cylinders satisfy the PDE

$$\frac{\partial^2 P}{\partial t^2} = c^2 \frac{\partial^2 P}{\partial x^2}, \quad 0<x<L, \ t>0$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \ t>0.$$

P - pressure, u - horizontal displacement of air molecules, and c - speed of sound in the ambient air.

Seperating variables & looking for solutions of the form $u(x,t) = v(x)\,w(t)$ leads to the following ODE s

$$\frac{d^2 v}{dx^2} = \lambda v, \quad \frac{d^2 w}{dt^2} = \lambda c^2 w$$

$$0<x<L \qquad\qquad t>0$$

Solving leads to a family of solutions $u_n(x,t) = v_n(x)\,w_n(t)$ for different values λ_n, which all satisfy the boundary conditions.

$$s^2 = \lambda \qquad\qquad s^2 = \lambda c^2$$

$$s = \pm\sqrt{\lambda} \qquad\qquad s = \pm\sqrt{\lambda}\,c$$

$$w = c_3 e^{\sqrt{\lambda} c} + c_4 e^{-\sqrt{\lambda} c}$$

$$v = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$$

Case: i) $\lambda > 0$, case (ii) $\lambda = 0$, case (iii) $\lambda < 0$.

From case : iii) → Ref previous problem

$$\lambda_n = -\frac{n^2\pi^2}{L^2} = -\left(\frac{n\pi}{L}\right)^2$$

$$\phi(t) = c_1 \sin\left(\frac{n\pi c}{L} t\right) + c_2 \cos\left(\frac{n\pi c}{L} t\right)$$

$$x(x) = c_1 \sin\left(\frac{n\pi x}{L}\right) + c_2 \cos\left(\frac{n\pi x}{L}\right)$$

$$\frac{d^2v}{dx^2} = \lambda v \qquad\qquad \frac{d^2w}{dt^2} = \lambda c^2 w$$

case: i) $\lambda > 0$

$$\frac{d^2v}{dx^2} = \lambda v$$

$s^2 = \lambda$

$s = \pm\sqrt{\lambda}$

$P(0,t) = 0$

$P(L,t) = 0$

$v = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$c_1 + c_2 = 0$

$v(L,t) = 0 = c_1 e^{\sqrt{\lambda}L} + c_2 e^{-\sqrt{\lambda}L}$

$\therefore \boxed{c_1 = c_2 = 0}$

(trivial solution)

case: ii) $\lambda = 0$

$s^2 = 0 \Rightarrow s = 0, 0$

$c_1 + c_2 x = 0$

$\boxed{c_1 = 0}$

$c_2 L = 0$

$c_2 = 0$

(trivial solution)

$\lambda < 0$

$s^2 = -\lambda$

$s = \pm i\sqrt{\lambda}$

$v = c_1 \cos(\sqrt{-\lambda}x) + c_2 \sin(\sqrt{-\lambda}x)$

$P(0,t) = 0, \quad P(L,t) = 0.$

$\therefore \lambda = -\frac{n^2\pi^2}{L^2}$

$\boxed{\therefore c_1 = 0}$ $\quad \boxed{c_2 = 0}$

$$v(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

our boundary condition: $\frac{\partial u}{\partial x}(0,t) = 0$

$\frac{\partial u}{\partial x}(L,t) = 0.$

$$w(t) = a\cos\left(\frac{n\pi c}{L}t\right) + b\sin\left(\frac{n\pi c}{L}t\right)$$

suppose,

$$u(x,0) = \frac{2x}{L} - 1, \qquad 0 < x < L$$

Let the initial velocity of displacement be zero for convenience.

The general solution takes the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(w_n t) + b_n \sin(w_n t)) \cos(\alpha_n x)$$

Determine what type of periodic extension is needed of this initial condition to solve for the co-efficients in the Fourier series. Then use Initial condt.

Hint: Triangular wave

$T(z)$ of period 2π has Fourier series

$$T(z) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nz)}{n^2}$$

Sawtooth wave $W(z)$ of period 2π has Fourier series

$$W(z) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin(nz)}{n}$$

Previous problem

Solu:

$$u(x,t) = \sum_{n \geq 0} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \right] +$$

$$\sum_{n \geq 0} \left[b_n \sin\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$u(x,0) = \sum_{n \geq 0} \left[a_n \cos\left(\frac{n\pi c}{L} (0)\right) \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$+ \sum_{n \geq 1} \left[b_n (0) \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$\frac{2x}{L} - 1 = \sum_{n \geq 0} \left[a_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{2}{L} \int_0^L \left(\frac{2x}{L} - 1\right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= \frac{2}{L} \left(\frac{L\,(2\sin(n\pi) - n\pi \cos(n\pi) - n\pi}{n^2 \pi^2} \right)$$

$$= \frac{2}{2}\left(\frac{2\sin(n\pi) - \pi n\cos(\pi n) - \pi n}{n^2\pi^2}\right)$$

$\boxed{\begin{matrix}\cos\pi = -1 \\ \cos 2\pi = 1\end{matrix}}$

odd:

$$2\left(\frac{0 + \pi n - \pi n}{n^2\pi^2}\right) = 0$$

even:

$$2\left(\frac{-\pi n - \pi n}{n^2\pi^2}\right) = \frac{-4\pi n}{n^2\pi^2} = \frac{-4}{n\pi}$$

Our problem

Incase:

$$u(x,t) = \sum_{n\geq 0}\left[a_n \cos\left(\frac{n\pi c}{L}t\right)\cos\left(\frac{n\pi x}{L}\right)\right] +$$

$$\sum_{n\geq 1}\left[b_n \sin\left(\frac{n\pi c}{L}t\right)\cdot\cos\left(\frac{n\pi x}{L}\right)\right]$$

$$u(x,0) = \sum_{n\geq 0}\left[a_n \cos\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

$$= \frac{2}{L}\left(L\,\frac{(\pi n \sin(\pi n) + 2\cos(\pi n) - 2)}{\pi^2 n^2}\right)$$

$$= 2\left(\frac{(\pi n\sin(\pi n)) + 2\cos(\pi n) - 2}{\pi^2 n^2}\right)$$

even:

$$2\left(\frac{0 + 2 - 2}{n^2\pi^2}\right) = 0$$

odd:

$$2\left(\frac{0 - 2 - 2}{n^2\pi^2}\right) = -\frac{8}{n^2\pi^2}$$

wave equation in MATLAB

Simple numerical method:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = f(t)$, $u(L,t) = g(t)$ $\longrightarrow$ Boundary conditions.

$u(x,0) = v(x)$, $\frac{\partial u}{\partial t}(x,0) = S(x)$ $\longrightarrow$ Initial conditions

we will use a centred time & space numerical scheme.

u_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$. Then the discrete (centered) second time derivative is

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\Delta t^2} + \text{terms of order } (\Delta t^2) \text{ and higher.}$$

and the discrete Second space derivative is

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{\Delta x^2} + \text{terms of order } (\Delta x^2) \text{ and higher.}$$

substituting the discrete time & space derivatives into the wave equation gives

$$\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{\Delta t^2} = c^2\left(\frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{\Delta x^2}\right)$$

$$u_j^{i+1} = \frac{c^2\Delta t^2}{\Delta x^2}\left(u_{j+1}^i - 2u_j^i + u_{j-1}^i\right) + 2u_j^i - u_j^{i-1}$$

$$u_j^{i+1} = r^2 u_{j+1}^i + 2(1-r^2)u_j^i + r^2 u_{j-1}^i - u_j^{i-1}$$

$$r = \frac{c\,\Delta t}{\Delta x}$$

In matrix notation:

$$\begin{pmatrix} u_1^{q+1} \\ u_2^{q+1} \\ \vdots \\ u_{N-1}^{q+1} \\ u_N^{q+1} \end{pmatrix} = \begin{bmatrix} 2(1-r^2) & r^2 & & & \\ r^2 & 2(1-r^2) & r^2 & & \\ & & \ddots & & \\ & & r^2 & 2(1-r^2) & r^2 \\ & & & r^2 & 2(1-r^2) \end{bmatrix} \begin{bmatrix} u_1^q \\ u_2^q \\ u_3^q \\ \vdots \\ u_{N-1}^q \\ u_N^q \end{bmatrix} - \begin{bmatrix} u_1^{q-1} \\ u_2^{q-1} \\ \vdots \\ u_{N-1}^{q-1} \\ u_N^{q-1} \end{bmatrix}$$

where at each time step q we impose the boundary conditions $u_1^q = f(q\Delta t)$ and $u_N^q = g(q\Delta t)$. Note that to compute u^{q+1}, we need information about u^q and u^{q-1}. So, how do we start the method (ensuring that our error has order less than Δt^2)?

$$\frac{u(x,\Delta t) - u(x,0)}{\Delta t} = \frac{\partial u}{\partial t}(x,0) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^2)$$

$$u(x,\Delta t) = u(x,0) + \Delta t \frac{\partial u}{\partial t}(x,0) + \frac{c^2 \Delta t^2}{2}\frac{\partial^2 u}{\partial x^2}(x,0) + O(\Delta t^2)$$

$\boxed{q\Delta t = (q=1) = \Delta t}$

Discretizing gives

$$u_g^1 = v_g + \Delta t\, s_g + \frac{r^2}{2}\left(v_{g+1} - 2v_g + v_{g-1}\right)$$

Condition for numerical stability:

$$\frac{c\Delta t}{\Delta x} \le 1.$$

$u(x,0) = v_g$

$u(x+1,0) = v_{g+1}.$

For a fixed real number c, how many solutions to $\ddot{y} = cy$ satisfy the conditions $y(0)=0$, $y(1)=1$.

Soln:

$c>0$

$s^2 = c$

$s = \pm\sqrt{c}$

$y = c_1 e^{\sqrt{c}\,t} + c_2 e^{-\sqrt{c}\,t}$

$y(0)=0$ $\qquad$ $y(1)=1$

$\boxed{0=a+b}$ $\qquad$ $1=ae^{\sqrt{c}}+be^{-\sqrt{c}}$

$\therefore$ Det $\begin{pmatrix} 1 & 1 \\ e^{\sqrt{c}} & e^{-\sqrt{c}} \end{pmatrix} \neq 0$, there is a unique

solution.

case:2 $c=0$:

$s=0,0$

$y=a+bt$

$\boxed{0=a}$

$1=a+b$

$\boxed{b=1}$

which has a unique solution.

case: 3

$c<0$, $\quad c=-\omega^2 \quad (\omega>0)$

$s=\pm i\omega$

$\therefore \quad y=a\cos\omega t+b\sin\omega t$

$\boxed{0=a}$

$1=b\sin\omega$

$\therefore \boxed{\sin\omega\neq 0}$

$\sin\omega=0$

$\boxed{\text{when} \quad \omega=n\pi}$

for $\omega=n\pi \rightarrow$ no solution. If $\sin\omega\neq 0$, there is a unique solution.

Conclusion: If $c=-(n\pi)^2$ for some +ve integer n, there is no solution; otherwise there is exactly one solution.

2) For a fixed real number c, how many solutions to $\ddot{y}=cy$ satisfy the conditions $y(0)=0$, $y'(1)=0$.

$c>0$ $\ddot{y}=cy$

$y=ae^{\sqrt{c}t}+be^{-\sqrt{c}t}$

$a+b=0 \rightarrow ①.$ $\therefore$ $\boxed{b=-a}$

$\dot{y}=a\sqrt{c}e^{\sqrt{c}t}-b\sqrt{c}e^{-\sqrt{c}t}$

$\dot{y}(1)=a\sqrt{c}e^{\sqrt{c}}-b\sqrt{c}e^{-\sqrt{c}}$

$0=a\sqrt{c}e^{\sqrt{c}}-b\sqrt{c}e^{-\sqrt{c}}$

$\therefore a\sqrt{c}e^{\sqrt{c}}+a\sqrt{c}e^{-\sqrt{c}}=0$

$e^{\sqrt{c}}+e^{-\sqrt{c}}=0$ (or) $\boxed{a=0}$

$\therefore \boxed{e^{\sqrt{c}}+e^{-\sqrt{c}}\neq 0}$

$\therefore$ $a=0$ is the only case that holds. There is only one solution.

case:2: $c=0$.

$y=a+bt$

$\boxed{0=a}$

$\dot{y}=b$

$\dot{y}(1)=b$

$\boxed{0=b}$

Not a valid solution.

$\therefore$ Trivial solution (a,b) is $(0,0)$. $y(t)$ has a unique solution 0.

case:3: $c<0$

$c=-\omega^2$ $(\omega>0)$

$y=a\cos\omega t+b\sin\omega t$

$y(0)=a$

$\boxed{0=a}$

$y'=-a\sin\omega t\cdot\omega+b\omega\cos\omega t$

$y'(1)=b\omega\cos\omega$

$0=b\omega\cos\omega$

$\cos\omega\neq 0$ (if $\omega\neq(2n+1)\frac{\pi}{2}$.

n - +ve integer.

$\therefore$ Zero solution occurs when $b=0$.

There is one solution, the zero solution when $b=0$;

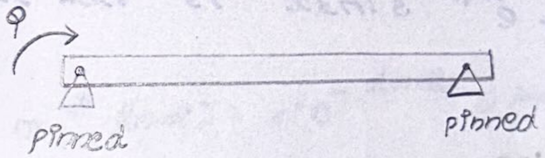
is $\cos w = 0$, (i.e if $w \neq (2n+1)\frac{\pi}{2}$ for some +ve integer n, is are infinitely many solutions

$V = b\sin wt$ when b any constant.

$c = -\left((2n+1)\frac{\pi}{2}\right)^2$. for some constant +ve integer n, there are only many solutions else there is one solution (zero solution)

Horizontal beam of length L with elasticity E and moment of Inertia I carries a load transverse to the beam axis (x). It is pinned (on a hinge) at the left ($x=0$) end with an applied torque (applied bending moment) Q in the +ve direction. It's pinned (on a hinge) at the right end ($x=L$).

soln:- Let x denote the vertical deflection of the beam.



The left end point at $x=0$ is pinned on a hinge, so $v(0)=0$.

Applied torque $\frac{d^2v}{dx^2}(0) \neq 0$

$$\boxed{\frac{d^2v}{dx^2} = Q}$$

Right end:

$$\left.\begin{aligned} v(L) &= 0 \\ \frac{d^2v}{dx^2}(L) &= 0 \end{aligned}\right\} \text{No applied moment}$$

1) For $\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}$, where α is +ve constant.

solu:

The set of solutions is a vector space.

The zero function is a solution, multiplying any solution by a scalar gives another solution, and adding any two solutions gives another solution, so the set of solutions is a vector space.

Although there is a basis of solutions consisting of functions of the form $v(x) w(t)$ there are other solutions that are linear combinations of these, and most of these are not products

$v(x) \quad w(t)$.

example:

$e^{-t} \sin x + e^{-9t} \sin 3x$ is not such a product.

∴ How we can write:

$$e^{-t} \sin x + e^{-9t} \sin 3x = v(x) w(t)$$

If such identity exists, setting $x = \frac{\pi}{3}$ would show that $w(t)$ is a scalar multiple of e^{-t} but then

$$\frac{e^{-t} \sin x + e^{-9t} \sin(3x)}{w(t)} = v(x)$$ is

proportional to $\sin x + e^{-8t} \sin(3x)$, which is a contradiction since $v(x)$ doesn't depends on t.

---

2) Function $\theta(x,t)$ for $x \in [0,1]$ and $t \geq 0$ is a solution to the heat equ $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ with

conditions $\theta(x,0) = 2\sin\pi x + 32\sin 2\pi x$, $\theta(0,t) = 0$, and $\theta(1,t) = 0$.

what's $\theta(x, \pi^{-2}\ln 2)$?

solu:

$$\theta(x,0) = 2\sin\pi x \cdot e^{0} + 32\sin 2\pi x\, e^{0}$$

$$\theta(x, \underbrace{\pi^{-2}\ln 2}_{t}) = 2e^{-\pi^2 t}\sin(\pi x) + 32e^{-4\pi^2 t}\sin 2\pi x$$

$\therefore$ For each +ve integer n, the function $e^{-n^2\pi^2 t}\sin(\pi n x)$ is the solution to the heat eqn satisfying $\theta(0,t) = 0$ and $\theta(1,t) = 0$. So linear combination

$$\theta(x,t) = 2e^{-\pi^2 t}\sin(\pi x) + 32e^{-4\pi^2 t}\sin(2\pi x)$$

is a solution that satisfies all the conditions,

then

$$\theta(x, \pi^{-2}\ln 2) = 2e^{-\ln 2}\sin\pi x + 32e^{-4\ln 2}\sin 2\pi x$$

$$= \sin\pi x + 2\sin(2\pi x)$$

Consider an insulated uniform metal rod of length π with exposed ends & with thermal diffusivity 1. suppose at $t=0$,

$$\theta(x,0) = 10 + \sin 3x + 20\sin 5x + 2\sin 7x.$$

when the ends are held at $0°C$, t is large, the temperature profile is closely approximated by a sinusoidal function of x whose amplitude is decaying to 0, what's the angular frequency of that sinusoidal function.

Hint: start with general heat eqn with BC.

$\theta(0,t)=0$, $\theta(\pi,t)=0$ [Ice at both ends]

$$\theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + b_3 e^{-9t}\sin 3x + \cdots$$

As $t\to\infty$, the sinusoidal functions of x have amplitude decaying at different rates, and it's the first non-zero term in the series that decays the slowest & that hence will eventually become most prominent.

substituting $t=0$,

$$10 + \sin 3x + 20\sin 5x + 2\sin 7x = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

for all x in $(0,\pi)$

To find all the numbers b_n, we need to figure out how to express the constant function 10 as a sum of sines on the interval $(0,\pi)$. To do so, extend it to an odd function on $(-\pi,\pi)$ and extend it to a period of 2π

$$10\,\mathrm{Sq}(t) = \frac{40}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

Thus

$$\frac{40}{\pi}\left(\sin x + \frac{\sin 3x}{5} + \frac{\sin 5x}{5} + \cdots\right) + \sin 3x + 20\sin 5x + 2\sin 7x = b_1\sin x + b_2\sin 2x + \cdots$$

$b_1 = \frac{40}{\pi}$ | $b_3 = \frac{40}{3\pi} + 1$ | $b_5 = \frac{40}{5\pi} + 20$

$b_2 = 0$ | $b_4 = 0$ | $\vdots$

In particular, the first nonzero b_n is b_1, so the dominant term in $\theta(x,t)$ for large t

is the first term,

$\frac{40}{\pi} e^{-t} \sin x$ a sinusoidal function of x of angular frequency 1 whose amplitude is decaying to 0.

$\theta(x,t)$ is the steady state solution to $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ for an insulated metal rod of 10 metres in length with left end held at $10^\circ c$ and right end held at $30^\circ c$. what's the value of $\theta(x,t)$ at a point 4 meters from the left end, in degree celsius?

solu:

In steady state solution, $\frac{\partial \theta}{\partial t} = 0$. which forces

$\therefore \frac{\partial^2 \theta}{\partial x^2} = 0$ (to be zero), which says

$s^2 = 0$ $s = 0, 0$

$\therefore \theta = ax + b$

$\therefore \theta(0) = 10^\circ c$, $\theta(10) = 30^\circ c$

$\boxed{10 = b}$ | $30 = 10a + 10$

$\boxed{a = 2}$

$\therefore \theta(x) = 2x + 10.$

$$\boxed{\theta(4) = 8 + 10 = 18^\circ c}$$

---

5) consider an insulated uniform metal rod of length π with insulated end (at $x = 0$) and with exposed right end (at $x = \pi$) held at $0^\circ c$. let $\theta(x,t)$ be its temperature in degrees celcius at a position x units from the left end after

t seconds. If $\theta(x, t)$ has the form $v(x)\,w(t)$ for some not-identically-zero functions $v(x)$ and $w(t)$. Which of the following must be true of $v(x)$?

solu:

$$v'(0) = 0, \quad v(\pi) = 0$$

Left end is insulated means, heat flux across $x = 0$ is 0.

$\therefore$ heat flux is proportional to $-\frac{\partial \theta}{\partial x}$, so $\frac{\partial \theta}{\partial x} = 0$ whenever $x = 0$. In other words $v'(x)\,w(t) = 0$ when ever $-v'(0)\,w(t) = 0$. But $w(t)$ is not identically zero, so $\boxed{v'(0) = 0}$

To say that the right end is exposed & held at $0°c$ means that $\theta(\pi, t) = 0$ for all $t > 0$. So $v(\pi)\ w(t)$ for all $t > 0$, but $w(t)$ is not identically zero, so $v(\pi) = 0$.

The heat equ with these boundary condt is going to have a solution of the form

$$\theta(x, t) = e^{-ct} \cos\left(\frac{x}{2}\right) \text{ for some}$$

constant $c > 0$. Thus it is possible that

$$v(x) = \cos\left(\frac{x}{2}\right). \text{ In which}$$

$$v'(x) = -\frac{1}{2} \sin\left(\frac{x}{2}\right); \text{ this shows}$$

$v(0)$, $v'(\pi)$ can be non zero.

---

consider, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with

B C $u(0, t) = 0$ and $u\left(\frac{\pi}{2}, t\right) = 0$ for all $t \geq 0$.

This models a vibrating string. Find the +ve number

r such that the general solution is a linear combination of the functions

$\sin(rKx)\cos(2KCt)$ and

$\sin(rKx)\sin(2KCt)$ as K ranges over +ve integers.

Soln ::

$u(x,t) = \sin(rKx)\cos(2KCt)$ into the P.DE yields

$$-(2KC)^2 \sin(rKx)\cos(2KCt) = c^2(-(rK)^2 \sin(rKx)\cos(2KCt))$$

Comparing,

$$-4K^2c^2 = c^2(-r^2K^2)$$

$$r^2 = 4$$

$$\boxed{r = 2}$$

Finding Eigen values & Eigen functions

Consider a thin insulated metal rod of length 1, which satisfies the differential equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} \quad 0 < x < 1, \; t > 0$$

Initially at $t = 0$, the temperature of the rod is given by $\theta(x,0) = f(x)$. Then the left end is placed in an ice bath & held at $0^\circ c$ and the right end is insulated.

By seperation of variables $\theta(x,t) = V(x)w(t)$ to reduce this PDE to the system

$$\frac{d^2 V(x)}{dx^2} = \lambda V(x) \quad \Bigg| \quad \frac{d}{dt} w(t) = \lambda w(t)$$

$\theta(0,t) = 0, \quad t > 0 \quad , \quad \frac{\partial\theta}{\partial x}(1,t) = 0, \quad t > 0.$

setting $\theta(x,t) = v(x)\, w(t)$ gives boundary condition

$$v(0) = 0 \qquad \frac{dv}{dx}(1) = 0$$

Start by solving $\frac{d^2}{dx^2} v(x) = \lambda v(x)$

$\lambda > 0$

$v(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$

$v(0) = c_1 + c_2 = 0$

$v'(1) = c_1\sqrt{\lambda}\, e^{\sqrt{\lambda}} - c_2\sqrt{\lambda}\, e^{-\sqrt{\lambda}}$

$0 = c_1\sqrt{\lambda}\,(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}})$

$\therefore e^{y} > 0$ for any values of y.

$\therefore c_1 = c_2 = 0 \rightarrow$ The Trivial Solution

$\lambda = 0$

$v(x) = c_1 x + c_2$

$v(0) = c_2 = 0$

$v'(1) = c_1 = 0$

Trivial solution

$\lambda < 0$

$v(x) = A\cos(-\sqrt{\lambda}\, x) + B\sin(\sqrt{-\lambda}\, x)$

$v(0) = A = 0$

$v'(1) = B\sqrt{-\lambda}\cos(\sqrt{-\lambda}) = 0$

$B = 0$ (or) $\cos(\sqrt{-\lambda}) = 0$

when $\sqrt{-\lambda} = \frac{2k+1}{2} \times \pi$

$k = 0, 1, 2, 3, ..$

$$\lambda = -\left(\frac{2k+1}{2} \times \pi\right)^2$$

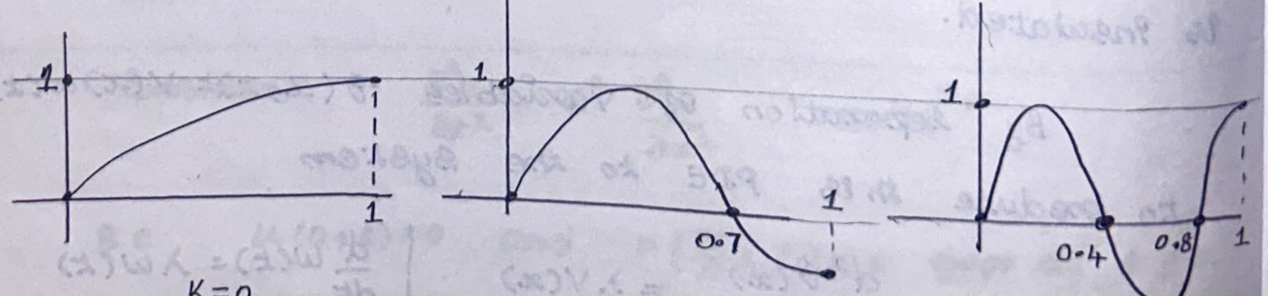
$$v_{(k)}(x) = \sin\left(\frac{(2k+1)\pi}{2} x\right)$$

Sketching Eigen values

$k = 0 \Rightarrow$ Amplitude 1 is $\sin\left(\frac{\pi x}{2}\right)$

$k = 1 \Rightarrow$ " " $\sin\left(\frac{3\pi x}{2}\right)$

$k = 2 \Rightarrow$ " " $\sin\left(\frac{5\pi x}{2}\right)$



K=0 K=1 K=2

<u>Normal modes</u>

$\theta_k(x,t) = v_k(x)\, w_k(t)$ for $k = 0, 1, 2, \ldots$ (Amplitude 1)

Soln: $\lambda = -\dfrac{(2k+1)^2 \pi^2}{4}$ → The functions $w_k(t)$ are

$$w_k(t) = e^{-\left(\frac{(2k+1)\pi}{2}\right)^2 t}$$

Hence the normal modes of amplitude 1 are

$$\theta_k(x,t) = v_k(x)\, w_k(t) = e^{-\left(\frac{(2k+1)\pi}{2}\right)^2 t} \cdot \sin\left(\frac{(2k+1)\pi}{2} x\right)$$

Appropriate Initial condition:

$$f(x) = \sum_{k=0}^{\infty} b_k v_k(x)$$ → which must be true of the type of periodic extension of $f(x)$?

<u>$t=0$:</u>

$$\theta(x,0) = f(x) = \sum_{k=0}^{\infty} b_k \sin\left(\frac{2k+1}{2}\pi x\right)$$

Because the Fourier series involves only sine terms of base period 4 (the period of $\sin\left(\frac{\pi x}{2}\right)$), we must extend $f(x)$ to be an odd function (periodic of period 4) in order to find co.effs b_k that solve this initial condition.

However, note that such a Fourier series has the form

$$\sum_{k=0}^{\infty} b_k \sin\left(\frac{k}{2}\pi x\right)$$

In order for the sine series to involve only the odd periodic terms, there must be a further symmetry. This further symmetry requires that $f(x)$ is a symmetric about the line $x = 1$. Note that in this case we can define $f(x)$ on the interval

$0 < x < 2$ by

$$F(x) = \begin{cases} f(x) & 0 < x < 1 \\ f(2-x) & 1 < x < 2. \end{cases}$$

computing the co.efficients for such a function using the fact that $f(x)$ is odd and of period $2L = 4$ we find

$$b_K = \frac{2}{2} \int_0^2 F(x) \sin\left(\frac{K}{2} \pi x\right) dx$$

$$= \int_0^1 f(x) \sin\left(\frac{K}{2} \pi x\right) dx + \int_1^2 f(2-x) \sin\left(\frac{K}{2}\pi x\right) dx$$

$$= \int_0^1 f(x) \sin\left(\frac{K}{2} \pi x\right) dx - \int_1^0 f(u) \sin\left(\frac{K}{2}\pi (2-u)\right) du$$

$$= \int_0^1 f(x) \sin\left(\frac{K}{2} \pi x\right) dx + \int_0^1 f(u) \sin\left(\frac{K}{2}\pi (2-u)\right) du$$

$$\therefore \quad u = 2 - x$$

$$\sin\left(\frac{K}{2}\pi (2-u)\right) = \sin(K\pi) \cos\left(\frac{K}{2}\pi u\right) - \sin\left(\frac{K}{2}\pi u\right) \cos(K\pi)$$

$$= -\sin\left(\frac{K}{2}\pi u\right) (-1)^K$$

$$= (-1)^{K+1} \sin\left(\frac{K}{2}\pi u\right)$$

Plugging back into the last line we get

$$b_K = \int_0^1 f(x) \sin\left(\frac{K}{2}\pi x\right) dx + (-1)^{K+1} \int_0^1 f(u) \sin\left(\frac{K}{2}\pi u\right) du$$

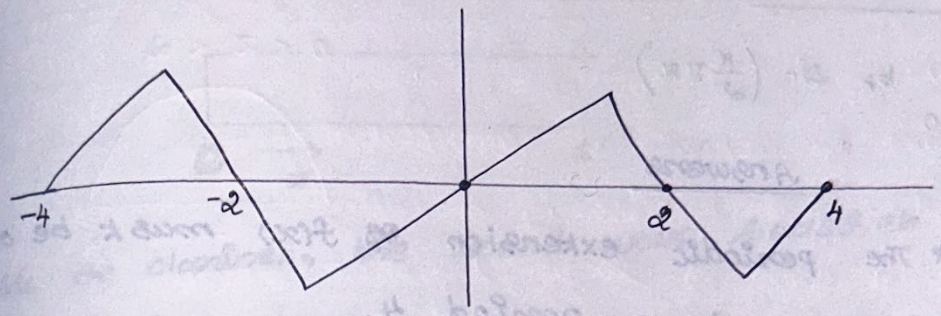
$$= \begin{cases} 2 \int_0^1 f(x) \sin\left(\frac{K}{2} \pi x\right) dx & K \text{ odd} \\ 0 & K \text{ even} \end{cases}$$

Thus the necessary condt is symmetry requires that $f(x)$ be odd, period 4, and symmetry about $x=1$.

Initially at $t=0$ $\theta(x,0) = x$

sketching $f(x)$ on $-4 \leq x \leq 4$.

$$b_K = 2 \int_0^1 x \sin\left(\frac{K}{2}\pi x\right) dx$$



$$\therefore \theta(x,t) = \sum_{K=0}^{\infty} C_K w_K(t) \vartheta_K(x), \quad \theta(x,0) = x$$

$$C_K = \frac{2}{1} \int_0^1 x \sin\left(\frac{2K+1}{2}\pi x\right) dx = \frac{8(-1)^K}{(\pi(2K+1))^2}$$

$$\boxed{\therefore \theta(x,0) = x}$$

Note: $\sin\left(\frac{2K+1}{2} x\pi\right)$

when $K = 0$

$$\sin\left(\frac{x\pi}{2}\right) + \sin\left(\frac{3x\pi}{2}\right) + \sin\left(\frac{5x\pi}{2}\right) + \sin\left(\frac{7\pi x}{2}\right) + \cdots$$

Base period:

$$= \frac{2\pi}{\left(\frac{\pi}{2}\right)} = 4 \quad \Bigg| \quad \frac{2\pi}{\left(\frac{3\pi}{2}\right)} = \frac{4}{3}, \quad \frac{2\pi}{\left(\frac{5\pi}{2}\right)} = \frac{4}{5}$$

$\therefore$ Base period $= 4$

$$\therefore \sin\left(\frac{K}{2}\pi x\right) = \sin\left(\frac{x\pi}{2}\right) + \sin\left(\frac{3x\pi}{2}\right) + \sin\left(\frac{5x\pi}{2}\right) + \cdots$$

$$\sum_{K=0}^{\infty} b_K \sin\left(\frac{K}{2}\pi x\right)$$

Extending $f(x)$ on the interval $0 < x < 2$.

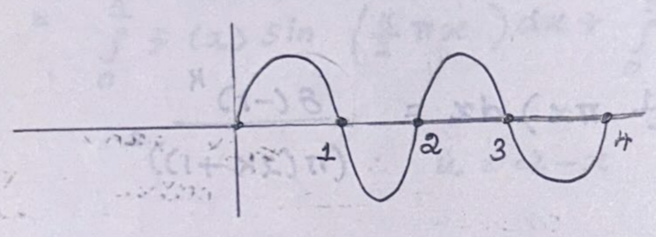
'convert our function to either odd or even'

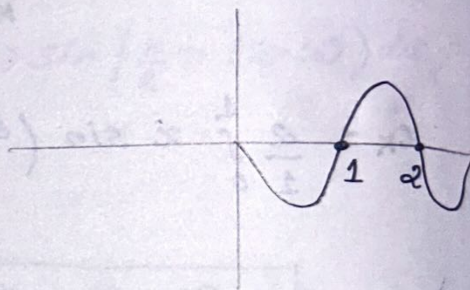
Convert it as odd:

$$f(x) = \sum_{K=0}^{\infty} b_K \sin\left(\frac{K}{2}\pi x\right)$$

Answers

* The periodic extension of $f(x)$ must be odd.
* must have period 4.
* must be symmetric about the line $x = 1$

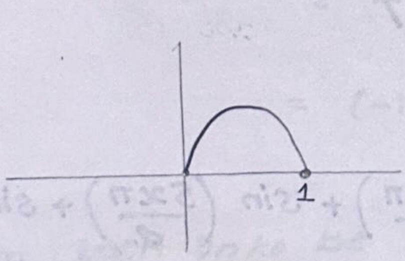




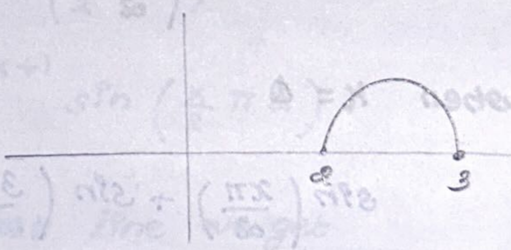
$\sin\left(\frac{K\pi x}{2}\right)$ $[K = 2]$

$\sin\left(K\pi \frac{(2-x)}{2}\right)$

$K = 2$



$f(x)$



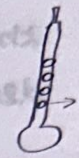
$f(2-x)$

'$b_n \rightarrow$ worked out in previous examples'

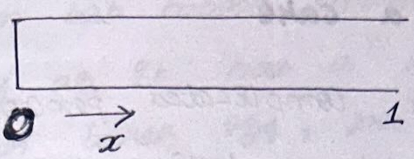
Modeling a clarinet

A clarinet is a curious instrument because its shape almost exactly a cylinder. You blow air through a reed at one end, and this creates a sound wave inside of the cylindrical clarinet. (Note: Clarinets are actually very close to being exactly cylindrical, which is why the following model works).

We model the geometry of the clarinet as a cylinder of length 1 with one (essentially) closed end (where the mouth piece is) and one open end.


→ clarinet

clariet Instrument



Inside the clarinet, the sound wave satisfies the diff eqn

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 P}{\partial t^2} = c^2 \frac{\partial^2 P}{\partial x^2}, \quad 0 < x < 1, \quad t > 0.$$

where $u(x,t)$ → Horizontal displacem

$u(0,t) = 0$ → No displacement at the closed end

$\frac{\partial u}{\partial x}(1,t) = 0$ → At open end (maximum displacement) → derivative will be zero

$P(1,t) = 0$ → open end (atmospheric pressure)

$\frac{\partial P}{\partial x}(0,t) = 0$ → At closed end pressure will be maximum (derivative will be zero).

Eigen functions:

Same kind of problem:

$$\lambda_n = -\left(\frac{(2K+1)\pi}{2}\right)^2, \quad v_K(x) = \sin\left(\frac{(2K+1)\pi}{2} x\right)$$

$K = 0, 1, 2, 3, \ldots$

$$v_K(x) = \cos\left(\frac{(2K+1)\pi}{2} x\right) \quad K = 0, 1, 2, \ldots$$

∴ In heat equation: time part will be in I order

Here 2nd order

$$w_K(t) = \gamma_K(t) = a\cos\left(\frac{(2K+1)}{2}\pi c t\right) + b\sin\left(\frac{(2K+1)}{2}\pi c t\right)$$

How does what you have found here compare the problem we did in Recitation & comparing the sound signals of a clarinet, guitar and human voice. Does this BVP explain the unique feature that you discovered in the frequency spectrum.

solu:

Baking a cake

Baking a cake is actually a complicated process. Here, we vastly simplify the process by modelling the baking of a 1-Dimensional cake of length L that starts at room temperature ($20°c$), one end is directly exposed to the oven temperature. ($200°c$). The thin 1-D cake is insulated and the other end is insulated by a pan.

solu:

Assume thermal conductivity $K(\theta)$ and heat capacity $c(\theta)$ vary with respect to temperature ignoring all the baking effects.

In this case the differential equation is

$$C(\theta) \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K(\theta) \frac{\partial \theta}{\partial x}\right) \quad 0<x<L, \ t>0.$$

The initial condition of the equ

$$c(\theta) \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K(\theta) \frac{\partial \theta}{\partial x}\right) \quad 0<x<L, \ t>0$$

is

$$\theta(x,0) = 20, \quad 0<x<L.$$

The boundary conditions are

$$\theta(0,t) = 200, \ \frac{\partial}{\partial x} \theta(L,t) = 0, \quad t>0.$$

The partial equ can't be solved by separation of variables. However it can be solved numerically. In the following problem, we will develop a function to model the thermal conductivity $K(\theta)$ and heat capacity $C(\theta)$, and then discretize & solve this PDE for cakes of different sizes to compare the baking time.

solu::

Model conductivity and capacity

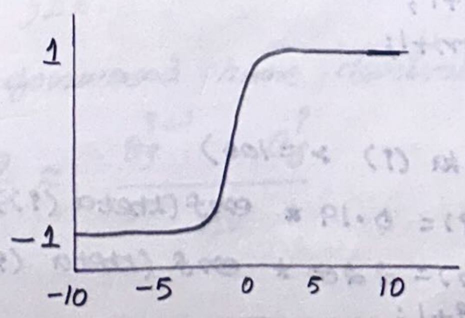
Raw cake batter has more thermal conductivity & a larger heat capacity as it has a large amount of water. As the water bakes off, the cake batter begins to resemble a foam, which has a lower thermal conductivity, and lower heat capacity. In this model, we assume that the cake is baked once it reaches 100°c and it's at this point that the thermal properties change as well.

Based on a paper that both experimentally & numerically modeled the baking of a cake, and measured these quantities, we will model a raw cake batter with a thermal conductivity 0.31 (watts / (meter . celcius)) at room temperature & heat capacity of 2800 (Joules / kg. celcius), and cooked cake batter with a thermal conductivity 0.19 (W/m°c) and a heat capacity (2200 (J/kg°c). we suppose that it varies continuously b/w these two values by using an error function. (we will ignore the steepness of this function to simplify our model)

The error function

$$erf(x)$$

varies smoothly, but sharply, between -1 and 1 at $x = 0$



our job is to scale & translate this error function. to create a function modeling the thermal conductivity $K(\theta)$ as a function of temperature that is 0.31 for temperatures less than $100°c$ and 0.19 for temperatures greater than $100°c$. we let the transition happens at $100°c$.

similarly, create a function $c(\theta)$ modeling the heat capacity that is 2800 for temperature less than $100°c$ and is 2200 for $T > 200°c$.

code:

```
theta = linspace (20,200, 1000);
```

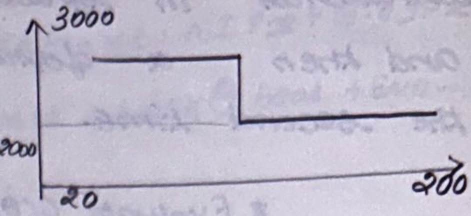
```
% create a vector k that is the correct scaled/translated
erf function
% which is evaluated at the vector theta and is 0.31
for theta < 100,
% and is 0.19 for theta >100
```

```
% create a vector c that is the correct scaled/translated
erf function
% which is evaluated at the vector θ and is 2800
for θ < 100
% and is 2200 for theta >100.
```

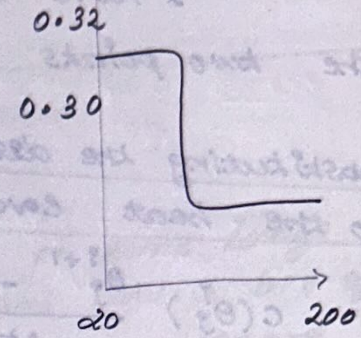
Method: 1:

```
theta = linspace (20,200, 1000);
i=1; m=1;
for i= 1:1000
    if (theta(i) <100)
        k(i) = 0.31 *erf (theta(i));
        c(m) = 2800 * erf (theta(i));
        i=i+1;
        m=m+1;
    end
    if (theta (i) >=100)
        k(i)= 0.19 * erf(theta(i));
        c(m)= 2200 * erf (theta (i));
        i=i+1;
```

```
m=m+1;
end
end   hold on
plot (theta, k)
plot (theta,c)
hold off
```

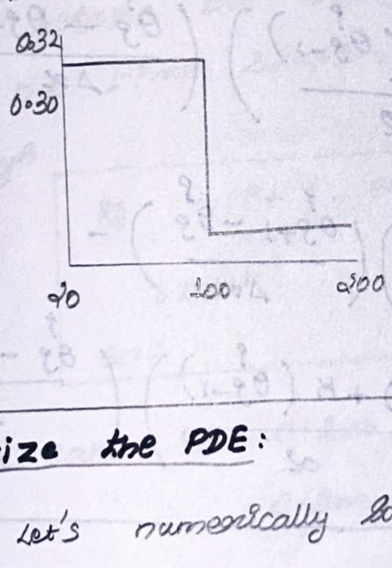


(or)

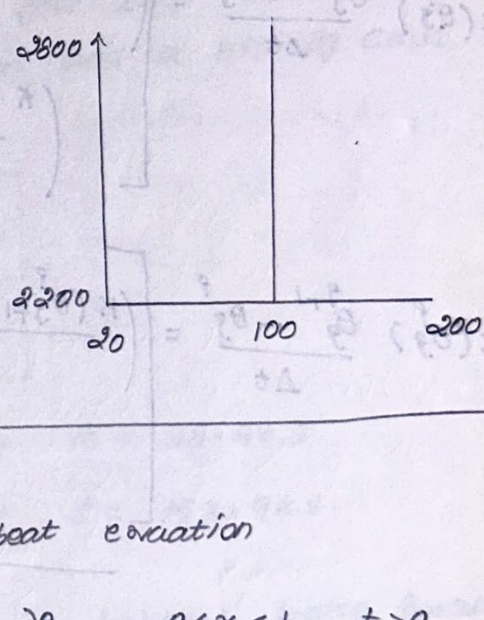



```
k= 0.25 - 0.06 * erf (theta - 100);
c = 2500 - 300 * erf (theta - 100);
plot (theta, k)
plot (theta,c)
```

∴ plot (theta, k)



plot(theta, c)



Discretize the PDE:

Let's numerically solve the heat equation

$$c(\theta)\ \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\ K(\theta)\ \frac{\partial \theta}{\partial x}\ ,\qquad 0<x<L\ ,\ t>0$$

with the boundary conditions $\theta(0,t) = 200$ and $\frac{\partial}{\partial x}\ \theta(L,t)=0$

and initial condition $\theta(x,0) = 20$.

we will use a forward in time, centered in space numerical scheme. Let θ_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$.

Then a discrete forward time derivative is

$$\frac{\partial \theta}{\partial t} \approx \frac{\theta_j^{i+1} - \theta_j^i}{\Delta t}$$

To create the centered space derivatives, we will take a backwards in space derivative the first time, and then a forwards in space derivative the second time.

* Evaluate $C(\theta)$ at the initial point in time
* take an average of the values of $K(\theta)$ b/w the two points in space.

Substituting the discrete time & space derivatives in to the heat eqn above gives:

$$C(\theta_j^i)\,\frac{\theta_j^{i+1}-\theta_j^i}{\Delta t} = \frac{\partial}{\partial x}\left(\frac{K(\theta_j^i)+K(\theta_{j-1}^i)}{2}\right)\left(\frac{\theta_j^i-\theta_{j-1}^i}{\Delta x}\right)$$

$$C(\theta_j^i)\,\frac{\theta_j^{i+1}-\theta_j^i}{\Delta t} = \left[\left(\frac{K(\theta_{j+1}^i)+K(\theta_j^i)}{2}\right)\left(\frac{\theta_{j+1}^i-\theta_j^i}{\Delta x}\right) - \left(\frac{K(\theta_j^i)+K(\theta_{j-1}^i)}{2}\right)\left(\frac{\theta_j^i-\theta_{j-1}^i}{\Delta x}\right)\right]\Big/\Delta x$$

$$C(\theta_j^i)\,\frac{\theta_j^{i+1}-\theta_j^i}{\Delta t} = \frac{\left[\left(\frac{K(\theta_{j+1}^i)+K(\theta_j^i)}{2}\right)\left(\frac{\theta_{j+1}^i-\theta_j^i}{\Delta x}\right) - \left(\frac{K(\theta_j^i)+K(\theta_{j-1}^i)}{2}\right)\left(\frac{\theta_j^i-\theta_{j-1}^i}{\Delta x}\right)\right]}{\Delta x}$$

$$\theta_j^{i+1} = \theta_j^i + \frac{\Delta t}{2C(\theta_j^i)(\Delta x)^2}\left[\left(K(\theta_{j+1}^i)+K(\theta_j^i)\right)\left(\theta_{j+1}^i-\theta_j^i\right) - \left(K(\theta_j^i)+K(\theta_{j-1}^i)\right)\left(\theta_j^i-\theta_{j-1}^i\right)\right]$$

where at each step time i we impose the boundary condition $\theta_1^i = 200$ and $\boxed{\theta_N^i = \theta_{N-1}^i}$

A cake is baked once it has reached $100°C$. Use the code to find the length of time it takes to bake

Q. small cake (L = 0.1m) and a large cake L = 0.2m)

Soln:

url = 'https://courses.edx.org/asset-v1:MITx+18.03Fx+3T2018+type@asset+block@heatEqn-cake-bake.m';

websave('heatEqn-cake-bake.m' url)

Small cake bake time: 38.46 seconds

Large cake bake time: 153.96 seconds

The easiest way to determine the time to some large number (tmax = 200 for example) and add a bit of code that stops the run time once the centre (which is last to the cook) reaches 100°C.

That is, before the end of the for loop that runs through the time steps, you can add a bit of code that says

```
if  u(i+1, end) > 100
    break
end.
```

For a cake of 0.1m to bake $t = 38.46s$

0.2m to bake $t = 153.96s$

∴ That's why professional cake bakers bake small cakes and assemble them into larger structures. The much longer baking time for ~~larger cakes~~ leads to over cooking or burning of the edges before the middle is cooked.

Matlab

1) Stormy Attaway, Matlab, A practical Introduction to Programming & Problem Solving. 3rd edition - 2013 Elsevier)

2) www.sciencedirect.com/science/book/9780124058767.

https //www.gnu.org/software/octave.

Octave: help function
doc function

semi-column or a New line represent a line seperator

comma or space → column seperator.

variable names must start with a letters

no. of char in the name must be less than 63 (= name length max)

* case sensitive.
* for, if, end, ... → Reserved words.

```
>> sin(3)
ans
0.1411
>> sin(3) = 2
00 2
>> sin(3)
2
```

used as variable name.

format changes screen output formatting. not the variable itself.

cf help format

long, long e, long g,

Short, Short e, long g, . . .

Octave

>>

* creating a new directory file.

>> clear all

>> mkdir introoctave ⟶ File will be created.

>> cd introoctave ⟶ opens that file

>> edit sphere ⟶ create a script sphere.m.

>> sphere.m

Enter: 1

ans:

3.14

sphere.m

```
r = input('Enter:')
sphere = pi * r ^ 2
```

---

save -mat workspace.mat

↳ saves workspace in this file.

load workspace.mat

↳ Load the variables stored in that file.

load. workspace.mat A
save -mat onevar.mat A
load -mat onevar.mat

} possible to load a single variable

In

gca - current figure

'save our plot : saveas (gca, 'myplot.pdf', 'pdf')

'use help plot
doc plot
help saveas'

Plot3 (, , , $\underbrace{\quad}_{}$) ⟶ plots 3-d graph.
(Properties)

Surf ⟶ command

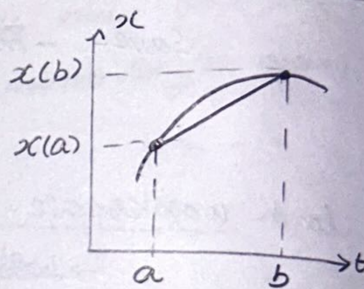
Integration

Mean value theorem states that one moment, your instantaneous speed is going to match your average speed.

Mean value theorem & applications

Average rate of change of a function $x(t)$ $= \dfrac{x(b) - x(a)}{b-a}$

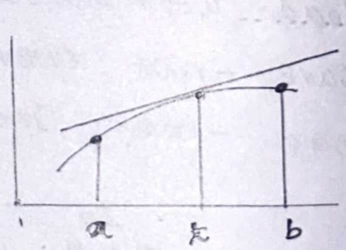
'Rate of (average) change over $[a,b]$ is the slope of the secant line through $(a,(x(a)),(b,x(b))$



Instantaneous rate of change:

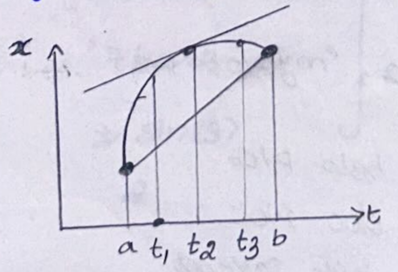
$$x'(t) = \lim_{\Delta x \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

'Tangent line through $(t, x(t))$



'Tangent line'

On the graph, the secant line through $(a, x(a)), (b, x(b))$ has the same slope as the tangent line(s) at which of the following point(s)?



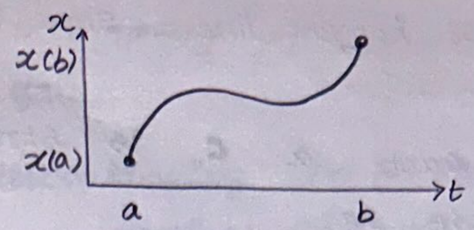
$\therefore$ t_2 (tangent) is parallel to the secant line.

'parallel lines have the same slope'

Mean value theorem

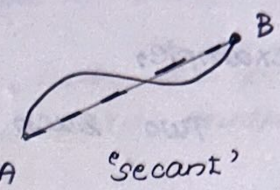
Mean value theorem relates the average rate of change and instantaneous rate of change.

Average rate of change $\xleftarrow{\quad} \text{MVT} \xrightarrow{\quad}$ Instantaneous rate of change.

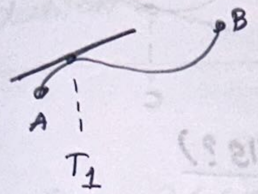


'Function $x(t)$'

'Average' rate of change' of the function $x(t)$ is the rate of change of the secant.



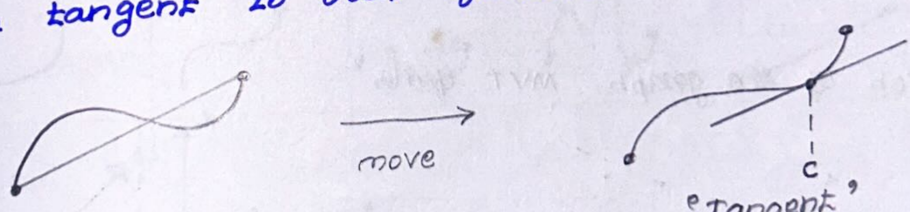
'Instantaneous → slope of secant'



Average rate of change → 1 value
Instantaneous " → can take diff values at diff positions.

How we are going to relate them?

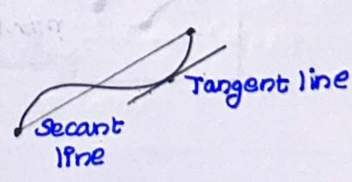
Let's move the secant line to a position where it is a tangent to our function $x(t)$.



'Note that position as c'.

(or)

At that point, the tangent line is parallel to the secant line (same slope).



At some point c, in b/w a and b

'end points'

Average rate of change from a to b = Instantaneous rate of change at c

At c, the slope of tangent line = slope of secant line.

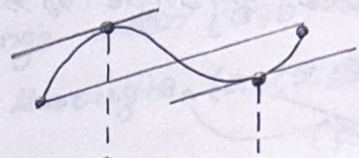
Key:

MVT says that, such a **c** is strictly in b/w the endpoints, but doesn't say

* where exactly such a c is (or)
* even how many such c's are possible.

In our example,

Two such c's are available



c c

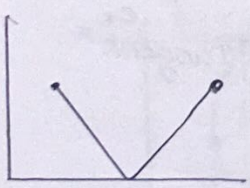
Hypothesis: when does this holds? (Fails?)

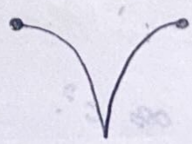
'Necessary hypothesis'

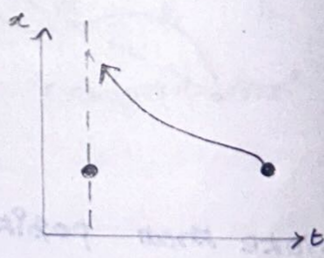
How the MVT goes wrong?

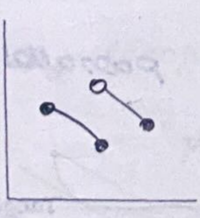
There is a point c, such that $a < c < b$, at which the tangent line is parallel to the secant line through $(a, x(a))$, and $(b, x(b))$.

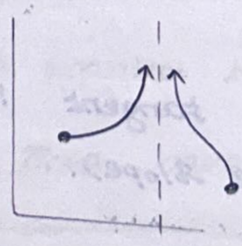
'For which of the graph MVT fails'

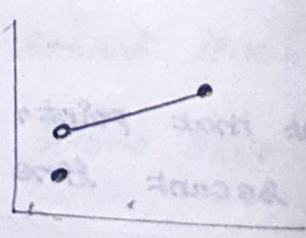


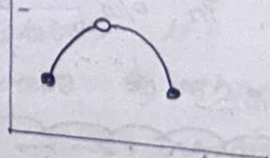












"Graphs with discontinuity or where derivative is not defined" → MVT fails.

The point of discontinuity can be removable or jump discontinuity, or a vertical asymptote, and can be at end point or within the interval. The point where the derivative is not defined can be either a corner or a cusp, and is within the interval.

"Graph of a lower semicircle has end points where the derivatives is undefined but still satisfies the MVT conclusion".

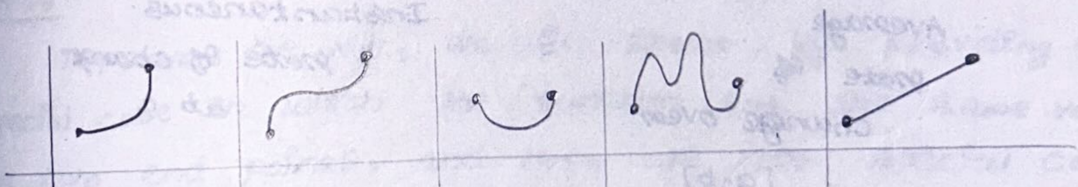
Conclusion

There is c, $a < c < b$, such that

Average rate of change over $[a, b]$ = Instanteous rate of change at c

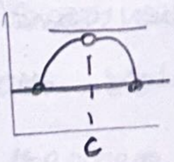
$\updownarrow$ Slope of secant

$\updownarrow$ Slope of tangent

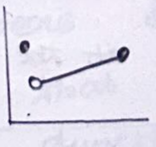


"Holds"

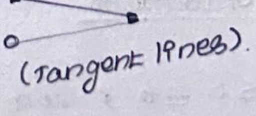
Fails:



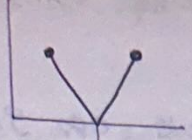
→ At c (undefined)



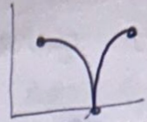
→ Jump discontinuity.



∴ "No c" → Slope of secant line ≠ Slope of tangent line



→ continuous



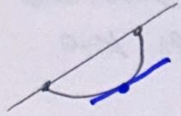
cusp.

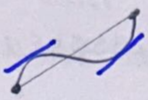
↳ Derivative doesn't exist

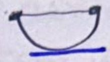
(corners)

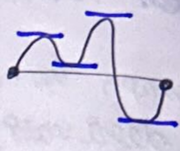
(Function is not differentiable)

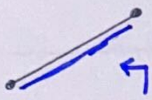
"successful".











'c' is at every point

"common → continuous & differentiable".

MVT conclusion holds:

If a function $x(t)$ is continuous in $[a,b]$ and differentiable in (a,b), then there is some point c in (a,b), such that

Average rate of change over $[a,b]$ = Instantaneous rate of change at c

$$\boxed{\frac{x(b)-x(a)}{b-a} = x'(c)}$$

In hypothesis → the function needs to be continuous in $[a,b]$ → including end points.

(a,b) → Differentiable

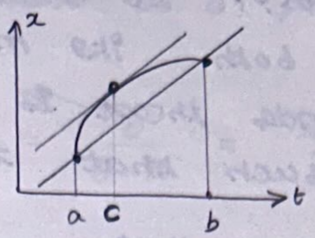
(No need not be differentiable at the endpoints).

Statement

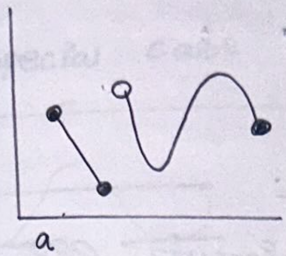
$x(t)$ is continuous $a \leq t \leq b$, and differentiable on $a<t<b$ that is $x'(t)$ is defined for all t, $a<t<b$, then

$$\frac{x(b)-x(a)}{b-a} = x'(c) \quad \text{for some } c, \text{ with } a<c<b$$

Geometrically:

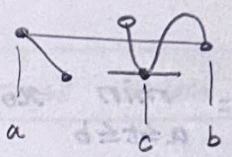


The tangent is parallel to the secant line through $(a, x(a))$ and $(b, x(b))$



Is there a point c $a<c<b$ which the tangent line is parallel to the secant line through $(a, x(a))$, $(b, (x(b))$.

yes



This graph doesn't satisfy the MVT hypothesis but satisfies the MVT conclusion. The example doesn't contradict the MVT conclusion.

Proof:

To prove the MVT, we will start by providing a special case in which the function has the same values at two end points, and then use this special case to prove the full theorem.

To prove this, we rely on the Extreme value theorem, which says that any function which is continuous on a closed interval must attain its maximum & minimum both values in that closed interval.

(These theorem requires deeper analysis of the real numbers & won't provide proof here). The point is that we need continuity to guarantee that the function attains both maximum & minimum.

Proof:

Suppose $x_0(t)$ satisfies the hypothesis of the MVT, that is

$x_0(t)$ is continuous on $[a,b]$, and differentiable on (a,b).

In this special case,

$x_0(a) = x_0(b)$. By the Extreme value theorem $x_0(t)$ attains both its maximum & minimum in $[a,b]$. In other words there is at least one point t_1 in $[a,b]$ such that $x_0(t_1) = \min\limits_{a \le t \le b} x_0(t)$

and at one point (t_2) such that

$$x_0(t_2) = \max_{a \le t \le b} x_0(t)$$

<u>There are only two possibilities.</u>

The maximum & minimum are either equal or not.

<u>case: 1:</u> $\max\limits_{a \le t \le b} x_0(t) = \min\limits_{a \le t \le b} x_0(t)$

$x_0(t)$ must be constant over $[a,b]$. So $x_0'(t) = 0$ for all $a < t < b$. In particular there is atleast one point c with $a < c < b$ at which $x_0'(c) = 0$.

<u>case: 2:</u> $\max\limits_{a \le t \le b} x_0(t) \neq \min\limits_{a \le t \le b} x_0(t)$

since $x_0(a) = x_0(b)$. They can't both be at the end points. Hence atleast one of $\max\limits_{a \le t \le b} x_0(t)$ (or) $\min\limits_{a \le t \le b} x_0(t)$ must be achieved in (a,b)

such that $x_0(c) = \max\limits_{(a \le t \le b)} x_0(t)$ (or)

$x_0(c) = \min\limits_{a \le t \le b} x_0(t)$.

Now recall the derivative of a differentiable function at a local maximum or minimum.

By the hypothesis, $x_0(t)$ is differentiable in (a,b)

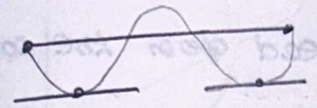
So $x_0'(c) = 0$. Since c is either a local minimum or maximum.

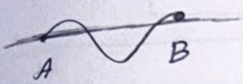
In both cases, there is a point c, with $a < c < b$, such that

$$x_0'(c) = 0 = \frac{0}{b-a} = \frac{x_0(b) - x_0(a)}{b-a}$$

$$\therefore x_0(b) = x_0(a)$$

This special case is known as Rolles theorem.



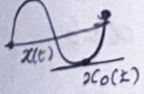


$\therefore$ At min (or) max (local or extremum) the slope $= 0$ as that is

$$\boxed{x(a) = x(b)}$$

Extending this to the case with different end points:

Suppose, $x(t)$ satisfies the hypothesis of the MVT, that is $x(t)$ is continuous on $[a, b]$ and differentiable on (a, b) let



$$x_0(t) = x(t) - \left(x(a) + \frac{x(b) - x(a)}{b-a}(t-a) \right)$$

That's construct a function $x_0(t)$ by subtracting from $x(t)$ the line that goes through $(a, x(a))$, $(b, x(b))$. Then $x_0(t)$ also satisfies the hypothesis of the MVT, and

$$x_0(a) = x_0(b) = 0$$

So we can apply rolles theorem to $x_0(t)$, and know that there is a c in (a, b), such that $x_0'(c) = 0$.

Now we can rearrange the eqn above & get $x(t)$ in terms of $x_0(t)$

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$$x(t) = x_0(t) + \left(x(a) + \frac{x(b) - x(a)}{b-a}(t-a) \right)$$

$f(b) = f(a) + f'(c)(b-a)$

Taking derivatives,

$$x'(t) = x_0'(t) + \frac{x(b) - x(a)}{b - a}$$

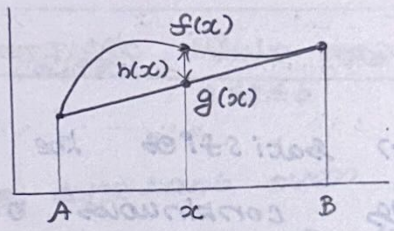
And at point c, in (a, b).

$$x_0'(c) = 0$$

$$x'(c) = x_0'(c) + \frac{x(b) - x(a)}{b - a}$$

$$= \frac{x(b) - x(a)}{b - a}$$

Thus, we have found a c we need for the conclusion of the MVT.



$$y - y_1 = m(x - x_1)$$

$$g(x) - f(a) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a)$$

$\therefore$ In our case $x = t$

$$g(x) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)$$

$h(a) = h(b) = 0$ [height].

$\therefore$ $\boxed{h(x) = f(x) - g(x)}$ on $[a, b]$

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right)$$

Notice, $h(a) = h(b) = 0$

As a result, Rolle's theorem applies.

For some $x = c$ in (a, b) such that $h'(c) = 0$

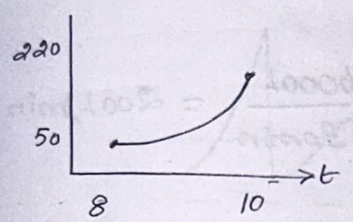
$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

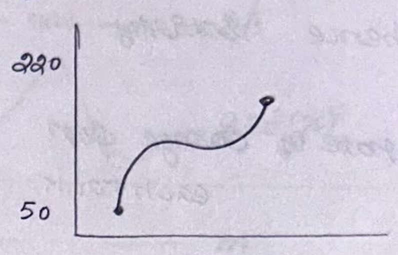
$$h'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b - a}\right)$$

$$0 = f'(c) - \left(\frac{f(b) - f(a)}{b - a}\right)$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Speeding





A car can't teleport. It's curve must be continuous & smooth looking. (Doesn't have discontinuity).

$\therefore$ velocity is continuous (Differentiable).

$\therefore$ $x(t)$ satisfies the hypothesis of MVT.

When 85mph?

$220 - 50 = 170$ miles $\left(\frac{170}{2\text{ hours}} = 85 \text{ mPH}\right)$

By MVT, the strongest conclusion the police officer could make about when the car is travelling at exactly 85mph is

There is / are at least one such moment(s) after 8, and before 10 and the police officer doesn't know when such moment(s) is (are).

Application to simultaneous rates

Let us use the MVT to obtain a qualitative result on simultaneous rates of change.

Suppose that we have two tanks each of volume 6000 liters. Initially, both are empty. They start getting filled at 1:00, although at variable rates that may be different. Both tanks become completely filled at exactly 1:30 (30 minutes later). Note that the rate of filling of each tank is continuous over time. We want to determine whether there was some moment when both tanks were being filled at the same rate.

Solu:-

The MVT tells that at some t_1 in the 30 minutes b/w 1:00 and 1:30, the instantaneous rate of filling of tank 1 is 200 L/min.

The rate of filling, $V_1'(t)$ and $V'(t)$ are continuous, the functions $V_1(t)$ and $V_2(t)$ are continuous & differenti-able, and hence satisfy the hypothesis of the MVT.

Average rate of change for each tank $\Big\}= \frac{6000L}{30min} = 200L/min$

By MVT, at some point t_1 b/w 1.00 and 1:30, the instantaneous rate of change of volume in tank 1 is equal to the average rate of change. Similarly, at some point t_2 b/w 1:00 and 1:30, $V_2'(t)$ is equal to this average rate of change. The MVT doesn't give any info about t_1 or t_2. So they could be the same or not the same, but we don't know.

MVT to a new function

suppose $V_1(t)$ is the volume of water in the tank 1 at time t, V_2 " " time t. of tank 2.

$$h(t) = V_1(t) - V_2(t). \text{ Then}$$

$h(1:00) = 0$ $\quad (0 - 0)$ litres $= 0$

$h(1:30) = 0$ $\quad (6000 - 6000) = 0$

Average change (height) $h = \frac{h(1:30) - h(1:00)}{30} = 0.$

$$h'(t) = V_1'(t) - V_2'(t) = 0.$$

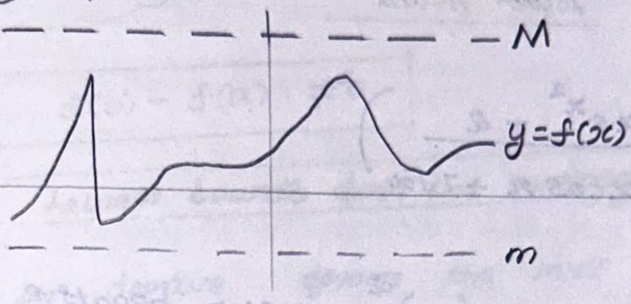
$\therefore$ $V_1'(t) = V_2'(t)$ at some point. (we don't know)

Upper & Lower bounds

A number M is an upper bound on a function $f(x)$ if
$$f(x) \leq M \text{ for all } x.$$

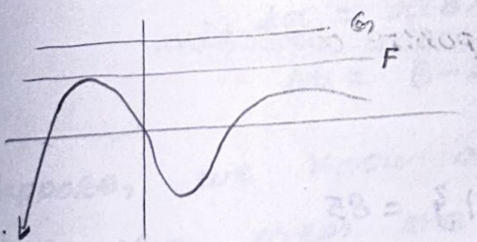
and a number m is a lower bound on a function $f(x)$ if
$$m \leq f(x) \quad \text{(all } x \text{ values)}$$

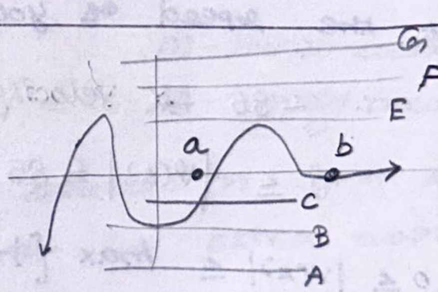
we can consider upper & lower bounds over the entire real number line, or over an interval.



$$m \leq f(x) \leq M$$

In other words, an upper bound on a function is a number that is larger than or equal to all values of the function. A lower bound on a function is a number which is smaller than or equal to all values of the function.





upper bounds: F, G

Lower bound: None ($-\infty$)

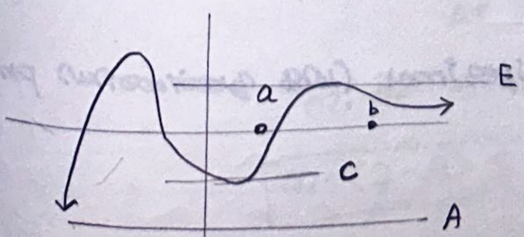
B/w $[a, b]$

UB = E, F, G

LB = C, B, A

Best upper bound & Lower bound

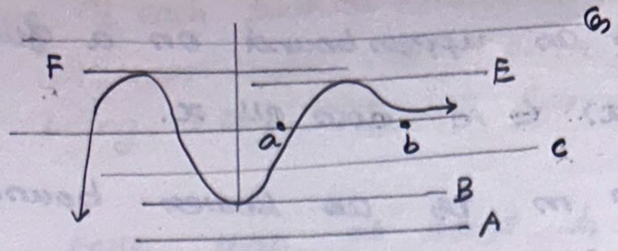
(Least UB & Greatest LB)



G.L.B = C

L.U.B = E

Greatest lower bound which is not a minimum



Lower bound

$[b, +\infty)$? $\longrightarrow$ A

Greatest lower bound on $f(x)$ over $[b, +\infty)$? $\longrightarrow$ 0 (x-axis)

Lower bound

Let:

$$f(x) = \left(\frac{xe^{x^2} + 2}{x\cos x + 7\sqrt{x}} \right)^2$$

Solu:

Function $f(x) = (\;)^2 \rightarrow$ Non negative

$\therefore$ Thus 0 is a lower bound.

Suppose you know a L.B & an upper bound of the velocity $v(t)$ of your car to be

$$-85 \le v(t) \le 65$$

Then what's the strongest statement you can make about the speed of your car.

$\therefore$ $-85 \rightarrow$ velocity in opposite direction.

$$\therefore \quad 0 \le |v(t)| \le 85$$

$$0 \le |v(t)| \le \max\{|-85|, |65|\} = 85$$

MVT backwards

$x' > 0 \rightarrow x$ is increasing $\rightarrow$ +ve velocity (Forward)

$x' < 0 \rightarrow x$ is decreasing

} Function.

$x' = 0 \rightarrow x$ is constant

Derivative $\rightarrow$ Infinitesimal (ശൂന്യ ദൂരത്തിൽ) property

Suppose f is a continuous, differentiable function.
to f' is never zero & $a \neq b$. Then show $f(a) \neq f(b)$

Ans:

MVT $\rightarrow$ $\dfrac{f(b) - f(a)}{b-a} = f'(c)$ ... $(a < c < b)$

$\boxed{f(b) - f(a) \neq 0}$

$f' \rightarrow$ never goes to zero (given)

$f(b) - f(a) = (b-a) f'(c)$

$\therefore$ $b \neq a$ (given)

$f'(c)$ never zero

$\boxed{\therefore \; f(b) - f(a) \neq 0}$

Lower bounds for the average rate of change

Three steps to derive from the MVT the fundamental fact that is the derivative of a function is non-negative, then the function is increasing or staying at the same value.

If $x(t)$ satisfies the hypothesis of MVT, $x(t)$ is continuous on $A \leq t \leq B$ & differentiable on $A < t < B$.

$\Delta x = x(B) - x(A)$ m - Lower bound.

$\Delta t = B - A$ M - upper bound

Suppose, we know a lower bound m for the derivative over the interval. In other words,

$m \leq x'(t)$ for all t such that $A < t < B$

By MVT:

Lower bound for the average rate of change & the total change.

$$m \leq \frac{\Delta x}{\Delta t}$$

$$m \cdot \Delta t \leq \Delta x$$

$$\boxed{m \cdot (B - A) \leq \Delta x}$$

If we know a lower bound for $x'(t)$ over the interval (A, B) we can conclude

As above, suppose $x(t)$ satisfies the hypothesis of the MVT, that's $x(t)$ is continuous on $A \le t \le B$, and differentiable on $A < t < B$, and suppose $m \le x'(t)$ for all t such that $A < t < B$.

Soln:

$$m \le \frac{x(b) - x(a)}{b - a} \text{ for all } b, a \text{ such that } A \le a < b \le B$$

'strong statement'

$x(t)$ satisfies the hypothesis of the MVT, $x(t)$ is continuous on $A \le t \le B$, and diff on $A < t < B$,

Now, let the upper bound of $x'(t)$ be zero. In other words, suppose

$$0 \le x'(t)$$

$x(a) \le x(b)$ for all a, b such that $A \le a \le b \le B$

$\therefore$ $x'(t) \ge 0$ ($\uparrow$) or const

This implies that over the big interval $[A, B]$, $x(t)$ is increasing or at the same value.

upper bound is zero

$$x'(t) \le 0.$$

$x(a) \ge x(b)$ for all a, b such that $A \le a < b \le B$

$\therefore$ $x'(t) \le 0$ for all t in (A, B) means $x(t)$ is decreasing or staying at the same value.

If $x'(t) = 0$ for all t in (A, B), then over $[A, B]$, $x(t)$ is staying at the same value.

old news

Recall,

$f(x)$ is $\uparrow$, if $a < b$, we have $f(a) \le f(b)$

$f(x)$ is $\downarrow$, if $a < b$, we have $f(a) \ge f(b)$

The conclusions we just made the MVT on all subintervals of $[A, B]$ are fundamental facts we have been relying on.

1) If $x'(t) \geq 0$ for all t in (A, B), then $x(t)$ is increasing or staying the same over $[A, B]$

2) If $x'(t) \leq 0$ for all t in (A, B), then $x(t)$ is decreasing or staying the same over $[A, B]$.

3) If $x'(t) = 0$ for all t in (A, B), then $x(t)$ is constant over $[A, B]$.

And the MVT gives the following consequences with strict inequalities as well:

1) If $x'(t) > 0$ for all t in (A, B) then $x(t)$ is strictly increasing over $[A, B]$

2) If $x'(t) < 0$ for all (A, B), then $x(t)$ is strictly decreasing over $[A, B]$.

These facts need proofs and their proofs are based on the MVT. The subtlety is that the MVT relates the infinitesimal behaviour of the function, the derivative, which is defined at each point, with the macroscopic behaviour, the total change over an interval.

Applications

At every moment in July, a wildfire is growing at a rate of at least 2 square kilometers / day. On July 15, it was 50 square kilometers in area. The strongest statement we can make about its area on July 25 is that it had atleast 70 kms.

July 5, atmost an area of 30 sq. kms.

Let $A(t)$ (with units km^2) be the area of the fire at time t (with units days), then

$$A'(t) \geq 2 \quad \text{for all} \quad 0 < t \leq 31$$

$\therefore$ $A(15) = 50$. The MVT says that average rate of change must be greater than or equal to any lower bound of $A'(t)$

$$\frac{\Delta A}{\Delta t} \geq 2$$

$$A(25) - A(15) \geq 2(25-15) = 20$$

$$A(25) \geq A(15) + 20$$

$$A(25) \geq 70.$$

from $f(b) - f(a) \geq m(b-a)$

Also,

$$A(15) - A(5) \geq 2(15-5) = 20$$

$$A(15) \geq A(5) + 20$$

$$30 \geq A(5).$$

Number of roots

$f(x) = -\frac{x^3}{6} - 3x - 2\cos x$, without graphing the function $f(x)$ is

Solu: As $x \to -\infty$, the cubic term $-\frac{x^3}{6}$ dominates

As $x \to \infty$, the cubic term " "

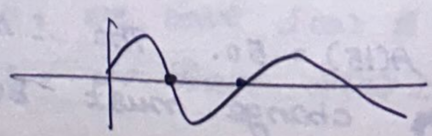
$\therefore$ $f(x) > 0$ as $x \to -\infty$

$f(x) < 0$ as $x \to \infty$

Some times +ve & sometimes (-) ve

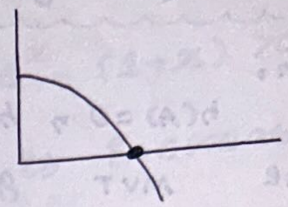
Based solely on that information, what we can say about the number of solution to the equation $f(x) = 0$?

$f(x)$ is sometimes +ve & some times (-)ve and $f(x)$ is a continuous function, the intermediate value theorem says that $f(x) = 0$ for atleast one x value.



$f(x) = -\frac{x^3}{6} - 3x - 2\cos x$

$f'(x)$ is negative for all x.



$$f'(x) = -\frac{x^2}{2} - 3 + 2\sin x$$

As ~~$x \to$~~ $-\frac{x^2}{2} \le 0$ for all x

$-3 + 2\sin(x) \le -3 + 2 = -1$ for all x.

$f'(x) \le -1 < 0$ for all x.

---

combining this new info, we can say that the number of solutions to the equation $f(x) = 0$?

The MVT shows that $f'(x) < 0$, the function $f(x)$ is strictly decreasing. $f(x)$ can only pass through the x-axis once.

Applying MVT to inequalities

using our old news to justify some inequalities b/w functions, such as $e^x > 1 + x$ for all $x > 0$.

Suppose our goal is to show $f(x) > g(x)$ on an interval. The following three steps are an argument to confirm the inequality.

Let $h(x) = f(x) - g(x)$ for $x \ge A$. $h(x)$ is continuous on $x \ge A$ and differentiable on $x > A$.

If $f(A) = g(A)$, then $\boxed{h(A) = 0}$

$$\therefore h(A) = f(A) - g(A) = f(A) - f(A) = 0$$

If $h'(x) > 0$ for all $x > A$

$h(x)$ is strictly increasing for all $x > A$.

What does the inequality $h(x) > 0$ for all $x > A$ say about the relation b/w $f(x)$ and $g(x)$ for $x > A$.

$\therefore h(x) > 0$ means $f(x) > g(x)$

$f(x) > g(x)$ for $x > A$.

$h(x) = f(x) - g(x)$, and $f(A) = g(A)$, then

$h(A) = f(A) - g(A) = 0$

Given,

$h(A) = 0$, h satisfies the hypothesis of the MVT, the MVT gives the old news that if

$h'(x) > 0$ for all $x > A$.

Then $h(x)$ is strictly increasing from $h(A) = 0$ for all $x > A$. This means that if $h'(x) > 0$ for all $x > A$, then $h(x) > 0$ for all $x > A$.

$\therefore f(x) > g(x)$ for all $x > A$.

$\therefore h'(x) > 0$

On the other hand, in the reign $x < A$, MVT gives the old news that if $h'(x) > 0$ for all $x < A$, then $h(x)$ is strictly increasing to $h(A) = 0$ for all $x < A$. This means that if $h'(x) > 0$ for all $x < A$, then $h(x) < 0$ for all $x < A$. Since addition on both sides of an inequality doesn't change the sign of the inequality, we can add $g(x)$ to both sides of the $h(x) > 0$ to get $f(x) - g(x) > 0$.

Based on the graphs of e^x and $1+x$ we may guess

$e^x > 1+x$ for all $x > 0$.

$\therefore e^x > 1$ for all $x > 0$ $\longrightarrow$ yes

$h(x) = e^x - (1+x)$

$h(0) = 0$

$h'(x) = e^x - 1$

which of the following would guarantee $e^x > 1 + x$ for all $x > 0$.

$h'(x) > 0$ for all $x > 0$

$h(x)$ is strictly $\uparrow$ for all $x > 0$.

$e^x > 1+x$ for all $x>0$ → True.

1) $e^x > 1+x+\frac{x^2}{2}$

$g(x) = e^x - (1+x+x^2/2)$ $\qquad$ $g'(x) = e^x - (1+x) > 0$

$g(0) = e^0 - 1$ $\qquad$ g is increasing.

$\boxed{g(0) = 0}$

$$\boxed{\therefore\ e^x > 1+x+\frac{x^2}{2}.}$$

$$\therefore\ e^x > 1+x+\frac{x^2}{2}+\frac{x^3}{3\times 2}+\frac{x^4}{4\times 3\times 2}+\cdots$$

$\frac{d}{dx}(\ln(1-x)) = -\frac{1}{1-x} = -\frac{1}{1-x}$ for all $x<1$

Thus the graph is strictly decreasing in all its domain.

Near end point 1

$\therefore \lim_{x\to 1^-} \ln(1-x) = \lim_{1-x\to 0^+} \ln(1-x) = -\infty$.

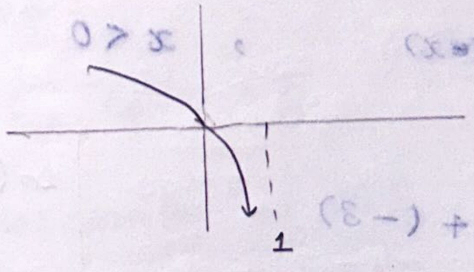
This means f must be strictly decreasing.

vertical asymptote at $x=1$.

$\lim_{x\to\infty} \ln(1-x) = \infty$

$\therefore \ln(1-x) = 0$ when $x=0$.

$\therefore$ graph passes through origin



Compare $\ln(1-x)$ with its linear & quadratic approximations near $x=0$.

Let us compare $\ln(1-x)$ with its linear & quadratic approximations near $x=0$.

$$\ln(1-x) < -x \quad \text{for all} \quad 0 < x < 1$$

$$\ln(1-x) < -x \quad \text{for all} \quad x < 0$$

$$\ln(1-x) < -x - \frac{x^2}{2} \quad \text{for all} \quad 0 < x < 1.$$

$\therefore$ 1) $\ln(1-x) < -x$ for all $x > 1$ $\longrightarrow$ False.

$$h_1(x) = \ln(1-x) - (-x)$$

$$h_1(x) = \ln(1-x) + x$$

$\boxed{h_1(0) = 0}$ $\qquad h_1'(x) = \dfrac{-1}{1-x} + 1$

case: 1: $x < 0$ $\qquad x < 0$

$1 - x > 1$ $\qquad \boxed{1-x > 1}$ $\qquad \therefore \boxed{x<0}$

$\dfrac{1}{1-x} < 1$ $\qquad \boxed{\dfrac{1}{1-x} < 1}$ $\qquad \therefore \dfrac{1}{1-x} \to$ Always less than 1.

$$\boxed{h_1'(x) = 1 - \frac{1}{1-x} > 0}$$

$h_1(x)$ is increasing to 0 in this reign. This means

$h_1(x) < 0$ for $x < 0$ $\qquad \boxed{h_1'(x) > 0}$ when $x<0$

$\ln(1-x) < -x$ for $x < 0$.

$\therefore$ $h_1'(x)$ is increasing for $\boxed{x<0}$,

$\therefore$ $h_1(x) = \ln(1-x) - (x)$ $\quad , \quad x < 0$

If $x = -3$

$h(x) = 1.38 + (-3)$

$= -1.613$

$x = -4$

$\ln(5) - 4 = -2.39056$

$\ln(6) - 5 = -3.208$

$h_1(x) < 0$ for $x < 0.$

$\ln(1-x) < -x$ for $x < 0.$

In $(-\infty, 0)$ interval $h_1(x)$ increases to $0.$

case 2: $0 < x < 1.$

$\therefore x$ is less than $1.$

$0 < x < 1$

$\boxed{1 - x > 0}$

$0 < 1 - x < 1$

$1 < \dfrac{1}{1-x}$

$$\boxed{h_1'(x) = 1 - \frac{1}{1-x} < 0}$$

$\therefore h_1'(x) < 0 \rightarrow$ Decreasing from 0 in this reign.

$h_1(x) < 0$ for $0 < x < 1$

$\ln(1-x) < -x$ for $0 < x < 1$

Now taking quadratic approximation:

$$h_2(x) = \ln(1-x) - \left(-x - \frac{x^2}{2}\right)$$

$\boxed{h_2(0) = 0}$

$$h_2'(x) = -\frac{1}{1-x} - (-1-x)$$

$$= \frac{-x^2}{1-x} < 0$$

case i) $x < 0$

$\therefore x^2$ will dominate.

$x < 0, \; h_2'(x) < 0$

$h_2(x)$ is decreasing to $0.$

$h_2(x) > 0$ for $x < 0$

$\ln(1-x) > -\dfrac{x^2}{2} - x$ for $x < 0.$

case: 2: $0 < x < 1$

$h_2'(x) < 0$, $h(x)$ is decreasing from 0 in this region

meaning

$$h_2(x) < 0 \quad \text{for } 0 < x < 1$$

$$\ln(1-x) < -x - \frac{x^2}{2} \quad \text{for } 0 < x < 1.$$

---

compare $\ln(1-x)$ with a cubic approximation polynomial which is the best cubic approximation at $x=0$

solu:

$$\boxed{\begin{array}{l} \ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} \quad \text{for all } 0 < x < 1 \\ \ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} \quad \text{for all } x < 0. \end{array}}$$

$$h_3(x) = \ln(1-x) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$\boxed{h_3(0) = 0}$$

$$h_3'(x) = -\frac{1}{1-x} - (-1 - x - x^2) = -\frac{x^3}{1-x}$$

case: 1 $x < 0$.

$$x < 0$$

$$1 - x > 0, \quad -x^3 > 0$$

$$h_3'(x) = \frac{-x^3}{1-x} > 0$$

$h_3(x)$ is increasing to zero in this region

$$h_3(x) < 0 \quad \text{for } x < 0$$

$$\ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} \quad \text{for } x < 0.$$

case: 2: $0 < x < 1$.

$$0 < x < 1$$

$$1 - x > 0, \quad -x^3 < 0.$$

$$h_3'(x) = \frac{-x^3}{1-x} < 0$$

$h_3(x)$ is decreasing from 0 in this reign.

$h_3(x) < 0$ for $0 < x < 1$

$$\ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} \quad \text{for } 0 < x < 1.$$

Note: we can generalize these inequalities to higher degree polynomials on the R.H.S. That's we can compare $\ln(1-x)$ with $-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \frac{x^n}{n}$

for any n let

$$h_n(x) = \ln(1-x) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n}\right)$$

$$\boxed{h_n(0) = 0}$$

$$h_n'(x) = \frac{-1}{-x+1} - \left(-1 - x - x^2 - \dots - x^{n-1}\right)$$

$$= \frac{-x^n}{1-x}$$

what will be the sign of h_n' ?

Denominator : $1-x$ will be +ve $x < 0$

$0 < x < 1$.

Numerator :

$-x^n < 0$ for all x, if $n \to$ even

$-x^n > 0$ for all $x<0$, if $n \to$ odd

and $-x^n < 0$ if $x > 0$

$-x^n \to$ +ve we need $n \to$ odd, $x < 0$

If n even $h_n'(x) < 0$ for all x

This implies $h_n(x) > 0$ for all $x < 0$

$h_n(x) < 0$ for $0 < x < 1$

Inequalities will be:

for n even.

$\ln(1-x) > -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} \rightarrow$ for $x < 0$ (n even)

$\ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{2} - \cdots - \frac{x^n}{n} \rightarrow$ for $0 \leq x < 1$ (n even)

If n odd, $h_n'(x) > 0$ if $x < 0$

$h_n'(x) < 0$ if $x > 0$

$h_n(x) < 0$ for both $x < 0$ and $0 < x < 1$.

Inequalities (when n is odd)

$\ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}$ for all $x < 0$ (n odd)

$\ln(1-x) < -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}$ for all $0 < x < 1$ (n odd)

Range of average rate of change & total change

Bounding the Average rate of change

Average velocity $\leq$ Maximum velocity

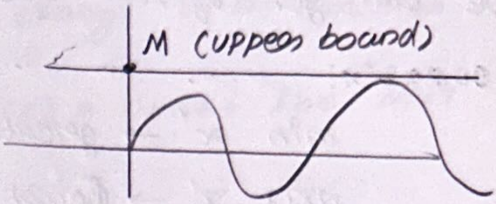
$$\frac{x(b) - x(a)}{b-a} \leq \max x' \text{ (from } a \text{ to } b)$$

Proof: Avg velocity = Instantaneous velocity (a, b) interval

MVT :

$$\frac{x(b)-x(a)}{b-a} = x' \text{ (At some point in } (a,b)) \leq \max x'$$

$\therefore$ $x'(c)$ where it's equal to the average will be either equal to (or) less than the maximum instaneous velocity.



$\therefore$

$$\frac{x(b)-x(a)}{b-a} = x'(c) \leq \max x' \leq M$$

$$\boxed{\therefore \frac{x(b)-x(a)}{b-a} \leq \max x' \leq M}$$

$$\boxed{m \leq x'(c) \leq M}$$
(lower bound)

Then,

$$m \leq \frac{x(b)-x(a)}{b-a}$$

At some point c,

$$x'(c) = \frac{x(b)-x(a)}{b-a}$$

If,

$m \leq x'(c) \leq M$ for all c's

$a < c < b$

then, $m \leq \dfrac{x(b)-x(a)}{b-a} \leq M$

$\boxed{b-a > 0}$

$$m.(b-a) \leq x(b)-x(a) \leq M(b-a)$$

$\therefore$

$$\min x' \leq x'(c) \leq \max x'$$

we can replace,

$$\min x' \leq \frac{x(b)-x(a)}{b-a} \leq \max x'$$

$\min x' (b-a) \leq x(b) - x(a) \leq \max x' \cdot (b-a)$.

$\therefore$ Average rate of change will be in b/w max & min.

<u>Note:</u>

minimum & maximum of x' is the tighest bounds we can get of the average rate of change this way.

Because,

min $x' \rightarrow$ greatest lower bound on x'

max $x' \rightarrow$ least upper bound on x'

<u>concluding</u>

$$m \leq \min_{a \leq t \leq b} x'(t) \leq x'(t) \leq \max_{a \leq t \leq b} x'(t) \leq M$$

$$m \leq \min_{a \leq t \leq b} x'(t) \leq \frac{x(b) - x(a)}{b-a} \leq \max_{a \leq t \leq b} x'(t) \leq M$$

$$m.(b-a) \leq \min_{a \leq t \leq b} x'(t)(b-a) \leq x(b) - x(a) \leq \max_{a \leq t \leq b} x'(t)(b-a)$$

$$\leq M(b-a)$$

<u>An inequality of sine.</u>

Goal: Determine the smallest constant C such that the following inequality holds.

$$|\sin(b) - \sin(a)| \leq C|b-a| \quad \text{for any } a, b.$$

Assume $b > a$.

In other words, what's the biggest number m

$$m \leq \frac{\sin(b) - \sin(a)}{b-a} \leq M \quad \text{for all choices of } a < b.$$

$m = -1, \; M = 1$

$$\left| \frac{\sin(b) - \sin(a)}{b-a} \right| \le c$$

what will the value of c if this condition need to be true? (smallest constant c)

solu:

$\frac{\sin b - \sin a}{b-a}$ → Average rate of change of the function $\sin(t)$ from a to b and $\sin'(t) = \cos t$, the MVT implies

$$\min_{a \le t \le b} \cos(t) \le \frac{\sin(b) - \sin(a)}{b-a} \le \max_{a \le t \le b} \cos(t)$$

for any $a<b$.

$\max\limits_{a \le t \le b} \cos(t)$ and $\min\limits_{a \le t \le b} \cos(t)$ are the minimum & maximum of $\cos(t)$ over the interval $[a,b]$, and depends on which interval $[a,b]$ is considered.

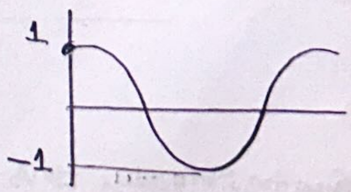
But we are looking for upper & lower bounds of $\frac{\sin(b) - \sin(a)}{b-a}$ that work for all choices of $[a,b]$, so

instead of $\max\limits_{a \le t \le b} \cos t$ and $\min\limits_{a \le t \le b} \cos(t)$,

we use the global maximum & global minimum of $\cos(t)$ in the inequalities.

$$-1 = \min_{-\infty < t < \infty} \cos(t) \le \frac{\sin(b) - \sin(a)}{b-a} \le \max_{-\infty < t < \infty} \cos(t) = 1$$

for any $a<b$.



$\max\limits_{-\infty < t < \infty} \cos(t) = 1$ → global maximum.

$\min\limits_{-\infty < t < \infty} = -1$ → global minimum.

$$\sin'(0) = \lim_{b \to 0} \frac{\sin(b) - \sin(0)}{b - 0}$$

As $b \to 0$, $\dfrac{\sin(b) - \sin(0)}{b-0} = \sin'(0) = 1.$

Hence, 1 is indeed the smallest number than can be an upper bound on $\dfrac{\sin b - \sin a}{b-a}$ for any $a < b$.

$\sin'(\pi)$ says as $b \to \pi$ $\dfrac{\sin(b) - \sin(\pi)}{b - \pi} \to \sin'(\pi) = -1.$

Lower bound $= -1$.

$$-1 \leq \frac{\sin(b) - \sin(a)}{b-a} \leq 1 \text{ for any } a < b.$$

where 1 is the least upper bound
-1 is the greatest lower bound.

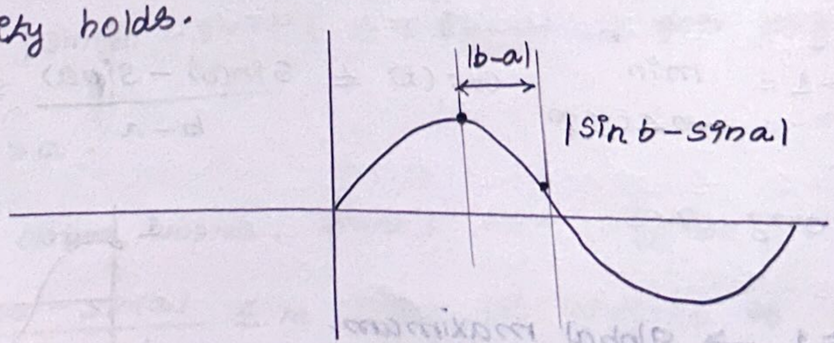
Also,
$$\frac{\sin(b) - \sin(a)}{b-a} = \frac{\sin(a) - \sin(b)}{a-b}.$$

So rewriting inequality with the fraction:

$$\boxed{\left| \frac{\sin(a) - \sin(b)}{a-b} \right| \leq 1 \text{ for any } a < b. \text{ for any } a, b}$$

$$\boxed{|\sin(b) - \sin(a)| \leq |b-a| \quad \text{for any } a, b.}$$

where the constant ~~coeff~~ $c = 1$. (smallest c where this inequality holds.



" Geometrically, this last inequality says that the vertical distance b/w any two points on the graph of sin(t) is at most equal to the horizontal distance b/w the two points.