18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1,2,3,4,5,6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

   1. Compute the condition probability $P[X = 4|Z = 6]$. **ANSWER:** $1/5$

   2. Compute the conditional expectation $E[Z|Y]$ as a function of Y. **ANSWER:** $Y + 7/2$.

2. (10 points) Janet is standing outside at time zero when it starts to drizzle. The times at which raindrops hit her are a Poisson point process with parameter $\lambda = 2$. In expectation, she is hit by 2 raindrops in each given second.

   (a) What is the expected amount of time until she is first hit by a raindrop? **ANSWER:** $1/2$ second

   (b) What is the probability that she is hit by exactly 4 raindrops during the first 2 seconds of time? **ANSWER:** $e^{-2\lambda}(2\lambda)^k/k! = e^{-4}4^4/4!$.

3. (10 points) Let X be a random variable with density function f, cumulative distribution function F, variance V and mean M.

   (a) Compute the mean and variance of $3X + 3$ in terms of V and M. **ANSWER:** Mean $3M + 3$, variance $9V$.

   (b) If $X_1, \ldots, X_n$ are independent copies of X. Compute (in terms of F) the cumulative distribution function for the largest of the X_i. **ANSWER:** $F(a)^n$. This is the probability that all n values are less than a.

4. (10 points) Suppose that X_i are i.i.d. random variables, each uniform on $[0,1]$. Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. **ANSWER:** $M_{aX_1} = E^{aX_1} = \int_0^1 e^{ax}dx = (e^a - 1)/a$. Moment generating function for sum is $(e^a - 1)^n/a^n$.

5. (10 points) Suppose that X and Y are outcomes of independent standard die rolls (each equal to $\{1,2,3,4,5,6\}$ with equal probability). Write $Z = X + Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $\log 6$ and $\log 6$

(b) Compute $H(X, Z)$. **ANSWER:** $\log 36 = 2\log 6$.

(c) Compute $H(10X + Y)$. **ANSWER:** $\log 36 = 2\log 6$ (since 36 sums all distinct).

(d) Compute $H(Z) + H_Z(Y)$. (Hint: you shouldn't need to do any more calculations.) **ANSWER:** $\log 36$

6. (10 points) Elaine's not-so-trusty old car has three states: broken (in Elaine's possession), working (in Elaine's possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.

(ii) Each morning the car starts out W, it has .5 chance of staying W, and a .5 chance of switching to B by the next morning.

(iii) Each morning the car starts out S, it has a .5 chance of staying S and a .5 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem. **ANSWER:** Markov chain matrix is

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .5 & .5 & 0 \\ 0 & .5 & .5 \end{pmatrix}$$

(b) If the car starts out B on one morning, what is the probability that it will start out B two days later? **ANSWER:** 1/4

(c) Over the long term, what fraction of mornings does the car start out in each of the three states, B, S, and W? **ANSWER:** Row vector π such that $\pi M = \pi$ (with components of π summing to one) is $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

7. Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of independent random variables which are each equal to 2 with probability 1/3 and .5 with probability 2/3. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^{n} X_i$ for $n \geq 1$.

(a) What is the the probability that Y_n reaches 8 before the first time that it reaches $\frac{1}{8}$? **ANSWER:** sequences is martingale, so $1 = EY_T = 8p + (1/8)(1-p)$. Solving gives $1 - 8p = (1-p)/8$, so $8 - 64p = 1 - p$ and $63p = 7$. Answer is $p = 1/9$.

(b) Find the mean and variance of $\log Y_{10000}$. **ANSWER:** Compute for $\log Y_1$, multiply by 10000.

(c) Use the central limit theorem to approximate the probability that $\log Y_{10000}$ (and hence Y_{10000}) is greater than its median value. **ANSWER:** About .5.

8. (10 points) Eight people toss their hats into a bin and the hats are redistributed, with all of the 8! hat permutations being equally likely. Let N be the number of people who get their own hat. Compute the following:

(a) $\mathbb{E}[N]$ **ANSWER:** 1

(b) $\mathrm{Var}[N]$ **ANSWER:** 1

9. (10 points) Let X be a normal random variable with mean μ and variance σ^2.

(a) $\mathbb{E}e^X$. **ANSWER:** $e^{\mu+\sigma^2/2}$.

(b) Find μ, assuming that $\sigma^2 = 3$ and $E[e^X] = 1$. **ANSWER:** $\mu + \sigma^2/2 = 0$ so $\mu = -9/2$.

10. (10 points)

1. Let $X_1, X_2, \ldots$ be independent random variables, each equal to 1 with probability 1/2 and -1 with probability 1/2. In which of the cases below is the sequence Y_n a martingale? (Just circle the corresponding letters.)

   (a) $Y_n = X_n$ **NO**

   (b) $Y_n = 1 + X_n$ **NO**

   (c) $Y_n = 7$ **YES**

   (d) $Y_n = \sum_{i=1}^n iX_i$ **YES**

   (e) $Y_n = \prod_{i=1}^n (1 + X_i)$ **YES**

2. Let $Y_n = \sum_{i=1}^n X_i$. Which of the following is necessarily a stopping time for Y_n?

(a) The smallest n for which $|Y_n| = 5$. **YES**

(b) The largest n for which $Y_n = 12$ and $n < 100$. **NO**

(c) The smallest value n for which $n > 100$ and $Y_n = 12$. **YES**

18.600 Probability and Random Variables
Fall 2019

Fall 2012 18.440 Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations.

1. (10 points) Lisa's truck has three states: broken (in Lisa's possession), working (in Lisa's possession), and in the shop. Denote these states B, W, and S.

(i) Each morning the truck starts out B, it has a 1/2 chance of staying B and a 1/2 chance of switching to S by the next morning.

(ii) Each morning the truck starts out W, it has 9/10 chance of staying W, and a 1/10 chance of switching to B by the next morning.

(iii) Each morning the truck starts out S, it has a 1/2 chance of staying S and a 1/2 chance of switching to W by the next morning.

Answer the following

(a) Write the three-by-three Markov transition matrix for this problem.

ANSWER: Ordering the states B, W, S, we may write the Markov chain matrix as

$$M = \begin{pmatrix} .5 & 0 & .5 \\ .1 & .9 & 0 \\ 0 & .5 & .5 \end{pmatrix}.$$

(b) If the truck starts out W on one morning, what is the probability that it will start out B two days later? **ANSWER:** $(9/10)(1/10) + (1/10)(1/2) = .09 + .05 = .14$

(c) Over the long term, what fraction of mornings does the truck start out in each of the three states, B, S, and W? **ANSWER:** We find the stationarity probability vector $\pi = (\pi_B, \pi_W, \pi_S) = (1/7, 5/7, 1/7)$ by solving $\pi M = \pi$ (with components of π summing to 1).

2. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 1 with probability 1/2 and -1 with probability 1/2. Write $Y_n = \sum_{i=1}^{n} X_i$. Answer the following:

(a) What is the probability that Y_n reaches 10 before the first time that it reaches -30? **ANSWER:** Probability p satisfies $10p + (-30)(1-p) = 0$, so $40p = 30$ and $p = 3/4$.

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = X_n + Y_n$ **ANSWER: NO**

(ii) $Z_n = \prod_{i=1}^n (2X_i + 1)$ **ANSWER: YES**

(iii) $Z_n = \prod_{i=1}^n (-X_i + 1)$ **ANSWER: YES**

(iv) $Z_n = \sum_{i=1}^n Y_i$ **ANSWER: NO**

(v) $Z_n = \sum_{i=2}^n X_i X_{i-1}$ **ANSWER: YES**

3. (10 points) Ten people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 10! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N^2]$ **ANSWER:** Let N_i be 1 if ith person gets own hat, zero otherwise. Then
$E[(\sum N_i)^2] = \sum_{i=1}^{10} \sum_{j=1}^{10} E[N_i N_j] = 90(1/90) + 10(1/10) = 2.$

(b) $P(N = 8)$ **ANSWER:** There are $\binom{10}{2}$ ways to pick a pair of people to have swapped hats. So answer is $\binom{10}{2}/10!$.

4. (10 points) When Harry's cell phone is on, the times when he receives new text messages form a Poisson process with parameter $\lambda_T = 3/\text{minute}$. The times at which he receives new email messages form an independent Poisson process with parameter $\lambda_E = 1/\text{minute}$. He receives personal messages on Facebook as an independent Poisson process with rate $\lambda_F = 2/\text{minute}$.

(a) After catching up on existing messages one morning, Harry begins to wait for new messages to arrive. Let X be the amount of time (in minutes) that Harry has to wait to receive his first text message. Write down the probability density function for X. **ANSWER:** time is exponential with parameter $\lambda_T = 3$, so density function is $f(x) = 3e^{-3x}$ for $x \geq 0$.

(b) Compute the probability that Harry receives 10 new messages total (including email, text, and Facebook) during his first two minutes of waiting. **ANSWER:** Number total in two minutes is Poisson with rate $\lambda = 2(\lambda_E + \lambda_T + \lambda_F) = 12$. So answer is $\lambda^k e^{-\lambda}/k! = 12^{10} e^{-12}/10!$.

(c) Let Y be the amount of time elapsed before the third email message. Compute $\text{Var}(Y)$. **ANSWER:** Variance of time till email message is $1/\lambda_E^2 = 1$. Memoryless property and additivity of variance of independent sums gives $\text{Var}(S) = 3$.

(d) What is the probability that Harry receives no messages of any kind during his first five minutes of waiting? **ANSWER:** Time till first message is exponential with parameter 6. Probability this time exceeds 5 is e^{-30}.

5. (10 points) Suppose that X and Y have a joint density function f given by

$$f(x,y) = \begin{cases} 1/\pi & x^2+y^2 < 1 \\ 0 & x^2+y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X. **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \frac{1}{\pi}2\sqrt{1-x^2} & -1 \leq x \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Express $E[\sin(XY)]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:**

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\sin(xy)dydx.$$

6. (10 points) Let X be the number on a standard die roll (i.e., each of $\{1,2,3,4,5,6\}$ is equally likely) and Y the number on an independent standard die roll. Write $Z = X + Y$.

(a) Compute the conditional probability $P[X = 6|Z = 8]$. **ANSWER:** $1/5$

(b) Compute the conditional expectation $E[Y|Z]$ as a function of Z (for $Z \in \{2,3,4,\ldots,12\}$). **ANSWER:**
$Z = E[Z|Z] = E[X+Y|Z] = E[X|Z] + E[Y|Z]$. By symmetry, $E[X|Z] = E[Y|Z] = Z/2$.

7. (10 points) Suppose that X_i are i.i.d. random variables, each of which assumes a value in $\{-1,0,1\}$, each with probability $1/3$.

(a) Compute the moment generating function for X_1. **ANSWER:** $Ee^{tX_1} = (e^{-t}+1+e^t)/3$.

(b) Compute the moment generating function for the sum $\sum_{i=1}^{n} X_i$.
ANSWER: $(e^{-t}+1+e^{t})^n/3^n$

8. (10 points) Let X and Y be independent random variables. Suppose X takes values in $\{1,2\}$ each with probability $1/2$ and Y takes values in $\{1,2,3,4\}$ each with probability $1/4$. Write $Z = X+Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $\log 2 = 1$ and $\log 4 = 2$.

(b) Compute $H(X,Z)$. **ANSWER:** $\log 2 + \log 4 = \log 8 = 3$.

(c) Compute $H(X+Y)$. **ANSWER:**

$$\sum_{i=1}^{6} P(X+Y=i)(-\log P(X+Y=i)) = 2\cdot\frac{1}{8}\log 8 + 3\cdot\frac{1}{4}\log 4 = 6/8+6/4 = 9/4.$$

9. (10 points) Let X be a normal random variable with mean 0 and variance 1.

(a) Compute $\mathbb{E}[e^X]$. **ANSWER:**

$$E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx =$$

$$e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx = e^{1/2}.$$

(b) Compute $\mathbb{E}[e^X 1_{X>0}]$. **ANSWER:**

$$\begin{aligned} E(e^X 1_{X>0}) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^x dx \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2-2x+1)/2+1/2} dx \\ &= e^{1/2} \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2} dx \\ &= e^{1/2} \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = e^{1/2}(1-\Phi(-1)) = e^{1/2}\Phi(1), \end{aligned}$$

where $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

(c) Compute $\mathbb{E}[X^2+2X-5]$. **ANSWER:**
$E[X^2] + 2E[X] - 5 = 1 + 0 - 5 = -4$.

10. (10 points) Let X be uniformly distributed random variable on $[0, 1]$.

(a) Compute the variance of X. **ANSWER:** $E[X^2] = \int_0^1 x^2 dx = 1/3$ and $E[X] = 1/2$ so $\text{Var}[X] = E[X^2] - E[X]^2 = 1/12$.

(b) Compute the variance of $3X + 5$. **ANSWER:** $9\text{Var}[X] = 3/4$.

(c) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z? **ANSWER:**

$$F_Z(a) = P\{Z \leq a\} = \prod_{i=1}^n P\{X_i \leq a\} = F_{X_1}(a)^n = \begin{cases} 0 & a < 0 \\ a^n & a \in [0, 1] \\ 1 & a > 1 \end{cases}$$

18.600 Probability and Random Variables
Fall 2019

Spring 2014 18.440 Final Exam Solutions

1. (10 points) Let X be a uniformly distributed random variable on $[-1,1]$.

(a) Compute the variance of X^2. **ANSWER:**

$$\mathrm{Var}(X^2) = E[(X^2)^2] - E[X^2]^2,$$

and

$$E[X^2] = \int_{-1}^{1} (x^2/2)dx = \frac{x^3}{6}\Big|_{-1}^{1} = 1/3,$$

$$E[(X^2)^2] = E[X^4] = \int_{-1}^{1} \frac{x^4}{2}dx = \frac{x^5}{10}\Big|_{-1}^{1} = 1/5,$$

so $\mathrm{Var}(X^2) = E[(X^2)^2] - E[X^2]^2 = 1/5 - (1/3)^2 = 1/5 - 1/9 = 4/45$.

(b) If $X_1, \ldots, X_n$ are independent copies of X, and $Z = \max\{X_1, X_2, \ldots, X_n\}$, then what is the cumulative distribution function F_Z? **ANSWER:** $F_{X_1}(a) = (a+1)/2$ for $a \in [-1,1]$. Thus

$$F_Z(a) = F_{X_1}(a)F_{X_2}(a)\ldots F_{X_n}(a) = \begin{cases} \left(\frac{a+1}{2}\right)^n & a \in [-1,1] \\ 0 & a < -1 \\ 1 & a > 1 \end{cases}$$

2. (10 points) A certain bench at a popular park can hold up to two people. People in this park walk in pairs or alone, but nobody ever sits down next to a stranger. They are just not friendly in that particular way. Individuals or pairs who sit on a bench stay for at least 1 minute, and tend to stay for 4 minutes on average. Transition probabilities are as follows:

(i) If the bench is empty, then by the next minute it has a 1/2 chance of being empty, a 1/4 chance of being occupied by 1 person, and a 1/4 chance of being occupied by 2 people.

(ii) If it has 1 person, then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 1 person.

(iii) If it has 2 people then by the next minute it has 1/4 chance of being empty and a 3/4 chance of remaining occupied by 2 people.

(a) Use E, S, D to denote respectively the states empty, singly occupied, and doubly occupied. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by E, S, and D. **ANSWER:**

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix}$$

(b) If the bench is empty, what is the probability it will be empty two minutes later? **ANSWER:** $\frac{1}{2}\frac{1}{2} + \frac{1}{4}\frac{1}{4} + \frac{1}{4}\frac{1}{4} = 6/16 = 3/8$.

(c) Over the long term, what fraction of the time does the bench spend in each of the three states? **ANSWER:** We know

$$\begin{pmatrix} E & S & D \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \\ 1/4 & 0 & 3/4 \end{pmatrix} = \begin{pmatrix} E & S & D \end{pmatrix}$$

and $E + S + D = 1$. Solving gives $E = S = D = 1/3$.

3. (10 points) Eight people throw their hats into a box and then randomly redistribute the hats among themselves (each person getting one hat, all 8! permutations equally likely). Let N be the number of people who get their own hats back. Compute the following:

(a) $E[N]$ **ANSWER:** $8 \times \frac{1}{8} = 1$

(b) $P(N = 7)$ **ANSWER:** 0 since if seven get their own hat, then the eighth must also.

(c) $P(N = 0)$ **ANSWER:** This is an inclusion exclusion problem. Let A_i be the event that the ith person gets own hat. Then

$$\begin{aligned} P(N > 0) &= P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_8) \\ &= \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \ldots \\ &= \binom{8}{1}\frac{1}{8} - \binom{8}{2}\frac{1}{8 \cdot 7} + \binom{8}{3}\frac{1}{8 \cdot 7 \cdot 6} \cdots \\ &= 1/1! - 1/2! + 1/3! + \ldots - 1/8! \end{aligned}$$

Thus,

$$P(N = 0) = 1 - P(N > 0) = 1 - 1/1! + 1/2! - 1/3! + 1/4! + 1/5! - 1/6! + 1/7! - 1/8! \approx 1/e.$$

4. (10 points) Suppose that $X_1, X_2, X_3, \dots$ is an infinite sequence of independent random variables which are each equal to 5 with probability 1/2 and -5 with probability 1/2. Write $Y_n = \sum_{i=1}^n X_i$. Answer the following:

(a) What is the probability that Y_n reaches 65 before the first time that it reaches -15? **ANSWER:** Y_n is a martingale, so by the optional stopping theorem, we have $E[Y_T] = Y_0 = 1$ (where $T = \min\{n : Y_n \in \{-15, 65\}\}$). We thus find $0 = Y_0 = E[Y_T] = 65p + (-15)(1-p)$ so $80p = 15$ and $p = 3/16$.

(b) In which of the cases below is the sequence Z_n a martingale? (Just circle the corresponding letters.)

(i) $Z_n = 5X_n$
(ii) $Z_n = 5^{-n} \prod_{i=1}^n X_i$
(iii) $Z_n = \prod_{i=1}^n X_i^2$
(iv) $Z_n = 17$
(v) $Z_n = X_n - 4$

ANSWER: (iv) only.

5. (10 points) Suppose that X and Y are independent exponential random variables with parameter $\lambda = 2$. Write $Z = \min\{X, Y\}$

(a) Compute the probability density function for Z. **ANSWER:** Z is exponential with parameter $\lambda + \lambda = 4$ so $F_Z(t) = 4e^{-4t}$ for $t \geq 0$.

(b) Express $E[\cos(X^2Y^3)]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:** $\int_0^\infty \int_0^\infty \cos(x^2y^3) \cdot 2e^{-2x} \cdot 2e^{-2y} dydx$

6. (10 points) Let X_1, X_2, X_3 be independent standard die rolls (i.e., each of $\{1, 2, 3, 4, 5, 6\}$ is equally likely). Write $Z = X_1 + X_2 + X_3$.

(a) Compute the conditional probability $P[X_1 = 6|Z = 16]$. **ANSWER:** One can enumerate the six possibilities that add up to 16. These are $(4, 6, 6), (6, 4, 6), (6, 6, 4)$ and $(6, 5, 5), (5, 6, 5), (5, 5, 6)$. Of these, three have $X_1 = 6$, so $P[X_1 = 6|Z = 16] = 1/2$.

(b) Compute the conditional expectation $E[X_2|Z]$ as a function of Z (for $Z \in \{3, 4, 5, \dots, 18\}$). **ANSWER:** Note that $E[X_1 + X_2 + X_3|Z] = E[Z|Z] = Z$. So by symmetry and additivity of conditional expectation we find $E[X_2|Z] = Z/3$.

7. (10 points) Suppose that X_i are i.i.d. uniform random variables on $[0,1]$.

(a) Compute the moment generating function for X_1. **ANSWER:** $E(e^{tX_1}) = \int_0^1 e^{tx}dx = \frac{e^t-1}{t}$.

(b) Compute the moment generating function for the sum $\sum_{i=1}^n X_i$. **ANSWER:** $\left(\frac{e^t-1}{t}\right)^n$

8. (10 points) Let X be a normal random variable with mean 0 and variance 5.

(a) Compute $E[e^X]$. **ANSWER:** $E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{5}\sqrt{2\pi}} e^{-x^2/(2\cdot 5)} e^{-x} dx$. A complete the square trick allows one to evaluate this and obtain $e^{5/2}$.

(b) Compute $E[X^9 + X^3 - 50X + 7]$. **ANSWER:** $E[X^9] = E[X^7] = E[X] = 0$ by symmetry, so $E[X^9 + X^3 - 50X + 7] = 7$.

9. (10 points) Let X and Y be independent random variables. Suppose X takes values $\{1,2\}$ each with probability $1/2$ and Y takes values $\{1,2,3\}$ each with probability $1/3$. Write $Z = X+Y$.

(a) Compute the entropies $H(X)$ and $H(Y)$. **ANSWER:** $H(X) = -(1/2)\log\frac{1}{2} - (1/2)\log\frac{1}{2} = -\log\frac{1}{2} = \log 2$. Similarly, $H(Y) = -(1/3)\log\frac{1}{3} - (1/3)\log\frac{1}{3} - (1/3)\log\frac{1}{3} = -\log\frac{1}{3} = \log 3$.

(b) Compute $H(X,Z)$. **ANSWER:** $H(X,Z) = H(X,Y) = H(X) + H(Y) = \log 6$.

(c) Compute $H(2^X 3^Y)$. **ANSWER:** Also $\log 6$, since each distinct (X,Y) pair gives a distinct number for $2^X 3^Y$.

10. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of independent random variables which are each equal to 2 with probability $1/3$ and $.5$ with probability $2/3$. Let $Y_0 = 1$ and $Y_n = \prod_{i=1}^n X_i$ for $n \geq 1$.

(a) What is the the probability that Y_n reaches 4 before the first time that it reaches $\frac{1}{64}$? **ANSWER:** Y_n is a martingale, so by the optional stopping theorem, $E[Y_T] = Y_0 = 1$ (where $T = \min\{n : Y_n \in \{1/64, 4\}\}$). Thus $E[Y_T] = 4p + (1/64)(1-p) = 1$. Solving yields $p = 63/255 = 21/85$.

(b) Find the mean and variance of $\log Y_{400}$. **ANSWER:** $\log X_1$ is $\log 2$ with probability 1/3 and $-\log 2$ with probability 2/3. So

$$E[\log X_1] = \frac{1}{3}\log 2 + \frac{2}{3}(-\log 2) = \frac{-\log 2}{3}.$$

Similarly,

$$E[(\log X_1)^2] = \frac{1}{3}(\log 2)^2 + \frac{2}{3}(-\log 2)^2 = (\log 2)^2.$$

Thus,

$$\mathrm{Var}(X_1) = E[(\log X_i)^2] - E[\log X_i]^2 = (\log 2)^2 - \big(\frac{-\log 2}{3}\big)^2 = (\log 2)^2(1 - \frac{1}{9}) = \frac{8}{9}(\log 2)^2.$$

Multiplying, we find $E[\log Y_{400}] = 400E[\log X_1] = -400(\log 2)/3$.
And $\mathrm{Var}[\log Y_{400}] = (3200/9)(\log 2)^2$.

(c) Compute $\mathbb{E}Y_{100}$. **ANSWER:** Since Y_n is a martingale, we have $E[Y_{100}] = 1$. This can also be derived by noting that for independent random variables, the expectation of a product is the product of the expectations.

Spring 2016 18.600 Final Exam Solutions

1. (10 points) Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence of normal random variables, each of which has mean 1 and variance 1.

(a) Compute the mean and variance of $Y = X_1 - 2X_2 + 3X_3 - 4X_4$. **ANSWER:** By additivity of expectation, mean is $1 - 2 + 3 - 4 = -2$. By additivity of variance for independent random variables (and fact that $\mathrm{Var}(X) = \mathrm{Var}(-X)$), variance is $1 + 4 + 9 + 16 = 30$.

(b) Compute the probability density function for Y. **ANSWER:** Y is normal with mean $\mu = -2$ and variance $\sigma^2 = 30$, so $f_Y(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{30}\cdot\sqrt{2\pi}} e^{-(x+2)^2/60}$.

(c) Compute $P(Y > 0)$ in terms of the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. **ANSWER:** $Y > 0$ if it is $2/\sqrt{30}$ standard deviations above its mean, so $P(Y > 0) = 1 - \Phi(2/\sqrt{30}) = \Phi(-2/\sqrt{30})$.

2. (10 points) Suppose that $X_1, X_2, X_3, \ldots$ is an infinite sequence of i.i.d. normal random variables, this time with mean 0 and variance 1.

(a) Write $Y_n = \sum_{i=1}^{n} X_i$. Is Y_n a martingale? **ANSWER:** Yes, since

$$E[Y_n|Y_0, Y_1, \ldots, Y_{n-1}] = E[Y_{n-1} + X_n|Y_0, Y_1, \ldots, Y_{n-1}] = E[Y_{n-1}|Y_0, Y_1, \ldots, Y_{n-1}] = Y_{n-1}$$

(b) Write $Z_n = Y_n^2 - n$. Compute $E[Z_n - Z_{n-1}|X_1, X_2, \ldots, X_{n-1}]$. You can use the following calculation to help you get started:

$$\begin{aligned} Z_n - Z_{n-1} &= (Y_n^2 - n) - (Y_{n-1}^2 - (n-1)) = Y_n^2 - Y_{n-1}^2 - 1 \\ &= (Y_{n-1} + X_n)^2 - Y_{n-1}^2 - 1 \\ &= 2Y_{n-1}X_n + X_n^2 - 1. \end{aligned}$$

ANSWER: By additivity of expectation, this is

$$E[2Y_{n-1}X_n|X_1, X_2, \ldots, X_{n-1}] + E[X_n^2 - 1|X_1, X_2, \ldots, X_{n-1}]$$

First term is zero since $E[X_n]$ is zero, and $2Y_{n-1}$ can be treated as a constant for this conditional expectation calculation (since it is known once $X_1, X_2, \ldots, X_{n-1}$ is given). Second term is same as $E[X_n^2 - 1]$ (by independence of X_n and $X_1, \ldots, X_{n-1}$) which is zero. Overall answer is thus zero.

(c) Is Z_n a martingale? **ANSWER:** By our definition, Z_n is a martingale if and only if $E[Z_n|Z_0, Z_1, Z_2, \ldots, Z_{n-1}] = Z_{n-1}$, or equivalently $E[Z_n - Z_{n-1}|Z_0, Z_1, Z_2, \ldots, Z_{n-1}] = 0$. We rewrite this as $E[2Y_{n-1}X_n + X_n^2 - 1|Z_0, Z_1, Z_2, \ldots, Z_{n-1}] = 0$. This is true by the analysis used in (b) and the fact that Y_{n-1} and X_n are conditionally independent of each other when $Z_0, Z_1, Z_2, \ldots, Z_{n-1}$ is given, with the conditional expectation of X_n being zero.

3. Suppose that $X_1, X_2, \ldots$ is an infinite sequence of i.i.d. Cauchy random variables, so that each has probability density function $\frac{1}{\pi(1+x^2)}$. Let W be an independent normal random variable with mean zero and variance one.

(a) Write $S = X_1 + X_2 + X_3 + X_4 + X_5$. Give an explicit formula for the probability density function f_S. **ANSWER:** We know that $T = S/5$ is a Cauchy random variable. So $f_S(x) = f_{5T}(x) = \frac{1}{5} f_T(x/5) = \frac{1}{5\pi\left(1+(x/5)^2\right)}$.

(b) Write the probability density function for $A = X_1 + W$. (Your answer will involve an integral that you do not have to try to evaluate explicitly.) **ANSWER:**

$$f_{X_1+W}(a) = \int_{-\infty}^{\infty} f_{X_1}(x) f_W(a-x)dx = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \frac{1}{\sqrt{2\pi}} e^{-(a-x)^2/2} dx.$$

(c) Compute (as an explicit rational number) the probability that $X_1 + X_2 > X_3 + 3$. (Hint: remember the spinning flashlight story.) **ANSWER:** Since $-X_3$ is also Cauchy, $B = (X_1 + X_2 - X_3)/3$ is Cauchy, and we are computing $P(B > 1)$, which is $1/4$.

4. Suppose that n people toss their shoes into a bin (two shoes—one left and one right—per person) and then the shoes are randomly shuffled and returned to the n people, with all ways of returning two shoes to each person being equally likely. Let B be the number of people who get one left and one right shoe (regardless of whether they match).

(a) Compute the expectation $E[B]$. **ANSWER:** Let A_i be 1 if ith person gets both a left and a right shoe, zero otherwise. If we imagine shoes handed out one by one, then whatever first person gets as first shoe, that person will have $n/(2n-1)$ chance of getting an opposite type shoe for the second, so $E[A_1] = 2/(2n-1)$. Similarly $E[A_i] = 2/(2n-1)$ for any i. Thus $E[B] = \sum_{i=1}^n E[A_i] = n^2/(2n-1)$.

(b) Compute $E[B^2]$. **ANSWER:** $E[B^2] = E[\sum_{i=1}^n A_i \sum_{j=1}^n A_j] = \sum_{i=1}^n \sum_{j=1}^n E[A_i A_j]$. There are $(n^2 - n)$ "off diagonal terms" with $i \neq j$ and n diagonal terms with $i = j$. Putting them together gives $n \cdot \frac{n}{2n-1} + (n^2 - n)\frac{n}{2n-1} \cdot \frac{n-1}{2n-3}$.

5. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the triangle $\{(x, y) : x \geq 0, y \geq 0, x + y \leq 1$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$. **ANSWER:** The triangle has area $1/2$, so $f_{X,Y}(x, y)$ is 2 if (x, y) is in the triangle and zero otherwise.

(b) Compute the conditional expectation $E[X|Y]$ as a function of Y. **ANSWER:** Once Y is given, X is conditionally uniform along the segment $[0, 1 - Y]$. So $E[X|Y] = (1 - Y)/2$.

(c) Compute $E[X]$. **ANSWER:**

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)x dx dy = \int_0^1\int_0^{1-y} 2x dx dy$$
$$= \int_0^1 (1-y)^2 dy = -(1-y)^3/3 \Big|_0^1 = 1/3.$$

6. (10 points) Let X and Y be independent uniform random variables on $[0,1]$.

   (a) Compute the moment generating function for $Z = X+Y$. **ANSWER:** We know $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} f_X(x)e^{tx}dx = \int_0^1 e^{tx}dx = (e^t-1)/t$. Thus $M_{X+Y}(t) = M_X(t)M_Y(t) = M_X(t)^2 = (e^t-1)^2/t^2$.

   (b) Compute the probablity density function for $W = X^3$. **ANSWER:** For $a \in (0,1)$, $F_W(a) = P(W \le a) = P(X^3 \le a) = P(x \le a^{1/3}) = a^{1/3}$. Thus, $f_W(a) = F_W'(a) = (1/3)a^{-2/3}$.

7. (10 points) Let X, Y, Z be i.i.d. uniform random variables on $[0,5]$.

   (a) Set $A = \min\{X,Y,Z\}$. Compute the probability density function f_A. **ANSWER:** For $a \in (0,5)$, $P(A > 5) = P(X > 5, Y > 5, Z > 5) = \left(\frac{5-a}{5}\right)^3$. So $F_A(a) = 1 - \left(\frac{5-a}{5}\right)^3$ and $f_A(a) = F_A'(a) = -3(-1/5)\left(\frac{5-a}{5}\right)^2$ for $a \in (0,5)$ and 0 for $a \notin (0,5)$.

   (b) Let B be the second largest of the three values in $\{X,Y,Z\}$. Compute $E[B]$. **ANSWER:** $E[B] = 5/2$ by symmetry. (In principle you could also compute this by rescaling to the interval $[0,1]$ and recalling some of our problem set problems on beta random variables.)

   (c) Let $C = \max\{X,Y,Z\}$. Compute the probability $P(1 < C < 4)$. **ANSWER:** $(4/5)^3 - (1/5)^3 = 63/125$.

8. (10 points)

   (a) Let X be the number of heads that come up when three independent fair coins are tossed. Compute the entropy $H(X)$. **ANSWER:**

$$H(X) = \frac{1}{8}\big(-\log\frac{1}{8}\big) + \frac{3}{8}\big(-\log\frac{3}{8}\big) + \frac{3}{8}\big(-\log\frac{3}{8}\big) + \frac{1}{8}\big(-\log\frac{1}{8}\big).$$

   (b) Suppose that X and Y are two (not necessarily independent or identically distributed) random variables, each of which takes values in the set $\{1,2,3,4,5,6\}$. Which of the following is *necessarily* true? (Just circle the corresponding letters.)

      (i) $H(X,Y) \ge H(X) + H(Y)$ **ANSWER:** Not necessarily true. For example, if $H(X) > 0$ and $X = Y$ with probability one, then $H(X,Y) = H(X) \le H(X) + H(Y) = 2H(X)$. The properties derived in lecture imply that the inequality actually *does* hold in the other direction, since $H(X,Y) = H(X) + H_X(Y) \le H(X) + H(Y)$.

(ii) $H(X,Y) = H(X) + H(Y)$ **ANSWER:** Not necessarily true, by reasoning above.

(iii) $H(X) \leq H(X,Y)$. **ANSWER:** True, since $H(X,Y) = H(X) + H_X(Y)$ and $H_X(Y) \geq 0$.

(iv) $H(X+Y) \leq \log(11)$. **ANSWER:** True, since $X+Y$ takes one of 11 possibilities, and (as shown on problem set) entropy is at most what if would be if all 11 possibilities were equally likely.

(v) $H(X-Y) \geq 0$. **ANSWER:** True, since entropy of any random variable is non-negative.

9. (10 points) A certain country has three distinct types of leaders: liberal, conservative, and insane. Every four years they elect a new leader.

(i) If the current leader is liberal, there is a 2/3 chance the next leader will be liberal also, a 1/4 chance the next leader will be conservative, and a 1/12 chance the next leader will be insane.

(ii) If the currently leader is conservative, there is a 2/3 chance the next leader will be conservative also, a 1/4 chance the next leader will be liberal, and a 1/12 chance the next leader will be insane.

(iii) If the currently leader is insane, then there is a 1/2 chance the next leader will be conservative and a 1/2 chance the next leader will be liberal. (They never allow two consecutive terms of insanity.)

(a) Use L, C, I to denote the three states. Write the three-by-three Markov transition matrix for this problem, labeling columns and rows by L, C, and I. **ANSWER:**

$$\begin{pmatrix} 2/3 & 1/4 & 1/12 \\ 1/4 & 2/3 & 1/12 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(b) If the current leader is insane, what is the probability that, after two elections, the leader will be insane again? **ANSWER:** 1/12

(c) Over the long term, what fraction of the time does the country spend under each of the three types of leaders? **ANSWER:** You can write

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 2/3 & 1/4 & 1/12 \\ 1/4 & 2/3 & 1/12 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}$$

and use $a + b + c = 1$ to solve and find $(a, b, c) = (6/13, 6/13, 1/13)$, which are the long term probabilities for states L, C and I respectively.

10. (10 points) A certain town (with constant climate) has had an average of one house fire per year for the past century. At the beginning of one calendar year, Jill moves to town, and during that year there are 6 house fires. A trial is held to determine whether the increase in fires is due to Jill being a witch. During the trial the judge asks a math expert the following (which you should answer):

(a) Suppose that house fire times in this town are a Poisson point process with parameter λ equal to 1 per year. Under this assumption, let p be the probability that there will be exactly 6 house fires during a single given year. What is p? **ANSWER:** $p = e^{-\lambda}\lambda^k/k!$ with $\lambda = 1$ and $k = 6$. Comes to $e^{-1}/6!$.

(b) Under the same assumption, what is the probability that, during the course of a century, there will be *at least* 1 calendar year during which there are exactly 6 house fires? Compute your answer in terms of the p computed in (a). **ANSWER:** $1 - (1-p)^{100}$.

When Jill first moved to town, Nora thought that there was a 1/100 chance that Jill was a witch. She also thought if Jill *wasn't* a witch the number of fires that year would be Poisson with parameter 1, and that if Jill *was* a witch the number would be 6 with probability 1. (Arranging for there to be exactly 6 fires per year is what arsonist witches in this world do.)

(c) *Given* that the number of observed fires during Jill's first year in town is 6, what is Nora's assessment of the *conditional* probability that Jill is a witch? Compute your answer in terms of the p computed in (a). **ANSWER:** Let F be event that there are six fires, W event Jill is a witch. Then

$$P(W|F) = \frac{P(WF)}{P(F)} = \frac{P(W)P(F|W)}{P(W)P(F|W) + P(W^c)P(F|W^c)} = \frac{(1/100)}{(1/100) + (99/100)p}.$$

Remark: In more realistic variants, "Jill" is a polluting industrial facility and "house fires" are health problems allegedly caused by pollution.

Spring 2017 18.600 Final Exam Solutions

1. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$.

(a) Compute $E[3X^{13}]$. **ANSWER:** Recall that $\int_0^\infty e^{-x}x^n = n!$ (which can be taken as one of the definitions of $n!$). This implies that $E[X^n] = n!$ and hence $E[3X^{13}] = 3 \cdot 13!$

(b) Compute the conditional probability $P[X > 10|X > 5]$ **ANSWER:** An exponential random variable of rate λ is greater than a with probability $e^{-\lambda a}$. Thus

$$P[X > 5, X > 10]/P[X > 5] = P[X > 10]/P[X > 5] = e^{-10}/e^{-5} = e^{-5}.$$

Alternatively, this follows from "memoryless" property of exponentials. (If X is the time a bus arrives, then *given* no bus for first five units, conditional probability of five more units without bus is same as original probability of five units without bus.)

(c) Let $Y = X^2$. Compute the cumulative distribution function F_Y. **ANSWER:**

$$F_Y(a) = P(Y \le a) = P(X^2 \le a) = P(X \le \sqrt{a}) = F_X(\sqrt{a}) = 1 - e^{-\sqrt{a}}.$$

2. (10 points) Sally is a pleasant texting companion. Each of her texts consists of a string of five emoji, each chosen independently from the same probability distribution. Precisely, if $X = (X_1, X_2, X_3, X_4, X_5)$ represents one of her texts, then each X_i is independently equal to

Person Shugging with probability 1/4,
Face with Tears of Joy with probability 1/4,
Face Blowing a Kiss with probability 1/8,
Face with Rolling Eyes with probability 1/8,
Love Heart with probability 1/16,
Thinking Face with probability 1/16,
See-No-Evil Monkey with probability 1/16,
Hula-hooping Statue of Liberty with probability 1/16.

(a) Compute the entropy of one of Sally's emoji. That is, compute $H(X_1)$. **ANSWER:** $\frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 + \frac{1}{16} \cdot 4 = 11/4$.

(b) Compute the entropy of an entire text, i.e., compute $H(X) = H(X_1, X_2, X_3, X_4, X_5)$. **ANSWER:** Since the X_i are i.i.d we have $H(X) = 5H(X_1) = 55/4$.

(c) Suppose you try to figure out the value of X_1 by asking a series of yes or no questions to someone who knows the value. Assume you use the strategy that minimizes the expected number of questions you need. How many questions do you expect to ask? **ANSWER:** We know that $H(X) = 11/4$ is the minimum number possible. Since we can divide the space exactly in half (in terms of probability) with each question, this minimum is achievable, and the answer is 11/4.

3. Let X_1, X_2, X_3 be i.i.d. random variables, each with probability density function $\frac{1}{\pi(1+x^2)}$.

(a) Assume that a and b are fixed positive constants and write $Y = aX_1 + b$. Compute the probability density function f_Y. **ANSWER:** Recall that in general $f_{aZ}(x) = a^{-1} f_Z(x/a)$ and $f_{Z+b}(x) = f_Z(x-b)$. Applying that here we have

$$f_Y(x) = a^{-1} f_{X_1}((x-b)/a) = a^{-1} \frac{1}{\pi\big(1 + (x-b)^2/a^2\big)}.$$

(b) Compute the probability that $X_1 \in [-1, 1]$. Give an explicit number. (Recall spinning flashlight story.) **ANSWER:** The flashlight at $(0,1)$ goes over a $\pi/2$ range of angles as it shifts from pointing to $(0,1)$ to $(0,-1)$, as compared to the total π range of angles. So the answer is $(\pi/2)/\pi = 1/2$.

(c) Compute the probability density function for $Z = (X_1 + X_2 + X_3)/3$. **ANSWER:** The average of Cauchy random variables is again Cauchy so $f_Z(x) = \frac{1}{\pi(1+x^2)}$.

(d) Compute the probability density function for $W = (X_1 + 2X_2)/3$. [Hint: use (c).] **ANSWER:** The average of X_2 and X_3 is Cauchy, so $X_2 + X_3$ has the same density function as $2X_2$. Hence the answer is the same as in (c), i.e., $f_W(x) = \frac{1}{\pi(1+x^2)}$

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables each of which is equal to 5 with probability 1/2 and -5 with probability 1/2. Write $Y_0 = 85$ and $Y_n = Y_0 + \sum_{i=1}^n X_i$ for $n > 0$.

(a) Find the probability that the sequence $Y_0, Y_1, Y_2, \ldots$ reaches 50 before it reaches 100. **ANSWER:** Write p_{50} for probability of reaching 50 first and p_{100} for probability of hitting 100 first. Then by the optional stopping theorem $50p_{50} + 100p_{100} = 85$ and $p_{50} + p_{100} = 1$. Solving gives $p_{50} = 3/10$.

(b) Let T be the smallest $n \geq 0$ for which Y_n is an integer multiple of 500. Compute $E[Y_T]$. **ANSWER:** Eventually Y_n will reach 0 or 500. By optional stopping theorem $E[Y_T] = Y_0 = 85$.

(c) Compute $E[3Y_{10} - 4Y_8 | Y_5]$ in terms of Y_5. In other words, recall that the conditional expectation $E[3Y_{10} - 4Y_8 | Y_5]$ can be understood as a random variable and express this random variable as a simple function of Y_5. **ANSWER:** $E[Y_{10} | Y_5 = Y_5]$ and $E[Y_8 | Y_5] = Y_5$. By additivity of conditional expectation $E[3Y_{10} - 4Y_8 | Y_5] = -Y_5$.

5. (10 points) Let X be a normal random variable with mean zero and variance one.

(a) Compute the moment generating function for X using the "complete the square" trick. (Show your work.) **ANSWER:**

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{tx} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2 - 2tx + t^2)/2 + t^2/2} dx = e^{t^2/2}.$$

(b) Fix positive constants a, b, and c and compute $E[aX^2 + bX + c]$ in terms of a, b, and c. **ANSWER:** Additivity of expectation gives

$$E[aX^2 + bX + c] = E[aX^2] + E[bX] + E[c] = a + c.$$

6. Ten people are taking a class together.

(a) Let K be the number of ways to divide the ten people into five groups, two people per group, so that each person in the class has a partner. Find K. **ANSWER:** $10!/2^5$ ways to get ordered set of five partnerships. Dividing by 5! we get $K = 10!/(2^5 5!)$ to get unordered division into five partnerships.. Alternatively, line people up, choose parner for first person (9 choices), partner for next un-partnered person (7 choices) etc. to get $K = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$.

(b) Assume that one of these K ways is chosen uniformly at random on one day, and another is chosen uniformly at random (independently) on a subsequent day. Let N be the number of people who have the same partner on both days. What is $E[N]$? **ANSWER:** Each person has 1/9 chance of getting same partner on second day. Additivity of expectation gives $E[N] = 10/9$.

(c) Compute $P(N = 6)$. You can use K in your answer if you did not compute an explicit value for K. **ANSWER:** Fix partnerships for first day. Then there are K ways to choose partnerships for second day. How many of these ways result in $N = 6$? Well, there are $\binom{5}{3}$ ways to decide which partnerships will stay unchanged on second day. Pick one of remaining four people, and there are two ways to assign that person a partner *different* from the previous day. So the number of ways to get $N = 6$ is $\binom{5}{3} \cdot 2$ and $P(N = 6) = \frac{\binom{5}{3} \cdot 2}{K}$.

7. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the semi-circle $\{(x, y) : x^2 + y^2 \leq 1, x \geq 0\}$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$. **ANSWER:** $f_{X,Y}(x, y) = 2/\pi$ if (x, y) in semi-circle, zero otherwise.

(b) Compute the conditional expectation $E[X|Y]$ as a function of Y. **ANSWER:** Given a value for Y in $[-1, 1]$, the value X is conditionally uniform on $[0, \sqrt{1 - Y^2}]$ so $E[X|Y] = \sqrt{1 - Y^2}/2$ for $Y \in [-1, 1]$.

(c) Express $E[X^3 \cos(Y)]$ as a double integral. You do not have to compute the integral. **ANSWER:**

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{2}{\pi} x^3 \cos(y) dx dy.$$

8. (10 points) Let X, Y, Z be i.i.d. exponential random variables, each with parameter $\lambda = 1$. Compute the probability density functions for the following random variables.

(a) $2X + 1$ **ANSWER:** $f_{2X+1}(x) = \frac{1}{2}e^{-(x-1)/2}$ for $x \in [1, \infty)$.

(b) $X + Y + Z$. **ANSWER:** This is a Γ distribution: $f_{X+Y+Z}(x) = x^2 e^{-x}/2$ on $[0, \infty)$.

(c) $\min\{X, Y, Z\}$. **ANSWER:** Exponential with $\lambda = 3$, so density is $3e^{-3x}$ on $[0, \infty)$.

9. (10 points) A certain athletically challenged monkey has difficulty climbing a ladder. The ladder has seven rungs. At each given second, the monkey can be on any of the seven rungs of the ladder — or on the "zeroth" rung, meaning on the ground). If $0 \le m \le 6$ and the monkey is on the mth rung of the ladder at a given second, then at the next second the monkey will be at rung $m + 1$ with probability $1/2$ and back at rung 0 with probability $1/2$. If the monkey is at level 7, then with probability 1 the monkey will go back to 0 at the next step.

(a) Give the 8 by 8 matrix describing the monkey's transition probabilities. (You don't have to write all 64 entries. Just write the entries that are non-zero.) **ANSWER:**

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) If the monkey is at state zero at a given time, which is the probability that the monkey is on the 7th rung of the ladder seven seconds later. (That is, what is the probability the monkey goes straight from the bottom to the top without falling once?) **ANSWER:** $2^{-7} = 1/128$.

(c) Compute $(\pi_0, \pi_1, \ldots, \pi_7)$ where π_i is the fraction of the time the monkey spends in state i, over the long term. (Hint: first see if you can express each π_j as an integer multiple of π_{j+1} for $j < 7$. Then see if you can express each π_j as a multiple of π_7.) **ANSWER:**

$$\begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 & \pi_6 & \pi_7 \end{pmatrix}.$$

We find that each $\pi_j = 2\pi_{j+1}$ for $j \in \{0, 1, 2, 3, 4, 5, 6\}$. Thus the row vector has the form $\begin{pmatrix} 128\pi_7 & 64\pi_7 & 32\pi_7 & 16\pi_7 & 8\pi_7 & 4\pi_7 & 2\pi_7 & \pi_7 \end{pmatrix}$. and since values add up to 1 we get $\pi_7 = 1/255$ and the answer is

$$\begin{pmatrix} 128/255 & 64/255 & 32/255 & 16/255 & 8/255 & 4/255 & 2/255 & 1/255 \end{pmatrix}.$$

10. (10 points) Let $X_1, X_2, \ldots, X_{300}$ be independent random real numbers, each chosen uniformly on the interval $[0, 100]$. Let $S = \sum_{i=1}^{300} X_i$.

(a) Compute $E[S]$ and $\mathrm{Var}(S)$. **ANSWER:** $E[S] = 100E[X_1] = 15000$. Recall that variance of uniform random variable on $[0, 1]$ is $1/12$, so $\mathrm{Var}[X_1] = 10000/12$ and $\mathrm{Var}[S] = 3000000/12 = 250000$.

(b) Give a Poisson approximation for the probability that $3 \le X_j < 4$ for exactly 3 of the values $j \in \{1, 2, \ldots, 300\}$. **ANSWER:** The number of values in that interval is roughly Poisson with parameter $\lambda = 3$, so probability we get $k = 3$ is

$$e^{-\lambda}\lambda^k/k! = e^{-3}3^3/3! = \frac{9}{2e^3}.$$

(c) Give an interval $[a, b]$ such that $E[S] = (a + b)/2$ and $P(S \in [a, b]) \approx .95$. (You may use the fact that $\int_{-2}^{2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx .95$.) **ANSWER:** Standard deviation of S is $\sqrt{250000} = 500$, and CLT implies chance of S being within two standard deviations of $E[S]$, i.e., in $[14000, 16000]$ is about .95.

18.600 Probability and Random Variables
Fall 2019

Spring 2018 18.600 Final Exam Solutions

1. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the unit circle $\{(x, y) : x^2 + y^2 \le 1\}$.

(a) Give the joint probability density function $f(x, y)$ for the pair (X, Y). **ANSWER:**

$$f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute the marginal law $f_X(x)$. **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi} & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

(c) Compute the conditional expectation $E[X^2|Y]$ as a function of Y. **ANSWER:** If $y \in [-1, 1]$, then $E[X^2|Y = y]$ equals $E[Z^2]$ for Z uniform on $[-a, a]$ with $a = \sqrt{1 - y^2}$. Here $E[Z^2] = \int_{-a}^{a} \frac{1}{2a} x^2 dx = x^3/3 \,\Big|_{-a}^{a} = a^2/3$. So $E[X^2|Y = y] = (1 - y^2)/3$ and $E[X^2|Y] = (1 - Y^2)/3$. (We do not define $E[X^2|Y]$ for $Y \notin [-1, 1]$.)

2. (10 points) Ivan is waiting for the bus. There are three kinds of buses, whose arrival times are three independent Poisson point processes. During each hour, on average one expects to see 3 yellow buses, 2 blue buses, and 1 red bus at the bus stop where Ivan is waiting.

(a) Give the probability density function for T where T is the length of time until the first bus of *any* kind shows up. **ANSWER:** This is exponential with parameter $\lambda = 6$, so density is $f_T(x) = 6e^{-6x}$ for $x \ge 0$.

(b) Compute the probability that there are exactly two yellow buses during the first half hour. **ANSWER:** Number of yellow buses in $1/2$ hour is Poisson with parameter $\lambda = 3/2$. So probability is $e^{-\lambda}\lambda^k/k!$ with $k = 2$, which is $e^{-3/2}(3/2)^2/2$.

(c) Suppose that a yellow bus would get Ivan home in 30 minutes (from the time it arrives at the bus stop until the time it gets to his house) while either a red or a blue bus would get Ivan home in 15 minutes (from the time it picks Ivan up at the bus stop until the time it gets to his house). If Ivan wants to minimize the expected amount of time until he gets home, and a yellow bus arrives first, should he get on the yellow bus or should he wait for a red/blue bus to come? Give a sentence or two of explanation. **ANSWER:** If he holds out for a red/blue bus, it will take 20 minutes to arrive (in expectation) but only shorten transit duration for 15 minutes. So he is better off (in expectation) taking the bird in hand (i.e., getting on the yellow bus).

3. (10 points) Harriet and Helen are real estate agents. Each week, each agent is sent to close a deal, and one of the two agents closes a deal 10 percent of the time, while the other closes it 20 percent of the time. To set notation, let X_j be 1 if the more capable agent closes the deal the jth week and 0 otherwise. Let Y_j be 1 if the less capable agent closes the deal on the jth week, and 0 otherwise. So each X_j is equal to 1 with probability 2/10 and 0 with probability 8/10. Each Y_j is equal to 1 with probability 1/10 and 0 with probability 9/10. Assume that the random variables $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ are all independent of each other. For each $j \geq 1$ write $Z_j = X_j - Y_j$. Write $S_n = \sum_{j=1}^{n} Z_j$.

(a) Compute the mean, variance and standard deviation of Z_1. **ANSWER:** $E[X_1] = .2$ and $E[Y_1] = .1$ so $E[Z_1] = E[X_1 - Y_1] = .1$. Using independence, $\mathrm{Var}(Z_1) = \mathrm{Var}(X_1 - Y_1) = \mathrm{Var}(X_1) + \mathrm{Var}(-Y_1) = \mathrm{Var}(X_1) + \mathrm{Var}(Y_1) = .09 + .16 = .25$. So $\mathrm{SD}(Z_1) = .5$.

(b) Compute the mean, variance and standard deviation of S_{100}. **ANSWER:** By additivity of mean and variance (for independent sums) we get $E[S_{100}] = 10$ and $\mathrm{Var}(S_{100}) = 25$ so $\mathrm{SD}(S_{100}) = 5$.

(c) Use the central limit theorem to approximate the probability that $S_{100} > 0$. (This is the probability that the more capable closer has closed more deals after 100 weeks.) You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** Note that 0 is two standard deviations below the expectation of S_{100}. So using CLT approximation, $P(S_{100} > 0) \approx P(N > -2)$ where N is a standard normal random variable. This is $1 - \Phi(-2)$ or equivalently $\Phi(2) \approx .977$. **Remark:** This is similar to a problem set problem in which the probabilities were .5 and .6 instead of .1 and .2. In the problem set problem, the analog of $\mathrm{Var}(Z_1)$ was $.25 + .24 = .49$ and the analog of $\mathrm{SD}(Z_1)$ was .7. It would have taken 196 weeks in the problem set scenario to be able to distinguish between the candidates with as much confidence as we achieved with 100 weeks in this problem. But the take home message is that in both situations it takes a long time to be able to determine (with .977 confidence) which candidate is stronger based on empirical results.

4. (10 points) A future society has exactly seven software companies. (You can guess what they are.) I will call them Company 1, Company 2, Company 3, Company 4, Company 5, Company 6, and Company 7. Every software engineer either works for one of these companies or is unemployed. At the end of each month, a survey asks what each software engineer is doing. It happens that...

> A software engineer who is unemployed one month has a 3/10 chance to be unemployed the next month, and (for each $j \in \{1, 2, \dots, 7\}$) a 1/10 chance to be at Company j.
>
> A software engineer at Company j one month (for any $j \in \{1, 2, \dots, 7\}$) has a 69/70 chance to remain at Company j the next month, and a 1/70 chance to be unemployed.

Complete the following:

(a) Represent the employment activity as a Markov chain by writing down an 8 by 8 transition matrix with states numbered 0 (unemployment) or 1 to 7 (the companies). You don't have to write out all 64 terms. Just write the non-zero terms. **ANSWER:**

$$\begin{pmatrix} 3/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 \\ 1/70 & 69/70 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/70 & 0 & 69/70 & 0 & 0 & 0 & 0 & 0 \\ 1/70 & 0 & 0 & 69/70 & 0 & 0 & 0 & 0 \\ 1/70 & 0 & 0 & 0 & 69/70 & 0 & 0 & 0 \\ 1/70 & 0 & 0 & 0 & 0 & 69/70 & 0 & 0 \\ 1/70 & 0 & 0 & 0 & 0 & 0 & 69/70 & 0 \\ 1/70 & 0 & 0 & 0 & 0 & 0 & 0 & 69/70 \end{pmatrix}$$

(b) If a software engineer is currently unemployed, what is the probability that he or she will be unemployed in two months? **ANSWER:** For this to happen, either stay in state 0 both months or transition out to company state and then back to state zero. Adding the two probabilities gives $(3/10)^2 + 7(1/10) \cdot (1/70) = .09 + .01 = 1/10$.

(c) Compute $(\pi_0, \pi_1, \pi_2, \ldots, \pi_7)$ where π_i is the fraction of months spent in state i over the long term. [Hint: start by working out π_0.] **ANSWER:** Imagine we just focus on two states (U for unemployed or E for employed), and note that this is still a Markov chain. So we have

$$\begin{pmatrix} \pi_U & \pi_E \end{pmatrix} \begin{pmatrix} 3/10 & 7/10 \\ 1/70 & 69/70 \end{pmatrix} = \begin{pmatrix} \pi_U & \pi_E \end{pmatrix},$$

and first equation we get from matrix is $(3/10)\pi_U + (1/70)\pi_E = \pi_U$, which implies $(1/70)\pi_E = (7/10)\pi_U$, so $\pi_E = 49\pi_U$. Combining with $\pi_E + \pi_U = 1$ we find $\pi_E = .98$ and $\pi_U = .02$. Now using symmetry (all seven companies are equally likely) we find $\pi_0 = .02$ and $\pi_j = .98/7 = .14$ for $j \in \{1, 2, \ldots, 7\}$.

5. (10 points) Each of the customers calling a certain toaster customer service number fits into one of seven predictable categories:

1/4 want to purchase a new toaster.

1/4 want to complain about toast being too dark and crispy even on light settings.

1/8 want to complain about a toaster that is not connecting to the internet properly.

1/8 want to complain about a hand being stuck in a toaster.

1/8 want to complain about a toaster having started a fire.

1/16 want to know how to make cinnamon toast.

1/16 want to know how to adjust the springs so that the toast really flies into the air.

Let X be the category of the next customer who will call. We do not yet know X so we model it as a "category-valued" random variable, which takes one of the above seven values with the indicated probabilities.

(a) Compute the entropy $H(X)$. **ANSWER:** $\sum p_i(-\log p_i) = (1/4)2 + (1/4)2 + (1/8)3 + (1/8)3 + (1/8)3 + (1/16)4 + (1/16)4 = 21/8$.

(b) Imagine that the call center determines the category by asking the customer a sequence of yes or no questions, using a strategy that minimizes the expected number of questions asked. What is the expected number of questions asked? **ANSWER:** Since all probabilities are powers of two, answer is $H(X) = 21/8$.

(c) Suppose $X_1, X_2, X_3, \ldots, X_{100}$ are i.i.d. random variables, each with the same law as X, representing the calls received over an entire day. Compute $H(X_1, X_2, X_3, \ldots, X_{100})$. **ANSWER:** By additivity of entropy (for independent random variables) this is $100H(X_1) = 2100/8$.

(d) Let Y be the total number of callers during the day who want to purchase a new toaster. So Y is a random integer between 0 and 100. Write $Z = Y^2$. Compute the entropy $H(Z)$. You can write this as a sum; no need to compute an explicit number. **ANSWER:** Observe that $H(Z) = H(Y)$ since the map $x \to x^2$ is one to one when we restrict to the set $\{0, 1, 2, \ldots, 100\}$. Next note that Y is a binomial random variable with $P(Y = k) = (1/4)^k(3/4)^{100-k}\binom{100}{k}$ for $k \in \{0, 1, \ldots, 100\}$ so

$$H(Z) = H(Y) = -\sum_{k=0}^{100}(1/4)^k(3/4)^{100-k}\binom{100}{k}\log\Big((1/4)^k(3/4)^{100-k}\binom{100}{k}\Big).$$

6. (10 points) Let $X_1, X_2, \ldots$ be i.i.d. random variables, each with density function $\frac{1}{\pi(1+x^2)}$.

(a) Write $S_0 = 0$ and (for $n \geq 1$) write $S_n = \sum_{j=1}^n X_j$. Is the sequence S_n a martingale? Explain why or why not. (Recall that the definition of martingale requires that $E[|S_n|] < \infty$ for all n, and explicitly note whether this is true.) **ANSWER:** This is not technically a martingale since $E[|S_1|] = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}|x|dx = \infty$.

(b) Compute the probability density function for S_{10}. **ANSWER:** We recall that $S_{10}/10$ has the same law as X_1, so S_{10} has the same law as $10X_1$. So density is $\frac{1}{10\pi(1+(x/10)^2)}$.

(c) Compute the probability $P(S_{10} > 10)$. **ANSWER:** This is same as probability that $X_1 > 1$, which is 1/4, as is easily seen from spinning flashlight story.

7. (10 points) Let $X_1, X_2, \ldots$ be an infinite sequence of i.i.d. random variables, each equal to -2 with probability 1/2 and 2 with probability 1/2. Write $S_0 = 0$ and (for $n \geq 1$) write $S_n = \sum_{j=1}^n X_j$.

(a) Compute the probability that the sequence $S_0, S_1, S_2, \ldots$ reaches 20 before the first time that it reaches -30. **ANSWER:** Let T be the first time one of these points is hit. Then by optional stopping theorem $S_0 = E[S_T] = 20P(S_T = 20) + (-30)P(S_T = -30)$. We also know $P(S_T = 20) + P(S_T = -30) = 1$. Solving for the two unknowns, we find $P(S_T = 20) = .6$ and $P(S_T = -30) = .4$.

(b) Compute the probability that there exists *some* positive integer n for which $S_n = -20$. **ANSWER:** This is 1. The probability that S_n hits -20 before hitting K (for large positive K) is $K/(K+20)$. This is a lower bound on the probability that -20 is hit *ever*. Since we can make this lower bound as close to 1 as we want (by taking K large enough) the answer is 1.

(c) Compute the moment generating function $M_{X_1}(t)$. **ANSWER:** $E[e^{tX_1}] = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$.

(d) Compute the moment generating function $M_{S_9}(t)$. **ANSWER:** $E[e^{t(X_1+\ldots+X_9)}] = \prod_{j=1}^0 E[e^{tX_j}] = \left(\frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}\right)^9$.

8. (10 points) Suppose there are 6 cards with labels $1, 2, 3, 4, 5, 6$. The cards are shuffled to a uniformly random permutation and are then turned over one at a time. Each time one sees a card that is higher in numerical value than all other preceding cards, one excitedly shouts "Record!" (The first card to be turned over is automatically a record, since it has no competition.) For example, if the cards are turned over in the order $1, 2, 3, 4, 5, 6$ then one will shout "Record!" all 6 times, since each card has a larger number than all previous numbers. If the cards are turned over in the order $6, 3, 4, 5, 1, 2$ then one will shout "Record!" only once. Let R be the total number of times "Record!" is shouted. (So R is an integer-valued random variable taking values between 1 and 6.) To further set notation, let A_i be 1 if the ith card turned over is a record and 0 otherwise. Compute the following:

(a) The expectation $E[R]$. **ANSWER:** $E[R] = E[\sum_{i=1}^6 A_i] = \sum_{i=1}^6 P(A_i = 1) = 1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 = 49/20$.

(b) The variance $\mathrm{Var}[R]$. **ANSWER:** The easiest way to do this is to observe that the A_j are independent of each other. Hence $\mathrm{Var}(R) = \mathrm{Var}(\sum_{i=1}^6 A_i) = \sum_{i=1}^6 \mathrm{Var}(A_i) = \sum_{i=1}^6 \frac{1}{i} \cdot \frac{i-1}{i} = 0/1 + 1/4 + 2/9 + 3/16 + 4/25 + 5/36$.

(c) Compute the conditional probability that the last three cards turned over are all records *given* that the first three cards are all records. **ANSWER:** Chance first three cards are in increasing order is $1/3!$. Chance all cards are records is $1/6!$. So conditional probability is $(1/6!)/(1/3!) = 3!/6! = 1/120$.

9. (10 points) Let $X_1, X_2, X_3 \dots$ be an infinite sequence of i.i.d. normal random variables, each with mean zero and variance one.

(a) Is the sequence $X_1, X_2, \dots$ a martingale? Explain why or why not in one sentence. **ANSWER:** No, because $E[X_n|\mathcal{F}_{n-1}] = 0$, which is not the same as X_{n-1}. In other words, a person at stage $n-1$ knows the true value of X_{n-1} but considers (given all information currently available) that the expectation of X_n is 0, and this is not generally the same as X_{n-1}.

(b) Compute the correlation coefficient of $A = \sum_{n=1}^{60} X_n$ and $B = \sum_{n=41}^{100} X_n$. **ANSWER:** Since A and B have mean zero, we know that

$$\mathrm{Cov}(A, B) = E[AB] = E[\sum_{j=1}^{60} X_j \sum_{k=41}^{100} X_k] = \sum_{j=1}^{60} \sum_{k=41}^{100} E[X_j X_k].$$

The only non-zero terms are when $j = k$, and there are 20 such terms, so $\mathrm{Cov}(A, B) = 20$. By additivity of variance for independent random variables, we have $\mathrm{Var}(A) = \mathrm{Var}(B) = 60$. So the correlation coefficient is

$$\mathrm{Cov}(A, B)/\sqrt{\mathrm{Var}(A)\mathrm{Var}(B)} = 20/60 = 1/3.$$

(c) Write down the probability density function for A. **ANSWER:** This is normal with mean zero and variance 60, so density is $\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ with $\sigma^2 = 60$, i.e., $\frac{1}{\sqrt{120\pi}} e^{-x^2/120}$

(d) Compute the variance of $A + B$. **ANSWER:** $\mathrm{Var}(A + B) = \mathrm{Var}(A) + \mathrm{Var}(B) + 2\mathrm{Cov}(A, B) = 60 + 60 + 40 = 160.$

10. (10 points) Let X, Y, Z be i.i.d. exponential random variables, each with parameter $\lambda = 1$.

(a) Compute the probability density function for $X + Y + Z$. **ANSWER:** This a gamma distribution with $n = 3$ and $\lambda = 1$. So density is $x^2 e^{-x}/2!$ on $[0, \infty)$.

(b) Compute the expectation and variance of $\max\{X, Y, Z\}$. **ANSWER:** This is the radioactive decay problem. Expectation is $1/3 + 1/2 + 1 = 11/6$. Variance is $1/9 + 1/4 + 1 = 49/36$.

(c) Compute the expectation and variance of $\min\{X, Y, Z\}$. **ANSWER:** The minimum is exponential with parameter $\lambda = 3$. So expectation is $1/3$ and variance $1/9$.

18.600 Probability and Random Variables
Fall 2019

Spring 2019 18.600 Final Exam Solutions

1. (10 points) A broccoli vendor is choosing a website photo: Photo 1 (attractive people eating broccoli at the beach) or Photo 2 (close-up of broccoli with salmon and quinoa). Assume that one of the photos is more "effective" and that a site visitor on average spends \$16 if shown the "more effective" photo and \$14 if shown the "less effective" photo (with standard deviation \$10 in each case). To find out which photo is best, the vendor implements an "A/B test" that involves trying each photo on 50 visitors. Denote the dollar amounts spent by those shown the more effective photo (whichever that is) by $X_1, X_2, \ldots, X_{50}$ and the amounts spent by those shown the less effective photo by $Y_1, Y_2, \ldots, Y_{50}$. Formally, for each $i \in \{1, 2, \ldots, 50\}$, we have $\mathrm{Var}[X_i] = \mathrm{Var}[Y_i] = 100$ and $E[X_i] = 16$ and $E[Y_i] = 14$, and the X_i are identically distributed (as are the Y_i) and all 100 random variables are independent of each other. Write $X = \sum_{i=1}^{50} X_i$ and $Y = \sum_{i=1}^{50} Y_i$.

(a) Use the central limit theorem to approximate the probability that $X > Y$ (so that the vendor correctly identifies the more effective photo). You may use the function $\phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** $(X - Y) = \sum_{i=1}^{50} (X_i - Y_i)$ is approximately normal with expectation $50(16 - 14) = 100$ and variance $\mathrm{Var}(X) + \mathrm{Var}(-Y) = 50 \cdot 100 + 50 \cdot 100 = 100000$ and hence standard deviation 100. Since 0 is one standard deviation below $E[(X - Y)]$ we have

$$P(X - Y) > 0 \approx \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \phi(-1) - \phi(1) \approx .84.$$

(b) Compute the conditional expectation $E\big[(X_1 - Y_1)|X\big]$ as a function of the random variable X. **ANSWER:** We know $E[X_1 + \ldots + X_{50}|X] = E[X|X] = X$. And $E[X_j|X]$ should be the same for each j by symmetry. Hence $E[X_1|X] = X/50$. Since Y_1 and X are independent $E[Y_1|X] = E[Y_1] = 14$. By additivity of conditional expectation, $E[(X_1 - Y_1)|X] = X/50 - 14$.

(c) Compute $E[(X - Y)^2]$. **ANSWER:** $\mathrm{Var}(X - Y) = E[(X - Y)^2] - E[(X - Y)]^2$, so $E[(X - Y)^2] = \mathrm{Var}(X - Y) + E[(X - Y)]^2 = 10000 + 10000 = 20000$.

2. (10 points) An explorer discovers an island with 10 islanders, each of whom has 10 apples. The explorer proposes a game. At each step of the game the explorer will do the following:

(i) First choose uniformly at random one of the $\binom{10}{2}$ possible *pairs* of islanders.

(ii) Then, if both islanders in the pair have *at least one* apple, toss a fair coin to declare one of them the "winner" and transfer one apple from the loser to the winner. (If either islander is already out of apples, do nothing.)

The above is repeated until one islander (the "overall winner") has all the apples. (This happens *eventually* with probability 1. You don't have to prove this; take it as given.) Let A_n^j be the number of apples the jth person has after n steps, and let T be the number of steps before the game ends. So $A_0^j = 10$ for $j \in \{1, 2, \ldots, 10\}$ and $A_T^j = \begin{cases} 100 & j \text{ is overall winner} \\ 0 & \text{otherwise} \end{cases}$

(a) When $j \in \{1, 2, \ldots, 10\}$ is fixed, is the sequence $A_0^j, A_1^j, A_2^j, \ldots$ a martingale? Explain why or why not. **ANSWER:** Yes. We put ourselves in the shoes of somebody who has seen the first n steps and knows A_n^j. The only way that we have $A_{n+1}^j \neq A_n^j$ is if a pair (j,k) is chosen and both jth and kth islanders have a positive number of apples. But if this happens, then the jth islander is just as likely to gain as to lose an apple. So $E[A_{n+1}^j|\mathcal{F}_n] = A_n^j$.

(b) Compute the expected number of islanders who *at some point* have exactly 25 apples. **ANSWER:** Write $S = \min\{n : A_n^j \in \{0, 25\}$. Write $p = P(A_S^j = 25)$. Then the optional stopping theorem implies $10 = 25p + (1-p)\cdot 0$ and solving gives $p = .4$. Each person has a .4 chance reach 25, so (by additivity of expectation) the expected number is $.4 \cdot 10 = 4$.

(c) Compute the expected number of islanders who *never* have more than 10 apples. **ANSWER:** Same analysis as (b) but replace 0 and 25 with 0 and 11. We find that the expected number who reach 0 before 11 is $10 \cdot \frac{1}{11} = 10/11$.

(d) Compute the probability that the overall winner is someone who at some point in the game only had one apple. (Hint: let B_j be the event that the jth islander's apple count drops to 1 before subsequently rising to 100. Observe that $B_1, B_2, \ldots, B_{10}$ are disjoint.) **ANSWER:** By similar analysis to (b) and (c), the jth islander has a $90/99 = 10/11$ chance to reach 1 before 100, and *given* that has a 1/100 chance to reach 100. Multiplying, we find $P(B_j) = 1/110$, and since the B_j are disjoint we have $P(\cup_{j=1}^{10} B_j) = 1/11$.

3. (10 points) Detective Irene has effective techniques for inducing people to confess to crimes. When Irene interrogates a guilty person, that person confesses with probability .9. Unfortunately, Irene's techniques (extended isolation, claiming confession in best interest, etc.) sometimes lead innocent people to confess. When Irene interrogates an innocent person, that person confesses with probability .1. There is a group of 10 people, and it is known that exactly one is guilty of a crime. Irene has a plan to catch the guilty party. Each day, she will pick one of the 10 people (uniformly at random) and interrogate that person. She will continue this every day until somebody confesses, at which point the investigation will end and the confessing individual will be locked up. (Note: the same person may be interrogated multiple times. But a person's probability of confessing during an interrogation is always the same — i.e., .9 if guilty, .1 if innocent — independently of what has happened before.)

(a) Compute the probability that a confession is obtained on the first day. **ANSWER:** $.9 \cdot .1 + .1 \cdot .9 = .18$

(b) Compute the conditional probability that the person interrogated on the first day is guilty *given* that the person confessed. **ANSWER:** $.9 \cdot .1/.18 = 1/2$.

(c) Let N be the total number of interrogations performed (including the final interrogation, the one that produces the confession). Compute $P(N = k)$ for $k \in \{1, 2, 3, \ldots\}$ and compute $E[N]$. **ANSWER:** $P(N = k) = (1 - .18)^{k-1} \cdot .18$. This is geometric with parameter $p = .18$ so $E[N] = 1/p = 100/18 = 50/9$.

(d) What is the overall probability that the person locked up at the end is guilty? **ANSWER:** *Given* that the confession occurred on the kth day, the conditional probability that the person was guilty is 1/2, by similar argument as in (b). Since this is true for any day, the overall probability is 1/2.

4. (10 points) Suppose 8 people toss their hats into a bin. The hats are randomly shuffled (all shufflings equally likely) and returned to the people, one hat per person. But there is an additional twist: while in

the bin, each hat has a 1/2 probability (independently of all else) of falling into a muddy corner of the bin and getting dirty.

(a) Let D be the number of hats that get dirty. Compute $E[D]$ and $\mathrm{Var}[D]$. **ANSWER:** D is binomial with $n = 8$ and $p = 1/2$. We have $E[D] = np = 8/2 = 4$ and $\mathrm{Var}[D] = npq = 8(1/2)(1/2) = 2$.

(b) Let N be the number of people who get back their own hat. Let N^* be the number of people who get their own hat back *and* find that hat to be clean (i.e., not dirty). Compute $E[N^*]$ and $\mathrm{Var}[N^*]$. (In case this notation helps: let N_i^* be 1 if the ith person gets own hat *and* finds it clean, and 0 otherwise, so that $N^* = \sum_{i=1}^{8} N_i^*$.) **ANSWER:** $E[N^*]^2 = \sum_{i=1}^{8}\sum_{j=1}^{8} E[N_i^* N_j^*]$. There are 64 terms, and 8 of them have $i = j$, 56 have $i \neq j$. So $E[N^*]^2 = \sum_{i=j} P(N_i^* = 1) + \sum_{i \neq j} P(N_i^* = N_j^* = 1) = 8 \cdot (1/8)(1/2) + 56(1/56)(1/2)(1/2) = 1/2 + 1/4 = 3/4$. Similar arguments give $E[N^*] = 8(1/8)(1/2) = 1/2$. So $\mathrm{Var}[N^*] = 3/4 - (1/2)^2 = 1/2$.

(c) Let C be the total number of hats that stay clean. Compute the correlation coefficients $\rho(C, D)$ and $\rho(N, C)$. (Hint: this problem should not require a lot of computation.) **ANSWER:** $\rho(C, D) = \rho(C, 8 - C) = \rho(C, -C) = -1$. (Note generally that if X and Y are random variables and c is constant then $\rho(X, Y) = \rho(X, Y + c)$.) And $\rho(N, C) = 0$ since N and C are independent.

5. (10 points) A certain biotech company has a distinctive corporate culture. Each employee has a "level" of 1, 2, 3, 4, or 5. At the end of each year, each employee of level j is assigned a new level in the following way:

1. If $j \in \{1, 2, 3, 4\}$ then the new level is j with probability 1/2 and $j + 1$ with probability 1/2. ("All non-top-level employees have even odds of being promoted each year," reads the company brochure.)

2. If $j = 5$ then the new level is 1 probability 1. ("All top-level employees return to their bottom-level roots.")

(a) Interpret this as a Markov chains and write the corresponding transition matrix. **ANSWER:**

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Over the long term, what fraction of the time does an employee spend in each of the 5 states?

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix}.$$

Solving gives

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} = \begin{pmatrix} 2/9 & 2/9 & 2/9 & 2/9 & 1/9 \end{pmatrix}$$

(c) If an employee starts out in level 1, how many promotion cyles will it take in expectation before the employee reaches state 5? More formally: if A_n is the employee's rank during year n and we are given that $A_0 = 1$ then what is $E[\min\{n : A_n = 5\}]$? **ANSWER:** The number of cycles required for each promotion is geometric with parameter $p = 1/2$, hence expectation $1/p = 2$. Expected number of cycles for four promotions is $4 \cdot 2 = 8$.

6. (10 points) Let X be a uniform random variable on the interval $[0, 10]$. For each real number K write $C(K) = E[\max\{X - K, 0\}]$.

(a) Compute $C(K)$ as a function of K for $K \in [0, 10]$. (Hint: you might find that the integral you need to compute to find $C(K)$ is the area of a triangle.) **ANSWER:**
$C(K) = \int_0^{10} \frac{1}{10} \max\{x - K, 0\}dx = \frac{1}{10}\int_K^{10}(x - K)dx = \frac{1}{10}(10 - K)^2/2$.

(b) Compute the derivatives C' and C'' on the interval $[0, 10]$. **ANSWER:** Taking derivative directly gives $C'(x) = -\frac{1}{10}(10 - x) = x/10 - 1$ and $C''(x) = \frac{1}{10}$. Alternatively, one could remember the call function formulas derived in lecture: $C'(x) = F_X(x) - 1$ and $C''(x) = f_X(x)$.

(c) Compute the expectation $E[X^3]$. **ANSWER:** $\int_0^{10} \frac{1}{10}x^3 dx = \frac{1}{10}x^4/4 \Big|_0^{10} = \frac{1}{10}10^4/4 = 250$.

7. (10 points) Let $X_1, X_2, X_3, \ldots$ be independent exponential random variables, each with parameter $\lambda = 1$.

(a) Let c be a fixed constant and write $Y_n = \left(\sum_{i=1}^n X_i^3\right) - cn$. (So $Y_0 = 0$.) For which (if any) values of c is the sequence $Y_0, Y_1, Y_2, \ldots$ a martingale? **ANSWER:** $Y_n = \sum_{i=1}^n (X_i^3 - c)$ is a cumulative sum of i.i.d. terms and is a martingale only if the terms have expectation zero. Since in general $E[X^n] = n!$ we have $E[X_i^3 - c] = 6 - c$ which is zero if and only if $c = 6$.

(b) Compute the probability $P\big(X_1 + X_2 + X_3 < 2 \text{ and } X_1 + X_2 + X_3 + X_4 > 2\big)$. (Hint: try to come up with a Poisson point process interpretation of the question.) **ANSWER:** The points $X_1, X_1 + X_2, X_1 + X_2 + X_3, X_1 + X_2 + X_3 + X_4, \ldots$ form a Poisson point process of rate 1. So answer is the probability that a Poisson point process of rate 1 has three points in $[0, 2]$ (with the fourth point coming after 2). This is given by $e^{-\lambda}\lambda^k/k!$ with λ set to 2 and k set to 3.

(c) Compute the correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. **ANSWER:** Each X_i has variance one. By independence and covariance bilinearity $\mathrm{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = 2$ and $\mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_2 + X_3 + X_4) = 3$ so answer is $2/\sqrt{3 \cdot 3} = 2/3$.

(d) Give the probability density function for $X_1 + X_2 + X_3$. **ANSWER:** This is gamma with $n = 3$ and $\lambda = 1$. Answer is $x^2e^{-x}/2$.

8. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the unit circle $\{(x, y) : x^2 + y^2 \le 1\}$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$. **ANSWER:** $1/\pi$ in the unit circle, 0 elsewhere.

(b) Compute the marginal probability distribution $f_X(x)$. **ANSWER:** $\frac{1}{\pi}2\sqrt{1 - x^2}$

(c) Compute $E[R]$ where $R = \sqrt{X^2 + Y^2}$. (Hint: maybe use a polar coordinates integral. Or maybe find a way to compute F_R and/or f_R without doing that.) **ANSWER:**
$F_R(a) = P(R \le a) = (\pi a^2)/\pi = a^2$ so $f_R(x) = 2x$ so $E[R] = \int_0^1 2x^2 dx = 2/3$.

9. (10 points) Andrew and Alyssa want to have children, and are eager to have at least one girl and at least one boy. So they decide they will have children (one at a time) until the first time they either have at least one child of each gender *or* they have four children total. Thus, if we let X denote the gender sequence for this family, then the possible values for X are $\{GB, GGB, GGGB, GGGG, BG, BBG, BBBG, BBBB\}$. Assume that each child born has a .5 chance to be a girl and .5 chance to be a boy, independently of what has happened before.

(a) Compute the entropy $H(X)$. (The answer is a rational number. Give it explicitly.) **ANSWER:** The log probability of an outcome is the number of children in that outcome. So the entropy is the expected number of children, which is $(1/2) \cdot 2 + (1/4) \cdot 3 + (1/4) \cdot 4 = 11/4$.

(b) Describe a strategy for asking a sequence of yes/no questions such that the *expected* number of questions one has to ask to learn the value of X is exactly $H(X)$. **ANSWER:** Optimal approach is to choose questions that have equal probability to be yes or no. There are many ways to do this, but one is to ask the questions "Is first child a girl? Is second child a girl? Is third child a girl? Is fourth child a girl?" but stop once X is known. (For example, if first answer is yes, second answer is no, then one can stop asking questions since it is then clear that $X = GB$ and there are no more children.)

(c) Let $Y \in \{G, B\}$ be the gender of the first child born. Compute $H(Y)$, $H_Y(X)$, and $H(X, Y)$. Is it true that $H(X, Y) = H(Y) + H_Y(X)$ in this setting? **ANSWER:** $H(Y) = 1$. And $H_Y(X) = \frac{1}{2}H_{Y=G}(X) + \frac{1}{2}H_{Y=B}(X)$. By symmetry of B and G this is the same as $H_{Y=G}(X)$ (which is the conditional entropy given the first child is a girl). By same logic as above, this is just the expected number of children (excluding the first) given that the first child is a girl, which comes out to $7/4$. And $H(X, Y) = H(X) = 11/4$ (since X determines Y, the pair (X, Y) contains no more information than X does alone). So the identity holds.

10. (10 points) Let X_1, X_2, X_3, X_4, X_5 be i.i.d. random variables, each with probability density function given by $f(x) = \frac{1}{\pi(x^2+1)}$.

(a) Compute the probability $P(\max\{X_1, X_2\} > \max\{X_3, X_4, X_5\})$. **ANSWER:** This just the probability that the largest of the five elements is either X_1 or X_2. Since each of the X_i is equally likely to be largest, the answer is $2/5$.

(b) Let N be the number of $j \in \{1, 2, 3, 4, 5\}$ for which $X_j > 0$. (So N is a random element of the set $\{0, 1, 2, 3, 4, 5\}$.) Compute the moment generating function $M_N(t)$. **ANSWER:** Let N_j be 1 if $X_j > 0$ and 0 otherwise. Then $M_{N_j}(t) = E[e^{tN_j}] = \frac{1}{2}e^0 + \frac{1}{2}e^t$. And $M_N(t) = M_{N_j}(t)^5 = \left(\frac{1}{2} + \frac{1}{2}e^t\right)^5$.

(c) Compute the probability $P(X_1 + X_2 > X_3 + X_4 + X_5 + 5)$. **ANSWER:** Note that by symmetry $X_1 + X_2 - X_3 - X_4 - X_5$ has same law as $X_1 + X_2 + X_3 + X_4 + X_5$ which in turn has same law as $5X_1$ (by a special property of Cauchy random variables). So answer is equivalent to $P(5X_1 > 5) = P(X_1 > 1)$ which in turn (recall spinning flashlight story) is $1/4$.

18.600 Probability and Random Variables
Fall 2019

Fall 2019 18.600 Final Exam Solutions

1. (10 points) A school has 60 students (30 boys and 30 girls) and these students are randomly divided into Classroom 1 and Classroom 2, with thirty students assigned to each class. Assume that each of the $\binom{60}{30}$ possible divisions is equally likely. A parent asks the principal, "Does Classroom 1 have at least 29 girls?" and is told that (surprisingly) the answer is yes. The following conversation ensues:

> **Parent zero:** Given what we have just learned, I wonder how likely it is that *all 30 students* in Classroom 1 are girls.
>
> **Parent one:** Well, any given child has a 1/2 chance to be a girl. So I'd say that if 29 are girls, the chance that the 30th is also a girl is 1/2.
>
> **Parent two:** No, that's silly. If we know the class has 29 girls, then there is only one girl left among the remaining 31 students, so the conditional probability is 1/31.
>
> **Parent three:** You are both formulating this the wrong way. The actual conditional probability is even lower than that.

You are summoned to resolve the dispute.

(a) Let A be the event that all 30 students in Classroom 1 are girls. Let B be the event that Classroom 1 has at least 29 girls. Compute the quantities $P(A)$ and $P(B)$. **ANSWER:** Only one of the $\binom{60}{30}$ divisions assigns all girls to Classroom 1, so $P(A) = 1/\binom{60}{30}$. To get exactly 29 girls, one has 30 choices for the girl not present, and 30 choices for the boy who is present—so the probability of exactly 29 girls is $900/\binom{60}{30}$ and $P(B) = 901/\binom{60}{30}$.

(b) Now compute the conditional probability $P(A|B)$. Based on this calculation, which (if any) of parents one, two and three is correct? **ANSWER:**

$$P(A|B) = P(AB)/P(B) = P(A)/P(B) = 1/901$$

so parent three is correct. **REMARK:** You can also consider a variant where there are 69 children and only 5 girls, and Classroom 1 has 5 students. Then the question "What is the probability Classroom 1 has 5 girls given it has at least 4?" is the same as the question "What is the probability you match all five regular Powerball balls given that you match at least four?" Again, it is somewhat surprising how much less likely 5 is than 4.

(c) Let G be the number of girls in Classroom 1 and B the number of boys in Classroom 1. Compute the expectation $E[GB]$. **ANSWER:** Let G_i (resp. B_i) be indicator function for event that ith girl (resp. boy) is in Classroom 1. Then

$$E[GB] = E\Big[\big(\sum_{i=1}^{30} B_i\big)\big(\sum_{i=1}^{30} G_i\big)\Big] = \sum_{i=1}^{30}\sum_{j=1}^{30} E[G_iB_j] = 900P(G_1 = B_1 = 1) = 900 \cdot \frac{1}{2} \cdot \frac{29}{59}$$

2. (10 points) A three-year-old child is vying for acceptance at two selective preschools. At each school, acceptance is determined by an entrance exam that measures important preschool skills but is not very reliable. The child's scores on the two exams take the form $S_1 = A + 2B_1$ and $S_2 = A + 2B_2$ where A, B_1, and B_2 are independent normal random variables, each with mean zero and variance one. Informally, A is the student's "entrance exam ability" while B_1 and B_2 are independent noise terms (encoding chance fluctuations).

(a) Compute the expectation $E[S_1 S_2]$. **ANSWER:**

$$E[S_1 S_2] = E[A^2 + 2B_1 A + 2B_2 A + 4B_1 B_2] = E[A^2] + E[2B_1 A] + E[2B_2 A] + E[4B_1 B_2].$$

Latter three terms are zero (by independence) so answer is $E[A^2] = 1$.

(b) Compute $\mathrm{Var}(S_1)$ and $\mathrm{Var}(S_2)$ and the correlation coefficient $\rho(S_1, S_2)$. **ANSWER:** By addivity of variance (for independent random variables) we have

$$\mathrm{Var}(S_1) = \mathrm{Var}(A) + \mathrm{Var}(2B_1) = \mathrm{Var}(A) + 4\mathrm{Var}(B_1) = 5.$$

Similarly $\mathrm{Var}(B_2) = 5$, and $\rho(S_1, S_2) = \frac{\mathrm{Cov}(S_1,S_2)}{\sqrt{\mathrm{Var}(S_1)\mathrm{Var}(S_2)}} = 1/5$.

(c) Compute the conditional expectation $E[S_2|S_1]$ in terms of S_1. That is, express the random variable $E[S_2|S_1]$ as a function of the random variable S_1. (Hint: if it helps, you can argue that $2B_1$ agrees in law with $\sum_{i=1}^4 Y_i$ where the Y_i are independent normal random variables, each with mean zero and variance one.) **ANSWER:** By independence, $E[A + 2B_2|A + 2B_1] = E[A|A + 2B_1]$. Following hint and writing $A = Y_0$, we can write this as $E[Y_0|S]$ where $S = \sum_{i=0}^4 Y_i$. We know that $S = E[S|S] = E[\sum_{i=0}^4 Y_i|S] = \sum_{i=0}^4 E[Y_i|S] = \sum_{i=0}^4 E[Y_i|\sum_{i=0}^4 Y_i]$ and since the Y_i are i.i.d. the latter five terms are all the same, hence equal to $S/5$. So answer is $S/5$.

3. Jill is an enthusiastic guitarist. Every song she plays has exactly 16 measures. During each measure, independently of all others, she randomly plays of one the five chords she knows:

1. **A major** with probability 1/2
2. **F# minor** with probability 1/8
3. **D major** with probability 1/8
4. **B minor** with probability 1/8
5. **E major** with probability 1/8

Let $X = (X_1, X_2, \ldots, X_{16})$ be the sequence of chords associated to one of Jill's songs.

(a) Compute the entropy $H(X_1)$, i.e., the entropy involved in choosing the first chord. **ANSWER:** $\frac{1}{2}(-\log(\frac{1}{2})) + 4 \cdot \frac{1}{8}(-\log(\frac{1}{8})) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$.

(b) Compute the entropy $H(X)$, i.e., the entropy involved in choosing an entire song. **ANSWER:** Since all 16 entries are independent, answer is $2 \cdot 16 = 32$.

(c) Describe a strategy for asking yes-or-no questions to determine X, such that the expected number of questions asked is as small as possible. What is the expected number of questions in this case? **ANSWER:** Ask "Is first chord A major?" If no, ask "Is first chord either F# minor or D major?" and "Is first chord either F# minor or B minor?" This determines first chord in 1 question half the time and 3 questions half the time. Repeat for all 16 chords. Expected number of question is $2 \cdot 16 = H(X)$ which is the smallest possible.

(d) Let $S = \{i : X_i = \text{A major}\}$. In other words, S is the collection of times at which an A major chord is played. Compute $H(S)$ and $H(S, X)$ and $H_S(X)$. **ANSWER:** $H(S) = 16$ (since S encodes 16 i.i.d. fair coin tosses) and $H(S, X) = H(X) = 32$ (since X determines S). By a general identity, $H(S, X) = H(S) + H_S(X)$ so $H_S(X) = 16$.

4. A certain small technical university has only five majors: Humanities, Science, Engineering, Business, and Math. During each given week, each student at this university is assigned to exactly one of the five majors. The students at this university change their majors frequently, but they tend to stay in Math a little longer than they do in the other majors. Here is how that works:

> If a student is majoring in Math one week, then the next week she stays in Math with probability 1/2 and transitions to each of the other majors with probability 1/8.
>
> If a student is majoring in any major other than Math one week, then the next week she stays in that major with probability 1/5 and also transitions to each of the other majors with probability 1/5.

Now answer the following:

(a) Represent the major transition process as a Markov chain and write out the five-by-five transition matrix. **ANSWER:** If first row and column correspond to Math, then the transition matrix is

$$\begin{pmatrix} 1/2 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$$

(b) Suppose a student is majoring in Math during the first week; compute the probability that he or she will be majoring in Business two weeks later. **ANSWER:** The probability of transition to Math followed by transition to Business is $\frac{1}{2} \cdot \frac{1}{8}$. The probability of transition to *some* other state followed by transition to Business is $\frac{1}{2} \cdot \frac{1}{5}$. So overall answer is $\frac{1}{2} \cdot (\frac{1}{5} + \frac{1}{8}) = \frac{1}{2} \cdot \frac{13}{40} = \frac{13}{80}$.

(c) Over the long haul, what fraction of the time does a student spend in each of the five majors? **ANSWER:** We can solve

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_5 \end{pmatrix}$$

We know that $\pi_2 = \pi_3 = \pi_4 = \pi_5 = c$ for some constant c and $\pi_1 = (1 - 4c)$ so we can write this as

$$\begin{pmatrix} 1-4c & c & c & c & c \end{pmatrix} \cdot \begin{pmatrix} 1/2 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix} = \begin{pmatrix} 1-4c & c & c & c & c \end{pmatrix}$$

Using just first column for above product, we get $\frac{1}{2}(1-4c) + \frac{4}{5}c = (1-4c)$ and solving gives $c = 5/28$. So over long haul a student spends a $5/28$ fraction of time in each major other than Math and a $(1-4c) = 8/28 = 2/7$ fraction in Math.

5. (10 points) Let X be an exponential random variable with parameter $\lambda = 1$. For each real number K write $C(K) = E[\max\{X-K, 0\}]$.

(a) Compute $C(K)$ as a function of K for $K \geq 0$. **ANSWER:**

$$\int_0^\infty e^{-x}\max\{x-K,0\}dx = \int_K^\infty e^{-x}(x-K)dx = \int_0^\infty e^{-(x+K)}xdx = e^{-K}\int_0^\infty e^{-x}xdx = e^{-K}$$

(b) Compute the derivatives C' and C'' on $[0,\infty)$. **ANSWER:** $C(x) = e^{-x}$ so $C'(x) = -e^{-x}$ and $C''(K) = e^{-x}$. Alternatively, if one simply remembers that the second derivative of the call function is the density function of X, and the first derivative is $F_X - 1$, then one can do this part without computation (and use it to derive part (a)).

(c) Compute the expectation $E[X^3 + 3X^2 + 3X + 1]$. **ANSWER:** Recall that $E[X^k] = k!$ so additivity of expectation gives the answer $3! + 3 \cdot 2! + 3 \cdot 1! + 1 = 6 + 6 + 3 + 1 = 16$.

6. (10 points) In a certain political party, there are 100 voters actively involved in the process of selecting a nominee. The are 23 candidates, which we number from 1 to 23. At any given time, each voter supports exactly one of these candidates. At the initial time $t = 0$

1. Candidates 1-15 are "lower tier candidates." Each has the support of exactly 2 voters.
2. Candidate 16-20 are "mid-tier candidates." Each has the support of exactly 3 voters.
3. Candidate 21 has the support of 11 voters.
4. Candidate 22 has the support of 21 voters
5. Candidate 23 has the support of 23 voters.

During each unit interval of time (after time $t = 0$) two voters are chosen (uniformly at random from the set of all $\binom{100}{2}$ possible pairs) to have a discussion with each other. If the two voters support the same candidate before the discussion, then the discussion changes nothing; but if they support *different* candidates then *one* of the two voters (chosen by a fair coin toss) switches support to the *other* voter's candidate. This continues until the time T at which all of the voters support the same candidate, at which point that candidate is declared the nominee. (This happens eventually with probability 1, but you don't have to prove that.)

(a) Let $A_i(t)$ be the number of voters supporting candidate i at time t. Is it true that for each $i \in \{1, 2, \ldots, 23\}$, the quantity $A_i(t)$ is a martingale? Explain why or why not. **ANSWER:** Yes, it is a martingale. The only way $A_i(t)$ can change is if two voters line up where one supports the ith candidate and one doesn't. When this happens, $A_i(t)$ goes up by one with probability 1/2 and down by one with probability 1/2, so the expected change (conditioned on everything known up to stage t) is 0.

(b) Compute the probability that for some t with $0 < t < T$ we have $A_{23}(t) = 1$ (so that the initially-leading candidate has lost the support of all but one voter). **ANSWER:** Since initial value of 23 is a $(100 - 23)/99 = 7/9$ fraction of the way from 100 toward 1, the answer (by shortcut from lecture) is $7/9$.

(c) Let N be the number of candidates who at *some* point before time T only have the support of a single voter. That is, N is the number of i values for which there is some t such that $0 < t < T$ and $A_i(t) = 1$. Compute the expectation $E[N]$. (Hint: you can use the fact that $\sum_{i=1}^{23} A_i(0) = 100$ and hence $\sum_{i=1}^{23}\big(100 - A_i(0)\big) = 2200$.) **ANSWER:** The probability for the ith candidate is $(100 - A_i(0))/99$. Adding these up to get the expectation, we have $2200/99 = 200/9$.

(d) Compute the probability that the candidate who wins the election is somebody who at some time $t > 0$ only had the support of a single voter. (This scenario is called an *epic comeback*.) **ANSWER:** The ith candidate has a $\frac{100-A_i(0)}{99} \cdot \frac{1}{100}$ of being an epic comeback winner. Since only one candidate do this, we can just add these probabilities to get $\frac{2200}{9900} = \frac{2}{9}$. That might seem surprisingly high: epic comebacks are not so unusual after all. (This is similar to the islander problem from the spring 2019 exam.) Note that the answer does not actually depend on the initial support levels for the candidates, as long as they are all at least one. You could also formulate the problem in terms of a betting market on which prices were known to be martingales changing by increments of ± 1.

7. (10 points) Suppose that the pair (X, Y) is uniformly distributed on the circle $\{(x, y) : x^2 + y^2 \le 1\}$.

(a) Compute the joint probability density $f_{X,Y}(x, y)$ and the marginal density function $f_X(x)$. **ANSWER:** $f_{X,Y}(x, y)$ is $1/\pi$ in the unit circle, 0 elsewhere and $f_X(x) = \frac{1}{\pi}2\sqrt{1 - x^2}$.

(b) Compute the probability $P(0 < X < Y)$. **ANSWER:** The intersection of the disk, the plane where $x > 0$, and the plane where $x < y$ together forms a wedge of angle $\pi/4$, which is one eighth of total disk, so answer is $1/8$.

(c) Write $Z = X^2 + Y^2$ and work out the density function f_Z. **ANSWER:** First work out

$$F_Z(a) = P(Z \le a) = P(X^2 + Y^2 < a) = P(\sqrt{X^2 + Y^2} < \sqrt{a}) = \pi(\sqrt{a})^2/\pi = a.$$

Differentiating we find that f_Z is equal to 1 on $[0, 1]$ so that Z is a uniform random variable.

8. (10 points) Sally the Spammer sends millions of emails every day encouraging the recipients (using various rationales) to grant her unrestricted access to their bank accounts. The times $X_1, X_2, X_3, \ldots$ (measured in years from some initial time) at which Sally is *granted* access to such a bank account form a Poisson point process with parameter $\lambda = 3$. So on average three people per year give Sally access to their bank accounts.

(a) Write $Y = X_3$ and compute the density function f_Y. **ANSWER:** The nth term in a Poisson point process with parameter λ (which is a sum of n independent exponentials with parameter λ) is a Gamma random variable with parameters λ and n. In this case X_3 is Gamma with parameters $n = 3$ and $\lambda = 3$ so answer is $f_Y(y) = 3 \cdot (3y)^2 e^{-(3y)}/2$.

(b) What is the probability that Sally is granted bank acount access at least three times during her first year of operation? **ANSWER:** Number of accesses during 1 year is Poisson with parameter $1 \cdot \lambda = 3$. So answer is $1 - \sum_{k=0}^{2} e^{-3}3^k/k!$

(c) Write $Y_0 = 0$ and $Y_k = X_k - \frac{k}{3}$. Is the sequence $Y_0, Y_1, \dots$ a martingale? Why or why not? **ANSWER:** Yes. The increments $I_k = X_k - X_{k-1}$ are independent exponential random variables with expectation $1/3$. So the increments $Y_k - Y_{k-1}$ are independent random variables with expectation 0. Alternatively, write $E[Y_{k+1}|\mathcal{F}_k] = E[Y_k + I_{k+1} - 1/3|\mathcal{F}_k] = Y_k$.

9. (10 points) There are 3 people, each of whom has 4 hats. All 12 hats are tossed into a bin and random divided evenly among the 3 people (so each person gets 4 hats back, with all ways of doing this being equally likely). Let A_i be the event that the ith person gets *all four* of his or her own hats back. Let N be the number of people who get *all four* of their own hats back.

(a) Compute the quantities $a = P(A_1)$ and $b = P(A_1A_2)$. **ANSWER:** There are $\binom{12}{4}$ possible collections of hats for person one, and only of them is right. So $a = P(A_1) = 1/\binom{12}{4}$. Similarly $b = P(A_1A_2) = (1/\binom{12}{4})(1/\binom{8}{4})$.

(b) Compute $E[N]$ and $\mathrm{Var}(N)$. (You can use the a and b from the previous part in your answer, if that helps.) **ANSWER:** $E[N] = \sum_{i=1}^{3} P(A_i) = 3a$. $E[N^2] = \sum_{i=1}^{3}\sum_{j=1}^{3} P(A_iA_j) = 3a + 6b$. So $\mathrm{Var}(N) = E[N^2] - E[N]^2 = 3a + 6b - 9a^2$.

(c) Compute $P(N > 0)$. (Again, you can use a and b in your answer, if that helps.) **ANSWER:** Note that $P(A_1A_2A_3) = P(A_1A_2)$. Then inclusion exclusion gives $P(N > 0) = 3a - 3b + b = 3a - 2b$.

10. (10 points) Suppose that $X_1, X_2, \dots$ are i.i.d. random variables, each equal to 0 with probability $1/8$, 1 with probability $3/4$ and 2 with probabiliy $1/8$. Write $S_n = \sum_{i=1}^{n} X_i$ and $A_n = S_n/n$.

(a) Compute the moment generating functions $M_{S_{50}}(t)$ and $M_{A_{50}}(t)$. **ANSWER:** $M_{X_1}(t) = E[e^{tX_1}] = \frac{1}{8} + \frac{3}{4}e^t + \frac{1}{8}e^{2t}$ and $M_{S_{50}}(t) = (\frac{1}{8} + \frac{3}{4}e^t\frac{1}{8}e^{2t})^{50}$ and $M_{A_{50}}(t) = (\frac{1}{8} + \frac{3}{4}e^{t/50}\frac{1}{8}e^{2t/50})^{50}$

(b) Compute $E[X_1]$ and $\mathrm{Var}(X_1)$. **ANSWER:** $E[X_1] = 1$ and $\mathrm{Var}[X_1] = 1/4$.

(c) Use the central limit theorem to approximate $P(S_{100} > 110)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** We have $E[S_{100}] = 100$ and $\mathrm{Var}(S_{100}) = 25$ and $\mathrm{SD}(S_{100}) = 5$. And chance to be at least two standard deviations above the mean is about $1 - \Phi(2) = \Phi(-2)$.

(d) Compute the correlation coefficient $\rho(S_{50}, S_{200})$. **ANSWER:** Using bilinearity of covariance we have $\mathrm{Cov}(S_{50}, S_{200}) = \mathrm{Cov}(S_{50}, S_{200} - S_{50}) + \mathrm{Cov}(S_{50}, S_{50})$. First term is zero (since it is a covariance of two independent random variables) and second term is $\mathrm{Var}(S_{50}) = 50/4$. So answer is $\frac{50/4}{\sqrt{(50/4)(200/4)}} = 1/2$.

18.600 Probability and Random Variables
Fall 2019

18.440 Midterm 1, Fall 2012: Solutions

1. (20 points) Toss 6 independent fair coins, and let X be the number that come up heads. Compute the following:

(a) $E[X]$ **ANSWER:** X is binomial $(n,p) = (6, 1/2)$ so $E[X] = np = 3$

(b) $\mathrm{Var}[X]$ **ANSWER:** $np(1-p) = 1.5$

(c) $P[X = 6|X \geq 5]$ **ANSWER:**
$P[X = 6]/P[X \geq 5] = 2^{-6}/(2^{-6} + \binom{6}{5}2^{-6}) = 1/7.$

2. (20 points) Four people have two hats each. They take off these hats and shuffle them up randomly and redistribute them among the four people (so that each again has two hats). Assume that all the ways of dividing the 8 hats among the 4 people (with each person getting two hats) are equally likely.

(a) How many possible outcomes are there (i.e., how many ways are there to divide 8 distinct hats among four distinct people, with each person getting two hats)? **ANSWER:** $\binom{8}{2\,2\,2\,2} = 8!/16$

(b) Label the people 1, 2, 3, and 4. Let E_i be the event that the ith person gets *both* of his or her own hats. Compute $P[E_1]$.
ANSWER: $\frac{1}{8} \cdot \frac{1}{7} \cdot 2 = \frac{1}{28}$

(c) Compute $P[E_1E_2]$. Then compute $P[E_1E_2E_3]$ (which is the same as $P[E_1E_2E_3E_4]$). **ANSWER:**

$$P[E_1E_2] = \frac{1}{8} \cdot \frac{1}{7} \cdot 2 \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot 2 = 4/(8 \cdot 7 \cdot 6 \cdot 5) = 1/420$$

$$P[E_1E_2E_3] = P[E_1E_2E_3E_4] = 16/8!$$

(d) Write $A = P[E_1]$, $B = P[E_1E_2]$, $C = P[E_1E_2E_3] = P[E_1E_2E_3E_4]$. Use inclusion-exclusion to express $P[E_1 \cup E_2 \cup E_3 \cup E_4]$ in terms of A, B, and C. **ANSWER:** $4A - 6B + 4C - C$

3. (20 points) Suppose that during each minute of a 90-minute soccer game there is a probability of 2/90 that one goal will be scored and a probability of 88/90 that no goal will be scored (independently of all other minutes). Let N be the total number of goals scored during the game.

(a) Compute $E[N]$ and $\mathrm{Var}[N]$. (Give exact answers, not approximate ones.) **ANSWER:** $E[N] = 90 \cdot \frac{2}{90} = 2$ and $\mathrm{Var}[N] = 90 \cdot \frac{2}{90} \cdot \frac{88}{90} = \frac{176}{90}$

(b) Compute the probability that there is exactly one goal. Give an *exact* answer. **ANSWER:** $\binom{90}{1}(2/90)^1(88/90)^{89}$

(c) Let E be the event that there are exactly 0 goals in the first half and exactly 2 goals in the second half. Use a Poisson random variable calculation to *approximate* the probability of E. **ANSWER:** Number in each half is approximately Poison(λ) with $\lambda = 1$. So $P(E) \approx e^{-\lambda}\lambda^1/1! \cdot e^{-\lambda}\lambda^2/2! = \frac{1}{2e^{-2}}$

4. (10 points) Roll 6 independent fair six-sided dice (so that each of the 6^6 outcomes is equally likely). Compute the probability that exactly three of the dice land on even numbers. **ANSWER:** $\binom{6}{3}2^{-6} = 20/64 = 5/16$.

5. (20 points) Roll 2 independent fair six-sided dice. Let X be the value on the first die, Y the value on the second die. Write $Z = Y + X$. Compute the following:

(a) $E[Z]$. **ANSWER:** $E[Z] = E[X] + E[Y] = 7$

(b) $E[Y^2]$. **ANSWER:** $E[Y^2] = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$

(c) $E[YZ]$. **ANSWER:** $[YZ] = E[Y^2 + XY] = (91/6) + (7/2)^2$

6. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability 1/3. Let X be such that the third heads occurs on the Xth toss.

(a) Compute $P[X = 9]$. **ANSWER:** $\binom{8}{2}(1/3)^2(2/3)^6(1/3)$

(b) Compute $E[X]$. **ANSWER:** $3/(1/3) = 9$

2

18.600 Probability and Random Variables
Fall 2019

18.440 Midterm 1, Spring 2014: 50 minutes, 100 points

1. (10 points) How many quintuples $(a_1, a_2, a_3, a_4, a_5)$ of non-negative integers satisfy $a_1 + a_2 + a_3 + a_4 + a_5 = 100$? **ANSWER:** This the number of ways to make a list of "100 stars and 4 bars", which is $\binom{104}{4} = \frac{104!}{4!100!}$.

2. (20 points) Thirty people are invited to a party. Each person accepts the invitation, independently of all others, with probability 1/3. Let X be the number of accepted invitations. Compute the following:

(a) $E[X]$ **ANSWER:** X is binomial with $n = 30$ and $p = 1/3$, so the expectation is $np = 10$.

(b) $\mathrm{Var}[X]$ **ANSWER:** X is binomial with $n = 30$ and $p = 1/3$, so the variance is $npq = np(1-p) = 20/3$.

(c) $E[X^2]$ **ANSWER:** $\mathrm{Var}(X) = E[X^2] - E[X]^2$. Using previous two parts and solving gives $E[X^2] = 20/3 + 100 = 320/3$.

(d) $E[X^2 - 4X + 5]$ **ANSWER:** By linearity of expectation, this is $E[X^2] - 4E[X] + 5 = 320/3 - 40 + 5 = 215/3$.

3. (20 points) Bob has noticed that during every given minute, there is a 1/720 chance that the Facebook page for his dry cleaning business will get a "like", independently of what happens during any other minute. Let L be the total number of likes that Bob receives during a 24 hour period.

(a) Compute $E[L]$ and $\mathrm{Var}[L]$. (Give exact answers, not approximate ones.) **ANSWER:** This is binomial with $n = 60 \times 24$ and $p = 1/720$. So $E[L] = np = 2$ and $\mathrm{Var}[L] = np(1-p) = 2\frac{719}{720}$.

(b) Compute the probability that $L = 0$. (Give an exact answer, not an approximate answer.) **ANSWER:** $(1-p)^n = \left(\frac{719}{720}\right)^{1440}$

(c) Bob is really hoping to get at least 2 more likes during the next 24 hours (because this would boost his cumulative total to triple digits). Use a Poisson random variable calculation to *approximate* the probability that $L \geq 2$. **ANSWER:** Note that L is approximately binomial with parameter $\lambda = E[L] = 2$. Thus $P\{L \geq 2\} = 1 - P\{L = 1\} - P\{L = 0\} \approx 1 - e^{-\lambda}\lambda^0/0! - e^{-\lambda}\lambda^1/1! = 1 - 3e^{-2} = 1 - 3/e^2$

4. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p. Compute (in terms of p) the

probability that the fifth head occurs on the tenth toss. **ANSWER:** This is the probability that exactly four of the first nine tosses are heads, and then the tenth toss is also heads. This comes to $\binom{9}{4}p^5(1-p)^5$.

5. (20 points) Let X be the number on a standard die roll (assuming values in $\{1,2,3,4,5,6\}$ with equal probability). Let Y be the number on an independent roll of the same die. Compute the following:

(a) The expectation $E[X^2]$. **ANSWER:** $(1+4+9+16+25+36)/6 = 91/6$.

(b) The expectation $E[XY]$. **ANSWER:** By independence of X and Y, we have $E[XY] = E[X]E[Y] = (7/2)^2 = 49/4$.

(c) The covariance $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$. **ANSWER:** Because of independence, $\mathrm{Cov}(X,Y) = 0$.

6. (20 points) Three hats fall out of their assigned bins and are randomly placed back in bins, one hat per bin (with all 3! reassignments being equally likely). Compute the following:

(a) The expected number of hats that end up in their own bins. **ANSWER:** Let X_i be 1 if ith hat ends up in own bin, zero otherwise. Then $X = X_1 + X_2 + X_3$ is total number of hats to end up in their own bins, and $E[X] = E[X_1] + E[X_2] + E[X_3] = 3\frac{1}{3} = 1$.

(b) The probability that the third hat ends up in its own bin. **ANSWER:** 1/3

(b) The conditional probability that the third hat ends up in its own bin *given* that the first hat does *not* end up in its own bin. **ANSWER:** Let A be event third hat gets own bin, B event that first hat does not end up in its own bin. Then $P(A) = 1/3$ and $P(B) = 2/3$. There is only one permutation that assigns the third hat to its own bin and does not assign first hat to its own bin, so $P(AB) = 1/6$. Thus $P(A|B) = P(AB)/P(B) = (1/6)/(2/3) = 1/4$.

18.600 Probability and Random Variables
Fall 2019

18.600 Midterm 1, Spring 2016 SOLUTIONS

1. (20 points) Let X_1, X_2, X_3 be independent random variables, each of which is equal to 1 with probability 1/4 and 0 with probability 3/4. Write $Y = X_1 + X_2$ and $Z = X_2 + X_3$. Compute the following:

(a) $E[Y]$ **ANSWER:** $E[X_1] + E[X_2] = 1/4 + 1/4 = 1/2$.

(b) $E[YZ]$ **ANSWER:** $E[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3] =$

$$E[X_1X_2] + E[X_1X_3] + E[X_2^2] + E[X_2X_3] = \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = 7/16$$

(c) $E[Y^2]$ **ANSWER:** $E[X_1^2] + 2E[X_1X_2] + E[X_2^2] = \frac{1}{4} + \frac{2}{16} + \frac{1}{4} = 5/8$.

(d) $\mathrm{Var}[Y]$ **ANSWER:** $E[Y^2] - E[Y]^2 = 5/8 - 1/4 = 3/8$. Alternate solution: Y is binomial (n, p) so $\mathrm{Var}(y) = npq = 2(1/4)(3/4) = 3/8$.

(e) $\mathrm{Cov}(Y, Z)$ **ANSWER:** $E[YZ] - E[Y]E[Z] = 7/16 - (1/2)^2 = 3/16$. Alternate solution: bilinearity of covariance and fact that independent random variables have covariance zero imply $\mathrm{Cov}(X_1 + X_2, X_2 + X_3) = \mathrm{Cov}(X_2, X_2) = \mathrm{Var}(X_2) = 3/16$.

2. (20 points) At a certain small college, an entering class has 400 first-year college students. Each student tries out for the fencing team with probabiity 1/200, independently of what the other students do. Let N be the total number of first-year students who try out for the fencing team.

(a) Compute the expectation $E[N]$. Give an exact answer, not an approximation. **ANSWER:** $np = 400 \cdot \frac{1}{200} = 2$

(b) Compute the variance $\mathrm{Var}[N]$. Give an exact answer, not an approximation. **ANSWER:** $npq = 2 \cdot \frac{199}{200} = 1.99$

(c) Write down an exact formula for the probability that $N = k$ (for $k \in \{0, 1, \ldots, 400\}$). **ANSWER:** $\binom{n}{k}p^k(1-p)^{n-k}$ where $p = 1/200$ and $n = 400$.

(d) Use a Poisson random variable to *approximate* the probability that $N = 3$. **ANSWER:** $e^{-\lambda}\lambda^k/k! = e^{-2}2^3/3!$.

3. (15 points) A deck of card contains 52 distinct card *types*, with exactly one card of each *type*. Suppose that one has two identical decks of cards (so 104 cards total) and that one accidentally loses 10 of these 104 cards

(chosen uniformly from the set of all possible 10-card subsets) so that one now has only 94 cards.

What is the probability that it is possible to form a single complete deck of cards from the 94 cards remaining? In other words, what is the probability that the set of 94 remaining cards includes *at least one* card of each of the 52 types? (Note: this is equivalent to the probability that the set of 10 lost cards includes *at most* one card of each type.)
ANSWER: $2^{10}\binom{52}{10}$ ways to choose the 10 types in the missing set, and which deck the card of each type came from. $\binom{104}{10}$ overall possibilities for missing set. Probability is the ratio $2^{10}\binom{52}{10}/\binom{104}{10}$.
4. (15 points) Seven people toss their hats in a bin and have them randomly shuffled and returned, one hat to each person. Let N be the number of people who get their own hat back. Compute the following:

(a) The expectation $E[N]$. **ANSWER:** By expectation additivity $E[N] = 7 \cdot \frac{1}{7} = 1$.

(b) The probability $P(N = 5)$. **ANSWER:** $\binom{7}{2}$ is number of ways to choose two people to have their hats swapped. So $P(N = 5) = \binom{7}{2}/7!$

(c) The conditional probability $P(N = 7|N \geq 5)$. **ANSWER:** $\frac{1/7!}{\binom{7}{2}/7!+1/7!} = \frac{1}{\binom{7}{2}+1} = 1/22$

5. (10 points) An urn contains 10 black balls and 10 white balls. If a collection of 8 balls is chosen uniformly at random from the urn, what is the probability that 4 of them are black and 4 of them are white?
ANSWER: $\binom{10}{4}\binom{10}{4}/\binom{20}{8}$
6. (20 points) Let X be the number on a standard die roll (assuming values in $\{1, 2, 3, 4, 5, 6\}$ with equal probability). Let Y be the number on an independent roll of the same die. Compute the following expectations:

(a) $E[X]$ **ANSWER:** $7/2$

(b) $E[X^2]$ **ANSWER:** $(1 + 4 + 9 + 16 + 25 + 36)/6 = 91/6$

(c) $E[5X^7 - 5Y^7 + 5]$ **ANSWER:** 5 by additivity of expectation, since $E[5X^7] = E[5Y^7]$.

(d) $E[XY]$ **ANSWER:** $E[X]E[Y]{=}49/4$, by independence of X and Y.

18.600 Midterm 1, Spring 2017 Solutions

1. (20 points) Let X be the number on a standard die roll (assuming values in $\{1,2,3,4,5,6\}$ with equal probability). Let Y and Z be the numbers on two independent rolls of the same die (so X, Y, and Z are independent of each other). Compute the following:

(a) $E[X]$ and $E[X^2]$ **ANSWER:** $E[X] = \frac{1}{6}(1+2+3+4+5+6) = 7/2$ and $E[X^2] = \frac{1}{6}(1+4+9+16+25+36) = 91/6$

(b) $\text{Var}[X]$ **ANSWER:** $E[X^2] - E[X]^2 = 91/6 - 49/4 = 35/12$

(c) $E[X^2 + 2Y^2 + 3Z^2]$ **ANSWER:** $E[X^2] + 2E[Y^2] + 3[Z^2]$ (by linearity of expectation) which is $6E[X^2] = 91$

(d) The conditional probablity $P[X = 1|X = Y]$ **ANSWER:** $P[X = 1, X = Y]/P[X = Y] = \frac{1/36}{1/6} = 1/6$

2. (10 points) An urn contains 5 black balls and 5 white balls and 5 red balls and 5 yellow balls. If a collection of 8 balls is chosen uniformly at random from the urn, what is the probability that the collection contains exactly 2 balls of each color? **ANSWER:** $\binom{5}{2}^4/\binom{20}{8}$

3. (20 points) Let M be the event that a person has a certain medical condition and let T be the event that a test of that condition comes back positive. Assume that the person has a 1/20 chance to have the condition and given that the person has the condition, there is a 90 percent chance the test will identify the condition correctly. Assume also that there is 20 percent chance the test will come back positive given that the person doesn't have the condition. In symbols, we can write the above assumptions as $P(M) = 1/20$ and $P(T|M) = 9/10$ and $P(T|M^c) = 1/5$. Now compute the following:

(a) $P(T)$ (i.e., overall likelihood test is positive) **ANSWER:** $P(M)P(T|M) + P(M^c)P(T|M^c) = 9/200 + 19/100 = 47/200$

(b) $P(M|T)$ (i.e., likelihood person has condition given test is positive) $P(MT)/P(T) = P(M)P(T|M)/P(T) = (9/200)/(47/200) = 9/47$

(c) $P(M|T^c)$ (i.e., likelihood person has condition given test is negative) $P(MT^c)/P(T^c) = P(M)P(T^c|M)/P(T^c) = (1/200)/(153/200) = 1/153$

4. (10 points) A standard deck of cards has 4 suits, and 13 different cards of each suit (52 total). We know that there are $\binom{52}{13}$ ways to choose an (unordered) set of 13 cards from this deck. How many ways are there to choose an (unordered) set of 13 cards with the property that 7 of the cards belong to the same suit (with the other 6 *not* belonging to that suit)? **ANSWER:** Choose the special suit (4 choices) then choose seven from that suit ($\binom{13}{7}$ choices), then choose 6 from the 39 not of that suit ($\binom{39}{6}$ choices). Overall number is $4\binom{13}{7}\binom{39}{6}$

5. (10 points) You have 12 pieces of pizza and 4 people at a party. How many ways are there to divide the 12 pieces among the four people? In other words, how many ordered four-tuples of non-negative integers (a_1, a_2, a_3, a_4) satisfy $a_1 + a_2 + a_3 + a_4 = 12$? **ANSWER:** Stars and bars argument gives $\binom{15}{3}$

6. (20 points) Harry is a (relatively untalented) basketball player practicing free throws. Each time Harry attempts a shot, Harry has a 1/100 probability of making the shot (independently of all other shots taken). Let X be the number of shots that Harry makes after 100 attempts.

(a) Compute the variance $\text{Var}[X]$. (Given an exact answer, not an approximation.) **ANSWER:** $npq = 100(1/100)(99/100) = 99/100$

(b) Use a Poisson approximation to give an estimate for the probability that $X = 2$. **ANSWER:** $e^{-\lambda}\lambda^k/k!$ with $\lambda = 1$ and $k = 2$ gives $\frac{1}{2e}$

(c) Harry is hoping the event $X > 1$ will occur, since this would mean that (thanks to good luck) Harry made *more* shots than he expected to make (which might impress any talent scouts who happened to present). Use a Poisson approximation to estimate $P\{X > 1\}$. **ANSWER:** Using same Poisson approximation with $k = 1$ and $k = 0$ we find $P\{X = 0\} \approx P\{X = 1\} \approx 1/e$. So $P\{X > 1\} = 1 - P\{X = 0\} - P\{X = 1\} \approx 1 - 2/e$

(d) Using the same Poisson approximation, estimate $P\{X < 1\}$. Is this larger or smaller than (or the same as) your estimate for $P\{X > 1\}$? **ANSWER:** As computed above $P\{X < 1\} = P\{X = 0\} \approx 1/e$. Since $e < 3$ we have $1 - 2/e < 1/e$. This means that Harry is more likely to be unlucky (and score less than expected) than to be lucky (and score more than expected), which sounds bad for Harry. But on the plus side, Harry can at worst score *one* less than expected, but Harry can at best score *a lot* more than expected.

7. (10 points) Fix an integer $n \geq 3$. Let σ be a random permutation chosen uniformly from the set of $n!$ permutations of $\{1, 2, \ldots, n\}$. Let $X = X(\sigma)$ be the number of cycles of length 3 in the permutation σ. Compute $E[X]$.

Note: Here is some notation you can use in your answer if it is helpful. Given integers $i < j < k$, let $E_{i,j,k}$ be the event that the integers i, j, k are part of a length three cycle (which means that *EITHER* $\sigma(i) = j$, $\sigma(j) = k$, and $\sigma(k) = i$ *OR* $\sigma(i) = k$, $\sigma(k) = j$, and $\sigma(j) = i$). Let $X_{i,j,k}$ be the indicator random variable $1_{E_{i,j,k}}$.
ANSWER: $E[X_{i,j,k}] = 2\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}$ for each $1 \leq i < j < k \leq n$. We have $E[X] = \sum E[X_{i,j,k}]$ where the sum ranges over $\binom{n}{3}$ such terms, so we end up with $2\frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}\binom{n}{3} = 2\frac{n(n-1)(n-2)}{n(n-1)(n-2)3!} = 1/3$

18.600 Probability and Random Variables
Fall 2019

18.600 Midterm 1, Spring 2018: Solutions

1. (20 points) Roll six standard six-sided dice independently, and let X be the number of dice that show the number 6.

(a) Compute the expectation $E[X]$. **ANSWER:** The number of heads is binomial with $n = 6$ and $p = 1/6$, so $E[X] = np = 1$.

(b) Compute the variance $\mathrm{Var}(X)$. **ANSWER:** $\mathrm{Var}(X) = npq = 5/6$ (where $q = 1 - p$).

(c) Compute $P(X = 5)$. **ANSWER:**
$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{6}{5}(1/6)^5(5/6) = 6 \cdot 5/6^6 = 5/6^5$

(d) Compute $P(X = 6|X \geq 5)$. **ANSWER:** $P(X = 6) = (1/6)^6$ so

$$P(X = 6|X \geq 5) = \frac{P(X = 6)}{P(X = 6) + P(X = 5)} = \frac{1/6^6}{30/6^6 + 1/6^6} = 1/31$$

2. (10 points) Suppose that E, F and G are events such that

$$P(E) = P(F) = P(G) = .4$$

and

$$P(EF) = P(EG) = P(FG) = .2$$

and

$$P(EFG) = .1.$$

Compute $P(E \cup F \cup G)$. **ANSWER:** Inclusion-exclusion tells us

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)$$

$$= 3(.4) - 3(.2) + .1 = 1.2 - .6 + .1 = .7$$

3. (10 points) Compute the following

(a)

$$\lim_{n\to\infty} (1 + 5/n)^n$$

ANSWER: General formula is $e^x = \lim_{n\to\infty}(1 + x/n)^n$. Plug in $x = 5$ and answer is e^5. Alternatively, one can just use definition of e (the special case $x = 1$). Write $n = 5m$ (which implies $5/n = 1/m$) and note that

$$\lim_{n\to\infty} (1 + 5/n)^n = \lim_{m\to\infty} (1 + 1/m)^{5m} = \lim_{m\to\infty} \left((1 + 1/m)^m\right)^5 =$$

$$\left(\lim_{m\to\infty} (1 + 1/m)^m\right)^5 = e^5.$$

(b)

$$\sum_{n=0}^{\infty} 5^n/n!$$

ANSWER: By Taylor expansion $e^x = \sum_{n=0}^{\infty} x^n/n!$. Setting $x = 5$ gives the answer e^5.

4. (20 points) Suppose that 10000 people visit Alice's new restaurant during its first few months of operation. Each person independently chooses to leave a positive Yelp review (e.g., "Great samosas!") with probability 5/10000, a negative Yelp review (e.g., "Rude servers!") with probability 1/10000 or no review at all with probability 9994/10000. Let X be total number of positive reviews received and Y the total number of negative reviews received.

(a) Compute $E(Y)$ and $\mathrm{Var}(Y)$. (Give exact values, not approximations.)
ANSWER: This is binomial with $n = 10000$ and $p = 1/10000$ so $E[Y] = np = 1$ and $\mathrm{Var}(Y) = np(1-p) = (1-p) = 9999/10000$.

(b) Use a Poisson random variable to approximate $P(X = 3)$.
ANSWER: $E[X] = 5$, so X is approximately Poisson with parameter $\lambda = 5$. This suggests

$$P(X = k) \approx e^{-\lambda}\lambda^k/k! = e^{-5}5^3/3! = \frac{125}{6e^5}.$$

(c) Use a Poisson random variable to approximate $P(Y = 0)$.
ANSWER: $e^{-\lambda}\lambda^k/k!$ with $k = 1$ and $\lambda = 1$ is $1/e$. Altenatively, just note directly that $P(Y = 0) = (1 - 1/10000)^{10000} \approx e^{-1} = 1/e$.

5. (10 points) Suppose that a deck of cards contains 120 cards: 30 red cards, 40 black cards, and 50 blue cards. A random collection of 12 cards is chosen (with all possible 12-card subsets being equally likely). What is the probability that this collection contains three red cards, four black cards, and five blue cards? **ANSWER:** Total number of ways to choose 12 cards is $\binom{120}{12}$. The number of ways to choose cards with desired color breakdown is $\binom{30}{3}\binom{40}{4}\binom{50}{5}$. So the ratio is

$$\frac{\binom{30}{3}\binom{40}{4}\binom{50}{5}}{\binom{120}{12}}.$$

6. (15 points) Ten people toss their hats in a bin and have them randomly shuffled and returned, one hat to each person. Let X_i be 1 if ith person

gets own hat back, 0 otherwise. Let $X = \sum_{i=1}^{10} X_i$ be the total number of people who get their own hat back. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $10 \cdot \frac{1}{10} = 1$

(b) The expectation $E[X_3 X_7]$. **ANSWER:** $X_3 X_7$ is 1 on the event that 3rd and 7th people both get own hats, and zero otherwise. So $E[X_3 X_7]$ is the probability that both 3 and 7 their own hats. There are 8! permuations in which 3 and 7 get own hats, so answer is $8!/10! = 1/90$.

(c) The expectation $E[X_1^2 + X_2^2 + X_3^2]$. **ANSWER:** Note that $X_j^2 = X_j$ for each j, so this is just $E[X_1 + X_2 + X_3] = 3E[X_1] = 3/10$.

7. (15 points) Bob is at an airport kiosk considering the purchase of a bottle of water with no listed price. Bob is thirsty but is too shy to ask about the price. He happens to know that the price in dollars (denoted X) is an integer between 3 and 7 and he considers each of the values in $\{3, 4, 5, 6, 7\}$ to be equally likely (probability 1/5 for each). According to Bob's probability measure, find the following:

(a) $E[X]$ **ANSWER:** $\frac{1}{5}(3 + 4 + 5 + 6 + 7) = 5$

(b) $\mathrm{Var}[X]$ **ANSWER:** $\mathrm{Var}(X) = E[(X - 5)^2]$. Note that $(X - 5)^2$ is 4 with probability 2/5 and 1 with probability 2/5 and zero otherwise. So $\mathrm{Var}[X] = E[(X - 5)^2] = \frac{2}{5} \cdot 4 + \frac{2}{5} \cdot 1 = 2$.

(c) $\mathrm{Var}[1.05X]$ (That is, the variance of the sales-tax-inclusive price assuming the airport is in a state with five percent sales tax.) **ANSWER:** $\mathrm{Var}(1.05(X)) = 1.05^2 \mathrm{Var}(X) = (21/20)^2 \cdot 2 = 441/200$

18.600 Midterm 1, Spring 2019 solutions

1. (20 points) A town has 2000 residents. An obscure film is playing in its only theater. Each resident decides independently whether to view the film, and each resident views the film with probability 1/1000. Let X be the number of people who view the film.

(a) Compute $E[X]$. Given an exact answer, not an approximation. **ANSWER:** X is binomial with $n = 2000$ and $p = 1/1000$, so $E[X] = np = 2$.

(b) Compute $\mathrm{Var}[X]$. Give an exact answer, not an approximation. **ANSWER:** $\mathrm{Var}(X) = np(1-p) = 2 \cdot 999/1000 = 1.998$ (which is approximately 2, for what it's worth)

(c) Compute $E[X^2]$. Give an exact answer, not an approximation. **ANSWER:** We know $1.998 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - 4$. Hence $E(X^2) = 5.998$. Alternatively, write $X = \sum_{i=1}^n X_i$ where X_i is 1 if ith person shows, 0 otherwise. Then

$$E(X^2) = E\big(\sum_{i=1}^n X_i \sum_{j=1}^n X_j\big) = \sum_{i=1}^n \sum_{i=1}^n E[X_i X_j].$$

Note that $E[X_i X_j] = p$ if $i = j$ and p^2 otherwise. Of the n^2 terms in the sum, we have n equal to p and $n^2 - n$ equal to p^2. So answer is

$$np + (n^2 - n)p^2 = 2 + (4000000 - 2000)/1000000 = 2 + 3.998 = 5.998.$$

(d) Use a Poisson random variable to *approximate* $P(X = 4)$. **ANSWER:** X should be approximately Poisson with $\lambda = E[X] = 2$. So $P(X = 4) \approx e^{-\lambda}\lambda^k/k! = e^{-2}2^4/4!$.

2.(10 points) Suppose that X is a Poisson random variable with parameter 2 and Y is a Poisson random variable with parameter 3.

(a) Compute the expectation $E(3X + 4Y + 5)$. **ANSWER:** By linearity of expectation, and fact Poisson of parameter λ has expectation λ, the answer is $3E[X] + 4E[Y] + 5 = 6 + 12 + 5 = 23$.

(b) Compute the variance $\mathrm{Var}(5X + 7)$. **ANSWER:** If a and b are constants, we have $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$. Poisson of parameter λ has variance λ so answer is $25\mathrm{Var}(X) = 50$.

3. (20 points) Alice, Bob, Carol, Dave, Eve, and Frank are gathered together for a night of pizza and dungeons and dragons. They order two large pizzas, each cut into 12 pieces, so there are 24 pieces altogether.

(a) How many ways are there to divide the 24 (indistinguishable) pieces among the six people? in other words, how many sequences $a_1, a_2, \ldots, a_6$ of *non-negative* integers satisfy $\sum_{i=1}^6 a_i = 24$? **ANSWER:** This is the stars and bars problem with $n = 24$ and $k = 6$, so answer is $\binom{29}{5}$.

(b) Eve proposes that, for the sake of fairness, only divisions in which each person gets *at least one slice* of pizza should be considered. How many sequences $a_1, a_2, \ldots, a_6$ of *strictly positive* integers satisfy $\sum_{i=1}^6 a_i = 24$? **ANSWERS:** First each person is given one piece, and then it is stars and bars with $n = 18$ and $k = 6$, so answer is $\binom{23}{5}$.

(c) Each of the six players pulls out a fair twenty-sided die (containing the numbers $\{1, 2, \ldots, 20\}$) and rolls it. (The six rolls are independent of each other.) What is the probability that the sum of the numbers on the dice is exactly 24? **ANSWER:** We realize that in part (b) each person gets a number of pieces of pizza between 1 and 19, so the number of ways to assign each person a die value (with total sum being 24) is exactly the answer in (b). The total number of die roll sequences is 20^6 so answer is $\binom{23}{5}/20^6$.

4. (20 points) An a capella group with 15 members (8 women and 7 men) is organizing a holiday gift exchange. Each member writes his or her name on a piece of paper and puts it in a bowl. Then the pieces of paper are randomly distributed among the 15 people, with all 15! arrangements being equally likely. Each person is assigned to buy a gift for the individual on the paper that he or she chose.

(a) Compute the expected number of people who will be assigned to buy gifts for themselves. **ANSWER:** Let X_i be 1 if ith person gets own name, 0 otherwise. Then $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = n \cdot 1/n = 1$.

(b) Compute the expected number of men who will be assigned to give gifts to women. **ANSWER:** Let X_i be 1 if ith man gives to woman, 0 otherwise. Then $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = 7 \cdot 8/15 = 56/15$.

(c) Compute the probability that *every* man is assigned to give a gift to a woman. **ANSWER:** $\frac{8}{15} \cdot \frac{7}{14} \cdot \frac{6}{13} \cdot \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} = \frac{8!8!}{15!}$

(d) Compute the probability that *every* individual is part of a cycle of length three (i.e., a group of people A, B, and C where A gives to B, B gives to C, and C gives to A). **ANSWER:** There are $\binom{15}{3,3,3,3,3}$ ways to divide 15 into a pile 1, pile 2, pile 3, pile 4, pile 5 with 3 per pile. If we don't care about ordering the piles, then we have $\binom{15}{3,3,3,3,3}/5!$ ways to divide 15 into the groups of 3. For each such division, there are two directions each cycle can go, so we end up with $2^5\binom{15}{3,3,3,3,3}/5!$, and the probability is $\frac{2^5\binom{15}{3,3,3,3,3}}{5!15!}$

5. (10 points) A standard deck of 52 cards has 13 cards of each suit (diamonds, hearts, clubs, or spades). The deck is randomly divided into 4 bridge hands with 13 cards each (with all divisions being equally likely). What is the probability that *each* of these hands contains cards from only a single suit? (So one hand is only hearts, one hand is only clubs, and so forth.) **ANSWER:** There are $\binom{52}{13,13,13,13}$ ways to give the players their hands, and 4! ways in which each player has a pure-suit hand. So answer is $4!/\binom{52}{13,13,13,13} = \frac{4!(13!)^4}{52!}$.

6. (20 points) Alicia is writing a paper for her history class. Whenever she writes a paper, there is a .7 chance it will be brilliant and a .3 chance it will be mediocre. A professor reading a brilliant paper gives it an A with probability .9. A professor reading a mediocre paper gives it an A with probability .3. Let B be the event that that the paper is brilliant and let A be the event that it gets an A grade, so that our assumptions can be stated as $P(B) = .7$ and $P(A|B) = .9$ and $P(A|B^c) = .3$. Now compute the following:

(a) $P(A)$ (i.e., overall likelihood she gets an A) **ANSWER:** $P(A) = P(BA) + P(B^cA) = P(B)P(A|B) + P(B^c)P(A|B^c) = .7 \cdot .9 + .3 \cdot .3 = .72$

(b) $P(B|A)$ (i.e., likelihood paper is brilliant given it got an A) **ANSWER:** $P(AB)/P(A) = .63/.72 = 7/8$.

(c) $P(B|A^c)$ (i.e., likelihood paper is brilliant given it did not get an A) **ANSWER:** $P(A^cB)/P(A^c) = P(B)P(A^c|B)/P(A^c) = .7 \cdot .1/.28 = 1/4$

18.600 Probability and Random Variables
Fall 2019

18.440 Midterm 1 Solutions, Fall 2011: 50 minutes, 100 points

1. (20 points) Jill goes fishing. During each minute she fishes, there is a 1/600 chance that she catches a fish (independently of all other minutes). Assume that she fishes for 15 hours (900 minutes). Let N be the total number of fish she catches.

(a) Compute $E[N]$ and $\text{Var}[N]$. (Give exact answers, not approximate ones.) **ANSWER:** By additivity of expectation $E[N] = 900/600 = 3/2$. By variance additivity for independent random variables $\text{Var}[N] = 900(1/600)(599/600)$

(b) Compute the probability she catches exactly 3 fish. Give an *exact answer.* **ANSWER:** $\binom{900}{3}(1/600)^3(599/600)^{897}$

(c) Now use a Poisson random variable calculation to *approximate* the probability that she catches exactly 3 fish. **ANSWER:** N is approximately Poisson with $\lambda = 900/600 = 3/2$. So $P\{N = 3\} \approx e^{-\lambda}\lambda^3/3! = e^{-3/2}\frac{9}{16}$.

2. (10 points) Given ten people in a room, what is the probability that no two were born in the same month? (Assume that all of the 12^{10} ways of assigning birthday months to the ten people are equally likely.) **ANSWER:** $\frac{\binom{12}{10}10!}{12^{10}}$

3. (10 points) Suppose that X, Y and Z are independent random variables such that each is equal to 0 with probability .5 and 1 with probability .5.

(a) Compute the conditional probability $P[X + Y + Z = 1 | X - Y = 0]$. **ANSWER:** *Both* events occur if and only if both $X = Y = 0$ and $Z = 1$. So $P\{X + Y + Z = 1, X - Y = 0\} = 1/8$ and $P\{X - Y = 0\} = 1/2$. Thus $P[X + Y + Z = 1 | X - Y = 0] = (1/8)/(1/2) = 1/4$.

(b) Are the events $\{X = Y\}$ and $\{Y = Z\}$ and $\{X = Z\}$ independent? Are they pairwise independent? Explain. **ANSWER:** Not independent. Each event has probability 1/2 but probability all events occur is $1/4 \neq (1/2)^3$. Are pairwise independent, since probability of any two occurring is $(1/2)^2 = 1/4$.

4. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p.

(a) Let X be such that the first heads appears on the Xth toss. In other words, X is the number of tosses required to obtain a heads. Compute (in terms of p) the expectation and variance of X. **ANSWER:** Recall or derive: $E[X] = \sum_{k=1}^{\infty} q^{k-1}pk$, where $q = 1-p$. Cute trick: write $E[X-1] = \sum_{k=1}^{\infty} q^{k-1}p(k-1)$. Setting $j = k-1$, we have $E[X-1] = q\sum_{j=0}^{\infty} q^{j-1}pj = qE[X]$. Thus $E[X]-1 = qE[X]$ and solving for $E[X]$ gives $E[X] = 1/(1-q) = 1/p$.

Similarly, recall or derive: $E[X^2] = \sum_{k=1}^{\infty} q^{k-1}pk^2$. Cute trick: $E[(X-1)^2] = \sum_{k=1}^{\infty} q^{k-1}p(k-1)^2$. Setting $j = k-1$, we have $E[(X-1)^2] = q\sum_{j=0}^{\infty} q^{j-1}pj^2 = qE[X^2]$. Thus $E[(X-1)^2] = E[X^2-2X+1] = E[X^2] - 2/p + 1 = qE[X^2]$. Solving for $E[X^2]$ gives $(1-q)E[X^2] = pE[X^2] = 2/p - 1$, so $E[X^2] = (2-p)/p^2$ and $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

(b) Let Y be such that the fifth heads appears on the Yth toss. Compute (in terms of p) the expectation and variance of Y. **ANSWER:** By additivity of expectation and variance (for independent random variables) we obtain $E[Y] = 5/p$ and $\mathrm{Var}[Y] = 5(1-p)/p^2$.

5. (20 points) Suppose that X is continuous random variable with probability density function $f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} e^{-x}xdx = 1$.

(b) The probability $P\{X \in [-50, 50]\}$. **ANSWER:** $P\{X \in [-50, 50]\} = \int_{-50}^{50} f_X(x)dx = \int_0^{50} e^{-x}dx = 1 - e^{-50}$

(c) The cumulative distribution function F_X. **ANSWER:** $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \le 0 \\ \int_0^a e^{-x}dx = 1 - e^{-a} & a > 0 \end{cases}$

6. (20 points) A group of 52 people (labeled $1, 2, 3, \ldots, 52$) toss their hats into a box, mix them up, and return one hat to each person (all 52! permutations equally likely). Compute the following:

(a) The probability that the first 26 people all get their own hats. **ANSWER:** $\frac{1}{52}\frac{1}{51}\cdots\frac{1}{27} = \frac{26!}{52!}$

2

(b) The probability that there are 26 pairs of people whose hats are switched: i.e., each pair can be labeled (a, b), such that a got b's hat and b got a's hat. **ANSWER:** Have $\binom{52}{2,2,2,\ldots,2} = 52!/(2^{26})$ ways to choose ordered list of 26 pairs. Dividing by 26! gives number of unordered collections of pairs. So we get $\frac{52!}{2^{26}26!}$ permutations of desired type. Dividing by 52! gives probability $\frac{1}{2^{26}26!}$.

3

18.440 Midterm 1, Spring 2011: 50 minutes, 100 points.
SOLUTIONS

1. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p.

(a) Let X be such that the first heads appears on the Xth toss. In other words, X is the number of tosses required to obtain a heads. Compute (in terms of p) the expectation $E[X]$. **ANSWER: geometric random variable with parameter p has expectation $1/p$.**

(b) Compute (in terms of p) the probability that exactly 5 of the first 10 tosses are heads. **ANSWER: binomial probability $\binom{10}{5}p^5(1-p)^5$**

(c) Compute (in terms of p) the probability that the 5th head appears on the 10th toss. **ANSWER: negative binomial. Need 4 heads in first 9 tosses, 10th toss heads. Probability $\binom{9}{4}p^4(1-p)^5p$.**

2. (20 points) Jill sends her resume to 1000 companies she finds on monster.com. Each company responds with probability 3/1000 (independently of what all the other companies do). Let R be the number of companies that respond.

(a) Compute $E[R]$. **ANSWER: binomial random variable with $n = 1000$ and $p = 3/1000$. $E[R] = np = 3$.**

(b) Compute $\text{Var}[R]$. **ANSWER: binomial random variable with $n = 1000$ and $p = 3/1000$. Var$[R] = np(1-p) = 3(1-3/1000)$.**

(c) Use a Poisson random variable approximation to estimate the probability $P\{R = 3\}$. **ANSWER: R is approximately Poisson with $\lambda = 3$. So $P\{R = 3\} \approx e^{-\lambda}\lambda^k/k! = e^{-3}3^3/3! = 9e^{-3}/2$.**

3. (10 points) How many four-tuples (x_1, x_2, x_3, x_4) of *non-negative* integers satisfy $x_1 + x_2 + x_3 + x_4 = 10$? **ANSWER: represent partition with stars and bars $**|**||******$. Have $\binom{13}{3}$ ways to do this.**

4. (10 points) Suppose you buy a lottery ticket that gives you a one in a million chance to win a million dollars. Let X be the amount you win. Compute the following:

(a) $E[X]$. **ANSWER: $\frac{1}{10^6}10^6 = 1$.**

(b) $\text{Var}[X]$. **ANSWER: $E[X^2] - E[X]^2 = \frac{1}{10^6}(10^6)^2 - 1^2 = 10^6 - 1$.**

1

5. (20 points) Suppose that X is continuous random variable with probability density function $f_X(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$. Compute the following:

(a) The expectation $E[X]$. **ANSWER:** $\int_{-\infty}^{\infty} f_X(x)xdx = \int_0^1 f_X(x)xdx = \int_0^1 2x^2dx = \frac{2}{3}x^3|_0^1 = 2/3$**.**

(b) The variance Var$[X]$. **ANSWER:** $E[X^2] = \int_{-\infty}^{\infty} f_X(x)x^2dx = \int_0^1 f_X(x)x^2dx = \int_0^1 2x^3dx = \frac{2}{4}x^4|_0^1 = 1/2$**.** **So variance is** $1/2 - (2/3)^2 = 1/2 - 4/9 = 1/18$**.**

(c) The cumulative distribution function F_X. **ANSWER:**
$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a < 0 \\ a^2 & a \in [0,1] \\ 1 & a > 1 \end{cases}.$$

6. (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all 52! permutations being equally likely). Compute the following:

(a) The probability that *all* of the top 4 cards in the deck are aces. **ANSWER:** 4! **ways to order aces,** 48! **ways to order remainder. Probability** 4!48!/52!

(b) The probability that *none* of the top 4 cards in the deck is an ace.**ANSWER: choose cards one at a time starting at the top and multiply number of available choices at each stage to get total number. Probability is** $48 \cdot 47 \cdot 46 \cdot 45 \cdot 48!/52!$**.**

(c) The *expected* number of aces among the top 4 cards in the deck. (There is a simple form for the solution.) **ANSWER: have probability** $4/52 = 1/13$ **that top card is an ace. Similarly, probability** $1/13$ **that** j**th card is an ace for each** $j \in \{1,2,3,4\}$**. Additivity of expectation gives answer:** $4/13$**.**

18.440 Midterm 2, Fall 2012: 50 minutes, 100 points

1. (10 points) Suppose that a fair die is rolled 18000 times. Each roll turns up a uniformly random member of the set $\{1, 2, 3, 4, 5, 6\}$ and the rolls are independent of each other. Let X be the total number of times the die comes up 1.

(a) Compute $\mathrm{Var}(X)$. **ANSWER:** $npq = 18000(5/6)(1/6) = 2500$

(b) Use a normal random variable approximation to estimate the probability that $X < 2900$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** Standard derivation is $\sqrt{2500} = 50$. Probability X more than 2 standard deviations below mean is approximately $\Phi(-2)$.

2. (20 points) Let X_1, X_2, and X_3 be independent uniform random variables on $[0, 1]$. Write $Y = X_1 + X_2$ and $Z = X_2 + X_3$.

(a) Compute $E[X_1X_2X_3]$. **ANSWER:** Independence implies $E[X_1X_2X_3] = E[X_1]E[X_2]E[X_3] = (1/2)^3 = 1/8$.

(b) Compute $\mathrm{Var}(X_1)$. **ANSWER:** $E(X_1^2) = \int_0^1 x^2 dx = 1/3$, so $\mathrm{Var}(X_1) = E(X_1^2) - E(X_1)^2 = 1/3 - 1/4 = 1/12$.

(c) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$. **ANSWER:** By bilinearity of covariance,

$$\mathrm{Cov}(Y, Z) = \mathrm{Cov}(X_1 + X_2, X_2 + X_3)$$

$$= \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_2, X_2) + \mathrm{Cov}(X_2, X_3).$$

All terms are zero by independence except $\mathrm{Cov}(X_2, X_2) = \mathrm{Var}(X_2) = 1/12$. Then $\rho(Y, Z) = \frac{1/12}{\sqrt{(2/12)(2/12)}} = 1/2$.

(d) Compute and draw a graph of the density function f_Y. **ANSWER:** $f_Y(a) = \int_{-\infty}^{\infty} f_X(a - y) f_X(y) dy$ where

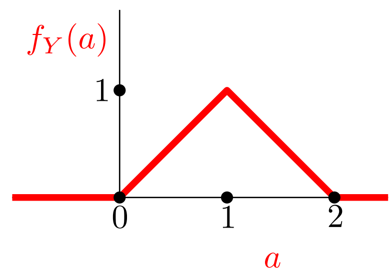
$$f_X(x) = \begin{cases} 1 & x \in (0, 1) \\ 0 & x \notin (0, 1) \end{cases}.$$

Then

$$f_X(a - y) f_X(y) = \begin{cases} 1 & a - y \in (0, 1), y \in (0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Now $a - y \in (0,1)$ is equivalent to $-y \in (-a, 1-a)$ or equivalently $y \in (a-1, a)$. Thus $f_Y(a)$ is equal to the length of the intersection of the intervals $(0,1)$ and $(a-1, a)$. This becomes

$$f_Y(a) = \begin{cases} 0 & a < 0 \\ a & 0 \le a < 1 \\ 2-a & 1 \le a < 2 \\ 0a \ge 2 & \end{cases}$$



3. (20 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on $[0,1]$.

(a) Write $Y = \min\{X_1, X_2, \ldots, X_n\}$. Compute the cumulative distribution function $F_Y(a)$ and the density function $f_Y(a)$ for $a \in [0,1]$. **ANSWER:** By independence, $P(\min\{X_1, X_2, \ldots, X_n\} > a) = P(X_1 > a)P(X_2 > a)\cdots P(X_n > a) = (1-a)^n$ So $F_Y(a) = 1 - (1-a)^n$, and $f_Y(a) = F_Y'(a) = n(1-a)^{n-1}$

(b) Compute $P(X_1 < .3)$ and $P\big(\max\{X_1, X_2, \ldots, X_n\}\big) < .3$. **ANSWER:** $.3$ and $.3^n$.

(c) Compute the expectation $E[X_1 + X_2 + \ldots + X_n]$. **ANSWER:** By additivity of expectation, this is $nE[X_1] = n/2$.

4. (20 points) Aspiring writer Rachel decides to lock herself in her room to think of screenplay ideas. When Rachel is thinking, the moments at which good new ideas occur to her form a Poisson process with parameter $\lambda_G = .5$/hour. The times when bad new ideas occur to her are a Poisson point process with parameter $\lambda_B = 1.5$ per hour.

(a) Let T be the amount of time until Rachel has her first idea (good or bad). Write down the probability density function for T.

ANSWER: T is exponential with parameter $\lambda = \lambda_G + \lambda_B = 2$, so $f_T(x) = 2e^{-2x}$.

(b) Compute the probability that Rachel has exactly 3 bad ideas total during her first hour of thinking. **ANSWER:** Number N of bad ideas is Poisson with rate $1 \cdot \lambda_B = 1.5$. So $P(N = 3) = \frac{(1.5)^3 e^{-1.5}}{3!}$.

(c) Let S be the amount of time elapsed before the fifth good idea occurs. Compute $\mathrm{Var}(S)$. **ANSWER:** Variance of time till one good idea is $1/\lambda_G^2$. Memoryless property and additivity of variance of independent sums gives $\mathrm{Var}(S) = 5/\lambda_G^2 = 20$.

(d) What is the probability that Rachel has no ideas at all during her first three hours of thinking? **ANSWER:** Time till first idea is exponential with $\lambda = 2$. Probability this time exceeds 3 is $e^{-2 \cdot 3} = e^{-6}$.

5. (20 points) Suppose that X and Y have a joint density function f given by

$$f(x, y) = \begin{cases} 1/\pi & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 \geq 1 \end{cases}.$$

(a) Compute the probability density function f_X for X. **ANSWER:**

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \frac{1}{\pi} 2\sqrt{1 - x^2} & -1 \leq x \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Compute the conditional expectation $E[X|Y = .5]$. **ANSWER:** Probability density for X given $Y = .5$ is uniform on $(-\sqrt{1 - .5^2}, \sqrt{1 - .5^2})$. So $E[X|Y = .5] = 0$.

(c) Express $E[X^3 Y^3]$ as a double integral. (You don't have to explicitly evaluate the integral.) **ANSWER:**

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} x^3 y^3 dy dx.$$

6. (10 points) Let X and Y be independent normal random variables, each with mean 1 and variance 9.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f. **ANSWER:**

$$f(x,y) = \frac{1}{3\sqrt{2\pi}}e^{-(x-1)^2/18}\frac{1}{3\sqrt{2\pi}}e^{-(y-1)^2/18} = \frac{1}{18\pi}e^{-\frac{(x-1)^2-(y-1)^2}{18}}.$$

(b) Compute $E[X^2]$ and $E[X^2Y^2]$. **ANSWER:**
$\mathrm{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - 1 = 9$, so $E[X^2] = 10$. By independence $E[X^2Y^2] = E[X^2]E[Y^2] = 100$.

18.600 Probability and Random Variables
Fall 2019

18.440 Midterm 2, Spring 2014: 50 minutes, 100 points

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 3/4 and tails with probability 1/4. Let X_i be 1 if the ith toss comes up heads and 0 otherwise.

(a) Compute $E[X_1]$ and $\mathrm{Var}[X_1]$. **ANSWER:** $E[X_1] = 3/4$ and $E[X_1^2] = 3/4$ so

$$\mathrm{Var}[X_1] = E[X^2] - E[X]^2 = (3/4) - (3/4)^2 = (3/4)(1/4) = 3/16.$$

(b) Compute $\mathrm{Var}[X_1 + 2X_2 + 3X_3 + 4X_4]$. **ANSWER:** Using previous problem, additivity of variance for independent random variables, and general fact that $\mathrm{Var}[aY] = a^2\mathrm{Var}[Y]$, we find that

$$\mathrm{Var}[X_1 + 2X_2 + 3X_3 + 4X_4] = (3/16)(1 + 4 + 9 + 16) = 90/16 = 45/8.$$

(c) Let Y be the number of heads in the first 4800 tosses of the biased coin, i.e.,

$$Y = \sum_{i=1}^{4800} X_i.$$

Use a normal random variable to approximate the probability that $Y \geq 3690$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** Y has expectation $4800E[X_1] = 3600$. It has variance $4800\mathrm{Var}[X_1] = 900$ and standard deviation 30. We are are looking for the probability that Y is more than three standard deviations above its mean. This is approximately the probability that standard normal random variable is three standard deviations above its mean, which is $1 - \Phi(3)$.

2. (10 points) Suppose that a fair six-sided die is rolled just once. Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the number that comes up. Let Y be 1 if the number on the die is in $\{1, 2, 3\}$ and 0 otherwise.

(a) What is the conditional expectation of X given that $Y = 0$? **ANSWER:** Given that Y is zero, X is conditionally uniform on $\{4, 5, 6\}$, so the conditional expectation is 5.

(b) What is the conditional variance of Y given that $X = 2$? **ANSWER:** Given that X is 2, the conditional probability that $Y = 1$ is one, so the conditional variance is 0.

3. (20 points) Let X be a uniform random variable on the set $\{-2,-1,0,1,2\}$. That is, X takes each of these values with probability $1/5$. Let Y be an independent random variable with the same law as X, and write $Z = X + Y$.

(a) What is the moment generating function $M_X(t)$? **ANSWER:** $M_X(t) = E[e^{tX}] = \frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t})$.

(b) What is the moment generating function $M_Z(t)$? **ANSWER:**

$$M_Z(t) = M_X(t)M_Y(t) = \left[\frac{1}{5}(e^{-2t} + e^{-t} + e^0 + e^t + e^{2t})\right]^2.$$

4. (20 points) Two soccer teams, the Lions and the Tigers, begin an infinite soccer games starting at time zero. Suppose that the times at which the Lions score a goal form a Poisson point process with rate $\lambda_L = 2/\text{hour}$. Suppose that the times at which the Tigers score a goal form a Poisson point process with rate $\lambda_T = 3/\text{hour}$.

(a) Write down the probability density function for the amount of time until the first goal by the Lions. **ANSWER:** This is an exponential random variable with parameter λ_L. So the density function on $[0,\infty)$ is $f(x) = \lambda_L e^{-\Lambda_L x} = 2e^{-2x}$.

(c) Write down the probability density function for the amount of time until the first goal by *either* team is scored. **ANSWER:** Recall that the minimum of two exponential random variables with parameters λ_L and λ_T is an exponential random variable with parameter $\lambda_L + \Lambda_T = 5$. So the density function on $[0,\infty)$ is $f(x) = 5e^{-5x}$

(c) Compute the probability that the Tigers score no goals at all during the first two hours. **ANSWER:** The probability that an exponential random of parameter λ is at least a is given by $e^{-\lambda a}$. Plugging in $\lambda = 3$ and $a = 2$ we get e^{-6}.

(d) Compute the probability that the Lions score exactly three goals during the first hour. **ANSWER:** The number of goals scored by the Lions during the first hour is a Poisson random variable with parameter $\lambda = \lambda_L = 2$. The probability that this is equal to a given k is given by $e^{-\lambda}\lambda^k/k!$. Plugging in $k = 3$ and $\lambda = 2$ we get

$$e^{-2}2^3/3! = \frac{4}{3e^2}.$$

5. (20 points) Let X and Y be independent uniform random variables on $[0,1]$. Write $Z = X + Y$. Write $W = \max\{X, Y\}$.

(a) Compute and draw a graph of the probability density function f_Z. **ANSWER:** This is given by

$$f_Z(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 2 - x & 1 < x \leq 2 \\ 0 & x \geq 2 \end{cases}$$

(b) Compute and draw a graph of the cumulative distribution function F_W. **ANSWER:** $F_W(a) = \begin{cases} 0 & a < 0 \\ a^2 & 0 \leq a \leq 1 \\ 1 & a > 1 \end{cases}$

(c) Compute the variances $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Var}(Z)$. **ANSWER:** $\text{Var}(X) = E[X^2] - E[X]^2 = \int_0^1 x^2 dx - (1/2)^2 = 1/3 - 1/4 = 1/12$. Then $\text{Var}(Y) = \text{Var}(X)$ and $\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y) = 2/12$.

(d) Compute the covariance $\text{Cov}(Y, Z)$ and the correlation coefficient $\rho(Y, Z)$. **ANSWER:** Using the linearity of covariance in its second argument, we find $\text{Cov}(Y, Z) = \text{Cov}(Y, X) + \text{Cov}(Y, Y)$. The first term is zero (since X and Y are independent) so this becomes $\text{Var}(Y) = 1/12$. The correlation coefficient is

$$\frac{\text{Cov}(Y, Z)}{\sqrt{\text{Var}(Y)\text{Var}(Z)}} = \frac{(1/12)}{\sqrt{(1/12)(2/12)}} = 1/\sqrt{2}.$$

6. (10 points) Let X and Y be independent exponential random variables, each with with parameter $\lambda = 5$.

(a) Let f be the joint probability density function for the pair (X, Y). Write an explicit formula for f. **ANSWER:** Since X and Y are independent, $f(x, y) = f_X(x) f_Y(y) = 5e^{-5x} \cdot 5e^{-5y} = 25e^{-5(x+y)}$.

(b) Compute $E[X^2 Y]$. **ANSWER:** First, note that X^2 and Y are independent, so this is $E[X^2]E[Y]$. Direct integration gives $E[Y] = 1/\lambda$ and $E[X^2] = 2/\lambda^2$, so the answer is $2/\lambda^3 = 2/125$.

18.600 Midterm 2, Spring 2016: Solutions

1. (20 points) Consider a sequence of independent tosses of a coin that is biased so that it comes up heads with probability 2/3 and tails with probability 1/3. Let X_i be 1 if the ith toss comes up heads and 0 otherwise. Write $S_n = \sum_{i=1}^n X_i$.

(a) Compute $E[X_1]$ and $\mathrm{Var}[X_1]$. **Answer:** 2/3 and 2/9

(b) Compute $E[S_n]$ and $\mathrm{Var}[S_n]$ as functions of n. **Answer:** $(2/3)n$ and $(2/9)n$

(c) Compute the covariance of S_5 and S_{10}. **Answer:**

$$\mathrm{Cov}\Big(\sum_{i=1}^5 X_i, \sum_{j=1}^{10} X_j\Big) = \sum_{i=1}^5 \sum_{j=1}^{10} \mathrm{Cov}(X_i, X_j).$$

Terms with $i \neq j$ are zero, so this is $\sum_{i=1}^5 \mathrm{Cov}(X_i, X_i) = 5\mathrm{Var}(X_i) = 10/9$.

(d) Using a normal approximation, estimate the probability that $S_{300} \leq 220$. You may use the function $\Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **Answer:** $\Phi(a)$ where $a = (220 - 200)/\sqrt{300 \times 2/9} = 20\sqrt{9/600} = \sqrt{3600/600} = \sqrt{6}$.

2. (20 points) Suppose that X and Y are the outcomes of independent fair die rolls. So each takes a value in $\{1, 2, 3, 4, 5, 6\}$, with all values being equally likely. Write $Z = X + Y$.

(a) Compute the moment generating function for X. **Answer:**

$$M_X(t) = E[e^{tX}] = \frac{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}.$$

(b) Compute the moment generating function for Z. **Answer:**

$$M_Z(t) = [M_X(t)]^2 = \Big(\frac{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}}{6}\Big)^2.$$

(c) Compute $E[Y|Z]$. (That is, express the random variable $E[Y|Z]$ as a function of the random variable Z.) **Answer:** $E[Y|Z] + E[X|Z] = E[Z|Z] = Z$ and by symmetry $E[Y|Z] = E[X|Z]$ so $E[Y|Z] = Z/2$. **Alternative answer:** check by hand that given $Z = k \in \{2, 3, \dots, 12\}$, conditional law of Y is uniform on a set of consecutive integers centered at $k/2$.

3. (20 points) Let X be a uniformly random variable on $[0, 5]$. Let Y be an independent uniformly random variable on $[0, 10]$. Write $Z = \min\{X, Y\}$.

(a) Compute the joint density function $f(x, y)$ for X and Y. **Answer:** $f(x, y) = 1/50$ if $(x, y) \in [0, 5] \times [0, 10]$, and 0 otherwise.

(b) Compute $P(Z > 0)$ and $P(Z > 3)$ and $P(Z > 5)$. **Answer:** $P(Z > 0) = 1$ and $P(Z > 3) = P(X > 3)P(Y > 3) = (2/5)(7/10) = 7/25$ and $P(Z > 5) = 0$.

(c) Compute the cumulative distribution function $F_Z(a)$. **Answer:** For $a \in [0,5]$, have $1 - F_Z(a) = P(X > a)P(Y > a) = (\frac{5-a}{5})(\frac{10-a}{10}) = \frac{(5-a)(10-a)}{50}$. So

$$F_Z(a) = \begin{cases} 0 & a < 0 \\ 1 - \frac{(5-a)(10-a)}{50} & a \in [0,5] \\ 1 & a > 5 \end{cases} .$$

.

4. (20 points) Alice's Pastry Shop is open from 7:00 a.m. until 10:00 p.m. Throughout those 900 minutes, Alice has an extremely steady business: customers show up according to a Poisson point process with parameter $\lambda = 1$, where time is measured in minutes. (That is, the expected number of customers per minute is one.) Let N be the total number of customers that arrive during the day.

(a) Compute the probability that there are exactly 3 customers during the first three minutes. **Answer:** $e^{-(\lambda t)}(\lambda t)^k/k! = e^{-3}3^3/6 = (9/2)e^{-3}$.

(b) Write a probability density function for the time it takes from the store opening until the arrival of the second customer. (Imagine that customers keep arriving after closing, so that with probability one a second customer comes *eventually*. In other words, don't worry about the 900 minute upper bound for this part of the problem.) **Answer:** this is Γ with parameters $\lambda = 1$ and $\alpha = 2$. Density function is $f(x) = xe^{-x}/\Gamma(2) = xe^{-x}$ for $x \geq 0$ (and zero if $x < 0$).

(c) Compute $E[N]$ and $\mathrm{Var}[N]$. **Answer:** $E[N] = \lambda = 900$ and $\mathrm{Var}[N] = \lambda = 900$.

(d) Compute the probability that the entire day goes by without a single customer. **Answer:** e^{-900}. (Probability λ-exponential random variable exceeds T is $e^{-\lambda T}$.)

5. (10 points) Suppose that $X_1, X_2, \dots, X_n$ are independent exponential random variables with parameter $\lambda = 1$.

(a) Write $Y = \min\{X_1, X_2, \dots, X_n\}$. Compute the density function f_Y. **Answer:** This is exponential with rate $\lambda = n$ so $f_Y(y) = \lambda e^{-\lambda y} = ne^{-ny}$.

(b) Compute $E[Y^k]$ as a function of n and k. You may assume that n and k are positive integers. **Answer:** Recall that if X is exponential with parameter 1 we have $E[X^k] = \int_0^\infty x^k e^{-x} dx = k!$. (This is one of the definitions of the factorial.) Note that Y has same law as X/n, so $E[Y^k] = E[(X/n)^k] = E[X^k]/n^k = k!/n^k$.

6. (10 points) Suppose that $X_1, X_2, X_3, X_4, \dots, X_n$ are independent random variables, each of which has a probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$. Compute the probability that $X_1 + X_2 + \ldots + X_n \geq n$. **Answer:** Each X_i is Cauchy and we seek the probability that $Z = \frac{1}{n}\sum_{i=1}^n X_i \geq 1$. Since Z is itself Cauchy, spinning flashlight story gives probability $P(Z) > 1 = (\pi/4)/\pi = 1/4$. (The angle between segment $[(0,1),(1,0)]$ and horizontal line $y = 1$ is $\pi/4$.)

18.600 Midterm 2, Spring 2017 Solutions

1. (10 points) Suppose that X, Y, and Z are independent random variables, each of which is equal to 1 with probability 1/3 and 5 with probability 2/3. Write $W = X + Y + Z$.

(a) Compute the moment generating function M_X. **ANSWER:** $M_X(t) = \frac{1}{3}e^t + \frac{2}{3}e^{5t}$

(b) Compute the moment generating function M_W. **ANSWER:** $M_W(t) = \left(\frac{1}{3}e^t + \frac{2}{3}e^{5t}\right)^3$

2. (15 points) Let $X_1, X_2, \ldots, X_7$ be independent normal random variables, each with mean 0 and variance 1. Write $Z = \sum_{j=1}^{7} X_j$.

(a) Give the probability density function for Z. **ANSWER:** Sum is normal with mean zero and variance $\sigma^2 = 7$ so

$$F_Z(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-x^2}{2\sigma^2}} = \frac{1}{\sqrt{14\pi}}e^{\frac{-x^2}{14}}.$$

b) Compute the probability that that the random variables are in increasing order, i.e., that $X_1 < X_2 < X_3 < \ldots < X_7$. **ANSWER:** All orderings are equally likely by symmetry, so the probability is $1/7!$.

3. (15 points) Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$.

(a) Compute the probability density function for $\min\{X_1, X_2, X_3\}$. **ANSWER:** This is exponential with rate $\lambda = 3$ so the density is $3e^{-3t}$ on $[0, \infty)$ (and zero elsewhere).

(b) Compute the expectation $E[\max\{X_1, X_2, X_3\}]$. **ANSWER:** This is the radioactive decay problem. Time first decay is exponential with rate 3, subsequent time until next decay is exponential with rate 2, and subsequent time until next decay is exponential with rate 1. Summing the expected times gives $\frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$.

4. (10 points) Suppose that X and Y are independent random variables, each of which has a probability density function given by $f(x) = \frac{1}{\pi(1+x^2)}$.

(a) Give the probability density function for $A = (X + Y)/2$. **ANSWER:** The average of independent Cauchy random variables is again Cauchy, so $f_A(x) = \frac{1}{\pi(1+x^2)}$

(b) Give the probability density function for $B = X - Y$. **ANSWER:** By symmetry, $-Y$ has same law as Y, so $X - Y$ has the same law as $2A = X + Y$. Tbus
$f_B(x) = f_{2A}(x) = \frac{1}{2}f_B(x/2) = \frac{1}{2\pi\left(1+(x/2)^2\right)}$.

5. (15 points) Let C be fraction of students in a very large population that will say (when asked) that they prefer curry to pizza. Imagine that you start out knowing nothing about C, so that your Bayesian prior for C is uniform on $[0, 1]$. Then you select a student uniformly from the population and ask what that student prefers. You independently repeat this experiment two more times. You find that two students prefer curry and one prefers pizza.

(a) *Given* what you have learned from these three answers, give a revised probabilty density function f_C for the unknown quantity C (i.e., a Bayesian posterior). **ANSWER:** This is Beta with parameters $a = 2 + 1 = 3$ and $b = 1 + 1 = 2$. Compute $B(3, 2) = 2!1!/4! = 1/12$. So $x^2(1 - x)B(2, 3) = 12x^2(1 - x)$ on the interval $[0, 1]$.

(b) According to your Bayesian prior, the expected value of C was $1/2$. *Given* what you learned from the three answers, what is your *revised expectation* of the value C? **ANSWER:** The expectation of a Beta (a, b) random variable is $a/(a + b)$ which in this case is $3/5$.

6. (15 points) Let $X_1, X_2, \dots, X_{100}$ be independent random variables, each of which is equal to 1 with probability 1/2 and 0 with probability 1/2. Write $S = \sum_{j=1}^{100} X_j$.

(a) Compute $E[S]$ and $\mathrm{Var}[S]$. **ANSWER:** S is binomial (n, p) and $E[S] = np = 50$ and $\mathrm{Var}[S] = npq = 25$.

(b) Use a normal random variable to approximate $P(S > 60)$. You may use the function $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answer. **ANSWER:** 60 is two standard deviations above the mean; by normal approximation $P(S > 60) \approx \int_2^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \Phi(2) = \Phi(-2)$.

7. (20 points) The residents of a certain planet are careful with nuclear weapons, but regional nuclear wars still occur. The times at which these wars occur form a Poisson point process with rate λ equal to one per thousand years. So the expected number of nuclear wars during any 1000 year period is 1.

(a) Compute the probability that there will be exactly three nuclear wars during the next 2000 years. **ANSWER:** This the probability a Poisson with parameter $\lambda = 2$ is equal to $k = 3$, which is $e^{-\lambda}\lambda^k/k! = e^{-2}2^3/3! = \frac{4}{3e^2}$.

(b) Let X be the number of millenia until the third nuclear war. (In other words, $1000X$ is the number of years until the third nuclear war.) Give the probability density function for f_X. **ANSWER:** This is Gamma with parameter $\lambda = 1$ and $n = 3$, which comes to $x^2 e^{-x}/2$.

(c) Let Y be the number of nuclear wars that will occur during the next 5000 years. Compute the variance of Y. **ANSWER:** This is the variance of a Poisson random variable with parameter $\lambda = 5$, which is 5.

18.600 Midterm 2, Spring 2018 Solutions

1. (20 points) Suppose that X is a random variable with probability density function given by

$$f(x) = \begin{cases} 0 & x \le 0 \\ x/2 & 0 < x \le 2 \\ 0 & x > 2 \end{cases}.$$

Suppose that Y is an independent random variable with the same probability density function. Write $Z = X^2 + Y^2$.

(a) Compute the joint density function $f_{X,Y}(x,y)$. **ANSWER:** $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ which is $xy/4$ when $0 < x \le 2$ and $0 < y \le 2$, and 0 otherwise.

(b) Compute $E[Z]$. (You should be able to get an explicit number.) **ANSWER:** $E[X^2+Y^2] = E[X^2] + E[Y^2] = 2E[X^2] = 2\int_0^2 x^2 x/2dx = 2x^4/8 \Big|_0^2 = 4$

(c) Compute $P(\max\{X,Y\} \le 1)$. **ANSWER:** $P(\max\{X,Y\} \le 1) = P(X \le 1, Y \le 1) = P(X \le 1)^2 = \left(\int_0^1 x/2dx\right)^2 = (1/4)^2 = 1/16$

2. (10 points) In a certain population, there are $n = 110000$ healthy people, each of whom has a $p = .01$ chance (independently of everyone else) of developing a certain disease during the course of a given decade. Let X be the number of people who develop the disease.

(a) Compute $E[X]$ and $\mathrm{Var}[X]$. **ANSWER:** $E[X] = np = 1100$ and $\mathrm{Var}[X] = npq = 1100 \cdot .99 = 1089$.

(b) Use a normal random variable to estimate $P(1100 < X < 1133)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** $SD(X) = \sqrt{\mathrm{Var}[X]} = 33$ so this is probability X is between 0 and 1 standard deviations above mean, which is approximately $\Phi(1) - \Phi(0)$ by de Moivre Laplace.

3. (20 points) Suppose that X_1, X_2 and X_3 are independent random variables, each of which has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \ge 0 \end{cases}$$

Write $X = X_1 + X_2 + X_3$. Write $A = \min\{X_1, X_2, X_3\}$. Write $B = \max\{X_1, X_2, X_3\}$.

(a) Give a probability density function for X. **ANSWER:** This is Gamma distribution with parameters $n = 3$ and $\lambda = 1$, so $f_X(x) = x^2 e^{-x}/2!$ for $x \ge 0$

(b) Give a probability density function for A. **ANSWER:** Minimum of three rate one exponentials is exponential with rate three, so $f_A(x) = 3e^{-3x}$ for $x \ge 0$.

(c) Compute $E[B]$ and $\mathrm{Var}[B]$. **ANSWER:** This is the "radioactive decay" problem. $E[B] = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$ and $\mathrm{Var}[B] = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$.

4. (10 points) Suppose that X, Y, and Z are independent random variables, each of which has probability density function $f(x) = \frac{1}{\pi(1+x^2)}$. Write $V = 3X$ and $W = X + Y + Z$.

(a) Compute the probability density function for V. **ANSWER:** $f_V(x) = \frac{1}{3} f_X(x/3) = \frac{1}{3\pi\left(1+(x/3)^2\right)}$.

(b) Compute the probability density function for W. **ANSWER:** W has the same law as V (by amazing property of Cauchy random variables) so $f_W(x) = f_V(x)$.

5.(10 points) Let X and Y be independent standard normal random variables (so each has mean zero and variance one).

(a) Compute $P(X^2 + Y^2 \leq 1)$. Give an explicit value. **ANSWER:** Since X and Y are independent we have $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi} e^{-x^2-y^2} dxdy$. Switching to polar coordinates, this is $\int_0^1 \int_0^{2\pi} \frac{1}{2\pi} e^{-x^2/2-y^2/2} d\theta dr = \int_0^1 re^{-r^2} = -e^{-r^2/2}\Big|_0^1 = 1 - e^{-1/2}$.

(b) Compute $P(\max\{|X|, |Y|\} \leq 1)$. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** By independence of X and Y we have $P(\max\{|X|, |Y|\} \leq 1) = P(|X| \leq 1)P(|Y| \leq 1) = \big(\Phi(1) - \Phi(-1)\big)^2$.

6. (20 points) Let X and Y be independent uniform random variables on $[0, 1]$ and write $Z = X + Y$.

(a) Compute the conditional expectation $E[X|Z]$. (That is, express the random variable $E[X|Z]$ as a function of the random variable Z.) **ANSWER:** Note that $E[X + Y|Z] = E[Z|Z] = Z$. By additivity of conditional expectation and symmetry $E[X + Y|Z] = E[X|Z] + E[Y|Z] = 2E[X|Z]$. So $E[X|Z] = Z/2$.

(b) Compute the conditional expectation $E[Z|X]$. (That is, express the random variable $E[Z|X]$ as a function of the random variable X.) **ANSWER:** $E[X + Y|X] = E[X|X] + E[Y|X] = X + E[Y|X]$. X and Y are independent so $E[Y|X] = E[Y] = 1/2$. Answer is $X + 1/2$.

(c) Compute the conditional variance $\mathrm{Var}[Z|Y]$. (That is, express the random variable $\mathrm{Var}[Z|Y]$ as a function of the random variable Y.) **ANSWER:** Given Y, the conditional law of Z is uniform on $[Y, Y + 1]$, and thus the conditional variance is $1/12$ (regardless of the Y value) so the answer is just $1/12$.

(d) Compute the correlation coefficient $\rho(X, Z)$. **ANSWER:**

$$\frac{\mathrm{Cov}(X, Z)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Z)}} = \frac{\mathrm{Cov}(X, X + Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(X + Y)}} = \frac{\mathrm{Var}(X)}{\sqrt{\mathrm{Var}(X)2\mathrm{Var}(X)}} = \frac{1}{\sqrt{2}}.$$

7. (10 points) Suppose that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on the interval $[0, 1]$. Write $X = X_1 + X_2 + \ldots + X_n$.

(a) Compute the characteristic function $\phi_{X_1}(t)$. **ANSWER:** $\phi_{X_1}(t) = E[e^{itX_1}] = \int_0^1 e^{itx} dx = e^{itx}/it\Big|_0^1 = (e^{it} - 1)/it$

(b) Compute the characteristic function $\phi_X(t)$. **ANSWER:** $\phi_X(t) = \big(\phi_{X_1}(t)\big)^n$.

18.600 Midterm 2 Solutions, Spring 2019: 50 minutes, 100 points

1. (10 points) Ramona enters a basketball free throw shooting contest and takes 100 shots. She makes each shot independently with probability .8 and misses with probability .2 Let X be the number of shots she makes.

(a) Compute the expectation and variance of X. **ANSWER:** $E[X] = np = 80$ and $\text{Var}(X) = npq = 16$

(b) Use a normal random variable to estimate the probability that she makes between 76 and 84 shots total. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** $\text{SD}(X) = 4$ and 76 is one SD below mean, 84 one SD above mean, so normal approximation gives $\Phi(1) - \Phi(-1) \approx .68$.

2. (20 points) Becky's Bagel Bakery does a brisk business. Customers arrive at random times, and each customer immediately purchases one type of bagel. The times $C_1, C_2, \ldots$ at which cinnamon raisin bagels are sold form a Poisson point process with a rate of 1 per minute. The times $P_1, P_2, \ldots$ at which pumpernickel bagels are sold form an independent Poisson point process with rate 2 per minute. And the times $E_1, E_2, \ldots$ at which everything bagels are sold form a Poisson point process with rate 3 per minute. Compute the following:

(a) The probability density function for C_3. **ANSWER:** Sum of three exponentials is Gamma with parameter $n = 3$ and $\lambda = 1$. So answer is $x^2 e^{-x}/2$ on $[0, \infty)$.

(b) The probability density function for $X = \min\{C_1, P_1, E_1\}$. **ANSWER:** Minimum of exponentials with rates $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ is itself exponential with rate $\lambda_1 + \lambda_2 + \lambda_3 = 6$. So answer is $6e^{-6x}$ on $[0, \infty)$.

(c) The probability that *exactly* 10 bagels (altogether) are sold during the first 2 minutes the bakery is open. **ANSWER:** The set of all bagels sale times is a Poisson point process with parameter 6. So number of points sold in first two minutes is Poisson with $\lambda = 12$. Probability to sell 10 is $e^{-\lambda}\lambda^k/k! = e^{-12}12^{10}/10!$.

(d) The expectation of $\cos(P_1 + E_1^2)$. (You can leave this as a double integral — no need to evaluate it.) **ANSWER:** P_1 exponential with parameter 2, and E_1 is exponential with parameter 3. So joint density is $2e^{-2x}3e^{-3y}$. So for general function $g(x, y)$ we can write

$$E[g(x, y)] = \int_0^\infty \int_0^\infty 2e^{-2x}3e^{-3y} g(x, y) dx dy$$

which in our case gives

$$\int_0^\infty \int_0^\infty 2e^{-2x}3e^{-3y} \cos(x + y^2) dx dy$$

3. (10 points) Suppose that the pair of real random variables X, Y has joint density function $f(x, y) = \frac{1}{\pi^2(1+x^2)(1+y^2)}$.

(a) Compute the probability density function for $\frac{X+Y}{2}$. **ANSWER:** $f(x,y) = \left(\frac{1}{\pi(1+x^2)}\right)\left(\frac{1}{\pi(1+y^2)}\right)$ so X and Y are independent Cauchy random variables. Hence their average is also Cauchy, with density $\frac{1}{\pi(1+x^2)}$.

(b) Compute the probability $P(X > Y + 2)$. **ANSWER:** Note that $(X - Y)/2$ has same probability density function as $(X + Y)/2$ (since density function for Y is symmetric) so it is Cauchy. Hence $P(X > Y + 2) = P(X - Y > 2) = P(\frac{X+Y}{2} > 1)$ is the probability that a Cauchy random variable is greater than 1. Recalling spinning flashlight story, this is probability that $\theta > \pi/4$ when θ is uniform on $[-\pi/2, \pi/2]$, and this is 1/4.

4. (20 points) Suppose that X_1, X_2, X_3, X_4 are independent random variables, each of which has density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Compute the following:

(a) The correlation coefficient $\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4)$. **ANSWER:**

$$\mathrm{Cov}(X_i, X_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

so bilinearity of covariance gives $\mathrm{Cov}(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = 2$. Variance additivity for independent random variables gives $\mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_2 + X_3 + X_4) = 3$. So

$$\rho(X_1 + X_2 + X_3, X_2 + X_3 + X_4) = \frac{2}{\sqrt{3 \cdot 3}} = \frac{2}{3}.$$

(b) The probability that $\min\{X_1, X_2\} > \max\{X_3, X_4\}$. **ANSWER:** This is the probability that X_1 and X_2 are the "top two". There are $\binom{4}{2}$ pairs which could be "top two" and by symmetry each such pair is equally likely, so answer is $1/\binom{4}{2} = 1/6$. Alternatively, one may consider that of 24 permutations of X_1, X_2, X_3, X_4, exactly four satisfy the constraint.

(c) The probability density function for $X_1 + X_2 + X_3$. **ANSWER:** Sum of independent normals is also normal (with mean and variance given by the sum of the respective means and variances of the individual terms). Thus $X_1 + X_2 + X_3$ is normal with mean $\mu = 0$, variance $\sigma^2 = 3$. So answer is $\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)} = \frac{1}{\sqrt{3}\sqrt{2\pi}}e^{-x^2/6}$.

(d) The probability $P(X_1^2 + X_3^2 \leq 2)$. Give an explicit value. **ANSWER:** The joint density of X_1 and X_3 is $f_{X_1,X_3}(x,y) = f_{X_1}(x)f_{X_3}(y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$. We have to integrate this over region where $x^2 + y^2 \leq 2$ which is the disk of radius $\sqrt{2}$. This can be done in polar coordinates: answer is

$$\int_0^{\sqrt{2}} \int_0^{2\pi} e^{-r^2/2} d\theta r dr = \int_0^{\sqrt{2}} e^{-r^2/2} r dr = -e^{-r^2/2}\Big|_0^{\sqrt{2}} = 1 - e^{-1}.$$

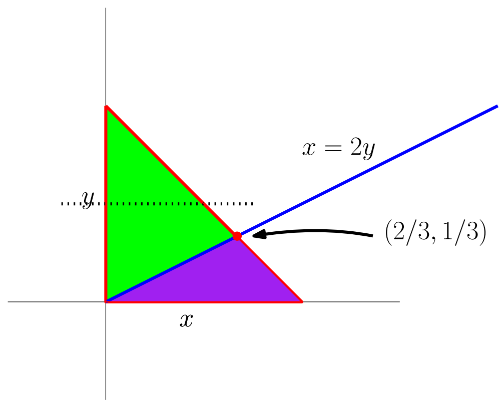
5. (10 points) Imagine that A, B, C and D are independent uniform random variables on $[0, 1]$. You then find out that A is the third largest of those random variables.

(a) Given this new information, give a revised probability density function f_A for A (i.e., a Bayesian posterior). **NOTE:** If you remember what this means, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$. **ANSWER:** Answer is Beta with $a - 1$ equal to number of points below A (that's 1) and $b - 1$ equal to number of points above A (that's 2). So $a = 2$ and $b = 3$ and answer is $x(1 - x)^2/B(2, 3)$ on $[0, 1]$. Can compute $B(2, 3) = 1!2!/4! = 1/12$, so answer is $12x(1 - x)^2$.

(b) According to your Bayesian prior, the expected value of A was $1/2$. *Given* that A was the third largest of the random variables, what is your *revised expectation* of the value A? **ANSWER:** $a/(a+b) = 2/5$, by the expectation formula given.

6. (15 points) Suppose that the pair (X, Y) is uniformly distributed on the triangle $T = \{(x, y) : 0 \le x, 0 \le y, x + y \le 1\}$. That is, the joint density function is given by

$$f_{X,Y}(x, y) = \begin{cases} 2 & (x, y) \in T \\ 0 & (x, y) \notin T \end{cases}.$$



(a) Compute the marginal density function f_X. **ANSWER:** $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$. If $x \in [0, 1]$, this value is 2 times length of intersection of vertical line through $(x, 0)$ with T, which is $1 - x$. So answer is $f_X(x) = \begin{cases} 2 - 2x & x \in [0, 1] \\ 0 & x \notin [0, 1] \end{cases}$

(b) Compute the probability $P(X < 2Y)$. **ANSWER:** Using figure shown, area of whole triangle is $1/2$, area of subtriangle on which $X < 2Y$ is $1/3$, so answer is $(1/3)/(1/2) = 2/3$.

(c) Compute the conditional density function $f_{X|Y=.5}(x)$. **ANSWER:** $f_Y(1/2) = f_X(1/2) = 1$ so

$$f_{X|Y=.5}(x) = f(x, 1/2)/f_Y(1/2) = f(x, 1/2) = \begin{cases} 2 & x \in [0, 1/2] \\ 0 & x \notin [0, 2] \end{cases}.$$

(Visually, given that (X, Y) is on horizontal dotted line, X is uniform on $[0, 1/2]$.)

7. (15 points) Suppose that X is an exponential random variable with parameter 1 and set $Z = X^5$.

(a) Compute the cumulative distribution function $F_Z(a)$ in terms of a. **ANSWER:** $F_X(a) = \int_0^a e^{-x}dx = 1 - e^{-a}$. And $F_Z(a) = P(Z \le a) = P(X \le a^{1/5}) = F_X(a^{1/5}) = 1 - e^{-a^{1/5}}$

(b) Compute the expectation $E[Z^2]$. **ANSWER:** $E[Z^2] = E[X^{10}] = \int_0^\infty e^{-x}x^{10}dx = 10!$. (Recall this is one of our definitions for $10!$.)

(c) Compute the conditional probability $P(Z > 32|Z > 1)$. **ANSWER:**

$$P(Z > 32|Z > 1) = P(X > 2|X > 1) = P(X > 1) = e^{-1}.$$

18.600 Probability and Random Variables
Fall 2019

18.440 Midterm 2 Solutions, Fall 2011: 50 minutes, 100 points

1. (20 points) Suppose that a fair die is rolled 72000 times. Each roll turns up a uniformly random member of the set $\{1, 2, 3, 4, 5, 6\}$ and the rolls are independent of each other. For each $j \in \{1, 2, 3, 4, 5, 6\}$ let X_j be the number of times that the die comes up j.

(a) Compute $E[X_3]$ and $\text{Var}[X_3]$. **ANSWER:** Take $n = 72000$, $p = 1/6$. Then $E[X_3] = np = 12000$ and $\text{Var}[X_3] = np(1-p) = 10000$.

(b) Compute $\text{Var}[X_1 + X_2]$. **ANSWER:** This counts the number of times that either a one or a two comes up. Each die roll has a 1/3 chance of being a 1 or 2. So $\text{Var}[X_1 + X_2] = n(1/3)(2/3) = 16000$.

(c) Use a normal random variable approximation to estimate the probability that $X_3 > 12100$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** 12100 is one standard deviation above the mean. Approximate probability is $1 - \Phi(1)$.

2. (20 points) Suppose that a fair die is rolled just once. Let Y be 1 if the die comes up 3 and zero otherwise. Let Z be 1 if the die comes up 2 and zero otherwise.

(a) Compute the covariance $\text{Cov}(Y, Z)$ and the variances $\text{Var}(Y)$ and $\text{Var}(Z)$. **ANSWER:**
$\text{Cov}(Y, Z) = E[YZ] - E[Y]E[Z] = 0 - 1/36 = -1/36$ and
$\text{Var}(Y) = \text{Var}(Z) = (1/6)(5/6) = 5/36$.

(b) Compute the covariance of $3Y + Z$ and $Y - 3Z$. **ANSWER:**
$\text{Cov}(3Y + Z, Y - 3Z) = 3\text{Var}(Y) - 8\text{Cov}(Y, Z) - 3\text{Var}(Z) =$
$-8\text{Cov}(Y, Z) = 2/9$.

(c) What is the conditional expectation of Y given that $Z = 0$?
ANSWER: 1/5.

3. (20 points) At a certain track competition, ten athletes take turns throwing javelins. Let X_i be the distance that the ith athlete throws the javelin. Suppose that each X_i is an exponential random variable with an expectation of 50 meters and that the X_i are independent of each other.

(a) What is the probability density function for X_1? What is the parameter λ of this exponential random variable? **ANSWER:**
$\lambda = 1/50$ and $f(x) = \lambda e^{-\lambda x}$ if $x > 0$, and 0 otherwise.

(b) Compute the probability that the first athlete throws the javelin more than 50 meters. **ANSWER:** $e^{-50\lambda} = e^{-1}$.

(c) Compute the probability that at least one athlete throws the javelin more than 50 meters. **ANSWER:** $1 - (1 - e^{-1})^{10}$.

(d) Compute $E[\min\{X_1, X_2, \ldots, X_{10}\}]$, i.e., the expectation of the distance that the last place athlete throws the javelin. **ANSWER:** Minimum of ten independent exponentials of rate $\lambda = 1/50$ is exponential of rate $10\lambda = 1/5$. Expectation is $1/(10\lambda) = 5$ meters.

4. (20 points) Let X be a uniformly random variable on $[0, 5]$.

(a) Write the probability density function f_X and the cumulative distribution function F_X. **ANSWER:** $f_X(x) = \begin{cases} 1/5 & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$.

(b) What is the moment generating function $M_X(t)$? **ANSWER:** $\frac{e^{5t}-1}{5t}$.

(c) Suppose that Y is a random variable for which $M_Y(0) = 1$ and $M'_Y(0) = 1$ and $M''_Y(0) = 2$. What are $E[Y]$, $E[Y^2]$ are $\mathrm{Var}[Y]$? **ANSWER:** $E[Y] = 1$, $E[Y^2] = 2$, and $\mathrm{Var}[Y] = 2 - 1^2 = 1$.

5. (20 points) Suppose that on a certain road, the times at which red cars go by a given spot are given by a Poisson point process with rate $\lambda = 2$/hour. Suppose that the times at which green cars go by are also given by a Poisson point process of rate $\lambda = 2$/hour. Similarly, the times at which blue cars go by are given by a Poisson point process of rate $\lambda = 2$/hour. Suppose that these three Poisson point processes are independent of each other.

(a) Write down the probability density function for the amount of time until the first red car goes by. **ANSWER:** Write $\lambda = 2$. Answer is $\lambda e^{-\lambda x} = 2e^{-2x}$ if $x > 0$, and 0 otherwise.

(c) Compute the expected amount of time until the first car of *any* of the three colors goes by. **ANSWER:** $1/6$ hour, or 10 minutes.

(c) Compute the probability that exactly three red cars go by during the first hour. **ANSWER:** $e^{-\lambda}\lambda^3/3! = 4/(3e^2)$.

(d) Compute the expected amount of time until at least one car of each of the three colors has gone by. (Hint: does this remind you of the radioactive decay problem?) **ANSWER:** $(\frac{1}{6} + \frac{1}{4} + \frac{1}{2}) = \frac{11}{12}$ hours, or 55 minutes.

2

18.440 Midterm 2 Solutions, Spring 2011

1. (20 points) Jill polishes her resume and sends it to 900 companies she finds on monster.com. Each company responds with probability .1 (independently of what all the other companies do). Let R be the number of companies that respond.

(a) Compute the expectation of R (give an exact number).

ANSWER: $900 \cdot .01 = 90$

(b) Compute the standard deviation of R (given an exact number).

ANSWER: $\sqrt{900 \cdot .1 \cdot .9} = 9$

(c) Use a normal random variable approximation to estimate the probability $P\{R > 113\} = P\{R \geq 114\}$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer.

ANSWER: 114 is $\frac{114-90}{9} = \frac{24}{9} = 8/3$ standard deviations above the mean. So $P\{R \geq 114\} \approx 1 - \Phi(8/3)$. (Could also replace 114 or 113 or by 113.5. Would the latter give a better approximation?)

2. (20 points) Let X_1, X_2, and X_3 be independent uniform random variables on $[0,1]$.

(a) Write $X = \max\{X_1, X_2, X_3\}$. Compute $P\{X \leq a\}$ for $a \in [0,1]$.

ANSWER:
$P\{X \leq a\} = F_X(a) = P\{X_1 \leq a\}P\{X_2 \leq a\}P\{X_3 \leq a\} = a^3$ for $a \in [0,1]$.

(b) Compute the probability density function for X on the interval $[0,1]$.

ANSWER: $f_X(a) = F'_X(a) = 3a^2$ for $a \in [0,1]$

(c) Compute the variance of the first variable X_1.

ANSWER: $\mathrm{Var}(X_1) = \frac{1}{12}$

(d) Compute the following covariance: $\mathrm{Cov}(X_1 + X_2, X_2 + X_3)$.

ANSWER: Using bilinearity of covariance, this is $\mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_1, X_3) + \mathrm{Cov}(X_2, X_2) + \mathrm{Cov}(X_2, X_3)$. All of these terms are zero except for $\mathrm{Cov}(X_2, X_2) = \frac{1}{12}$.

3. (10 points) Toss 3 fair coins independently.

(a) What is the *conditional* expected number of heads given that the first coin comes up heads?

ANSWER: Given first coin heads, each of second and third has .5 chance to be heads. Conditional expectation is 2.

(b) What is the *conditional* expected number of heads given that there are at least two heads among the three tosses.

ANSWER: A priori, have 3/8 chance to have 2 heads and 1/8 chance to have 3 heads. Conditioned on having either 2 or 3, there is a 3/4 chance to have 2 heads and a 1/4 chance to have three heads. So conditional expectation is $\frac{9}{4}$.

4. (10 points) Suppose that the amount of time until a certain radioactive particle decays is exponential with parameter λ. If there are three such particles, and their decay times are independent of each other, what is the expected amount of time until all three particles have decayed?

ANSWER: Time till first one decays is exponential with parameter 3λ. Subsequent time till next one decays is exponential with parameter 2λ. Subsequent time until last one decays is exponential with parameter λ. Expected sum of these three times is $\frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda}$.

5. (10 points) Let X be the number on a standard die roll (so X is chosen uniformly from the set $\{1, 2, 3, 4, 5, 6\}$).

(a) What is the moment generating function $M_X(t)$?

ANSWER: $M_X(t) = E[e^{Xt}] = \frac{1}{6}\big(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\big)$.

(b) Suppose that ten dice are rolled independently and Y is the sum of the numbers on all the dice. What is the moment generating function $M_Y(t)$?

ANSWER:
$M_Y(t) = (M_X(t))^{10} = \Big(\frac{1}{6}\big(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\big)\Big)^{10}$.

6. (20 points) On a certain hiking trail, it is well known that the lion, tiger, and bear attacks are independent Poisson point processes with respective λ values of .1/hour, .2/hour, and .3/hour. Let T be the number of hours until the first animal of any kind attacks.

(a) What is the probability that there are no lion attacks during the first hour?

ANSWER: $e^{-.1}$

(b) What is the probability density function for T?

ANSWER: Set of all attacks is a Poisson point process with $\lambda = .1 + .2 + .3 = .6$. So density is $f_T(t) = 0.6e^{-0.6t}$ for $t > 0$.

(c) What is the expected amount of time until the first tiger attack?

ANSWER: Expected amount of time until the first tiger attack is $1/.2 = 5$ hours.

(d) What is the distribution of the time until the fifth attack by any animal? (Give both the name of the distribution and an explicit formula.)

ANSWER: Sum of five independent exponentials of parameter .6 is a Gamma distribution with parameters $\alpha = 5$ and $\lambda = .6$. The density is $f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}$ for $x > 0$.

18.600 Midterm 1, Fall 2019, Solutions

1. (15 points) A *super-eruption* is a volcanic eruption producing more than 1000 cubic kilometers of deposits (and maybe enough ash to change the global climate for several years). Assume that each year (independently of all other years) there is a $\frac{1}{25{,}000}$ probability that there will be a single super-eruption somewhere in the world. (To simplify matters, assume that the probability of more than one super-eruption during the same year is zero.)

(a) Compute the expected number of super-eruptions that will take place during the next 100,000 years. **ANSWER:** Binomial with $n = 100,000$, $p = \frac{1}{25{,}000}$ has expectation $np = 4$.

(b) Use a Poisson approximation to estimate the probability that there will be exactly 3 super-eruptions during the next 100,000 years. **ANSWER:** Setting $\lambda = np = 4$, probability is approximately $e^{-\lambda}\lambda^k/k! = e^{-4}4^3/3!$.

(c) Use a Poisson approximation to estimate the probability that there will be *at least* one super-eruption at some point during the next 100 years. **ANSWER:** This is binomial with $n = 100$ and $p = \frac{1}{25{,}000}$, hence roughly Poisson with $\lambda = np = 1/250$. Chance of at least one super-eruption is one minus chance of zero. So roughly $1 - e^{-\lambda}\lambda^k/k!$ with $k = 0$. That is, $1 - e^{-1/250}$. If interested, note by Taylor that this is roughly $1 - (1 - 1/250) = 1/250$ to say somebody living 100 years has roughly $1/250$ chance of living through super-eruption.

2. (15 points) Two teams are playing a soccer game (in a league with no overtime or shootouts). The first team's score is a Poisson random variable X with parameter $\lambda_X = 1$. The second team's score is an independent Poisson random variable Y with parameter $\lambda_Y = 2$.

(a) Compute the probability that the game ends in a tie. That is, compute $P(X = Y)$. (You can leave your answer as an infinite sum.) **ANSWER:** $P\sum_{k=0}^{\infty} P(X = k, Y = k)$ is

$$\sum_{k=0}^{\infty}\big(e^{-\lambda_X}\lambda_X^k/k!\big)\big(e^{-\lambda_Y}\lambda_Y^k/k!\big) = \sum_{k=0}^{\infty}\big(e^{-1}/k!\big)\big(e^{-2}2^k/k!\big).$$

FYI, summing this on computer gives about 21 percent. Ties happen.

(b) Compute the probability the underdog team wins. That is, compute $P(X > Y)$. (You can leave your answer as a double infinite sum.) **ANSWER:** $\sum_{j=1}^{\infty}\sum_{k=0}^{j-1} P(X = j, Y = k)$ is

$$\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\big(e^{-1}/j!\big)\big(e^{-2}2^k/k!\big).$$

FYI, summing this on a computer gives about 18 percent. Underdog has a fighting chance!

(c) Compute the probability that exactly two goals are scored overall. That is compute $P(X + Y = 2)$. **ANSWER:** $P(X = 0, Y = 2) + P(X = 1, Y = 1) + P(X = 2, Y = 0)$ is

$$\big(e^{-1}/0!\big)\big(e^{-2}2^2/2!\big) + \big(e^{-1}/1!\big)\big(e^{-2}2^1/1!\big) + \big(e^{-1}/2!\big)\big(e^{-2}2^0/0!\big) = (2 + 2 + 1/2)/e^3 = 9/(2e^3)$$

3. (20 points) 14 students are taking a chemistry class, and the professor plans to assign each person a partner — so that there are 7 (unordered) partnerships with two people per partnership.

(a) How many ways are there to do that? **ANSWER:** If pairs were ordered (pair one, pair two, etc.) number would be $\binom{14}{2,2,2,2,2,2,2} = 14!/2^7$. Dividing by 7! gives answer $14!/(2^7 \cdot 7!)$. Alternatively, line people up in a row. First person has 13 choices for partner, next unpartnered person in row has 11 choices for partner, etc. So answer is $13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$.

(b) Two of the students are good friends and are hoping they will get to be partners. Assuming the professor chooses the partner division randomly (with all possible ways of forming the 7 partnerships being equally likely) what is the probability that they will be partners? **ANSWER:** 1/13 (since first student equally likely to be paired with any of 13 others)

(c) Suppose that 7 of the people are men and 7 are women. Let N be the number of partnerships with exactly one man and one woman and compute the expectation $E[N]$. (If it helps, you can write N_i for the random variable that is 1 if the ith female has a male partner and 0 otherwise.) **ANSWER:** Expectation additivity gives $\sum_{j=1}^{7} E[N_j] = 49/13$.

(d) Compute the expectation $E[N^2]$. **ANSWER:** Additivity of expectation gives $E\big[(\sum_{j=1}^{7} N_j)(\sum_{k=1}^{7} N_k)\big] = \sum_{j=1}^{7}\sum_{k=1}^{7} E[N_j N_k]$. The 7 terms with $j = k$ contribute 7/13 each, and the 42 with $j \neq k$ contribute (7/13)(6/11) each. Answer is $49/13 + 42(7/13)(6/11)$

4.(20 points) Compute the following:

(a) $\lim_{n\to\infty}(1 - \frac{1}{4n})^n$ **ANSWER:** Generally $\lim_{n\to\infty}(1 + x/n)^n = e^x$ so this is $e^{-1/4}$

(b) $\sum_{k=0}^{9}\Big(2^k 8^{9-k}\binom{9}{k}\Big)$ **ANSWER:** Binomial theorem gives $(2 + 8)^9 = 10^9$.

(c) $\sum_{k=0}^{\infty}\frac{1}{2^k \cdot k!}$ **ANSWER:** Generally $\sum_{k=0}^{\infty} x^k/k! = e^x$ so this is $e^{1/2}$.

(d) $\sum_{k=0}^{\infty}\Big((\frac{5}{6})^{k-1}(\frac{1}{6})k\Big)$ **ANSWER:** This is expectation of geometric random variable with $p = 1/6$ which is $1/p = 6$.

5. (15 points) A baking competition has ten contestants. The judges are allergic to most baking ingredients, so instead of tasting the food, they select three winners at random (with all possible three-person subsets of the 10 contestants being equally likely). Contestants Alice, Bob and Carol are good friends who are hoping to all be winners, but who feel it will be awkward if two of them are winners and the third one isn't. Let A be the event that Alice, Bob and Carol are all winners and let B be the event that *at least two* of these three people are winners.

(a) Compute $P(A)$. **ANSWER:** $1/\binom{10}{3}$

(b) Compute $P(B)$. **ANSWER:** Either all three are chosen or one of three left out and one of other seven chosen. So $1 + 3 \cdot 7 = 22$ possibilities. Answer is $22/\binom{10}{3}$

(c) Compute the conditional probability $P(A|B)$. **ANSWER:** $P(AB)/P(B) = P(A)/P(B) = 1/22$. Note: it may seem surprising that even though each person has a $3/10$ chance of winning separately, it is still the case that *given* that two win, the chance that all three win is only $1/22$. This is similar to the powerball problem (where the chance of matching all five white balls is surprisingly small compared to chance of matching 4 or 3).

6. (15 points) Janet thinks she might have a fever. Or maybe just a headache or a cold. She is not really sure. She prepares to take her temperature with a digital thermometer which reports Fahrenheit temperature (rounded to the nearest integer) and she thinks she will see one of the values in $\{98, 99, 100, 101, 102\}$ each with probability $1/5$. Let X be number she actually sees.

(a) Compute the variance $\text{Var}(X)$. Simplify your expression to give an exact value (i.e., an explicit rational number). **ANSWER:** $E[X] = 100$ so $\text{Var}(X) = E[(X-100)^2]$ which is $\frac{1}{5}(4+1+0+1+4) = 2$.

(b) Use the answer in (a) to compute $E(X^2)$. **ANSWER:** $2 = \text{Var}(X) = E[X^2] - E[X]^2 = E[X^2] - 10000$ so $E[X^2] = 10002$.

(c) After taking her temperature, regardless of what the thermometer shows, Janet plans to a roll a three-sided die (which takes values in $\{0, 1, 2\}$ each with equal probability). If N is the number that comes up, Janet will take N ibuprofen tablets. Compute the expectation $E\big[2N^3 + 3X^2\big]$. **ANSWER:** $E[N^3] = \frac{1}{3}(0+1+8) = 3$. Additivity of expectation gives answer: $2 \cdot 3 + 3 \cdot 10002 = 30012$.

18.600 Midterm 2, Fall 2019 Solutions

1. (20 points)

(a) Melissa is applying to 20 different out-of-state medical schools. Because of her excellent GPA/MCAT/essays, her chance of being accepted to each school is 1/20, and the decisions at the 20 schools are independent of each other. Using a Poisson approximation, estimate the probability that Melissa will be accepted to at least two of these schools. **ANSWER:** Number X of acceptances is roughly Poisson with parameter $\lambda = 20 \cdot \frac{1}{20} = 1$. Thus $P(X \geq 2) = 1 - P(X = 1) - P(X = 0) \approx 1 - e^{-\lambda}\lambda^1/1! - e^{-\lambda}\lambda^0/0! = 1 - 2/e \approx .26424$. **Remark:** If we compute the exact value using a binomial distribution, we get $P(X \geq 2) \approx .26416$, so the approximation is quite good.

(b) Jill is applying to 25 different out-of-state medical schools and has a 1/5 chance (independently) of being invited for an interview at each school. Let X be the number of medical schools at which she is invited to interview. Compute $E[X]$ and $\mathrm{Var}[X]$. **ANSWER:** The number of interviews is binomial with parameter $n = 25$ and $p = 1/5$. So $E[X] = np = 5$ and $\mathrm{Var}[X] = np(1-p) = 4$.

(c) Using a normal approximation, roughly approximate the probability that Jill is invited to interview at fewer than 2.5 schools. You may use the function

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

in your answer. **ANSWER:** Since the standard deviation of X is 2, the value 2.5 is 5/4 standard deviations below the mean. Hence the probability is approximately $\Phi(-5/4) \approx .10565$. **Remark:** The true probability is .098 which is pretty close.

2. (20 points) A room has four lightbulbs, each of which will burn out at a random time. Let X_1, X_2, X_3, X_4 be the burnout times, and assume they are independent exponential random variables with parameter $\lambda = 1$. Write

1. $X = X_1 + X_2 + X_3 + X_4$.
2. $Y = \min\{X_1, X_2, X_3, X_4\}$, i.e., Y is time when first bulb burns out.
3. $Z = \max\{X_1, X_2, X_3, X_4\}$, i.e., Z is time when last bulb burns out.

Compute the following:

(a) The probability density function f_X. **ANSWER:** This is a Gamma distribution with parameters $\lambda = 1$ and $n = 4$. So $f_X(x) = x^3 e^{-x}/3!$ for $x \in [0, \infty)$.

(b) The probability density function f_Y. **ANSWER:** The minimum of four exponentials of parameter 1 is exponential with parameter 4. Hence $f_Y(x) = 4e^{-4x}$ for $x \in [0, \infty)$.

(c) The expectation $E[Z]$. **ANSWER:** This is basically the radioactive decay problem from lecture. Answer is $1/4 + 1/3 + 1/2 + 1$.

(d) The covariance $\mathrm{Cov}(Y, Z)$. (Hint: use memoryless property.) **ANSWER:** The memoryless property implies that Y and $Z - Y$ are independent and hence $\mathrm{Cov}(Y, Z) = \mathrm{Cov}(Y, Y + (Z - Y)) = \mathrm{Cov}(Y, Y) = \mathrm{Var}(Y)$. Since Y is exponential with parameter $\lambda = 4$ its variance is $1/\lambda^2 = 1/16$.

3. (20 points) Five applicants are applying for a job, and an interviewer gives each applicant a score between 0 and 1. Call these scores $X_1, X_2, \ldots, X_5$ and assume that they are i.i.d. uniform random variables on $[0, 1]$. The top applicant has score $Y = \max\{X_1, X_2, \ldots, X_5\}$, and the second to the top has score Z, which we define to be the *second* largest of the X_i. Compute the following:

(a) The cumulative distribution function $F_Y(r)$ for $r \in [0, 1]$. **ANSWER:** $P(Y \le r) = P(\max\{X_1, X_2, \ldots, X_5\}) \le r) = P(X_1 \le r, X_2 \le r, \ldots) = P(X_1 \le r)^5 = r^5$.

(b) The density function f_Y. **ANSWER:** $f_Y(r) = F'_Y(r) = 5r^4$ for $r \in [0, 1]$ (and zero if $r \notin [0, 1]$).

(c) The density function f_Z and the value $E[Z]$. **NOTE:** If you remember what this means, you may use the fact that a Beta (a, b) random variable has expectation $a/(a + b)$ and density $x^{a-1}(1 - x)^{b-1}/B(a, b)$, where $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$. **ANSWER:** The ordering of candidates is independent of the set of scores obtained by the candidates. This means that the density of Z is the same that of a uniform random variable conditioned on three people being smaller, one being larger. This is a Beta (a, b) random variable with $a - 1 = 3$ and $b - 1 = 1$. So it comes to $x^3(1 - x)/B(4, 2) = 20x^3(1 - x)$ and $E[X] = 4/(4 + 2) = 2/3$.

(d) The probability $P(X_2 > 2X_1)$ (i.e., probability second candidate's score is more than than double first candidate's score). **ANSWER:** Note that joint density $f_{X_1,X_2}(x, y)$ is 1 on the unit square $[0, 1]^2$ and zero elsewhere. Therefore the probability is the area of the subset of $[0, 1]^2$ where $y > 2x$, which comes to $1/4$. So the answer is $1/4$.

4. (15 points) Let X and Y be independent random variables with density function given by $\frac{1}{\pi(1+x^2)}$.

(a) Compute $P(X < 1)$. **ANSWER:** X is a Cauchy random variable, so the answer is $3/4$ by our spinning flashlight story. Recall that in that story, we draw a line from $(0, 1)$ with a uniformly chosen angle and its intersection with $\mathbb{R}$ is a Cauchy random variable. The angle range corresponding to $(-\infty, 1)$ is $3/4$ of the total range, so the answer is $3/4$.

(b) Compute the probability density function for the random variable $Z = (X - Y)/2$. **ANSWER:** If Y is Cauchy then $-Y$ is also Cauchy. The average of two independent Cauchy random variables it itself Cauchy, so the answer is $\frac{1}{\pi(1+x^2)}$.

(c) Compute $E[e^{-X^2-Y^2}]$. You can leave your answer as a double integral—no need to evaluate it explicitly. **ANSWER:** $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi(1+x^2)}\frac{1}{\pi(1+y^2)}e^{-x^2-y^2}dxdy$

5. (10 points) Let $X_1, X_2, X_3, \ldots, X_{10}$ be the outcomes of independent standard die rolls—so each takes one of the values in $\{1,2,3,4,5,6\}$, each with equal probability. Write $S = X_1 + X_2 + \ldots + X_{10}$. Compute the following:

(a) The moment generating function $M_{X_1}(t)$. **ANSWER:** $M_{X_1}(t) = E[e^{tX_1}] = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$.

(b) The moment generating function $M_S(t)$.**ANSWER:** The moment generating function of a sum of independent random variables is the product of the moment generating functions of the individual random variables. Hence $M_S(t) = \left(\frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})\right)^{10}$.

6. (15 points) Let X and Y be be random variables with joint density function $f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$. Write $Z = X + Y$.

(a) Compute $E[XY]$. **ANSWER:** X and Y are independent normal random variables, each with mean zero and variance one. Since they are independent we have $E[XY] = E[X]E[Y] = 0$. Alternatively, write $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\frac{1}{2\pi}e^{-(x^2+y^2)/2}dxdy$. Then there are various ways to argue by symmetry that this must be zero.

(b) Compute the conditional expectation $E[Y|Z]$. That is, express the random variable $E[Y|Z]$ in terms of Z. **ANSWER:** We have $Z = E[Z|Z] = E[X|Z] + E[Y|Z]$. Since $E[X|Z]$ and $E[Y|Z]$ are the same by symmetry, the answer must be $Z/2$.

(c) Compute the probability $P(X^2 + Y^2 \leq 4)$. **ANSWER:** This can be computed using polar coordinates. The integral becomes $\int_0^2\int_0^{2\pi}\frac{1}{2\pi}e^{-r^2/2}rd\theta dr = \int_0^2 e^{-r^2/2}rdr = -e^{-r^2/2}|_0^2 = -e^{-2} - (-1) = 1 - e^{-2} \approx .86466$

18.440 Practice Final Exam: 100 points
Carefully and clearly show your work on each problem (without writing anything that is technically not true) and put a box around each of your final computations. This practice exam deals only with the portion of the course after the second midterm, while the actual final exam will cover the entire course.

I. (20 points) Let X be a random variable with finite mean μ and variance σ^2. The central limit theorem states that if X_i are independent instances of X, then the quantities

$$T_n := \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}}$$

converge in law to a standard normal random variable. That is

$$\lim_{n\to\infty} P\{T_n < a\} = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Prove, using the following steps, that the moment generating functions of the T_n (assuming they exist and are well defined) converge to the moment generating function of a standard normal random variable.

1. Explain why it suffices to consider mean zero, variance one random variables X. You may then assume below that $\mu = 0$ and $\sigma^2 = 1$.
2. Let $M_X(t) = \mathbb{E}e^{tX}$. Show that $M_{T_n}(t) = M_X(t/\sqrt{n})^n$. Show also that $M''_X(0) = 1$ and $M'_X(0) = 0$.
3. Let $L_X(t) = \log M_X(t)$ and show that $L_{T_n}(t) = nL_X(t/\sqrt{n})$.
4. Show that $L'_X(0) = 0$ and $L''_X(0) = \sigma^2 = 1$.
5. Show that if N is a standard normal random variable then $L_N(t) = t^2/2$.
6. Use Taylor approximation to show that, for each fixed t, $\lim_{n\to\infty} L_{T_n}(t) = L_N(t)$. [Recall that Taylor approximation states that if R is twice differentiable at zero then $R(t) = R(0) + R'(0)t + R''(0)t^2/2 + o(t^2)$, for small t.]

Solution: Check the central limit theorem derivation in the textbook and slides.

II. (15 points) State the strong and weak laws of large numbers and explain why the strong law of large numbers implies the weak law of large numbers.

Solution: Check the law of large number explanations in the textbook and slides.

III. (10 points) Let X and Y be the outcomes of independent die rolls, and let $Z = X + Y$. (Assume that these are 3-sided dice, taking the values 1, 2, and 3 with equal probability.) Compute the following:

1. The entropies $H(X)$, $H(Y)$, $H(Z)$, and $H(X, Y)$.
2. Show that $H(X, Y) = H(X, Z)$.
3. Verify by explicitly calculating both sides that
   $H(X, Z) = H(Z) + H_Z(X)$.

Solution:

1. $H(X) = H(Y) = -\log \frac{1}{3} = \log 3$ and
   $H(X, Y) = H(X) + H(Y) = 2 \log 3$.
2. Like the pair (X, Y), the pair (X, Z) takes 9 values, all with equal probability. So $H(X, Z) = -\log \frac{1}{9} = 2 \log 3$.
3. The variable Z takes 5 values: $2, 3, 4, 5$ and 6 with probabilities $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, $\frac{2}{9}$, and $\frac{1}{9}$. Now,

$$\begin{aligned} H_Z(X) &= \sum_{j=2}^{6} P\{Z = j\} H_{Z=j}(X) \\ &= \frac{1}{9} \log 1 + \frac{2}{9} \log 2 + \frac{3}{9} \log 3 + \frac{2}{9} \log 2 + \frac{1}{9} \log 1 \\ &= \frac{4}{9} \log 2 + \frac{1}{3} \log 3. \end{aligned}$$

And

$$
\begin{aligned}
H(Z) &= \frac{1}{9}(-\log\frac{1}{9}) + \frac{2}{9}(-\log\frac{2}{9}) + \frac{3}{9}(-\log\frac{3}{9}) + \frac{2}{9}(-\log\frac{2}{9}) + \frac{1}{9}(-\log\frac{1}{9}) \\
&= \frac{4}{9}\log 3 + \frac{4}{9}(2\log 3 - \log 2) + \frac{1}{3}\log 3 \\
&= \frac{5}{3}\log 3 - \frac{4}{9}\log 2.
\end{aligned}
$$

So indeed $H(Z) + H_Z(X) = 2\log 3 = H(X, Z)$.

IV. (10 points) Elaine's trusty old car has three states: broken (in Elaine's possession), working (in Elaine's possession), and in the shop. Denote these states B, W, and S.

1. Each morning the car starts out B, it has a .5 chance of staying B and a .5 chance of switching to S by the next morning.
2. Each morning the car starts out S, it has a .75 chance of staying S and a .25 chance of switching to W by the next morning.
3. Each morning the car starts out W, it has a .75 chance of staying W, and a .25 chance of switching to B by the next morning.

Over the long term, on what percentage of days does the car start out in state W?

Solution: Ordering the states B, W, S, we may write the Markov chain matrix as

$$
M = \begin{pmatrix} .5 & 0 & .5 \\ .25 & .75 & 0 \\ 0 & .25 & .75 \end{pmatrix}.
$$

We find the stationarity probability vector $\pi = (\pi_B, \pi_W, \pi_S) = (.2, .4, .4)$ by solving $\pi M = \pi$ (with components of π summing to 1). So $\pi_W = .4$.

V. (20 points)

1. Let $X_1, X_2, \ldots$ be independent random variables with expectation one. In which of the cases below is the sequence Y_i necessarily a martingale?

   (a) $Y_n = X_n - 1$ **NO**
   (b) $Y_n = X_n^2 - 1$ **NO**
   (c) $Y_n = 7$ **YES**
   (d) $Y_n = n - \sum_{i=1}^n X_i$ **YES**
   (e) $Y_n = n - \sum_{i=1}^n i^2 X_i$ **NO**
   (f) $Y_n = \prod_{i=1}^n X_i$ **YES**
   (g) $Y_n = \sum_{i=1}^n X_n^2 - 2n$ **NO**
   (h) $Y_n = \prod_{i=1}^n X_i^2$ **NO**

2. Let Y_n be a martingale. Which of the following is necessarily a stopping time for Y_n?

   (a) The smallest n for which $Y_n > 17$. **YES (assuming it's almost surely finite)**
   (b) The largest n for which $Y_n = 17$. **NO**
   (c) The smallest value n for which $Y_n = Y_{n+1}$. **NO**

3. Give an example of a martingale M_n and a stopping time T such that with probability one, $M_0 = 1$ but $\mathbb{E}M_T = 0$. Explain why your example does not violate the optional stopping theorem.
   SOLUTION: Write $Y_n = 1 + \sum_{i=1}^n X_i$ where each X_i is independently -1 with probability $.5$ and 1 with probability $.5$ and let T be the smallest n for which that $Y_n = 0$. This does not violate the optional stopping theorem because there is no bound on how large Y_n can become before time T.

VI. (15 points) This problem asks you to complete a few calculations that arise in the derivation of the Black-Scholes formula. Let X be a mean zero normal random variable with variance σ^2. Compute the following (you may use the function $\phi(a) := \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ in your answers):

1. $\mathbb{E}e^X$.

2. $\mathbb{E}e^X 1_{X>A}$ for a fixed constant A.

3. $\mathbb{E}1_{X>A}$.

4. $\mathbb{E}G(e^X)$ where $G(a) := \begin{cases} 0 & a < K \\ a - K & a \geq K \end{cases}$.

5. How do the answers above change if X is a normal random variable with mean $-\sigma^2/2$ and variance σ^2?

6. BONUS: Can you explain how the answers above imply the Black-Scholes formula for the price of a European call option on a stock whose risk neutral probability density is log-normal?

Solution: Check the homework solutions on the Black Scholes derivation (as well as the slides on Black Scholes and the review problem in the last lecture).

VII. (10 points) Assume zero dividends/interest and that the strike price for a European call option on a stock at a fixed maturity date T and strike price K is given by $C(K)$. Suppose that $C(K) = e^{-K}$ for all $K \geq 0$, and answer the following:

1. What must the present value of the stock be?

2. What is the risk neutral probability that the stock price will lie in the interval $[5, 10]$ at maturity?

3. What is the present value of a contract that pays X^2 at maturity if the stock price at maturity is X?

Solution: Let X be the value of the stock at time T. Let f_X be the risk neutral probability density function and F_X the corresponding cumulative distribution function. Derive (or recall from lecture) the important facts

1. $C'(x) = F_X(x) - 1$

2. $C''(x) = f_X(x)$.

Thus,

1. $F_X(x) = 1 - e^{-x}$ on $[0, \infty)$
2. $f_X(x) = e^{-x}$ on $[0, \infty)$

Assuming no arbitrage, the present value of the stock is $E[X] = 1$, the risk neutral probability it will belong to $[5, 10]$ is $F_X(10) - F_X(5) = e^{-5} - e^{-10}$ and the present value of the contract paying X^2 at maturity is the expectation $E[X^2] = 2$. We remark that the form of f_X we find here (the exponential density function) is not typical for the risk neutral probability of an ordinary stock (more common is for f_X to be approximately log-normal, as in Black-Scholes theory, but with fatter tails).

BASIC DISCRETE RANDOM VARIABLES X (using $q = 1 - p$)

1. **Binomial** (n, p)**:** $p_X(k) = \binom{n}{k} p^k q^{n-k}$ and $E[X] = np$ and $\mathrm{Var}[X] = npq$.
2. **Poisson with mean** λ**:** $p_X(k) = e^{-\lambda}\lambda^k/k!$ and $\mathrm{Var}[X] = \lambda$.
3. **Geometric** p**:** $p_X(k) = q^{k-1}p$ and $E[X] = 1/p$ and $\mathrm{Var}[X] = q/p^2$.
4. **Negative binomial** (n, p)**:** $p_X(k) = \binom{k-1}{n-1} p^n q^{k-n}$, $E[X] = n/p$, $\mathrm{Var}[X] = nq/p^2$.

BASIC CONTINUOUS RANDOM VARIABLES X

1. **Uniform on** $[a, b]$**:** $f_X(x) = 1/(b - a)$ on $[a, b]$ and $E[X] = (a + b)/2$ and $\mathrm{Var}[X] = (b - a)^2/12$.
2. **Normal with mean** μ **variance** σ^2**:** $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$.
3. **Exponential with rate** λ**:** $f_X(x) = \lambda e^{-\lambda x}$ (on $[0, \infty)$) and $E[X] = 1/\lambda$ and $\mathrm{Var}[X] = 1/\lambda^2$.
4. **Gamma** (n, λ)**:** $f_X(x) = \frac{\lambda}{\Gamma(n)} e^{-\lambda x}(\lambda x)^{n-1}$ (on $[0, \infty)$) and $E[X] = n/\lambda$ and $\mathrm{Var}[X] = n/\lambda^2$.
5. **Cauchy:** $f_X(x) = \frac{1}{\pi(1+x^2)}$ and both $E[X]$ and $\mathrm{Var}[X]$ are undefined.
6. **Beta** (a, b)**:** $f_X(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$ on [0,1] and $E[X] = a/(a + b)$.

MOMENT GENERATING / CHARACTERISTIC FUNCTIONS

1. **Discrete:** $M_X(t) = E[e^{tX}] = \sum_x p_X(x)e^{tx}$ and $\phi_X(t) = E[e^{itX}] = \sum_x p_X(x)e^{itx}$.
2. **Continuous:** $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} f_X(x)e^{tx}dx$ and $\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} f_X(x)e^{itx}dx$.
3. **If** X **and** Y **are independent:** $M_{X+Y}(t) = M_X(t)M_Y(t)$ and $\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$.
4. **Affine transformations:** $M_{aX+b}(t) = e^{bt}M_X(at)$ and $\phi_{aX+b}(t) = e^{ibt}\phi_X(at)$
5. **Some special cases:** if X is normal $(0, 1)$, complete-the-square trick gives $M_X(t) = e^{t^2/2}$ and $\phi_X(t) = e^{-t^2/2}$. If X is Poisson λ get "double exponential" $M_X(t) = e^{\lambda(e^t-1)}$ and $\phi_X(t) = e^{\lambda(e^{it}-1)}$.

STORIES BEHIND BASIC DISCRETE RANDOM VARIABLES

1. **Binomial** (n, p)**:** sequence of n coins, each heads with probability p, have $\binom{n}{k}$ ways to choose a set of k to be heads; have $p^k(1 - p)^{n-k}$ chance for each choice. If $n = 1$ then $X \in \{0, 1\}$ so $E[X] = E[X^2] = p$, and $\mathrm{Var}[X] = E[X^2] - E[X]^2 = p - p^2 = pq$. Use expectation/variance additivity (for independent coins) for general n.
2. **Poisson** λ**:** $p_X(k)$ is $e^{-\lambda}$ times kth term in Taylor expansion of e^λ. Take n very large and let Y be # heads in n tosses of coin with $p = \lambda/n$. Then $E[Y] = np = \lambda$ and $\mathrm{Var}(Y) = npq \approx np = \lambda$. Law of Y tends to law of X as $n \to \infty$, so not surprising that $E[X] = \mathrm{Var}[X] = \lambda$.
3. **Geometric** p**:** Probability to have no heads in first $k - 1$ tosses and heads in kth toss is $(1 - p)^{k-1}p$. If you think about repeatedly a tossing coin forever, it makes intuitive sense that if you have (in expectation) p heads per toss, then you should need (in expectation) $1/p$ tosses to get a heads. Variance formula requires calculation, but not surprising that $\mathrm{Var}(X) \approx 1/p^2$ when p is small (when p is small X is kind like of exponential random variable with $p = \lambda$) and $\mathrm{Var}(X) \approx 0$ when q is small.

4. **Negative binomial** (n, p)**:** If you want nth heads to be on the kth toss then you have to have $n-1$ heads during first $k-1$ tosses, and then a heads on the kth toss. Expectations and variance are n times those for geometric (since were're summing n independent geometric random variables).

STORIES BEHIND BASIC CONTINUUM RANDOM VARIABLES

1. **Uniform on** $[a, b]$**:** Total integral is one, so density is $1/(b-a)$ on $[a, b]$. $E[X]$ is midpoint $(a+b)/2$. When $a = 0$ and $b = 1$, w know $E[X^2] = \int_0^1 x^2 dx = 1/3$, so that $\text{Var}(X) = 1/3 - 1/4 = 12$. Stretching out random variable by $(b-a)$ multiplies variance by $(b-a)^2$.

2. **Normal** (μ, σ^2)**:** when $\sigma = 1$ and $\mu = 0$ we have $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The function $e^{-x^2/2}$ is (up to multiplicative constant) *its own Fourier transform*. The fact that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ came from a cool and hopefully memorable trick involving passing to two dimensions and using polar coordinates. Once one knows the $\sigma = 1, \mu = 0$ case, general case comes from stretching/squashing the distribution by a factor of σ and then translating it by μ.

3. **Exponential** λ**:** Suppose $\lambda = 1$. Then $f_X(x) = e^{-x}$ on $[0, \infty)$. Remember the integration by parts induction that proves $\int_0^\infty e^{-x} x^n = n!$. So $E[X] = 1! = 1$ and $E[X^2] = 2! = 2$ so that $\text{Var}[X] = 2 - 1 = 1$. We think of λ as rate ("number of buses per time unit") so replacing 1 by λ multiplies wait time by $1/\lambda$, which leads to $E[X] = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.

4. **Gamma** (n, λ)**:** Again, focus on the $\lambda = 1$ case. Then f_X is just $e^{-x} x^{n-1}$ times the appropriate constant. Since X represents time until nth bus, expectation and variance should be n (by additivity of variance and expectation). If we switch to general λ, we stretch and squash f_X (and adjust expecation and variance accordingly).

5. **Cauchy:** If you remember that $1/(1 + x^2)$ is the derivative of arctangent, you can see why this corresponds to the spinning flashlight story and where the $1/\pi$ factor comes from. Asymptotic $1/x^2$ decay rate is why $\int_{-\infty}^{\infty} f_X(x) dx$ is finite but $\int_{-\infty}^{\infty} f_X(x) x dx$ and $\int_{-\infty}^{\infty} f_X(x) x^2 dx$ diverge.

6. **Beta** (a, b)**:** $f_X(x)$ is (up to a constant factor) the probability (as a function of x) that you see $a-1$ heads and $b-1$ tails when you toss $a+b-2$ p-coins with $p = x$. So makes sense that if Bayesian prior for p is uniform then Bayesian posterior (after seeing $a-1$ heads and $b-1$ tails) should be proportional to this. The constant $B(a, b)$ is by definition what makes the total integral one. Expectation formula (which you computed on pset) suggests rough intuition: if you have uniform prior for fraction of people who like new restaurant, and then $(a-1)$ people say they do and $(b-1)$ say they don't, your revised expectation for fraction who like restaurant is $\frac{a}{a+b}$. (You might have guessed $\frac{(a-1)}{(a-1)+(b-1)}$, but that is not correct — and you can see why it would be wrong if $a-1 = 0$ or $b-1 = 0$.)