定义1: 设$A \in C^{n\times n}$，若 $\lim\limits_{k\to\infty} A^k = 0$ (k为正整数)，则称A为收敛矩阵.

定理1 设$A \in C^{n\times n}$，则A为收敛矩阵的充要条件是$r(A) < 1$.

定理2(Neumann定理) 方阵A的Neumann级数

$$\sum_{k=0}^{\infty} A^k = I + A + A^2 + \cdots + A^k + \cdots$$

收敛的充要条件是$r(A) < 1$，且收敛时，其和为$(I - A)^{-1}$.

定理3 设幂级数

$$f(z)=\sum_{k=0}^{\infty}c_k z^k$$

的收敛半径为 r

(1) 如果$r(A)<r$,则矩阵幂 级数 $\sum_{k=0}^{\infty}c_k A^k$ 绝对收敛（$\sum_{k=0}^{\infty}c_k A^k$收敛）

(2) 如果$r(A)>r$,则矩阵幂数 $\sum_{k=0}^{\infty}c_k A^k$发散.

定义2　设幂级数 $\sum_{k=0}^{\infty} c_k z^k$ 收敛半径为r，且当

$|z|<r$时，幂级数收敛于$f(z)$，即

$$f(z)=\sum_{k=0}^{\infty} c_k z^k, \quad |z|<r$$

如果$A\in C^{n\times n}$满足$r(A)<r$，则称收敛的矩阵幂级数 $\sum_{k=0}^{\infty} a_k A^k$的和为矩阵函数，记为$f(A)$，即

$$f(A)=\sum_{k=0}^{\infty} c_k A^k,$$

把 $f(A)$的方阵A换为At, t为参数, 则得到

$$f(At)=\sum_{k=0}^{\infty} c_k (At)^k.$$

1.常用的矩阵函数:

$$(1)\ \ e^A=\sum_{k=0}^{\infty}\frac{1}{k!}A^k,\quad A\in C^{n\times n}$$

$$(2)\ \ \sin A=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}A^{2k+1},\quad A\in C^{n\times n}$$

$$(3)\ \cos A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} A^{2k}, \quad A \in C^{n\times n}$$

$$(4)\ (E - A)^{-1} = \sum_{k=0}^{\infty} A^k, \quad r(A) < 1$$

$$(5)\ \ln(E + A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} A^{k+1}, \quad r(A) < 1$$

2、矩阵函数值的计算

(1)、利用相似对角化:

设$P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) = D$ $\Rightarrow$ $A = PDP^{-1}$ $\Rightarrow$

$$f(A)=\sum_{k=0}^{\infty}c_k A^k=\sum_{k=0}^{\infty}c_k(PDP^{-1})^k=P\left(\sum_{k=0}^{\infty}c_k D^k\right)P^{-1}$$

$$=P\begin{pmatrix}\sum_{k=0}^{\infty}c_k{\lambda_1}^k & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty}c_k{\lambda_n}^k\end{pmatrix}P^{-1}$$

$$=P\begin{pmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}P^{-1}$$

同理

$$f(At) = P\mathrm{diag}(f(\lambda_1 t), f(\lambda_2 t), \cdots, f(\lambda_n t))P^{-1}.$$

例1

设 $A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix}$, 求$e^{At}$.

解: 1) $\det(\lambda E - A) = (\lambda + 2)(\lambda - 1)^2$

$\lambda_1 = -2, \lambda_2 = \lambda_3 = 1$

2) 对应的特征向量

$$\lambda_1 = -2:\ \xi_1 = (-1,1,1)^T$$

$$\lambda_2 = \lambda_3 = 1:\ \xi_2 = (-2,1,0)^T, \xi_3 = (0,0,1)^T \Rightarrow$$

$$P = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$e^{At}=P\begin{pmatrix} e^{-2t} & & \\ & e^{t} & \\ & & e^{t} \end{pmatrix}P^{-1}$$

$$=\begin{pmatrix} 2e^{t}-e^{-2t} & 2e^{t}-2e^{-2t} & 0 \\ e^{2t}-e^{-t} & 2e^{-2t}-e^{t} & 0 \\ e^{-2t}-e^{t} & 2e^{-2t}-2e^{t} & e^{t} \end{pmatrix}$$

(2)、Jordan 标准形法:

$$A = PJP^{-1} = Pdiag(J_1, J_2, \cdots, J_s)P^{-1}$$

$$f(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k=0}^{\infty} a_k (PJP^{-1})^k = \sum_{k=0}^{\infty} a_k PJ^k P^{-1} = P(\sum_{k=0}^{\infty} a_k J^k)P^{-1}$$

$$= P\sum_{k=0}^{\infty} a_k \begin{pmatrix} J_1^{\ k} & & \\ & \ddots & \\ & & J_s^{\ k} \end{pmatrix} P^{-1} = P\sum_{k=0}^{\infty} \begin{pmatrix} a_k J_1^{\ k} & & \\ & \ddots & \\ & & a_k J_s^{\ k} \end{pmatrix} P^{-1}$$

$$= P\begin{pmatrix} \sum\limits_{k=0}^{\infty} a_k J_1^{\ k} & & \\ & \ddots & \\ & & \sum\limits_{k=0}^{\infty} a_k J_s^{\ k} \end{pmatrix} P^{-1} = P\begin{pmatrix} f(J_1) & & \\ & \ddots & \\ & & f(J_s) \end{pmatrix} P^{-1}$$

$$\text{(I)}\quad J_i=\begin{pmatrix}\lambda_i & 1 & \cdots & 0\\ & \lambda_i & \ddots & \vdots\\ & & \ddots & 1\\ & & & \lambda_i\end{pmatrix}_{m_i\times m_i}=\begin{pmatrix}\lambda_i & & & \\ & \lambda_i & & \\ & & \ddots & \\ & & & \lambda_i\end{pmatrix}+\begin{pmatrix} & 1 & \cdots & 0\\ & & \ddots & \vdots\\ & & & 1\\ & & & \end{pmatrix}=\lambda_i E+T\Rightarrow$$

$$J_i^k=(\lambda_i E+T)^k$$

$$=\sum_{l=0}^{k}C_k^l(\lambda_i E)^{k-l}T^l=\sum_{l=0}^{m_i-1}C_k^l\lambda_i^{k-l}T^l=\begin{pmatrix}\lambda_i^k & C_k^1\lambda_i^{k-1} & \cdots & C_k^{m_i-1}\lambda_i^{k-(m_i-1)}\\ & \lambda_i^k & & \vdots\\ & & \ddots & C_k^1\lambda_i^{k-1}\\ & & & \lambda_i^k\end{pmatrix}$$

$$\text{(II)}\quad f(J_i)=\sum_{k=0}^{\infty}a_kJ_i^k=\sum_{k=0}^{\infty}a_k\begin{pmatrix}\lambda_i^k & C_k^1\lambda_i^{k-1} & \cdots & C_k^{m_i-1}\lambda_i^{k-(m_i-1)}\\ & \lambda_i^k & & \vdots\\ & & \ddots & C_k^1\lambda_i^{k-1}\\ & & & \lambda_i^k\end{pmatrix}\Rightarrow$$

$$f(J_i)=\begin{pmatrix} \sum_{k=0}^{\infty}a_k\lambda_i^k & \sum_{k=0}^{\infty}a_kC_k^1\lambda_i^{k-1} & \cdots & \sum_{k=0}^{\infty}a_kC_k^{m_i-1}\lambda_i^{k-(m_i-1)} \\ & \sum_{k=0}^{\infty}a_k\lambda_i^k & & \vdots \\ & & \ddots & \sum_{k=0}^{\infty}a_kC_k^1\lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty}a_k\lambda_i^k \end{pmatrix} (\text{其中}C_k^l=0(k<l))$$

$$=\begin{pmatrix} \sum_{k=0}^{\infty}a_k\lambda_i^k & \sum_{k=1}^{\infty}a_kC_k^1\lambda_i^{k-1} & \cdots & \sum_{k=m_i-1}^{\infty}a_kC_k^{m_i-1}\lambda_i^{k-(m_i-1)} \\ & \sum_{k=0}^{\infty}a_k\lambda_i^k & & \vdots \\ & & \ddots & \sum_{k=1}^{\infty}a_kC_k^1\lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty}a_k\lambda_i^k \end{pmatrix}$$

$$f(J_i) == \begin{pmatrix} \sum_{k=0}^{\infty} a_k \lambda_i^k & \sum_{k=1}^{\infty} a_k C_k^1 \lambda_i^{k-1} & \cdots & \sum_{k=m_i-1}^{\infty} a_k C_k^{m_i-1} \lambda_i^{k-(m_i-1)} \\ & \sum_{k=0}^{\infty} a_k \lambda_i^k & & \vdots \\ & & \ddots & \sum_{k=1}^{\infty} a_k C_k^1 \lambda_i^{k-1} \\ & & & \sum_{k=0}^{\infty} a_k \lambda_i^k \end{pmatrix}$$

(III) $$f(z) = \sum_{k=0}^{\infty} a_k z^k \Rightarrow f(J_i) = \begin{bmatrix} f(\lambda_i) & \frac{1}{1!} f'(\lambda_i) & \cdots & \frac{1}{(m_i-1)!} f^{(m_i-1)}(\lambda_i) \\ & f(\lambda_i) & \cdots & \frac{1}{(m_i-2)!} f^{(m_i-2)}(\lambda_i) \\ & & \ddots & \vdots \\ & & & f(\lambda_i) \end{bmatrix}$$

注：

$$f(z)=\sum_{k=0}^{\infty}a_k z^k \Rightarrow f'(z)=\left(\sum_{k=0}^{\infty}a_k z^k\right)'=\sum_{k=0}^{\infty}\left(a_k z^k\right)'=\sum_{k=1}^{\infty}k a_k z^{k-1}$$

$$f''(z)=\left(\sum_{k=1}^{\infty}k a_k z^{k-1}\right)'=\sum_{k=2}^{\infty}k(k-1)a_k z^{k-2}$$

$$f^{l}(z)=\sum_{k=l}^{\infty}k(k-1)\cdots(k-l+1)a_k z^{k-l}$$

$$\sum_{k=1}^{\infty}a_k C_k^1\lambda_i^{k-1}=\sum_{k=1}^{\infty}k a_k \lambda_i^{k-1}=f'(z)|_{z=\lambda_i}=f'(\lambda_i)$$

$$\sum_{k=l}^{\infty}a_k C_k^l\lambda_i^{k-l}=\sum_{k=l}^{\infty}\frac{k!}{l!(k-l)!}a_k\lambda_i^{k-l}=\frac{1}{l!}\sum_{k=l}^{\infty}k(k-1)\cdots(k-l+1)a_k\lambda_i^{k-l}$$

$$=\frac{1}{l!}f^{(l)}(z)|_{z=\lambda_i}=\frac{1}{l!}f^{(l)}(\lambda_i)$$

例 2

$$A=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}, \text{ 求} \sin A.$$

解： 1) 化为*Jordan*标准形

$$A \longrightarrow J_1=1, J_2=\begin{pmatrix}1&1\\0&1\end{pmatrix}$$

2) 计算$\sin J_i$

$$\sin J_1=\sin 1, \sin J_2=\begin{pmatrix}\sin 1&\frac{1}{1!}\cos 1\\0&\sin 1\end{pmatrix}$$

$$\therefore \sin A = \begin{pmatrix} \sin 1 & 0 & 0 \\ 0 & \sin 1 & \cos 1 \\ 0 & 0 & \sin 1 \end{pmatrix}$$

三、矩阵函数的一些性质

性质1：如果$AB = BA$，则$e^A e^B = e^B e^A = e^{A+B}$.

性质2：如果$AB = BA$，则

(1) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(2) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(3) $\cos(2A) = \cos^2 A - \sin^2 A$

(4) $\sin(2A) = 2\sin A \cos A$

四、矩阵函数的几种特殊情形

(1) $A^2 = A$

$$A^2 = A \Rightarrow A^k = A (k \geq 1) \longrightarrow f(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k=1}^{\infty} a_k A^k + a_0 E$$

$$= a_0 E + \sum_{k=1}^{\infty} a_k A$$

若 $\sum_{k=0}^{\infty} a_k z^k = f(z), \quad |z| < R (R > 1)$，则

$$f(A) = a_0 E + (\sum_{k=1}^{\infty} a_k \cdot 1) A = a_0 E + (f(1) - a_0) A$$

(2) $A^2 = E$

$$A^2 = E \Rightarrow \begin{cases} A^{2k} = E \\ A^{2k+1} = A \end{cases} \longrightarrow f(A) = \sum_{k=0}^{\infty} a_k A^k = \sum_{k=0}^{\infty} a_{2k} A^{2k} + \sum_{k=0}^{\infty} a_{2k+1} A^{2k+1}$$

$$= \sum_{k=0}^{\infty} a_{2k} E + \sum_{k=0}^{\infty} a_{2k+1} A$$

$$(1)\quad A^2 = A \Rightarrow e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = E + \sum_{k=1}^{\infty} \frac{1}{k!} A = E + (\sum_{k=1}^{\infty} \frac{1}{k!} \cdot 1) A$$

$$= E + (e^1 - 1) A = E + (e - 1) A$$

$$A^2 = A \Rightarrow \sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A = (\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot 1) A = (\sin 1) A$$

$$(2)\quad A^2 = E \Rightarrow \sin A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A = (\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot 1) A = (\sin 1) A$$

$$A^2 = E \Rightarrow \cos A = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} A^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} E = (\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot 1) E = (\cos 1) E$$

例1: 设$A=\begin{pmatrix} 0 & c & c \\ c & 0 & c \\ c & c & 0 \end{pmatrix}(c \in R)$, 讨论$c$取何值时$A$为收敛矩阵.

解: $\det(\lambda E - A) = (\lambda - 2c)(\lambda + c)^2 \Rightarrow r(A) = 2|c|$

A为收敛矩阵$\Leftrightarrow r(A) < 1 \Leftrightarrow 2|c| < 1 \Leftrightarrow -\frac{1}{2} < c < \frac{1}{2}$.

例2：求$\sum_{k=0}^{\infty}\begin{pmatrix}0.1 & 0.7\\ 0.3 & 0.6\end{pmatrix}^k$

$$r(A)\le\|A\|_\infty=0.9<1\Rightarrow\sum_{k=0}^{\infty}\begin{pmatrix}0.1 & 0.7\\ 0.3 & 0.6\end{pmatrix}^k=$$

$$\left(E-\begin{pmatrix}0.1 & 0.7\\ 0.3 & 0.6\end{pmatrix}\right)^{-1}=\begin{pmatrix}0.9 & -0.7\\ -0.3 & 0.4\end{pmatrix}^{-1}$$

$$=\left(\frac{1}{10}\begin{pmatrix}9 & -7\\ -3 & 4\end{pmatrix}\right)^{-1}=\frac{1}{0.15}\begin{pmatrix}0.4 & 0.7\\ 0.3 & 0.9\end{pmatrix}$$

例3：$\|A\|<1$，求$\sum_{k=1}^{\infty} kA^{k-1}$

$$\sum_{k=1}^{\infty} kz^{k-1} = \sum_{k=1}^{\infty}\left(z^{k}\right)' = \left(\sum_{k=0}^{\infty} z^{k}\right)' = \left(\frac{1}{1-z}\right)'$$

$$= \frac{1}{\left(1-z\right)^{2}} \qquad (|z|<1).$$

$$r(A) \le \|A\| < 1 \Rightarrow \sum_{k=1}^{\infty} kA^{k-1} = \left[(E-A)^{-1}\right]^{2}$$