§5 $M-P$ 广义逆矩阵 A^+

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定义 1 设 $\boldsymbol{A} \in \mathbf{C}^{m\times n}$，如果存在矩阵 $\boldsymbol{G} \in \mathbf{C}^{n\times m}$，使得

$$\boldsymbol{AGA}=\boldsymbol{A},\ \boldsymbol{GAG}=\boldsymbol{G},\ (\boldsymbol{GA})^H=\boldsymbol{GA},\ (\boldsymbol{AG})^H=\boldsymbol{AG},$$

同时成立，则称 $\boldsymbol{G}$ 为 $\boldsymbol{A}$ 的 $\boldsymbol{M-P}$ 广义逆矩阵，记为 $\boldsymbol{G}=\boldsymbol{A}^+$.

定理 1 设 $\boldsymbol{A} \in \mathbf{C}^{m\times n}$，$\boldsymbol{A}=\boldsymbol{BD}$ 是 $\boldsymbol{A}$ 的最大秩分解，则

$$\boldsymbol{G}=\boldsymbol{D}^H\left(\boldsymbol{DD}^H\right)^{-1}\left(\boldsymbol{B}^H\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H$$

是 $\boldsymbol{A}$ 的广义逆矩阵 $\boldsymbol{A}^+$.

证明： (1) 若 $\mathbf{rank}(\boldsymbol{A})=\mathbf{0}$，即 $\boldsymbol{A}=\boldsymbol{O}$，则 $\boldsymbol{A}^+=\boldsymbol{O}$.

(2) 若 $\mathbf{rank}(\boldsymbol{A})>\mathbf{0}$,则存在最大秩分解 $\boldsymbol{A}=\boldsymbol{BD}$. 又因 $\mathbf{rank}(\boldsymbol{B}^H\boldsymbol{B})=\mathbf{rank}(\boldsymbol{DD}^H)=\mathbf{rank}(\boldsymbol{A})$，所以 $\boldsymbol{B}^H\boldsymbol{B}$，$\boldsymbol{DD}^H$ 都是可逆的. 下面直接验证.

$$\begin{aligned}\boldsymbol{AGA}&=\boldsymbol{BDD}^H\left(\boldsymbol{DD}^H\right)^{-1}\left(\boldsymbol{B}^H\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H\boldsymbol{BD}\\&=\boldsymbol{BD}\\&=\boldsymbol{A}.\end{aligned}$$

$$\begin{aligned}\boldsymbol{GAG}&=\boldsymbol{D}^H\left(\boldsymbol{DD}^H\right)^{-1}\left(\boldsymbol{B}^H\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H\boldsymbol{BDD}^H\left(\boldsymbol{DD}^H\right)^{-1}\left(\boldsymbol{B}^H\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H\\&=\boldsymbol{D}^H\left(\boldsymbol{DD}^H\right)^{-1}\left(\boldsymbol{B}^H\boldsymbol{B}\right)^{-1}\boldsymbol{B}^H\\&=\boldsymbol{G}.\end{aligned}$$

$$(\boldsymbol{GA})^H=\boldsymbol{A}^H\boldsymbol{G}^H$$

$$
\begin{aligned}
&= \boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B}\left[\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\right]^{H}\left[\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\right]^{H}\boldsymbol{D}\\
&= \boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\boldsymbol{D}\\
&= \boldsymbol{D}^{H}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\boldsymbol{D}\\
&= \boldsymbol{D}^{H}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D}\\
&= \boldsymbol{G}\boldsymbol{A}.
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{A}\boldsymbol{G})^{H} &= \boldsymbol{G}^{H}\boldsymbol{A}^{H}\\
&= \boldsymbol{B}\left[\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\right]^{H}\left[\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\right]^{H}\boldsymbol{D}\boldsymbol{D}^{H}\boldsymbol{B}^{H}\\
&= \boldsymbol{B}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\boldsymbol{D}\boldsymbol{D}^{H}\boldsymbol{B}^{H}\\
&= \boldsymbol{B}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{H}\\
&= \boldsymbol{B}\boldsymbol{D}\boldsymbol{D}^{H}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{H}\\
&= \boldsymbol{A}\boldsymbol{G}.
\end{aligned}
$$

综上，$\boldsymbol{G}$ 是 $\boldsymbol{A}$ 的 $\boldsymbol{M-P}$ 广义逆矩阵 $\boldsymbol{A}^{+}$.

定理 2 设 $\boldsymbol{A} \in \mathbf{C}^{m\times n}$，则 $\boldsymbol{A}^{+}$ 是唯一的.

证明： 令 $\boldsymbol{A}_1^{+}$，$\boldsymbol{A}_2^{+}$ 都是 $\boldsymbol{A}$ 的 $\boldsymbol{M-P}$ 广义逆，则

$$
\begin{aligned}
\boldsymbol{A}_1^{+} &= \boldsymbol{A}_1^{+}\boldsymbol{A}\boldsymbol{A}_1^{+}\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}\boldsymbol{A}_2^{+}\boldsymbol{A})\boldsymbol{A}_1^{+}\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}\boldsymbol{A}_2^{+})(\boldsymbol{A}\boldsymbol{A}_1^{+})\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}\boldsymbol{A}_2^{+})^{H}(\boldsymbol{A}\boldsymbol{A}_1^{+})^{H}\\
&= \boldsymbol{A}_1^{+}\left(\boldsymbol{A}_2^{+}\right)^{H}\boldsymbol{A}^{H}\left(\boldsymbol{A}_1^{+}\right)^{H}\boldsymbol{A}^{H}\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}_2^{+})^{H}\left(\boldsymbol{A}\boldsymbol{A}_1^{+}\boldsymbol{A}\right)^{H}\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}_2^{+})^{H}\left(\boldsymbol{A}\right)^{H}\\
&= \boldsymbol{A}_1^{+}(\boldsymbol{A}\boldsymbol{A}_2^{+})^{H}\\
&= \boldsymbol{A}_1^{+}\boldsymbol{A}\boldsymbol{A}_2^{+} \quad \text{(考虑左侧结合，重复上述步骤.)}\\
&= \boldsymbol{A}_2^{+}\boldsymbol{A}\boldsymbol{A}_2^{+}\\
&= \boldsymbol{A},
\end{aligned}
$$

得证.

定理 3 设 $\boldsymbol{A} \in \mathbf{C}^{m\times n}$，则

(1) $\left(\boldsymbol{A}^{+}\right)^{+}=\boldsymbol{A}$*;*

(2) $\left(\boldsymbol{A}^{T}\right)^{+}=\left(\boldsymbol{A}^{+}\right)^{T}$，$\left(\boldsymbol{A}^{H}\right)^{+}=\left(\boldsymbol{A}^{+}\right)^{H}$*;*

(3) $\boldsymbol{A}^{+}=\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{+}\boldsymbol{A}^{H}=\boldsymbol{A}^{H}\left(\boldsymbol{A}\boldsymbol{A}^{H}\right)^{+}$*;*

(4) $\mathbf{R}(\boldsymbol{A}^{+})=\mathbf{R}(\boldsymbol{A}^{H})$*;*

(5) $\boldsymbol{A}\boldsymbol{A}^{+}=\mathbf{P}_{\mathrm{R}(\boldsymbol{A})}$，$\boldsymbol{A}^{+}\boldsymbol{A}=\mathbf{P}_{\mathrm{R}(\boldsymbol{A}^{H})}$*;*

(6) $\mathbf{R}(\boldsymbol{A})=\mathbf{R}(\boldsymbol{A}^{H})$ *的充要条件是* $\boldsymbol{A}\boldsymbol{A}^{+}=\boldsymbol{A}^{+}\boldsymbol{A}$*.*

证明： (1) 显然成立；

(2) 设 $\boldsymbol{A}=\boldsymbol{B}\boldsymbol{D}$ 是矩阵 $\boldsymbol{A}$ 的最大秩分解，则 $\boldsymbol{A}^{T}=\boldsymbol{D}^{T}\boldsymbol{B}^{T}$ 是矩阵 $\boldsymbol{A}^{T}$ 的最大秩分解. 于是有

$$\begin{aligned}
\left(\boldsymbol{A}^{+}\right)^{T}&=\left[\boldsymbol{D}^{H}\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{H}\right]^{T}\\
&=(\boldsymbol{B}^{H})^{T}\left[\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{-1}\right]^{T}\left[\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{-1}\right]^{T}(\boldsymbol{D}^{H})^{T}\\
&=(\boldsymbol{B}^{T})^{H}\left[\left(\boldsymbol{B}^{H}\boldsymbol{B}\right)^{T}\right]^{-1}\left[\left(\boldsymbol{D}\boldsymbol{D}^{H}\right)^{T}\right]^{-1}(\boldsymbol{D}^{T})^{H}\\
&=(\boldsymbol{B}^{T})^{H}\left[\boldsymbol{B}^{T}(\boldsymbol{B}^{T})^{H}\right]^{-1}\left[\boldsymbol{D}^{T}(\boldsymbol{D}^{T})^{H}\right]^{-1}(\boldsymbol{D}^{T})^{H}\\
&=(\boldsymbol{A}^{T})^{+}.
\end{aligned}$$

类似可证明 $\left(\boldsymbol{A}^{H}\right)^{+}=\left(\boldsymbol{A}^{+}\right)^{H}$.

(3) 设 $\boldsymbol{A}=\boldsymbol{B}\boldsymbol{D}$ 是矩阵 $\boldsymbol{A}$ 的最大秩分解，则 $\boldsymbol{A}^{H}\boldsymbol{A}=\boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D}=\boldsymbol{B}_1\boldsymbol{D}_1$ 是矩阵 $\boldsymbol{A}^{H}\boldsymbol{A}$ 的最大秩分解，其中 $\boldsymbol{B}_1=\boldsymbol{D}^{H}$ 列满秩，$\boldsymbol{D}_1=\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D}$ 行满秩. 进而

$$\begin{aligned}
\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{+}\boldsymbol{A}^{H}&=\left[\boldsymbol{D}_1^{H}(\boldsymbol{D}_1\boldsymbol{D}_1^{H})^{-1}(\boldsymbol{B}_1^{H}\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^{H}\right](\boldsymbol{B}\boldsymbol{D})^{H}\\
&=(\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D})^{H}[(\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D})(\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D})^{H}]^{-1}(\boldsymbol{D}\boldsymbol{D}^{H})^{-1}\boldsymbol{D}\boldsymbol{D}^{H}\boldsymbol{B}^{H}\\
&=\boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B}(\boldsymbol{B}^{H}\boldsymbol{B}\boldsymbol{D}\boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B})^{-1}\boldsymbol{B}^{H}\\
&=\boldsymbol{D}^{H}\boldsymbol{B}^{H}\boldsymbol{B}(\boldsymbol{B}^{H}\boldsymbol{B})^{-1}(\boldsymbol{D}\boldsymbol{D}^{H})^{-1}(\boldsymbol{B}^{H}\boldsymbol{B})^{-1}\boldsymbol{B}^{H}\\
&=\boldsymbol{D}^{H}(\boldsymbol{D}\boldsymbol{D}^{H})^{-1}(\boldsymbol{B}^{H}\boldsymbol{B})^{-1}\boldsymbol{B}^{H}\\
&=\boldsymbol{A}^{+}.
\end{aligned}$$

类似，可证另外一个等式.

(4) 因 $\boldsymbol{A}^{+}$ 是 $\boldsymbol{A}$ 的自反广义逆，则 $\mathbf{rank}(\boldsymbol{A}^{+})=\mathbf{rank}(\boldsymbol{A})=\mathbf{rank}(\boldsymbol{A}^{H})$. 又由 (3) 知，$\mathbf{R}(\boldsymbol{A}^{+})\subset\mathbf{R}(\boldsymbol{A}^{H})$，进而 $\mathbf{R}(\boldsymbol{A}^{+})=\mathbf{R}(\boldsymbol{A}^{H})$.

(5) 因 $\boldsymbol{A}^{+}$ 是 $\boldsymbol{A}$ 的自反广义逆，则 $\boldsymbol{A}\boldsymbol{A}^{+}$ 和 $\boldsymbol{A}^{+}\boldsymbol{A}$ 都是幂等矩阵，即

$$\boldsymbol{A}\boldsymbol{A}^{+}=\mathbf{P}_{\mathrm{R}(\boldsymbol{A}\boldsymbol{A}^{+})},\quad \boldsymbol{A}^{+}\boldsymbol{A}=\mathbf{P}_{\mathrm{R}(\boldsymbol{A}^{+}\boldsymbol{A})}.$$

又因 $\mathbf{R}(\boldsymbol{AA}^+) = \mathbf{R}(\boldsymbol{A})$，$\mathbf{R}(\boldsymbol{A}^+\boldsymbol{A}) = \mathbf{R}(\boldsymbol{A}^+)$，$\mathbf{R}(\boldsymbol{A}^+) = \mathbf{R}(\boldsymbol{A}^H)$，故

$$\boldsymbol{AA}^+ = \mathbf{P}_{\mathbf{R}(\boldsymbol{A})},\quad \boldsymbol{A}^+\boldsymbol{A} = \mathbf{P}_{\mathbf{R}(\boldsymbol{A}^H)}.$$

(6) 充分性：因 $\boldsymbol{AA}^+ = \boldsymbol{A}^+\boldsymbol{A}$，结合 (5) 的证明，则

$$\mathbf{R}(\boldsymbol{A}) = \mathbf{R}(\boldsymbol{AA}^+) = \mathbf{R}(\boldsymbol{A}^+\boldsymbol{A}) = \mathbf{R}(\boldsymbol{A}^+) = \mathbf{R}(\boldsymbol{A}^H).$$

必要性：因 $\mathbf{R}(\boldsymbol{A}) = \mathbf{R}(\boldsymbol{A}^H)$，$\boldsymbol{AA}^+ = \mathbf{P}_{\mathbf{R}(\boldsymbol{A})}$，$\boldsymbol{A}^+\boldsymbol{A} = \mathbf{P}_{\mathbf{R}(\boldsymbol{A}^H)}$，则

$$\boldsymbol{AA}^+ = \boldsymbol{A}^+\boldsymbol{A}.$$

定理 4 设 $\boldsymbol{A} \in \mathbf{C}^{m\times n}$，则有

(1) $(\boldsymbol{A}^H\boldsymbol{A})^+ = \boldsymbol{A}^+(\boldsymbol{A}^H)^+$，$(\boldsymbol{AA}^H)^+ = (\boldsymbol{A}^H)^+\boldsymbol{A}^+$*;*

(2) $(\boldsymbol{A}^H\boldsymbol{A})^+ = \boldsymbol{A}^+(\boldsymbol{AA}^H)^+\boldsymbol{A} = \boldsymbol{A}^H(\boldsymbol{AA}^H)^+(\boldsymbol{A}^H)^+$*;*

(3) $\boldsymbol{AA}^+ = (\boldsymbol{AA}^H)(\boldsymbol{AA}^H)^+ = (\boldsymbol{AA}^H)^+(\boldsymbol{AA}^H)$*;*
$\boldsymbol{A}^+\boldsymbol{A} = (\boldsymbol{A}^H\boldsymbol{A})(\boldsymbol{A}^H\boldsymbol{A})^+ = (\boldsymbol{A}^H\boldsymbol{A})^+(\boldsymbol{A}^H\boldsymbol{A})$.

证明： (1) 设 $\boldsymbol{A} = \boldsymbol{BD}$ 是矩阵 $\boldsymbol{A}$ 的最大秩分解，则 $\boldsymbol{A}^H\boldsymbol{A} = \boldsymbol{D}^H\boldsymbol{B}^H\boldsymbol{BD} = \boldsymbol{B}_1\boldsymbol{D}_1$ 是矩阵 $\boldsymbol{A}^H\boldsymbol{A}$ 的最大秩分解，其中 $\boldsymbol{B}_1 = \boldsymbol{D}^H$ 列满秩，$\boldsymbol{D}_1 = \boldsymbol{B}^H\boldsymbol{BD}$ 行满秩. 于是有

$$\begin{aligned}
\left(\boldsymbol{A}^H\boldsymbol{A}\right)^+ &= \boldsymbol{D}_1^H(\boldsymbol{D}_1\boldsymbol{D}_1^H)^{-1}(\boldsymbol{B}_1^H\boldsymbol{B}_1)^{-1}\boldsymbol{B}_1^H\\
&= (\boldsymbol{B}^H\boldsymbol{BD})^H[(\boldsymbol{B}^H\boldsymbol{BD})(\boldsymbol{B}^H\boldsymbol{BD})^H]^{-1}(\boldsymbol{DD}^H)^{-1}\boldsymbol{D}\\
&= \boldsymbol{D}^H\boldsymbol{B}^H\boldsymbol{B}(\boldsymbol{B}^H\boldsymbol{BDD}^H\boldsymbol{B}^H\boldsymbol{B})^{-1}(\boldsymbol{DD}^H)^{-1}\boldsymbol{D}\\
&= \boldsymbol{D}^H\boldsymbol{B}^H\boldsymbol{B}(\boldsymbol{B}^H\boldsymbol{B})^{-1}(\boldsymbol{DD}^H)^{-1}(\boldsymbol{B}^H\boldsymbol{B})^{-1}(\boldsymbol{DD}^H)^{-1}\boldsymbol{D}\\
&= \boldsymbol{D}^H(\boldsymbol{DD}^H)^{-1}(\boldsymbol{B}^H\boldsymbol{B})^{-1}(\boldsymbol{DD}^H)^{-1}\boldsymbol{D}\\
&= \left[\boldsymbol{D}^H(\boldsymbol{DD}^H)^{-1}(\boldsymbol{B}^H\boldsymbol{B})^{-1}\boldsymbol{B}^H\right]\left[\boldsymbol{B}(\boldsymbol{B}^H\boldsymbol{B})^{-1}(\boldsymbol{DD}^H)^{-1}\boldsymbol{D}\right]\\
&= \boldsymbol{A}^+(\boldsymbol{A}^H)^+.
\end{aligned}$$

类似，可证另外一个等式.

(2) 定理 3 知，$\left(\boldsymbol{A}^H\right)^+ = \left(\boldsymbol{A}^+\right)^H = \left[\boldsymbol{A}^H\left(\boldsymbol{AA}^H\right)^+\right]^H = \left(\boldsymbol{AA}^H\right)^+\boldsymbol{A}^H$. 又由 (1) 知，$(\boldsymbol{A}^H\boldsymbol{A})^+ = \boldsymbol{A}^+(\boldsymbol{A}^H)^+$，进而

$$(\boldsymbol{A}^H\boldsymbol{A})^+ = \boldsymbol{A}^+(\boldsymbol{A}^H)^+ = \boldsymbol{A}^+\left(\boldsymbol{AA}^H\right)^+\boldsymbol{A}^H.$$

类似，可证另外一个等式.

(3) 由定理 3(3)，知

$$\boldsymbol{AA}^+ = \boldsymbol{A}\left[\boldsymbol{A}^H\left(\boldsymbol{AA}^H\right)^+\right] = \left(\boldsymbol{AA}^H\right)\left(\boldsymbol{AA}^H\right)^+.$$

又因为

$$\begin{aligned}
\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)^{+} &= \left[\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)^{+}\right]^{\boldsymbol{H}} \\
&= \left[\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)^{+}\right]^{\boldsymbol{H}}\left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)^{\boldsymbol{H}} \\
&= \left(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}\right)^{+}\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}} \\
&= \boldsymbol{A}\boldsymbol{A}^{+}.
\end{aligned}$$

故有

$$\boldsymbol{A}\boldsymbol{A}^{+} = (\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}})(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}})^{+} = (\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}})^{+}(\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}).$$

类似，可证另外一个等式.

定理 5 设 $\boldsymbol{A} \in \mathbf{C}^{m\times l}$，$\boldsymbol{B} \in \mathbf{C}^{l\times n}$，则

$$(\boldsymbol{A}\boldsymbol{B})^{+} = \boldsymbol{B}^{+}\boldsymbol{A}^{+} \Leftrightarrow \mathrm{R}(\boldsymbol{A}^{\boldsymbol{H}}\boldsymbol{A}\boldsymbol{B}) \subset \mathrm{R}(\boldsymbol{B})\text{且}\mathrm{R}(\boldsymbol{B}\boldsymbol{B}^{\boldsymbol{H}}\boldsymbol{A}^{\boldsymbol{H}}) \subset \mathrm{R}(\boldsymbol{A}^{\boldsymbol{H}}).$$