$$(mr^2 + br + K)\, e^{rt} = 0 \qquad r = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

where there are two distinct char complex roots,

$$b^2 - 4mK < 0$$

$$r = -\frac{b}{2m} \pm i\sqrt{\frac{4mK}{4m^2} - \frac{b^2}{4m^2}}$$

$$r = -\frac{b}{2m} \pm i\sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

$$\therefore\ x(t) = e^{\left(-\frac{b}{2m}\right)t}\left(c_1 \cos \omega_d t + c_2 \sin \omega_d t\right) \qquad c_1, c_2 \in \mathbb{R}.$$

where,

$$\omega_d = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} \longrightarrow \text{Damped frequency of the system}$$

It is useful to contrast the solutions of the general spring-mass dashpot system with the solutions to the system with no dashpot (damping). $(b=0)$, $(< \sqrt{4mK})$. If the system has no damping,

$$m\ddot{x} + Kx = 0 \qquad m, K > 0.$$

The general real solution reduces to

$$(mr^2 + K) = 0 \qquad = \frac{0 \pm \sqrt{-4mK}}{2m}$$

$$\omega_n = \sqrt{\frac{K}{m}} \rightarrow \text{Natural frequency} \qquad = \pm i\sqrt{\frac{K}{m}}$$

$$x(t) = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t) \qquad \boxed{c_1, c_2 \in \mathbb{R}}$$

What do these general solutions look like? Both of these general solutions involve linear combinations of $\cos\omega t$ and $\sin\omega t$. For some angular frequency ω that is the same for both the sine & cosine functions. What do the graphs of such linear combinations look like?

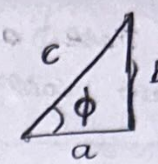
(To make graphing these solutions easy, we will first rewrite the linear combination

$$a\cos\theta + b\sin\theta \text{ in polar form.})$$

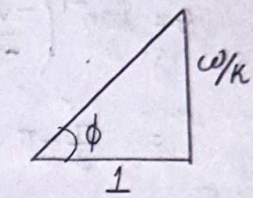
Real part:

$$\frac{1}{1+\left(\frac{\omega}{K}\right)^2}\left(\cos\omega t + \frac{\omega}{K}\sin\omega t\right) \longrightarrow \text{Convert it.}$$

$$\boxed{a\cos\theta + b\sin\theta = C\cos(\theta-\phi)}$$



$$\frac{1}{1+\left(\frac{\omega}{K}\right)^2}\sqrt{\left(1+(\omega/K)\right)^2}\cos\omega t-\phi$$



$$C=\sqrt{1+(\omega^2/K^2)}$$

There are two ways of expressing any sinusoidal signal function.

1) In rectangular form
2) In polar form.

They are related as follows:

Rectangular form Polar form

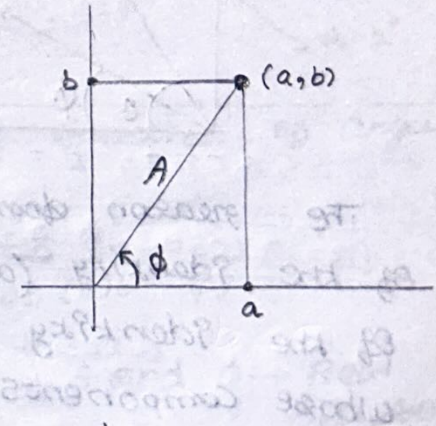
$a\cos\theta + b\sin\theta = A\cos(\theta-\phi)$, $a, b, \phi \in R$, $A \geq 0 \in R$.

$$a\cos\theta + b\sin\theta = \underbrace{A\cos\phi}_{a}\cos\theta + \underbrace{A\sin\phi}_{b}\sin\theta$$

where A and ϕ in terms of a and b are given implicitly by the following diagram.

$A=\sqrt{a^2+b^2}$

ϕ = Angle b/w the +ve horizontal axis & the ray of the point (a,b)



Note: (A,ϕ) → Polar coordinates of the point with rectangular coordinates (a,b)

In practice, the argument of the cosine & sine terms is often a function rather than a constant angle. In this course, we usually have

$\theta = \omega t$ (for some $\omega > 0$)

Exam: 5.1 Convert $-\cos 5t - \sqrt{3}\sin 5t$ to polar form.

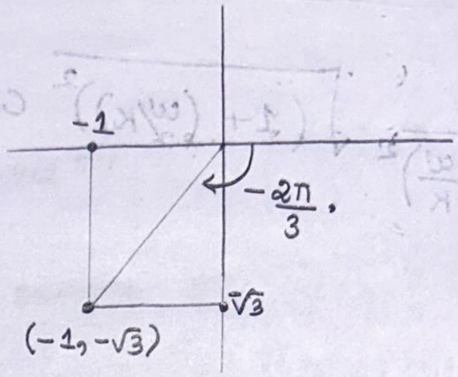
Solu:

$a = -1,\ b = -\sqrt{3},\ \theta(t) = \omega t = 5t,\quad \omega = 5$

Want: A, ϕ. These are polar coordinates of the point with rectangular coordinates (a, b).

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) - \pi$$

$$= \frac{\pi}{3} - \pi = \frac{-2\pi}{3}$$



$$\therefore A = \sqrt{(-1)^2 + (-\sqrt{3})^2}$$

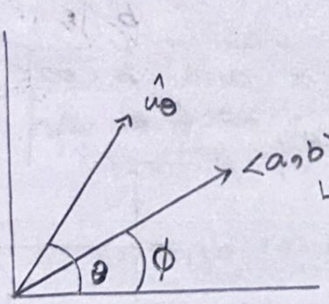
$$\boxed{A = 2}$$

$$\therefore A\cos(\theta - \phi) = 2\cos\left(5t + \frac{2\pi}{3}\right)$$

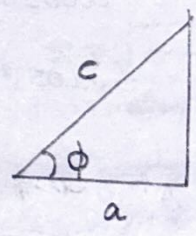
$\therefore \frac{4\pi}{3}$ or $-\frac{2\pi}{3} + 2\pi n$) for any integer n) also works since ϕ is well defined up to addition of 2π.

The equivalence of the three forms:

Proof:



$\hookrightarrow a\hat{i} + b\hat{j}$ (vector)



$a\cos\theta + b\sin\theta = A\cos(\theta - \phi)$

The reason for doing this is the left hand side of the identity $(a\cos\theta + b\sin\theta)$ is the dot product of the identity vector $\langle a, b\rangle$, ~~which~~ ~~the~~ the vector whose components are cosine theta & $\sin\theta$.

It's a unit vector because $\cos^2\theta + \sin^2\theta = 1$.

$$\langle a, b\rangle \cdot \langle \cos\theta, \sin\theta\rangle = (a\hat{i} + b\hat{j})(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

$\downarrow$ unit vector

$$= a\cos\theta + b\sin\theta.$$

By formula

$\langle a,b\rangle \cdot \langle \cos\theta, \sin\theta\rangle = |\langle a,b\rangle| \cdot |\langle \cos\theta, \sin\theta\rangle| \cdot \cos$ (angle b/w $\hat{u}$ and $\langle a,b\rangle$)

$= |\langle a,b\rangle| \times 1 \times \cos(\theta-\phi)$

---

① Highschool proof (Take right side then prove R.H.S)

② 18.02 method

③ 18.03 method.

$= (a-bi)(\cos\theta + i\sin\theta)$ [why minus sign]

$\boxed{-i \times -i = 1.}$

$\boxed{y = ce^{i\theta}}$

To get R.H.S

$\theta = \tan^{-1}\left(-\frac{b}{a}\right)$

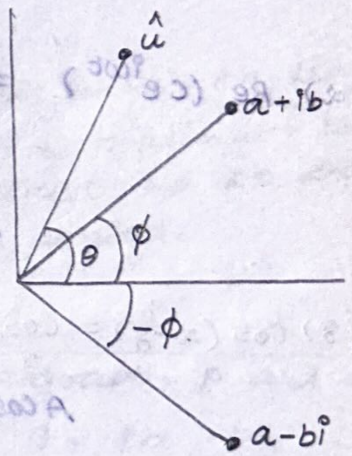
$= \sqrt{a^2+b^2} \cdot e^{-i\phi} \cdot e^{i\theta}$

$= \sqrt{a^2+b^2}\, (e^{i(\theta-\phi)})$

$= \sqrt{a^2+b^2}\, (\cos(\theta-\phi) + i\sin(\theta-\phi))$

$= Re$ (" ")

$= \sqrt{a^2+b^2}\, \cos(\theta-\phi)$



Three ways to write a sinusoid function of angular frequency ω:

1) Amplitude phase form: $A\cos(\omega t - \phi)$

A and $\phi \to$ Real numbers

$A \geq 0.$

2) Complex form: $Re(ce^{i\omega t})$,

$c \to$ complex numbers.

3) Linear Combination : $a\cos\omega t + b\sin\omega t$,

$a, b \to$ Real numbers.

Proposition 6.1

If constants A, ω, ϕ, a, b, c are set so that the equations

$$\bar{c} = Ae^{i\phi} = a+bi \qquad c \in \mathbb{C},\ A \geq 0, a, b, \phi \in \mathbb{R}$$

hold, then

$$Re\,(ce^{i\omega t}) = A\cos(\omega t - \phi) = a\cos\omega t + b\sin\omega t$$

$\omega > 0 \in \mathbb{R}$

warning: Don't forget that it is $\bar{c}$ and not c itself that appears in the key equations. An equivalent form of the key equations.

$$c = Ae^{-i\phi} = a - bi.$$

Constant c

$c \in \mathbb{C}$

(complex number)

proofs:

1) $Re\,(ce^{i\omega t}) = Re\,(Ae^{-i\phi} \cdot e^{i\omega t})$

$= Re\,(Ae^{i(\omega t - \phi)})$

$= A\cos(\omega t - \phi)$

2) $Re\,(ce^{i\omega t}) = Re\,((a-bi)(\cos\omega t + i\sin\omega t))$

$= Re\,(a\cos\omega t + b\sin\omega t + i(\ldots))$

$= a\cos\omega t + b\sin\omega t.$

3) $\cos(x-y) = \cos x\cos y + \sin x \sin y$

$A\cos(\omega t - \phi) = A\cos\omega t\cos\phi + A\sin\omega t\sin\phi$

$= a\cos\omega t + b\sin\omega t$

$a = A\cos\phi,\ b = A\sin\phi,$ A, ϕ polar coordinates of (a,b)

$\therefore \boxed{a = x\cos\theta}, \boxed{b = y\sin\theta}$

4) $a\cos\omega t + b\sin\omega t$

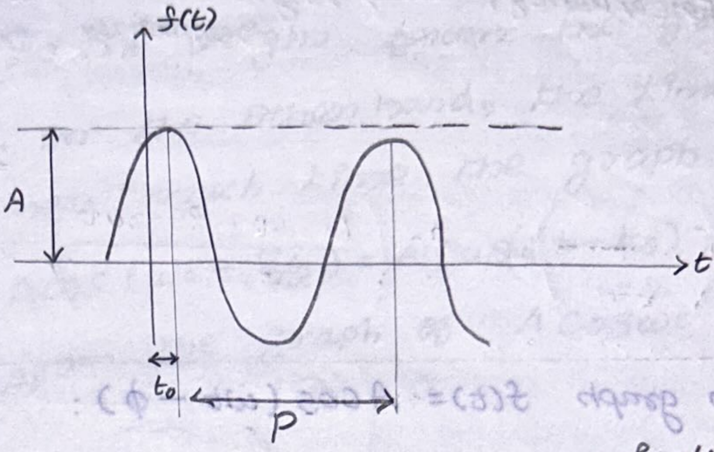
$= \langle a, b\rangle \cdot \langle\cos\omega t, \sin\omega t\rangle$

$= |\langle a,b\rangle|\,|\langle\cos\omega t, \sin\omega t\rangle|\ |\cos(\text{angle b/w})|$

$= A\cos(\omega t - \phi)$

Graphing sinusoidal functions:

In polar form, It is easy to see that the graph of the sinusoidal function $f(t) = A\cos(\omega t - \phi)$ is a rescaled & shifted version of the cosine graph.



The graph of f(t) Can be described geometrically in terms of

i) A → Amplitude, how high the graph rises above the t axis at its maximum

ii) P → (In seconds or seconds per cycle), Its period, the time for one complete osciuation, (width b/w two successive maxima)

iii) t_0 (In seconds), Its time lag (Relative to the cosine curve), a t-value at which a maximum is attained. The time lag is well defined up to the addition of integer multiples of a period.

How are the parameters describing the graph, P and t_0, related to the parameters ω and ϕ, in the function f(t)?

Solu:

★ $P = \frac{2\pi}{\omega}$ (∵ Adding $\frac{2\pi}{\omega}$ to t increases the angle $\omega t - \phi$ by 2π)

★ $t_0 = \frac{\phi}{\omega}$ ∵ t_0 is the t value where

$$\omega t - \phi = 0$$

$$\omega t = \phi$$

$$\boxed{t = \frac{\phi}{\omega}}$$

There is also frequency $\nu = \frac{1}{P}$ (In Hertz = cycles per second), the number of complete oscillations per second.

$$\nu\text{(Frequency)} \xrightarrow{\text{Convert}} \text{Angular frequency} = \frac{2\pi \text{ radians}}{1 \text{ cycle}} \times \nu$$

$$= 2\pi\nu = \frac{2\pi}{P}.$$

$$P = \frac{2\pi}{\omega}$$

Steps to graph $f(t) = A\cos(\omega t - \phi)$:

start with the curve $y(\theta) = \cos\theta$ (Then work from the outside in)

1. Amplify (stretch vertically) by a factor of A, the amplitude.
we now have a graph $y(\theta) = A\cos\theta$.

2) Shift the graph by ϕ (In radians) to the right.
Now $y(\theta) = A\cos(\theta - \phi)$

The angle ϕ is called the phase lag or shift (relative to the cosine curve). The first maximum of the graph $A\cos(\theta - \phi)$ is at $\theta = \phi$, shifted from $\theta = 0$ in the graph $A\cos(\theta)$

3) Compress the result horizontally by dividing by the scale factor ω (In radians/second), the angular frequency. The result is the graph of
$f(t) = A\cos(\omega t - \phi)$

with this horizontal rescaling, the horizontal axis changes from the θ-axis (In radians) to the t-axis (In seconds). The first maximum of the rescaled graph will be at $t_0 = \frac{\phi}{\omega}$ (instead of $\theta = \phi$). and one complete oscillation take $\frac{2\pi}{\omega}$ instead of 2π.

Remark: The phase lag ϕ tells us by what fraction of a cycle that the graph of $f(t)$ is shifted from a cosine graph of the same frequency. For example, the graph of $f(t) = A\cos(\omega t - \frac{\pi}{2})$ is shifted by $\frac{1}{4}$ of a cycle to the right from the graph of $A\cos(\omega t)$. On the other hand, the time lag t_0 tells us by how much time the graph of

$$f(t) = A\cos(\omega t - \pi/2) = A\cos(\omega(t - t_0))$$ is shifted

$A\cos\omega t$.

to the right from the graph of

$$\frac{\phi}{2\pi} = \frac{t_0}{p}.$$

$\therefore\ l = r\theta$

$\theta = \frac{l}{r}$ (unitless)

$$\boxed{\text{unit of } \phi = \text{unitless (or radians)}}$$

$$\cos\omega t - 2\sin\omega t = A\cos(\omega t - \phi)$$

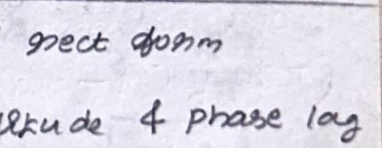
(using matlab graph)

1) set $a = 1, b = -2$

Find A, ϕ $\qquad A = 2.23,\ \phi = -63^\circ$

2) At what (A, ϕ), the two functions allign

$A\cos(\omega t - \phi)$



$a\cos\omega t + b\sin\omega t$

$(a, b) = (1, -2)$ $\quad A = \sqrt{1 + 2^2} = \sqrt{5} \approx 2.24.$

$$\phi = \arctan\left(\frac{-2}{1}\right) \approx -63^\circ.$$

---

circular frequency $\rightarrow$ Angular Frequence.

a) write real part of $\dfrac{e^{i\omega t}}{2 + 3i}$ in polar & rect form

b) what's the circular frequency, Amplitude & phase lag

c) sketch ~~Real~~ (vs) time
part

Solu

$$\frac{e^{i\omega t}}{2 + 3i}$$

$r = \sqrt{4 + 9}$

$= \sqrt{13}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 0.9827$

~~$2+3i$~~ $=$

$$\frac{e^{i\omega t}}{\sqrt{13}\, e^{i\phi}} = \frac{1}{\sqrt{13}} e^{i(\omega t - \phi)}$$

$\phi = \tan^{-1}\left(\frac{3}{2}\right)$

Real part:

$$Re\left(\frac{1}{\sqrt{13}} e^{i(\omega t-\phi)}\right) = \frac{1}{\sqrt{13}} \cos(\omega t - \phi)$$ → polar form.

Rectangular form:

$$Re\left(\frac{e^{i\omega t}}{2+3i}\right) = Re\left(\frac{\cos\omega t + i\sin\omega t}{2+3i}\right)$$

$$= Re\left(\frac{\cos\omega t + i\sin\omega t}{4-9i^2} \times (2-3i)\right)$$

$$= Re\left(\frac{2\cos\omega t - 3i\cos\omega t + 2i\sin\omega t + 3\sin\omega t}{4+9}\right)$$

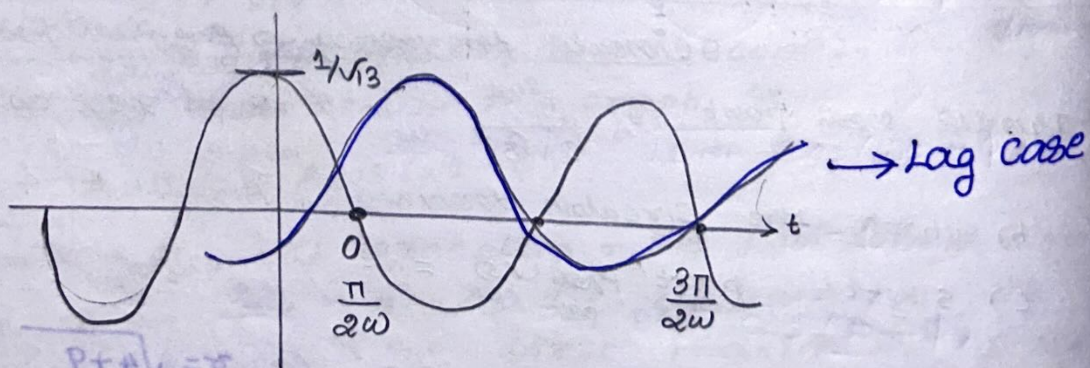
$$= \left(\frac{2\cos\omega t + 3\sin\omega t}{13}\right)$$

b) circular frequency: $\omega = ?$,

From polar form $A = \frac{1}{\sqrt{13}}$ (since $\phi = 0$ at max when $e^0 = 1$ $= \frac{1}{\sqrt{13}}$)

ϕ (phase lag) $= \tan^{-1}\left(\frac{3}{2}\right) = 0.9827.$

Sketch:



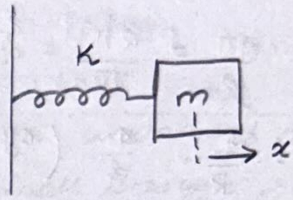
our case: lag

Application to spring – mass system

Solution in Amplitude phase form,

A mass of 1kg is attached to a spring with spring constant of (approximately) π^2 N/m. It is pulled to 1m to the left of the equilibrium & then pushed and released at a velocity of π m/s to the right.

What's the position $x(t)$ of the mass at time t seconds after being released? Answer in amplitude-phase form $x(t) = A\cos(\omega t - \phi)$ by entering the values of A, ω & ϕ.



Solu:

$m\ddot{x} + Kx = 0$

$\ddot{x} + \pi^2 x = 0$

$(r^2 + \pi^2) = 0$ $\qquad r = \pm i\pi$

$\therefore x(t) = e^0 \left(c_1 \cos(\pi t) + c_2 \sin(\pi t)\right) \qquad c_1, c_2 \in R$

$x(0) = -1$, $\dot{x}(0) = \pi$ m/s

$\therefore \boxed{-1 = c_1}$, $\qquad \dot{x}(t) = -\pi c_1 \sin\pi t + \pi c_2 \cos\pi t$

$\pi = \pi c_2$

$\boxed{c_2 = 1}$

$x(t) = -1\cos\pi t + \sin\pi t$ $\qquad a\cos\theta + b\sin\theta = A\cos(\theta - \phi)$

In polar form:

$x(t) = \sqrt{(1) + 1}$

$= \sqrt{2}$

$\omega = \pi$, $A = \sqrt{2}$

$\cancel{\nu = 2\pi\omega}$ $\cancel{= 2\pi^2}$ $\qquad \omega = 2\pi\nu$

$\nu = \frac{2\pi}{\pi}$

$\boxed{\nu = 2}$

$\therefore A\cos(\omega t - \phi) =$

$\phi \to$ Angle of the function.

$A\cos(\omega t - \phi) = \sqrt{2}\cos\left(\pi t - \frac{3\pi}{4}\right)$.

$= \tan^{-1}\left(\frac{1}{-1}\right) + \pi$

$= -\frac{\pi}{4} + \pi$

$= 3\pi/4$

giving $A = \sqrt{2}$, $\omega = \pi$, $\phi = \frac{3\pi}{4}$ · $\left(\frac{3\pi}{4} + 2\pi n\right)$ for any integer n would work as the value of ϕ as well.

In amplitude-phase form the solution $x(t) = A\cos(\omega t - \phi)$

Units: $A \rightarrow$ meters, ω = radians per second, ϕ = Radians.

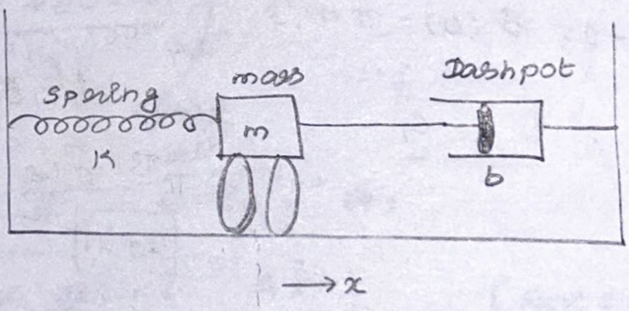
$P \rightarrow$ seconds, $\nu \rightarrow$ cycles per second, $t_0 \rightarrow$ seconds

$$P = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2 \text{ sec.}$$

$$\nu = \frac{1}{P} = \frac{1}{2} \text{ (cycles / seconds)}$$

$$t_0 = \frac{\phi}{\omega} = \frac{3\pi/4}{\pi} = \frac{3}{4}.$$

Amplitude phase forms of solutions to spring-mass-dashpot system



I. $m=1$, $b=4$, $K=5$ $(y(0)=1,\ y'(0)=0)$

Solution $y(t) = e^{-2t}(\cos t + 2\sin t)$ Changing to Amplitude phase form

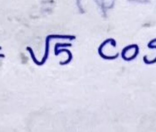
soln: $y(t) = e^{-2t}(\cos t + 2\sin t)$

$$\cos t + 2\sin t = \sqrt{1+4}\,\cos(\omega t - \phi)$$

$$\omega = 1$$

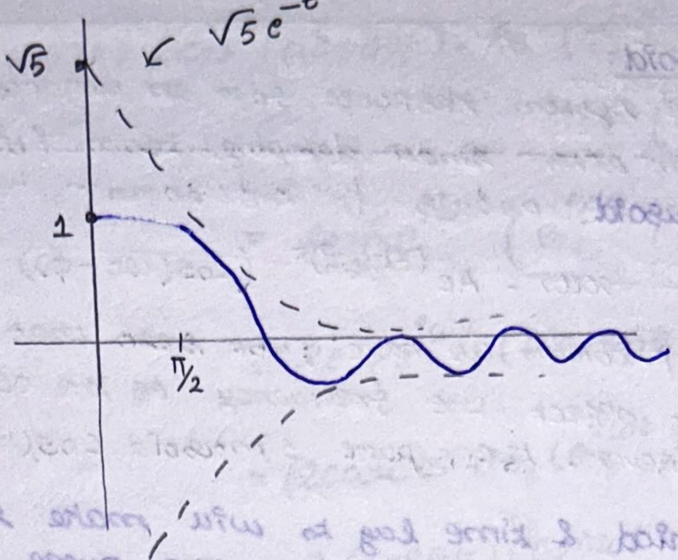
$$\phi = \tan^{-1}\left(\frac{2}{1}\right) \qquad = \sqrt{5}\cos(t - \phi)$$

$$\phi = \tan^{-1}(2) = 63.43^\circ$$



$$\therefore\ y(t) = e^{-2t}\sqrt{5}(\cos t - 2\sin t)$$

$$= e^{-2t}\sqrt{5}\,(\cos(t - 63.43^\circ))$$



The main difference b/w solutions to an undamped system & a damped system (where b is small enough so there are two complex roots). is that real solutions to the damped system have an additional overall exponential factor.

General solution to $m\ddot{y} + b\dot{y} + ky = 0$ $(b^2 < 4km)$ is

$$y(t) = e^{-b/2m\,t}\,(c_1 \cos \omega_d t + c_2 \sin \omega_d t)$$

$$= e^{-b/2m\,t}\,(A \cos(\omega_d t - \phi))$$

$$= \left(Ae^{-b/2m\,t}\right) \cos(\omega_d t - \phi)$$

∴ Since cosine oscillates b/w 1 and −1, the function $y(t)$ oscillates b/w $Ae^{-b/2m\,t}$ and $-Ae^{-b/2m\,t}$. we call the graphs of $\pm Ae^{(-b/2m)t}$ the envelope of the oscillations.

How many times does $x(t)$ crosses the t-axis as $t\uparrow$?

$Ae^{(-\frac{b}{2m})t} \neq 0$ for all t, $x(t) = 0$ (whenever) $\cos(\omega_d t - \phi) = 0$. Therefore the zeros of $x(t)$ are the same as the zeros of pure sinusoid $\cos(\omega_d t - \phi)$, which are at the t-values satisfying

$$\omega_d t - \phi = \pi/2 + n\pi \quad (n - \text{Any Integer})$$

$$t = \frac{\pi/2 + n\pi + \phi}{\omega_d}$$

$x(t) \rightarrow$ has ∞ many equally spaced zeros. The exp factor

factor changes the amplitude but not the frequency

Damped sinusoid:

The system response for an unforced spring-mass dashpot system with small damping term $(b^2 < 4mk)$ is called a damped sinusoid and is in the form

$$x(t) = Ae^{(-b/2m)t}\,(\cos(\omega t - \phi))$$

In the last problem, we have just seen that the exponential factor doesn't affect the frequency of the oscillations: $x(t)$ crosses whenever the pure sinusoid $\cos(\omega t-\phi)$ does.

$\therefore$ still period & time lag t_0 will make sense & have the same formula as for the pure sinusoid $\cos(\omega t - \phi)$

$$t_0 = \phi/\omega$$

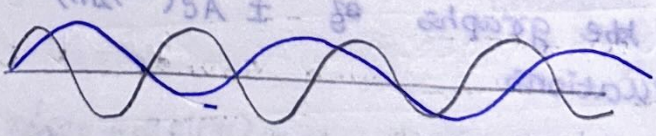
$$P = 2\pi/\omega$$

Because the graph of $x(t)$ is not truly periodic - its amplitude is $\downarrow$. we call P the pseudo-period.

For damped sinusoids, the time lag t_0 is easier to see from the shifts of zeros. The value of t_0 is the time difference b/w a zero of $x(t)$ and a corresponding zero of $\cos(\omega t)$

Beats

occurs when two nearby pitches are sounded simultaneously



$g(t) = A\sin(\omega t - \phi)$

$f(t) = \sin t$

Problem:

Consider two sinusoid sound waves of angular frequencies $\omega + \epsilon$ and $\omega - \epsilon$, say $\cos((\omega+\epsilon)t)$ and $\cos((\omega-\epsilon)t)$,

$\epsilon \rightarrow$ much smaller than ω

what happens when they are superimposed

soln:

The sum is

$$\cos((\omega+\epsilon)t)+\cos((\omega-\epsilon)t) = Re\left(e^{i(\omega+\epsilon)t}\right) + Re\left(e^{i(\omega-\epsilon)t}\right)$$

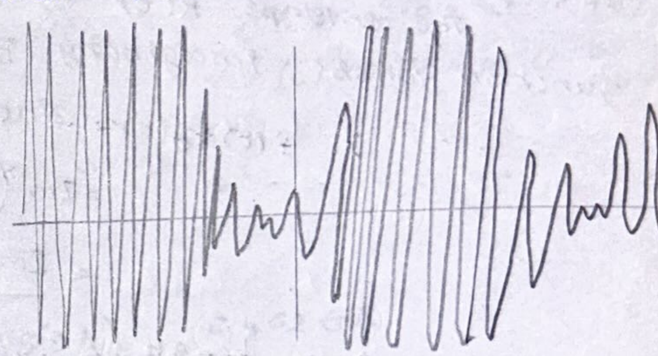
$$= Re\left(e^{i\omega t}\left(e^{i\epsilon t}+e^{-i\epsilon t}\right)\right)$$

$$= Re\left(e^{i\omega t}(2\cos\epsilon t)\right)$$

$$= (2\cos\epsilon t)\, Re\left(e^{i\omega t}\right)$$

$$= 2(\cos\epsilon t)(\cos\omega t)$$

The function $\cos\omega t$ oscillates rapidly b/w ± 1. multiplying it by the slowly varying function $2\cos\epsilon t$ produces a rapid oscillation b/w $\pm 2\cos\epsilon t$, so one hears a sound wave of angular frequency ω whose amplitude is the slowly varying function $|2\cos\epsilon t|$.



$2(\cos\epsilon t)(\cos\omega t)$ for $\epsilon = \frac{1}{8}$, $\omega = 4$

<u>Practical application:</u>

You hear beats when tuning the strings of an instrument, or in tuning one instrument to another. The waa waa waa sound you hear is exactly these beats. The higher the frequency of the beats, the more out of tune. As the instruments or strings become closer & closer in tune, the frequency of the beats diminish until you can't hear them at all.

∴ (Changing the phase of one signal w.r.t the other doesn't change the frequency of the beats! This is important. If you could hear phase shifts, it would be very difficult to tune your instrument by ear.

<u>Envelope of the beats:</u>

In the mathlet above, use the slider at the very bottom to set the amplitude A of the sinusoid

$g(t) = A\sin(\omega t - \phi)$ to $A = 0.5$.

<u>Note</u>: The two curves comprising the envelope of the beats are no longer sinusoids.

In general, the sum of the two sinusoids with different frequencies & different amplitudes can be written as the imaginary part of a complex valued function as follows

$g(t) = A\sin(\omega t - \phi)$ to $A = 0.5$.

<u>solu</u>:

$$f(t) + g(t) = \sin t + A\sin \omega t$$
$$= Im\left(R(t)\, e^{i\theta(t)}\right)$$
$$= R(t) \cdot \sin(\theta(t))$$

$\therefore |f(t) + g(t)| \leq R(t)$. $R(t) \rightarrow$ Envelope of the beats.

<u>$R(t) = ?$</u>

The envelope $R(t)$ is the modulus of the complex function whose imaginary part is $f(t) + g(t)$.

$$\therefore f(t) + g(t) = \sin t + A\sin \omega t$$
$$= Im\left(e^{it} + Ae^{i\omega t}\right)$$
$$= Im\left(\cos t + A\cos\omega t + i(\sin t + A\sin\omega t)\right)$$

$$R(t) = |\cos t + A\cos\omega t + i(\sin t + A\sin\omega t)|$$
$$= \sqrt{(\cos t + A\cos\omega t)^2 + (\sin t + A\sin\omega t)^2}$$
$$= \sqrt{1 + A^2 + 2\cos t \cdot A\cos\omega t + 2\sin t\,\sin\omega t \cdot A}$$
$$= \sqrt{1 + A^2 + 2A(\cos t\cos\omega t) + 2A\sin t\sin\omega t}$$
$$= \sqrt{1 + A^2 + 2A\left(\cos(1-\omega)t\right)}$$

When $A = 1$, the amplitudes of the two sinusoids in $f(t)$ are the same, and $R(t)$ reduces to what we have computed.

$$R(t) = \sqrt{2 + 2(\cos(1-\omega)t}$$
$$= \sqrt{2\left(1 + \cos(1-\omega)t\right.}$$

$\cos^2\theta = \cos 2\theta + 1$

$$= \sqrt{2 \times 2\left(\cos^2\left(\frac{1-\omega}{2}\right)t\right)}$$
$$= 2\left|\cos\left(\frac{1-\omega}{2}t\right)\right|$$

1) $\cos 2t + \sin 2t$ (in the form $A \cos(\omega t - \phi)$, A, ω, ϕ

Solu:

$$r = \sqrt{a^2+b^2} = \sqrt{2}$$

$$\cos 2t + \sin 2t = \sqrt{2} \cos(2t - \pi/4)$$

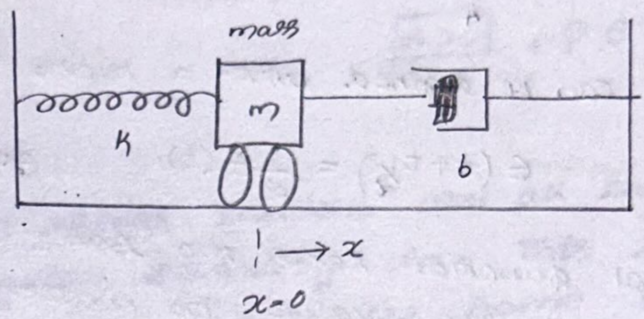
$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$\theta = \frac{\pi}{4}$$

$\therefore$ $A = \sqrt{2}$

$\omega = 2$

$\phi = \pi/4$



Solu.

$m = 1\,\text{kg}$, $k = \frac{17}{16}\,\text{N/m}$, $b = \frac{1}{2}\,\text{Ns/m}$, $x(0) = 1\,\text{m}$, (to the right)

$\dot{x}(0) = 0.75\,\text{m/s}$ (to the right)

$$\ddot{x} + \frac{1}{2}\dot{x} + \frac{17}{16}x = 0$$

$$r = -\frac{1}{4} \pm i$$

$$x(t) = e^{-t/4}(c_1 \cos t + c_2 \sin t) \quad c_1, c_2 \in \mathbb{R}$$

$x(0) = c_1 \Rightarrow c_1 = 0$

$x'(0) = 0.75 \Rightarrow -\frac{1}{4} + c_2 = \frac{3}{4} \Rightarrow c_2 = 1$

$$\therefore x(t) = \sqrt{2}\, e^{-t/4} \cos(t - \pi/4)$$

$P = \frac{1}{4}$, $A = \sqrt{2}$, $\omega = 1$, $\phi = \frac{\pi}{4}$ (or) $\frac{\pi}{4} + 2\pi n$ for $n \to$ integer

Let us consider the envelope $\pm E(t) = Ae^{-pt}$ of the solution $x(t)$ that you found above. The decaying exponential envelope is a main feature that distinguishes a damped sinusoid from a pure sinusoid.

How long does it take for the envelope to decay by half? In other words $t_{1/2}$ in (seconds) such that $E(t_{1/2}) = \frac{1}{2}E(0)$. This is called the half-time of the decaying exponential envelope.

Solu:

$$E(t) = \sqrt{2}\, e^{-t/4} \quad \text{(From previous problem)}$$

$$A e^{-t_{1/2}/4} = \frac{A}{2} e^{0t}$$

$$e^{-t_{1/2}/4} = \frac{1}{2}$$

$$-\frac{t_{1/2}}{4} = \ln\left(\frac{1}{2}\right)$$

$$\frac{t_{1/2}}{4} = +\ln(2) \quad \Rightarrow \boxed{t_{1/2} = 4\ln 2}$$

Half time can be defined with a more general eqn:

$$E(t + t_{1/2}) = \frac{1}{2} E(t) \quad \text{for any } t.$$

The exponential function takes the same amount of time to decay to half its value no matter when you start measuring.

The position function $x(t) = \sqrt{2}\, e^{-t/4} \cos\left(t - \frac{\pi}{4}\right)$, crosses the t-axis exactly when $\cos\left(t - \frac{\pi}{4}\right) = 0$.

$$\therefore t_1 = \frac{\pi}{2} + t_0 = \frac{3\pi}{4}.$$

$$t_2 = \frac{3\pi}{2} + t_0 = \frac{7\pi}{4}$$

$$t_3 = \frac{5\pi}{2} + t_0 = \frac{11\pi}{4}$$

$$t_4 = \frac{7\pi}{2} + t_0 = \frac{15\pi}{4}.$$

Damped Harmonic Oscillators

1) Translate properties of the roots of the char eqn for a second order mass/spring/dashpot system into corresponding damping conditions.

2) Solve II order homogeneous linear ODE with a repeated char root.

3) Graph the solutions to a damped harmonic oscillator.

4) Compare different qualitative behaviors of an underdamped, damped & a critically damped spring-mass-dashpot system.

5) Understand the effects of damping on both the frequency & Amplitude of oscillations.

No damping: $\ddot{x} + \frac{K}{m} x = 0 \Rightarrow \ddot{x} + \omega_n^2 x = 0$ $\boxed{\omega_n = \sqrt{\frac{K}{m}}}$

Char eqn: $P(r) = mr^2 + K$

Roots: $\pm i\omega_n$ $\left(\omega_n = \sqrt{\frac{K}{m}}\right)$

Basis of solution space: $e^{i\omega_n t}, e^{-i\omega_n t}$

Realvalued basis: $\cos\omega_n t, \sin\omega_n t$.

General real valued solutions: $a\cos\omega_n t + b\sin\omega_n t$

$a, b \to$ Real constants

$= A\cos(\omega_n t - \phi)$

$\boxed{A > 0}$, $\phi \in R$ constants

In other words,

the real valued solutions are all the sinusoidal functions of angular frequency ω_n. This system, or any other system governed by the same DE, is also called a simple harmonic oscillator.

$\omega_n \to$ Angular Frequency (Natural or resonant frequency) of the oscillator

A spring system that is undamped is called the harmonic oscillator, or an ideal spring. What's the period of a nonzero solution of $\ddot{x} + 4x = 0$

Solu:

$r^2 + 4 = 0$

$r = \pm 2i$

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$

$z_1 = e^{i(2t)}, z_2 = e^{-i(2t)}$

$Re(z_1) = \cos 2t$

$Im(z_1) = \sin 2t$.

$x = C_1\cos 2t + C_2\sin 2t$

$= A\cos(2t - \phi)$

Effect of spring constant:

How does the stiffness of the spring affect the oscillations frequency of the oscillations? In other words, If the $K \uparrow$, what happens to $?$ of oscillations.

Solu:

$\omega_n = \sqrt{\frac{K}{m}}$. If $K\uparrow$, ω_n also $\uparrow$. (The stiffer the spring, the faster the oscillations.

$2p = \frac{b}{m}, \omega_0^2 = \frac{K}{m}$ (or) $\omega_n^2 = \frac{K}{m}$

$m\ddot{x} + b\dot{x} + Kx = 0$

$\ddot{x} + \frac{b}{m}\dot{x} + \frac{K}{m}x = 0$ (or) $y'' + 2py' + \omega_0^2 y = 0$

char eqn: $r^2 + 2pr + \omega_0^2 = 0$

$r = -p \pm \sqrt{\frac{4p^2}{4} - \frac{4(1)\omega_0^2}{4}}$

$= -p \pm \sqrt{p^2 - \omega_0^2}$

(i) where $p = 0$ [undamped]

$p = \frac{b}{2m}$ $[m \neq 0, b = 0]$

$\therefore y'' + \omega_0^2 y = 0$ [simple harmonic motion]

$\omega_0 \rightarrow$ circular (or) natural frequency.

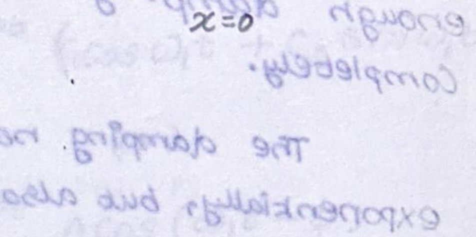
Soln:

$r = \pm i\omega_0$

$\therefore y(t) = c_1 \cos\omega_0 t + c_2 \cos\omega_0 t$

$= A\cos(\omega_0 t - \phi)$

Damped Harmonic oscillator.



$mr^2 + br + K = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$

$= -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{K}{m}}$

$= -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_n^2}$

There are three cases, depending upon the sign of $b^2 - 4mK$. (Behaviour of solutions will be different)

Case: 1: $b^2 < 4mK$ (under damped)

There are two complex roots, and will give names to the real & imaginary parts. Since the real part is always (−)ve, we call it as $-p$, with

$p = \frac{b}{2m}$. The imaginary part is either +ve or (-)ve of the damped frequency ω_d, given by.

$$\omega_d = \frac{\sqrt{4mk - b^2}}{2m}$$

$$= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\omega_n^2 - p^2} \qquad \omega_n^2 = \frac{k}{m}.$$

Note: Both ω_n and p are +ve.

<u>Summary:</u>

Roots: $-p \pm i\omega_d$

Basis of solution space: $e^{(-p+i\omega_d)t}, e^{(-p-i\omega_d)t}$.

Real valued basis: $e^{-pt}(\cos\omega_d t), e^{-pt}(\sin\omega_d t)$

General real solution: $e^{-pt}(a\cos\omega_d t + b\sin\omega_d t)$

$a, b \in$ Real

$$= Ae^{-pt}\cos(\omega_d t - \phi)$$

This is a sinusoid multiplied by a decaying exp. Each nonzero solution tends to zero, but changes sign ~~only~~ many times along the way. The system is called underdamped. Because there was not enough damping to eliminate the oscillation completely.

The damping not only causes the solution to decay exponentially, but also changes the frequency of the sinusoid. The new angular frequency ω_d is called damped (angular) frequency (sometimes pseudo (angular) frequency).

The damped frequency ω_d is less than the natural (undamped) frequency ω_n. As evident from the formula $\omega_d = \sqrt{\omega_n^2 - p^2}$ (when $b = 0$)

$$\omega_d = \omega_n$$

The damped solutions are not actually periodic: they don't repeat exactly. (Because of the decay).

$\therefore T = \frac{2\pi}{\omega_d}$ (pseudo-period).

Here $\omega_1 = \omega_d$, $\omega_0 = \omega_n$

(we get oscillations, when roots are really complex)

$\omega_d = \sqrt{\omega_0^2 - p^2}$ $\quad (p^2 - \omega_0^2) < 0$

$r = -p \pm \sqrt{p^2 - \omega_0^2}$ $\quad$ (k-m-b system).

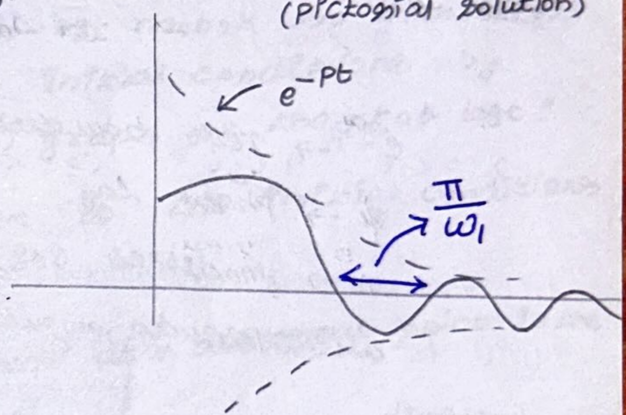
when $p^2 - \omega_0^2 < 0$

$\therefore \boxed{p < \omega_0}$ $\quad$ (complex root is possible)

Damping must be less than angular frequency.

From previous problem. ←

(pictorial solution)



ω_1: pseudo frequency.

T: pseudo circular period.

If damping goes up, what happens?

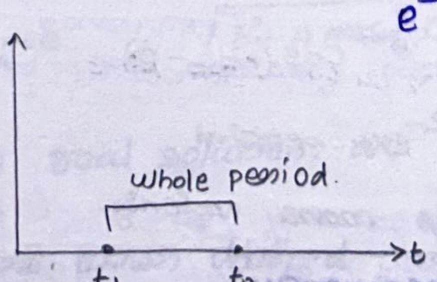
If damping c ↑, the frequency or ω_1 goes down.

$r = -p \pm \sqrt{-(\omega_0^2 - p^2)}$ $\quad e^{-pt}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t)$

$r = -p \pm \sqrt{-\omega_1^2}$

(or)

$e^{-pt} A \cos(\omega_1 t - \phi)$



curve crosses the t axis

$$\omega_1 t_1 - \phi_2 = \pi/2$$

$$\omega_1\left(t_1 + \frac{2\pi}{\omega_1}\right) - \phi = \frac{\pi}{2} + 2\pi$$

Solution is $e^{-pt} A \cos(\omega_1 t - \phi)$

Parameters:

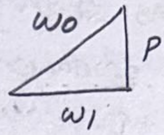
$p \rightarrow$ only on ODE $(c/2m)$

$\phi \rightarrow$, $A \rightarrow$ } ~~only on ODE~~ Depends on initial conditions.

$\omega_1 \rightarrow$ only on ODE $\therefore \omega_1 = 2\pi\nu_1$

$$\omega_1^2 = \omega_0^2 - p^2 \quad (\text{or}) \quad \boxed{\omega_d^2 = \omega_n^2 - p^2}$$

$\omega_1 \rightarrow$ depends on damping & spring constant.



$e^{-pt} \rightarrow$ Tells how fast it's coming down

$\phi \rightarrow$ phase lag. $\frac{\pi}{|\omega_1|}$

$A \rightarrow$ modifies exp function (where it starts & goes)

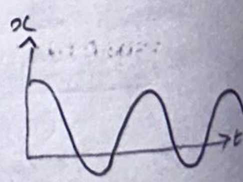
$\omega_1 \rightarrow$ pseudo angular frequency (How the system is bobbing).

Effect of damping on the decay rate:

Fix a set of values for m, b, k. So that $x(t)$ is oscillating. Then slowly ↑ the b using slider (mathlets.org)

As the damping constant b ↑ (But such that the solution remains oscillatory), the Amplitude of the oscillation

when $m=1, k=1,$ $b=0$

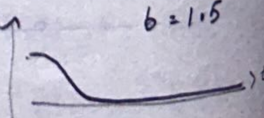


Ans: Decays more quickly.

$$x(t) = Ae^{-bt/2m} \cos(\omega_d t - \phi)$$

As b ↑, exponential factor $e^{-bt/2m}$ decays more quickly

when $m=1, k=1$ $b=1.5$



Effect of damping on frequency:

$m = 1.5$, fix b and k so that $x(t)$ is oscillating.

To ↑ the frequency of the oscillations, that is, to ↑ ω_d, which of the following could you do?

(To see the effect of b on the frequency. You could use small values of K, and move the point on the $(x, \dot{x})$ - plane on the upper left to modify initial conditions.

Ans: For maximum oscillations increase K and decrease b.

$$\omega_d = \sqrt{\omega_n^2 - p^2}$$

$$\therefore \omega_d = \sqrt{\frac{K}{m} - \frac{b^2}{4m^2}} \quad (\downarrow b \ \& \uparrow K)$$

Dependence on Initial Conditions:

Fix a set of values of m, b, K so that $x(t)$ is oscillating. Then change the initial conditions by moving the point on the $(x, \dot{x})$ graph at the top left: which of the following can happen to the initial conditions to the damped oscillator change?

Ans: The oscillations can start at a different point in the cosine cycle.

The maximum amplitude of the oscillation can increase.

Case-2: Overdamped:

$$b^2 > 4mK \text{ (overdamped)}$$

$$\frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$ are real & distinct. Both roots are negative since $\sqrt{b^2 - 4mK} < b$. Call them $-s_1$ & $-s_2$.

Solu: General real solution: $ae^{-s_1 t} + be^{-s_2 t}$.

As in all the other damped cases, all solutions tend to zero as $t \to \infty$. The term corresponding to the less negative root eventually controls the rate of return to equilibrium. The system is called overdamped. There is so much damping that is slowing the return to equilibrium.

1) Assume an unforced overdamped spring mass dashpot started at $\dot{x}(0)=0$. Show that it never crosses the equilibrium position $x=0$ for $t>0$.

2) Show that regardless of IC an overdamped oscillator can cross equilibrium at most once. (more than once)

Soln.

$$m\ddot{x}+b\dot{x}+Kx=0 \qquad (m, b, K>0)$$

$$(mr^2+br+K)=0$$

$$r=\frac{-b\pm\sqrt{b^2-4mK}}{2m}$$

(Real constants are possible)

$$b^2-4ac$$

$$b^2>4ac$$

$$x=c_1e^{\omega_1 t}+c_2e^{\omega_2 t}$$

In overdamped unforced damping system, the roots will be real & (–)ve.

$$\therefore \boxed{-b>\sqrt{b^2-4mK}}$$

$\omega_2<\omega_1<0$ (Both real).

$$\boxed{\frac{\omega_1}{\omega_2}<1}$$

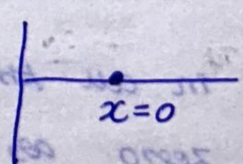
IC: $\dot{x}(0)=0$.

$$0=c_1(\omega_1)e^0+c_2(\omega_2)e^0$$

$$\omega_1c_1+\omega_2c_2=0$$

Let at t^* the system crosses the equilibrium position.

$\frac{\text{equilibrium}}{x(0)}$



$$x(t^*)=0=c_1e^{\omega_1t^*}+c_2e^{\omega_2t^*}$$

$$-c_1e^{\omega_1t^*}=c_2e^{\omega_2t^*}$$

$$-\frac{c_1}{c_2}=e^{(\omega_2-\omega_1)t^*}$$

$$-\frac{c_2}{c_1} = e^{(\omega_1-\omega_2)t^*}$$

$$\ln\left(-\frac{c_2}{c_1}\right) = (\omega_1-\omega_2)t^*$$

$$t^* = \frac{\ln\left(-\frac{c_2}{c_1}\right)}{(\omega_1-\omega_2)}$$

Irrespective of Initial conditions, the system will never cross the equilibrium more than once.

$\omega_1, \omega_2 \rightarrow$ Fixed in a system.

ii) $\rightarrow$ only one t^* is available

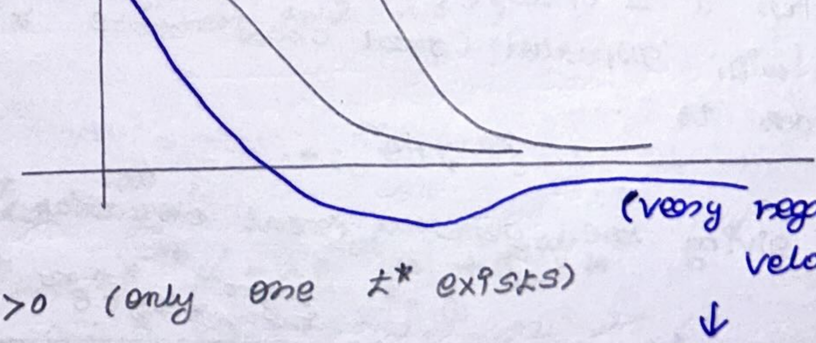
i) $\dot{x}(0)=0 \Rightarrow -\frac{c_2}{c_1} = \frac{\omega_1}{\omega_2} < 1.$ ($\because$ Negative)

$$t^* = \frac{\ln\left(-\frac{c_2}{c_1}\right)}{\omega_1-\omega_2} \quad \begin{matrix}\rightarrow \text{Negative} \\ \rightarrow +\text{ve}\end{matrix}$$

$t^* = (-)$ve.

So For $t>0$, the system never crosses zero.

ii) and part of the problem.



when $-\frac{c_2}{c_1} < 0$

(very negative velocity)

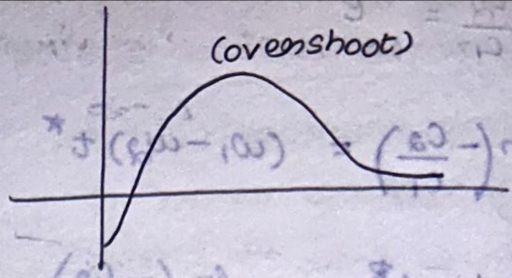
$\downarrow$

crosses 0 only once.

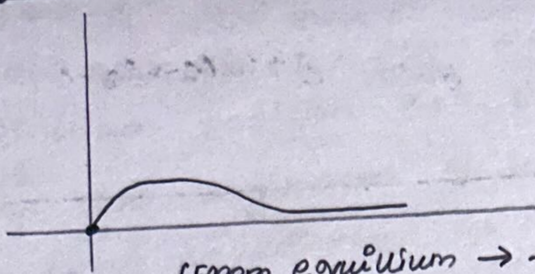
If $-\frac{c_2}{c_1} > 0$ (only one t^* exists)

(or)

(overshoot)



Starting from equilibrium point:



(From equilibrium → ∴ The gn system is unforced)

Typical behaviour of a damped oscillator → No oscillation [Just attracted towards rest]

In the underdamped case → The oscillations continue forever (Theoretically)

In the overdamped case, it's not that oscillations start & then stop at a later time. There are no oscillations at all.

Critically damped.

$b^2 = 4mK$

The critically damped case happens when $b^2 = 4mK$, at the border b/w the underdamped case when $b^2 < 4mK$ & the overdamped case, when $b^2 > 4mK$.

There is a repeated (−) ve real root, which we denote by $-p$ $\left(p = \frac{b}{2m}\right)$. The repeated root gives only one exponential solution

$e^{-bt/2m}$ (what's the other solution)

Solu:

For a II order (homogeneous linear Constant co.eff) ODE with repeated real char roots r, a basis of solutions is

$$e^{rt},\ te^{rt}, ..$$

giving the general (real equation

$$c_1 e^{rt} + c_2 t e^{rt}$$

7.1: $\ddot{x} + 4\dot{x} + 4x = 0$

Solu: $(r+2)^2 = 0$ $\quad r = -2, -2.$

The only exponential solution is e^{-2t}. Another solution is not a constant multiple of e^{-2t}, is given by

$$te^{-2t}.$$

So, $x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ (or) $e^{-2t}(c_1 + c_2 t)$.

Summary: (critically damped)

Basis of solution space: e^{-pt}, te^{-pt}

General real solution: $e^{-pt}(a+bt)$

$a, b \to$ Real constants.

What happens to the solutions as $t \to +\infty$?

The solution e^{-pt} tends to 0.

So does $te^{-pt} = \frac{t}{e^{pt}}$ = Even though the numerator is tending to $+\infty$, the denominator e^{pt} is tending to $+\infty$ faster (In a contest b/w exponentials & polynomials, exponentials always win). Thus all solutions eventually decay.

(This case is when there is just enough damping to eliminate oscillation. The system is called critically damped.

Critically damped:

$r = -a$ (the root)

$$(r+a)^2 = 0$$

$$r^2 + 2ar + a^2 = 0$$

$$(y'' + 2ay' + a^2 y) = 0 \quad \text{(a in } 2ay' \text{ neglect damping)}$$

Solution: $y = e^{-at}$ (other solution = ?)

Let: $y = e^{-at} \cdot u$

$$y' = -ae^{-at}u + e^{-at}u'.$$

$$y'' = a^2 e^{-at} u - ae^{-at}u' - ae^{-at}u' + e^{-at}u''$$

$$y'' = a^2 e^{-at} u - 2ae^{-at}u' + e^{-at}u''$$

$$y'' + 2ay' + a^2 y = (\cancel{a^2 e^{-at} u} - \cancel{2ae^{-at} u'} + e^{-at}u'') - \cancel{2a^2 e^{-at} u} + \cancel{2ae^{-at} u'}$$
$$+ \cancel{a^2 e^{-at} u}$$

$y'' + 2ay' + a^2 y = e^{-at} u''$

$0 = e^{-at} u''$

$u'' = 0$

$\int u'' = 0 \Rightarrow u' = c_1$

$\int u' = \int c_1 \, dt \Rightarrow u = c_1 t + c_2$ (Just t is enough)

other solution $= e^{-at} \cdot t$ → second solution

<u>Summary</u>

mKb system: $m\ddot{x} + b\dot{x} + Kx = 0$ $(m, b, K > 0)$

case	Roots	Situation
$b = 0$	two complex roots $\pm i\omega_n$	undamped (simple harmonic oscillator)
$b^2 < 4mK$	Two complex roots $-p \pm i\omega_d$	underdamped (Damped oscillator)
$b^2 = 4mK$	repeated real root $-p, -p$	critically damped.
$b^2 > 4mK$	distinct real roots $-s_1, -s_2$	overdamped.

$p = \frac{b}{2m}$ and $\omega_d = \frac{\sqrt{4mK - b^2}}{2m}$

$= \sqrt{\omega_n^2 - p^2}$ $\qquad \omega_n = \sqrt{\frac{K}{m}}$.

$m = 1, b = 2, K = 4$

$r^2 + 2r + 4 = 0$

$(r+1)^2 + 3 = 0$

$r = -1 \pm i\sqrt{3}$ (underdamped)

Basis of the solution space: $e^{(-1+i\sqrt{3})t}, e^{(-1-i\sqrt{3})t}$

Real valued basis: $e^{-t}\cos(\sqrt{3}t), e^{-t}\sin(\sqrt{3}t)$.

General real solutions: $e^{-t}(a\cos\sqrt{3}t + b\sin\sqrt{3}t)$ where a, b → Real solutions.

The damped frequency is $\sqrt{3}$.

Damped oscillator:

For mkb system, any non zero solution $x = e^{-t}(a\cos\sqrt{3}t + b\sin\sqrt{3}t)$ crosses the equilibrium position $x=0$ infinitely many times. How much time elapses b/w consecutive crossings?

solu: Ans: $\frac{\pi}{3}$. ∴ The solution has the same zeros as the sinusoid $a\cos(\sqrt{3}t) + b\sin(\sqrt{3}t)$ of $\omega = \sqrt{3}$. $P = \frac{2\pi}{\sqrt{3}}$. But a sinusoid crosses 0 at each half (pseudo period).

So Answer $= \frac{\pi}{3}$.

Aiming for critical damping

In mathlet, $m = 0.25$, $k = 0.5$. (use slider b to ↑ the damping constant from 0 and find b when char roots are repeated & real.

$b^2 = 4mk$

$$b = \sqrt{4mk} = \sqrt{4(0.25)(0.5)} = \frac{1}{\sqrt{2}}.$$

The frequency of oscillation decreases as b ↑; the oscillations are so far apart we can't see it anymore on the mathlet by eye.

when $b > 0.71$, $x(t)$ crosses t-axis either once or not at all, (Depending on the IC)

Quadrupling the mass:

Now ↑ the mass by 4 times $m=1$, $k=0.5$ b the same $b = 0.71$ (answer. use mathlet to find new b_1. (ratio of $b_1/b = ?$)

$$b_1 = \sqrt{4mk} = \sqrt{2} \approx 1.42$$

$$\frac{b_1}{b} = \frac{1.42}{0.71} = 2$$

(If m ↑ by 4 times, damping ↑ by 2 times).

Damping

3

$\ddot{x} + \dot{x} + 3x = 0$

$(r^2 + r + 3) = 0$

$$\frac{-1 \pm \sqrt{1-12}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{11}}{2} \rightarrow \text{underdamped.}$$

Solutions: $e^{-t/2}\cos\frac{t\sqrt{11}}{2}$, $e^{-t/2}\sin\frac{t\sqrt{11}}{2}$

$$x(t) = \left(c_1 \cos t\frac{\sqrt{11}}{2} + c_2 \sin t\frac{\sqrt{11}}{2}\right) e^{-t/2}$$

$$= Ae^{-t/2} \cos\left(\frac{t\sqrt{11}}{2} - \phi\right) \qquad \omega_d = \frac{\sqrt{11}}{2}$$

$x(0) = 1,\ \dot{x}(0) = 0$

$\boxed{c_1 = 1}$, $\dot{x}(t) = \left(c_1\left(-\frac{\sqrt{11}}{2}\right)\sin t\frac{\sqrt{11}}{2} + c_2\left(\frac{\sqrt{11}}{2}\right)\cos t\right)e^{-t/2}$

$$+\left(c_1 \cos t\frac{\sqrt{11}}{2} + c_2 \sin t\frac{\sqrt{11}}{2}\right)\left(-\frac{1}{2}\right)e^{-t/2}$$

$$0 = c_1\left(-\frac{1}{2}\right)e^{-t/2} + c_2\left(\frac{\sqrt{11}}{2}\right)\cos t \cdot e^{-t/2}$$

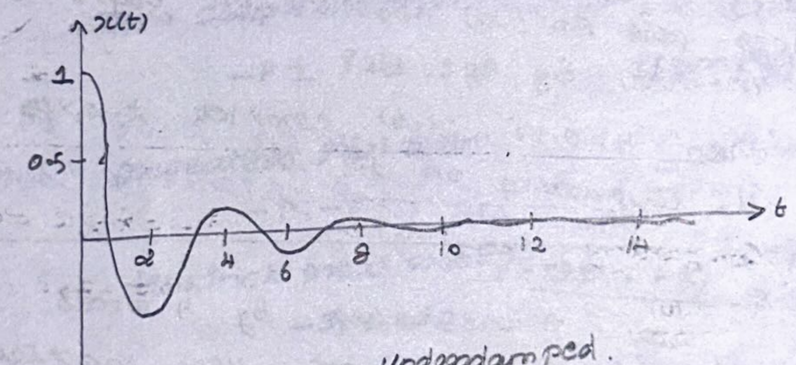
$$0 = -\frac{c_1}{2} + c_2\left(\frac{\sqrt{11}}{2}\right)$$

$\phi = \tan^{-1}\left(\frac{1}{\sqrt{11}}\right)$

$$\frac{1}{2} = c_2\left(\frac{\sqrt{11}}{2}\right)$$

$$\boxed{c_2 = \frac{1}{\sqrt{11}}}$$

Graph:



$\ddot{x} + 4\dot{x} + 3x = 0 \Rightarrow (r^2 + 4r + 3) = 0 \quad r = \frac{-4 \pm \sqrt{16 - 12}}{2}$

$$= \frac{-4 \pm 2}{2} = -2 \pm 1$$

$$= -1, -3.$$

(Both (-)ve roots)

↳ overdamped.

$x(0) = 1,\ \dot{x}(0) = 0.$

Solu:

$x(t) = c_1 e^{-t} + c_2 e^{-3t}$

$1 = c_1 + c_2 \rightarrow (1)$

$0 = c_1(-1)e^0 + c_2(-3)e^0$

$\dot{x}(0)\ 3c_2 + c_1 = 0$

$c_1 + c_2 = 1$

$c_1 + 3c_2 = 0$

$-2c_2 = 1$