函数逼近

一般多项式函数逼近

切比雪夫多项式

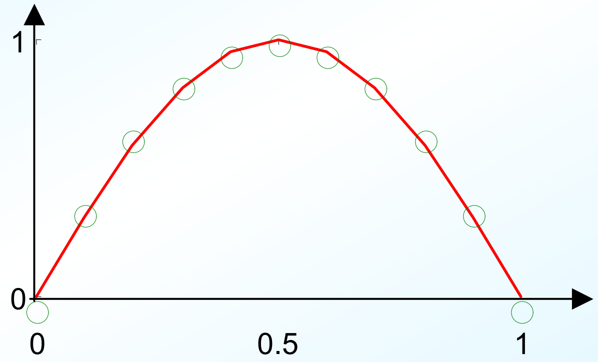
勒让德多项式

正交多项式的应用

一般多项式函数逼近

问题： 求二次多项式 $P(x)=a_0+a_1x+a_2x^2$ 使

$$\int_0^1[P(x)-\sin(\pi x)]^2dx=\min$$



连续函数的最佳平方逼近.

已知 $f(x)\in C[0,1]$, 求多项式

$$P(x)=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n$$

使得

$$L=\int_0^1[P(x)-f(x)]^2dx=\min$$

令 $$L(a_0,a_1,\cdots,a_n)=\int_0^1[\sum_{j=0}^{n}a_jx^j-f(x)]^2dx$$

$$L=\int_0^1[\sum_{j=0}^{n}a_jx^j]^2dx-2\sum_{j=0}^{n}a_j\int_0^1x^jf(x)dx+\int_0^1[f(x)]^2dx$$

$$\boxed{L=\int_0^1[\sum_{j=0}^{n}a_jx^j]^2dx-2\sum_{j=0}^{n}a_j\int_0^1x^jf(x)dx+\int_0^1[f(x)]^2dx}$$

$$\frac{\partial L}{\partial a_k}=2\sum_{j=0}^{n}a_j\int_0^1x^{j+k}dx-2\int_0^1x^kf(x)dx$$

令 $\dfrac{\partial L}{\partial a_k}=0$　　记 $b_k=\int_0^1x^kf(x)dx$

$$\begin{bmatrix} 1 & 1/2 & \cdots & 1/(n+1) \\ 1/2 & 1/3 & \cdots & 1/(n+2) \\ \cdots & \cdots & \cdots & \cdots \\ 1/(n+1) & \cdots & \cdots & 1/(2n+1) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}=\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}$$

系数矩阵被称为Hilbert矩阵

定义6.3 设 $f(x)$, $g(x)\in C[a, b]$, $\rho(x)$是区间$[a,b]$上的权函数,若等式

$$(f,g)=\int_a^b \rho(x)f(x)g(x)dx=0$$

成立,则称$f(x)$, $g(x)$在$[a, b]$上带权$\rho(x)$正交. 当$\rho(x)=1$时,简称正交。

例1 验证 $\varphi_0(x)=1$, $\varphi_1(x)=x$ 在$[-1, 1]$上正交, 并求二次多项式 $\varphi_2(x)$ 使之与$\varphi_0(x)$, $\varphi_1(x)$正交

解:

$$\int_{-1}^1 \varphi_0(x)\varphi_1(x)dx=\int_{-1}^1 1\cdot xdx=0$$

设 $\varphi_2(x) = x^2 + a_{21}x + a_{22}$

$$\int_{-1}^{1} 1 \cdot \varphi_2(x)dx = 0 \qquad \int_{-1}^{1} x\varphi_2(x)dx = 0$$

$$\int_{-1}^{1} (x^2 + a_{21}x + a_{22})dx = 0 \qquad \int_{-1}^{1} x(x^2 + a_{21}x + a_{22})dx = 0$$

$2/3+2a_{22} = 0$ $2a_{21}/3=0$	$a_{22}= - 1/3$ $a_{21}=0$

所以， $\varphi_2(x) = x^2 - \dfrac{1}{3}$

切比雪夫多项式

$$T_0(x)=1,\ T_1(x)=\cos\theta=x,$$

$$T_2(x)=\cos 2\theta \cdots\cdots$$

$$T_n(x)=\cos(n\theta),\cdots\cdots\cdots$$

1.递推公式:

由 $\cos(n+1)\theta+\cos(n-1)\theta=2\cos\theta\cos(n\theta)$

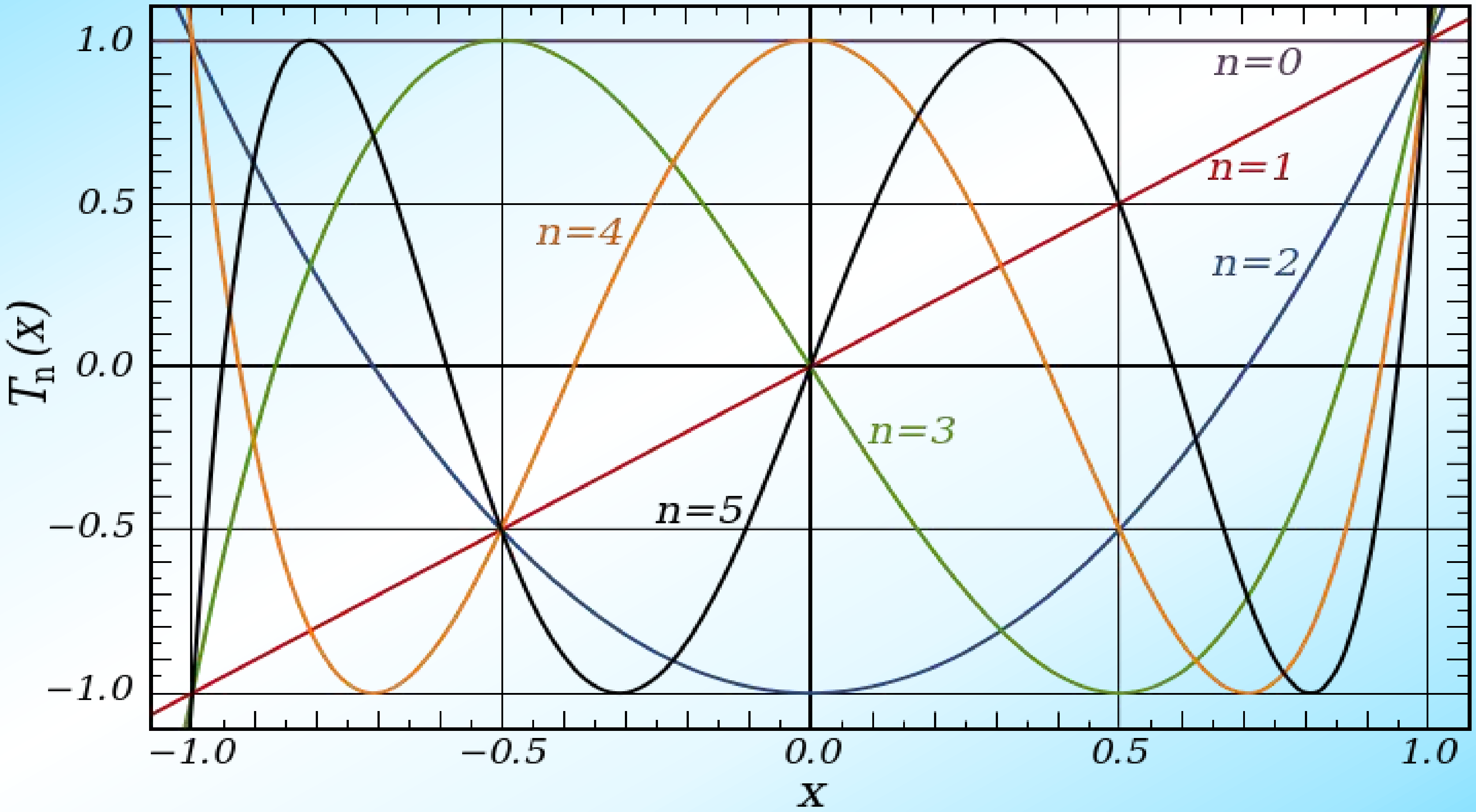
有 $\cos(n+1)\theta=2\cos\theta\cos(n\theta)-\cos(n-1)\theta$，从而

$$T_{n+1}(x)=2\,x\,T_n(x)-T_{n-1}(x)\qquad(n\geq 1)$$

所以, $T_0(x)=1,\ T_1(x)=x,\ T_2(x)=2x^2-1,\ \cdots,$

$$T_n(x)=\cos(n\arccos(x)),\cdots\cdots\cdots$$

切比雪夫多项式



2.切比雪夫多项式的正交性

$$\int_0^{\pi} \cos(m\theta)\cos(n\theta)d\theta = 0 \qquad (m \neq n)$$

$$(T_m, T_n) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x) T_n(x) dx$$

$$= \int_0^{\pi} \cos m\theta \cos n\theta d\theta = 0$$

所以, 切比雪夫多项式在[– 1 , 1]上带权

$$\rho(x) = \frac{1}{\sqrt{1-x^2}}$$ 正交

3.切比雪夫多项式零点

$T_1=\cos\theta=x$

n阶Chebyshev多项式: $T_n=\cos(n\theta)$,

或 $T_n(x)=\cos(n\arccos x)$

取 $$n\arccos x=\frac{(2k+1)\pi}{2} \qquad (k=0,1,\cdots,n-1)$$

即 $$x_k=\cos\left(\frac{(2k+1)\pi}{2n}\right) \qquad (k=0,1,\cdots,n-1)$$

4.切比雪夫多项式的极性

$T_n(x)$ 的最高次项 x^n 的系数为 2^{n-1}.

若$P_n(x)= 2^{1-n}\,T_n(x)$，则在所有最高次项系数为1的n次多项式$Q_n(x)$中，有

$$\max_{-1\le x\le 1}|P_n(x)| = \min\{\max_{-1\le x\le 1}|Q_n(x)|\}$$

例如 $t_k = -1+0.2k \quad (k = 0, 1, 2, \cdots, 10)$

$$x_k = \cos\left(\frac{(2k+1)\pi}{22}\right) \quad (k = 0, 1, 2, \cdots, 10)$$

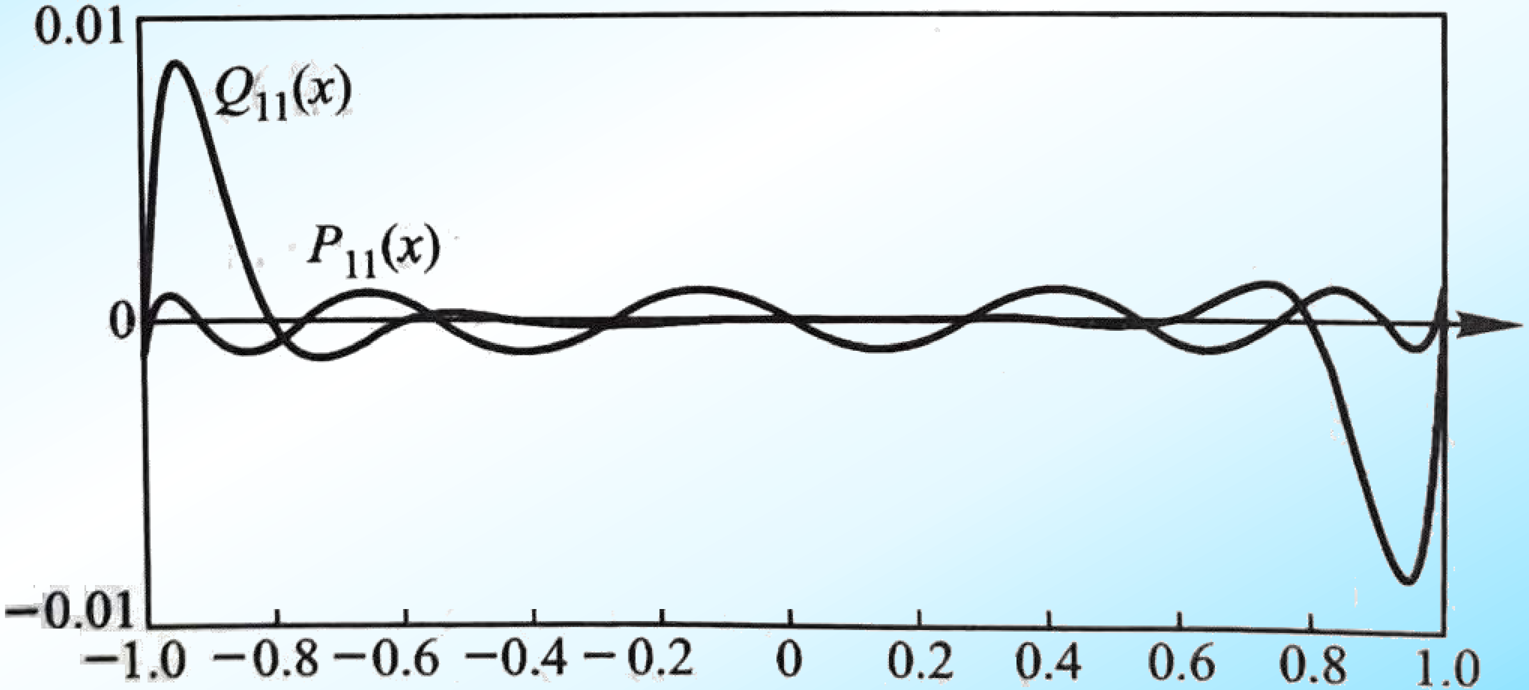
➔ $P_{11}(x)=(x-x_0)(x-x_1)\cdots\cdots(x-x_{10})$

$Q_{11}(x)=(x-t_0)(x-t_1)\cdots\cdots(x-t_{10})$

4.切比雪夫多项式的极性

$$P_{11}(x)=(x-x_0)(x-x_1)\cdots\cdots(x-x_{10})$$

$$Q_{11}(x)=(x-t_0)(x-t_1)\cdots\cdots(x-t_{10})$$



勒让德(Legendre)多项式

1.表达式 $P_0(x)=1,\ P_1(x)=x$

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n] \qquad (n\geq 1)$$

2. 正交性

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0, & m\neq n\\ \dfrac{2}{2n+1}, & m=n\end{cases}$$

3.递推式

$$\begin{cases} p_0 = 1, \ p_1 = x, \\ p_{n+1} = \dfrac{2n+1}{n+1} x p_n - \dfrac{n}{n+1} p_{n-1} \end{cases}$$

$$p_2(x) = \frac{1}{2}(3x^2 - 1) \qquad p_3(x) = \frac{1}{2}(5x^3 - 3x)$$

4.零点分布

$P_n(x)$ 的n 个零点,落入区间$[-1, 1]$中

$P_2(x)$的两个零点: $x_1 = -\dfrac{1}{\sqrt{3}} \qquad x_2 = \dfrac{1}{\sqrt{3}}$

$P_3(x)$的三个零点: $x_1 = -\sqrt{\dfrac{3}{5}} \qquad x_2 = 0 \quad x_3 = \sqrt{\dfrac{3}{5}}$

用正交多项式作最佳平方逼近

设$P_0(x), P_1(x), \cdots, P_n(x)$为区间$[a, b]$上的正交多项式, 即

$$(P_k, P_j) = \int_a^b P_k(x) P_j(x) dx = 0$$

$$(k \neq j,\ k, j = 0, 1, \cdots, n)$$

求 $P(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x)$

使
$$L = \int_a^b [P(x) - f(x)]^2 dx = \min$$

$$L(a_0, a_1, \cdots, a_n) = \int_a^b [\sum_{j=0}^{n} a_j P_j(x) - f(x)]^2 dx$$

$$\frac{\partial L}{\partial a_k}=2\int_a^b P_k(x)[\sum_{j=0}^{n}a_jP_j(x)-f(x)]dx$$

由于 $(P_k,P_j)=\int_a^b P_k(x)P_j(x)dx=0,\ (k\neq j)$

令 $\dfrac{\partial L}{\partial a_k}=0$ 记 $(P_k,f)=\int_a^b P_k(x)f(x)dx$

则有 $(P_k,P_k)a_k=(P_k,f)\quad (k=0,1,2,\cdots,n)$

$$a_k=\frac{(P_k,f)}{(P_k,P_k)}\qquad (k=0,1,2,\cdots,n)$$

$f(x)$的平方逼近 $P(x)=\sum_{k=0}^{n}\dfrac{(P_k,f)}{(P_k,P_k)}P_k(x)$

例　求二次多项式 $P(x)= a_0 + a_1x + a_2x^2$ 使

$$\int_0^1 [P(x)-\sin(\pi x)]^2 dx = \min$$

构造区间[0，1]上的正交多项式

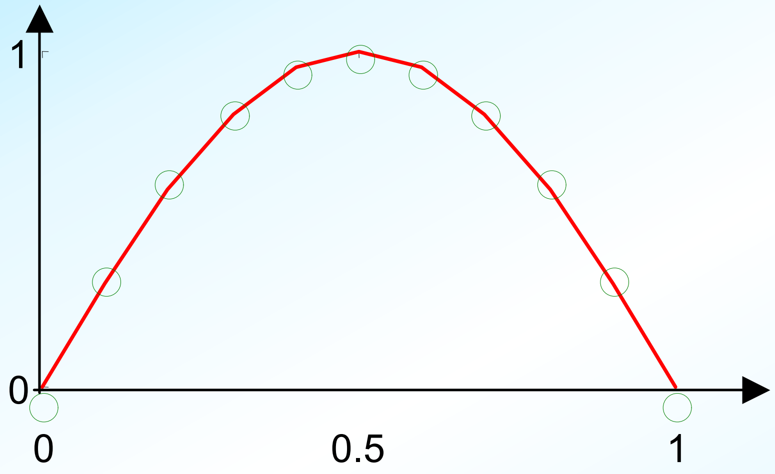
$P_0(x)= 1$，$P_1(x)= x - 1/2$，$P_2(x)= x^2 - x + 1/6$

$$\sin(\pi x) \approx \frac{(P_0,\sin(\pi x))}{(P_0,P_0)} + \frac{(P_1,\sin(\pi x))}{(P_1,P_1)} P_1(x) + \frac{(P_2,\sin(\pi x))}{(P_2,P_2)} P_2(x)$$

$$\frac{(P_0,\sin(\pi x))}{(P_0,P_0)} = \frac{2/\pi}{1} \qquad \frac{(P_1.\sin(\pi x))}{(P_1,P_1)} = \frac{0}{1/12}$$

$$\frac{(P_2.\sin(\pi x))}{(P_2,P_2)} = \frac{(\pi^2-12)/3\pi^3}{1/180}$$

最佳平方逼近： $\sin(\pi x)\approx\dfrac{2}{\pi}-4.1225(x^2-x+\dfrac{1}{6})$



$$P(x)=\frac{2}{\pi}-4.1225(x^2-x+\frac{1}{6})$$

$$f(x)=\sin(\pi x)$$

一、两角和与差的三角函数

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$a\sin\alpha+b\cos\alpha=\sqrt{a^2+b^2}\,\sin(\alpha+\varphi)$$

二、和差化积

$$\sin\alpha+\sin\beta=2\sin[(\alpha+\beta)/2]\cdot\cos[(\alpha-\beta)/2]$$

$$\sin\alpha-\sin\beta=2\cos[(\alpha+\beta)/2]\cdot\sin[(\alpha-\beta)/2]$$

$$\cos\alpha+\cos\beta=2\cos[(\alpha+\beta)/2]\cdot\cos[(\alpha-\beta)/2]$$

$$\cos\alpha-\cos\beta=-2\sin[(\alpha+\beta)/2]\cdot\sin[(\alpha-\beta)/2]$$