associated homogeneous function, finding one particular solution & applying superposition.

* Solve IVP for I order linear ODEs.

Seperation of variables:

Seperation of variables is a technique that reduces the problem of solving certain I-order ODES to the problem of evaluating two integrals.

Seperation of variables works when we can write the equ in the form.

$$\frac{dy}{dt} = g(t) \cdot f(y).$$

warning: we divided by y, so at some point we will have to check $y=0$ as a potential solution.

eg: $\dot{y} - 2ty = 0$

Solu:

$$\frac{dy}{dt} = 2ty \Rightarrow \frac{dy}{y} = 2t\,dt$$

$$\int \frac{dy}{y} = 2t\,dt$$

$$c_1 + \ln|y| = t^2 + c_2$$

$$\ln|y| = t^2 + (c_2 - c_1)$$

$$\ln|y| = t^2 + c$$

$$|y| = e^{t^2} \cdot e^c$$

$$y = \pm e^c \cdot e^{t^2}$$

$$\boxed{y = ce^{t^2}} \quad c \neq 0$$

Note: The method reduces to evaluating two integrals seperately, one in y and one in t. Observe that we combined the two constant c_1 & c_2 of integration into one constant c. We will combine the two constants to form

$c = c_2 - c_1$ (immediately from now on)

As c runs over all real numbers, the co.eff $\pm e^c$ runs over all non zero real numbers. Thus the solution we have found so far are

$$y = ce^{t^2} \quad \boxed{c \neq 0}$$

Because we divided by y, there is possibly an exceptional solution $y=0$. checking directly, it turns out that it is indeed a solution. It can be considered as the function ce^{t^2} for $c=0$

As $y=0$

$$y = ce^{t^2}$$

$$0 = ce^{t^2} \quad (e^{t^2} \neq 0)$$

So $\boxed{c=0}$ But $\boxed{c \neq 0}$ → As per $\therefore c \to$ all real numbers except 0.

The general solution to $\dot{y} - 2ty = 0$

$$y = ce^{t^2} \text{ (where } c \text{ — Any real number)}$$

Double check:

we can put $y = ce^{t^2}$ in $\dot{y} - 2ty = 0$

$$\frac{d}{dt}(ce^{t^2}) - 2tce^{t^2} = 0$$

$$2t \cdot c \cdot e^{t^2} - 2tce^{t^2} = 0$$

$$\boxed{0 = 0}$$

Seperation of variables, the systematic procedure:

1. Check that the DE is a I-order ODE (If not – give up & try another method). Suppose that the function to be solved for is $y = y(t)$.

2. Rewrite $\dot{y}$ as $\frac{dy}{dt}$.

3. Express $\frac{dy}{dt}$ as a function of t and y only

$$\frac{dy}{dt} = h(t, y)$$

4. If you are lucky, you can write $h(t,y)$ as a product of two function (purely t, purely y).

$$h(t,y) = g(t) f(y)$$

5. Seperate y's & t's. (mul or div) → equality

$$\frac{dy}{f(y)} = g(t)\, dt$$

If there are factors involving both variables such as $y+t$, then it is impossible to seperate variables; in this case give up & try a diff method.

warning: Dividing by $f(y)$ invalidates the calculation if $f(y)=0$, so at the end, check what happens when $f(y)=0$; This may add to the list of solutions.

6. Integrate both sides to get an eqn. of the form

$$F(y) = G(t) + C$$

These are implicit eqn for the solutions, in terms of parameter c.

7. If possible (and is desired), solve for y in terms of t.

8. Check for extra solutions coming from dividing by zero (The solution in the previous step & the step comprise the general solution).

9. (optional) → But recommended (check by verifying the answer.)

---

a) Solve IVP $\frac{dy}{dx} = y^2$, $y(0) = 1$, General solution to the DE $\frac{dy}{dx} = y^2$

a) Solu:

$$\frac{dy}{y^2} = dx$$

$$\int y^{-2}\, dy = \int dx$$

$$\frac{y^{-1}}{-1} + C_1 = x + C_2$$

$$\boxed{C = C_2 - C_1}$$

$$-\frac{1}{y} = x + C$$

$y(0) = 1$

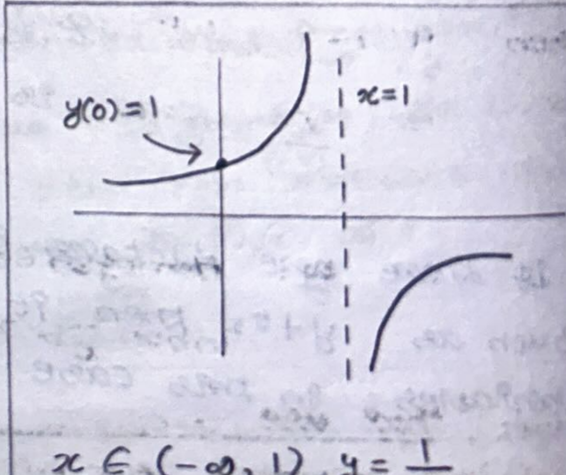
$$-\frac{1}{1} = C$$

$$\boxed{C = -1}$$

$$\therefore -\frac{1}{y} = x - 1$$

$$\frac{1}{y} = 1 - x$$

$$y = \frac{1}{1-x}$$



$x \in (-\infty, 1)$, $y = \frac{1}{1-x}$

$x \in (1, +\infty)$ $y = \frac{1}{1-x}$

General solution:

$$y = \frac{1}{-c-x}$$

$$y = -\frac{1}{x+c}$$

$\begin{cases} y = -\dfrac{1}{x+c}, \; y = 0 & \rightarrow \text{Solutions available} \\ y \neq 0 & \rightarrow \text{undef.} \end{cases}$

I order linear ~~diff~~ eqn:

$$a(x)y' + b(x)y = c(x)$$

Linear: $ay_1 + by_2 = c$ $\quad$ ($c = 0 \rightarrow$ homogeneous).

Standard form: (Linear)

$$y' + P(x)\,y = q(x)$$

$$P(x) = \frac{b(x)}{a(x)}$$

$$q(x) = \frac{c(x)}{a(x)}$$

Std. linear form:

Every I order linear ODE can be written in std. linear form as follows

$$\dot{y} + P(t)\,y = q(t)$$

$P(t), q(t) \rightarrow$ any functions of t.

when $q(t) = 0$, we call the eqn homogeneous

An eqn. that is not homogeneous is inhomogeneous.

eg: Homogeneous: $\dot{y} + P(t)\,y = 0$

Inhomogeneous: $\dot{y} + P(t)\,y = q(t)$

Models that use I order linear ODES

I order linear eqn are common in modelling. Here are some examples.

1. Temperature conduction.

2. Concentration diffusion

3. Radioactive decay

4. Bank account interest

5. Simple population growth.

Put the following in std. linear form.

$$x^2(y' - xe^x) + (\ln x)y = 0$$

solu:

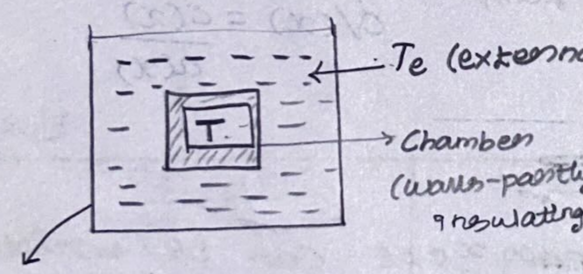
$$x^2y' - x^3e^x + (\ln x)y = 0$$

$$x^2y' + (\ln x)y = x^3e^x$$

$$y' + \left(\frac{\ln x}{x^2}\right)y = xe^x$$

Modelling Conduction & diffusion models:

Conduction | Diffusion



(Transmission of heat is only by Conduction)

t - time, T = Temperature.

Newton's law of cooling:

$$\frac{dT}{dt} \propto (T_e - T)$$

(∴ K have significance. (phy) ↳ K → +ve)

$$\frac{dT}{dt} = K(T_e - T)$$

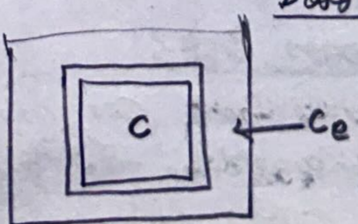
∴ $K > 0$ → Conductivity of the system.

If $T_e > T$, $\frac{dT}{dt}$ → raise

If $T_e < T$, $\frac{dT}{dt}$ → Decrease.

$$\boxed{T(0) = T_0}$$

Diffusion model



c → Salt Concentration

C_e → Salt Conc. outside.

Through a semipermeable membrane. - Salt may diffuse (takes hard time)

(membrane wall).

$$\frac{dc}{dt} \propto (C_e - C)$$

$\therefore$ If $C_e > C$, $\frac{dC}{dt} \rightarrow$ Raise.

$$\frac{dc}{dt} = k(C_e - C)$$

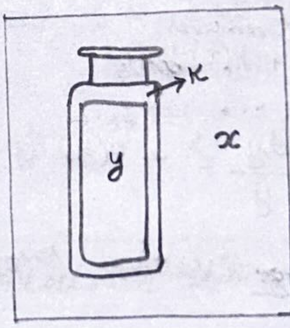
Minestrone soup - Italian origin (vegetable soup often with pasta or rice - some time both) + ingredients

My minestrone soup is in an insulating thermos. Model its temperature as a function of time.

Simplifying assumptions:

* The insulating behaviour (ability) of the thermos doesn't change with time.
* The rate of cooling depends only on the difference b/w the soup temp & external temp.

1. Draw a picture:



2. Identify variables:

$t \rightarrow$ time (minutes)

$x \rightarrow$ external temp (°c)

$y \rightarrow$ soup temp (°c)

$k \rightarrow$ Conduction of thermos (to be determined by env)

$t \rightarrow$ Independent variable . x, y - Functions of t. $k \rightarrow$ System parameter det by env

3. Identify input, response, and any Initial condt.

System: Thermos full of soup

Input x, the temp outside of the thermos.

* Response y – temp of the soup inside the thermos
* No IV's were given

4. Differential equation:

$$\dot{y} = K(x-y) \quad \text{(Newton's Cooling law)}$$

$K > 0$, unit of $K \to min^{-1}$.

Smaller $K \to$ Better insulation.

Std. form:

$$\dot{y} + Ky = Kx$$

Here $P(t) = K$
$q(t) = Kx(t)$.

Here,
$q = Kx(t)$ is not the input. The input signal is $x(t)$ and the response is $y(t)$.

Solving homogeneous I-order linear ODES

Homogeneous I-order linear ODE can be solved by seperation of variables.

$$\dot{y} + p(t)\,y = 0$$

$$\frac{dy}{dt} + p(t)\,y = 0$$

$$\frac{dy}{y} = -p(t)\,dt$$

Choose any anti derivative $P(t)$ of $p(t)$ then

$$\ln|y| = -P(t) + c$$

$$|y| = e^{-P(t)}\, e^{c}$$

$$y = \pm e^{c} \cdot e^{-P(t)}$$

$$\boxed{y = c\,e^{-P(t)}}$$

$c \to$ Any number (we brought back the solution $y = 0$ corresponding to $c = 0$)

Note: If you choose a different antiderivative, it will have the form $P(t) + d$ for some constant d, and then the new $e^{-P(t)}$ is just a constant e^{-d} times the old one. So the set of all scalar multiples of the function $e^{-P(t)}$ is the same as before.

Conclusion:

General solution to I-order homogeneous linear ODE

Let $p(t)$ be a continuous function on an open interval I. This ensures that $p(t)$ has an antiderivative $P(t)$.

General solution to $\dot{y} + p(t)y = 0$ is

$$y = c\,e^{-P(t)}$$

$c \rightarrow$ Any real number. The parameter c can be determined from an initial condition.

Suppose that x_h is a solution to the differential equ. $\dot{x} + p(t)x = 0$ and that $x_h(a) < 0$ at some time $t = a$. What does this tell us about the solution function x_h. Ans: The function x_h has a (-)ve value at every t.

solu:

The solution has the form $x_h = c e^{-P(t)}$ for some constant c and some antiderivative $P(t)$ of $p(t)$. The value of $e^{-P(t)}$ is +ve, but $x_h(a) < 0$. So c must be (-)ve

Then, $x_h = c e^{-P(t)} < 0$ for all t.

---

minestrone soup continuation.

Initially, $y = 100°c$ $(t = 0)$

$x = 0°c$ (refrigerated room - large)

Find solution.

$$\dot{y} = k(x - y)$$

$$\frac{dy}{dt} = k(x - y)$$

$$\frac{dy}{dt} = -ky \qquad (x = 0)$$

↳ Always.

$$\frac{dy}{y} = -k\,dt$$

$$|y| = e^{-kt} \cdot e^{c}$$

$$y = c\,e^{-kt}$$

$y = ce^{-Kt}$ $\qquad y = 100^{\circ}c\,(t=0)$

$100 = c\,e^{0}$ $\qquad y = 100\,e^{-Kt}$ → Actual solution

$\boxed{C = 100}$

Solving Inhomogeneous equations: variation of parameters:

variation of parameters is for solving (method) inhomogeneous linear ODE s.

$$\dot{y} + P(t)\,y = q(t) \rightarrow \text{standard form}$$

eg: 6.1 : Solve $t\dot{y} + 2y = t^5$ on the interval $(0,\infty)$

solu:

$t\dot{y} + 2y = t^5$ on the interval $(0,\infty)$

solu: The associated homogeneous equ is $t\dot{y} + 2y = 0$

$$\dot{y} + \frac{2}{t}y = 0$$

$$\frac{dy}{dt} = -\frac{2}{t}y$$

$$\frac{dy}{y} = -\frac{2}{t}dt$$

$\ln|y| = -2\ln t + c \qquad (t>0)$

$y = e^{-2\ln t}\cdot e^{c}$ $\qquad$ Here we've recovered the $y=0$ solution by allowing $C=0$.

$y = c\times(t^{-2})$

$\boxed{y = ct^{-2}}$

Choose one non-zero solution, solution say $\boxed{y_h = t^{-2}}$

Step:2: Sub $y = u(t)\cdot t^{-2}$ in to the inhomogeneous equ

$$t\dot{y} + 2y = t^5$$

$$t\frac{d}{dt}(u(t)\cdot t^{-2}) + 2(u(t)\cdot t^{-2}) = t^5$$

$$t\left(\dot{u}t^{-2} + u(t)\times -2t^{-3}\right) + 2u(t)t^{-2} = t^5$$

$$\dot{u}t^{-1} - 2ut^{-2} + 2ut^{-2} = t^5$$

$$\dot{u}t^{-1} = t^5 \Rightarrow \frac{du}{dt} = t^6$$

$$\dot{u} = t^6$$

$$\int \frac{du}{dt} = \int t^6$$

$$\int du = \int t^6 . dt$$

$$\boxed{u = \frac{t^7}{7} + c}$$

The general solution to the inhomogeneous equ is

$$y = ut^{-2} = \left(\frac{t^7}{7} + c\right) t^{-2}$$

$$= \frac{t^5}{7} + ct^{-2}$$

<u>Variation of parameters general solution:</u>

1. Find a non zero solution, say y_h, of the associated homogeneous ODE

$$\dot{y}_h + P(t) y_h = 0$$

2. Substitute $y = u y_h$ in to the inhomogeneous equ. $\dot{y} + P(t) y = q(t)$ to find an equ for the unknown function $u = u(t)$.

$$\frac{d}{dt}(u y_h) + P u y_h = q$$

$$\dot{u} y_h + u \dot{y}_h = P u y_h = q$$

$$\dot{u} y_h + u \underbrace{(\dot{y}_h + P y_h)}_{=0} = q$$

$$\dot{u} y_h = q$$

Note that the term in parentheses is zero because $y_h \rightarrow$ solution to the homogeneous equ.

3) Solve $\dot{u} = \frac{q}{y_h}$ for $u(t)$ by integration

4) Once the general $u(t)$ is found, don't forget to multiply it by the homogeneous equ. maybe we can get solutions to the inhomogeneous equ by allowing the parameter c to vary (i.e) if we replace it by a non constant function $u(t)$

$\dot{y} + ty = t$; $y(0) = 3$ (Linear ODE — IVP).

solu: $\dot{y}+ty=t$ $\rightarrow$ by separation of variables

$\dot{y}+ty=t$

$dy = t(1-y)\,dt$

$\frac{dy}{1-y} = t\,dt$

$-\ln|1-y| = \frac{t^2}{2}+c$

$1-y = \pm c e^{-t^2/2}$

$\boxed{y = 1+ce^{-t^2/2}}$

$c = \pm e^{c}$

<u>variation of parameters method:</u>

I $\rightarrow$ ① homogeneous solution.

$\dot{y}+ty=0$

$\frac{\dot{y}}{y} = -t$

$\frac{dy}{y} = -t\,dt$

$\ln|y| = -\frac{t^2}{2}+c$

$|y| = e^{-t^2/2}\cdot e^{c}$

$\boxed{y = c\,e^{-t^2/2}}$

one nonzero homogeneous solution $y_h = e^{-t^2/2}$

$\boxed{c=1}$ $\downarrow$ For simplicity

sub $y = u(t)\cdot e^{-t^2/2}$

$\dot{y}+ty = \frac{d}{dt}(ue^{-t^2/2}) + t(ue^{-t^2/2}) = t$

$\dot{u}e^{-\frac{t^2}{2}} + u\times -\frac{2t}{2}(e^{-t^2/2}) + tue^{-t^2/2} = t$

$\dot{u}e^{-\frac{t^2}{2}} - \cancel{tu}\,e^{-t^2/2} + tue^{-t^2/2} = t$

$\dot{u}e^{-t^2/2} = t$

$\dot{u} = t\cdot e^{t^2/2}$

$\frac{du}{dt} = te^{t^2/2}$

$\int du = \int te^{t^2/2}\cdot dt$

$\int du = \int e^{v}\cdot dt$

$v = \frac{t^2}{2}$

$\frac{dv}{dt} = \frac{2t}{2}$

$dv = t\cdot dt$

$u = e^{v}+c$

$\boxed{u = e^{t^2/2}+c}$

$y = ue^{-t^2/2}$

$= (e^{t^2/2}+c)\,e^{-t^2/2}$

$= 1+ce^{-t^2/2}$

use IVP

$y(0)=3$

$3 = 1+c$

$\boxed{c=2}$

$\therefore$ $\boxed{y = 1+2e^{-t^2/2}}$

---

minestrone soup example

$\dot{y}+ky = kx.$

External temperature T is constant $y(0) = 80$

Soln:
$$\dot{y} + Ky = Kx$$

Homogeneous solution:
$$y = Ce^{-Kt}$$
$$\boxed{y_h = e^{-Kt}}$$

Sub $y = ue^{-Kt}$
$$\dot{y} + Ky = Kx$$
$$\dot{u}e^{-Kt} + u(-K)e^{-Kt} + Ky = Kx$$
$$\dot{u}e^{-Kt} - Kue^{-Kt} + Kue^{-Kt} = Kx$$
$$\dot{u}e^{-Kt} = Kx$$

$\boxed{x = T}$ Constant

$$\dot{u}e^{-Kt} = KT$$
$$\dot{u} = e^{Kt} \cdot KT$$
$$du = e^{Kt} \cdot KT\, dt$$
$$u = KT\left(\frac{e^{Kt}}{K}\right) + C$$
$$u = Te^{Kt} + C$$

$$y = ue^{-Kt}$$
$$= (Te^{Kt} \times e^{-Kt}) + Ce^{-Kt}$$
$$= T + Ce^{-Kt}$$

$\therefore$ $y(0) = 80$
$$80 = T + C$$
$$C = (80 - T)$$

$\therefore$ $y = T + (80 - T)e^{-Kt}$

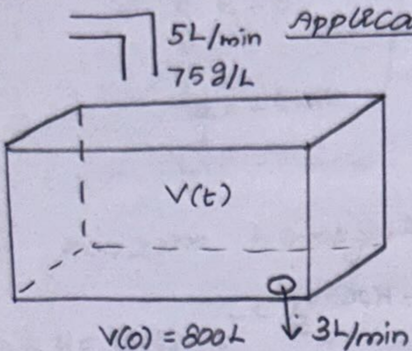
Note:
A particular solution to the inhomogeneous ODE is $y_p = T$. This is the solution in which the soup temperature is already in equilibrium with the external temperature.

The general solution to the homogeneous ODE is $y_h = Ce^{-Kt}$.

The general solution to the homogeneous ODE

As $y = T + Ce^{-kt}$, as $t \to \infty$, the soup temperature approaches T no matter what the initial temp.
This makes sense.

why $y = u(t) \cdot y_h$ $\therefore u(t) \to$ correction factor.



Applications of variation of parameters

$x(t)$ = Total grams salt in tank

$V(t)$ = Total litres fluid in tank.

V(0) = 800L

Concentration = $\dfrac{x(t)}{V(t)}$

$\boxed{\dfrac{dx}{dv} \cdot \dfrac{dv}{dt}}$

$\dfrac{dx}{dt}$ = (Rate in) − (Rate out)

$V(t) = 800 + (5-3)t$

$$= \left(5\frac{L}{min}\right)\left(75\frac{g}{L}\right) - \left(\frac{x(t)}{V(t)} \times 3\frac{L}{min}\right)$$

$$= 375\frac{g}{min} - \frac{3x(t)}{800+2t}$$

$(x(0) = 0)$
↳ pure 800L water.

$$\frac{dx}{dt} = 375 - \frac{3x}{800+2t}$$

(IN) (OUT)

Long-time prediction:

Concentration = $\dfrac{x(t)}{V(t)} = \dfrac{x(t)}{800+2t}$

Salt in
our fish lives in the environment 34 g/L.

over a long time,

$$\frac{x(t)}{800+2t} \approx 75\, g/L \quad \text{(For large t)}$$

$$x(t) \approx 75(800) + 150t$$

$75 \neq 34\, g/L$
(Not close)

we need to find the time when this (34 g/L) happens.

$$\frac{dx}{dt} = 375 - \frac{3x}{800+2t}$$

$$\dot{x} + \frac{3}{800+2t} x = 375 \qquad \left[\dot{x} + P(t)\, x = q(t) \right)$$

$P(t)$ $\qquad$ $q(t)$ $\qquad$ (Inhomogeneous)

Variation of parameters:

① $$\dot{x} = -\frac{3x}{800+2t}$$

$$\frac{dx}{x} = -\frac{3\,dt}{800+2t}$$

$$\ln|x| = -3\int \frac{dt}{800+2t}$$

$$\ln|x| = -\frac{3}{2}\int \frac{dt}{400+t}$$

(we are not considering when $x<0$) So using parenthesis

$$c_2 + \ln|x| = -\frac{3}{2}\ln(400+t) + c_1$$

$$\ln x = -\frac{3}{2}\ln(400+t) + c$$

$$x = e^{-\frac{3}{2}\ln(400+t)}\, e^{c}$$

$$x = (400+t)^{-3/2} \cdot e^{c}$$

$$x = e^{c}(400+t)^{-3/2}$$

$$\boxed{e^{c} = 1}$$

Choosing 1 non zero solution

$$x_h = (400+t)^{-3/2}$$

$$x = u\, x_h = u\,(400+t)^{-3/2} \rightarrow \text{Substituting}$$

$$\dot{x} + \frac{3x}{800+2t} = 375$$

$$\frac{d}{dt}\left(u\,(400+t)^{-3/2}\right) + \frac{3}{(800+2t)}(400+t)^{-3/2} = 375$$

$$\dot{u}\,(400+t)^{-3/2} + u \times \left(\frac{-3}{2}\right)\left(u\,(400+t)^{-3/2-1}\right)(1) + \frac{3}{(800+2t)}u(400+t)^{-3/2} = 375$$

$$\dot{u}\,(400+t)^{-3/2} - \frac{3}{2}u\,(400+t)^{-5/2} + \frac{3}{2}u(400+t)^{-5/2} = 375$$

$$\dot{u}\,(400+t)^{-3/2} = 375$$

$$\frac{du}{dt} = 375\,(400+t)^{3/2}$$

$$\int du = 375 \int (400+t)^{3/2} . dt$$

$$u = 375 \left[\frac{(400+t)^{5/2}}{\frac{5}{2}} \cdot \frac{1}{1} \right]$$

$$5\,\overline{)\,375}\quad 75;\quad 35;\quad 25$$

$$u = \frac{375}{5} \times 2\,(400+t)^{5/2} + C_2$$

$$\boxed{u = 150\,(400+t)^{5/2} + C_2}$$

$$x = u(t) . x_h$$

$$= (150\,(400+t)^{5/2}) . (400+t)^{-3/2} + C_2\,(400+t)^{-3/2}$$

$$= 150\,(400+t) + \frac{C_2}{(400+t)^{3/2}}$$

$$= 60000 + 150t + \frac{c}{(400+t)^{3/2}}$$

$$x(0) = 0$$

$$0 = 60000 + \frac{c}{(400)^{3/2}}$$

$$0 = 60000 + \frac{c}{8000}$$

$$x(t) = 60000 + 150t - \frac{4.8 \times 10^8}{(400+t)^{3/2}}$$

$$-6000 \times 8000 = c$$

$$\boxed{-4.8 \times 10^8 = c}$$

$\hookrightarrow$ Required

As $t \to \infty$ $\quad x(t) \approx 60000 + 150t \quad$ (when $x = 75\,g/L$).

we need 34 g/L concentration:

The concentration is always ↑.

$$\therefore C(t) = \frac{x(t)}{V(t)}$$

$$\dot{c} = \frac{V\dot{x} - x\dot{V}}{V^2} \qquad \dot{x} = 375 - \frac{3x}{V} \quad \left(\frac{dV}{dt} = 2\right)$$

$$\dot{c} = \frac{V\left(375 - \frac{3x}{V}\right) - 2x}{V^2} = \frac{375V - 5x}{V^2}$$

∴ Denominator is square.

Considering Numerator alone:

$$V = 800 + 2t$$

$$= 375V - 5x$$

$$= 375(800 + 2t) - 5\left(60000 + 150t - \frac{4.8 \times 10^8}{(400+t)^{3/2}}\right)$$

$$= 5\left(\frac{4.8 \times 10^8}{(400+t)^{3/2}}\right)$$

∴ +ve. ∴ Concentration is always increasing.

Is the tank too small:

Size of our fish tank is 1800L. will it overflow before 34 g/L conc achievement?

∴ concentration is continuous function of time.

$$V(t) = 800 + 2t$$

$$1800 = 800 + 2t$$

$$t = 500 \text{ minutes.}$$

$$x(500) = 60000 + 150(500) - \frac{(4.8 \times 10^8)}{(400+500)^{3/2}}$$

$$= 135000 - 4.8 \times 10^8 / 2.7 \times 10^4.$$

$= 117{,}222.2$

$Q(500) =$ $\quad C = \dfrac{x(t)}{V(t)} = \dfrac{117222}{1800}$

$= 65.123$

$\therefore$ Tank can hold up to 65.123 level of conc.

(The tank is indeed large enough to prepare the salt solution)

Solving inhomogeneous equ by integrating factor:

This method is exactly the same as variation of parameters. It is algebraically equivalent, but comes at the approach from a slighty different angle. we are adding it here because some texts & courses will refer to the integrating factor).

$$\dot{y} + P(t)y = q(t)$$

To find an integrating factor:

1. Find an antiderivative $P(t)$ of $p(t)$. The integrating factor is $e^{P(t)}$.

2. Multiply both sides of the ODE by the Integrating factor $e^{P(t)}$.

$$e^{P(t)}\dot{y} + e^{P(t)} \underbrace{p(t)}_{\text{Small } p} y = q(t)e^{P(t)}$$

we do this multiplication. Because it is now possible to express the left side as the derivative of something

$$e^{P(t)}\dot{y}(t) + e^{Pt} \cdot p(t)\, y(t) = \frac{d}{dt}\left(e^{P(t)} y(t)\right)$$

$$x'y + y'x = \frac{d}{dt}(xy)$$

$$\therefore \quad \frac{d}{dt}\left(e^{P}y\right) = qe^{P}$$

Integrating,

$$e^{P}y = \int qe^{P}\, dt$$

$e^{P} y = e^{-P} \int q e^{P} \, dt.$

The indefinite integral $\int q(t) e^{P(t)} \, dt$ represents a family of solutions because there is a constant of integration. If we fix one antiderivative, say $R(t)$, then the others are

$$R(t) + C$$

$$\therefore \quad y = e^{-P} (R(t) + C)$$

$$\boxed{y = e^{-P} . R(t) + Ce^{-P}}$$

Remarks 8.1: The integrating factor $e^{\int P(t)}$ is the reciprocal of a solution to the homogeneous equation.

Remarks 8.2: This process is equivalent to variation of parameters. The connection b/w the methods is that the unknown in variation by parameters is

$$u = \frac{y}{y_h} = ye^{P}$$

Integrating factor $u(x)$:

multiplying with gn diff equ

$$u\dot{y} + uPy = uq$$

we can consider,

$$(uy)' = u\dot{y} + Puy$$

only when:

$$\dot{u} = pu$$

Solving,

$$\frac{du}{u} = p \, dx$$

$$\int \frac{du}{u} = \int P(x) \, dx$$

$$\ln u = \int P dx$$

$$u = e^{\int P dx}$$

we can solve a equ (earlier)

$$\boxed{\begin{array}{l} \dot{u} = e^{kt} . kt \\ \dot{u} = te^{t^2/2} \end{array}}$$

some solved eg.

$\therefore$ only one u is possible (only 1 integration factor).

method: $y' + Py = Q$

1) (Notice the std form)
2) Calculate $e^{\int P dx}$: integrating factor
3) Multiply both sides by $e^{\int P dx}$
4) Integrate.

$$xy' - y = x^3$$

$$y' - \frac{y}{x} = x^2$$

multiply by u:

$$e^{\int P dx} = e^{-\int \frac{1}{x} dx}$$

$$= e^{-\ln x}$$

$P = -\frac{1}{x}$, $Q = x^2$

$$= e^{\ln x^{-1}}$$

$$= \frac{1}{x}$$

$$xy' - y = x^3$$

multiply all by u

$$uy' - u\frac{y}{x} = x^2 \cdot u$$

$$(uy)' = x^2 \cdot u$$

only when

$$\dot{u} = -\frac{u}{x}$$

$$\frac{du}{u} = -\frac{dx}{x}$$

$$\ln u = -\ln x$$

$$\ln u = \ln x^{-1}$$

$$u = x^{-1}$$

$$\boxed{u = \frac{1}{x}}$$

$\therefore$ Multiplying.

$$uy' - \frac{uy}{x} = ux^2$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}x^2$$

$$\frac{1}{x}y' - \frac{1}{x^2}y = x \qquad (uv)' = u'v + v'u$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\boxed{\frac{1}{x}\left(y' - \frac{1}{x}y\right) = x \qquad y' - \frac{1}{x}y = x^2}$$

$$\left(\frac{1}{x}y\right)' = x$$

Integrating,

$$\frac{y}{x} = \frac{x^2}{2} + c$$

$$\boxed{y = \frac{x^3}{2} + cx}$$

Linear Combinations:

A linear combination of a list of functions is any function that can be built from them by scalar multiplication & addition.

linear Combinations of $f(t)$: the functions $c\,f(t)$,

$c \to$ Any number.

linear Combinations of $f_1(t)$ & $f_2(t)$: the functions of the form $c_1 f_1(t) + c_2 f_2(t)$, where

c_1, c_2 – Any numbers.

Examples:

* $2\cos t + 3\sin t$ is a linear combination of the functions $\cos t$ and $\sin t$.
* $9t^5 + 3$ is a linear Combinations.

$3\cos^2 t - 4 = 3\cos^2 t + (-4)\cdot 1 \rightarrow$ Linear combination

$\sin^2 t = (-1)\cos^2 t + 1\cdot 1 \rightarrow$ Linear combination

$\cos 2t = 2\cos^2 t + (-1)\cdot 1 \rightarrow$ Linear combination

$5 = 0\cos^2 t + 5\cdot 1 \rightarrow$ Linear combination

$0 = 0\cos^2 t + 0\cdot 1 \rightarrow$ Linear combination

$\sin 2t = 2\sin t \cos t \rightarrow$ Not a linear combination.

Every linear combination of $\cos^2 t$ and 1 has the form

$$C_1\cos^2 t + C_2$$

for some numbers C_1 and C_2. All such functions are even functions.

But $\sin 2t \rightarrow$ odd function.

Warning: This trick might not work in other situations.

A linear combination of function f_1 and $f_2(t)$ is anything of the form

$$C_1 f_1(t) + C_2 f_2(t)$$

$C_1, C_2 \rightarrow$ constants

Say: $f_1(t) = e^t$

$f_2(t) = t.$

$\therefore C_1 e^t + C_2 t$ is a linear combination of f_1 & f_2.

$t\dot{y} + 2y = 0$

$\dot{y} + \frac{2}{t}y = 0$

$\frac{dy}{dt} = -\frac{2}{t}y$

$\frac{dy}{y} = -\frac{2}{t}dt$

$\ln y = -2\ln t$

$\boxed{y = t^{-2} + c}$

$t\dot{y} + 2y = t^5$

$y_h = \frac{1}{t^2}$

$\boxed{y = u y_h}$

$t\left(\frac{d}{dt}\left(u \times \frac{1}{t^2}\right)\right) + 2\left(u\cdot\frac{1}{t^2}\right) = t^5$

$t\left(\dot{u}\cdot\frac{1}{t^2} + u\cdot\left(-2t^{-2-1}\right)\right) +$

$2\left(\frac{u}{t^2}\right) = t^5$

$t\left(\dot{u}\,\frac{1}{t^2}\right) - \frac{2ut}{t^3} + \frac{2u}{t^2} = t^5$

$\dot{u}\cdot\frac{1}{t} = t^5$

$\dot{u} = t^6$

$\int du = \int t^6\cdot dt$

$u = \frac{t^7}{7} + c$

$$y = \left(\frac{t^7}{7} + c\right)\left(\frac{1}{t^2}\right)$$

$$\boxed{y = \frac{t^5}{7} + ct^{-2}}$$

General solution to $t\dot{y} + 2y = 0$: ct^{-2}

One solution to $t\dot{y} + 2y = t^5$: $\frac{t^5}{7}$

General solution to $t\dot{y} + 2y = t^5$: $\frac{t^5}{7} + ct^{-2}$

one solution to $t\dot{y} + 2y = 1$: $\frac{1}{2}$

General solution to $t\dot{y} + 2y = 1$: $\frac{1}{2} + ct^{-2}$

For each one solution above, scalar multiply to get:

one solution to $t\dot{y} + 2y = 9t^5$: $9\left(\frac{t^5}{7}\right)$

one solution to $t\dot{y} + 2y = 3$: $\frac{3}{2}$

one solution to $t\dot{y} + 2y = 9t^5 + 3$: $\frac{9t^5}{7} + \frac{3}{2}$.

The general principle, which works for all linear ODEs,

Superposition principle:

1) Multiplying a solution to $P_n(t)y^n + \dots + P_0(t)y = q(t)$ by a number a gives a solution to $P_n(t)y^n + \dots + P_0(t)y = aq(t)$

2) Adding a solution of $P_n(t)y^n + \dots + P_0(t)y = q_1(t)$ to a solution of $P_n(t)y^n + \dots + P_0(t)y = q_2(t)$ gives a solution of $P_n(t)y^n + \dots + P_0(t)y = q_1(t) + q_2(t)$

Together these two prop. show that linear combinations of y's solve the ODE with correspond linear combination of q's.

Consequence of superposition for I order linear ODE:

To understand the general solution $y(t)$ to an inhomogeneous linear ODE

Inhomogeneous: $\dot{y} + P(t)\,y = v(t)$

Do the following:

1. Find the general solution to the associated homogeneous eqn.

homogeneous: $\dot{y} + P(t)\,y = 0$;

write down the general solution y_h.

2. Find (in some way) any one particular solution y_p to the inhomogeneous ODE.

3. Add y_p to the general solution of the homogeneous ODE to get the general solution to the inhomogeneous ODE.

Summary: $y = y_p + y_h$

y_p → particular solution

y_h → general homogeneous solution.

y (General inhomogeneous Solution)

Note: we are using y_h for the gn. homogeneous solu rather than a specific homo solution.

Why? (This work)

superposition says that adding y_p to a homoge. solution gives a solution to the DE with right hand side $v(t) + 0 = v(t)$. All solutions to the DE with right hand side $v(t)$ arise this way.

$\therefore$ subtracting y_p from any solution gives a solution to the DE with right hand side 0.

Use the following general homogeneous solution & particular solutions to the gn differential eqn to find the solution to the DE below.

Diff eqn	General homoge solution
$\dot{x} + 2x = 0$	$x = ce^{-2t}$

Diff equ

$\dot{x} + 2x = 1$

$\dot{x} + 2x = t$

$\dot{x} + 2x = e^{-2t}$

partial solution

$x_p = 1/2$

$x_p = t/2 - 1/4$

$x_p = te^{-2t}$

Find general solution to the differential equation.

$$\dot{x} + 2x = 5 + 6t - 7e^{-2t}.$$

$$\text{General solution} = ce^{-2t} + \frac{5}{2} + 6\left(\frac{t}{2} - \frac{1}{4}\right) - 7\left(te^{-2t}\right)$$

$$= ce^{-2t} + \frac{5}{2} + 3t - \frac{3}{2} - 7te^{-2t}$$

$$= ce^{-2t} + 1 + 3t - 7te^{-2t}$$

$$= e^{-2t}(c - 7t) + 1 + 3t.$$

__Superposition fails for nonlinear ODES:__

The superposition principle doesn't hold for non linear differential equations.

Suppose $x_1(t)$ and $x_2(t) \rightarrow$ 2 solutions that satisfy the nonlinear differential equation.

$$\dot{x} + P(t)x^2 = 0$$

what rh side $q(t)$ is needed for the function $x_1(t) + x_2(t)$ to be a solution to the differential equation $\dot{x} + P(t)x^2 = q(t)$

__Let's put our solution:__

$$\frac{d}{dt}(x_1 + x_2) + P(t)(x_1 + x_2)^2 = \dot{x}_1 + \dot{x}_2 + P(t)(x_1^2 + x_2^2 + 2x_1x_2)$$

$$= (\dot{x}_1 + Px_1^2) + (\dot{x}_2 + Px_2^2) + 2P(t)x_1(t)x_2(t)$$

__Considering:__

$$\dot{x} + P(t)x^2 = 0$$

$$= 2P(t)x_1(t)x_2(t)$$

$$\therefore \dot{x}_1 + P(t)x_1^2 = 0$$

$$\dot{x}_2 + P(t)x_2^2 = 0$$

If the differential equation had been linear, the answer would have been $r(t) = 0+0$.

$$\frac{d}{dt}(x_1+x_2) + P(x_1+x_2) = (\dot{x}_1 + Px_1) + (\dot{x}_2 + Px_2)$$

Linear: $\dot{x} + Px = 0$

$x_1 + x_2 \rightarrow$ Solutions

$= 0+0 = 0$.

(which is different from non-linear case).

Linear Equations

a) which of the following ODE's are linear?

i) $\dot{y} = ky \longrightarrow$ Linear $\quad (\dot{y} - ky = 0)$

ii) $\dot{y} + y^2 t = y \longrightarrow$ Non linear.

iii) $y' + \cos(x) y = x^3 \longrightarrow$ Linear

iv) $\frac{\dot{y}}{y} = t^2 \longrightarrow$ Linear

v) $x^2 y y' + 4x = x^3 \longrightarrow$ Non linear

b) show $\dot{y} + y^2 = r(t)$ doesn't satisfy superposition.

Solu:

$$\frac{dy}{dx} + P(x) y = r(x) \longrightarrow \text{I order linear ODE.}$$

ii) $\dot{y} + y^2 t = y \rightarrow$ (Also known as Bernoulli equation)

iii) $\dot{y} + \cos(x) \cdot y = x^3$ $\quad P(x) = \cos x$

$r(x) = x^3$

iv) $\frac{\dot{y}}{y} = t^2$

$P(t) = -t^2$

$\dot{y} - t^2 y = 0 \longrightarrow$ Linear.

$r(t) = 0$

v) $x^2 y y' + 4x = x^3$

$$y' + \frac{4}{xy} = \frac{x}{y} \longrightarrow \text{Non linear}$$

Through proper substitution like $u = \frac{1}{y}$ in (ii) and $u = y^2$ in (v) we can make it linear

---

b) $\dot{y} + y^2 = r(t) \longrightarrow$ Non linear.

Consider equ, $\dot{y}_1 + y_1^2 = 1$

$\dot{y}_2 + y_2^2 = t$.

Superposition:

$$y = y_1 + y_2$$

$$\dot{y} + y^2 = 1+t.$$

$$\frac{d}{dt}(y_1+y_2) + (y_1+y_2)^2 = 1+t.$$

$$(\dot{y}_1 + \dot{y}_2) + y_1^2 + y_2^2 + 2y_1 y_2 = 1+t$$

$$(\dot{y}_1 + y_1^2) + (\dot{y}_2 + y_2^2) + 2y_1 y_2 = 1+t$$

$$(1+t) + 2y_1 y_2 \neq 1+t.$$

$\boxed{2y_1y_2} \rightarrow$ is the extra term.

Newton's law of cooling. and CSI

CSI - crime scene Investigation.

Newton's law of cooling:

$T_e \rightarrow$ ambient temperature (சூழ்வெப்பநிலை)

$$\frac{dT}{dt} = K(T_e - T)$$

$T(t)$ - Temperature as a function of time.

where $\boxed{T_0 = T(0)}$

solving,

$$\frac{d}{dt}(-T) = -1$$

$$\frac{dT}{dt} = KT_e - KT$$

$$\frac{dT}{(T_e - T)} = K\,dt$$

$$-\ln(T_e - T) = Kt + C$$

$$T_e - T = e^{-Kt} \cdot e^{-C}$$

$$T_e - T = Ce^{-Kt}$$

$$T = T_e - Ce^{-Kt}$$

$\boxed{T_0 = T(0)}$

IVP:

$$T_0 = T_e - Ce^0$$

$$\boxed{C = T_e - T_0}$$

$$T = T_e - (T_e - T_0)e^{-Kt}$$

$$= T_e + (T_0 - T_e)e^{-Kt}$$

This eqn shows a good application in real world temp changes over time. It's used by law enforcement in criminal cases to determine the time of death in a murder case.

we assume that up until a person is murdered the body temperature is constant at 37°c. After the murder, the body cools, via Newton Cooling, to the ambient temp when investigators arrive at the scene to find a corpse(unknown), one of the first things done is to measure of the core temperature of the body & the ambient temperature. This gives two immediate places of info.

1. $T(t_1)$ where $t_1 \rightarrow$ unknown time since the murder.

2. T_e - Ambient temperature.

Assume: $\boxed{T_0 = 37^\circ c}$

solu:

Is the above details enough? For measurement.

$$T(t_1) = T_e + (T_0 - T_e)\, e^{-Kt_1}$$

still t_1 and K are unknown.

K- constant of conductivity.

Assume: The body is not disturbed, that for a gn case the value of K won't change over time.

If so, we can find K by taking a second measurement at t_2.

we can then write a system of eqn using our data,

$$T(t_1) = T_e + (T_0 - T_e)\, e^{-Kt_1}$$

$$T(t_2) = T_e + (T_0 - T_e)\, e^{-Kt_2}$$

we can solve the above eqn, ∴ we know $(t_2 - t_1)$.

You arrive on the scene of a crime & find a body. The victim was found in a room that is at a constant temperature of 25°c all day. You examine body at 7:00 pm right after arriving and find the core temp is 34°c. At 8:00 pm, you take a second measurement. & find that the core temp has dropped to 31°c. Death time (closest)

solu:

1) $T_e = 25^\circ c$

$T_0 = 37^\circ c$, $T_1 = 34^\circ c$, $T_2 = 31^\circ c$.

$T(t_1) = T_e + (T_0 - T_e)\, e^{-Kt_1}$

$T(t_2) = T_e + (T_0 - T_e)\, e^{-Kt_2}$

$34 = 25 + (37-25)\, e^{-Kt_1}$

$31 = 25 + (37-25)\, e^{-Kt_2}$

$t_2 = t_1 + 1$

$\therefore$ ① $\Rightarrow$ $9 = 12 e^{-Kt_1}$

② $\Rightarrow$ $6 = 12 e^{-K(1+t_1)}$

$\frac{3}{1} = \frac{4}{2} \frac{e^{-Kt_1}}{e^{-K(1+t_1)}}$

$3 = 2 e^{-Kt_1 + Kt_1 + K}$

$3 = 2e^{K}$

$\frac{3}{2} = e^{K}$

$K = \ln\left(\frac{3}{2}\right)$

$\boxed{K = 0.405465108}$

when $T = 37^\circ C$

$37 = 25 + 12 e^{-0.405465108t}$

$12 = 12 e^{-0.405465108t}$

$1 = e^{-0.405465108t}$

$e^{0.405465108t} = 1$

$0.405465108t = \ln 1$

$\boxed{t = 0}$

when $T = 34^\circ C$

$34 = 25 + 12 e^{-0.405465108 t_1}$

$9 = 12 e^{-0.405465108t_1}$

$3 e^{+0.405465108 t_1} = 4$

$e^{0.405465108t_1} = \frac{4}{3}$

$0.405465108t_1 = \ln\left(\frac{4}{3}\right)$

$0.405465108t_1 = 0.287682072$

$\boxed{t_1 = 0.709513} = 7.00\text{ AM}$ $\left(\begin{array}{l}0.7095 \\ \times 60 = 42.57\end{array}\right)$

100 = 60min

$T = 31^\circ C$

$6 = 12 e^{-0.405465108 t_2}$

$\frac{1}{2} = e^{-0.405465108t_2}$

$0.405465108 t_2 = \ln 2$

$\boxed{t_2 = 1.709511292} = 8:00\text{ AM}.$

So t_0 will be about , $1.709511292 \times 60 = 102.57065$

$= 1\text{hr } 42\text{min}.$

∴ Deadth happened before 42.57 minutes ago.

Cell division

Yeast growth:

$$\dot{y}(t) = ay.$$ $a \rightarrow$ proportionality constant.

$\boxed{y_h(t) = y(0)\,e^{at}} \rightarrow$ Solution to the differential equation.

Extending complexity:

Suppose yeast cells are living in bread dough. A baker usually doesn't keep one batch of yeast growing indefinitely, but instead will remove portions of dough over time to bake into bread, while maintaining some of the original yeast in the batch so that this so-called "mother" can grow again for the next day's bread.

To model the amount of yeast in a batch using the differential equation method, let us express this process in terms of a rate at which the yeast is being removed from the batch, with units millions of cells / hour. Suppose the rate of removal is $b(t)$. Then we can just modify our diff equ.

$$\frac{dy}{dt} = ay - b(t)$$

Particular solution:

$$\dot{y} - ay = -b$$

Homogeneous equ:

$$\dot{y} - ay = 0$$

$$\dot{y} = ay$$

$$\frac{dy}{y} = a\,dt$$

$$\ln|y| = at + c$$

$$|y| = e^{at} \cdot e^{c}$$

$$y = c\,e^{at}$$

$\boxed{y_h = e^{at}}$ $\boxed{c=1}$

$$y = u(t) \cdot y_h$$

$$\frac{d}{dt}(u e^{at}) - a(u e^{at}) = -b$$

$$\dot{u} e^{at} + a u e^{at} - a u e^{at} = -b$$

$$\dot{u} e^{at} = -b$$

$$\dot{u}\, e^{at} = -b$$

$$\frac{du}{dt} = -be^{-at}$$

$$\int du = \int -be^{-at}\, dt$$

$$u = -b\left[\frac{e^{-at}}{-a}\right] + c$$

$$\boxed{u = \frac{b}{a}e^{-at} + c}$$

$$y = \frac{b}{a}e^{-at} \times e^{at} + ce^{at}$$

$$y = \frac{b}{a} + ce^{at}$$

∴ Solutions can be like

$\frac{b}{a}$, $\frac{b}{a} + e^{at}$, $\frac{b}{a} - 3e^{at}$

Existence & uniqueness of solutions:

Using separation of variables (in the homogeneous case) and variation of parameters (in the heterogeneous case). To nail down a specific solution in this family, we need one initial condt eg: $y(0)$. ∴ A family of solutions are available for that parameter.

Existence & uniqueness theorem for a linear ODE:

Let $p(t)$ and $q(t)$ be continuous functions on an open interval I. Let $a \in I$, and let b be a given number. Then there exists a unique solution defined on the entire interval I to the I order ODE.

$$\dot{y} + p(t)y = q(t)$$

satisfying the initial condition,

$$y(a) = b$$

* Existence means that there is atleast one solution.
* Uniqueness means that there is only one solution.

$x \rightarrow$ Independent
$y \rightarrow$ Dependent } variable.

What's standard linear form?

$$y' - (\tan x)y = 1.$$