Non linear differential equations: Graphical method

1) Use slope fields & isoclines to aid in the drawing of solution curves for the first order ODE.

2) Draw integral curves aided by existence & uniqueness theorems, and assess long term behaviours of solution.

Intro

Linear (VS) Nonlinear

$y' = y(1-y) \Rightarrow y' = y - y^2$ (Non linear)

$y' + 2y = x \Rightarrow$ (linear)

$y' - y^2 = -x^2 \Rightarrow$ (Non linear)

$y' - y^3 + 3y = -x \Rightarrow$ (Non linear)

$y' = xy \Rightarrow \frac{dy}{y} = x \cdot dx$ (linear)

$y' = -\frac{x}{y} \Rightarrow y \cdot dy = y' + \frac{x}{y} = 0$ [Non linear] ↳ Degree of y is -1.

$y' = y/x \Rightarrow \frac{dy}{y} = \frac{1}{x} dx$ (linear)

$yy' = x^3 - y \Rightarrow yy' + y = x^3$ (Non linear)

Linear $=$ $y' + P(x) y = q(x)$. $\Rightarrow$ $y' - xy = 0$

$y' + 2y = x$

$y' - \frac{1}{x} y = 0$

$y' = f(x, y) \rightarrow$ First order ODE's.

$y' = \frac{x}{y} \rightarrow$ Solvable by seperation of variables.

$y' = x - y^2 \rightarrow$ (Not solvable in ordinary sense).

Standard form:

$y' = f(x, y)$

$f(x,y) \rightarrow$ is a function of the two variables x and y.

Notice: we have two std. forms for the I order diff equ.

A I order diff equ (Linear) can be written as

$$y' + P(x)y = Q(x)$$

In this unit → we are going to solve some nonlinear I order diff equ. (Qualitative behaviour of solutions using geometric methods of approximation).

Geometric view of DE

(Analytic method).

Explicitly write the equations: $\boxed{y' = (f(x,y))}$

↳ we'll look for the function which solves it (previous parts).

Geometric method

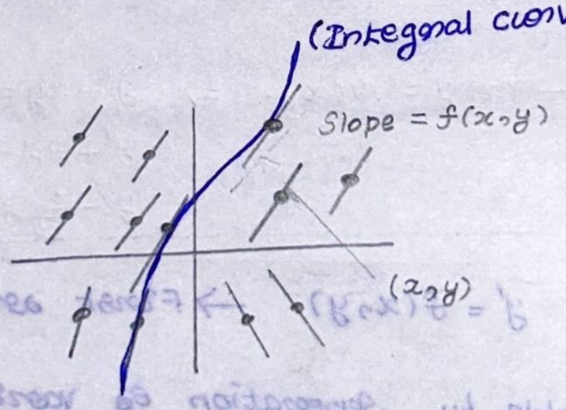
ODE → ★ Direction field. (version of writing equ in std. form)

Solution → Integral curve

(In geometric point of view — known as)

(Integral curve).

Direction field:



(Plane) (Random - Here).

Integral curve: (curve — which goes through the plane and at every point is tangent to the element there)

→ Integral curve

(If I have the element line here — I would find that the curve had exactly the right slope there.)

what distinguishes the integral curve is that everywhere it has the direction - that's the way I'll indicate that it's tangent has the direction of the field everywhere at all points,

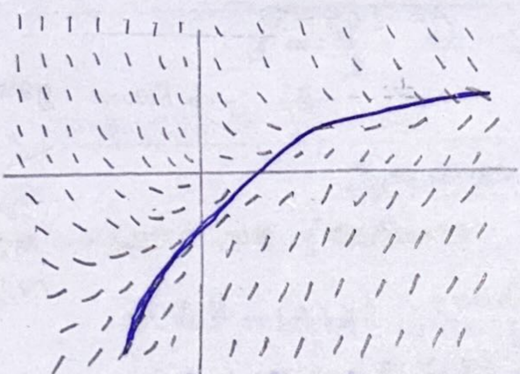
Integral curve is the solution graph to the ~~diff~~ eqn.

<u>Geometrically</u>

Drawing direction field — Same as writing ODE (Analytically)

Integral curve (Drawing) — Solving an ODE analytically)

$y' = f(x, y)$



Solution curve (or) Integral curve

An integral curve must be tangent to the slope field at every point : $y_1'(x) = f(x, y_1(x))$.

<u>Slope field.</u>

<u>Exam: 4.1</u>

Sketch the slope field for $y' = y^2 - x$.

<u>Solu:</u> Let $f(x,y) = y^2 - x$

$f(1,2) = 3$ → draw a short segment of slope = 3

$f(0,0) = 0$ → slope = 0

$f(1,0) = -1$ → " slope = −1

$f(0,1) = 1$ → " slope = 1.

The diagram of all these short segments is the slope field. You can see how tedious this process is; a computer will sketch the slope field much more quickly. You can see the slope field for this example in the mathlet.

$\boxed{y' = y^2 - x}$ → mathlets.org

Isoclines.

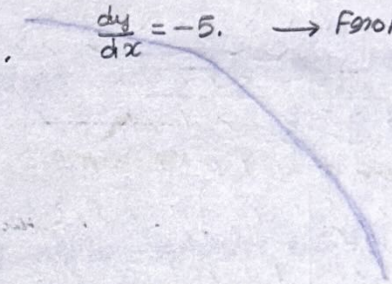
why slope field?

∴ The ODE is telling us that the slope of the solution curve at each point is the value of $f(x, y)$. So the short segment is, to first approximation, a little piece of the solution curve. To get an entire solution curve follow the solutions curves next.

---

1) $\frac{dy}{dx} = x + y$ At $(0, -5)$

$\frac{dy}{dx} = -5.$ → From graph

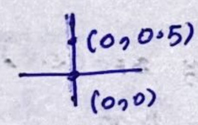


(Decrease) −ve slope near $(0, -5)$

$y' = y(1-y)$

solu:

If $y(0) = \frac{1}{2}$



The lines approaching / / /

∴ solution approaches 1.

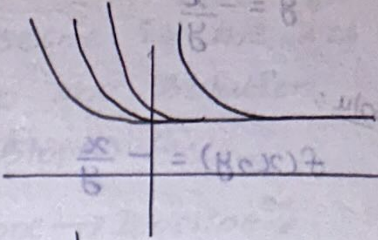
$y(0) = -1$ \ \ \ \ \ \ \

Solution approaches $-\infty$.

$y(0) = 2$ → solution approaches 1.

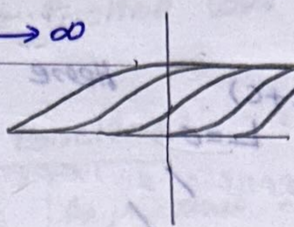
$y' = y - y^2$	$y(0) = \frac{1}{2}$	$y(0) = 2$
$= -1 - 1$	$y' = \frac{1}{2} - \frac{1}{4}$	$y' = 2 - 4$
$= -2$	$= \frac{1}{4}$	$= -2$
↓ goes to ∞	(Solution approaches $\frac{1}{4}$	Solution approaches 1

$1 < y(a) < \infty$ (Irrespective of a) → the solution curve decreases to 1. as $x \to \infty$.

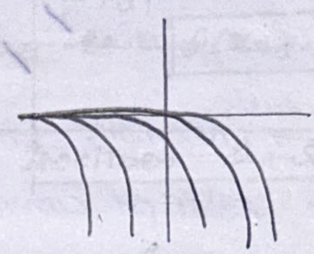


↳ click the point $(0, \frac{1}{2})$.

$0 < y(a) < 1$ – the solution curve ↑ to 1 as $x \to \infty$



$y(a) < 0$ – the solution curve ↓ to $-\infty$ as $x \to \infty$



$y' = \frac{y}{x}$ At $(1,1)$ → A ray coming out of the origin

---

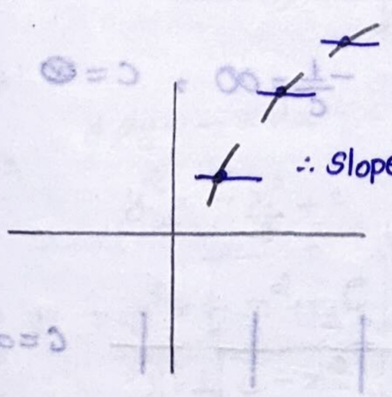
Isoclines

Computer — Naive one (ஐகுளா)

1) pick (x, y) (equally spaced)
2) Computes $f(x,y)$ at that point.
3) Draws (Line element having slope $f(x,y)$)

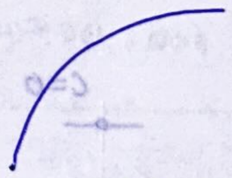
Human:

1) pick slope c
2) $f(x,y) = c$: plot this equ.



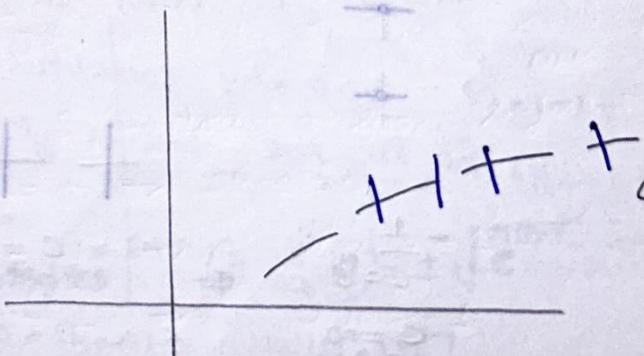
c=0

∴ Slope $c = 0$



Isocline
$C = 0$

(curve points where slope $c = 0$)



$c = -1$

$y' = -\frac{x}{y}$

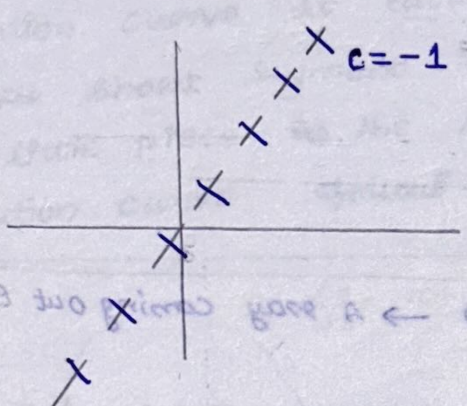
solu:

$f(x,y) = -\frac{x}{y}$

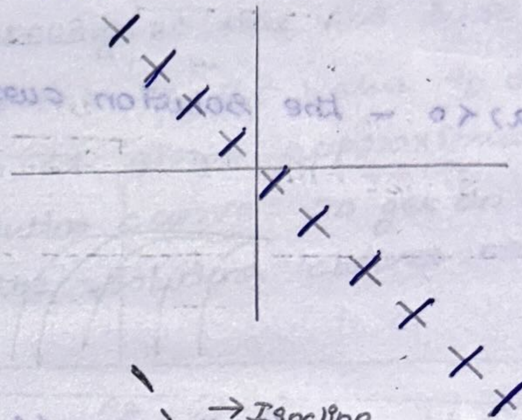
$C = -\frac{x}{y}$

$\boxed{y = -\frac{1}{C}x}$ $(y=mx+C)$ Here C = slope

$L=0.$



$c=-1$



$C=+1.$

\ → Isocline

/ → Line segment.

$y = x$

Note: They are always going to be perpendicular.

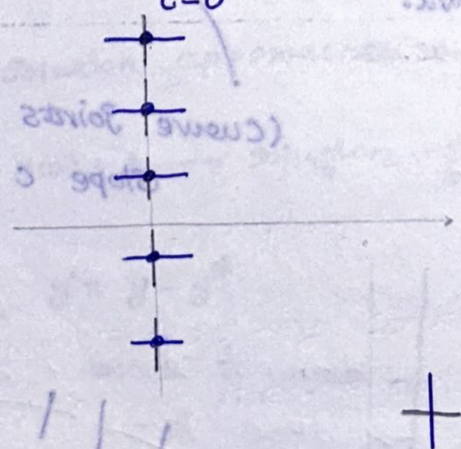
∴ Slope $= \frac{-1}{c}$ (Slope of the line)

Slope $= C$. (Slope of the line element)

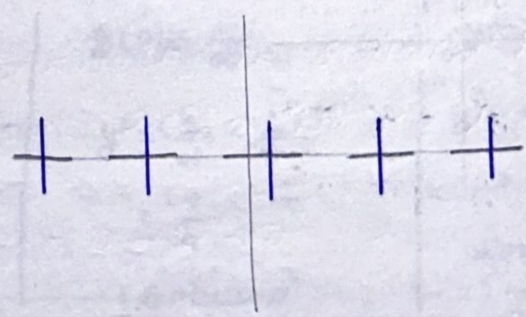
∴ $\frac{-1}{c}$ and $+c$ are negative reciprocals.

$\boxed{m_1 \times m_2 = -1}$

$C=0$



$-\frac{1}{c} = \infty$, $C = 0$



$C=\infty$

$-\frac{1}{C}=0$

$c=\infty$

weaselly thing → resembling weasel (small, slender carnivorous mammal)

Definition: 6.1:

For a number c, the c-isocline is the set of points in the (x, y)-plane such that the solution curve through that point has slope c.

(Same incline (or) same slope $\rightarrow$ Isocline)

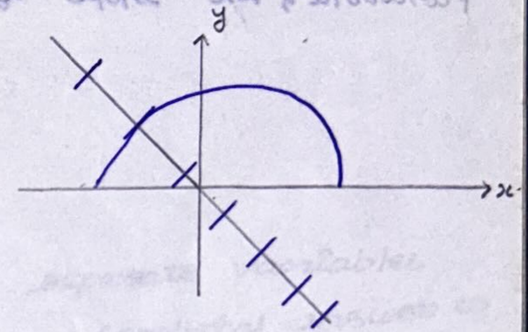
Eqn of the c-isocline:

The ODE says that the slope of the solution curve through a point (x, y) is $f(x, y)$, so the solution of the c-isocline is $\boxed{f(x,y) = c}$

Isoclines & Solution curves.

Solution curve: 

Isocline: 



Equation of Isocline: $y' = -\frac{x}{y}$. (Find the eqn of the 2-isocline)

Soln:

$$f(x,y) = -\frac{x}{y} = 2 \Rightarrow y = -\frac{1}{2}x.$$

$y' = -\frac{x}{y}$ (Solve)

Soln:

$$y\,dy = -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c \qquad \text{At } (0, 2)$$

$$y^2 = -x^2 + C$$

$$y = \pm\sqrt{C - x^2}$$

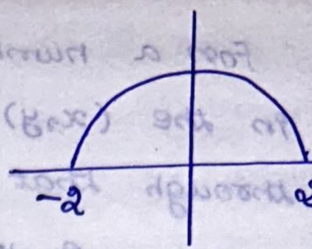
$(x, y) = (0, 2)$ also satisfies $\Rightarrow$ $2 = \pm\sqrt{C}$

$$\boxed{C = 4}$$

$$y = \pm\sqrt{4 - x^2}$$

Domain $(-2, 2)$

Food for thought : why are the solution curves not a full circle?



Zero Isoclines

(For human assistance).

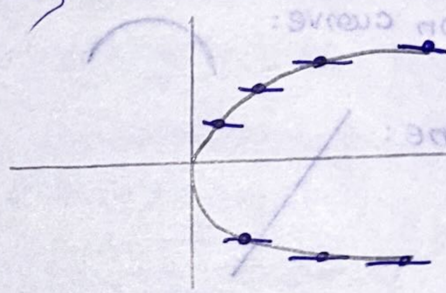
Isoclines organize the slope field. The 0- isocline is known as the null isocline, is specially helpful.

7.1 :-

$$y' = y^2 - x \qquad (0\text{-isocline})$$

Solu :-

$\boxed{C=0} \rightarrow 0 = y^2 - x \longrightarrow$ parabola (At every point of this parabola, the slope of the solution curve is 0.



7.2 $y' = y^2 - x$ (where are the points at which the slope of the solution curve is +ve.)

Solu :-

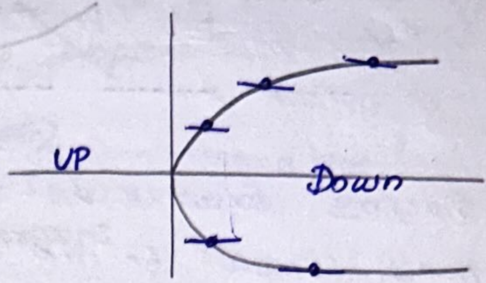
$f(x,y) > 0$. The 0-isocline $f(x,y) = 0$ divides the plane in to regions and $f(x,y)$ has constant sign on each region. To test the sign, check one point in each region.

$$f(x,y) = y^2 - x$$

$f(-1,0) = 1 > 0$, by continuity it follows that $f(x,y) > 0$ in the entire region to the left of the parabola

III^{ly} $f(1,0) = 0 - 1 < 0$

It allows $f(x,y) < 0$ in the region right of the parabola. Therefore, the answer is that the slope of the solution of the curve is (–)ve in the region to the right of the parabola. Therefore, the answer is that the slope of the solution curve is negative in the region to the right of the parabola, & the slope of the solution is +ve in the region to the left.



Isocline mathlet

use mathlets.org for complete analysis

Example

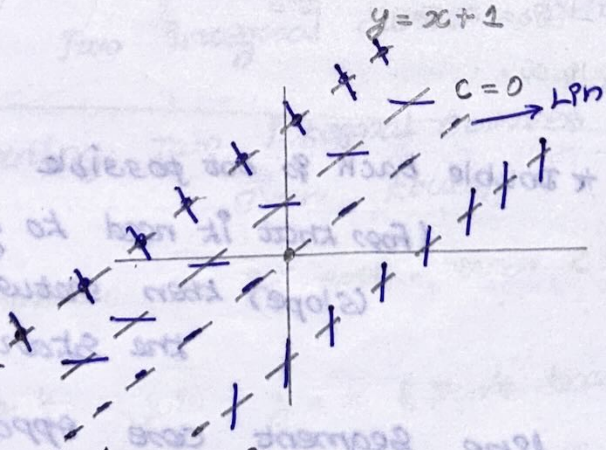
$y' = 1 + x - y$

Solu:

$y = 1 + x - c$

$\boxed{y = (1-c) + x}$ → we can't seperate variables (Analytical answer is tougher).

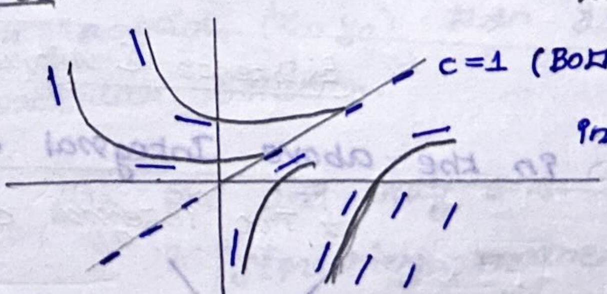
Geometrically: when $c = 0$



→ Line itself. | → slope (Line seg)

Solution curve will be:

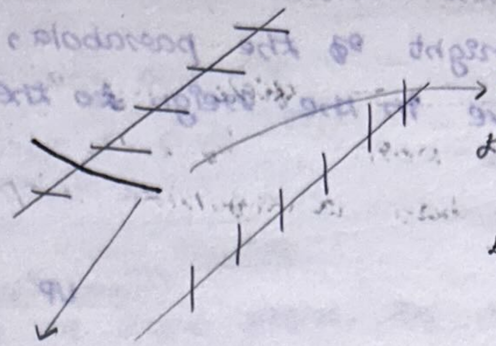
At $c = 1$ where



c = 1 (Both Isocline & Integral curve)

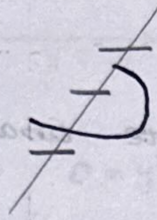
(By that line segment, the solution curve follow it.)

what's happening ~~in the~~ corridor b/w $c=0$ & $c=2$ isoclines:



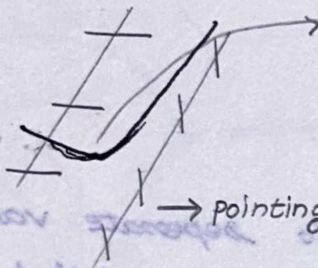
Integral curve.

For escaping (from) this corridor it needs to return to the starting position. For that it need to adjust its position.



(For doing this it have a (−)ve slope at some point)

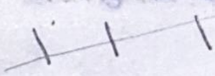
But the slope of isoclines are +ve.



The solution can't escape.

pointing upwards. (Have a Negative slope)

Not possible. (For doing that it must have lesser steepness − slope than the isoclines.



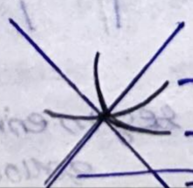
★ Double back is not possible (For that it need to go horizontal (slope) then return to the starting)

∴ The slopes of line segment are opposing it to return & blocks it.

Existence & uniqueness theorem

principles in the above Integral curves:

★ Two integral curves can't cross at an angle.



Integral curve 1

2

Line segment 1, 2

why :

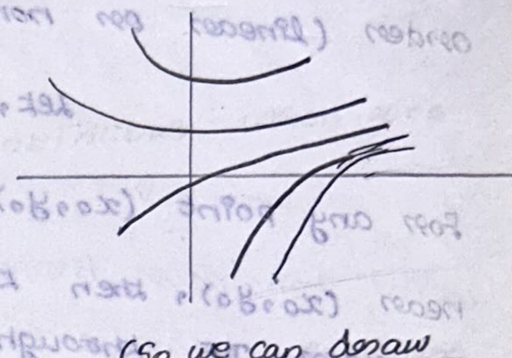
They fight eachother? what will be the true slope at that point.

∴ The direction field only allows to have a single slope at a point? If there is a line element at that point, it has a definite slope.

(So it can't have two slopes)

↙

So we can make,



All the integral curves are trying to reach a particular solution as

$y(x) \sim x$ as $x \to \infty$

(So we can draw them closer & closer)

($x - a$ solution)

↳ They can't escape & intersect eachother

They move towards a particular solution. → Here $\boxed{y = x}$

Principle: 2: (Less obvious)

Two integral curves can't be a tangent.

meaning: Two integral curves don't even touch.

(Not possible!)

(Asymptotic to it even, even closer).

They can't join $y = x$ (Don't have a common solution)

why: Existence & uniqueness theorem.

which says through a point (x_0, y_0) then $y' = f(x, y)$ has only one particular solution.

↳ Has one and only one solution

[Stressing- mathematically]

Hypothesis: $f(x,y)$ → should be continuous function.

eg: Polynomials, Sines, (Continuous in the vicinity (near) of that point)

That guarantees existence. We can't guarantee uniqueness. It's based on the partial derivative w.r.t y which should be continuous near x_0, y_0.

Theorem: Existence & uniqueness theorem for a first order (linear or nonlinear) ODE.

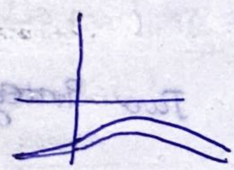
Let, $y' = f(x,y)$

For any point (x_0, y_0), if $f(x,y)$ & $\frac{\partial f}{\partial y}$ are continuous near (x_0, y_0), then there is a unique solution to the I order DE through the point (x_0, y_0)

As a consequence of uniqueness, we have the following two geometric features:

1) Solutions can't intersect (can't cross)

2) Solution curves can't become tangent to one another; (They can't touch).



(They don't)

---

when Existence & uniqueness fails.

1) $xy' = y-1$

Solu:

$$x\frac{dy}{dx} = y-1$$

$$\frac{dy}{y-1} = \frac{1}{x}dx$$

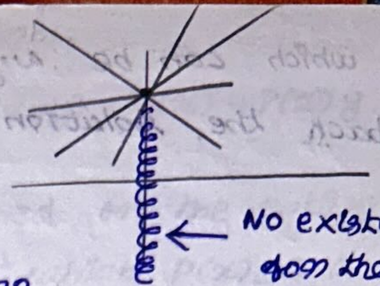
$$\ln|y-1| = \ln|x| + c$$

$$|y-1| = c|x|$$

$$y-1 = cx$$

$y = 1+cx$ Here $y \neq 1$ $\therefore$ $\boxed{c \neq 0}$

→ This point is oversupplied.
(uniqueness fails)



Is it a violation?

No! Because a theorem has no exceptions, it wouldn't be a theorem. (At $y=1$)

Then what's wrong:

Writing in standard form

$$\frac{dy}{dx}=\frac{1-y}{x} \rightarrow \text{Not continuous when } x=0$$

the y-axis.

(Failed).

As a practical manner, the way existence & uniqueness fails in ordinary ~~life~~ work with diff eqn fails in not through sophisticated examples that mathematicians can construct, but normally because $f(x,y)$ will fail to be defined somewhere,

---

At the points where the hypotheses of the existence & uniqueness theorem fail, the conclusion of the theorem may also fail. Here is another example demonstrating this.

11.2: $y'=\frac{2y}{x}$

Solu: $f(x,y)=\frac{2y}{x}$ (undefined at $x=0$) → So things might go wrong along the y-axis, and in fact they do go wrong.

Solve the ODE by seperation of variables:

$$\frac{dy}{dx}=\frac{2y}{x} \quad (x\neq 0)$$

$$\frac{dy}{y}=\frac{2dx}{x}$$

$$\ln|y| = 2\ln|x| + c$$

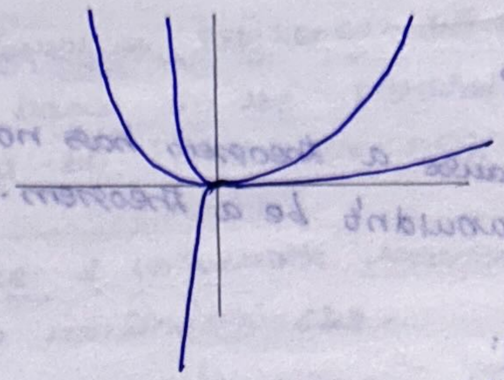
$$y=\pm e^{2\ln|x|+c}$$

$$y=\pm|x|^2 e^c$$

$y=Cx^2 \quad (x\neq 0)$

$C=\pm e^c$

which can be any non zero real numbers. To bring back the solution $y=0$, we allow $c=0$ as well.



several half parabolas as solution curves to $y' = \frac{2y}{x}$.

weird behaviour happens along $x=0$ where $y' = \frac{2y}{x}$ is not even defined:

* Through any point $(0,b)$ on the y-axis, there is no solution curve. The existence theorem doesn't apply.

* Geometrically, the parabolas become tangent at the origin. This would be ruled out by uniqueness if the uniqueness theorem applied. The full parabolas are not solutions; the solution curves are half parabolas defined for either all $x<0$ or all $x>0$.

Both existence & uniqueness apply to every point outside the y-axis. The rest of the plane (outside the y axis) is covered with good solution curves, one through each point, none touching or crossing the others.

(There is no connection b/w any half parabola on the left & any half parabola on the right.)

Interval of validity

The characteristics of a solution curves that we've been discussing are true for both linear & non linear (conv.) But there is often a big difference b/w the domain of definition of solutions to linear & Non linear ODEs.

For I order linear DEs of the form,

$$y' = f(x, y) \quad \text{where} \quad f(x,y) = q(x) - p(x)y.$$

Any solution defined at $x=a$ is defined on the entire interval which contains a and on which $p(x)$ and $q(x)$ are continuous. (Note that p and q being continuous is equivalent to the hypothesis of the general existence / uniqueness theorem presented in this lecture, namely that

f and $\frac{\partial f}{\partial y}$ are continuous.

For non-linear DEs, solutions don't have to be defined on the entire interval on which f and $\frac{\partial f}{\partial y}$ are continuous.

Draw the solution curve to $y' = y^2$ that satisfies $y(0) = 1$. What's the domain of definition of this curve?

Solu. This DE is non linear. (solving it by seperation of variables)

$$\frac{dy}{dx} = y^2$$

$$y^{-2}\,dy = dx \quad (y \neq 0)$$

$$\int y^{-2}\,dy = \int dx$$

$$-y^{-1} = x + c$$

$$y = -\frac{1}{x+c} \qquad (y(0) = 1)$$

$$1 = -\frac{1}{c} \qquad \Rightarrow y = \frac{-1}{x-1}$$

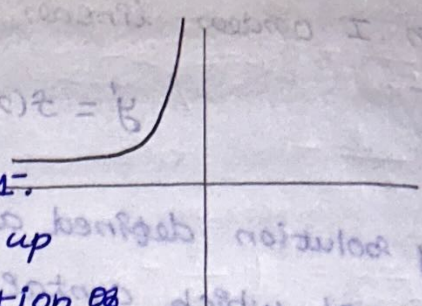
$$\boxed{c = -1} \qquad \boxed{x \neq 1}$$

which gives the solution

$$y = \frac{1}{1-x} \quad (\text{where } -\infty < x < 1)$$

Solution curve

At $x = 1 \rightarrow$ Blows up (The solution).



The solution approaches ∞ as $x \to 1^-$.
We say that the solution blows up at $x = 1$. The domain of definition of the solution is $-\infty < x < +1$.

It is also called the interval of validity of the solution. The interval of validity of a function is the largest interval on which it can be defined.

The graph of $y = \frac{1}{1-x}$ (consists of both branches of the hyperbola. But be careful, the full hyperbola is not a solution. Only the branch through the point $(x,y) = (0,1)$ is.

For non-linear DE's, the interval of validity of a solution can't be read off from the eqn. It can be much smaller than the domain on which the eqn is defined.

Interval of validity:

$$y' = xy^4 \quad (y(0) = 1)$$

$a < x < b$ of $y(x)$

Solu:

$$y' = xy^4 \Rightarrow y^{-4}dy = xdy$$

$$-3y^{-3} = \frac{x^2}{2} + c$$

$$\frac{1}{-3}y^{-3} = \frac{x^2}{2} + c$$

$$y^{-3} = -\frac{x^2+c}{3}$$

$$y^{-3} = -\frac{3x^2}{2} + c$$

$$y^3 = \frac{3}{x^2+c}$$

$$y = \frac{1}{\left(-\frac{3}{2}x^2 + c\right)^{1/3}}$$

$$y = \sqrt[3]{3/x^2+c}$$

when $y(0) = 1$

$$1 = \sqrt[3]{\frac{8}{c}}$$

$$1 = \frac{8}{c}$$

$\boxed{c = 8}$

$$y = \cancel{\sqrt[3]{\frac{8}{x^2+3}}}$$

$$y = \frac{1}{\left(-\frac{3}{2}x^2+1\right)^{1/3}}$$

we see that y is not defined at $x = \pm\sqrt{\frac{2}{3}}$.

$\therefore$ validity of the solution satisfying $y(0) = 1$ is

$-\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$. Infact $y \to \infty$ as $x \to \pm\sqrt{\frac{2}{3}}$; that's the solution blows up at $x = \pm\sqrt{\frac{2}{3}}$.

Direction fields:

$$y' = \frac{-y}{x^2+y^2}$$

i) sketch the direction field

ii) For solution through $y(0) = 1$

why is $y(x) > 0$ for $x > 0$?

why is $y(x) \downarrow$ for $x > 0$?

Solu:

Looking at isocline: curve where the derivative $y' = m = \text{constant}$.

$$\boxed{y' = \frac{-y}{x^2+y^2} = m}$$

Null isocline: $m = 0$

$$0 = \frac{-y}{x^2+y^2}$$ (only possible when $y = 0$)

The Nullcline for this ODE is $y = 0$.

General isoclines,

$$-\frac{1}{m}y = x^2+y^2.$$

$$x^2+y^2+\frac{1}{m}y=0$$

$$x^2+\left(\frac{1}{m}y+y^2\right)=0$$

$$x^2+\left(y+\frac{1}{2m}\right)^2=\frac{1}{4m^2}$$

$\boxed{x^2+y^2=a^2}\rightarrow$ eqn of o^{le}.

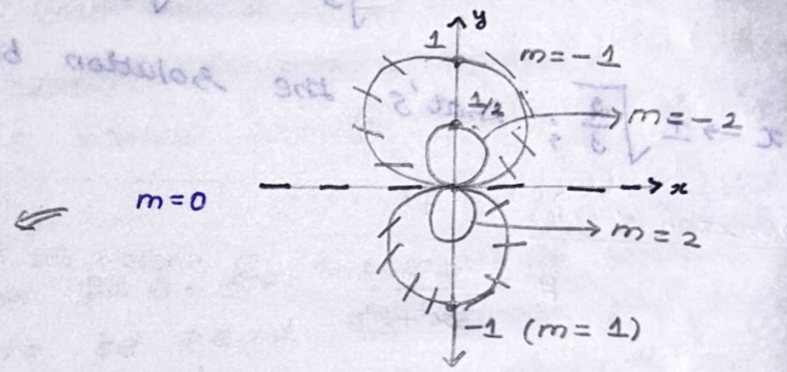
$$y^2+2\left(\frac{1}{2m}y\right)+\frac{1}{4m^2}-\frac{1}{4m^2}=0$$

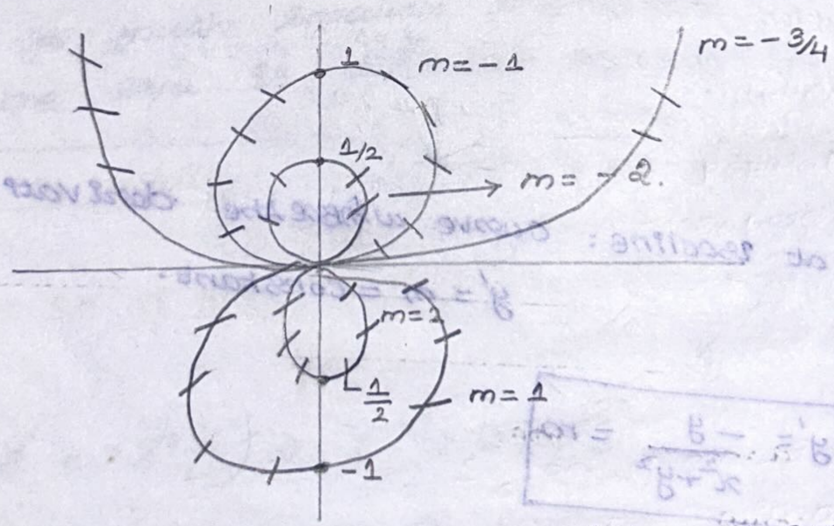
$$\left(y+\frac{1}{2m}\right)^2=\frac{1}{4m^2}$$

circle having coordinates

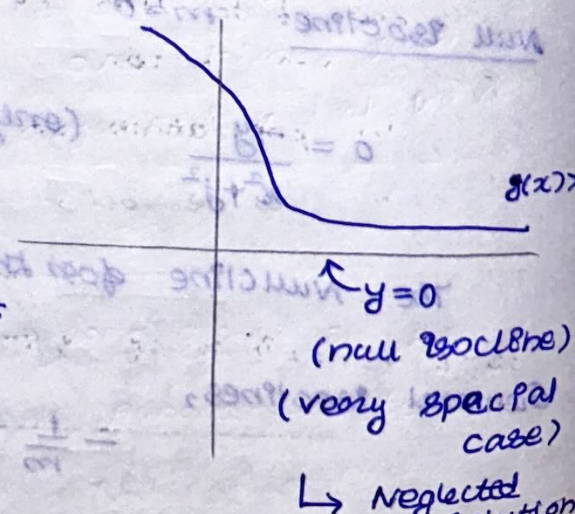
$$(x-x_1)+(y-y_1)=a^2$$

The centre is $\left(0, -\frac{1}{2m}\right)$, radius $=\frac{1}{2|m|}$





The integral curve will be



No way (Not necessary) that every nullcline need to be solution to the DE

★ Solution curves can't cross or touch each other.

ii) $y(x) > 0$ for $x > 0$.

$$y' = -\frac{y}{x^2+y^2} \qquad y \to >0, \quad -y \to <0 \quad \Big| \quad x^2+y^2 > 0$$

$\therefore \frac{-ve}{+ve} = (-ve) \to$ monotonically decreasing

The integral curve, y always stays in the upper half for $x > 0$. So the $y(x) > 0$.

Change of variables

$\dot{y} + y^2 t - y = 0$ (known as Bernoulli's equation)

Solu: Make the change of variable $y = \frac{1}{u}$.

$\dot{u} = f(u,t)$

$$\dot{y} = -u^{-2} = -\frac{1}{u^2} \cdot \dot{u}$$

Solu:
$$-\frac{\dot{u}}{u^2} + \frac{1}{u^2} t - \frac{1}{u} = 0$$

$$-u^2 \cdot \frac{\dot{u}}{u^2} + t - u = 0 \Rightarrow -\dot{u} + t - u = 0$$

$$\boxed{\dot{u} = t - u}$$

Notice that this equ is linear.

Autonomous equation

1) Identify autonomous ODEs
2) Critical points
3) Use the phase line to determine stability of solutions to autonomous ODEs.
4) Model population growth with the logistic equ & find qualitative behaviour of solutions.
5) Determine how varying parameters in a system can change behaviour of solutions using a bifurcation diagram.

$$\frac{dy}{dt} = f(y) \rightarrow \text{No } t \text{ on RHS}$$

(Time Independent)

∴ Sometimes the independent variable is not time. So we are using the term autonomous. (Generic word for being no independent variable on R.H.S)

How to get useful info from the equ without solving the equ:

* Because, its fast
* Gives you a lot of insight

Get qualitative info without solving it.

An autonomous ODE is a diff equ that doesn't explicitly depend on the independent variable. If time is the independent variable, this means that the ODE is time invariant.

The standard form for a I order autonomous equ is

$$\frac{dy}{dt} = f(y) \quad (\text{Instead of } f(t, y))$$

where R.H.S doesn't depend on t, $\dot{y} = \frac{dy}{dt}$

For example, the (non-linear) equ:

$$\boxed{\dot{y} = y - 3y^2}$$ is autonomous. We'll findout later that this DE models population growth in an environment with limited resources. It's called a logistic equ.

why is this called autonomous?

In ordinary English, a machine or robot is called autonomous, if it operates without human input. A differential equ is called autonomous if its co.eff are not changed over time, such as might happen if a human adjusted a dial on a machine & let it run.

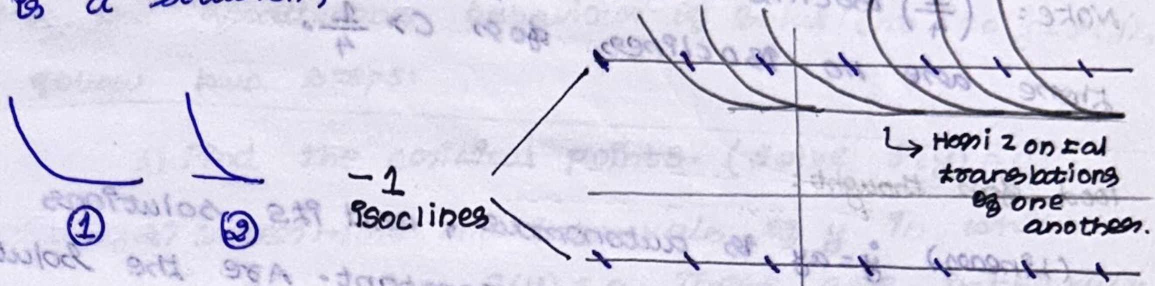
$\dot{x} = 2x - 5x^2$, $\dot{y} = \frac{3}{y}$, $\dot{y} = \cos y$ → don't depend on the independent variable t. Therefore Autonomous, because of this fact

First properties

Here are two consequences of the time invariance of an automation eqn:

* Each isocline in the (t, y)-plane has one or more horizontal lines.

* Solution curves (in the (t,y)-plane) are horizontal translations of one another. That is, if $y(t)$ is a solution, then so $y(t-a)$ for any a.



① & ② are same - just horizontally translated.

Why are all translates of solutions also solutions? a nongeometric argument:

$y(t)$ be solution to $\dot{y} = f(y)$

Let $u = t - a$, then

$$\frac{d}{dt} y(t-a) = \frac{d}{dt} y(u(t)) = \frac{dy}{du} \cdot \frac{du}{dt} \quad \text{(chain rule)}$$

Function of $y \to$ has u, $u \to$ has t

$$= f(y(u))(1)$$

$$= f(y(t-a))$$

$\therefore$ $y(t-a)$ is also a solution.

Horizontal isoclines

Find the (-1)-isocline for the DE $\dot{y} = y(1-y)$. Answer by giving the eqn of the lines that the isocline consists of.

Soln:

$$\dot{y} = y(1-y)$$

$$c = y - y^2$$

$$y^2 - y + c = 0 \qquad y = \frac{1 \pm \sqrt{1-4c}}{2}$$

when $c < \frac{1}{4}$, the c-isocline has two horizontal lines $y = (1 \pm \sqrt{1-4c})/2$. As expected, the isoclines are horizontal for this autonomous equation. In particular

(-1) isocline is gn by the horizontal lines

$$y = \left(\frac{1 \pm \sqrt{1+4}}{2}\right)$$

$$y = \frac{1 \pm \sqrt{5}}{2}$$

Note: $\left(\frac{1}{4}\right)$ isocline has a single horizontal line $y = \frac{1}{2}$ and there are no isoclines for $c > \frac{1}{4}$.

Food for thought:

(Linear) $\dot{y} = ay$, is autonomous, and its solutions are $y(t) = ce^{at}$. where c - any constant. Are the solutions horizontal translations (or time shifts of one another).

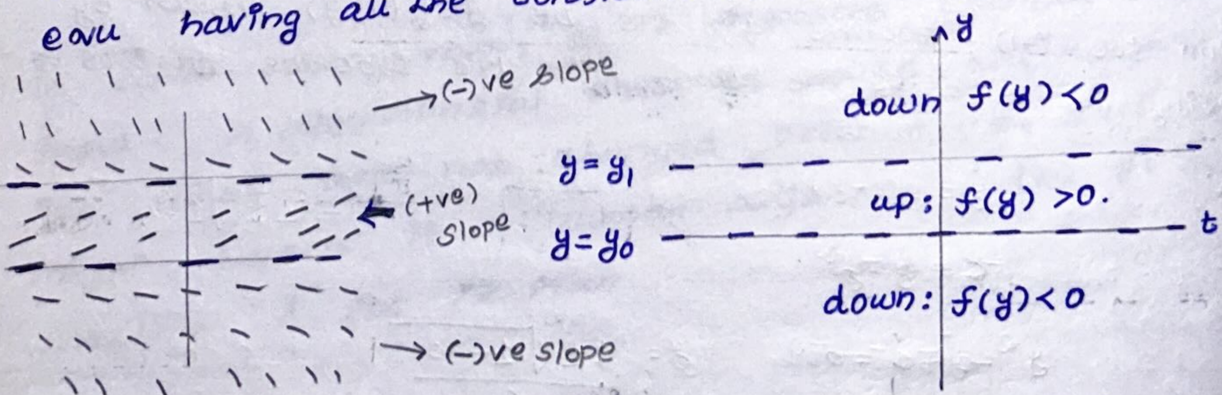
Critical points

Def: 4.1

The values of y at which $f(y) = 0$ are called critical points or equilibria of the automomous equation.

$$\dot{y} = f(y)$$

If y_0 is a critical point of an autonomous eqn, then $y = y_0$ is a constant (or horizontal or equilibrium) solution. Because the derivative of a constant solution is 0. The zero isocline of an autonomous function is eqn having all the constant solutions.



The constant solutions $y = y_0$ & $y = y_1$ (where y_0 and y_1 are critical points) divide the (t, y) plane into 'up' & 'down' regions.

Recall that for any I order DE, the 0-isocline divides the (t,y) plane into 'up' regions, where $f>0$ and solutions are increasing, and 'down' regions, where $f<0$ and solutions are decreasing. For autonomous eqn, the qualitative behaviour of all solutions is encoded by the critical points and the signs of $f(y)$ in the intervals b/w the critical points.

To find the qualitative behaviour of solutions to $\dot{y}=f(y)$, we follow two steps:

1) Find the critical points. (Solve $f(y)=0$)

2) Determine the intervals of y in which $f(y)>0$ and in which $f(y)<0$. These are intervals in which solutions are increasing & decreasing respectively.

<u>Example: 1</u>

<u>Critical points and qualitative behaviour of solutions:</u>

y – money in the bank account

r – continuous interest.

Autonomous eqn ← $\dfrac{dy}{dt} = ry - w$ y – principle amount.

W = Rate of embezzlement. (By shifty tellers)
(கையாடல்)

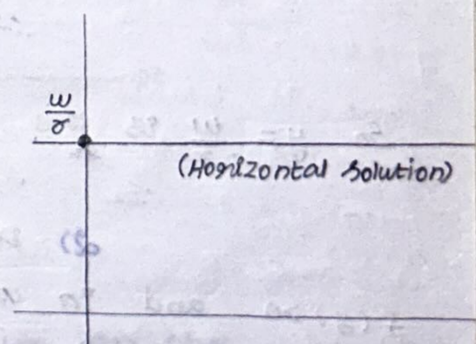
↳ Reduced from your account

<u>Analyzing using the method of critical points:</u>

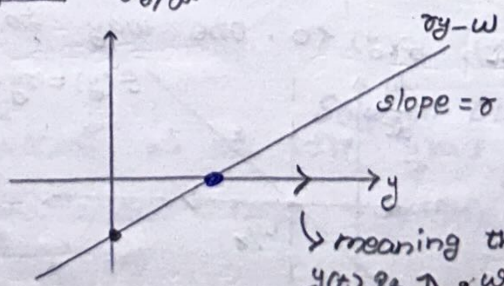
<u>Step: 1:</u> $\dfrac{dy}{dt}=0 \Rightarrow ry-w$

$w = ry$

$\boxed{y=\dfrac{w}{r}}$



<u>Graph f(y): Step: 2:</u>

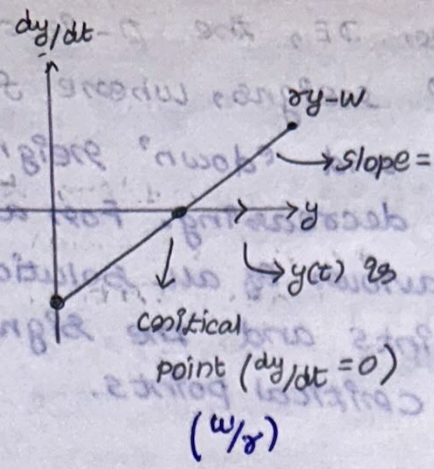


↳ meaning that $y(t)$ is ↑, when

$y = mx + c$

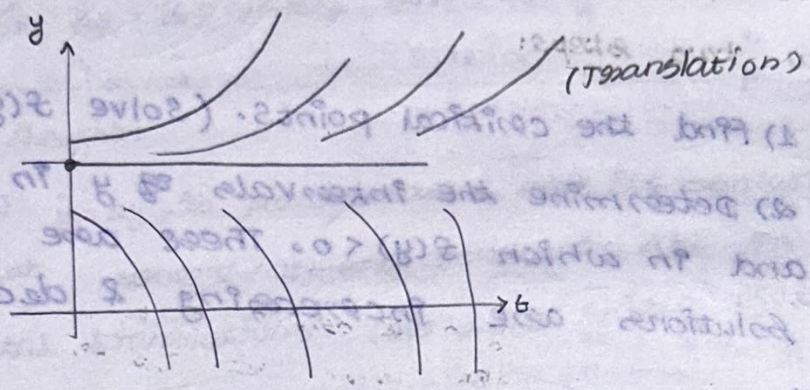
$\dfrac{dy}{dt} = ry - w$

$f(y)$ is +ve



∴

Solutions:



Above the horizontal line, the Function (solution curve is) ↑

Below the solution curve is ↓.

<u>phase line:</u>

$\dot{y} = f(y) = ry - w$ $(r, w$ are $+ve)$.

The variable y models the amount of money in (dollars) in a saving account; r is the interest rate (In units of years^{-1}.) and w (in dollars is the rate of money being stolen out of the account.

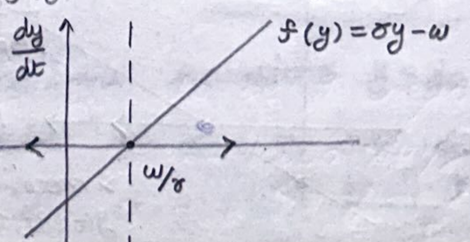
To find qualitative behaviour of the solutions, we follow the two steps described here:

1) Find the critical points

$$f(y) = ry - w = 0 \Rightarrow y = \frac{w}{r}.$$

So $y = \frac{w}{r}$ is the constant solution.

2) Determine the intervals of y in which $f(y) > 0$ and in which $f(y) < 0$. One way to do this is by graphing $f(y)$.



For $y > \frac{w}{\gamma}$, $f(y) > 0$ and any solution $y(t)$ is increasing

for $y < \frac{w}{\gamma}$, $f(y) < 0$ and any solution $y(t)$ is decreasing

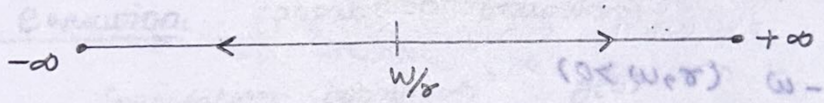
We can summarize the above info by adding two arrows on the horizontal (y-axis as in the picture). The critical points divides the y-axis into intervals. We place a right arrow (in the $+y$ direction) to each interval of y where $f(y) > 0$ & any solution $y(t)$ is increasing.

III^{ly}, we place a left arrow (in the $-y$ direction) to each interval where $f(y) < 0$ and solutions are decreasing. The y-axis with this extra info is called a phase line.

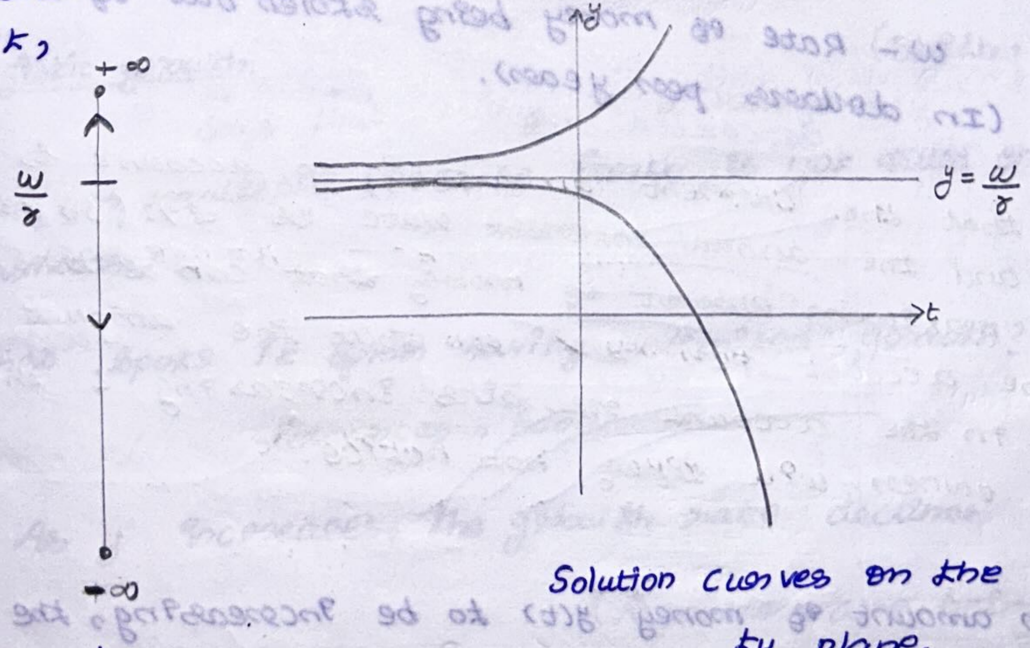
<u>Definition: 1</u>

The phase line of a I order autonomous DE $\dot{y} = f(y)$ is a plot of the y-axis with all critical points and with an arrow in each interval b/w the critical points indicating whether solutions $\uparrow$ or $\downarrow$.

Here,



If the phase line is drawn vertically, the qualitative behaviour of solution curves can be read directly from it,



Solution Curves on the ty plane.

Vertical phase line

5.2 Use the phase line above to determine what happens to solutions with different initial conditions as time passes.

Solu:

- $y(0) = w/r$ (if), which the critical ~~position~~, then the solution will be $y(t) = \frac{w}{r}$ for all t. This corresponds to the constant solution show on (t, y) in the above figure.

- If $y(0) > w/r$, then the up arrow (the +ve y direction) on the phase line tells us that $t \uparrow$, $y(t) \uparrow$ and will tend to $+\infty$. This corresponds to a solution curve above the horizontal solution, such as the one shown on the (t, y) plane above.

- If $y(0) < w/r$, then the down arrow (in the (-)ve direction of y) on the phase line says that as $t \uparrow$, $y(t)$ decreases and will tend to $-\infty$. This corresponds to a solution curve below the horizontal solution, such as the one shown on the (t, y)-plane above.

---

maximum rate of embezzlement.

$\dot{y} = f(y) = ry - w \quad (r, w > 0)$

where y - Amount of money in a savings account;
r - Interest rate (in $year^{-1}$)
w - Rate of money being stolen out of the account (In dollars per year).

Suppose that the current amount in an account is \$10000, and the annual interest rate is 2%. (so $r = 0.02$) what is maximum amount of money that can be stolen from the account per year so that the amount of money in the account remains increasing & the account owner will likely not notice.

Solu:

For amount of money $y(t)$ to be increasing, the solution has to satisfy $y(0) = 10000 > w/r$, where $r = 0.02$.

$w = 10000r = 200$. So the maximum amount of money stolen per year has to be less than 200.

Consider $y' = ry - w$

If $y(0) < \frac{w}{r}$, what is $\lim_{t \to -\infty} y(t)$

soln:

$y(0) < \frac{w}{r}$, the solution curve is below the horizontal solution.

As $t \to -\infty$, $y(t) \to w/r$ → from (t, y) graph.

We can also use the phase line to find the answer. To run time backwards, reverse the arrows in the phase line. As $t \to -\infty$, we have $y(t) \to w/r$.

Warning: The phase line can only be used for autonomous equations.

(The phase line doesn't capture the exact shape of solution curves. However, since all solutions are translations of one another, the phase line does retain the qualitative behaviour of all solutions.

Logistic equation: (population equation)

'population behaviour' $y(t)$

$\frac{dy}{dt} = ky$.

k - Growth rate.

(Net birth rate)

(Birth - dying).

Logistic growth:

Says that telling k a constant is non realistic. (Because Earth is not filled entirely with humans.)

What spots it from having unlimited growth?

Resources, food

As y increases, the growth rate declines

(Resources are being used up).

simplest choice, $k = a - by$

$$\frac{dy}{dt} = (a-by)y.$$

$= ay - by^2$ (Logistic equ - Many applications).

The simplest model for population $y(t)$ is the ODE $\dot{y} = ky$ for a positive growth constant k, which is the birth rate - death rate of the population. The DE says that the rate of population growth is proportional to the current population, and we know $y(t) = ce^{kt}$ [solutions]

But by reality, If $y(t)$ is too large - the competition for food & space will grow. The population will grow less quickly. One simple way to model this is to adjust the growth rate from a constant function of y to a linearly decreasing function of y.

$K(y) = a - by$ $(a, b > 0)$ constants.

$$\dot{y} = K(y)y$$
$$= (a-by)y = ay - by^2 \quad (a, b > 0)$$

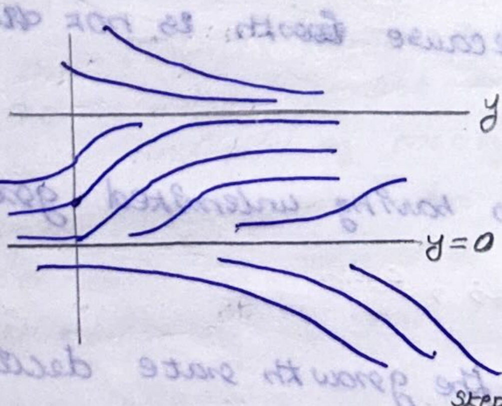
This diff equ is called the logistic equ. It is a non linear, autonomous equation.

Qualitative behaviour

Critical points:

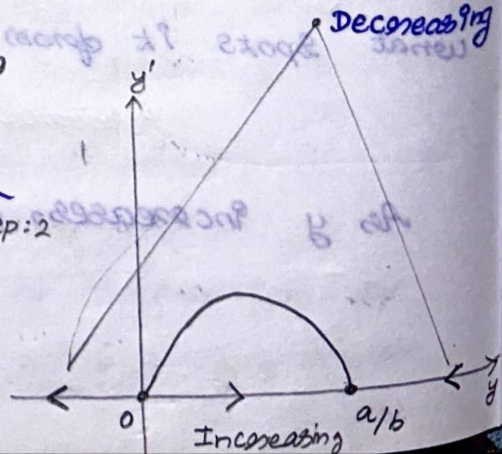
$y(a-by) = 0$ $y = 0$, $y = \frac{a}{b}$.

Step: 1



(Two constant solutions

Step: 2



As y^2 is larger than y.

Exam: 7.1

Describe qualitatively the solutions to $\dot{y} = 3y - y^2$

(special case of the logistic equ $\dot{y} = ay - by^2$)

Solu:

To find qualitative behaviours of the solutions, we follow the two steps described before.

1. Find critical points

$$0 = 3y - y^2 \qquad\qquad y = 0, 3$$

$$(3y - y^2) = 0$$

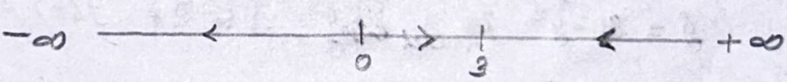
$$3y = y^2$$

$$\boxed{y = 3}$$

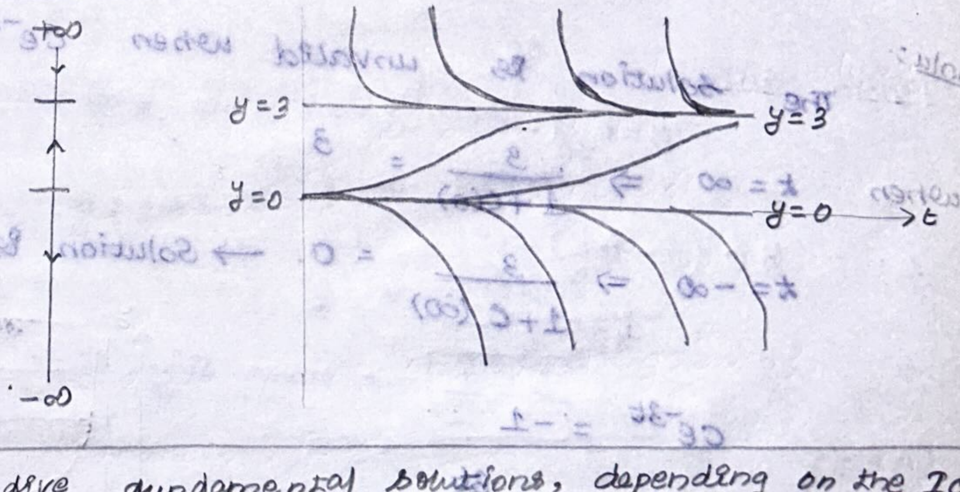
2) Determine the intervals of y in which $f(y) > 0$ and in which $f(y) < 0$. To do this, we can graph $f(y)$.

- $f(-1) = -2 < 0$ $(y < 0)$ is a down sign
- $f(1) = 2 > 0$ (up sign) $0 < y < 3$
- $f(4) = 12 - 16 < 0$ $(y > 3)$ $\rightarrow$ down sign.

Phase line:



Sketch some solutions:



There are five fundamental solutions, depending on the IC:

- If $y(0) = 0$, the solution will be $y(t) = 0$ for all t.
- If $y(0) = 3$, the solution will be $y(t) = 3$ for all t.

• If $0 < y(0) < 3$, $y(t)$ will increase as t increases. It tends to 3 as $t \to +\infty$ and tends to 0 as $t \to -\infty$.

• If $y(0) > 0$, then $y(t)$ decreases as $t \uparrow$, tending to 3 without reaching 3.

• If $y(0) < 0$ then $y(t) \downarrow$ as $y(t) \to -\infty$ as t grows.

For a solution $y(t)$ with $0 < y(t) < 3$, is it possible that $y(t)$ tends to a limit less than 3.

Ans: No, if $y(t) \to$ tends to any number a b/w 0 and 3, then as the solution curve ~~levels off~~ while approaching a, $\dot{y}(t)$ must tend to zero.

But we know that $f(y) > 0$ for all $0 < a < 3$,

So this is impossible.

$\therefore$ up to (b/w) 2.9... and 0.000...1 (the curve is $\uparrow$) B/w that the curve doesn't decrease or settles).

Interval of validity of a solution

By separation of variables, the general solution to

$\dot{y} = 3y - y^2$ is $y(t) = 0$ (or) $y(t) = \frac{3}{1+ce^{-3t}}$

(constant)

This is valid $a < t < b$

Solu:

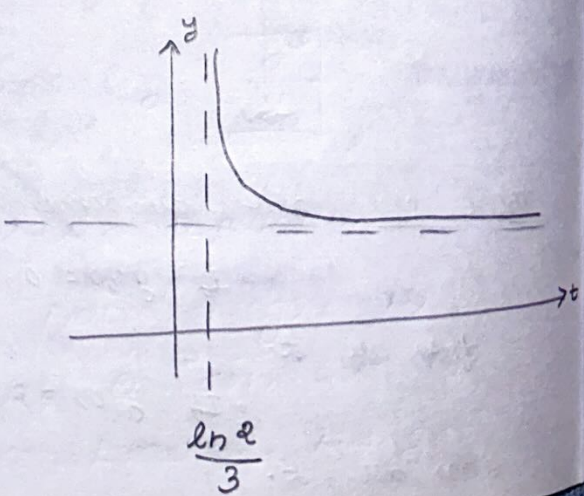
The solution is unvalid when $ce^{-3t} = -1$

when $t = \infty \Rightarrow \frac{3}{1+c(0)} = 3$

$t = -\infty \Rightarrow \frac{3}{1+c(\infty)} = 0. \to$ Solution is valid

$ce^{-3t} = -1$

$c = -e^{3t}$



$\dot{y} = 3y - y^2$

$\frac{dy}{3y - y^2} = dt.$