Hint: collecting in to a vector.

$$u_K = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}, \quad u_{K+1} = A u_K$$

$$u_{K+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_K$$

$$u_{K+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$$

Eigen values:

$$\det \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = -\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}$$

$$= \frac{1+2.236}{2}, \quad \frac{1-2.236}{2}$$

$$= \frac{3.236}{2}, \quad -\frac{1.236}{2}$$

$\lambda_1 + \lambda_2 = 1$ (Trace)

$\lambda_1 \lambda_2 = -1.$

$$= 1.618, \quad -0.618.$$

$\downarrow$ >1 $\quad$ $\downarrow$ <1

Diagonalizable? Yes

How fast fibonacci sequence increases?

'Two - distinct Eigen values'

$F_{100} = ?$

what's controlling the growth of fibonacci numbers?

Eigen numbers!

$\downarrow$

which: The big ones.

$$F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)^{100}$$

$$u_{100} = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2$$

great

$\downarrow$

$0.6^{100} \longrightarrow$ very small.

(disappearing)

'Dynamic problem: instead of $Ax = b$'

Eigen vectors:

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = 0 \qquad x = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \rightarrow \therefore$$

Then only (Row 2)x = 0

when $x_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \therefore C_1 x_1 + C_2 x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Again, $F_{K+2} = F_{K+1} + F_K$ $\qquad u_{K+1} = A u_K$

$u_K = A u_0$ → Solution

Now, $F_{K+2} = F_{K+1} + F_K$

A → matrix.

In matrix form:

$$\underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} F_{K+1} \\ F_K \end{pmatrix}}_{u_K} = F_{K+1} + F_K.$$

Adding another eqn & making it as a system:

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} F_{K+1} \\ F_K \end{pmatrix}}_{u_K}$$

① $F_{K+2} = F_{K+1} + F_K$

② $F_{K+1} = F_{K+1}$

Idea: F_{K+1}, F_K → Into a matrix u_K

$$u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2$$

$$u_{100} = c_1 (\lambda_1)^{100} x_1 + c_2 (\lambda_2)^{100} x_2$$

$\hookrightarrow > 1$ (Has effect) $\qquad \hookrightarrow < 1$ (very small = vanishing).

Dynamic: As the system is evolving to the powers.

Continuation:

$$u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 x_1 + c_2 x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c_1 \begin{pmatrix} \frac{1+\sqrt5}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1-\sqrt5}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

↓ x_1 $\qquad$ ↓ x_2

$x_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$

$c_1 + c_2 = 0$

$c_1 = -c_2$

$$c_1 \left(\frac{1+\sqrt5}{2}\right) + c_2 \left(\frac{1-\sqrt5}{2}\right) = 1$$

$$c_1 \left(\frac{1+\sqrt5}{2}\right) - c_1 \left(\frac{1-\sqrt5}{2}\right) = 1$$

$$c_1\left(\frac{2\sqrt{5}}{2}\right) = 1 \qquad c_2 = -c_1$$

$$c_1 = \frac{1}{\sqrt{5}} \qquad c_2 = -\frac{1}{\sqrt{5}}.$$

$$u_K = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix} = c_1 \lambda_1^K x_1 + c_2 \lambda_2^K x_2.$$

$$\therefore \quad u_K = \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^K \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^K \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$$F_K = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^K (1) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^K (1)$$

Real answer ←

$$\boxed{F_K = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^K - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^K}$$

$$F_{K+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^K \left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^K \left(\frac{1-\sqrt{5}}{2}\right)$$

$$\boxed{F_{K+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{K+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{K+1}}$$

$$F_K \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^K$$

→ Ref Gilbert Strang's book for other method.

Key:

* start from u_0
* write in matrix form
* find Eigen values
* find Eigen vectors
* solve c_1 for the known values like $\begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
* Now we can take F_K out of the matrix.

Recitation : Powers of a matrix

Find a formula for C^K where $C = \begin{pmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{pmatrix}$

Calculate C^{100} when $a=b=-1$

solu:

$$\boxed{A^{100} = S \Lambda^{100} S^{-1}}$$

$\hookrightarrow$ Eigen vector matrix.

$$\det \begin{pmatrix} (2b-a)-\lambda & a-b \\ 2b-2a & 2a-b-\lambda \end{pmatrix} = 0$$

$$(2b-a-\lambda)(2a-b-\lambda) - (2b-2a)(a-b) = 0.$$

$$(4ab - 2b^2 - 2b\lambda - 2a^2 + ab + a\lambda - 2a\lambda + b\lambda + \lambda^2) + 2(a-b)(a-b) = 0$$

$$(\lambda^2 + \lambda(a - 2a + b - 2b) - \cancel{2a^2} - \cancel{2b^2} + 5ab) + \cancel{2a^2} + \cancel{2b^2} - 4ab = 0$$

$$\lambda^2 + \lambda(-a-b) + ab = 0$$

$$\lambda^2 - \lambda(a+b) + ab = 0$$

$$(\lambda - a)(\lambda - b) = 0$$

$ab\lambda^2$

$-a\lambda \quad -b\lambda$

$-(a+b)\lambda$

Eigen vectors: $\lambda = a$

$$\begin{pmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = 0$$

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda = b$

$$\begin{pmatrix} b-a & a-b \\ 2b-2a & 2a-2b \end{pmatrix} x = 0$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$C = S \Lambda S^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$$

$\downarrow$

Diagonal matrix having Eigen values in the diagonals.

$$C^K = S \Lambda^K S^{-1}$$

$$C^K = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a^K & 0 \\ 0 & b^K \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$$

when $a = b = -1$.

$$C^{100} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1^{100} & 0 \\ 0 & -1^{100} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & 1-1 \\ -2+2 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Lecture - 24   Differential equations & exp(At)

we can copy Taylor's series for e^x to define e^{At} for a matrix A. If A is diagonalizable, we can use Λ to find the exact value of e^{At}. This allows us to solve systems of differential equations

$\frac{du}{dt} = Au$ the same way we solved equations

like $\frac{dy}{dt} = ky$.

★ Differential equ $\frac{du}{dt} = Au$

★ Exponential e^{At} of a matrix.

---

1) How to solve 1st order constant coefficient differential equations!

2) Solutions to them are exponentials.

what's there in the exponential? → Linear algebra.

Last lect: powers of a matrix

Now: exponential of a matrix.

Initial condition $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

① $\frac{du_1}{dt} = -u_1 + 2u_2$

$\frac{du_2}{dt} = u_1 - 2u_2$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \rightarrow \text{Singular (we can say one of the } \lambda = 0)$$

Other Eigen value — from trace $\lambda = 0, -3$

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} = (1+\lambda)(2+\lambda) - 2$$

$$= 2 + \lambda^2 + \lambda + 2\lambda - 2$$

$$0 = \lambda^2 + 3\lambda$$

$$\lambda(\lambda+3) = 0 \qquad \lambda = 0, -3$$

$\lambda = 0$	$\lambda = -3$
This part will be steady	Negative Eigen value → going to disappear. (e^{-3t})

$$\boxed{\text{Solution} = \underbrace{(\lambda=0)\text{ part}}_{\text{steady}} + \underbrace{(\lambda=-3)\text{ part}}_{\text{Disappearing}}}$$

Eigen vectors:

$\lambda = 0$

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(Nullspace)

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A x_1 = 0 x_1$$

$\lambda = -3$

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A x_2 = -3 x_2$$

"Eigen values & Eigen vectors" → are found.

Solution:

2 Eigen values — 2 pure exponential solutions.

$$\boxed{u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2} \rightarrow \text{we are combining two eqn in the matrix form.}$$

checking:

$$\frac{du}{dt} = Au \quad (\text{plug in } e^{\lambda_1 t} x_1)$$

$$\frac{d}{dt}(e^{\lambda_1 t} x_1) = A(e^{\lambda_1 t} x_1)$$

$$(\lambda_1 e^{\lambda_1 t} x_1) = A e^{\lambda_1 t} x_1$$

$$\lambda_1 x_1 = A x_1 \rightarrow \text{Eigen vectors. form.}$$

---

pure exponentials are the analogous of differential equations of the pure powers.

---

we found that

$$u_{k+1} \approx c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = Au_k$$

Plugging λ's

$$u_t = c_1 (1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

↓

Solving using Initial conditions

$u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ At $t = 0$

$$c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$2c_1 + c_2 = 1$$
$$c_1 - c_2 = 0$$

$$3c_1 = 1$$
$$c_1 = 1/3$$

$c_1 = c_2 = 1/3$

$$u_t = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3} e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

↳ stable ↳ vanishing.

$$u(\infty) = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Stability: $u(t) \to 0$

"Negative Eigen values" $\to e^{\lambda t} \to 0$ when $\lambda < 0$.

Complex:

$$e^{(-3+6i)t} = e^{-3t}(\cos 6t + i\sin 6t)$$

$$|e^{(-3+6i)t}| = e^{-3t}\sqrt{\cos^2 6t + \sin^2 6t}$$

$$\boxed{|e^{(-3+6i)t}| = e^{-3t}} \to \text{Real part must be zero.}$$

else blows up.

Steady state.

$\lambda_1 = 0$, other Eigen values are negative (Real part)

Blow up

when any real part of the $\lambda > 0$

Sign change: what about $-A$:

* Trace $\to$ will change (Blew up - In this case)
* zero Eigen value will stay at zero.

For 2x2 matrix: $Re\ \lambda_1 < 0$, $Re\ \lambda_2 < 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Trace $= a+d$ will be negative.

The complex part of λ_1 and λ_2 will be conjugate complex of each other.

What about determinant?

$\boxed{\det > 0}$ $\boxed{\lambda_1 \lambda_2 = +ve}$

matrix: Negative trace : still not stable (Blow up)

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \quad \lambda_1 = -2, \lambda_2 = 1$$

Trace $= -1$

$$\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

Key

when $Re(\lambda_1), Re(\lambda_2) < 0$

Trace $= -ve$

$\det = \lambda_1 \lambda_2 = +ve$.

Stability picture

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \lambda_1 = -2, \ \lambda_2 = 1$$

$$\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} = -2 , \qquad \text{Trace} = -2$$

when both determinant & trace are negative - solution blew up. (presence of $\lambda_2 = 1$)

Stable

positive determinant & negative trace → yields stable solution.

---

continuing $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

In to matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad [\text{As } t = 0]$$

$S \rightarrow$ Eigen vector matrix.

$$SC = u(0)$$

Note: Each pure exponential goes on its own way once started from u_0. what about the relation b/w them and u_0 ?

In terms of λ and S

$\frac{du}{dt} = Au \longrightarrow$ Idea: uncoupling by diagonalization.

matrix A couples them

Eigen vectors - uncouple them

Set $u = Sv$ → Diagonalize them

$$\frac{d}{dt}(Sv) = ASv$$

$$S\frac{dv}{dt} = ASv$$

$$S \frac{dv}{dt} = ASv$$

↳ Eigen vector matrix.

$$\frac{dv}{dt} = S^{-1} A S V$$

$$\frac{dv}{dt} = \Lambda V \longrightarrow \left.\begin{aligned}\frac{dv_1}{dt} &= \lambda_1 v_1 \\ \frac{dv_2}{dt} &= \lambda_2 v_2 \\ &\vdots\end{aligned}\right) \text{ particular ones from } \frac{dv}{dt} = \Lambda v$$

$$\left.\begin{aligned} V(t) &= e^{\Lambda t} v(0) \\ &\boxed{u(t) = S e^{\Lambda t} S^{-1} u(0)}\end{aligned}\right) \text{ solutions to } \frac{dv}{dt} \text{ \& } \frac{du}{dt}$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

what does this mean?

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

Matrix exponential:

$e^{At} \rightarrow$ powerseries (defined by)

$$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$\boxed{e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots + \frac{(At)^n}{n!} + \cdots} \longrightarrow ①$$

$$= \sum_{0}^{\infty} \frac{x^n}{n!} \quad \text{[one type of taylor's series]}$$

Geometric series which has $\sum_{0}^{\infty} x^n$

$$\boxed{(I - At)^{-1} = I + At + (At)^2 + (At)^3 + \cdots + (At)^n} \longrightarrow ②$$

(If t is small $\rightarrow$ then Ans $\approx I + At$)

① $\rightarrow$ Always converge (due to $n!$)

② $\rightarrow$ Blew up when t is large.

$$A = S\Lambda S^{-1}$$

$$e^{At} = I + \underbrace{S\Lambda S^{-1}}_{A} t + \frac{S\Lambda^2 S^{-1} t^2}{2} + \frac{S\Lambda^3 S^{-1} t^3}{6} + \dots + \frac{S\lambda^n S^{-1} t^n}{n!}$$

$$e^{At} = SS^{-1} + S\Lambda S^{-1} t^2 + \dots + \frac{S\Lambda^n S^{-1} t^n}{n!}$$

$$e^{\lambda t} = 1 + \lambda t + (\lambda t)^2 + (\lambda t)^3 + \dots (\lambda t)^n$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$ → Does this always work?

'Inveresible' 'A - diagonalizable'

'n-independent Eigen vectors'

$$\frac{dv}{dt} = S^{-1} A S v \qquad \frac{du}{dt} = Au$$

$$\frac{dv}{dt} = \Lambda v$$

$$v(t) = e^{\Lambda t} v(0) \qquad u(t) = e^{At} u(0)$$

$$\boxed{u(t) = S e^{\Lambda t} S^{-1} u(0)}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

$$\therefore \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

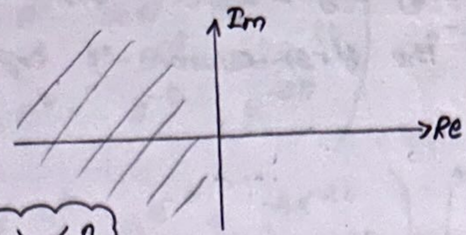
<u>when did this exponential matrix goes to zero:</u>

$e^{\Lambda t}$ → smaller & smaller.

$\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ → every Eigen value negative.

So exponential will be smaller.

(goes to ...



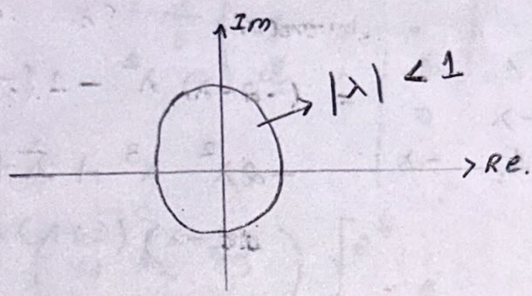
$\boxed{Re\ \lambda < 0}$

$e^{\Lambda t}$ goes to zero.

Then $e^{At} = S\, e^{\Lambda t} S^{-1} \rightarrow$ goes to zero.

Eigen values for powers:

powers of a matrix goes to zero.



'stability region for powers'.

II - order eqn $\quad y'' + by' + ky = 0$

↳ 2nd order eqn into 1st order 2x2 matrix

$$u = \begin{pmatrix} y' \\ y \end{pmatrix}$$

$$u' = \begin{pmatrix} y'' \\ y' \end{pmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{pmatrix} y' \\ y \end{pmatrix}$$

$y'' = -by' - ky \rightarrow (1)$

$y' = y' \rightarrow (2)$

so we can convert 5th order differential eqn in 5 x 5 matrix.

$y' = y'$
↳ Trivial case

$$\begin{bmatrix} - & - & - & - & - \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}$$

5th order → 1st row → Equation
4 trivial case like $y' = y'$

Solve the diff eqn $y''' + 2y'' - y' - 2y = 0$ for the general soluto. what is matrix A? Find the first column of $\exp(At)$

solu:

$$y''' + 2y'' - y' - 2y = 0$$

$$y''' = -2y'' + y' + 2y$$

$$u = \begin{pmatrix} y'' \\ y' \\ y \end{pmatrix}, \quad u' = \begin{pmatrix} y''' \\ y'' \\ y' \end{pmatrix} = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} y'' \\ y' \\ y \end{pmatrix}$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

Eigen values:

$$\det \begin{vmatrix} -2-\lambda & 1 & 2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = (-2-\lambda)\lambda^2 - 1(-\lambda - 2) = 0$$

$$-2\lambda^2 - \lambda^3 + \lambda + 2 = 0$$

$$(1-\lambda)(1+\lambda)(2+\lambda) = 0$$

$$\lambda = -1, 1, -2$$

Eigen vectors:

$\lambda = -1$

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} (x) = 0 \qquad x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} (x) = 0 \qquad x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} (x) = 0 \qquad x = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

$$u(t) = c_1 e^{t} x_1 + c_2 e^{-t} x_2 + c_3 e^{-2t} x_3$$

From last column. $(1)\ (1)\ (1)$

$$\boxed{y(t) = c_1 e^{t}(1) + c_2 e^{-t} + c_3 e^{-2t}} \rightarrow \therefore \text{we are not provided with initial condt.}$$

$$\exp(At) = S e^{\Lambda t} S^{-1}$$

$$S = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad e^{\Lambda t} = \begin{pmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{-2t} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{t} & e^{-t} & 4e^{-2t} \\ e^{t} & -e^{-t} & -2e^{-2t} \\ e^{t} & e^{-t} & e^{-2t} \end{pmatrix} S^{-1}$$

$$e^{At} = \begin{pmatrix} e^{t} & e^{-t} & 4e^{-2t} \\ e^{t} & -e^{-t} & -2e^{-2t} \\ e^{t} & e^{-t} & e^{-2t} \end{pmatrix} \begin{pmatrix} 1/6 & \cdot & \cdot \\ -3/6 & \cdot & \cdot \\ 2/6 & \cdot & \cdot \end{pmatrix}$$

$$S^{-1} = \frac{1}{\det S} C^{T} = \frac{1}{6} \begin{pmatrix} 1 & \cdot & \cdot \\ -3 & \cdot & \cdot \\ 2 & \cdot & \cdot \end{pmatrix} \qquad \det S = 6$$

our area: first column of e^{At}

$$= \frac{1}{6} \begin{pmatrix} 1 & \cdot & \cdot \\ -3 & \cdot & \cdot \\ 2 & \cdot & \cdot \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} e^{t} & & \\ & e^{-t} & \\ & & e^{-2t} \end{bmatrix} S^{-1}$$

$$= \begin{pmatrix} e^{t} x_1 & e^{-t} x_2 & e^{-2t} x_3 \end{pmatrix} \begin{pmatrix} 1/6 & \cdot & \cdot \\ -3/6 & \cdot & \cdot \\ +2/6 & \cdot & \cdot \end{pmatrix}$$

1st column of $\exp(At)$ $= \begin{bmatrix} \frac{e^{t}}{6} x_1 & \cdot & \cdot \\ \frac{e^{-t}}{2} x_2 & \cdot & \cdot \\ \frac{e^{-2t}}{3} x_3 & \cdot & \cdot \end{bmatrix}$

Markov matrices; Fourier series

Like differential eqn, Markov matrices describes changes over time. once again, the eigen values & Eigen vectors describe the long term behaviour of the system.

Fourier series - Describe periodic functions as points in an infinite dimensional vector space.

Applications of Eigen values.

$$A = \text{markov matrix} = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

Properties: 1) Every Entry is greater than or equal to zero.

2) Add down the columns gives 1.

"markov matrices are related to probabilities - which is never negative"

Powers of my matrix → Also markov matrices.

Steady State:

$\lambda = 0$ → Diff equ

Power case → Eigen value of 1. (steady)

∴ Power case $\lambda < 1$ (vanishing)

$\lambda > 1$ (Blow up)

markov: $\lambda = 1$ (steady)

Sum of column elements = 1

(Guarantees Eigen value $\lambda = 1$)

1) $\lambda = 1$ is an Eigen value.

2) All others $|\lambda_i| < 1$.

Power case

$$u_k = A^k u_0$$

$= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots$

$\lambda_1 = 1$, $\lambda_2, \lambda_3 < 1$

↓ (vanish)

stable

Eigenvectors:

$x_1 \geq 0$ [steady state is +ve]

↳ (Rare) → But not negative.

$\lambda = 1$

$$\begin{pmatrix} 0.1-\lambda & 0.01 & 0.3 \\ 0.2 & 0.99-1 & 0.3 \\ 0.7 & 0 & 0.4-\lambda \end{pmatrix} x = \begin{pmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{pmatrix} x = 0$$

"Singular?" → columns add up to zero.

"why? - dependent (Rows) — Linearly.

eg: $R_1+R_2+R_3=0$

Because (1 1 1) is in the null space of A^T.

$X = n(A) = \begin{pmatrix} 6 \\ 330 \\ 7 \end{pmatrix} \geq 0$ (or) $\begin{pmatrix} 0.6 \\ 33 \\ 0.7 \end{pmatrix}$

↳ steady state

$4.2 - 4.2 = 0$

$1.2 + 2.1 = 3.3 - 330(0.01) = 0$

$-5.4 + 2.1 = -3.3 + 330(0.01) = 0.$

Note: Eigen values of A & A^T are same!

$\therefore \det A = \det A^T$

But Eigen vectors are different.

Application

1) $u_{K+1} = Au_K$

A → markov matrix.

Example: 2×2 → 2 states (california, massachusetts) [population]

A → In a year some movement happened.

"cal → mas
mas → cal
stayed there") Add up to one. [All people]

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=K+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_K$$

0.9 cal → stayed there
0.1 cal → moved to masschusetts

0.8 mas → stayed
0.2 mas → moved to cal

↳ A is the same forever.
(Limitation)

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

ucal → start growing
umass → start reducing

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix} \quad \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix} = \begin{bmatrix} 340 \\ 660 \end{bmatrix}$$

$\therefore 0.2 \, (800) = 160$

$0.1 \, (200) = 20$

Eigen values:

$$\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$$

$\lambda_1 = 1$

$\lambda_1 + \lambda_2 = \text{Trace}(A)$

$1.7 - 1 = \lambda_2$

$|A| = 1 \times 0.7 = 0.7$

$\lambda_2 = 0.7$

when $\lambda = 1$

$$\begin{pmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{pmatrix} x = 0$$

$$x_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

when $\lambda = 0.7$

$$\begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{pmatrix} x = 0$$

$$x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

population at ∞:

$$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2$$

$$= c_1 (1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 (0.7) \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

After 100 times:

$$u_k = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 (0.7)^k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\therefore u_0 = \begin{pmatrix} 0 \\ 1000 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 (0.7)^0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$c_1 + c_2 = 1000$

$2c_1 - c_2 = 0$

$2c_1 = c_2$

$c_1 = c_2/2$

$\frac{c_2}{2} + c_2 = 1000$

$c_2 = \frac{2}{3} \times 1000$

$c_1 = \frac{1000}{3}$

$$u_k = \frac{1000}{3}\begin{pmatrix}2\\1\end{pmatrix} + \frac{2000}{3}(0.7)^k\begin{pmatrix}-1\\1\end{pmatrix}$$

↳ Stable. ↳ Disappearing

Adding: modelling people movement without lose or gain.

Comment:

In Electrical Engineering: Some times we are preferred to work with rows: But Eigen Values → column vectors.

vector × matrix → we use transpose of matrix

(Rows add up to 1).

Projections: Fourier Series

Projections with orthonormal basis:

$q_1, q_2, \dots, q_n$ → Basis of $\mathbb{P}$ (projection matrix)

Any vector

$$v = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$$

(combinations of basis (orthogonal))

what's $x_1, x_2, \dots, x_n$

'Taking Inner products'

$$q_1^T v = x_1 q_1^T q_1 + x_2 q_1^T q_2 + \dots + x_n q_1^T q_n$$

$$= x_1(1) + 0 + 0 \dots$$

$$q_1^T v = x_1$$

$$v = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{v = Q \quad X}$$

Solution: $X = Q^{-1}v = Q^T v$ → $x_1 = q_1^T v$, $x_2 = q_2^T v$

Columns are orthonormal: $Q^T = Q^{-1}$

Key: orthonormal v's

Fourier series: Built on orthonormality.

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

$$= a_0 + \sum_{n=1} (a_n \cos nx + b_n \sin nx)$$

(infinite)

sines & cosines are orthogonal.

(Instead of vectors - fourier understood by using functions)

Basis:

$1, \sin x, \cos x, \cos 2x, \sin 2x, \ldots$

success: orthogonal.

orthogonal vectors:

$$\boxed{y^T x = 0}$$

Dot products of functions?

$\cos x \cos 2x = 0$
$\cos x \sin x = 0$) How?

vectors

$$v^T w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

$$(v_1 \cdots v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

Functions

$$f^T g = \int_0^{2\pi} f(x)\, g(x)$$

'continuous'

Limits = ? (Periodic) 2π-period.

$$\boxed{\therefore \; f(x) = f(x + 2\pi)}$$

'Inner products'

$$\int_0^{2\pi} \sin x \cos x \, dx = 0$$

How do I get a_1?

'Inner product of everything with $\cos x$'

$$\int_0^{2\pi} f(x) \cos x \, dx.$$

only term survives is $a_1 \cos x$. $= a_1 \int_0^{2\pi} (\cos x)^2 dx.$ $= \pi$

$= a_1 \pi$

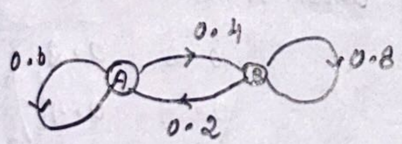
$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x \, dx$$

Co. efficient of Fourier series

'Expansion in orthonormal basis'

Recitation

A particle jumps b/w positions A and B with the following probabilities



Starts at A, what's the probability it is at A and B after i) 1 step ii) n step iii) ∞ steps

soln:

$$\begin{array}{ccc} & A & B \\ A = & \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} & \begin{matrix} A \\ B \end{matrix} \end{array}$$ positions

Stay at A $\rightarrow$ 0.6

Stay at B $\rightarrow$ 0.8

move to A $\rightarrow$ 0.2

move to B $\rightarrow$ 0.4.

* All entries ≥ 1

* sum of column entries $= 1$.

$$P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So problem is

$$P = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$P_1 = A P_0$$

$$= \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

ii) n steps:

$$P_1 = AP_0 , \quad P_2 = AP_1 , \quad P_n = AP_{n-1}$$

$$P_2 = A(AP_0) \qquad P_n = A^n P_0$$

$$P_2 = A^2 P_0$$

Recalling

$$A = S \Lambda S^{-1}$$

$$A^n = S \Lambda^n S^{-1}$$

Eigen value

$\lambda_1 = 1$ (one)

$\lambda_2 = \text{Trace} - \lambda_1$

$= 1.4 - 1$

$\boxed{\lambda_2 = 0.4}$

det $= \lambda_1 \lambda_2 = 1 \times 0.4 = 0.4.$

Eigen vectors

when $\lambda = 1$

$$\begin{pmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{pmatrix} x_1 = 0$$

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

when $\lambda = 0.4$

$$\begin{pmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{pmatrix} x_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \cdot -\frac{1}{3}$$

$$A^n = -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$P_n = A^n P_0 = S \Lambda^n S^{-1} P_0$$

$$P_n = -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 0.4^n \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P_n = -\frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}1^n & 0\\ 0 & 0.4^n\end{bmatrix}\begin{pmatrix}-1\\ -2\end{pmatrix}$$

For simplifying work
Starting from right to left

$$= -\frac{1}{3}\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{pmatrix}-1^n\\ -2(0.4)^n\end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix}-1^n - 2(0.4)^n\\ -2(1)^n + 2(0.4)^n\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}2(0.4)^n + 1\\ -2(0.4)^n + 2\end{pmatrix}$$

iii) $n = \infty$

As $n \to \infty$ $2(0.4)^n \to$ vanishes

$$P_\infty = \frac{1}{3}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}1/3\\ 2/3\end{pmatrix} \longrightarrow \text{stable } (\lambda_1 = 1) \dashrightarrow \text{uniform distribution.}$$

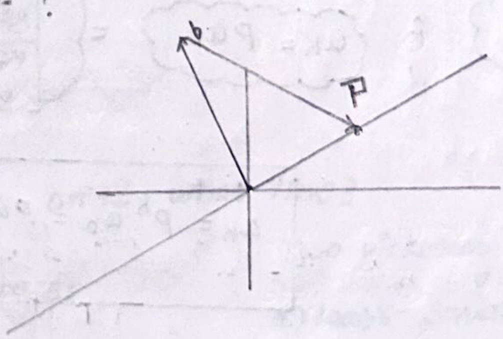
Exam review - 2

①	② det A	③ Eigen value
$Q = [q_1 \cdots q_n]$	properties	$Ax = \lambda x$
Projection - Least squares	Big formula ($n!$ terms, $\pm$)	$\det(A - \lambda I) = 0$
Gram - Schmidt	cofactors / A^{-1}	Diagonalize $S^{-1}AS = \Lambda$
		powers A^k

$\boxed{Q^TQ = I}$

$a = \begin{bmatrix}2\\ 1\\ 2\end{bmatrix}$, projection matrix P ?

'Line'



Solu:

$$P = A(A^TA^{-1})A^T$$

$$P = \frac{aa^T}{a^Ta} \quad [\text{Just a line}]$$

$$= \frac{\begin{pmatrix}2\\1\\2\end{pmatrix}(2\ 1\ 2)}{(2\ 1\ 2)\begin{pmatrix}2\\1\\2\end{pmatrix}} = \frac{1}{9}\begin{pmatrix}4 & 2 & 4\\ 2 & 1 & 2\\ 4 & 2 & 4\end{pmatrix}$$

Eigen values of P: (Rank = 1)

column space: $C\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\lambda = 0, 0, 1$ [Trace $= \lambda_1 + \lambda_2 + \lambda_3$]

Eigen vector = ? → of P that has Eigen value 1

• vector doesn't move'

$Ax = \lambda x$

↳ here for P

Relating P and λ:

$\therefore$

$P = \frac{aa^T}{a^Ta}$

$Pa = \frac{aa^Ta}{a^Ta}$

$\lambda = 1$

$Pa = a$

Solve difference eqn

$u_{k+1} = Pu_k$

$u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$, find u_k

solu:

$u_1 = Pu_0$

$= \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}$

(or) $u_1 = Pu_0 = a\,\frac{a^Tu_0}{a^Ta} =$

$= a\left(\frac{(2\ 1\ 2)\begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix}}{9}\right)$

$= a\left(\frac{27}{9}\right)$

$= \underbrace{3a}_{\hat{x}}$

$= \begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$

$u_2 = Pu_1 = P^2u_0$

$u_k = P^k u_0$

$P^k = P$ $\quad u_k = Pu_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$

$$\boxed{u_k = P^k u_0 = Pu_0 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}}$$

$$\boxed{\begin{aligned} &\therefore u_0 = c_1x_1 + c_2x_2 + c_3x_3 + \dots \\ &A^k u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \dots + c_n\lambda_n^k x_n \end{aligned}}$$

→ Normal case $A \neq P$

---

u_0 = vector combination of Eigen vectors.

$u_{k+1} = Au_k$

$u_1 = Au_0$

$u_2 = Au_1 = A(Au_0) = A^2u_0$

$u_k = A^k u_0$

$$A u_0 = A c_1 x_1 + A c_2 x_2 + \cdots$$

$$\boxed{A x_1 = \lambda_1 x_1} \quad A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots$$

$$\boxed{A^K u_0 = c_1 \lambda_1^K x_1 + c_2 \lambda_2^K x_2 + \cdots}$$

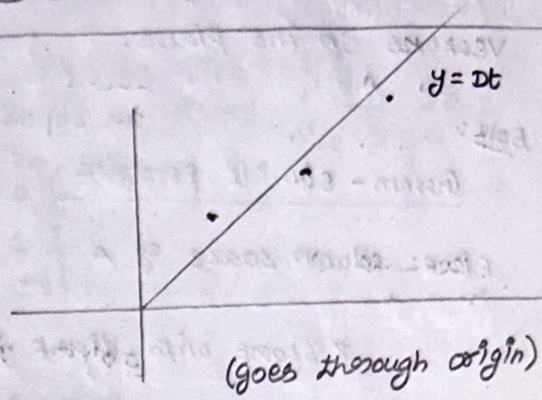
---

2) <u>Data points:</u>

$t = 1, \; y = 4$

$t = 2, \; y = 5$

$t = 3, \; y = 8$



(goes through origin)

Only one unknown, D.

<u>Solu:</u>

1. $D = 4$
2. $D = 5$
3. $D = 8$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} D = \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix}$$

$$A x = b$$

<u>Find the best D:</u>

$$A^T A \hat{D} = A^T b$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 14.$$

$$14 \hat{D} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix}$$

$$\boxed{\hat{D} = \frac{38}{14}}$$

b) what vector did we project onto what line?

Two pictures

1) Least-Square

2) projection. (line) on to Column Space of A

projecting b on to the Column Space:

$$\boxed{p = Pb}$$

$$p = a \frac{a^T b}{a^T a}$$

$$= \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix}}{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot (4 + 10 + 24)}{1 + 4 + 9}$$

$$= \frac{38}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\hat{x} \to$ Best combination to give projection

3) $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ determine a plane.. Find two orthogonal vectors in the plane.

solu:

Gram - schmidt process:

Plane: column space of A

* Start with first vector: $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$B \perp A \to$ Aim

$$B = a_2 - \frac{a_1^T a_2}{a_1^T a_1} a_1$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(1 \;\; 2 \;\; 3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{(1 \;\; 2 \;\; 3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\cancel{6}^{3}}{\cancel{14}\,7} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3/7 \\ 1 - 6/7 \\ 1 - 9/7 \end{pmatrix} = \begin{pmatrix} 4/7 \\ 1/7 \\ -2/7 \end{pmatrix}$$

Eigen

4×4 matrix $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (condition - matrix is invertible)

solu:

Invertible if : None of them are zero.
(No zero Eigen values)

$$\det A^{-1} = \left(\frac{1}{\lambda_1}\right)\left(\frac{1}{\lambda_2}\right)\left(\frac{1}{\lambda_3}\right)\left(\frac{1}{\lambda_3}\right)$$

$$\text{Trace}(A+I) = (\lambda_1 + 1) + (\lambda_2 + 1) + (\lambda_3 + 1) + (\lambda_4 + 1)$$

$$= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4.$$

Determinant

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Let $D_n = \det A_n$

Solu:
use cofactors to show that $D_n = \_\_ D_{n-1} + \_\_ D_{n-2}$

$$= 1 \underbrace{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}}_{D_{n-1}} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

Further

$$= 1 \underbrace{\left(1(0) - 1(1)\right)}_{D_{n-1}} - 1 \underbrace{\left(1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right)}_{D_{n-2}}$$

$$= 1(-1) - 1\left(1(1-1)\right)$$

$$= -1.$$

$$\therefore \quad D_n = D_{n-1} - D_{n-2} \rightarrow (1)$$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$D_3 = ,$

Set as a system:

$D_{n-1} = D_{n-1} \rightarrow (2)$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

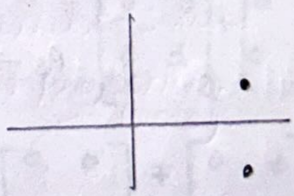
$$-\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$\lambda_1 = \frac{1 + \sqrt{3}\, i}{2}$$

$$\lambda_2 = \frac{1 - \sqrt{3}\, i}{2}$$



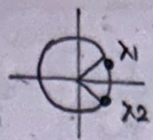
Trace = (-)ve

determinant = (+)ve.

magnitude:

$$|\lambda| = \sqrt{\frac{1+3}{4}}$$

$$|\lambda| = 1.$$



$\theta = \tan^{-1}(\sqrt{3})$

$= \pi/3$

So Answers will be $e^{i\pi/3}$ and $e^{-i\pi/3}$

'stability: ?'

$\lambda_1^6 = \lambda_2^6 = 1$ → what about matrix?

$(A^6) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

'periodic with period 6'

$D_1 = 1,\ D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,\ D_3 = 1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1$

Here, $D_n = D_{n-1} - D_{n-2}$

$D_4 = -1 - 0$

$D_4 = -1$

$D_5 = -1 + 1 = 0$

$\boxed{D_6 = 0 + 1 = 1}$

---

family of symmetric matrices

$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} = A_4^T, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

Q) Find P on to the column space! of A^3.

$A_3 \to$ singular.

$P = A(A^TA)^{-1}A^T$

Eigen values:

$|A_3 - \lambda I| = 0$

$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$

$-\lambda^3 - 4\lambda - 1(-\lambda) = 0$

$\lambda^3 - 5\lambda = 0$

$\lambda = 0,\ (\lambda^2 - 5 = 0)$

$\lambda^2 = 5$

$\boxed{\lambda = \pm\sqrt{5}}$

$\boxed{\text{Trace: } 0 + \sqrt{5} - \sqrt{5} = 0}$

Determinant $= 0$

2) c) what's P on to the column space of A^4?

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix} \rightarrow \text{Invertible}$$

So it's column space is R^4, → Entire (b is also in that space)

↳ $P = I$

How invertible:

$|A_4| = -1(-9) = 9$

$|A_4| = 9$

Summary

A_4, A_2 → even numbers (Invertible)

A_1, A_3 → odd numbers (singular).

Recitation

1) Find all the non-zero terms in the big formula $\det A = \Sigma \pm a_{1\alpha}\, a_{2\beta}\, a_{3\gamma}\, a_{4\delta}$ and compute $\det A$.

2) Find cofactors $C_{11}, C_{12}, C_{13}, C_{14}$.

3) Find column 1 of A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}$$

Solu:

$\det A = \Sigma \pm a_{1\alpha}\, a_{2\beta}\, a_{3\gamma}\, a_{4\delta}$

↓

$4! = 24$ factors (meaningful)

(we care about non-zero terms)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$$

↓

In row 3 & 4, we will have 4 combinations like this - out of 4 two have $\begin{bmatrix} 0 & \cdot \\ 0 & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 0 \\ \cdot & 0 \end{bmatrix}$ → nullity. So two here $\begin{bmatrix} 9 & 0 \\ 0 & 12 \end{bmatrix}, \begin{bmatrix} 0 & 10 \\ 11 & 12 \end{bmatrix}$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 12 \end{bmatrix}, \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 11 & 0 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\det = 0 \qquad \underbrace{\det \neq 0 \qquad \det = 0 \qquad \det \neq 0}_{\text{2 choices.}}$$

First two rows:

$$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix} \rightarrow \text{delta} \rightarrow \text{non-negative}$$

$$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix} \rightarrow \text{delta} \rightarrow \text{zero}$$

$$\begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix} \rightarrow \text{delta} \rightarrow \text{zero}$$

Same zero for $\begin{bmatrix} 3 & 4 \\ 7 & 8 \end{bmatrix}$ entries

so we can get $\det \neq 0$ for only $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$

$$\text{2choices} \leftarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & 2 \\ 5 & 6 \end{matrix}} & \begin{matrix} 3 & 4 \\ 7 & 8 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 9 & 10 \\ 11 & 12 \end{matrix}} \end{bmatrix}$$

$$\downarrow$$

2choices

$4! = 24$ reduced to 4.

computing

$$\det A = \underset{\substack{\downarrow \ \downarrow \ \downarrow \ \downarrow \\ A_{11}\ A_{22}\ A_{33}\ A_{44}}}{(1, 2, 3, 4)} + \underbrace{\overset{A_{11}\ A_{12}\ A_{34}\ A_{43}}{(1, 2, 4, 3)} + \overset{A_{12}\ A_{21}\ A_{33}\ A_{44}}{(2, 1, 3, 4)} + \overset{A_{12}\ A_{21}\ A_{34}}{(2, 1, 4, 3)}}_{\text{Combinations.}}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 12 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 11 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 12 \end{vmatrix}$$

$$+\begin{vmatrix} 0 & 2 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 11 & 0 \end{vmatrix}$$

$$= A_{11}(A_{22}(A_{33}A_{44})) + (A_{11}(A_{22}(A_{34}A_{43})))$$
$$+ (A_{12}(A_{21}(A_{33}A_{44}))) + (A_{12}(A_{21}(A_{34}A_{43})))$$

$$= A_{11}A_{22}A_{33}A_{44} - A_{11}A_{22}A_{43}A_{34} + A_{21}A_{12}A_{43}A_{34}$$
$$- A_{21}A_{12}A_{33}A_{44}$$

$$= (1\times 6\times 9\times 12) - (1\times 6\times 11\times 10) - (5\times 2\times 9\times 12) + (5\times 2\times 11\times 12)$$

$$= 8$$

Key: Identify the non-zero entries. (Take their combination)

$$C_{11} = \det\begin{pmatrix} 6 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{pmatrix} = 6\times(9\times 12 - 10\times 11)$$
$$= -12$$

Factor.

$$C_{12} = -\det\begin{pmatrix} 5 & 7 & 8 \\ 0 & 9 & 10 \\ 0 & 11 & 12 \end{pmatrix} = -[5(9\times 12 - 10\times 11)]$$
$$= +10$$

$$C_{13} = \det\begin{pmatrix} 5 & 6 & 8 \\ 0 & 0 & 10 \\ 0 & 0 & 12 \end{pmatrix} = 5(0) = 0$$

$$C_{14} = \det\begin{pmatrix} 5 & 6 & 7 \\ 0 & 0 & 9 \\ 0 & 0 & 11 \end{pmatrix} = 0$$

verify: det A : rows · cofactors

$$: 1(-12) + 2(+10)$$
$$: 8$$

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3) $$A^{-1} = \frac{1}{|\det A|} C^T = \frac{1}{8}\begin{pmatrix} -12 \\ 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 5/4 \\ 0 \\ 0 \end{pmatrix}$$