$$v = c_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 3 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\qquad\qquad}_{S} \qquad \underbrace{\qquad\qquad}_{S^{\perp}}$$

$$\begin{pmatrix} 1 & 1 & 0 & -5 \\ 2 & 3 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = v$$

↳ Linearly independent - Invertible.

For every v, we can have unique c_1, c_2, c_3, c_4

$$A = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 - x_3 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A^T y = 0 \Rightarrow \begin{pmatrix} x_1 - x_2 & x_2 - x_3 & x_1 - x_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

For what y?

<u>Solu:</u>

$$y_1(x_1 - x_2) + y_2(x_2 - x_3) + y_3(x_1 - x_3) = 0$$

$$y_1 = 1, \; y_2 = 1, \; y_3 = -1.$$

<u>Also</u>

$$y^T b = 1$$

$$(y_1 \;\; y_2 \;\; y_3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = y_1 + y_2 + y_3 = 1 + 1 - 1 = 1.$$

<u>16.2</u>: r, n, c - Non-zero vectors and l are the four non-zero vectors in R^2.

a) What are the conditions for those to be bases for the four fundamental subspaces $C(A^T)$, $N(A)$, $C(A)$, and $N(A^T)$ of a 2 by 2 matrix?

b) What is one possible matrix A?

<u>Solu:</u>

In order for r and n to be bases for $N(A)$ and $C(A^T)$, we must have

$$r \cdot n = 0$$

(Row & null space - must be orthogonal). Similarly, in order for c and 1 to form bases for C(A) and N(A^T) we need,

$$c \cdot 1 = 0$$

In addition,

$$\dim N(A) + \dim C(A^T) = n$$

$$\dim N(A^T) + \dim C(A) = m$$

∴ In our case,

2×2 - matrix

As the four vectors are non-zero, they automatically satisfy 1+1 = 2.

b) $A \rightarrow$ possible matrix?

$A = cr^T$

Each column of A will be a multiple of c, so it will have the desired column space. On the other hand the row space will be the multiple of r, A will have the desired row space.

Desired row & column space $\longrightarrow$ Results Null space.

A = 2×2

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix} = a \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

(2×1) (1×2)

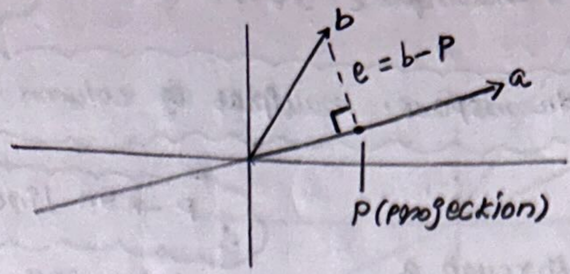
Projections onto subspaces

Lecture - 16

Projections!

least squares

... matrix.



To find the point along a - closest to b.

$$\boxed{p = xA \rightarrow \text{(some multiple of a)} \quad \hookrightarrow \text{like half of a}}$$

a is perpendicular to e.

$a^T e = 0$

$a^T (b - p) = 0$

$a^T (b - xa) = 0$

$x a^T a = a^T b$

$$x = \frac{a^T b}{a^T a}$$

$$\boxed{p = xa}$$

$$p = a \cdot \frac{a^T b}{a^T a}$$

when $b = 2b$

$p = 2p$

when a is doubled

$\boxed{p = p}$

when a is $-a$

$\boxed{p = p}$

projection is the same.

proj $p = Pb$

P — projection $p \rightarrow$ That projects b. P - matrix

$$P = \frac{a a^T}{a^T a}$$

matrix multiplication is

$$\therefore p = a \frac{a^T b}{a^T a}$$

$p = \mathbb{P}b$ → column space of the line a.

$\mathbb{P} = \frac{aa^T}{a^Ta}$ — columnspace: multiples of column vectors.

∴ $p \rightarrow$ on line a

$C(\mathbb{P}) \rightarrow$ Line through a

↳ For any b, $\mathbb{P}b$ lies on a.

Rank $(\mathbb{P}) \rightarrow 1$.

"column space of $\mathbb{P}$ is spanned by a because for any b, pb lies on the line determined by a.

∴ $p = xa$

column space: "multiply any vector b, we always land in the column space".

$$\mathbb{P} = \left(a\frac{a^T}{a^Ta}\right)$$

∴ $p = a\frac{a^Tb}{a^Ta}$ → multiplying with the vector b.

↳ column space of $\mathbb{P}$ is on the line.

Rank $= 1$ ⟶ aa^T (a-vector)

$(1\times3)(3\times1) = $ Rank 1

Column × row

↳ Basis for column space (1d)

Is it symmetric:

"yes" $\mathbb{P}^T = \mathbb{P}$

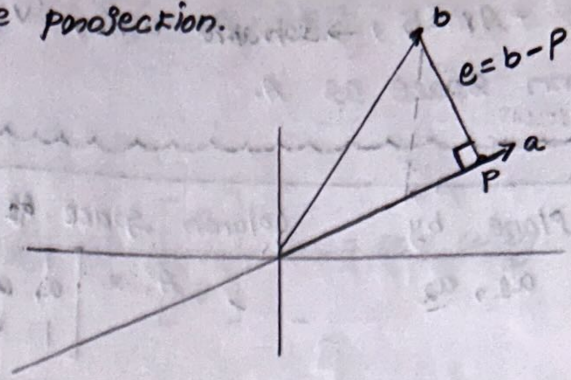
$$aa^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}(a_1\ a_2\ a_3) = \begin{pmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_3a_2 & a_3^2 \end{pmatrix}$$

↳ symmetric

Projection twice:

$p = \mathbb{P}b$

$\boxed{P^2 = P}$ → Same projection.



P once projects b to p. Again will yield the same result.

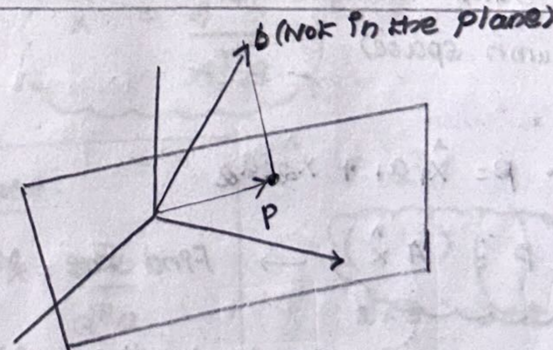
$P = \frac{aa^T}{a^Ta}$	$p = Pb$	Property: $P^T = P$, $P^2 = P$	$x = \frac{a^Tb}{a^Ta}$

more dimensions.

Why projection: our chapter deals with $Ax=b$ may have no solutions. (more eqn than unknowns). So I will solve the closest problem.

* change b to the closest vector in the column space of A to solve $A\hat{x} = p$.

↳ projection of b on to column space.



Project b (not in the plane) to p nearest to b in the plane.

what the plane is?

'Basis a_1 and a_2'

a_1 & a_2 → Independent (No need to be orthogonal)

plane – column space.

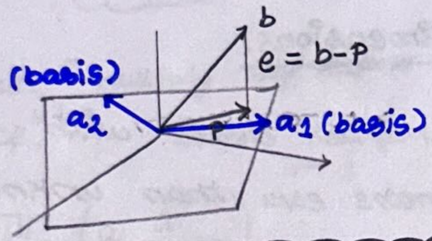
$Ax = b \rightarrow$ Solvable when vector b is in the column space of A.

Plane by a_1, a_2	=	Column space of $A = \begin{bmatrix} \vert & \vert \\ a_1 & a_2 \\ \vert & \vert \end{bmatrix}$

If b is in the column space, then it's p.

But most likely,

we have an error which is not zero.



e is perpendicular to the plane. $\rightarrow$ vector (e)

what is p?

Some multiple of the columns of A (basis of the column space)

$\therefore p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$p = A\hat{x}$ $\rightarrow$ Find the right combinations of the columns, so that the error vector is $\perp^r$ to the plane.

$p = A\hat{x} \rightarrow$ Find $\hat{x}$

Key: $(b - A\hat{x}) = e \rightarrow \perp^r$ to the plane

meaning: vector e is orthogonal to the column space of the A

$a_1^T (b - A\hat{x}) = 0$, $a_2^T (b - A\hat{x}) = 0$

As a matrix,

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{one way.}$$

$$A^T (b - A\hat{x}) = 0$$

on line: A has only one column.

Connecting Subspaces:

$(b - A\hat{x}) = e \rightarrow$ what subspace. $N(A^T)$ (Null space of A)

Null space of A^T

e in $N(A^T)$
$e \perp C(A)$

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = \frac{A^T b}{A^T A}$$

1 dimensional case:

$$x = \frac{a^T b}{a^T a}$$

$x = \frac{\text{number}}{\text{number}}$ (Ratio)

$\therefore$ $\underset{1\times 3}{\overset{a^T}{(\quad)}} \underset{3\times 1}{\overset{b}{(\quad)}} = \underset{1\times 1}{(\,)}$

(Number)

$\underset{1\times 3}{\overset{a^T}{(\quad)}} \underset{3\times 1}{\overset{a}{(\quad)}} = \underset{\text{number}}{(\,)}$

3-dimension case:

In this case

$A^T A \rightarrow$ matrices $(n\times n)$

Formulas:

* what about projection matrix?

$\hat{x} = (A^T A)^{-1} (A^T b)$

Projection:

$P = A\hat{x}$

$P = \underbrace{A (A^T A)^{-1} (A^T}_{\mathbb{P}} b)$

1d case

$\mathbb{P} = \frac{aa^T}{a^T a}$

$aa^T \to m \times m$

$a^T a \to$ numbers

Projection matrix $\mathbb{P} = A(A^T A)^{-1} A^T$

If A is a square matrix:

$A A^{-1} (A^T)^{-1} A^T = I$

But our case may be rectangular matrix.

So $A \to$ may not have an inverse.

$\therefore A^T A \to$ square matrix

So we are not allowed to do so.

If $A \to$ perfectly square, invertible

$\hookrightarrow$ Then column space is entire R^n

(Full matrix)

Projection matrix: Identity matrix.

$\therefore$ b is already in R^n (Already in column space).

$\mathbb{P}^T = \mathbb{P}$, $\mathbb{P}^2 = \mathbb{P}$

$\mathbb{P} \to$ Leaves us in P (closest point) on the plane.

Again projecting using $\mathbb{P}$ will leave me on the same point.

$\therefore$ P is the closest point from $\vec{b}$.

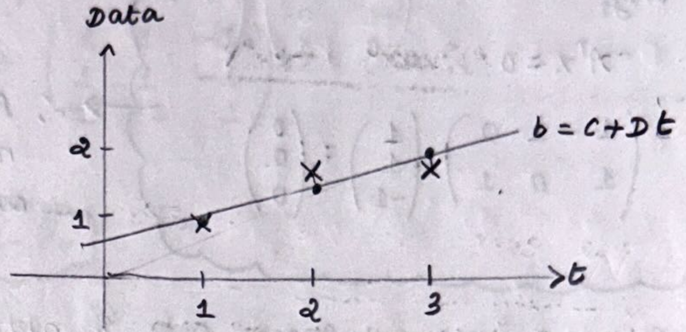
$\mathbb{P}^2 = [A A^{-1} (A^T)^{-1} A^T] [A A^{-1} (A^T)^{-1} A^T]$

In case of General case:

$$P^2 = \underbrace{\left[A\,(A^TA)^{-1}\,A^T\right]\left[A\right.}_{\text{cancels}}\left.(A^TA)^{-1}A^T\right]$$

$$= \left[A\,(A^TA)^{-1}\,A^T\right] = P$$

Least square: Fitting by a line: (Application)



(1, 1), (2, 2), (3, 2)

Line: going close to the three points. Line equ ($mx+c=y$)

we can't solve
$$\begin{pmatrix} C+D=1 \\ C+2D=2 \\ C+3D=2 \end{pmatrix}$$

So what's the closest answer:

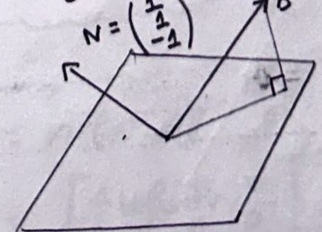
$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}} \rightarrow \text{No solution.}$$

'best solution?'

$$\boxed{A^TA\hat{x} = A^Tb} \rightarrow \text{we can solve for } \hat{x}$$

Find the orthogonal projection matrix on to the plane.

$x+y-z=0$



$\mathbb{P} = A\,(A^T A)^{-1} A^T$

$A \rightarrow$ columns $a_1\ a_2$

$A = \begin{pmatrix} | & | \\ a_1 & a_2 \\ | & | \end{pmatrix}$

<u>one choice:</u>

$$a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},\quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Then only, ~~because~~

$$A^T x = 0$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow$ $\therefore$ Row & null spaces are in orthogonal.

Column & Left null spaces are in orthogonal.

<u>Computing:</u>

$$A^T A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{4-1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\mathbb{P} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

<u>verification:</u>

projection matrix $\times$ Normal $= 0$

$$\mathbb{P} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \text{verified}$$

we can choose A as per our wish in accordance with our problem.

$$\boxed{Ib = \mathbb{P}b + \mathbb{P}_N b}$$

$$\boxed{I = \mathbb{P} + \mathbb{P}_N} \qquad \boxed{\therefore \mathbb{P} = I - \mathbb{P}_N}$$

$$\mathbb{P}_N = N(N^T N)^{-1} N^T$$

$$\mathbb{P}_N = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$$

$$\mathbb{P}_N = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\boxed{\mathbb{P} = I - \mathbb{P}_N}$$

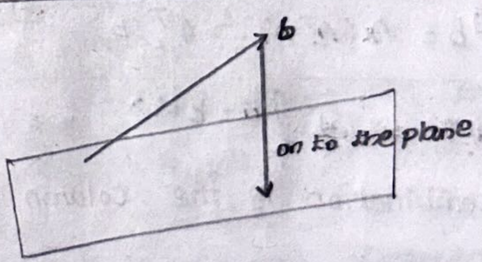
Any vector is a sum of two components.

1) projection on to the plane $\rightarrow \mathbb{P}b$

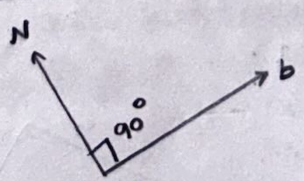
2) projection on to the orthogonal component of the plane.

---

projection on to the plane: $\mathbb{P}b$



Projection on to the orthogonal component of the plane:



1) Projection is easier in 1-d.

2) we have given N.

$$\boxed{b = Pb + P_N b}$$

$$\mathbb{P} = A(A^T A)^{-1} A^T \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Shape of P = 4 by 4. 4-d vector (w, x, y, z)

Projection of b:

$$P = Pb = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

Column space of A → wxy plane (not in z plane)

Lecture-17

$Pb = b$ → when b is already in the column space

$Pb = 0$ → when b is perpendicular to the column space

$Pb = 0$

$$P = A(A^T A)^{-1} A^T$$

↓

what vectors are $\perp^r$ to the column space exactly

↓

nullspace of A^T

∴ b → Is in null space → of A^T

∴ $A^T b = 0$

$$p = Pb = A(A^T A)^{-1} \underbrace{A^T b}_{=0}$$

$Pb = b$?

b → combination of the column space,

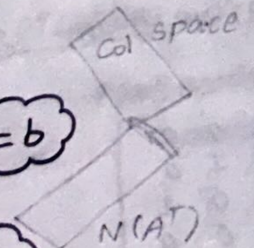
$\boxed{p = Ax}$

$$p = Pb = A(A^T A)^{-1} A^T (Ax)$$

$$= Ax$$

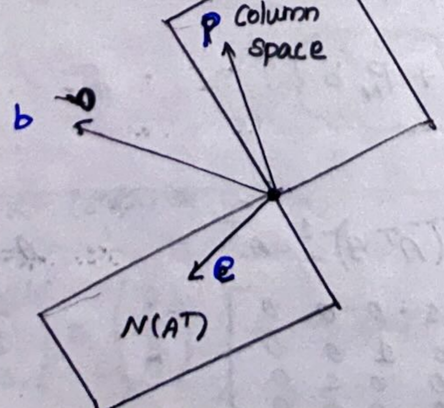
∴ $P = b$ $\boxed{P = P = b}$

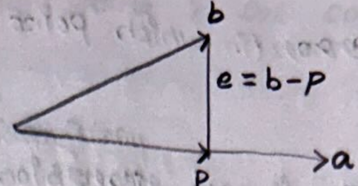
Geometrically,



$p + e = b$

$p = Pb$





$\therefore p = xa$

what about e? → e is also a projection of $N(A^T)$ space.

$\therefore e = b - p$

$e = b - \bar{P}b$

$e = (I - \bar{P})b$

↳ projection on to the $\perp$ space.

If $\bar{P}$ is a projection $(I-\bar{P})$ is also a projection

$\bar{P}$ is a symmetric $(I-\bar{P})$ is also a symmetric

$\bar{P}^2 = \bar{P}$ then $(I-\bar{P})^2 = (I-\bar{P})$

$\therefore$ e is perpendicular to the column space.

$\bar{P} = A(A^TA)^{-1}A^T$ → Basis for our column space.

$\therefore$ p is a vector in the column space → 1-d case.

<u>Application</u> (Find the best straight line)

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

↳ b doesn't in the column space.

↳ Basis for the column space (Independent)

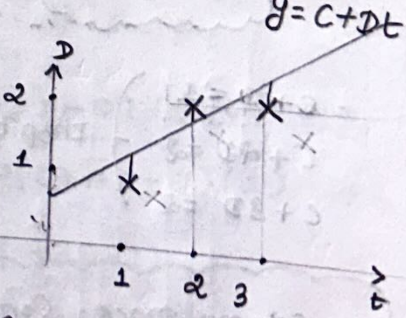
<u>minimize</u> (square & add up: Least squares:)

$$\|Ax - b\|^2 = \|e^2\|$$

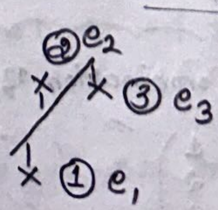
↳ Length.

$\|Ax-b\|^2 = e_1^2 + e_2^2 + e_3^2$

e is 0 only when b is in the column vector space. In this case, we have errors.



(3 errors)

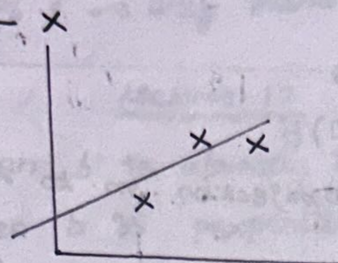


$e_1, e_2, e_3 \rightarrow$ error (in which point missed the solution line).

Fitting straight line: linear regression: (பின்நோக்கி)

If I have a fourth point away (far), is still the line same.

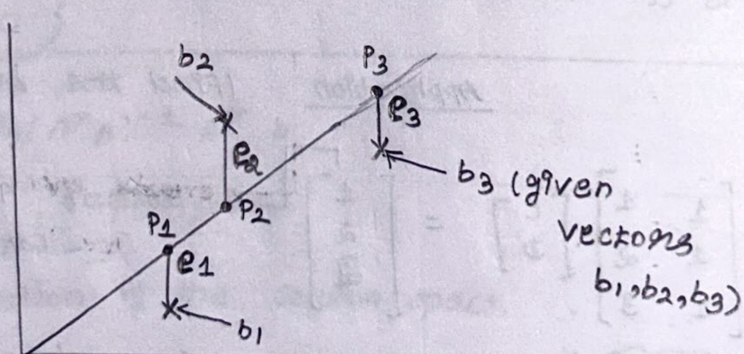
outlier ← (far away) mistake.



→ Any one don't want their solution completely turn up due to this one outlier.

Least Square: overcompensates the outliers (errors)

↳ we need to minimize them. (worst case).



$$e_1 = P_1 - b_1$$
$$e_2 = P_2 - b_2$$
$$e_3 = P_3 - b_3$$

P_1, P_2, P_3 lies on the line.

$$\left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\}$$ Inspite of $b_1, b_2, b_3 \rightarrow$ with P_1, P_2, P_3 we can solve the eqn.

Column space: Combination of Columns of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

$b : \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

p is the closest point in the column space.

Fine $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p $\hat{x} \rightarrow$ Estimated.

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$ $\rightarrow$ Symmetric Invertible positive.

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\left.\begin{aligned} 3C + 6D &= 5 \\ 6C + 14D &= 11. \end{aligned}\right]$$ Normal equations.

$\therefore \quad \|Ax - b\|^2 = \|e^2\|$

$= e_1^2 + e_2^2 + e_3^3$

$= (C + D - 1)^2 + (C + 2D - 2)^2 + (C + 3D - 2)^2$

$\rightarrow$ overall squared error (Need to minimize).

$$\therefore \left.\begin{aligned} C + D &= 1 \\ C + 2D &= 2 \\ C + 3D &= 2 \end{aligned}\right\}$$ some errors

$\therefore C + D - 1 = e_1$

$C + 2D - 2 = e_2$

$C + 3D - 2 = e_3$

<u>using calculus:</u>

$e \rightarrow$ Function ~~of~~ two variables. (minimum = ?)

w.r.t C

$$2\left(\frac{\partial e}{\partial c}\right) = 2(C + D - 1)(1) + 2(C + 2D - 2)(1) + 2(C + 3D - 2)(1)$$

$$0 = (C + D - 1) + (C + 2D - 2) + (C + 3D - 2)$$

$$0 = 3C + 6D - 5$$

$3C + 6D = 5 \rightarrow$ our 1st normal equ.

w.r.t d

$$0 = (C+D-1)2 + 4(C+2D-2) + 4(C+3D-2)$$

$\boxed{11 = 6C + 14D}$ → 2nd Normal eqn

conclusion: calculus & linear algebra both giving the same P

① ×2 $6C + 12D = 10$

$6C + 14D = 11$

$-2D = -1$

$D = 1/2$

$3C = 5 - 6(1/2)$

$3C = 2$

$C = 2/3$

∴ Best line: $\frac{2}{3} + \frac{1}{2}t$

P_1 = (At t = 1)

$y = \frac{2}{3} + \frac{1}{2}$

$= \frac{4+3}{6} = \frac{7}{6}$

P_2 = (At t = 2)

$y = \frac{2}{3} + \frac{1}{2}(2)$

$= \frac{2}{3} + 1$

$= \frac{5}{3}$

P_3 = (At t = 3)

$y = \frac{2}{3} + \frac{3}{2}$

$= \frac{4+9}{6} = \frac{13}{6}$

e_1 (error 1):

$\frac{7}{6} - 1 = \frac{1}{6}$

$(P_1) - b_1$

e_2 (error 2):

$\frac{5}{3} - 2 = -\frac{1}{3}$

(other direction)

∴ $b_2 > P_2$

e_3 (error 3)

$\frac{13}{6} - 2 = \frac{1}{6}$

∴ $p + e = b$

$$\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}7/6\\5/3\\13/6\end{bmatrix} + \begin{bmatrix}-1/6\\1/3\\-1/6\end{bmatrix}$$ → changed sign.

$$\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

what about p and e:

'They are perpendicular'

$$\begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↳ In the column space.

Is e is $\perp^r$ to the whole column space?

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow$$ yes $\perp^r$ to all vectors in $C(A)$.

↳ In the column space

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$p = A\hat{x}$

$$p = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 + 1/2 \\ 2/3 + 1 \\ 2/3 + 3/2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}$$

$A^T A \rightarrow$ Announced as invertible.

If A has independent columns then $A^T A$ is invertible.

'Back-up theory'

To prove:

$A^T A x = 0 \rightarrow$ show $A^T A$ is invertible.

show $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only solution.

Solu:

matrix is invertible, when it's null space is $\{0\}$.

TRICK:

$x^T A^T A x = 0$ [multiply by x^T]

$(Ax)^T Ax = 0$

If $y^T y = 0$ what does this tells me:

$\therefore$ y must be zero.

$\therefore Ax = 0 \Leftarrow (Ax)^T Ax = 0 \Rightarrow x = 0$

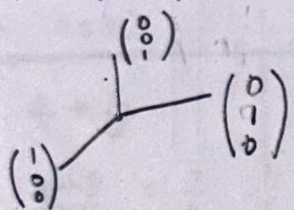
If A is independent, $Ax = 0$, then $x = 0$

---

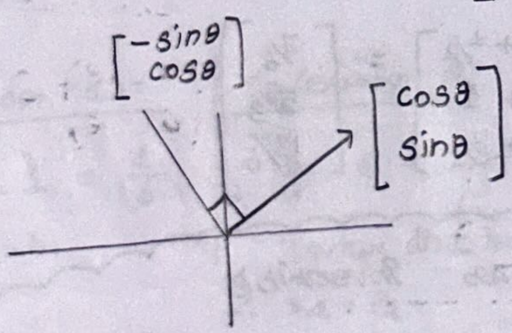
case: columns are certainly independent.

↳ If they are perpendicular unit vectors.

eg. $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$



orthonormal vectors: matrices with their columns $\perp^r$ to each other.

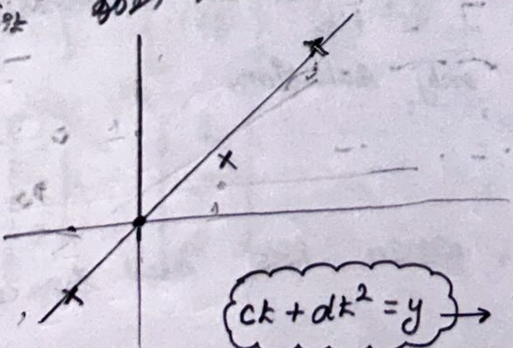


orthonormal vectors are easy to solve.

Recitation - Least square approximation

Find the quad curve through the origin that is a best fit for the points $(1,1)$, $(2,5)$, $(-1,-2)$

solu:



$ct + dt^2 = y$ → Through origin → Constant term $= 0$

At $t=1, y=1$	$t=0, y=0$	$t=2, y=5$	$t=-1, y=-2$
$c+d=1$	$0+0=0$	$2c+4d=5$	$-c+d=-2$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}, \quad \hat{x} = \begin{pmatrix} c \\ d \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$$

we can't solve $Ax = b$

Best approximation:

$$A^T A \hat{x} = A^T b$$

$$A^T A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 8 & 10 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 8 \\ 8 & 10 \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = \begin{pmatrix} 13 \\ 19 \end{pmatrix}$$

$$6\hat{c} + 8\hat{d} = 13$$
$$8\hat{c} + 10\hat{d} = 19$$

(or)

Elimination

$$\begin{pmatrix} 6 & 8 \\ 0 & -2 \end{pmatrix} \hat{x} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

$$d = -\frac{5}{2}, \quad c = \frac{11}{2}$$

$$\boxed{\therefore \quad y = \frac{11}{2}t - \frac{5}{2}t^2}$$

$$Pe = P(b-p)$$
$$= Pb - Pp$$
$$Pe = p - p = 0$$

$p = Pb$

$Pp = p$

$\therefore$ multiply by P in the point p will project to the same p.

orthogonal matrices & Gram - schmidt

many calculations become simpler when performed using orthonormal vectors or orthonormal matrices. (Procedure does converting any basis to orthonormal one).

Lecture-18

orthonormal vectors:

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \rightarrow \text{ortho} \\ 1 & \text{if } i = j \rightarrow \text{Normal (unit length - normalize)} \end{cases}$$

* No overflow
* No underflow.

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \dots & q_n \\ | & | & & | \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & \dots \\ \dots & & & 1 \end{bmatrix}$$

$$Q^T Q = I$$

New-type of matrices:

*like projection matrices - orthonormal matrices

orthonormal matrix Q : when it's square.

we get inverse.

when Q is square, $Q^T Q = I$

$$Q^T = Q^{-1}$$

Examples:

Any permutation matrix.

$$P_{31} P_{23} = Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$$ Another ortho-normal matrix

$$\boxed{QQ^T = I}$$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow$$ Is that orthogonal?

(columns are orthogonal)

↳ But rows are not.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$$ diving by $\frac{1}{2}$ gives I.

$$\therefore \quad Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \text{orthogonal.}$$

$$\boxed{Q^T Q = I}$$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad QQ^T = I \text{ when divide by } \frac{1}{4}.$$

So, $Q = \frac{1}{2}Q \rightarrow$ yield orthogonal matrix.

↳ Named after Hadamard.

we know how to do it for two, four, sixteen, 64, ...
But nobody knows exactly which size matrices have (allows) orthogonal matrices of 1's and -1's.

<u>why it is good to have orthogonal matrices?</u>

matrix could be rectangular!

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

↳ orthonormal basis for this 2-d space.

'Independent'

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

↳ No square root.

Gram - Schmidt: we have square roots.

what's the good of having Q?

Q has orthonormal columns.
project onto its column space.

what's the projection matrix?

$$P = Q(Q^TQ)^{-1}Q^T$$

$$P = Q(I)^{-1}Q^T$$

$$P = QQ^T$$

$Q^TQ^{-1} = I$

$Q \quad Q^T$
$(2\times3)\ (3\times2) = 2\times2$

$Q^T \quad Q$
$(3\times2)\ (2\times3) = (3\times3)$

If Q is square,

$P = I$

In case of Independent square matrix → The column space is their (column's) combination. span the entire space (R^n). So $b = P$, $P = I$

$\therefore$ b is already in the column space.

In case of not square → we don't have $b = P$

Properties of projection matrix:

* Symmetric
* Again projection won't move the point

$$(QQ^T)(QQ^T) = QQ^T$$

$\therefore$ In case of orthonormal:

$Q^TQ = I$ $\left[(QQ^T)(QQ^T) = QQ^T\right]$

our equations become trivial - Incase of orthonormal basis.

eg: Normal eqn: $A^TA\hat{x} = A^Tb$

Now A is Q

$$Q^TQ\hat{x} = Q^Tb \Rightarrow I\hat{x} = Q^Tb$$

$$\boxed{\hat{x} = q^T b}$$

$$\boxed{\hat{x}_q = q_q^T b}$$

$$q^T b = \begin{bmatrix} q_1 & - \\ q_2 & - \\ \vdots & \\ q_n & - \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

dot product.

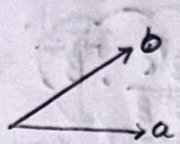
Independent vectors $\xrightarrow[\text{make}]{}$ ortho-normal vectors

<u>Gram – Schmidt :</u>

'Goal : make the matrix orthogonal'

↳ columns : orthonormal.

<u>vectors a, b</u>



(Independent)

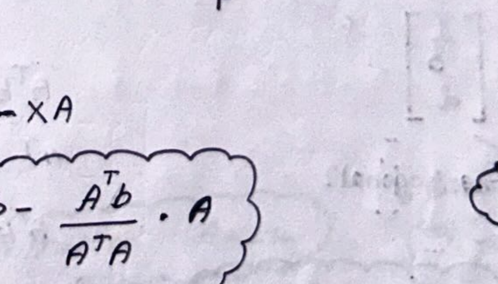
produce q_1 & q_2

$a, b \rightarrow A, B \rightarrow$ orthonormal

orthogonal $\quad q_1 = \dfrac{A}{\|A\|} \quad q_2 = \dfrac{B}{\|B\|}$

Problem: direction of b is not fine (not orthogonal to a)

Idea: vector: starts with b & makes it orthogonal to a



(ok, we are taking as it is)

$B = b - xA$

$$\boxed{B = b - \frac{A^T b}{A^T A} \cdot A} \qquad \boxed{A \perp B}$$

$A^T B = 0$

$$A^T \left(b - \frac{A^T b}{A^T A} \cdot A \right) = A^T b - A^T b \cdot \frac{A^T A}{A^T A}$$

$$= A^T b - A^T b = 0 \quad \text{So } \boxed{A \perp B}$$

Now $a, b, c \rightarrow A, B, C \rightarrow$ orthonormal

orthogonal

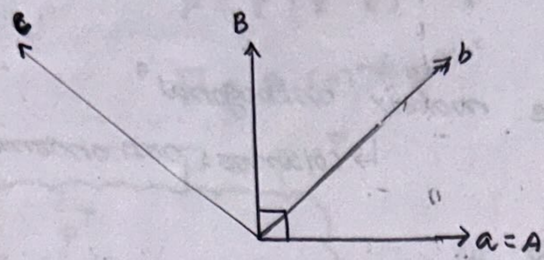
$v_1 = \frac{A}{\|A\|}$, $v_2 = \frac{B}{\|B\|}$,

$v_3 = \frac{C}{\|C\|}$

↳ divided by its length.

we have brough A and B $\perp^r$

Now: c to $\perp^r$ to both A and B.



$$C = c - \frac{A^T c}{A^T A} \cdot A - \frac{B^T c}{B^T B} \cdot B$$

↳ from b direction.

↳ It's projection from a direction

$$\boxed{\begin{matrix} C \perp A \\ C \perp B \end{matrix}}$$

∴ In $A \perp B \rightarrow$ we subtract the projection from a

∴ In this case of C, we have two errors. That affects a from being orthogonal to a & c from being orthogonal to b.

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Not orthogonal.

$A^T b = (1\ 1\ 1)\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$= 3$

$A^T A = 3$

Solu:

$\boxed{A = a}$

$$B = b - \frac{A^T b}{A^T A} \cdot A$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

<u>Is $A \perp B$:</u>

$$A^T B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = 0 \rightarrow \text{yes.}$$

$$Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \rightarrow \text{Normalized.}$$

↳ orthonormal.

<u>Column space of Q related with A:</u>

plane - vectors in 3d.

column space of un-normalized $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

is one and the same as the column space of

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

∴ B → combination of b and a [working in the same space ↳ To get 90°]

<u>Final point:</u>

In elimination $A = LU$

$q_1 \rightarrow$ unit vector in the direction of a_1.

<u>In gram-Schmidt:</u>

$A = QR \rightarrow$ Some Combinations.

$$\begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ a_1^T q_2 & a_2^T q_2 \end{bmatrix}$$

$R \rightarrow$ upper triangular

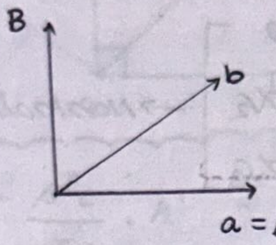
why $a_1^T a_2 = 0$?

Idea: constructed these later a-s to be perpendicular to the earlier vectors.

$A = \hat{Q} R \rightarrow$ connection b/w A and Q is a triangular matrix.

$A \downarrow$ matrix with Independent columns

$Q \downarrow$ ortho-normal columns,



$\therefore$ Normalized $A \rightarrow a_1$

Normalized $B \rightarrow a_2$.

a_2 is $\perp^r$ to a_1 = 0.

Recitation

Find a_1, a_2, a_3 (orthonormal) from a, b, c (columns of A)

Then write A as QR (Q → orthogonal, R – upper triangular)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Solu:

$$a_1 = \frac{a}{\|a\|} = \frac{a}{1} = a$$

$$\boxed{a_1 = a}$$

$$B = b - \frac{A^T b}{A^T A} \cdot A$$

$$= \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \frac{2}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A^T b = (1\ 0\ 0)\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 2$$

$$A^T A = (1\ 0\ 0)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$$

$$= \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$A \perp B \rightarrow$ verify $(1\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0$

$$a_2 = \frac{B}{\|B\|} = \frac{B}{3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{a_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

a_3:

$$C = c - (c \cdot a_1)\, a_1 - (c \cdot a_2) \cdot a_2$$

$$= \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$\boxed{\therefore C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B}$$

$$a_3 = \frac{C}{\|C\|} = \frac{1}{5} \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Q-R decomposition:

$$A = QR$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} R$$

↳ permutation matrix changes R_2 and R_1

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Analyze

Matrix multiplication by columns

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} \\ a \quad\; b \quad\; c \end{array} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ a_1 \quad a_2 \quad a_3 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$a = a_1$

$b = 2a_1 + 3a_2$

$c = 4a_1 + 6a_2 + 5a_3$.

$\boxed{a = 1a_1 = A}$

$B = 3a_2$

$C = 5a_3$

$$C = c - 4\alpha_1 - 6\alpha_2$$

$$\boxed{5\alpha_3 + 4\alpha_1 + 6\alpha_2 = c}$$

$$B = b - 2\alpha_1$$

$$\boxed{3\alpha_2 + 2\alpha_1 = b}$$

$$\boxed{a = \alpha_1}$$

Determinant - Lecture 19

single number : which encodes a lot of info about the matrix.

Three simple properties completely describe the determinant.

'Square matrices'

Determinants : Eigen values → Associated with every square matrix.

$\det A = |A|$

Invertible → $\det A \neq 0$.

singular → $\det A = 0$

Properties :

1) $\det I = 1$

2) Exchange rows : Reverse the sign of the determinants.

3)

Relating : 1 and 2

Exchanging rows in I → permutation matrix.

$\det P = 1$ (or) -1

(Depending on - No. of exchanges → odd $= -1$
Even $= 1$)

---

1 $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

2 $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

3 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3a) $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t(ad - bc) = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3b) $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = (a+a')d - (b+b')c = (ad-bc) + a'd - b'c$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$.

'linear for each row - 1st row'

↳ Not in both row.

(combinations - linear in any one row).

---

④ If two rows are equal - Determinant is zero.

$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

why it's true for nxn?

Exchange those two rows (equal) → same matrix

↳ same determinant

$\det A = 0$

Two equal rows: Not invertible.

---

⑤ Prop: Subtract (some multiple) $l \times$ Row i from row k → determinant doesn't change.

$\det A = \det U$

exam:

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \to \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = ad - alb - bc + bla$

$= ad - bc$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix}$

↳ property ④

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix}$

∴ Row 2 → combination of Row 1.

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

---

⑥ Prop: Row of zeros → $\det A = 0$

$$5\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 5.0 & 5.0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$ when $t=0$, $\det A = 0$

---

7) Prop: Matrix → triangular

$$a = \begin{vmatrix} d_1 & * & * & * & * \\ 0 & d_2 & * & * & * \\ 0 & 0 & d_3 & * & * \\ 0 & 0 & 0\,0\,0 & & d_n \end{vmatrix} = d_1 \times d_2 \times d_3 \times \dots \times d_n$$

matlab: Elimination → Triangular → Product of pivots (determinant)

↳ In case of row exchange = odd → $-\det u$
Even → $+\det u$..

<u>case: pivots: non-zero:</u>

<u>diagonal matrix:</u> → Row reduced form.

$$\begin{vmatrix} d_1 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & 0 & \dots & & d_n \end{vmatrix}$$

(By property → Row reduction doesn't change determinants)

$$d_n \dots d_3\, d_2\, d_1 \begin{vmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & \dots & & \\ 0 & 1 & 1 & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$ → Factoring out

↳ By property ④, By property ① $\det I = 1$.

'Product of pivots'

---

8) <u>case: pivot entry = 0 (diagonal = 0):</u>

$\det A = 0$ → singular case.

$\det A \neq 0$

A → Invertible.

<u>Determinant</u>

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix} = ad - a\left(\frac{cb}{a}\right) = ad - bc$$

(property)

diagonal $d_1 \times d_2$

---

9) $\det(AB) = (\det A)(\det B)$ / $\det A^{-1} = ?$

$A^{-1}A = I$

$(\det A^{-1})(\det A) = \det$