Let $r \to r_0$ Therefore since $p(r_0) = 0$ $\longrightarrow$ Given

$$p(D)(te^{r_0 t}) = p'(r_0)\, e^{r_0 t} + p(r_0)\, te^{r_0 t}$$

$$= p'(r_0)\, e^{r_0 t}$$

If $p'(r_0) \neq 0$ then, we can divide

$$p(D)\left(\frac{te^{r_0 t}}{p'(r_0)}\right) = e^{r_0 t}$$

and $y_p = \dfrac{te^{r_0 t}}{p'(r_0)}$ is a particular solution to

$p(D)y = e^{r_0 t}$.

Example: 10.1: $\ddot{x} - 4x = e^{-2t}$

solu:

$p(r) = r^2 - 4$ thus $p(-2) = 0$ But $p' = 2r$

$p'(-2) = -4 \neq 0$

$\therefore$ $x_p = \dfrac{te^{rt}}{p'(r)} = \dfrac{te^{-2t}}{-4}$

Generalized exponential response formula.

If p is a polynomial and r_0 is a number

such that

$p(r_0) = p'(r_0) = \dots = p^{(m-1)}(r_0) = 0$ $(p^{(m)}(r_0) \neq 0)$

then $p(D)(t^m e^{r_0 t}) = p^{(m)}(r_0)\, e^{r_0 t}$

$y_p = \dfrac{1}{p^{(m)}(r_0)}\, t^m e^{r_0 t}$ is a particular solution

to $p(D)\, y = e^{r_0 t}$

In other words, multiply the input signal by t^m, and then multiply by the number $1/p^{(m)}(r_0)$

$p^{(m)} \rightarrow m^{th}$ derivative of p.

Proof: Generalized ERF.

$$P(D)\, e^{r_0 t} = P(r_0)\, e^{r_0 t}$$ for all x, How ever

$P(r_0) = 0$, we can't devide it

Solu.

$$P(D)\, e^{rt} = P(r)\, e^{rt}$$

$$\frac{\partial}{\partial r}\left(P(D)\, e^{rt}\right) = \frac{\partial}{\partial r}\left(P(r)\, e^{rt}\right)$$

$$P(D)\left(\frac{\partial}{\partial r} e^{rt}\right) = P'(r)\, e^{rt} + P(r)\cdot t e^{rt}$$

In that case that r_0 is a repeated root with multiplicity m, then $P(r) = Q(r)(r-r_0)^m$ and

$$\frac{d^m}{dr^m} P(r_0) \neq 0$$

$$\therefore P(r) = Q(r)(r-r_0)^m$$

however, all lower derivatives are zero at r.

$$\frac{d^{m-1}}{dr^{m-1}} P(r_0) = 0$$

$$\vdots \qquad \vdots$$

$$P(r_0) = 0$$

Diff both sides to get a relationship (m times)

$$P(D)(t^m e^{rt}) = \left(P^{(m)}(r) + m t P^{(m-1)}(r) + \frac{m(m-1)}{2} t^2 P^{(m-2)}(r) + \dots + t^m P(r)\right) e^{rt}$$

Evaluating at r_0, only the m^{th} derivative term survives

$$P(D)(t^m e^{r_0 t}) = P^{(m)}(r_0)\, e^{r_0 t}$$

$P(D)\, y = e^{r_0 t}$ of the form $y_p = \dfrac{t^m e^{r_0 t}}{P^{(m)}(r_0)}$

Binomial part.

$$P(D)(t^m e^{rt}) = \Big(P^{(m)}(r) + mt\,P^{(m-1)}(r) + \frac{m(m-1)}{2} t^2 P^{(m-2)}(r) + \dots + t^m P(r)\Big)e^{rt}$$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$$= 1t^0 P^{(4)}(r) + 4t\,P^{(3)}(r) + 6t^2 P^{(2)}(r) + 4t^3 P^{(1)}(r) + t^4 P^{(0)}(r).$$

$\ddot{x} + x = e^{it}$ with IC $x(0)=1$, $\dot{x}(0)=0$

Soln:

$r^2+1 \Rightarrow$ char polynomial $\Rightarrow$ roots $\pm i$.

From generalized ERF tells us to find the smallest integer s for which $p^{(s)}(i) \neq 0$.

In this case

$$p'(r) = 2r, \quad p'(i) = 2i \neq 0 \quad (s=1)$$

$$x_p(t) = \frac{1}{p'(i)} t^1 e^{it} = \frac{1}{2i} t e^{it} \qquad \left(\frac{1}{i} = e^{-i\pi/2}\right)$$

$$x_p(t) = \frac{1}{2} t e^{it - \frac{i\pi}{2}}$$

$\therefore$ The char polynomial has $\boxed{\text{roots}}$ $\pm i$, we know that (the) pair e^{it}, e^{-it} form a basis for the space of solutions to the homogeneous eqn

$\ddot{x} + x = e^{it}$ is

$$x(t) = \underbrace{c_1 e^{it} + c_2 e^{-it}} + \underbrace{\frac{1}{2} t\, e^{it - \frac{i\pi}{2}}}.$$

$x(0) = 1$, $\dot{x}(0) = 0$

$$x(0) = 1 = c_1 + c_2$$

$$c_2 = 1 - c_1$$

$$x(t) = c_1 e^{it} + (1-c_1)e^{-it} + \frac{1}{2} t e^{it - \frac{i\pi}{2}}.$$

we see that

$$\dot{x}(t) = ic_1 e^{it} - i(1-c_1)e^{-it} + \frac{1}{2} e^{it - \frac{i\pi}{2}} + \frac{1}{2} t e^{it}$$

we see that,

$$\dot{x}(0) = ic_1 - i(1-c_1) + \frac{1}{2} e^{-i\pi/2}$$

$$= ic_1 - i(1-c_1) - \frac{i}{2}$$

By $\dot{x}(0) = 0 \Rightarrow c_1 = \frac{3}{4}, \quad c_2 = 1 - \frac{3}{4}$

$$= \frac{1}{4}.$$

$$x(t) = \frac{3}{4} e^{it} + \frac{1}{4} e^{-it} + \frac{1}{2} t e^{it - \frac{i\pi}{2}}$$

Linear, Constant Co-efficient, Inhomogeneous ODES:

a) $\dot{x} + Kx = 1$

b) $\dot{x} + Kx = e^{-5t}$

c) $\dot{x} + Kx = 4 + 7e^{-5t}$

Solu:

$\dot{x} + Kx = 0$ $\quad \boxed{r + K = 0}$

$\boxed{r = -K}$

$\therefore \quad \boxed{x = c_1 e^{-Kt}}$ $\quad$ Let $x_h = e^{-Kt}$ $(c_1 = 1)$

Takeng a particular one.

$y = u(t) x_h.$

$$\dot{x} + Kx = 1$$

$$u'(t) e^{-Kt} + u(t)(-K) e^{-Kt} + K(e^{-Kt} u(t)) = 1$$

$$u'(t) e^{-Kt} = 1$$

$$\dot{u}(t) = e^{Kt}$$

$$u(t) = \frac{e^{Kt}}{K}$$

General solution $y = \left(\frac{e^{Kt}}{K}\right) \cdot e^{-Kt} + C_1 e^{-Kt} \Rightarrow y = \frac{1}{K} + C_1 e^{-Kt}$

For quick calculations:

Let $\dot{x} = 0$

$Kx = 1$

$x = \frac{1}{K}$

$$\boxed{y = \frac{1}{K} + ce^{-Kt}}$$

Not a systematic way

b) $\dot{x} + Kx = e^{-5t}$

$\therefore \boxed{r = -5}$

$x_b = \frac{e^{-5t}}{P(-5)}$

$P = S + K$ (or)

$r + K$ $(r = -5)$

$x_b = \frac{e^{-5t}}{(K-5)}$

$\therefore y = \frac{e^{-5t}}{K-5} + ce^{-Kt}$ $\Rightarrow K \neq 5.$

we can also use ERF

Systematic approach

$\dot{x} + 5x = e^{-5t}$ $(K = -5)$

$u'(t)\, e^{-5t} = e^{-5t}$

$u'(t) = e^{(-5+5)t}$

$$\boxed{\cancel{u(t) = \frac{e^{(K-5)t}}{K-5}}}$$

$u'(t) = 1$

$u(t) = t + C$

$x(t) = u(t)\, x_h$

$x = (t+c)\, e^{-5t}$

$\cancel{= \frac{e^{(K-5)t}}{K-5}}\; e^{-Kt}$

$\boxed{\text{when } K = 5 \text{ This true}}$

$x =$

c) $\dot{x} + Kx = 4 + 7e^{-5t}$

$\dot{x} + Kx = 4(1) + 7(e^{-5t})$

$= 4\left(\frac{1}{K} + ce^{-Kt}\right) + 7(t+c)e^{-5t}$

Basis of homogeneous solutions with linear operators

1) $y''' + 10y'' + 31y' - 30y = 0$

Soln:

$P(r) = (r^3 - 10r^2 + 31r - 30)$

$= (r-2)(r-3)(r-5)$

(order → 3 – Dimension in vector space of solutions = 3)

e^{2t} is a solution, since

$P(D)e^{2t} = P(2)e^{2t}$

$P(D)e^{2t} = 0$

e^{3t} " " $P(D)e^{3t} = 0$

e^{5t} " " $P(D)e^{5t} = 0$

Just because we wrote down 3 solutions doesn't mean that they form a basis ∴ if we had written down $e^{2t}, e^{3t}, 4e^{2t} + 6e^{3t}$, then they wouldn't need to form a basis, They are linearly dependendent (span = 2 dimensional)

Is e^{2t}, e^{3t} and e^{5t} linearly independent?

$e^{5t} = c_1e^{2t} + c_2e^{3t}$ (say) c_1, c_2 – some numbers

$(D-2)(D-3)e^{5t} = (D-2)(D-3)(c_1e^{2t} + c_2e^{3t})$

$= c_1(D-2)(D-3)e^{2t} + c_2(D-2)(D-3)e^{3t}$

$= c_1(0) + c_2(0)$ (By linearity)

$(D-2)(D-3)e^{3t} = (5-2)(5-3)e^{5t} \neq 0$

The contradiction implies that e^{2t}, e^{3t} and e^{5t} are linearly independent. Forms a basis in 3d – dim solution space.

Repeated roots with linear operators:

$D^3 y = 0 \quad (r^3 = 0, \quad r = 0, 0, 0)$

$D^2 y = c_1$

$Dy = c_1 t + c_2$

$y = c_1 \frac{t^2}{2} + c_2 t + c_3$ $\qquad C_1 = c_1/2$

$\boxed{y = C_1 t^2 + C_2 t + C_3}$ $\qquad e^0, te^0, t^2 e^0.$

$(D-5)^3 y = 0$

soln:

$(r-5)^3 \Rightarrow r = 5, 5, 5$

roots. $y = c_1 e^{5t} + t c_2 e^{5t} + t^2 c_3 e^{5t}$

$(D-5)ue^{5t} = (\dot{u}e^{5t} + u \cdot 5e^{5t}) - 5ue^{5t}$

$= \dot{u}e^{5t}$

Replacing $(D-1)$ by $(D-5)^2$ & $(D-5)^3$

$(D-5)^2 ue^{5t} = \ddot{u}e^{5t}$

$(D-5)^3 \ddot{u}e^{5t} = u^{(3)} e^{5t}$

14.1 $\quad 2\ddot{x} + \dot{x} + x = 1 + 2e^t$ $\qquad P(s) = 2s^2 + s + 1$

roots $= \left(\frac{-1 \pm \sqrt{7} i}{4} \right)$

$x_h(t) = e^{-t/4} \left(c_1 \cos\left(\frac{t\sqrt{7}}{4}\right) + c_2 \sin\left(\frac{\sqrt{7}t}{4}\right) \right)$

$= Ae^{-t/4} \left(\frac{\sqrt{7}t}{4} - \phi \right)$

The inhomogeneous solutn

$P(D)x = 1 + 2e^t$

$x_p = x_1 + 2x_2$

Thus constant function 1 is exponential: $1 = e^{0t}$

$P(D)x = 1$ has a particular solution.

$x_1 = \frac{1}{P(0)} = 1.$

Is a particular solution to $P(D)x = e^t$. Thus

$x_p = 1 + 2\left(\frac{1}{4}e^t\right)$

$= 1 + \frac{1}{2}e^t$

Is a particular solution to $P(D)x = 1 + 2e^{t}$, so

$2\ddot{x} + \dot{x} + x + 1 = 1 + 2e^{t}$ is $x(t) = 1 + \frac{1}{2}e^{t} + Ae^{-t/4}\cos\left(\frac{t\sqrt{7}}{4} - \phi\right)$

$\ddot{x} + 8\dot{x} + 15x = e^{-5t}$

$p(r) = r^2 + 8r + 15$, $P(-5) = 0$.

Generalizing, $p'(r) = 2r + 8$, $\boxed{p'(-5) = -2}$

$$x_p = \frac{te^{-5t}}{p'(-5)} = -\frac{te^{-5t}}{2}$$

ERF: $\frac{d^5 y}{dt^5} - y = e^{at}$

solu: $r^5 - 1 = 0 \Rightarrow 1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}$.

Comparison of ERF & variation of parameters

ERF works for	Variation of parameters work for
nth order DES	1st order DE's (so far)
Constant Coefficient DE s (LTI s)	Any linear DE
Exponential input	Any system input.

---

1) $(D^3 - D)x = 0 \Rightarrow x_1, -x_2, ax_1, x_2 + 1, ax_1 + bx_2$ are some solutions

---

$(D^3 - D)x = e^{2t}$

solu: $r^3 - r = 0$

$P(D)e^{2t} = P(2)e^{2t}$

$P(D)\left(\frac{e^{2t}}{P(2)}\right) = e^{2t}$

$P(2) = 8 - 2 = 6$

$$y_p = \frac{e^{2t}}{6}$$

$$(D^3 - D)x = e^{-t}$$

Soln:- $p(r) = r^3 - r$

$p(-1) = -1+1 = 0$

$$\frac{\partial}{\partial r}\left(P(D) \cdot e^{rt}\right) = \frac{\partial}{\partial r}\left(P(r) e^{rt}\right) \qquad \left(P'(r) e^{rt} + P(r) t e^{rt}\right)$$

$$P(D)\left(\frac{t e^{rt}}{P'(r)}\right) = e^{rt} \qquad \boxed{P(r) = 0}$$

$$y_p = \frac{t e^{-t}}{(3r^2 - 1)_{r=-1}} = \frac{te^{-t}}{3(1)-1} = \frac{te^{-t}}{2}$$

Complex replacement -

1. Use complex replacement to solve any inhomogeneous LTI system with sinusoidal input.

2. Find the complex gain of an LTI system in terms of the complex system response, & the complex valued system input.

3. Describe the phase shift & amplitude - gain of any LTI system with sinusoidal input signal in terms of the complex gain.

4. Describe conditions for stability in physical system & distinguish b/w long term (steady state) & transient behaviour in a stable system.

Boston Harbor example:

Suppose we are studying in the tides in Boston harbor. Let $x \rightarrow$ The water level in Boston harbor. Let y be the input (ocean level), which is responsible for the changing tides x in Boston harbor, the system response

Physics:

we assume that the ocean and the harbours are connected by a narrow channel so that the flow is slow and turbulent. This allows us to assume that the flow rate is pressure driven, and is linearly proportional to the pressure difference. Furthermore the pressure difference is linearly proportional to the difference in water level b/w the ocean and the harbor.

solu:
The rate of change in water level in the harbor is proportional to the difference in water level b/w the ocean & the harbor.

$$\dot{x} \propto (y-x)$$

$$\dot{x} = K(y-x)$$

NOT $(x-y)$ → we have the habit of choosing +ve parameter.

Complex replacement method:

$$y' + Ky = K\,q_e(t) \qquad (\text{when } q_e(t) = \cos\omega t)$$

↳ physical input

ω → Angular (or) circular frequency.
→ No. of comple osciuations.

$q_e = \cos\omega t$ → physical input. (Find the response)

solu:
complexify the problem (change in to complex domain):
↳ since it's easier to integrate exponentials.

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

The complexified equ is

$$\tilde{y}' + K\tilde{y} = Ke^{i\omega t}$$

we can't use the same y in the complexified problem since y has real solutions.

$$\tilde{y} = y_1 + iy_2 \quad (\text{complex solution})$$

Find $\tilde{y}$, then y_1 → Real part will solve the original ODE.

complex replacement:

It is a method for finding a particular solution to an inhomogeneous linear ODE.

$$P(D)x = \cos\omega t$$

where $P \to$ Real polynomial, and ω is a real number.

1) write the right hand side of the equ $\cos\omega t$ as $Re(e^{i\omega t})$:

$$P(D)x = Re(e^{i\omega t})$$

2) Replace the right hand side of the differential equ with the complex exponential $e^{i\omega t}$. we need a new variable for the solution, which will be a complex function. Give the name z for the unknown complexified diff equ

$$P(D)z = e^{i\omega t}$$

(complex replacement)

3) Use ERF (or generalized ERF if $P(i\omega)=0$) to find a particular solution z_p to the complexified ODE.

$$z_p = \frac{e^{i\omega t}}{P(i\omega)}$$

4) compute $x_p = Re(z_p)$. Then x_p is a particular solution to the original ODE.

The ocean tides are periodic with angular frequency ω. The input can be modeled as the sinusoidal function

$$y = A\cos(\omega t)$$

Solu- what's the complexified ODE for the tide problem?

$A\cos\omega t = Re(Ae^{i\omega t})$, the complexified diff equ

$$\dot{z} + Kz = KAe^{i\omega t}$$

---

why does complex replacement work?

If $z = x_1 + ix_2$ is a solution to the complex function replacement ODE.

$$(\text{I.D.E}) \quad P(D)z = e^{i\omega t}$$

$$P(D)(x_1 + i x_2) = \cos\omega t + i \sin\omega t$$

since P has real coefficients, taking the real parts of both sides gives

$$P(D) x_1 = \cos\omega t$$

which says that x_1 is a solution to the original ODE.

Notice:

For free we've actually solved another eqn. Taking the imaginary parts of both gives us

$$P(D) x_2 = \sin\omega t.$$

Complex replacement is helpful also with other real input signals, with any real valued function that can be written as the real part of a reasonably simple complex input signal. Here are some single examples that would be helpful to have memorized:

Real input signal	Complex replacement
$\cos\omega t$	$e^{i\omega t}$
$A\cos(\omega t - \phi)$	$Ae^{i(\omega t - \phi)}$
$e^{at}\cos\omega t$	$e^{(a+i\omega)t}$

Using complex arithmetic, there is a more complicated formula which can be derived as well.

Real input signal	Complex replacement
$a\cos\omega t + b\sin\omega t$	$(a-bi)\, e^{i\omega t}$

Each function in the first column is the real part of the corresponding function in the second column. The nice thing about these examples is that the complex replacement is a constant times a complex times a complex exponentions.

So ERF (or) Generalized ERF applies.

$$\ddot{x} + 4x = \cos 2t.$$

Taking complexicity:

$$\ddot{z} + 4z = e^{2it}$$

$$z_p = \frac{te^{2it}}{p'(2i)} = \frac{te^{2it}}{4i}$$

$$x_p = Re\left(\frac{te^{2it}}{4i}\right)$$

$$= Re\left(\frac{t\cos 2t + ti\sin 2t}{4i}\right)$$

$$= \frac{t}{4}\sin 2t$$

5.2 $\ddot{x} + \dot{x} + 2x = \cos 2t$

$$\ddot{z} + \dot{z} + 2z = e^{2it}$$

$$r^2 + r + 2 = p(r)$$

$$z_p = \frac{1}{p(2i)} e^{2it} = \frac{1}{-2+2i} e^{2it}$$

$$x_p = Re(z_p) = Re\left(\frac{1}{-2+2i} e^{2it}\right)$$

Converting to Amplitude-phase form

$$-2 + 2i = 2\sqrt{2}\, e^{i(3\pi/4)}$$

$$\tan^{-1}(-1) + \pi$$

$$= -\frac{\pi}{4} + \pi$$

$$= \frac{3\pi}{4}.$$

$$z_p = \frac{e^{2it}}{2\sqrt{2}\, e^{i(3\pi/4)}}$$

$$= \frac{1}{2\sqrt{2}} e^{i\left(2t - \frac{3\pi}{4}\right)}$$

$$x_p = \frac{1}{2\sqrt{2}} \cos\left(2t - \frac{3\pi}{4}\right)$$

Converting to a linear combination of cos and sin we have

$$z_p = \frac{1}{-2+2i} e^{2it}$$

$$e^{2it} = \cos 2t + i\sin 2t$$

$$\frac{\cos 2t + i\sin 2t}{-2+2i} = \frac{\cos 2t + i\sin 2t}{-2+2i}\left(\frac{-2-2i}{-2-2i}\right)$$

$$= (\cos 2t + i\sin 2t)\,\frac{-2-2i}{8}$$

$$= \frac{(-\cos 2t + \sin 2t) - i(\cos 2t + \sin 2t)}{4}$$

$$x_p = -\frac{1}{4}\cos(2t) + \frac{1}{4}\sin(2t)$$

Damped sinusoidal inputs.

we can use complex replacement to solve any ODE of the form

$$P(D)x = e^{at}\cos(\omega t - \phi)$$

1) solve $\ddot{x} + 2x = e^{-t}\cos(3t - \phi)$

$\phi \to$ Any Real numbers.

solu:

Replace the right hand side with constant exponential

$$e^{-t} e^{i(3t-\phi)} = e^{-t} e^{i3t - i\phi}$$

$$= e^{-i\phi} e^{+i3t - t}$$

$$= e^{-i\phi} e^{(-1+3i)t}$$

The complexified eqn is

$$\ddot{z}+2z = e^{-i\phi}\, e^{(-1+3i)t}$$

Apply ERF:

$$P(r) = r^2+2$$

$$P(-1+3i) = (1-6i+9i^2)+2$$

$$= -6-6i$$

From

$$P(D)\,x = P(r)\,x$$

From this,

$$z_p = \frac{e^{-i\phi}\, e^{(-1+3i)t}}{(-6-6i)}$$

A solution to the original problem is then

$x_p = Re(z_p)$. The solution involves a sinusoid polar form is the most convenient way to solve.

$$-6-6i = 6\sqrt{2}\, e^{-i3\pi/4}$$

$$z_p = \frac{e^{-i\phi}\, e^{(-1+3i)t}}{-6-6i} = \frac{1}{6\sqrt{2}}\, e^{-i\phi+i\frac{3\pi}{4}}\, e^{(-1+3i)t}$$

$$x_p = Re(z_p)$$

$$= \frac{1}{6\sqrt{2}}\, e^{-t} \cos\left(3t-\phi+\frac{3\pi}{4}\right)$$

Complex replacement of sine

$$y''-y'+2y = 10e^{-x}\sin x$$

Solu:

$$r^2-r+2 = P(r)$$

$$(r-2)(r+1) = P(r)$$

$$P(r) = (r-2)(r+1).$$

Particular solution

$10e^{-x}\sin x$ is the R.H.S.

$$Im(e^{ix}) = \sin x.$$

Let

$$10e^{-x}\cdot e^{ix} = 10\,e^{(-1+i)x}$$

The complexified solution is

$$\ddot{z}-\dot{z}+2z = 10e^{(-1+i)x}$$

z (or) $\tilde{y}_p$ (tilde) $= \dfrac{10e^{(-1+i)x}}{D^2-D+2}$

$$= \frac{10e^{(-1+i)x}}{(-1+i)^2-(-1+i)+2}$$

$$y_p = Im(z)$$

$$= Im\left(\frac{10\, e^{(-1+i)x}}{3-3i}\right)$$

$$= Im\left(\frac{10}{3}\, \frac{e^{(-1+i)x}}{1-i}\right)$$

$$= Im\left(\frac{10}{3}\, \frac{e^{(-1+i)x}(1+i)}{1-i^2}\right)$$

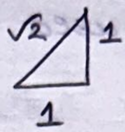
$$= Im\left(\frac{10}{3}\, \frac{e^{(-1+i)x}+i(e^{(-1+i)x})}{2}\right)$$

$$Im\left(\frac{5}{3}\left(e^{(-1+i)x}+i(e^{(-1+i)x})\right)\right)$$

$$= \frac{5}{3}\, e^{-x}(\cos x+\sin x)$$

$= \frac{5}{3} e^{-x} (\sqrt{2}) \cos\left(x - \frac{\pi}{4}\right))$

$\cos + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$



Complex gain

our goal is to explain how the amplitude & phase lag of the system response depend on system parameters & the input frequency. To do so, we will use our method of complex replacement & introduce the complex gain.

we have an LTI system modeled by the differential equ

$$P(D)x = Q(D)y$$ with input signal y

and system response x. The ERF together with complex replacement shows that if $y = \cos\omega t$, then a particular solution is given by

$$x_p = Re\left(G(\omega) e^{i\omega t}\right) \longrightarrow \text{ERF}$$

where

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)} \quad \text{if} \quad P(i\omega) \neq 0.$$

$$P(D)\left(x\right) = Q(D)y$$

The complexified solution

$$P(D)z = Q(D)e^{i\omega t}$$

$$P(D)x = Q(D)\cos\omega t$$

$$\boxed{x_p = Re\left(\frac{Q(i\omega)}{P(i\omega)}\right)e^{i\omega t}}$$

If $P(i\omega) \neq 0$, the ERF gives a particular solution of the form

$$z_p = \frac{Q(i\omega)}{P(i\omega)} e^{i\omega t}$$

$$x_p = Re\left(G(\omega) e^{i\omega t}\right)$$

$$z_p = G(\omega) e^{i\omega t}$$

$$\therefore G(\omega) = |G(\omega)| e^{-i\phi} \quad \text{(In polar form)}$$

$$z_p = |G(\omega)| e^{i(\omega t - \phi)}$$

so the original solution is

$$x_p = |G(\omega)| \cos(\omega t - \phi)$$

This form leads directly to the polar form of the sinusoidal function x_p. Let's see how this works with the eqn we used to model the tide in Boston Harbor.

$$\dot{x}+Kx = K\cos\omega t$$

Complexing:

$$\dot{z}+Kz = Ke^{i\omega t} \quad (\text{with input signal } e^{i\omega t})$$

$\therefore$ one particular response determined by ERF is

$$z_p = \frac{\phi(i\omega)}{p(i\omega)}e^{i\omega t} = \frac{K e^{i\omega t}}{D+K}$$

$$= \frac{K}{i\omega+K}e^{i\omega t}$$

The ERF shows that the system response to a complex exponential input signal is a constant multiple of the input signal. The constant is the complex gain

$$G(\omega) = \frac{\text{Complexified system response}}{\text{Complexified system input}}$$

In the present case,

$$G(\omega) = \frac{K}{i\omega+K} \quad \text{and}$$

$$z_p = G(\omega)e^{i\omega t} = \frac{K}{i\omega+K}e^{i\omega t}$$

This complex number G, expressed as a ratio of two functions of time is constant. It depends upon the system parameters, of course, but we regard them as fixed. We are interested in how it varies with the input frequency ω, write $G(\omega)$ to stress that functional dependence.

Now comes the best part of this method. Working the particular solution in terms of G leads directly to the polar form for x_p. To find it, write out the polar expression for the complex number $G(\omega)$

$$G(\omega) = |G(\omega)| e^{-i\phi}$$

$$|G(\omega)| = \sqrt{\frac{K^2}{\omega^2+K^2}} \Rightarrow z_p = \frac{K}{\sqrt{\omega^2+K^2}} e^{i(\omega t-\phi)}$$

$$= \frac{K}{\sqrt{\omega^2+K^2}}$$

$$x_p = Re(z_p)$$

$$= \frac{K}{\sqrt{\omega^2+K^2}} \cos(\omega t - \phi)$$

$$\text{Gain} = g(\omega) = \frac{K}{\sqrt{\omega^2+K^2}}$$

General rule.

$$\therefore \text{gain} = g(\omega) = |G(\omega)|$$

$$\text{phase lag} = \phi = -\arg G(\omega)$$

$$\phi = -\arg G = -\left(\arg\left(\frac{K}{K+i\omega}\right)\right) = -\arg K + \arg(K+i\omega)$$

$$\boxed{\text{gain} = |G| = \frac{K}{\sqrt{K^2+\omega^2}}}$$

$$= \arg(K+i\omega)$$

$$\therefore \arg K = 0 \quad \therefore K \to \text{Real} \ \& \ +\text{ve}.$$

$$\text{complex Gain} = G = \frac{Q(i\omega)}{P(i\omega)}$$

$$\text{phase lag} = -\arg G = \arg P(i\omega) - \arg Q(i\omega)$$

$$\therefore z = a+bi \qquad \tan^{-1}\left(\frac{0}{a}\right) = 0$$

when $z = a$

<u>Complex gain & LTI</u>

why is it enough to consider input signals of the type $\cos(\omega t)$

(what about an input signal $A\cos(\omega t - \theta)$)

<u>solu</u>:-

$x_p = g\cos(\omega t - \theta)$ is a steady state response

to

$$P(D)x = \cos\omega t$$

what's a s.s. response to

$P(D)x = A\cos(\omega t - \theta)$?

Solu:

If $x_p = g\cos(\omega t - \phi) \longrightarrow$ SS to $P(D)x = \cos\omega t$

$x_p = gA\cos(\omega t - \theta - \phi) \rightarrow$ SS to $P(D)x = A\cos(\omega t - \theta)$

---

mkb system $\ddot{x} + b\dot{x} + kx = ky$ $y(t) = \cos\omega t$

Find gain (when $y = \cos\omega t$, $k = 2$)

$\ddot{x} + b\dot{x} + 2x = 2\cos(t)$

$\cos t \rightarrow$ Input signal

$x(t) \rightarrow$ Output signal

Solu:

$\ddot{z} + b\dot{z} + 2z = 2e^{it}$

$$z_p = \frac{2e^{it}}{i^2 + bi + 2} = \frac{2e^{it}}{bi + 1}$$

Complex Gain = magnitude of particular solution over the input

$$\frac{z_p}{2e^{it}} = \frac{2}{1+bi}$$

Here

$$\text{Gain} = |G| = \frac{2(1)}{\sqrt{1+b^2}}.$$

since $\sqrt{\cos^2 x + \sin^2 x} = 1$.

---

phase lag: when $b = 1$ (what about phase lag.)

$\phi = -\text{Arg}(G)$

$$= -\text{Arg}\left(\frac{2}{P(i)}\right)$$

$$= -(\text{Arg}(2) - \text{Arg}(P(i)))$$

$$= +\text{Arg}(P(i))$$

$$= \text{Arg}(bi + 1)$$

when $b \uparrow$, the argument $\text{Arg}(1 + bi)$ increase

Amplitude: $\ddot{x} + b\dot{x} + 2x = 2\cos t$ (when b starts at 1, and ↑, what happens to A)

Solu:
$$A = |G| \text{ (or) } g = \frac{2}{|(p(i))|} = \frac{2}{\sqrt{1+b^2}} =$$

As b ↑, $\frac{2}{\sqrt{1}} = 2$, $\frac{2}{\sqrt{2}} = \sqrt{2}$, $\frac{2}{\sqrt{5}} = 0.894$

(It decreases)

---

meaning of LTI.

LTI - Linear Time-invariant systems. 'Time-invariant' means that the system parameters are not changing, or are changing very slowly relative to the time scale we are interested in. The implication for our input/output analysis is this: If $x(t)$ is a system response to the input signal $f(t)$. Then if we delay the input signal by t_0 seconds, the output signal is the same as before but delayed by t_0 seconds as well: $x(t-t_0)$ is a system response to the input $f(t-t_0)$. The system parameters are the coeff in the diff equ, so time-invariant is the same as constant coeff.

Because our systems are also linear. If $x(t)$ is a system response to $f(t)$, then $A\,x(t)$ is a system response to $Af(t)$. Thus when studying a linear-time-invariant with sinusoidal input $A\cos(\omega t - \phi)$, it is enough to consider the input $\cos(\omega t)$ since linearity & time-invariance give the response to all sinusoidal inputs for free.

Exam: 9.1

$P(D)\,x = \cos\omega t$ has $x_p = A\cos(\omega t - \phi)$ as a particular solution, then shifting time by $a = \frac{\alpha}{\omega}$ shows that

$$P(D)\,x = \cos(\omega t - \alpha)$$

has $x_p = A\cos(\omega t - \alpha - \phi)$ as a particular solution.

The gain and phase lag represent a relationship b/w the input & the output (both sinusoidal). The gain is the ratio of the output amplitude to the input amplitude, and the phase lag is the number of radians of the output signal falls behind the input signal. Because our system is time-invariant, those relationships b/w input & responses are unchanged by replacing the input signal $\cos(\omega t)$ with $\cos(\omega t - \alpha)$: the gain is A and the phase lag is ϕ.

If $x_p = A\cos(\omega t - \phi)$ is a particular solution to $P(D)x = \cos(\omega t)$, what about $P(D)x = \sin(\omega t)$

Solu:

$$\sin\omega t = \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\therefore \quad x_p = A\cos\left(\omega t - \frac{\pi}{2} - \phi\right) = A\sin(\omega t - \phi)$$ as the gain & phase lag are unchanged

---

Review of Complex replacement & time-invariance.

a) use complex techniques to solve $\dot{x} + Kx = \cos\omega t$, $K, \omega \to$ constant

b) use work from part (a) to solve

$$\dot{x} + Kx = F\sin(\omega t), \quad F \text{ const}$$

c) use the superposition principle to solve

$$\dot{x} + Kx = \cos(\omega t) + 3\sin\omega t$$

d) use work from (a) to solve.

$$\dot{x} + Kx = \cos(\omega t - \phi), \quad \phi \text{ Const.}$$

Solu: General solution $= X = x_h + x_p$

particular solution:

a) $\dot{x} + Kx = \cos\omega t$ (complex replacement)

Complexing:

$$\dot{z} + Kz = e^{i\omega t}$$

$$z_p = \frac{e^{i\omega t}}{D+K} = \frac{e^{i\omega t}}{i\omega + K}$$

$$\therefore \quad x_p = Re(z_p)$$

$$= Re\left(\frac{e^{i\omega t}}{i\omega + K}\right) = \frac{1}{K^2+\omega^2}\left(K\cos(\omega t) + \omega \sin(\omega t)\right)$$

b) $\dot{x} + Kx = F\sin\omega t$

$$x = F\left(Im\left(\frac{e^{i\omega t}}{i\omega + K}\right)\right)$$

$$z_p = \frac{F(K - i\omega)}{K^2+\omega^2}\left[\cos\omega t + i\sin\omega t\right]$$

$$x_p = Im(z_p)$$

$$x_p = \frac{F}{K^2+\omega^2}\left[-\omega\cos\omega t + K\sin\omega t\right]$$

c) $\dot{x} + Kx = \cos(\omega t) + 3\sin\omega t$

solu:

$$x_p = \frac{1}{K^2+\omega^2}(K\cos\omega t + \omega\sin\omega t) + \frac{3(K - i\omega)}{K^2+\omega^2}\left[\cos\omega t + i\sin\omega t\right]$$

$$\frac{1}{K^2+\omega^2}\left[-\omega\cos\omega t + K\sin\omega t\right]$$

$$\therefore \boxed{x = x_a + x_b}$$

$$x_p = \frac{1}{K^2+\omega^2}\left[(K\cos\omega t + \omega\sin\omega t) - 3\omega\cos\omega t + 3K\sin\omega t\right]$$

$$= \frac{1}{K^2+\omega^2}\left[(K - 3\omega)\cos\omega t + (\omega + 3K)\sin\omega t\right]$$

d) $\dot{x} + Kx = \cos(\omega t - \phi)$

solu:

$$\dot{x} + Kx = \cos\left(\omega\left(t - \phi/\omega\right)\right)$$

applying

$$x_p = \frac{1}{K^2+\omega^2}\left(K\cos\left(\omega\left(t - \frac{\phi}{\omega}\right)\right) + \omega\sin\left(\omega\left(t - \frac{\phi}{\omega}\right)\right)\right)$$

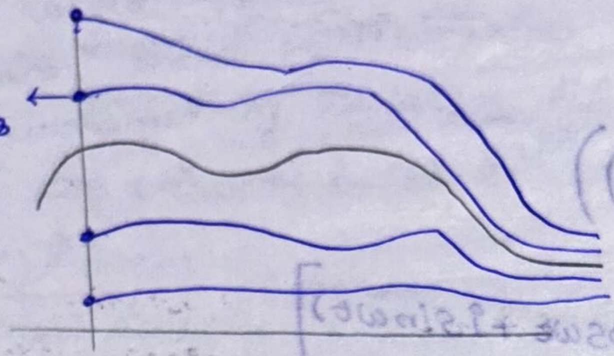
Stability

$\dot{y} + ky = v(t)$

Solu: Here, $y = e^{-kt}\int v(t) e^{kt}\, dt + ce^{-kt}$ (say)

Steady state (Long term solution)

Transient (It goes to zero (vanish) (As $t \to \infty$))



After some time, the transient state will begin to closely follow the steady state curve. $\bullet \longrightarrow$ Transient Solution.

Peculiar $\rightarrow$ characteristic of only one person or group

Here we can say ce^{-kt} a steady state (But $c=0$)

$\hookrightarrow$ Looks simpler.

(The $\int v(t)\,dt$ function)

If this is a peculiar function, we can get more simplified solution. So the better option is to integrate then choose the simple

$\hookrightarrow$ Don't use an $\boxed{\text{arbitrary constant.}}$

A certain spring/mass/dashpot system is modeled by the LTI ODE

$$\ddot{x} + 7\dot{x} + 12x = 12f(t) \qquad f(t) = \cos\omega t$$

(Input signal)

Solu: Suppose that the solution to this equ with initial conditions $(x(0), \dot{x}(0)) = (2, 3)$ is $x(t)$. What can you say about the solution $y(t)$ to the same ODE with initial condition $(y(0), \dot{y}(0)) = (3, -8)$?

Solu: It isn't related to $x(t)$ in any way

It oscillates.

Its graph becomes asymptotic to that of $x(t)$ as $t \to \infty$

Its graph diverges from that of $x(t)$ as $t \to \infty$.

solu: $r^2+7r+12=p(r)$

$(r+4)(r+3)=p(r)$

$x_h = c_1 e^{-3t} + c_2 e^{-4t}$

$c_1, c_2 \to$ constant (Determined by IC)

General solution to the original inhomogeneous ODE

$$x = x_h + x_p$$

$$= Re\left(\frac{12}{8+14i} e^{2it}\right) + c_1 e^{-3t} + c_2 e^{-4t}$$

In general, a damped oscillator forced with a sinusoidal input produces a sinusoidal output signal. That output is a particular solution called a steady-state solution, because this is what the solution looks like as $t \to \infty$. Every other solution is the steady state solution plus a transient, where the transient is a decaying function which decays to 0 as $t \to +\infty$.

Changing Ic changes c_1 & c_2 alone. So the ss is the same. A system like this, in which changes in the initial conditions have vanishing effect on the long term behaviour of the solution, is called stable.

Tests for stability in a second-order system:

Stability means that the long-term system behaviour is independent of initial conditions.

stability test in terms of roots.

A constant coeff linear ODE of any order is stable if and only if every root of the char polynomial has (-)ve real part.

Explaining this using (II order system)

$$m\ddot{x} + b\dot{x} + kx = 0$$

3 cases:

1) complex conjugate roots $\Rightarrow a \pm bi$

2) Roots are real & repeated: s, s

3) Roots are distinct real numbers r_1, r_2

If the roots are complex conjugates then the general solution to the homogeneous equ takes the form $e^{at}(A\cos(bt-\phi))$. The homogeneous solution is transient only if the real part is negative $(a<0)$. If the real part is zero $a=0$, the homogeneous solution oscillates forever with constant amplitude. If the real part is +ve $(a>0)$, the homogeneous solution osciluates & the amplitude grows exponentially as time goes on.

Roots are real & repeated - then homogeneous solution will be as $(A+Bt)e^{st}$, and these solutions are transient only if $s<0$.

If the roots are real & distinct, then the homogeneous solution takes the form $c_1e^{r_1t}+c_2e^{r_2t}$, and these solutions are transient only if $r_1, r_2<0$.

Roots	General solution x_h	Condition for stability
Complex $a \pm bi$	$e^{at}(c_1\cos bt + c_2\sin bt)$ (since periodical) → all depends on e^{at}	$a<0$
Repeated real s, s	$e^{st}(c_1+c_2t)$	$s<0$
Distinct real r_1, r_2	$c_1e^{r_1t}+c_2e^{r_2t}$	$r_1, r_2<0$

$$y''+Ay'+By=f(t)$$

$$y = \underset{\text{Steady State solution}}{y_p} + \underbrace{c_1y_1+c_2y_2}_{\text{uses initial condt. (Transient)}}$$

when does $c_1y_1+c_2y_2 \rightarrow$ goes to zero as $t\rightarrow\infty$ for all c_1, c_2

If this is so, then ODE $\longrightarrow$ Stable (called)

Case by Case Analysis

(Roots) $\longrightarrow$ See above

Statement: If all the characteristic roots have (-)ve real part. $\longrightarrow$ Function is stable.

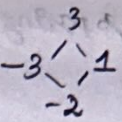
$$x^{(3)}+\ddot{x}+\dot{x}-3x=\sin 7t$$

$$\begin{array}{c|cccc} 1 & 1 & 1 & 1 & -3 \\ & & 1 & 2 & 3 \\ \hline & 1 & 2 & 3 & \boxed{0} \end{array}$$

soln: $r^3+r^2+r-3=0 \quad \Rightarrow (r-1)(r^2-2r+3)$

$(r-1)(r-3)(r+1)$

$x_h = c_1 e^{+t} + c_2 e^{+3} + c_2 e^{-1}$



Coefficients & tests for stability.

mkb system (you can change m, k, b to be a variety of +ve constants).

In 2nd order case:

Stability test in terms of coefficients, 2nd order case. Assume that a_0, a_1, a_2 are real numbers with $a_0 > 0$. $(a_0 D^2 + a_1 D + a_2)x = F(t)$ is stable if and only if $a_1 > 0$ and $a_2 > 0$

Proof:- By dividing by a_0, we can assume $a_0 = 1$. Break into cases according to the table.

Char polynomial

$r^2 - 2ar + (a^2+b^2)$

$r^2 - 2sr + s^2$

$r^2 - (r_1 + r_2) + r_1 r_2$

* when roots are complex, $a \pm bi$, $a < 0$ if and only if the coeff $-2a$ and a^2+b^2 (both +ve)

* when the roots are s, s, we have $s < 0$ if & only if the coeff $-2s$ and s^2 are both +ve.

* when the roots are distinct & real ($r_1 \neq r_2$). Both roots are less than zero if & and only if the coeff $-(r_1+r_2)$ and $r_1 r_2$ are both +ve. Knowing that $(-(r_1+r_2))$ is +ve means that at least one of r_1, r_2 is $(-)$ve; Let moreover the product $r_1 r_2$ is +ve, then the other root must be $(-)$ve too

---

Remark: There is a generalization of the coefficient test to higher order ODEs, called the Routh-Hurwitz conditions for stability. but the conditions are much more complicated.

All spring (mass) dashpot systems,

$$m\ddot{x} + b\dot{x} + kx = F_{ext} \quad ; \quad m, b, k > 0$$

Soln: All $m \times b$ systems are stable. This is required by our physical model which came up with the eqn

$$\boxed{m\ddot{x} + b\dot{x} + kx = F_{ext}}$$

and required the coeffs m, b, k to be +ve real numbers physically this makes sense as well. w.r.t there can be no such thing as a perpetual motion machine, so in the absence of external forces, we expect energy to dissipate in the form of friction or heat, and thus our system will eventually be unmoving. Any good diff eqn model should also reflect this behaviour.

Upshot of Stability

Why are stable systems so great? If we are interested in understanding the long term behaviour w.r.t an exponential input, it is enough to find one particular solution. All other solutions will tend asymptotically to any particular solution. And with ERF and its versions, we have a simple way of determining one nice particular solution to any exponential input.

Keep in mind: That while the long term behaviour is enough to answer a question, there are some situations where the IC and behaviour as a system reaches steady state are important. For instance as in the example from Lecture 2 about mixing saline water) to create a salt water solution habitable for ocean fish.

Damped Sinusoidal Signals

Complex replacement & the exponential response formula can be used to solve eqn of the form

$$P(D)\,x = e^{at}\cos(\omega t - \phi)$$

where $P(D)$ is any LTI differential operator.

15.1: $2\ddot{x}+\dot{x}+x=e^{-t}\cos t$

solu: $2\ddot{z}+\dot{z}+z=e^{(-1+i)t}$ (complexifying)

$$z_p=\frac{e^{(-1+i)t}}{P(-1+i)}$$

To get x_p we have to extract the real part of z_p. We have the choice to put x_p in rectangular form or polar form:

$$P(-1+i)=2(-1+i)^2+(-1+i)+1=-3i$$

So, $z_p=\dfrac{ie^{(-1+i)t}}{3}$, As $x_p=Re(z_p)$

$$x_p=-\left(\frac{1}{3}\right)e^{-t}\sin t.$$

To put x_p in polar form,

$$\frac{1}{P(-1+i)}=ge^{-i\phi}$$

So that,

$$z_p=ge^{-i\phi}e^{(-1+i)t}$$

$$=ge^{-t}e^{i(t-\phi)}$$

$$x_p=ge^{-t}\cos(t-\phi)$$

$$\ddot{x}+\omega_n^2x=A\cos(\omega t)$$

solu: $A\cos(\omega t)$ is the input. Find a the complex gain, the gain, phase lag & particular solution. (Is this stable)

solu:

$$G(\omega)=\frac{1}{P(i\omega)}$$

$$P(i\omega)=(i\omega)^2+\omega_n^2=\omega_n^2-\omega^2$$

Thus a particular solution is given by

$$x_p=Re\left(\frac{1}{\omega_n^2-\omega^2}\cdot Ae^{i\omega t}\right)$$ as long as

the input frequency is different from the natural frequency of the harmonic oscillator. Since the denominator is real, the real part is easy to find:

$$x_p = A \frac{\cos(\omega t)}{\omega_n^2 - \omega^2}$$

Gain

$g = G(\omega) : \frac{1}{\omega_n^2 - \omega^2}$ Since the complex gain is real, the phase lag is either 0 or π depending on whether $\omega_n > \omega$ or $\omega_n < \omega$.

In particular, the same eqn forced with a sine input curve

$$\ddot{y} + \omega_n^2 y = A \sin \omega t$$

has response

$$y_p = A \frac{\sin \omega t}{\omega_n^2 - \omega^2} \quad \text{(by time invariance)}$$

This solution puts in precise form some of the things we can check from experimentation with vibrating systems. when the frequency of the signal is smaller than the natural frequency of the system,

$\omega < \omega_n$, the denominator is +ve.

The system response is a +ve multiple of the signal: the vibration of the mass is in sync with the impressed force. As $\omega \uparrow$ towards ω_n, the denominator of the particular solutions nears zero. So the amplitude of the solution grows arbitrarily large.

when $\omega = \omega_n$ the system is in resonance with the signal: the exponential response formula fails because the gain would be infinite there & there is no periodic (or even bounded) solution. This phenomenon is called resonance. and will be discussed in more detail later.

when $\omega > \omega_n$, the denominator is (-)ve.

The system response is a (-)ve multiple of the signal: the vibration of the mass is perfectly out of sync with the impressed force.

∴ The coeff are constant here, a time-shift of the signal results in the same time-shift of the solution.

$$\ddot{x} + \omega_n^2 x = A\cos(\omega t - \phi)$$

has the periodic solution

$$x_p = A \frac{\cos(\omega t - \phi)}{\omega_n^2 - \omega^2}$$

The eqn $x_p = \frac{A \cos \omega t}{\omega_n^2 - \omega^2}$ and $y_p = \frac{A \sin \omega t}{\omega_n^2 - \omega^2}$ will be

very useful in fourier series.

$\ddot{x} + 8x = \cos \omega t$ (undamped, Forced - Reason)

Solu:

$r^2 + 8 = 0$

$r = \pm i 2\sqrt{2}$

$x_h = c_1 \cos \sqrt{8} t + c_2 \sin(\sqrt{8} t)$

Polar form:

$x_h = A \cos(\sqrt{8} t - \psi)$

General solution

$\cos(\omega t) = Re(e^{i\omega t})$

$\ddot{z} + 8z = e^{i\omega t}$

$$z_p = \frac{e^{i\omega t}}{P(i\omega)} = \frac{e^{i\omega t}}{(i\omega)^2 + 8(i\omega)} = \frac{e^{i\omega t}}{-\omega^2 + 8i\omega}$$

$$\cancel{= \frac{e^{i\omega t}(-\omega^2 + 8i\omega)}{\omega^4 - 64 i^2 \omega^2}}$$

$$= \frac{e^{i\omega t}}{8 - \omega^2}$$

when $\boxed{\omega^2 = 8}$ $\rightarrow$ Blows up [Resonance]

(when Input frequency is the same as natural freq)

$(\omega^2 \neq 8)$

$$x_p = \frac{\cos \omega t}{8 - \omega^2}$$

$x = x_h + x_p$

$$= c_1 \cos \sqrt{8} t + c_2 \sin \sqrt{8} t + \frac{\cos \omega t}{8 - \omega^2}$$

why is this called undamped, force?

Since we doesn't have a damping term. Also we are giving an input $\cos\omega t \rightarrow$ so Forced

2) $\ddot{x} + 2\dot{x} + 4x = \cos(3t)$

Soln:

$$r^2 + 2r + 4 = 0$$

$$r = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= -1 \pm \sqrt{1-4}$$

$$= -1 \pm \sqrt{3}i$$

$$\boxed{x_h = e^{-t}\left(c_1 \cos\sqrt{3}t + c_2 \sin(\sqrt{3}t)\right)}$$

Find a particular solution:

$$\ddot{z} + 2\dot{z} + 4z = e^{i3t}$$

$$z_p = \frac{e^{i3t}}{9i^2 + 2\times 3i + 4}$$

$$= \frac{e^{i3t}}{-5 + 6i}$$

$$= \frac{e^{i3t}(-5 - 6i)}{25 - 36i^2}$$

$$= \frac{-5e^{i3t} - 6ie^{i3t}}{61}$$

$$x_p = Re(z_p)$$

$$= Re\left(\frac{-5(\cos 3t + i\sin 3t) - 6i(\cos 3t + i\sin 3t)}{61}\right)$$

$$= \frac{-5\cos 3t + 6\sin 3t}{61}$$

$$x = e^{-t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t) + \left(\frac{-5\cos 3t + 6\sin 3t}{61}\right)$$

As we have damping, the ~~general~~ solution has a decaying term