
\end{bmatrix}
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求解三对角方程组$Ax=f$等价于解两个三角形方程组

（1）求解$Ly=f$，得向量y；

（2）求解$Ux=y$，得方程组的解x。

样条插值函数的极性

设$f(x)\in C^2[a, b]$, 对于$a = x_0 < x_1 < \ldots < x_n = b$, 有$f(x_j)=y_j$ $(j=0,1,\cdots,n)$. $S(x)$是满足$S(x_j)=y_j$ $(j=0,1,\cdots,n)$ 的三次自然样条. 则有

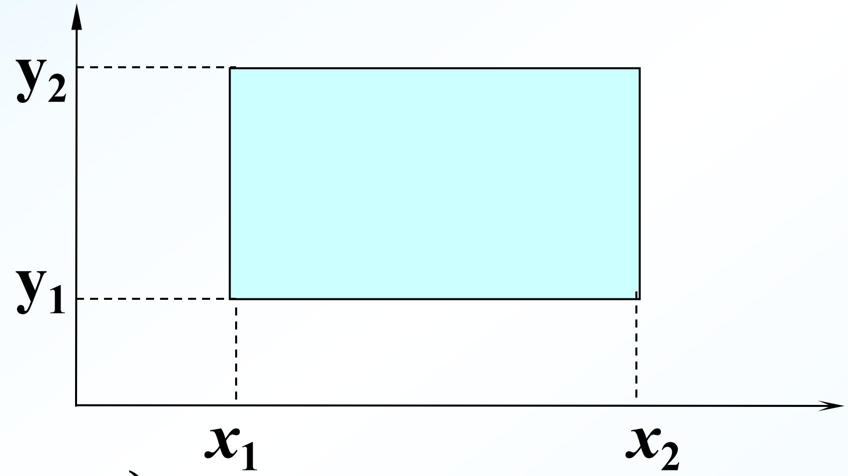
$$\|S''(x)\|\le\|f''(x)\|$$

即 $$\int_a^b [S''(x)]^2 dx \le \int_a^b [f''(x)]^2 dx$$

$$K = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}.$$

样条函数S(x)在[a, b]上的总曲率最小.

二元函数插值简介



矩形区域上函数$f(x, y)$的双线性插值.

$$P(x, y) = ax + by + cxy + d$$

插值条件：$P(x_1, y_1) = z_1$，$P(x_2, y_1) = z_2$，

$P(x_2, y_2) = z_3$，$P(x_1, y_2) = z_4$

$$P(x, y) = z_1(1-u)(1-v) + z_2\, u(1-v) + z_3\, u\, v + z_4\,(1-u)v$$

其中

$$u = \frac{x - x_1}{x_2 - x_1}$$

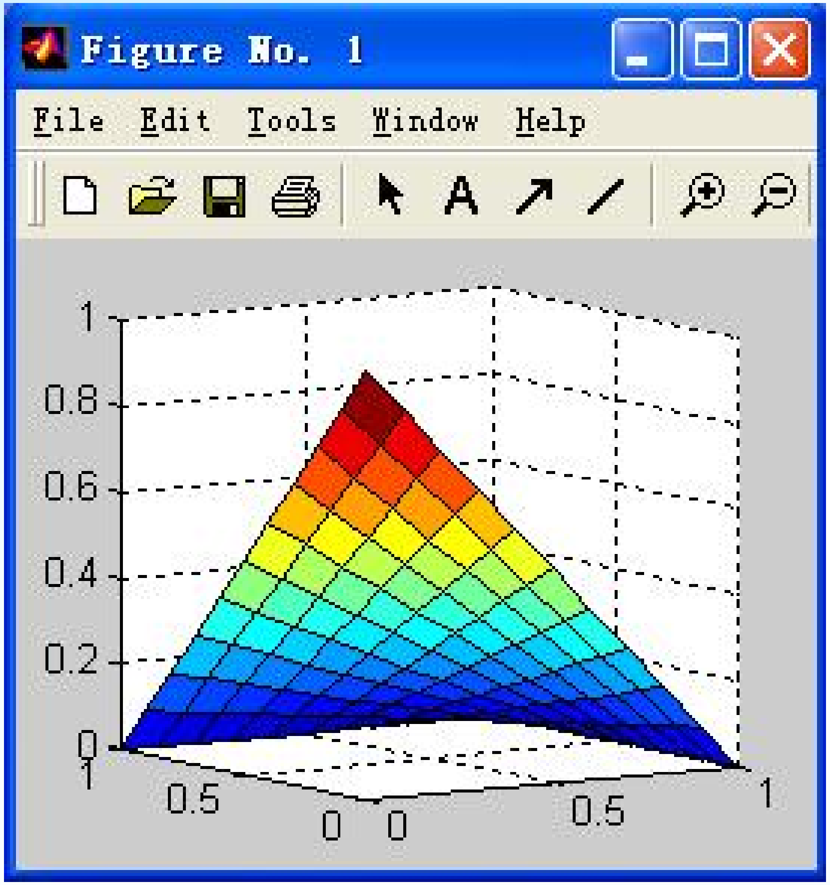
$$v = \frac{y - y_1}{y_2 - y_1}$$

$$l_1(u,\ v) = (1-u)(1-v)$$

$$l_2(u,\ v) = u(1-v)$$

$$l_3(u,\ v) = u\, v$$

$$l_4(u,\ v) = (1-u)\, v$$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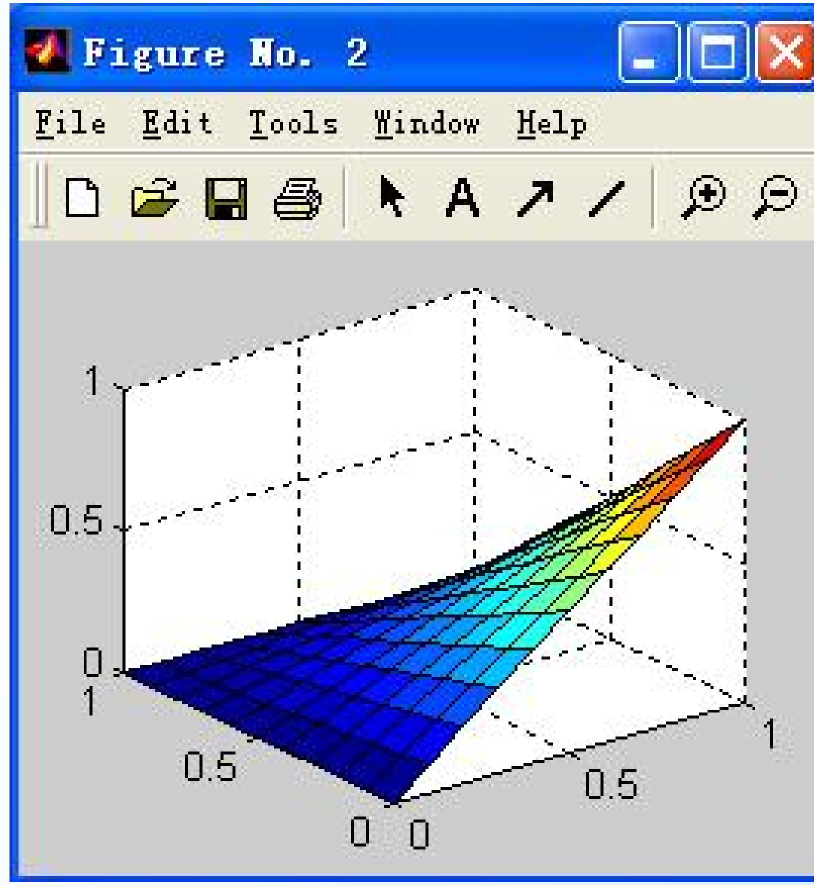
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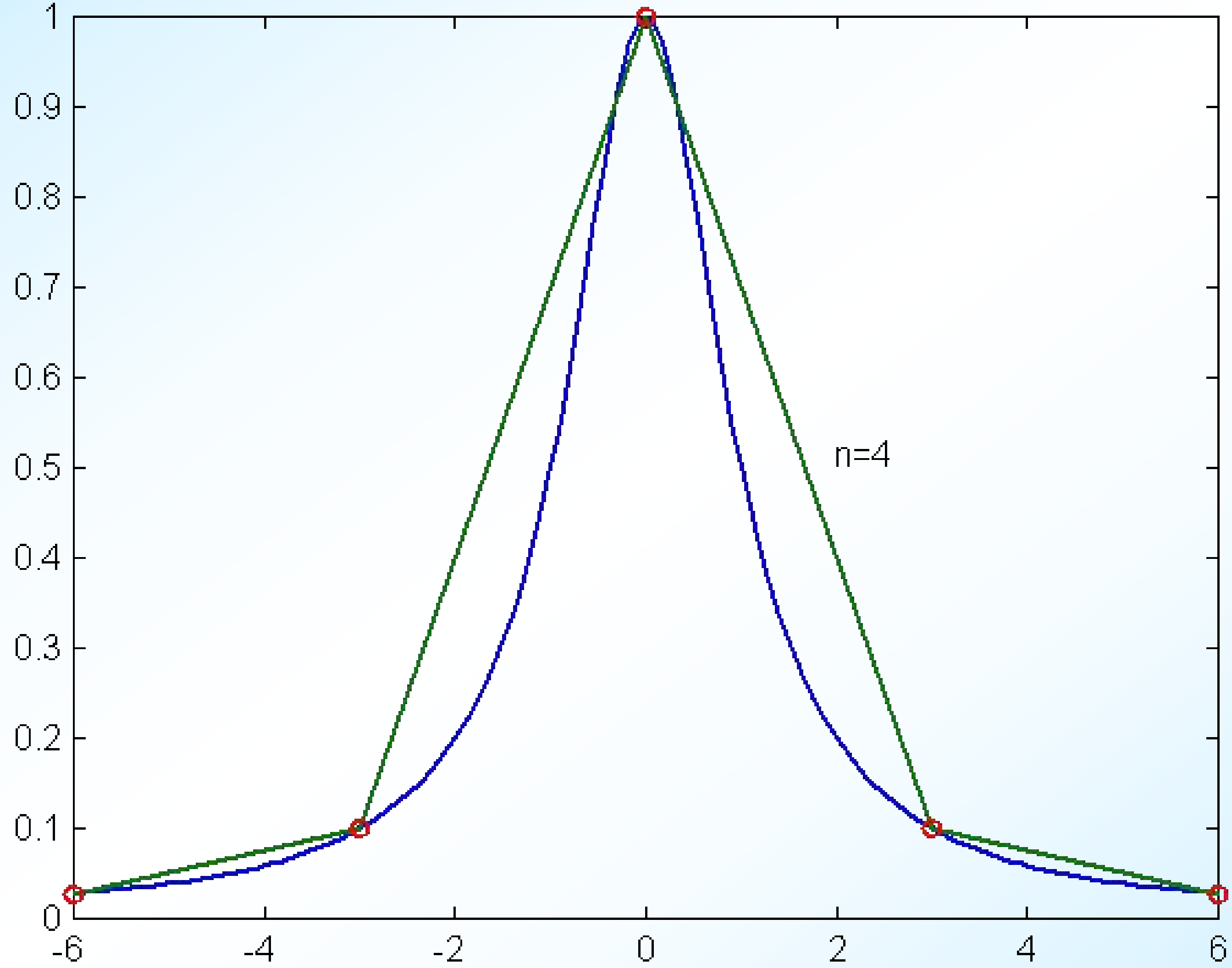
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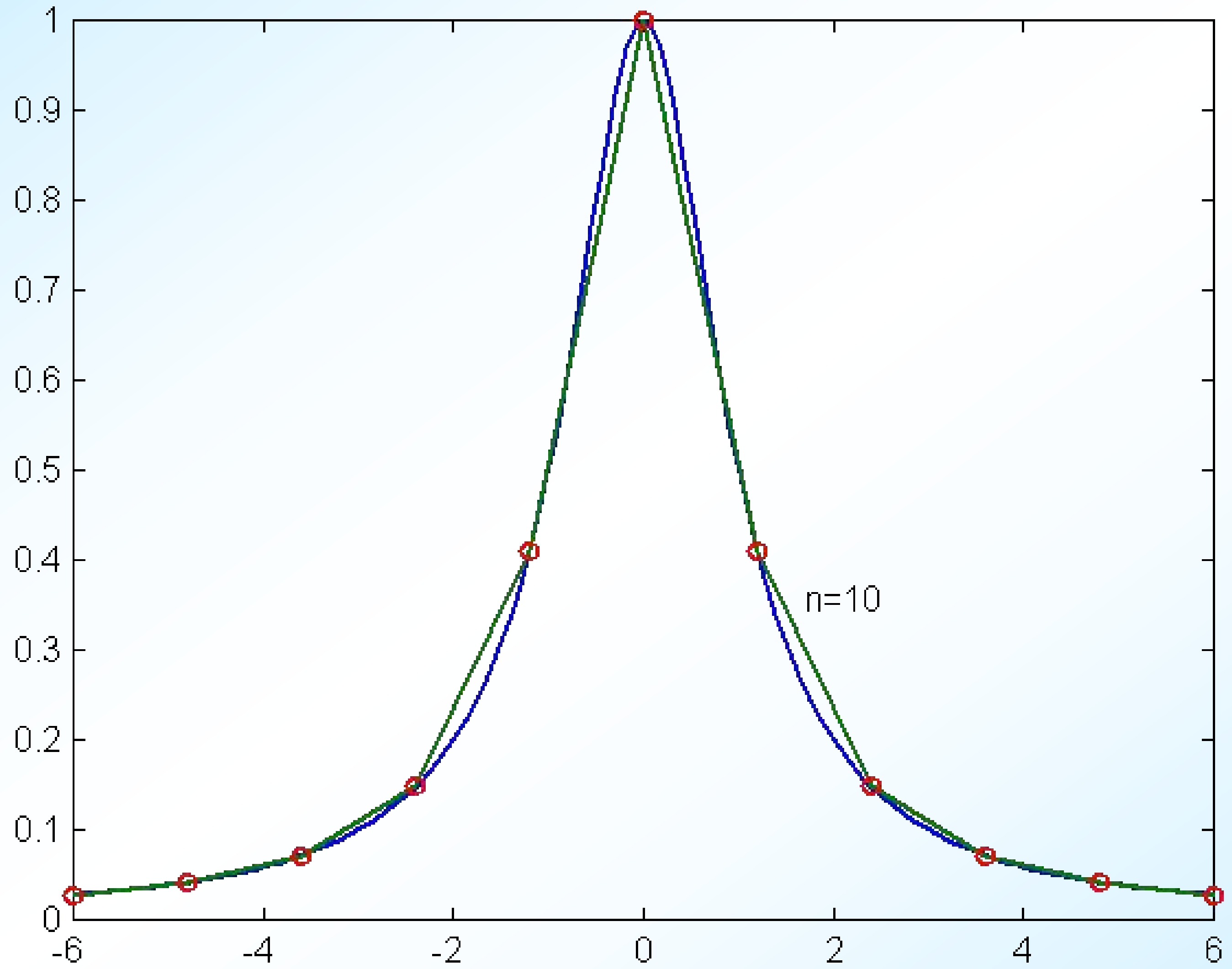

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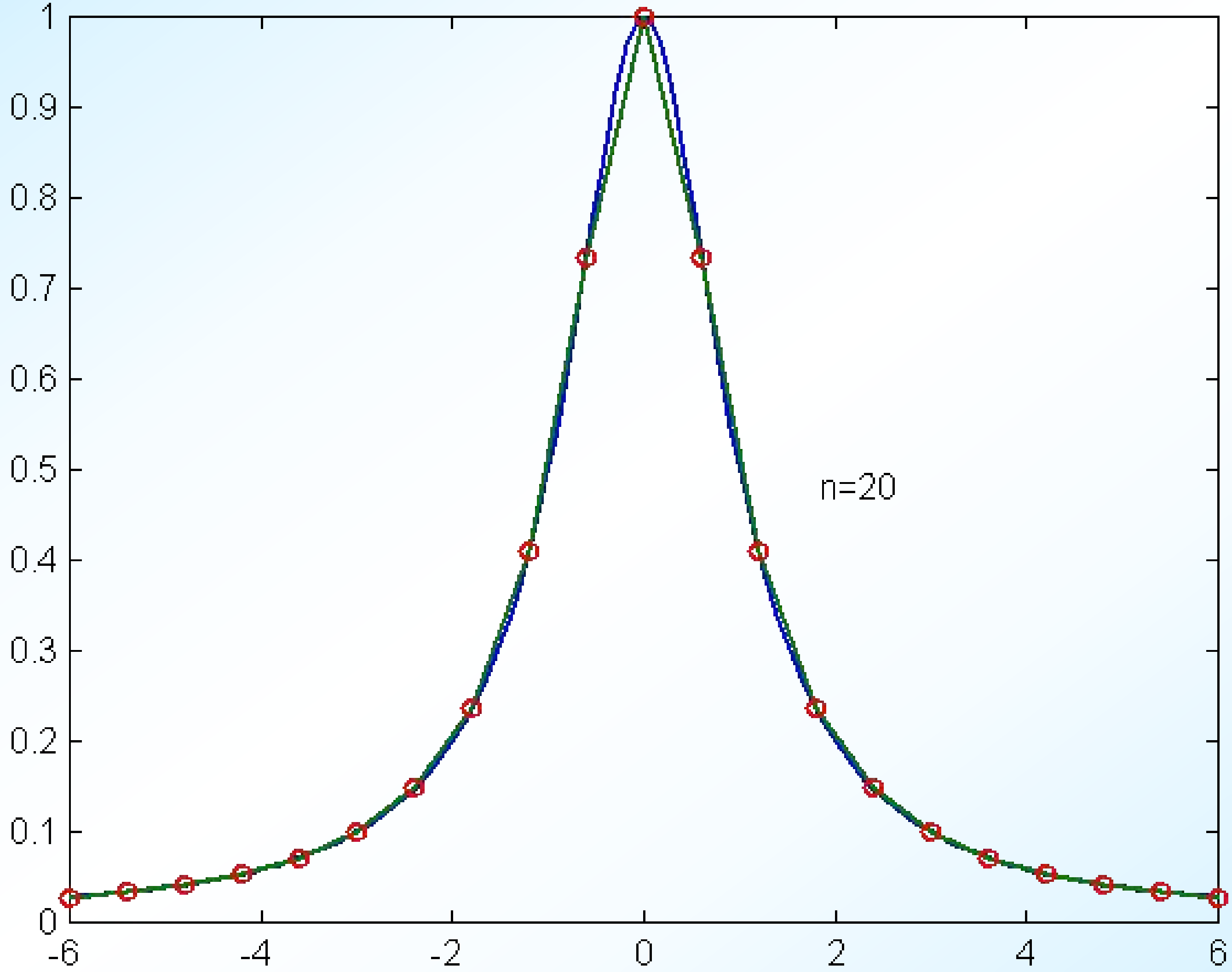

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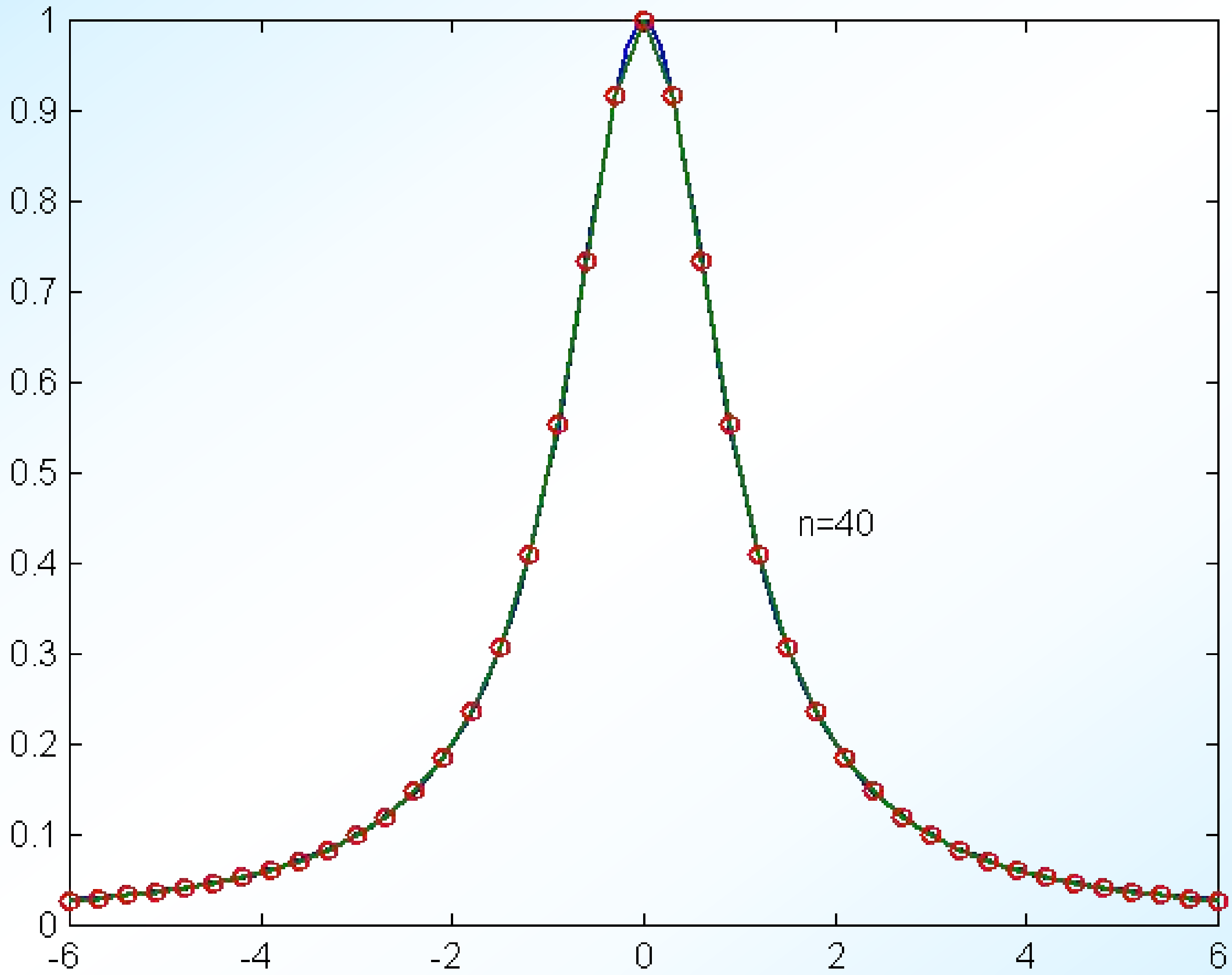

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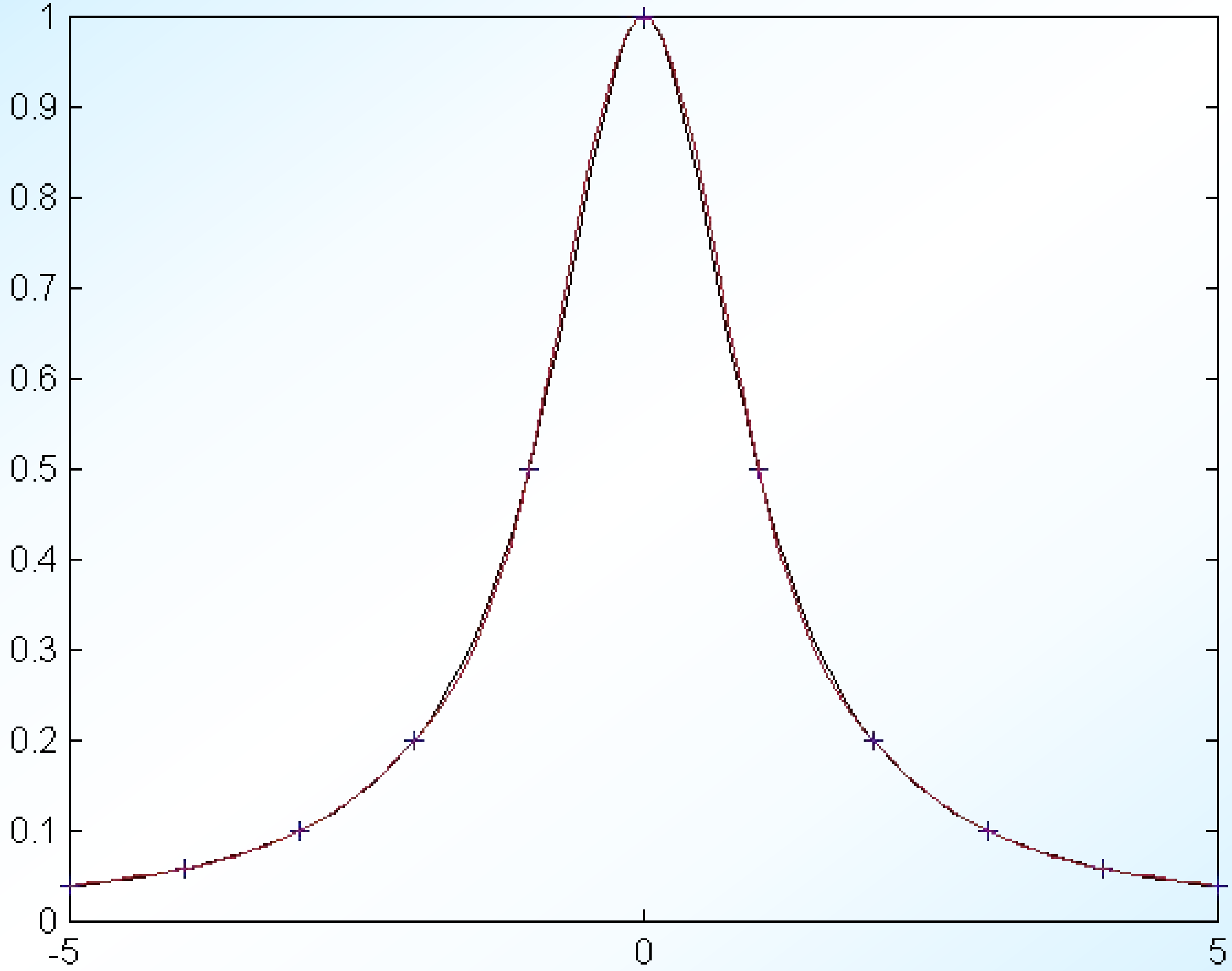
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