of terms of the form

$$\frac{a}{s+b} \quad \text{and} \quad \frac{as+c}{s^2+bs+k},$$

a, b, c, k are all real

4.1: Find P.I decomposition of

$$\frac{s^2}{s^2-1} = \cancel{\frac{a}{(s+1)}} + \cancel{\frac{b}{s-1}} \quad 1 + \frac{1}{s^2-1}$$

The denominator s^2-1 factors in to a product of linear factors

$$s^2-1 = (s-1)(s+1)$$

Decomposed form:

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

To find A, multiply through by $s-1$:

$$\frac{1}{s+1} = A + \frac{B(s-1)}{s+1}$$

$\boxed{\text{set} \quad s=+1}$

$$A = \frac{1}{2}.$$

To find B, mul by $s+1$

$$\frac{1}{s-1} = \frac{A(s+1)}{s-1} + B$$

$$B = \frac{1}{s-1} \Rightarrow B = \frac{-1}{2}$$

$(s=-1)$

$$\therefore \frac{s^2}{s^2-1} = 1 + \frac{1/2}{s-1} - \frac{1/2}{s+1}$$

$$\frac{s^2}{s^2-1} = 1 + \frac{1}{2(s-1)} - \frac{1}{2(s+1)}$$

Another method:

$$\cancel{\frac{s^2}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1}}$$

$$\cancel{\frac{s^2}{s^2-1} = \frac{A(s-1)+B(s+1)}{(s+1)(s-1)}}$$

$$\frac{s^2}{s^2-1} = 1 + \frac{1}{s^2-1}$$

Partial fractions

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

Let $s=0$

$$1 = A(s+3) + Bs$$

$$A(s+3) = 1$$

$$A = \frac{1}{s+3} \quad (\text{set } s=0)$$

$$A = \frac{1}{3}$$

$s=-3$

$$\frac{1}{s} = \frac{A(s+3)}{s} + B$$

$$B = \frac{1}{s}$$

$$B = -\frac{1}{3}$$

$$\frac{1}{s(s+3)} = \frac{1}{3s} - \frac{1}{3(s+3)}$$

P.I in algebraic form

$$\xrightarrow{\;}$$

d^{-1}

$$\frac{s^2+3s+8}{(s-1)(s-2)(s+5)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+5}$$

Solu:

$$A(s-2)(s+5) + B(s-1)(s+5) + C(s-1)(s-2)$$

$$(As-2A)(s+5) + (Bs-B)(s+5) + (Cs-C)(s-2)$$

$$(As^2+5As-2As-10A) + (Bs^2-Bs-5B+5Bs) + (Cs^2-Cs+2C-2Cs)$$

$$A+B+C = 1$$

$$3A+4B-3C = 3$$

$$-10A-5B+2C = 8$$

$A = -2,\ B = \frac{18}{7},\ C = \frac{3}{7}.$

Exam: 4.2: Find a, b, c

$$\frac{s^2+1}{s^3+2s^2+2s} = \frac{a}{s} + \frac{bs+c}{s^2+2s+2}$$

Solu: multiply through by s^2+2s+2 $\cdot(s = -1+i)$

$$bs + c + \frac{a}{s}(s^2+2s+2) = \frac{s^2+1}{s}$$

$$b(-1+i) + c = \frac{(-1+i)^2+1}{-1+i}$$

gives

$$(c-b) + bi = \frac{(1-2i-1)+1}{-1+i}$$

Rationalize the denominator by multiplying numerator & denominator by $\overline{-1+i} = -1-i$:

$$(c-b) + bi = \frac{(1-2i)(-1-i)}{2} = -\frac{3+i}{2}$$

Implies $b = \frac{1}{2},\ c = b - \frac{3}{2} = -1$ so

$$\frac{s^2+1}{s^3+2s^2+s} = \frac{1}{2s} + \frac{\frac{1}{2}s - 1}{s^2+2s+2}$$

Cover-up

1) write out the proposed partial fractions decomposition

2) pick one of the linear or quadratic terms in the denominator; cover up all the other terms in the denominator; cover up all the other terms in the partial fractions sum & the choosen factor in the original denominator (and in the chosen term's denominator).

3) set s equal to a root of the chosen factor & solve.

$$\frac{1}{s^3-1} = \frac{a}{s-1} + \frac{bs+c}{s^2+s+1}$$

Solu:

$$\frac{1}{s^3-1} = \frac{as^2+as+a+bs^2-bs-c+cs}{(s-1)(s^2+s+1)}$$

$$a+b=0$$

$$a-b+c=0$$

$$a-c=1$$

$$a=\frac{1}{3},\ b=\frac{-1}{3},\ c=\frac{-2}{3}$$

Partial fractions : Repeated roots

$$\frac{a}{s+3} + \frac{b}{(s+3)^2} = \frac{1}{(s+3)^2}$$

If $(s^2+2s+2)^2$ occurs, we have to allow

$$\frac{as+b}{s^2+2s+2} + \frac{cs+d}{(s^2+2s+2)^2}$$

we need to include all powers b/w the I and the power that occurs in the denominator.

$$\frac{3s+2}{s(s+1)^2} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

Solu: covering up s: $(s=0)$

$$\frac{3s+2}{(s+1)^2} = a + \frac{bs}{s+1} + \frac{cs}{(s+1)^2}$$

$$a = \frac{2}{(s+1)^2}$$

$$a = \frac{2}{1} = 2.$$

Covering $(s+1)^2$

$$\frac{(3s+2)}{s(s+1)^0} = \frac{a(s+1)^2}{a} + b + \frac{c(s+1)}{(s+1)^2}$$

$$\cancel{b = \frac{3(-1)+2}{-1(-1+1)}}$$

$$\frac{3s+2}{s} = \frac{a(s+1)^2}{s} + b(s+1) + \frac{c(s+1)^2}{(s+1)^2}$$

$s = -1$

$$c = \frac{3s+2}{s}$$

$$= \frac{-1}{-1} = 1$$

$s^2 + 2s + 1$

$\therefore$ set $s = 1$

$$\frac{a}{1} + \frac{b}{2} + \frac{c}{4} = \frac{5}{4}$$

$$b = -2$$

$$\frac{s^2}{(s+1)^2(s-1)} = \frac{a}{s-1} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

solu:

$$\frac{s^2}{(s+1)^2} = a + \frac{b(s-1)}{s+1} + \frac{c(s-1)}{(s+1)^2}$$

$(s = 1)$

$$a = \frac{1}{4}$$

$$\frac{s^2}{s-1} = \frac{a(s+1)^2}{s-1} + b(s+1) + c$$

$(s = -1)$

$$c = \frac{1}{-2}$$

say $s = 2$

$$\frac{4}{9(1)} = \frac{1}{4(1)} + \frac{b}{3} - \frac{1}{2(9)}$$

$$\frac{4}{9} = \frac{1}{4} + \frac{b}{3} - \frac{1}{18}$$

$$b = 3\left(\frac{4}{9} - \frac{1}{4} + \frac{1}{18}\right) = 3/4$$

Complete decomposition

A significant feature of the complex numbers is that any nonzero polynomial factors in to a product of linear factors. Each quadratic term we saw above factors as a product $(s-r)(s-\bar{r})$. This lets us write out partial fraction expansions involving only terms of the form.

$$\frac{a}{(s-r)^k}$$

Exam: 5.2

$$\frac{s+1}{s^2+1} = \frac{a}{s-i} + \frac{b}{s+i}$$

solu:

$$\frac{s+1}{(s-i)(s+i)} = \frac{a}{s-i} + \frac{b}{s+i}$$

coverup $(s-i)$ $(s=i)$

$$\frac{s+1}{s+i} = a + \frac{b(s-i)}{s+i}$$

$$\frac{i+1}{2i} = a + \frac{b(i-i)}{s+i}$$

$$a = \frac{i+1}{2i}$$

cover up $(s+i)$ $(s=-i$

$$\frac{s+1}{s-i} = \frac{a(s+i)}{s-i} + b$$

$$b = \frac{-i+1}{-2i}$$

$\therefore$ a and b are complex conjugates; this reflects the fact that the coefficients in the original expression are real.

$$\frac{s+2i}{s^2+1} = \frac{a}{s+i} + \frac{b}{s-i}$$

solu:

$$\frac{s+2i}{s+i} = \frac{a(s-i)}{s+i} + \frac{b(s-i)}{s-i} \quad (s=i)$$

$$b = \frac{3i}{2i}$$

$$\boxed{b = \frac{3}{2}}$$

cover up $(s-i)$ $\qquad s=-i$

$$\frac{s+2i}{s-i} = a + \frac{b(s+i)}{s-i}$$

$$a = \frac{i}{-2i} = -\frac{1}{2}$$

complex coverup with ODE

automobile suspension is modeled by the equation

$$m\ddot{x} + b\dot{x} + Kx = b\dot{y} + Ky$$

$(m=1,\ b=2,\ K=3)$ $\qquad y = 4\sin(5t)$

$x(0) = 0,\ \dot{x}(0) = 0$

solu:

$$(D^2 + 2D + 3I)x = (2D + 3I)\,4\sin 5t$$

Applying Laplace:

$$(s^2 + 2s + 3)X = (2s+3)\,4 \times \left(\frac{5}{s^2+25}\right)$$

$$X = \frac{20(2s+3)}{(s^2+25)(s^2+2s+3)}$$

we need to find a function that has this as its Laplace transform. It's a rational function – a quotient of a polynomial – but a pretty complicated one, and it certainly show up in our look-up table.

This is where the technique of partial fractions comes in handy. The degree of numerator is one, which is less than the degree of the denominator (which is 4), so partial fractions assures us that we can write the right hand side as a sum of fractions with denominators s^2+25 and s^2+2s+3. The entries in our table with quadratic denominators have those denominators written as completed squares - (in the form $(s+a)^2+b$ - so let's smart & start by completing the squares.

$$s^2+2s+3 = (s+1)^2+2$$

So the partial fractions guarantees that are real constants such that (a, b, c & d)

$$\frac{20\,(2s+3)}{(s^2+25)(s^2+2s+3)} = \frac{as+b}{s^2+25} + \frac{c(s+1)+d}{(s+1)^2+2}$$

we can discover the values of a, b, c and d by cross-multiplying & equating co.eff. But it will be more fun to use Coverup method.

why

$$\frac{1}{s^2+25} = \frac{as+b}{s^2+b}$$

$$s^2 = -25$$

$$s = \sqrt{-25}$$

$$= \pm 5i$$

$$\frac{1}{(s+5i)(s-5i)} = \frac{a}{(s+5i)} + \frac{b}{s-5i}$$

$$\frac{1}{s-5i} = a + \frac{b(s+5i)}{s-5i}$$

$(s=-5i)$

$$a = \frac{1}{s-5i}$$

$$= \frac{1}{-10i}$$

$$\frac{1}{s+5i} = \frac{a}{\frac{(s-5i)}{s+5i}} + \frac{b(s-5i)}{s-5i}$$

$(s=5i)$

$$b = \frac{1}{s+5i}$$

$$b = \frac{1}{10i}$$

$$\frac{1}{s^2+25} = \frac{\frac{-1}{10i}}{s+5i} + \frac{\frac{1}{10i}}{s-5i}$$

$$= \frac{-1}{10i(s+5i)} + \frac{1}{10i(s-5i)}$$

$$\frac{s^2+1}{s^3+2s+s} = \frac{s^2+1}{s(s^2+2s+1)}$$ $$\frac{-2\pm\sqrt{4-4(1)(1)}}{2}$$

$$= \frac{s^2+1}{s(s+1)(s+1)}$$ $-1.$

$$= \frac{s^2+1}{s(s+1)(s+1)}$$

For $\frac{s+1}{s^2+1} = \frac{i+1}{2i(s-i)} + \frac{-i+1}{-2i(s+i)}$

$$= \frac{i+1(s+i) + (i-1)(s-i)}{2i(s-i)(s+i)}$$

$$= \frac{si + \cancel{i^2} + \cancel{s} + \cancel{i} + si - \cancel{s} \cancel{+i} \cancel{-i^2}}{2i(s^2+1)}$$

$$= \frac{2si+2i}{2i} \frac{1}{s^2+1}$$

$$= \frac{s+1}{s^2+1} = \frac{bs+c}{s^2+1}.$$

Complex Cover up:

So multiply by s^2+25

$$\frac{20(2s+3)}{s^2+2s+3} = as+b + \frac{(c(s+1)+d)(s^2+25)}{(s+1)^2+d}$$

$s^2=-25$

$s = 5i$ (set)

$$\frac{20(2s+3)}{s^2+2s+3} = as+b$$

$$\frac{20(10i+3)}{-25+10i+3} = 5ai+b$$

Taking congugate

$$\frac{20(10i+3)}{(-22)^2-(10i)^2} \times (-22+10i) = 5ai+b$$

$$\frac{(200i+60)(-22+10i)}{584} = 5ai+b.$$

$$= 20\ \frac{(-66+100i)+(-220-30)i}{584}$$

$20 \cdot$

$$b = 20\left(\frac{34}{584}\right) = \frac{85}{73} \quad , \quad a = -4\left(\frac{250}{584}\right) = -\frac{125}{73}$$

$$\frac{20(2s+3)}{(s^2+25)(s^2+2s+3)} = \frac{as+b}{s^2+25} + \frac{c(s+1)+d}{(s+1)^2+2}$$

Solu:

$$\frac{20(2s+3)}{s^2+2s+3} = as+b + \frac{(c(s+1)+d)(s^2+25)}{(s^2+2s+3)}$$

$s = 5i$

$$as+b = \frac{20(2\times 5i+3)}{s^2+2s+3}$$

$$a(5i)+b = \frac{20(10i+3)}{25i^2+2(5i)+3}$$

$$a5i+b = \frac{200i+60}{-22+10i}$$

$$= \frac{(200i+60)(-22-10i)}{484+100}$$

$$= \frac{-4400i - 2000i^2 - 1320 - 600i}{584}$$

$$= \frac{2000-1320 - i(5000)}{584}$$

$$5ai = \frac{5000i}{584}$$

$$a = \frac{5000}{5(584)}$$

$$= -\frac{125}{73}$$

$$b = \frac{680}{584} = \frac{85}{73}$$

$$c(s+1)+d = \frac{20(2s+3)}{s^2+25} + \frac{A}{s+3}$$

$$s^2+2s+3$$

$$\frac{-2 \pm \sqrt{4-12}}{2}$$

$$= -1 \pm \sqrt{2}i$$

$$c(-1+\sqrt{2}i+1)+d = \frac{20(-2+2\sqrt{2}i+3)}{(-1+\sqrt{2}i)^2+25}$$

$$c(\sqrt{2}i)+d = \frac{20(1+2\sqrt{2}i)}{24-2\sqrt{2}i}$$

$$c = \frac{1}{\sqrt{2}} \operatorname{Im}\left(\frac{20(1+2\sqrt{2}i)}{24-2\sqrt{2}i}\right)$$

$$\therefore \quad c = \frac{1000}{24^2+8} = \frac{125}{73} \quad \Big| \quad d = \frac{20(16)}{24^2+8} = \frac{40}{73}$$

Solve a differential equ

$1 \rightsquigarrow \frac{1}{s}$ $(\operatorname{Re} s > 0)$

$e^{rt} \rightsquigarrow \frac{1}{s-r}$ $\operatorname{Re} s > \operatorname{Re} r$

$\cos\omega t \rightsquigarrow \frac{s}{s^2+\omega^2}$ $\operatorname{Re} s > 0$

$\sin\omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}$ $\operatorname{Re} s > 0$

Solve, $t \rightsquigarrow \frac{1}{s^2}$ $\operatorname{Re} s > 0$

$\dot{x} + 3x = 2\cos(4t)$ with $x(0) = 1$ $x(0)=1$

Solu:

$$\left((sX-1)+3X = \frac{2s}{s^2+16}\right)$$

$$(sX+3X) = 2\left(\frac{s}{s^2+16}\right)$$
$$-1$$

$$(s+3)X = 2\left(\frac{s}{s^2+16}\right)$$
$$-1$$

$$X = \frac{2s}{(s^2+16)(s+3)} + \frac{1}{s+3}$$

$$\frac{2s}{(s^2+16)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+16}$$

Solving,

$$A = -\frac{6}{25}, \quad B = \frac{6}{25}, \quad C = \frac{32}{25}$$

$$\therefore \frac{2s}{(s^2+16)(s+3)} = \frac{-\frac{6}{25}}{s+3} + \frac{\frac{6s}{25} + \frac{32}{25}}{s^2+16}$$

$$\frac{2s}{(s^2+16)(s+3)} + \frac{1}{s+3} = \frac{19}{25(s+3)} + \left(\frac{6s/25}{s^2+16} + \frac{32/25}{s^2+16}\right)$$

using Laplace table

$$x(t) = \frac{1}{25}\left(19e^{-3t} + 6\cos 4t + 8\sin 4t\right)$$

This is the expected form of the solution. The sinusoidal part is the steady state solution, and can be obtained from complex replacement and ERF; we expect a sinusoid of angular frequency 4. The exponential term is the transient; we expect an exponential with constant -3. It's worth checking that the initial condition is satisfied.

Rest initial conditions

$$f^{(n)}(t) \rightsquigarrow s^n F(s) - \left(f(0)s^{n-1} + f'(0)s^{n-2} + \cdots + f^{(n-1)}(0)\right)$$

The most important part of the formula for $\mathcal{L}(f^{(n)}(t); s)$ is $s^n F(s)$. The whole business would be easier without the other terms. This can be arranged by assuming that the function & the relevant derivatives are all zero at $t=0$. This is the hypothesis of 'Rest initial conditions'.

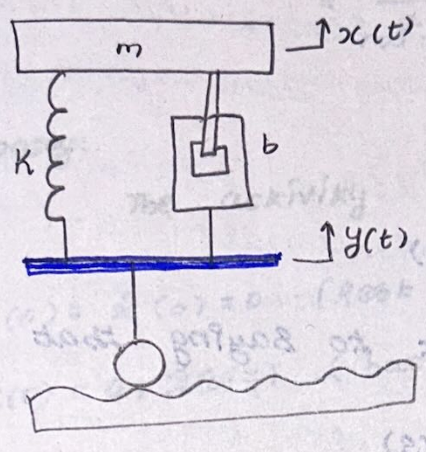
observation

If $f(0) = \cdots = f^{(n-1)}(0) = 0$, then

$$f^{(n)}(t) \rightsquigarrow s^n F(s)$$

If we are working with an LTI system that is stable, the initial conditions don't matter much anyway, and we should feel free to let them be whatever is convenient. Here we will study ZSR (zero-state response) → solution with rest initial conditions.

Suspension system



One quarter of the cabin of the car is represented by the mass at the top of the picture. As the car moves along a road, potholes cause the wheels to move up & down, this is represented by vertical motion of the platform at the bottom.

$$m\ddot{x} + b\dot{x} + Kx = b\dot{y} + Ky$$

$y(t)$ - Displacement of the blue platform upwards from some arbitrarily chosen zero position & the system response $x(t)$ is the displacement of a point in the car cabin. (Mass). The modelling assumption is that $x = y$ when the spring is relaxed. $\dot{x} = \dot{y}$ (when the platform & mass are moving together.)

Let's study this using Laplace transform. The rest initial condition assumption of x is that $x(0) = \dot{x}(0) = 0$. Since only the first derivative of y enters the eqn, we need only assume $y = 0$. Then we apply Laplace transform

$$ms^2 X + bsX + KX = bsY + KY$$

$$P(s) = ms^2 + bs + K \qquad X(s) = \frac{bs + K}{ms^2 + bs + K} Y(s)$$

The factor on the right is precisely the transfer function of the system. from the general formula

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Theorem: 8.1:

$$P(D)\, x = q(D)\, y$$

with rest conditions, then

$$P(s) X(s) = q(s) Y(s)$$

Note that this is equivalent to saying that

$$X(s) = H(s)\, Y(s)$$

$H(s)$ = Transfer function.

This is a fantastic result in the s-domain, the system simply multiplies the input signal by the transfer function. (Subject to rest Initial conditions). This is the key observation of this entire course.

Practice with rest initial conditions

A mkb system

$$m\ddot{x} + b\dot{x} + Kx = Ky$$

1) $x(0) = 0$, $\dot{x}(0) = 0$

Solu:

$$ms^2 X + bXs + KX = KY$$

$$(ms^2 + bs + K) X = KY$$

$$X = \frac{K}{ms^2 + bs + K} Y$$

ii) $x(0) = 0, \ \dot{x}(0) = 1$

$$m(X s^2 - 1(s)) + bXs + KX = KY$$

$$X(ms^2 + bs + K) - m = KY$$

$$X = \frac{K}{ms^2+bs+K} \cdot Y + \frac{m}{ms^2+bs+K}$$

$$= \frac{K \cdot 1}{ms^2+bs+K} \cdot Y + \frac{m}{ms^2+bs+K}$$

summary:

The activity really has 3 cases.

1) $x(0) = \dot{x}(0) = 0$ (Rest IC or zero state)

2) $x(0) = 0, \dot{x}(0) = 1$ (but input $y=0$ (homogeneous or zero input))

3) $x(0) = 0, \dot{x}(0) = 1$ but y arbitrary.

Case 3 is the sum of the first two cases: case-1 (zero steady state), case-2 (zero input response). The Laplace transform of the solution is the sum of these 2 cases.

In case 3 (By linearity) $x(t)$ is the sum of the rest initial conditions solution & the solution with zero input signal but given IC).

Foot note

ZIR/ZSR:

$\dot{x} + Kx = Ke^{-t}$ with arbitrary constant $x(0)$ and got

$$x(t) = x(0)e^{-Kt} + \frac{K}{K-1}(e^{-t} - e^{-Kt})$$

our method dealt seperately with the term involving the IC $x(0)$, and so our formula for

$x(t)$ has two parts: the first part gives a transient with the correct initial conditions; the second part consists in the steady state solution plus a transient designed to make it vanish at $t=0$.

This is a standard & useful decomposition of the system response of any LTI system, with given initial conditions: Say $P(D)x = Q(D)y$. with initial values of x and some of its derivatives given.

The zero state response (or) ZSR → solution to $P(D)x = Q(D)y$ with rest initial conditions. (The 'state' here refers to the initial condition.)

The zero input response or ZIR → solution to the homogeneous equation $P(D)x = 0$, with the same IC. It is a transient. (Assuming that the system is stable.)

The solution to the IVP:

$$x = ZSR + ZIR.$$

<u>Rules</u>

1) Use partial fraction decomposition to find the inverse Laplace transform of any function (rational) $Q(s)/P(s)$ with $\deg Q < \deg P$.

2) Use the pole diagram of $\mathcal{L}(f(t);s)$ to describe the long term behaviour of a function $f(t)$.

3) Analyze the stability of an LTI system by means of the pole diagram of its transfer function.

4) Find Laplace transforms using the s-derivative rule & Exponential shift formula.

Review:

The Laplace transform carries functions of time, $f(t)$ for $t \geq 0$, to complex functions of a complex variable s. The integral definition is

$$\mathcal{L}(f(t); s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Laplace transform technique depends upon developing two lists: A list of rules & a list of computations.

Rules

Linearity:

$$af(t) + bg(t) \xrightarrow{\text{Laplace}} aF(s) + bG(s)$$

Inverse transform:

$F(s)$ essentially determines $f(t)$.

t-derivative rules:

For a function $f(t)$ of exponential type,

$$f'(t) \rightsquigarrow sF(s) - f(0)$$

$$f^{(n)}(t) \rightsquigarrow s^n F(s) - \left(f(0)s^{n-1} + f'(0)s^{n-2} + \cdots + f^{(n-1)}(0)\right)$$

Calculations:

$$1 \rightsquigarrow \frac{1}{s}, \quad Re(s) > 0$$

$$e^{rt} \rightsquigarrow \frac{1}{s-r}, \quad Re(s) > Re(r)$$

$$\cos\omega t \rightsquigarrow \frac{s}{s^2+\omega^2}, \quad Re(s) > 0$$

$$\sin\omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}, \quad Re(s) > 0$$

s-derivative rule

We definitely need to expand our table of calculations: at this point we don't even know how to solve

$\ddot{x} = 1$ with rest initial conditions.

It leads to $s^2 x = 1$ and we don't have $\frac{1}{s^3}$ in our table. It's not different from $\frac{1}{s^2}$ though in fact:

$$\frac{1}{s^3} = -\frac{1}{2} \frac{d}{ds}\left(\frac{1}{s^2}\right)$$

Let's go back to integral definition

$$F'(s) = \frac{d}{ds} \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\therefore F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$F'(s) = \int_0^\infty \left(\frac{d}{ds}\left(e^{-st}\right)\right) f(t)\, dt$$

$$= \int_0^\infty f(t)\,(-t)\, e^{-st}\, dt$$

$$= -\int_0^\infty t \cdot f(t)\, e^{-st}\, dt$$

$$= -\mathcal{L}(t f(t); s)$$

This is a new entry to our table of rules.

<u>The s-derivative rule:</u>

$$t\, f(t) \rightsquigarrow -F'(s)$$

In general,

$$t^n f(t) \rightsquigarrow (-1)^n F^{(n)}(s)$$

<u>For example:</u>

Starting with $1 \rightsquigarrow s^{-1}$, we find in sequence,

$$1 \rightsquigarrow s^{-1}$$

$$(t\cdot 1)\quad t \rightsquigarrow -\frac{d}{ds}s^{-1} = s^{-2}$$

$$(t\cdot t)\quad t^2 \rightsquigarrow -\frac{d}{ds}s^{-2} = 2s^{-3}$$

$$(t\cdot t^2)\quad t^3 \rightsquigarrow -\frac{d}{ds}2s^{-3} = 3(2s^{-4}) = 6s^{-4}$$

$$(t f(t) \rightsquigarrow -F'(s))$$

and in general

$$t^n \rightsquigarrow \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3, \ldots$$

Note that these all converge for $\operatorname{Re} s > 0$

So, to return to $\ddot{x} = 1$ with rest initial conditions

$$s^2 X = \frac{1}{s} \qquad \left(\mathcal{L}(\ddot{x}(t), s) = s^2 X\right.$$

$$\mathcal{L}(1, s) = \frac{1}{s}$$

$$X = \frac{1}{s^3}$$

$$x = \frac{1}{2}t^2 \quad \text{(from our derivation)}$$

$$\left[t^2 (t\cdot t) \rightsquigarrow 2s^{-2}\right]$$

Laplace

$$\frac{1}{2}t^2 \rightsquigarrow s^{-2}$$

which is indeed the solution to $\ddot{x} = 1$ with rest initial conditions $x(0) = 0, \dot{x}(0) = 0$

$$\mathcal{L}(te^{at}; s)$$

Solu:

$$t\,f(t) \rightsquigarrow -F'(s)$$

$$\cancel{t\cdot e^{at} \rightsquigarrow -\frac{d}{dt}(e^{at}) = -ae^{at}}$$

$$\cancel{t^2 e^{at} \rightsquigarrow -\frac{d}{dt}}$$

$$e^{at} \rightsquigarrow \frac{1}{s-a}$$

$$t\,e^{at} \rightsquigarrow -\frac{d}{ds}\left(\frac{1}{s-a}\right) = -\frac{(s-a)(\cancel{1}) - 1(1)}{(s-a)^2}$$

$$= \frac{+1}{(s-a)^2}$$

By applying generality

$$t^n e^{at} \rightsquigarrow (-1)^{n+1} / (s-a)^{2n+1} \cdot n!$$

$\mathcal{L}(t^5 e^{at})$

solu:

$\frac{5!}{(s-a)^6}$ (From generalized formula)

$e^{at} \rightsquigarrow \frac{1}{s-a}$

$t^5 e^{at} \rightsquigarrow -\frac{d^5}{ds^5} \frac{1}{s-a}$

$= -\frac{d^4}{ds^4}\left(-\frac{1}{(s-a)^2}\right)$

$= -\frac{d^3}{ds^3}\frac{2}{(s-a)^3} = \dots \frac{5!}{(s-a)^6}$

$\mathcal{L}^{-1}\left(\frac{4}{s^3}\right)$

solu:

$\therefore t^2 \rightsquigarrow \frac{2}{s^3}$

$2t^2 \rightsquigarrow \frac{1}{s^3}$

Resonance

It occurs when the input frequency is close to a natural frequency of the system. In that case the periodic system response will have exceptionally large amplitude. In the complete absence of damping (a mathematical possibility) there is no periodic response if the two frequencies coincide.

Exam: 4.1

(There is negligible damping) $\rightarrow$ Assume

$\ddot{x} + 9x = 9\cos(3t)$

Because the input frequency coincides with the natural frequency of the harmonic oscillator, we are in resonance. There are still solutions, but none of them are periodic. One can be found using complex replacement & the generalized ERF.

But we can also find solutions using Laplace transform. As usual with the Laplace transform, we need to specify the initial condt.

Say $x(0) = 2, \dot{x}(0) = 0$

Applying Laplace:

$$9\cos 3t \rightsquigarrow \frac{9s}{s^2+9}$$

$$x \rightsquigarrow X$$

$$\ddot{x} \rightsquigarrow s^2X - sx(0) - x'(0)$$
$$= s^2X - 2s$$

we get the algebraic equation in X:

$$(s^2+9)X - 2s = \frac{9s}{s^2+9}$$

$$X = \frac{9s}{(s^2+9)^2} + \frac{2s}{s^2+9}$$

using partial fractions ($\because$ Already in std. partial form)

Finding inverse:

$$\sin\omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}$$

$$t\sin\omega t \rightsquigarrow \frac{-d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$\omega = 3$

$$\frac{3}{2}t\sin(3t) \rightsquigarrow \frac{9s}{(s^2+9)^2}$$

and we find

$$x = \frac{3}{2}t\sin(3t) + 2\cos(3t)$$

$\because x(0) = 2, \dot{x}(0) = 0$

The presence of a repeated factor in the denominator X(s) is the mark of the Laplace transform of the resonance in the s-domain..

presence of the

As you see, in this example one factor came from a characteristic polynomial of the system, the other from the Laplace transform of the input signal. Repeated factors produce repeated roots, and when they occur in the denominator they produce poles of higher multiplicity.

The s-derivative rule & the multiplicity of poles:

$$t f(t) \rightsquigarrow -F'(s)$$

$$t^k f(t) \rightsquigarrow (-1)^k F^{(k)}(s)$$

example:

$$e^{rt} \rightsquigarrow (s-r)^{-1}$$

$$t^{k-1} e^{rt} \rightsquigarrow \frac{(k-1)!}{(s-r)^k}$$

The s-derivative rule associates a pole at $s=r$ of multiciplicity k with a polynomial of degree $k-1$ times e^{rt}.

Computations with quadratic denominators

The previous one used the s-derivative to compute $\mathcal{L}(t \sin\omega t\,; s)$. The s-derivative rule also leads to computation of $\mathcal{L}(t\cos\omega t\,; s)$. and we have two new calculations.

$$t \sin\omega t \rightsquigarrow \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$t \cos\omega t \rightsquigarrow \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Because we are often using our Laplace table to go backwards, that is to compute

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\,; t\right)$$ on our table as we

find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+w^2)^2} ; t\right)$ by combining

$$t\cos(wt) \rightsquigarrow \frac{s^2-w^2}{(s^2+w^2)^2}$$

$$\sin wt \rightsquigarrow \frac{w}{s^2+w^2} = \frac{(s^2+w^2)w}{(s^2+w^2)^2}$$

solu:-

$$\frac{\sin wt}{w} \rightsquigarrow \frac{s^2+w^2}{(s^2+w^2)^2}$$

$$\frac{\sin(wt)}{w} - t\cos(wt) \rightsquigarrow \frac{s^2+w^2-s^2+w^2}{(s^2+w^2)^2}$$

$$= \frac{2w^2}{(s^2+w^2)^2}$$

$$\therefore \frac{1}{2w^2}\left(\frac{\sin wt}{w} - t\cos wt\right) \rightsquigarrow \frac{1}{(s^2+w^2)^2}$$

S-shift rule

we have found the Laplace transform of sinusoidal functions. Damped sinusoids are just as important & occurs as transients of an LTI system response.

Laplace transform of $e^{at}\cos wt$ by writing $e^{at}\cos wt$ as the real part of exponentials (complex), using this expression to find its Laplace transform as a rational function with real coefficients

solu:-

$$e^{at}\cos wt = Re\left(e^{at} \cdot e^{iwt}\right)$$

$$= Re\left(e^{(a+iw)t}\right)$$

From

$$e^{rt} \rightsquigarrow \frac{1}{s-r}$$

$$e^{(a+iw)t} \rightsquigarrow \frac{1}{s-a-iw} \times \frac{(s-a)+iw}{(s-a)+iw}$$

$$= \frac{s-a+iw}{(s-a)^2+w^2}$$

$$Re\left(e^{(a+iw)t}\right) = \frac{s-a}{(s-a)^2+w^2}$$

$$\sin wt = Im\left(e^{iwt}\right)$$

$$e^{at}\sin wt \rightsquigarrow \frac{w}{(s-a)^2+w^2}$$

By forming linear combinations, we can find the Laplace transforms of any damped sinusoid. But the form of the answer here gives us pause; It's very close to the Laplace transform of the undamped sinusoid $\cos wt$. Let's see if this is something general.

$$f(t) \rightsquigarrow F(s) \qquad [say]$$

Using integral,

$$\mathcal{L}\left(e^{at} f(t); s\right) = \int_0^\infty \left(e^{at} f(t)\right) e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$$

$$= F(s-a)$$

That is,

<u>The s-shift rule:</u>

$$e^{at} f(t) \rightsquigarrow F(s-a)$$

This is consistent with the calculation just we did,

$$e^{at}\sin(wt) \rightsquigarrow \frac{w}{(s-a)^2+w^2}$$

Consistency

$\mathcal{L}(1) = \frac{1}{s}$ (so using s-shift $\mathcal{L}(e^{at}\cdot 1) = \frac{1}{s-a}$

matches $\mathcal{L}(e^{at})$ formula.

Exponential shift.

$$\int_0^\infty e^{at} f(t)\, e^{-st}\, dt = \int_0^\infty f(t) \cdot e^{-(s-a)t}\, dt$$

$$= F(s-a)$$

e^{-st} → known as kernal part.

$$\therefore \int_0^\infty f(t) e^{-st} dt = F(s)$$

s is replaced by $(s-a)$

$$e^{at} f(t) \rightsquigarrow F(s-a) \qquad \therefore f(t) \rightsquigarrow F(s)$$

---

1) $e^{-t} \cos(3t)$

Solu:

$$e^{-t} \cos 3t \rightsquigarrow \frac{s}{s^2+\omega^2}$$

s replaced by $s-(-1) = s+1$

$$= \frac{s+1}{(s+1)^2+\omega^2}$$

$\boxed{\omega = 3}$

$$\mathcal{L}(e^{-t} \cos 3t \, ; s) = \frac{s+1}{(s+1)^2+9}$$

$e^{2t}\ t \cos 3t$

Solu:

$$t \cos 3t \rightsquigarrow -\frac{d}{ds}\left(\frac{s}{s^2+9}\right) = -\left(\frac{s^2+9-s(2s)}{(s^2+9)^2}\right)$$

$$e^{2t}\ t \cos t \rightsquigarrow \frac{(s-2)^2-9}{((s-2)^2+9)^2} \qquad = -\left(\frac{s^2+9-2s^2}{(s^2+9)^2}\right)$$

(By the formula)

$$= -\left(\frac{-s^2+9}{(s^2+9)^2}\right)$$

$$= \frac{s^2-9}{(s^2+9)^2}$$

Laplace table

$1 \rightsquigarrow \frac{1}{s}$ Re $s > 0$

$e^{rt} \rightsquigarrow \frac{1}{s-r}$ Re $s >$ Re r

$\cos \omega t \rightsquigarrow \dfrac{s}{s^2+\omega^2}$ $\quad Re\ s > 0$

$\sin \omega t \rightsquigarrow \dfrac{\omega}{s^2+\omega^2}$ $\quad Re\ s > 0$

$t \to \dfrac{1}{s^2}$ $\quad Re\ s > 0$

$t^n \to \dfrac{n!}{s^{n+1}}$ $\quad Re\ s > 0$

$t \sin \omega t \rightsquigarrow \dfrac{2\omega s}{(s^2+\omega^2)^2}$, $\quad Re\ s > 0$

$t \cos \omega t \rightsquigarrow \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$, $\quad Re\ s > 0$

$\dfrac{1}{2\omega} t \sin(\omega t) \rightsquigarrow \dfrac{s}{(s^2+\omega^2)^2}$ $\quad Re\ s > 0$

$\dfrac{1}{2\omega^2}\left(\dfrac{1}{\omega}\sin \omega t - t\cos \omega t\right) \rightsquigarrow \dfrac{1}{(s^2+\omega^2)^2}$ $\quad Re\ s > 0$

<u>Rules:</u>

$f'(t) \rightsquigarrow sF(s) - f(0)$ $\to$ t-derivative rule

$t f(t) \rightsquigarrow -F'(s)$ $\to$ s- " "

$e^{at} f(t) \rightsquigarrow F(s-a)$ $\to$ s-shift rule

$\mathcal{L}^{-1}\left(\dfrac{4}{(s-5)^3}\right)$

$t^2 \rightsquigarrow \dfrac{2}{s^2}$

<u>Solu:</u>

$2\dfrac{t^2}{2} \rightsquigarrow \dfrac{2}{s^2} \times 2 = \dfrac{4}{s^2}$

$2e^{5t}\cdot t^2$

$2e^{5t}\cdot\left(\dfrac{t^2}{2}\right) \rightsquigarrow \dfrac{4}{(s-5)^3}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s^2+4s+13}\right)$

<u>Solu:</u>

$\mathcal{L}^{-1}\left(\dfrac{1}{(s^2+4s+4)+9}\right) = \mathcal{L}^{-1}\left(\dfrac{1}{(s+2)^2+9}\right)$

$= \dfrac{e^{-2t}}{\omega}\cdot \sin \omega t \quad (\omega = 3)$

$= \cancel{e^{-2t}\cdot \dfrac{1}{2(9)}\left(\dfrac{1}{3}\sin 3t - t\cos 3t\right)}$

$= \frac{e^{-2t}}{3} \sin 3t$ $\therefore \sin\omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}$

$\frac{1}{3}\sin 3t \rightsquigarrow \frac{1}{s^2+9}$

$e^{-2t}\frac{1}{3}\sin 3t \rightsquigarrow \frac{1}{(s+2)^2+9}$

Pole diagram & Homogeneous Solutions

From previous lectures, we have learnt how to read stability information of an LTI system from the pole diagram of its transfer function. We also learnt how to take the laplace function of a function $f(t)$ in the time domain, and express it instead in the frequency domain, $F(s)$. This gave us a new method to find a system response of an LTI system to a wide variety of input signals using algebra, partial fractions & inverse laplace transforms read off of a table.

In practice, the real power of laplace transform is that it gives you very useful information about the solutions without requiring you to solve it out explicitly. In particular, the pole diagram of the transfer function is the pole diagram of a generic zero input response (ZIR). (Recall: The zero input response is the solution when the input is zero) → on the solution to the associated homogeneous equ). Meaning that important information about the homogeneous solutions can be read off directly from the pole diagram of the transfer function.

- a real pole of $F(s)$ at $s=a$ comes from a term in $f(t)$ of the form e^{at}.

- A conjugate pair $a \pm i\omega$ comes from a term in $f(t)$ of the form $e^{at}\cos(\omega t - \phi)$

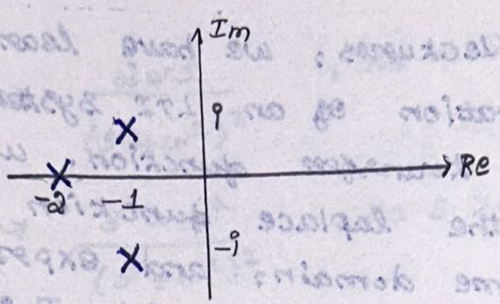
Examples:

1) An LTI system has the following transfer function.

$$H(s) = \frac{q(s)}{p(s)} = \frac{1}{(s^2+2s+2)(s+2)}$$

$$= \frac{As+B}{s^2+2s+2} + \frac{C}{s+2}$$

The system is stable. $\therefore$ poles $s = -1 \pm i, -2$ have negative real parts.



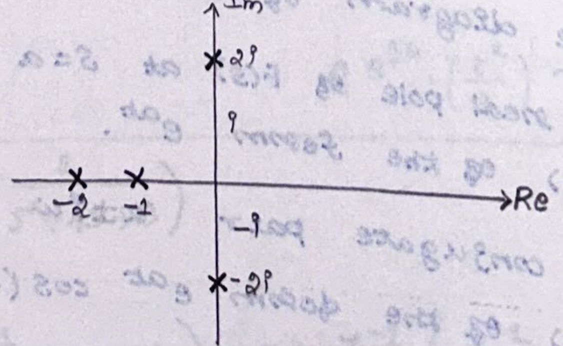
The pole diagram implies that the general zero input response is

$$Ae^{-t}\cos t + Be^{-t}\sin(t) + Ce^{-2t} = De^{-t}\cos(t-\phi) + Ce^{-2t}$$

we can read this homogeneous solution immediately ~~of~~ of the pole diagram, which is the pole diagram of the Laplace transform of a generic homogeneous solution.

The poles at $-1 \pm i$ contribute to a transient term $De^{-t}\cos(t-\phi)$. The pole at $s=-2$ contributes to a transient term Ce^{-2t}.

2. A certain LTI system has the following pole diagram



From the pole diagram, we see that a generic homogeneous solution for this system takes the form

$$A_1e^{-t} + Be^{-2t} + C\cos(2t) + D\sin(2t)$$

The first two components from poles in the left half plane are transient & tend to zero as $t \to \infty$. The poles along the imaginary axis correspond to

$$C \cos 2t + D \sin 2t$$

This system is not stable. Since the poles don't have negative real part.

Summary of new terminology (Relates to old terminology)

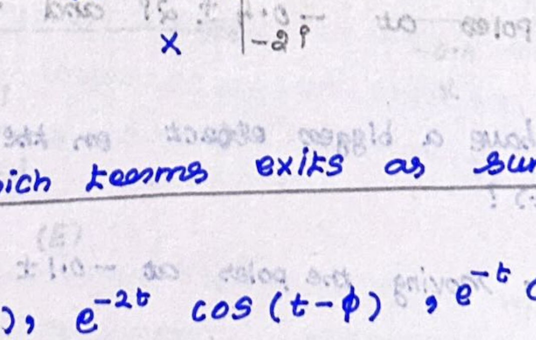
Then the transfer function is

$$H(s) = \frac{Q(s)}{P(s)}$$

$\because P(D)x = Q(D)y$ (where P and Q have no common factors)

New terminology	old terminology
poles of $H(s)$	The zeros of $P(D)$
ZIR's (zero input response)	Homogeneous solutions (solutions to $P(D)x = 0$)

Determine long-term behaviour (From pole-diag)



Determine which terms exits as summands of ZIR:

$$\cos(t-\phi),\ e^{-2t}\cos(t-\phi),\ e^{-t}\cos(2t-\phi)$$

$$A\cos(t-\phi_1) + Be^{-t}\cos(2t-\phi_2) + Ce^{-2t}\cos(t-\phi_3)$$

True or False: All ZIR's of the system with the given pole diagram tend to 0 as time grows large.

solu.

Note that a ZIR tend to zero if the real part of all poles is negative. Since there are two purely imaginary poles of this system, many ZIRs have a dominant term $A\cos(t-\phi_1)$ which doesn't tend to zero as t gets large. So the answer is large.

'False'

Poles & vibrations

$$e^{at}\cos(bt) = Re\left(e^{(a\pm bi)t}\right)$$

$$e^{at}\sin(bt) = Im\left(e^{(a\pm bi)t}\right)$$

Find the poles of $\mathcal{L}(e^{-4t}\cos 5t; s)$

Solu:

$$e^{-4t}\cos 5t \rightsquigarrow \frac{(s+4)}{(s+4)^2+25}.$$

$$e^{-4t}\sin 5t \rightsquigarrow \frac{\omega}{\omega^2+s^2}$$

$$\rightsquigarrow \frac{5}{25+(s+4)^2}$$

$$= s^2+8s+16+25$$

$$= s^2+8s+41$$

$$= \frac{-8\pm\sqrt{64-4(41)}}{2}$$

$$= -4\pm\sqrt{16-41}$$

$$= -4\pm\sqrt{-25}$$

$$= -4\pm 5i$$

$$x(t) = Ae^{at}\cos(bt) + Be^{ct}\cos(dt)$$

(Let $B=0$)

Set $A=2, B=2$ (Poles at $-0.4\pm 2i$ and $-0.1\pm 2i$)

Solu:

which will have a bigger effect on the long-term behaviour of $x(t)$?

Ans: Moving the poles at $-0.1\pm 2i$ a little.

The long-term behaviour is determined by the right-most pole. So moving the poles at $-0.1\pm 2i$ will have the biggest affect on the long-term behaviour of x. Moving them left or right will affect the decay rate and moving them up or down affect the frequency of oscillation.