10　Kronecker积

定义1:设$A=(a_{ij})\in P^{m\times n}$, $B=(b_{ij})\in P^{p\times q}$,

$$A\otimes B=\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

则$A\otimes B$称为A与B的*Kronecker*积.

(直积direct product、张量积tensor product) .

*Kronecker*积的最代表性的应用在信号处理与系统理论中的多变元时间序列的高阶统计量理论与方法中的应用.

多变元时间序列的高阶累积量、高阶谱(多谱)和描述输入、输出与多信道冲击响应三者关系变得非常简洁.

例1 设$A=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$与$B=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$. 则

$$A\otimes B=\begin{pmatrix}B & 2B & 3B\\ 3B & 2B & B\end{pmatrix}=\begin{pmatrix}2 & 1 & 4 & 2 & 6 & 3\\ 2 & 3 & 4 & 6 & 6 & 9\\ 6 & 3 & 4 & 2 & 2 & 1\\ 6 & 9 & 4 & 6 & 2 & 3\end{pmatrix}$$

$$A\otimes B\neq B\otimes A.$$

例2 $I_2 \otimes B = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$

例3 $B \otimes I_2 = \begin{bmatrix} b_{11} & 0 & b_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ b_{21} & 0 & b_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix}$

例4　设$x \in R^m, y \in R^n$

$$x \otimes y = \left[x_1 y^T, \quad \cdots, \quad x_m y^T \right]^T \in R^{mn}$$

例5　设$x \in R^m, y \in R^n$

$$x \otimes y^T = \left[x_1 y, \quad \cdots, \quad x_m y \right]^T$$

$$= \begin{bmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{bmatrix} \in R^{m \times n}$$

定理1：Kronecker积的性质：

设$A \in P^{m\times n}, B \in P^{p\times q}, C \in P^{r\times s}, D \in P^{k\times h}$

(1) $E_m \otimes E_n = E_{mn}$

(2) $\lambda(A\otimes B) = (\lambda A)\otimes B = A\otimes(\lambda B)$

(3) $(A\pm B)\otimes C = (A\otimes C)\pm(B\otimes C)$

$A\otimes(B\pm C) = (A\otimes B)\pm(A\otimes C)$

(4) $(A\otimes B)\otimes C = A\otimes(B\otimes C)$

(5) $(A\otimes B)^T = A^T\otimes B^T, \overline{A\otimes B} = \overline{A}\otimes\overline{B},$

$(A\otimes B)^H = A^H\otimes B^H.$

(6) 当$n = r, q = k$时,$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

(7) 当$m = n, p = q$时,且A, B可逆,则

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

(8) $rank(A \otimes B) = rankA \cdot rankB;$

(9) 当$m = n, p = q$时,

$$tr(A \otimes B) = trA \cdot trB$$

$$det(A \otimes B) = (detA)^{p} \cdot (detB)^{n}$$

证明:

(6) $(A \otimes B)(C \otimes D)$

$$= \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mnB} \end{bmatrix} \begin{bmatrix} c_{11}D & \cdots & c_{1s}D \\ \vdots & \ddots & \vdots \\ c_{n1}D & \cdots & c_{ns}D \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}c_{k1}BD & \cdots & \sum_{k=1}^{n} a_{1k}c_{ks}BD \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}c_{k1}BD & \cdots & \sum_{k=1}^{n} a_{mk}c_{ks}BD \end{bmatrix} = AC \otimes BD$$

$$(9)A = P^{-1}\begin{pmatrix} \lambda_1 & & & * \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}P = P^{-1}J_1P$$

$$B = Q^{-1}\begin{pmatrix} \mu_1 & & & * \\ 0 & \mu_2 & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \mu_p \end{pmatrix}Q = Q^{-1}J_2Q$$

$$A \otimes B = (P^{-1} J_1 P) \otimes (Q^{-1} J_2 Q)$$

$$= (P \otimes Q)^{-1} (J_1 \otimes J_2)(P \otimes Q) \Rightarrow$$

$$det(A \otimes B) = det(J_1 \otimes J_2)$$

$$= (\prod_{j=1}^{p} \lambda_1 \mu_j)(\prod_{j=1}^{p} \lambda_2 \mu_j)\cdots(\prod_{j=1}^{p} \lambda_n \mu_j)$$

$$= (\lambda_1)^p (\prod_{j=1}^{p} \mu_j) \cdot (\lambda_2)^p (\prod_{j=1}^{p} \mu_j) \cdots\cdots (\lambda_n)^p (\prod_{j=1}^{p} \mu_j)$$

$$\therefore det(A \otimes B) = (detA)^p \bullet (detB)^n$$

定义2

$Kronecker$ 积的乘幂：$A^{[k]} = A \otimes A \otimes \cdots \otimes A.$

定理2

(1) 当$A^T = A, B^T = B$时, $A \otimes B$也是对称矩阵;

当$A^H = A, B^H = B$时, $A \otimes B$也是$Hermite$矩阵;

(2) 当U, V均为酉矩阵时, $U \otimes V$也是酉矩阵;

(3) $(AB)^{[k]} = A^{[k]}B^{[k]}.$

例6： 以1或−1为元素的m阶矩阵H, 如果有

$$HH^T = mE_m$$

则称H为m阶Hadamard矩阵. 设H_m, H_n分别为m, n阶Hadamard矩阵,则 $H_m \otimes H_n$为mn阶Hadamard矩阵.

$$(6)(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

证：

$$(H_m \otimes H_n)(H_m \otimes H_n)^T$$

$$= (H_m \otimes H_n)(H_m^T \otimes H_n^T)$$

$$= H_m H_m^T \otimes H_n H_n^T$$

$$= mE_m \otimes nE_n = mnE_{mn}$$

二、Kronecker积的特征值

定理3： 设$\lambda_i(i=1,2,\cdots,m)$为$A\in C^{m\times m}$, $x_i(i=1,2,\cdots,m)$为相应的特征向量;$\mu_j(j=1,2,\cdots,n)$为$B\in C^{n\times n}$, $y_j(j=1,2,\cdots,n)$为相应的特征向量, 则$A\otimes B$有mn个特征值$\lambda_i\mu_j$, 对应的特征向量为$x_i\otimes y_j$.

$(6)(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$

证： $Ax_i=\lambda_i x_i, By_j=\mu_j y_j \Rightarrow$

$$(A\otimes B)(x_i\otimes y_j)=Ax_i\otimes By_j=\lambda_i x_i\otimes \mu_j y_j$$

$$=\lambda_i\mu_j(x_i\otimes y_j)$$

定义3 m阶矩阵A与n阶矩阵B的*Kronecker* 和：

$$A\oplus_k B = A\otimes E_n + E_m\otimes B$$

$$\left(A\oplus_k B = E_n\otimes A + B\otimes E_m\right)$$

例7 设$A=\begin{pmatrix}1&2&3\\3&2&1\\1&1&4\end{pmatrix}$与$B=\begin{pmatrix}2&1\\2&3\end{pmatrix}$. 计算$A\oplus_k B$.

解 $A\oplus_k B = A\otimes E_2 + E_3\otimes B$

$$=\begin{bmatrix}1&0&2&0&3&0\\0&1&0&2&0&3\\3&0&2&0&1&0\\0&3&0&2&0&1\\1&0&1&0&4&0\\0&1&0&1&0&4\end{bmatrix}+\begin{bmatrix}2&1&0&0&0&0\\2&3&0&0&0&0\\0&0&2&1&0&0\\0&0&2&3&0&0\\0&0&0&0&2&1\\0&0&0&0&2&3\end{bmatrix}$$

定理4： 设$\lambda_i(i=1,2,\cdots,m)$为$A\in C^{m\times m}$, $x_i(i=1,2,\cdots,m)$为相应的特征向量; $\mu_j(j=1,2,\cdots,n)$为$B\in C^{n\times n}$, $y_j(j=1,2,\cdots,n)$为相应的特征向量, 则$\lambda_i+\mu_j$是$A\oplus_k B$的特征值, $x_i\otimes y_j$为对应的特征向量.

$$(6)(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$$

证： $(A\oplus_k B)(x_i\otimes y_j)$

$$=(A\otimes E_n)(x_i\otimes y_j)+(E_m\otimes B)(x_i\otimes y_j)$$

$$=(Ax_i)\otimes y_j+x_i\otimes(By_j)=(\lambda_i+\mu_j)x_i\otimes y_j$$

2、向量化算符

$$\text{设}\quad A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

记A的列为 $A_{c1}, A_{c2}, \ldots, A_{cn} \Rightarrow A=(A_{c1}, A_{c2}, \ldots, A_{cn})$

向量化算符: $\mathrm{Vec}\ A=\begin{pmatrix} A_{c1} \\ A_{c2} \\ \vdots \\ A_{cn} \end{pmatrix}$

例：$A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \Rightarrow Vec(A)=\begin{pmatrix}1\\3\\2\\4\end{pmatrix}$

性质1： $Vec(kA+lB)=kVec(A)+lVec(B)$

若矩阵$A_{m\times n}=ab^T$,则 $\text{Vec}\ (ab^T)=b\otimes a.$

定理5： 设$A\in C^{m\times n}, X\in C^{n\times r}, B\in C^{r\times s}$,则

$$\text{Vec}\,(AXB)=(B^T\otimes A)\text{Vec}\,X$$

证明：设$X_{n\times r}=\left[x_1,x_2,\cdots,x_r\right]$,且$e_1,e_2,\cdots,e_r$是$r\times 1$的单位向量,

则 $X_{n\times r}=\sum_{i=1}^{r}x_ie_i^T$,

$$\text{Vec}\,(AXB)=\text{Vec}\,(\sum_{i=1}^{r}Ax_ie_i^TB)\quad=\sum_{i=1}^{r}\text{Vec}\left[(Ax_i)(e_i^TB)\right]$$

由$\text{Vec}\,(ab^T)=b\otimes a$可得

$$=\sum_{i=1}^{r}\text{Vec}\left[(Ax_i)(B^Te_i)^T\right]=\sum_{i=1}^{r}(B^Te_i\otimes Ax_i)$$

$$=(B^T\otimes A)\sum_{i=1}^{r}(e_i\otimes x_i)=(B^T\otimes A)\sum_{i=1}^{r}\text{Vec}(x_ie_i^T)$$

$$=(B^T\otimes A)\text{Vec}(X).$$

推论1： 设$A \in C^{m\times m}, B \in C^{n\times n}, X \in C^{m\times n}$,则

(1) $\text{Vec}(AX) = (E_n \otimes A)\text{Vec}\, X;$

(2) $\text{Vec}(XB) = (B^T \otimes E_m)\text{Vec}\, X;$

(3) $\text{Vec}(AX+XB) = \left[(E_n \otimes A)+(B^T \otimes E_m)\right]\text{Vec}\, X$

性质2 $A_1, A_2, \cdots, A_k \in C^{m\times n}$线性无关

$\Leftrightarrow \text{Vec}(A_1), \text{Vec}(A_2), \cdots, \text{Vec}(A_k)$线性无关

例8： 解矩阵方程

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$AX = C \Rightarrow Vec\,(AX) = (E \otimes A)Vec\ X = Vec\,(C) \Rightarrow$$

$$\begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & a_{11} & a_{12} \\ & & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} c_{11} \\ c_{21} \\ c_{12} \\ c_{22} \end{pmatrix}$$

3.*Kronecker* 乘积的应用

矩阵方程的求解

$$(1): AX + XB = D\left(Sylvester\ equation\right)$$

$$\Leftrightarrow \mathrm{Vec}(AX + XB) = [(E \otimes A) + (B^T \otimes E)]\mathrm{Vec}(X) = \mathrm{Vec}(D)$$

$$(2): AXB = D$$

$$\Leftrightarrow \mathrm{Vec}(AXB) = (B^T \otimes A)\mathrm{Vec}(X) = \mathrm{Vec}(D)$$

$$(3): A_1XB_1 + A_2XB_2 = D$$

$$\Leftrightarrow [(B_1{}^T \otimes A_1) + (B_2{}^T \otimes A_2)]\mathrm{Vec}(X) = \mathrm{Vec}(D)$$

线代基础知识复习

补子空间(不唯一)；正交补子空间(唯一).

维数定理　　和与直和(主要区别：向量分解是否唯一)

线性子空间($W \subset V, W$对V中的$+,\times$封闭)

$\Uparrow$

线性空间　$(V,P,+,\cdot)$ $P=R(C)$实(复)空间.

$\Downarrow$　　$\cup$

线性变换T

$T:\alpha$-基下矩阵A;

β-基下矩阵B.

α-基到β-基矩阵P;

$B=P^{-1}AP$.

投影变换$T^2=T$

内积空间(欧式空间，酉空间)

内积由α-基下度量矩阵A唯一确定.

β-基下的度量矩阵为B　$B=P^TAP$.

正交投影$T^2=T, R^{\perp}(T)=N(T)$

保距变换T：正交变换、酉变换.

*Householder*变换；*Givens*变换；*Gauss*变换

$\Downarrow\Rightarrow$初等矩阵

Kronecker乘积