$$1.(1) A \in C^{m\times n}, G \in C^{n\times m}, \begin{cases} GA = E_n \Rightarrow G = A_L^{-1} \\ AG = E_m \Rightarrow\Rightarrow G = A_R^{-1} \end{cases}$$

$$(2) A \in C^{m\times n},\ G \in C^{n\times m}, AGA = A \Rightarrow G = A^{-}$$

$$(3) A \in C_n^{m\times n} \Leftrightarrow A_L^{-1}\text{存在} \Leftrightarrow N(A) = 0 \Leftrightarrow A^{-}A = E_n$$

$$A \in C_m^{m\times n} \Leftrightarrow A_R^{-1}\text{存在} \Leftrightarrow R(A) = C^m \Leftrightarrow AA^{-} = E_m$$

$$(4) A \in C^{m\times n},\ G \in C^{n\times m}, \begin{cases} AGA = A \\ GAG = G \end{cases} \Rightarrow G = A_r^{-}$$

（5）初等变换求左(右)逆矩阵:

$$\text{(I)}\quad P(A \quad E_m) = \begin{pmatrix} E_n & G \\ 0 & * \end{pmatrix}, G = A_L^{-1} \qquad G = (A^H A)^{-1} A^H = A_L^{-1}$$

$$\text{(II)}\quad \begin{pmatrix} A \\ E_n \end{pmatrix} Q = \begin{pmatrix} E_m & 0 \\ G & * \end{pmatrix}, G = A_R^{-1} \qquad G = A^H (AA^H)^{-1} = A_R^{-1}$$

$$(6)\, A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} (A_{11} \in C_r^{r\times r}) \Rightarrow \begin{cases} A^- = \begin{pmatrix} A_{11}^{-1} & X \\ Y & C \end{pmatrix} \\ A_r^- = \begin{pmatrix} A_{11}^{-1} & X \\ Y & YA_{11}X \end{pmatrix} \\ A^+ = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{cases}$$

$$(7)A\in C^{m\times n}, P\in C_m^{m\times m}, Q\in C_n^{n\times n} \Rightarrow Q(PAQ)^- P\in A\{1\}$$

$$A\in C^{m\times n}, S\in C_m^{m\times m}, T\in C_n^{n\times n}, B=SAT \Rightarrow T^{-1}A^- S^{-1}\in B\{1\}$$

$$\begin{cases} AXA=A \\ AYA=A \end{cases} \Rightarrow Z = XAY = A_r^-$$

$$\boldsymbol{X=(A^H A)^- A^H = A_r^-} \qquad \boldsymbol{Y=A^H(AA^H)^- = A_r^-}$$

$$(8)rank(A_r^-)=rank(A)=rank(A^-A)=rank(AA^-)\le rank(A^-)$$

$$
(9)\begin{cases}
R(AA^{-}) = R(A),\ N(A^{-}A) = N(A) \\
R(AA_r^{-}) = R(A),\ N(AA_r^{-}) = N(A_r^{-}) \\
R(A_r^{-}A) = R(A_r^{-}),\ N(A_r^{-}A) = N(A) \\
R(A^{+}) = R(A^{H})
\end{cases}
$$

$$
(10)\begin{cases}
AA^{-},\ A^{-}A(AA_L^{-1}, A_L^{-1}A) \\
AA_r^{-},\ A_r^{-}A \\
AA^{+},\ A^{+}A \\
(AA^{+})^{H} = AA^{+}, (A^{+}A)^{H} = A^{+}A
\end{cases}
$$

幂等

（11）. 幂等矩阵的性质

$$A\in C^{n\times n},\ \ A=A^2 \Rightarrow \begin{cases} (1) & A^H=\left(A^H\right)^2, E-A=\left(E-A\right)^2 \\ (2) & \sigma(A)=\{\lambda \mid Ax=\lambda x, x\neq 0\}=\{0,1\} \\ (3) & rank(A)=tr(A) \\ (4) & A(E-A)=(E-A)A=0 \\ (5) & A\alpha=\alpha \Leftrightarrow \alpha\in R(A) \\ (6) & N(A)=R(E-A), R(A)=N(E-A) \end{cases}$$

$$2.A\in C^{m\times n},\quad G\in C^{n\times m},\begin{cases} AGA=A,\ GAG=G, \\ (GA)^H=GA,(AG)^H=AG, \end{cases}\Rightarrow G=A^+.$$

$$G\in A\{1,3\}\Rightarrow (GA)^2=GA,(GA)^H=GA$$

$$G\in A\{1,4\}\Rightarrow (AG)^2=AG,(AG)^H=AG$$

$$(1)\ \begin{cases} (A^HA)^+=A^+(A^H)^+,(AA^H)^+=(A^H)^+A^+; \\ A^+=(A^HA)^+A^H=A^H(AA^H)^+; \\ rank(A)=rank(A^+),(UAV)^+=V^HA^+U^H \end{cases}$$

$$(2)A\in C_r^{m\times n},A=BD(B\in C_r^{m\times r},D\in C_r^{r\times n})$$

$$A^+=D^+\bullet B^+$$

(其中$D^+=D^H(DD^H)^{-1},B^+=(B^HB)^{-1}B^H$)

$$(3) A \in C_r^{m\times n}, \quad A = U\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V = UDV, \quad D_r = diag(\sigma_1, \cdots, \sigma_r)$$

$$\Rightarrow \begin{cases} (1) \quad A^+ = V^H D^+ U^H = V^H \begin{pmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^H; \\ (2) \| A \|_F^2 = \sum_{i=1}^{r} \sigma_i^2, \quad \| A^+ \|_F^2 = \sum_{i=1}^{r} \frac{1}{\sigma_i^2}; \\ (3) \| A \|_2 = \max_{1\le i\le r} \{\sigma_i\} \quad \| A^+ \|_2 = \frac{1}{\min_{1\le i\le r}\{\sigma_i\}} \end{cases}$$

$$(4) A \in C_r^{m\times n}, AA^H \in C_r^{m\times m},$$

$$\begin{cases} AA^H \alpha_i = \lambda_i \alpha_i (i = 1, \cdots, r), \quad \alpha_i^H \alpha_j = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \\ \Delta_r = diag(\lambda_1, \cdots, \lambda_r), \quad U_1 = (\alpha_1, \alpha_2, \cdots, \alpha_r) \end{cases}$$

$$\Rightarrow A^+ = A^H (AA^H)^+ = A^H U_1 \Delta_r^{-1} U_1^H$$

(5) 几个结论

(I): $(AB)^{+}=B^{+}A^{+}$ ($\times$)

(II) $(A^{k})^{+}=(A^{+})^{k}$, (其中k是正整n) ($\times$)

(III)若P,Q为可逆矩阵, $(PAQ)^{+}=Q^{-1}A^{+}P^{-1}$ ($\times$)

(6) $(AB)^{+}=B^{+}A^{+}\Leftrightarrow\begin{cases}R(A^{H}AB)\subset R(B),\\ R(BB^{H}A^{H})\subset R(A^{H}).\end{cases}$

(I) $A\in C_{r}^{m\times n}, A=BD(B\in C_{r}^{m\times r}, D\in C_{r}^{r\times n})$

$$(BD)^{+}=A^{+}=D^{+}\bullet B^{+}$$

3. $Ax=b$有解 $\Leftrightarrow AA^{-}b=b$（$AA^{+}b=b$）

$$\Rightarrow x=A^{-}b+(E_n-A^{-}A)u \quad \forall \quad u\in C^n.$$

(1)相容方程$Ax=b$

$$AA^{+}b=b \Leftrightarrow rank(A)=rank(A|b)$$

$$\Rightarrow \text{通解} x=Db+(E_n-DA)u \quad \forall \quad u\in C^n.(D\in A\{1,3\})$$

$$=A^{+}b+(E_n-A^{+}A)u \quad \forall \quad u\in C^n.$$

$Db=A^{+}b$—————— **最小范数解**

⑵不相容方程$Ax=b$

$$AA^{+}b\neq b\Leftrightarrow rank(A)\neq rank(A|b)$$

$\Rightarrow$ 最小二乘解的通解$x=Gb+(E_n-A^{-}A)u\quad\forall\ \ u\in C^{n}.(G\in A\{1,4\})$

$$=A^{+}b+(E_n-A^{+}A)u\quad\forall\ \ u\in C^{n}.$$

$A^{+}b$------------ **最佳逼近解**

例1：已知A的M-P逆A^{+}，求$\begin{pmatrix} A & A \\ A & A \end{pmatrix}^{+}$

$$A=BD \Rightarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} B \\ B \end{pmatrix} \begin{pmatrix} D & D \end{pmatrix} \Rightarrow \begin{pmatrix} A & A \\ A & A \end{pmatrix}^{+} = \frac{1}{4} \begin{pmatrix} A^{+} & A^{+} \\ A^{+} & A^{+} \end{pmatrix}$$

$$\begin{pmatrix} B \\ B \end{pmatrix}^{+} = \left(\begin{pmatrix} B \\ B \end{pmatrix}^{H} \begin{pmatrix} B \\ B \end{pmatrix} \right)^{-1} \begin{pmatrix} B \\ B \end{pmatrix}^{H} = \left((B^{H}, B^{H}) \begin{pmatrix} B \\ B \end{pmatrix} \right)^{-1} (B^{H}, B^{H})$$

$$= (2B^{H}B)^{-1}(B^{H}, B^{H}) = \frac{1}{2}(B^{+}, B^{+})$$

$$\begin{pmatrix} D & D \end{pmatrix}^{+} = \begin{pmatrix} D & D \end{pmatrix}^{H} \left(\begin{pmatrix} D & D \end{pmatrix} \begin{pmatrix} D & D \end{pmatrix}^{H} \right)^{-1} = \begin{pmatrix} D^{H} \\ D^{H} \end{pmatrix} \left(\begin{pmatrix} D & D \end{pmatrix} \begin{pmatrix} D^{H} \\ D^{H} \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} D^{H} \\ D^{H} \end{pmatrix} (2DD^{H})^{-1} = \frac{1}{2} \begin{pmatrix} D^{+} \\ D^{+} \end{pmatrix}$$

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}^{+} = \frac{1}{2}\begin{pmatrix} D^{+} \\ D^{+} \end{pmatrix}\frac{1}{2}(B^{+}, B^{+}) = \frac{1}{4}\begin{pmatrix} D^{+}B^{+} & D^{+}B^{+} \\ D^{+}B^{+} & D^{+}B^{+} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} A^{+} & A^{+} \\ A^{+} & A^{+} \end{pmatrix}$$

类例：已知A的$M-P$逆A^{+}，求$\begin{pmatrix} A \\ A \end{pmatrix}^{+}$

$$A = BD \Rightarrow \begin{pmatrix} A \\ A \end{pmatrix} = \begin{pmatrix} B \\ B \end{pmatrix} D \Rightarrow \begin{pmatrix} A \\ A \end{pmatrix}^{+} = \frac{1}{2}\begin{pmatrix} A^{+} & A^{+} \end{pmatrix}$$

类例：已知A的逆A^{-1}，求$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}^{+}$

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E \\ 0 \end{pmatrix}\begin{pmatrix} 0 & A \end{pmatrix} = BD \Rightarrow A^{+} = D^{+}B^{+} = \begin{pmatrix} 0 & 0 \\ A^{-1} & 0 \end{pmatrix}$$

例2：设A是元素全为1的$m\times n$矩阵，则$A^{+}=$

$$A=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{m\times 1}\begin{pmatrix}1 & 1 & \cdots & 1\end{pmatrix}_{1\times n}=BD\Rightarrow A^{+}=D^{+}B^{+}=\frac{1}{mn}A^{T}$$

例3：已知$A\in C_n^{m\times n}\Rightarrow\left\|AA^+\right\|_2=1$

$$\left(AA^+\right)^2=AA^+\Rightarrow\lambda(AA^+)=0\text{或}1$$

$$A\in C_n^{m\times n}\Rightarrow\forall x\neq 0, Ax\neq 0\Rightarrow AA^+Ax=Ax=1\bullet Ax\Rightarrow 1\in\lambda(AA^+)$$

$$\Rightarrow\left\|AA^+\right\|_2=\sqrt{r(\left(AA^+\right)^H AA^+)}=\sqrt{r(AA^+)}=1$$

类例：已知$A\in C^{m\times n}, A\neq 0\Rightarrow\left\|A^+A\right\|_2=1$

$$B=A^+A, B^2=B\Rightarrow\lambda(B)=0\text{或}1; rank(B)=rank(A)\geq 1\Rightarrow B\neq 0.$$

$$B^H=B, B\neq 0\Rightarrow\text{存在}\lambda(B)\neq 0\Rightarrow\text{存在}\lambda(B)=1$$

$$\Rightarrow\left\|B\right\|_2^2=r(B^HB)=r(B)=1\Rightarrow\left\|B\right\|_2=1.$$

例:用广义逆矩阵方法判断线性方程组

$$\begin{cases} 2x_1+4x_2+x_3+x_4=3 \\ x_1+2x_2-x_3+2x_4=0 \\ -x_1-2x_2-2x_3+x_4=3 \end{cases}$$

是否有解?如果有解,求通解和最小范数解;

如果无解,求最小二乘解和最佳逼进解.

解 $A=\begin{pmatrix} 2 & 4 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ -1 & -2 & -2 & 1 \end{pmatrix}, b=\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

(1)：求A的最大秩分解：$A=BD$

$$B=\begin{pmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}, D=\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(2)：求$A^{+}=D^{H}(DD^{H})^{-1}(B^{H}B)^{-1}B^{H}$

$$=\frac{1}{33}\begin{pmatrix} 2 & 1 & -1 \\ 4 & 2 & -2 \\ 1 & -5 & -6 \\ 1 & 6 & 5 \end{pmatrix}$$

(3)：检验 $AA^{+}b=b$是否成立.

$$AA^{+}b=\begin{pmatrix}1\\2\\1\end{pmatrix}\neq b$$

故$Ax=b$是不相容的方程.

(4): 求最小二乘解的通解及最佳逼近解.

通解:

$$x = A^{+}b + (E - A^{+}A)u = \frac{1}{11}\begin{pmatrix} 1 \\ 2 \\ -5 \\ 6 \end{pmatrix} + (E - A^{+}A)u$$

最佳逼近解:

$$x = A^{+}b = \frac{1}{11}\begin{pmatrix} 1 \\ 2 \\ -5 \\ 6 \end{pmatrix}$$

例 $A_i \in C^{m\times n}, A_i A_j^H = 0, A_i^H A_j = 0 (i \neq j, i, j = 1, 2, \cdots, r)$,

记 $A = \sum_{i=1}^{r} A_i, G = \sum_{i=1}^{r} A_i^+$; 则 $\left(\sum_{i=1}^{r} A_i\right)^+ = A^+ = G = \sum_{i=1}^{r} A_i^+$

证明: (1) $AG = \left(\sum_{i=1}^{r} A_i\right)\left(\sum_{j=1}^{r} A_j^+\right) = \left(\sum_{i=1}^{r} A_i\right)\left(\sum_{j=1}^{r} A_j^H (A_j A_j^H)^+\right)$

$$= \sum_{i=1}^{r} (A_i A_i^H)(A_i A_i^H)^+ = \sum_{i=1}^{r} A_i [A_i^H (A_i A_i^H)^+] = \sum_{i=1}^{r} A_i A_i^+$$

(2) $GA = \left(\sum_{i=1}^{r} A_i^+\right)\left(\sum_{j=1}^{r} A_j\right) = \left(\sum_{i=1}^{r} (A_i^H A_i)^+ A_i^H\right)\left(\sum_{j=1}^{r} A_j\right)$

$$= \sum_{i=1}^{r} (A_i^H A_i)^+ (A_i^H A_i) = \sum_{i=1}^{r} [(A_i^H A_i)^+ A_i^H] A_i = \sum_{i=1}^{r} A_i^+ A_i$$

$(A^+ = (A^H A)^+ A^H = A^H (AA^H)^+)$

$$(1)\Rightarrow (AG)^H=\left(\sum_{i=1}^{r}A_iA_i^+\right)^H=\sum_{i=1}^{r}(A_iA_i^+)^H=\sum_{i=1}^{r}A_iA_i^+=AG$$

$$(2)\Rightarrow (GA)^H=\left(\sum_{i=1}^{r}A_i^+A_i\right)^H=\sum_{i=1}^{r}(A_i^+A_i)^H=\sum_{i=1}^{r}A_i^+A_i=GA$$

$$(3)AGA=\left(\sum_{i=1}^{r}A_i\right)\left(\sum_{j=1}^{r}A_j^+\right)\left(\sum_{k=1}^{r}A_k\right)=\left(\sum_{i=1}^{r}A_iA_i^+\right)\left(\sum_{k=1}^{r}A_k\right)$$

$$=\left(\sum_{i=1}^{r}A_i(A_i^HA_i)^+A_i^H\right)\left(\sum_{k=1}^{r}A_k\right)=\sum_{i=1}^{r}A_i(A_i^HA_i)^+A_i^HA_i=\sum_{i=1}^{r}A_iA_i^+A_i=\sum_{i=1}^{r}A_i=A$$

$$(4)GAG=\left(\sum_{i=1}^{r}A_i^+\right)\left(\sum_{j=1}^{r}A_j\right)\left(\sum_{k=1}^{r}A_k^+\right)=\sum_{i=1}^{r}A_i^+=G$$

故 $\left(\sum_{i=1}^{r}A_i\right)^+=\sum_{i=1}^{r}A_i^+$