$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → This guy is not in the $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

2 families.

$$M \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} M^{-1} = 4 M I M^{-1}$$

$$= 4 M M^{-1} = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$ for any invertible matrix M.

$$\boxed{\therefore B = MAM^{-1}}$$

"Lone family: $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is the only member of this family.

Other big family

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → Is a big family.

Conclusion

"Two families with Eigen values 4 and 4"

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ is not in the family of $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

∴ Small family has only $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ alone.

The only matrix similar to $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is the $4I \rightarrow \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → All other matrices having Eigen values 4 and 4.

↳ Not diagonalizable "Only one Eigen Vector"

$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ → Jordan form

"Most closely diagonalizable"

"Not completely"

Jordan form: 'Most diagonalizable form - as best as possible'

It's good for repeated Eigen value case not for a general matrix (changing a number in matrix will completely change Eigen values, vectors, rank and all)

more members:

$$\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 17 & 4 \end{bmatrix}, \begin{bmatrix} a & b \\ \frac{(8a-a^2-16)}{b} & 8-a \end{bmatrix}$$

↳ If it's diagonalizable, the diagonalized form will be $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.

Same λ's → Same no. of independent Eigen Vectors.

'But this λ count is not just enough to count Eigen vectors'

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → Eigen values = 0, 0, 0, 0

Eigen Vectors (null space) = 2 (Rank)

$Ax = 0x$

$\therefore \dim(N(A)) = 2$

$$\begin{bmatrix} 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → Eigen values $\lambda = 0, 0, 0, 0$

Same no. of Eigen vectors.

↳ similar to the one above but not beautiful.

'Jordan put a one above the diagonal for every missing eigen vector'

Here 2 are missing, we got 2.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → $\lambda = 0, 0, 0, 0$

Rank = 2, 2 missing, 2 Eigen vectors.

these could be

"A count of Eigen vectors looks like similar but they are not"

$$\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \& \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

2 (2 by 2) blocks — Are not similar — 1 (3 by 3) block.

Jordan block J_i:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ 0 & & & \ddots \end{bmatrix}$$

'Jordan block has one Eigen vector'

$\lambda_i \rightarrow$ repeated (one Eigen vector)

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2 blocks (2 Eigen vectors) — 2 blocks (2 Eigen vectors)

2 Blocks are not of same size.

Jordan's theorem:

Every square matrix A is similar to a Jordan matrix J.

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_d \end{bmatrix}$$

$J_1, J_2, \ldots \rightarrow$ Jordan blocks

No. of Jordan blocks = Number of Eigen vectors ummary:

1) $\lambda \to$ non repeated (independent Eigen vectors) $\to$ means diagonalizable (Jordan block forming Jordan matrix is diagonalizable)

$$J = \Lambda$$

2) $\lambda \to$ Repeated (not independent Eigen vectors) $\to$ Not diagonalizable A is similar to a Jordan matrix J.

Recitation

Similar matrices:

Which are the following statements are true? Explain

a) If A and B are similar matrices, then $2A^3 + A - 3I$ and $2B^3 + B - 3I$ are similar.

b) If A and B are 3×3 matrices with Eigen values $1, 0, -1$, then A and B are similar.

c) The matrices $J_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $J_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Solu.

a) $MAM^{-1} = B$

$$M(2A^3 + A - 3I)M^{-1} = 2(MAM^{-1}\ MAM^{-1}\ MAM^{-1}) + MAM^{-1} - 3MM^{-1}$$

$$= 2B^3 + B - 3I$$

'True'

b) Distinct Eigen values $\longrightarrow$ 'Diagonalizable'

$A = S\Lambda S^{-1}$, $\Lambda = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$B = T \Lambda T^{-1}$

S, T → Eigen vector matrices of A and B

∴ $\Lambda \rightarrow$ same for A and B

• Similarity is a transitive relation.

If it hold for two objects, then it also hold for third one too (example)

$$(TS^{-1}) A (TS^{-1})^{-1} = B$$

Both A and B are similar to Λ.

∴ A and B are similar to each other.

c) False – • similarity preserves Eigen vectors & Eigenvalues

$$J_1 + I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow N(J_1) = 1 \text{ (Rank} = 2)$$

1 independent Eigen vectors

$$J_2 + I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad N(J_2) = 2 \text{ (Rank} = 1)$$

So they can't possibly be similar.

Note: $B = M^{-1} A M$

Singular value decomposition

Lecture-30

If A is both symmetric & positive definite, there is an orthogonal matrix Q for which $A = Q\Lambda Q^T$. Here Λ is the matrix of eigen values. Singular value decomposition lets us write any matrix A as a product

$$U \Sigma V^T$$

U, V – orthogonal, Σ – Diagonal matrix whose non-zero entries are square roots of the Eigen values of A^TA.

The columns U and V give bases for the four fundamental subspaces.

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

'SVD – Best factorization'

$A = S\Lambda S^{-1}$

$\boxed{A = Q\Lambda Q^T \rightarrow \text{In case of orthogonal matrix } Q}$

$\therefore$ Because of symmetric $\rightarrow$ Eigen vectors are orthogonal.

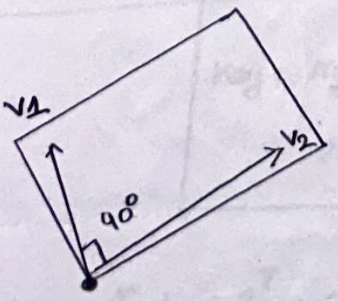
In usual case:

$A = S\Lambda S^{-1}$

↳ Not orthogonal (Eigen vectors are not orthogonal)

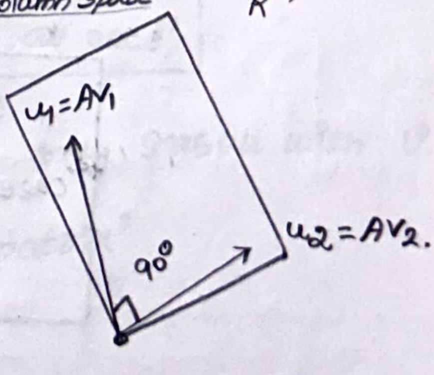
Row space

R^n



column space

R^m



Looking for an orthogonal basis at Row space to knocked over as an orthogonal basis at the column space.

1) we can find an orthogonal basis for the row space

2) Any of the orthogonal basis multiplied by A need to give an orthogonal vector of column space.

'special set up'

A takes these row basis (orthogonal) of Row space in to the orthogonal basis of column space.

<u>Null spaces:</u>

zeros of Σ will take care of them.

<u>workout:</u>

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots \end{bmatrix}$$

$u_1 \downarrow$ orthogonal

<u>making it orthonormal:</u>

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots \end{bmatrix}$$

$\sigma_1, \sigma_2, \cdots \rightarrow$ stretching factors to make it orthonormal

$$A\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \cdots \sigma_r \end{bmatrix}$$

$\therefore$ $Av_1 = \sigma_1 u_1$

$AV = U\Sigma \longrightarrow$ Matrix representation

symmetric +ve definite

case: where $AV = V\Sigma$

$V = U$

$A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$, $v_1, v_2 \rightarrow$ in row space R^2 | $\sigma_1 > 0$

$u_1, u_2 \rightarrow$ in column space R^2 | $\sigma_2 > 0$

Completing

$A \begin{bmatrix} v_1 & v_2 \cdots & v_r & v_{r+1} \cdots v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \cdots & u_r & u_{r+1} \cdots u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \end{bmatrix}$

$\downarrow$

dim (Rowspace) = dim (column space) = r

So we have vectors up to v_r, then the remaining vectors of R^n will be in the Null space.

'we can complete that with zeros'

Problem:

our case: $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \rightarrow$ we can't use Eigen vectors.

$\therefore$ They are not orthogonal.

'Not symmetrical'

Solution: make orthogonal guys to make it work!

$Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

$AVV^{-1} = U\Sigma V^{-1}$

$AI = U\Sigma V^{-1}$

$A = U\Sigma V^{-1}$

$\boxed{A = U\Sigma V^T}$

$U, V \rightarrow$ orthogonal matrices

'we need to find both of them at once'

Key: make U disappear only result with V.

'A^TA - Great matrix'

$A^TA = A^TU\Sigma V^T$

$A^TA = (U\Sigma V^T)^T U\Sigma V^T$

$A^TA = V\Sigma^T u^T u \Sigma V^T$

$A^TA \to$ positive definite - atleast +ve semidefinite.

$\therefore$ In orthogonal case:

$u^T = u^{-1}$

$\boxed{\therefore\ u^Tu = I}$

$\Sigma \to$ diagonal matrix

$\Sigma = \Sigma^T$

$A^TA = V\Sigma^T\Sigma V^T$

$$A^TA = V\begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots & \\ & & & \sigma_r^2 \end{pmatrix}V^T$$

we need to find V alone

other way: Knock out Vs and use u s by

$AA^T = (u\Sigma V^T)(u\Sigma V^T)^T$

Looking at

$$A^TA = V\begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}V^T$$

similar to $Q\Lambda Q^T$

so $\Sigma \to$ (In u knocked out case) $\to$ Eigen values of A^TA

$\Sigma \to$ (In V knocked out case) $\to$ Eigen values of AA^T

what about $\Sigma \to$ Square positive root of $\begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$

Compute A^TA & Eigen values & vectors

$$\begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

Eigen vectors:

$$\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \qquad \text{Eigen values } \lambda = 32, 18$$

making them orthonormal:

$$32\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}, \quad 18\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

$$\underset{A}{A = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}} = \underset{u}{\begin{bmatrix} & \\ & \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix}\sqrt{32} & \\ & \sqrt{18}\end{bmatrix}} \underset{V^T}{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}^T}$$

Find u's by AA^T method:

$$AA^T = u\Sigma V^T V \Sigma^T u^T$$

$$AA^T = u\Sigma^2 u^T$$

$$\begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}\begin{bmatrix}4 & -3\\4 & 3\end{bmatrix} = \begin{bmatrix}32 & 0\\0 & 18\end{bmatrix}$$

Eigen vectors: AA^T $\begin{pmatrix}1\\0\end{pmatrix}$ = Eigen value = 32

Eigen vector: AA^T $\begin{pmatrix}0\\1\end{pmatrix}$ = Eigen value = 18

$$\boxed{\text{Eigen values of } A^TA = \text{Eigen values of } AA^T}$$

$\hookrightarrow \therefore$ Eigen values of $AB = BA$.

$$\therefore\ A = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix} = \underset{u}{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}} \underset{\Sigma}{\begin{bmatrix}\sqrt{32} & \\ & \sqrt{18}\end{bmatrix}} \underset{V^T}{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}$$

Because of this fact $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ 6 u_2

$\therefore\ u_2 = AV_2$

$$AV_2 = \begin{bmatrix}4 & 4\\-3 & 3\end{bmatrix}\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix} = \begin{bmatrix}0\\-6/\sqrt{2}\end{bmatrix} = \begin{bmatrix}0\\-3\sqrt{2}\end{bmatrix} = 3\sqrt{2}\begin{bmatrix}0\\-1\end{bmatrix}$$

we found $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$ as E-vector for AA^T. we have

taken $u = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $\therefore$ $AV_2 = u_2$

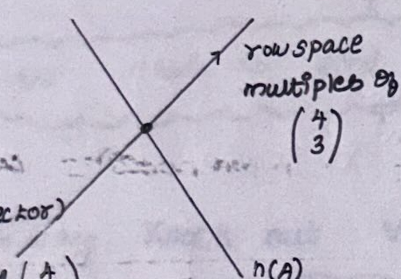
Example 2

Singular matrix: Rank 1

$A = \begin{bmatrix}4 & 3\\8 & 6\end{bmatrix} \rightarrow$ 1 dimensional Row & Column Space.

Rowspace: multiples of $\begin{pmatrix}4\\3\end{pmatrix}$

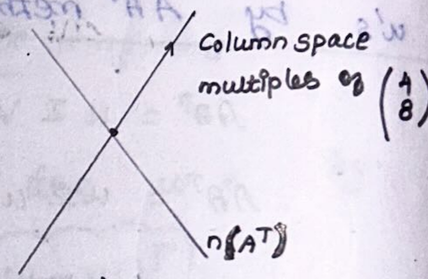
Columnspace: multiples of $\begin{pmatrix}4\\8\end{pmatrix}$



(unit vector)

$v_1 = \frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}$

$v_1 = \begin{pmatrix}0.8\\0.6\end{pmatrix}$



$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$

SVD

$A = u \Sigma V^T$

$$\begin{bmatrix}4 & 3\\8 & 6\end{bmatrix} = \begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\2/\sqrt{5} & -1/\sqrt{5}\end{bmatrix}\begin{bmatrix}\sqrt{125} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0.8 & 0.6\\0.6 & -0.8\end{bmatrix}^T$$

$$A^TA = \begin{bmatrix}4 & 8\\3 & 6\end{bmatrix}\begin{bmatrix}4 & 3\\8 & 6\end{bmatrix} = \begin{bmatrix}16+64 & 60\\60 & 45\end{bmatrix} = \begin{bmatrix}80 & 60\\60 & 45\end{bmatrix}$$

Eigen values: $0, (125-0) = 125$

$0, 125$

u_2 and u_1 are orthogonal

v_1 and v_2 are orthogonal

[we know u_1 and v_1]

u_2 and v_2 are in null space.

$$\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

'we are choosing the four subspaces basis'

$v_1, \dots, v_r \rightarrow$ orthonormal basis for row space

$u_1, \dots u_r \rightarrow$ orthonormal basis for column space

$v_{r+1}, \dots v_n \rightarrow$ " " null space

$u_{r+1}, \dots u_m \rightarrow$ " " $n(A^T)$

$$\boxed{Av_i = \sigma_i u_i}$$

Av_i is in the direction of corresponding u's.

$\dim(\text{column space}) = \dim(\text{row space}) = \text{rank}$

$\dim(n(A)) = n - r$

$\dim(n(A^T)) = m - r$

'Idea: relate vectors from R^n and R^m spaces'

R^n — Row, Null

R^m — Column, Left null.

$$Av_i = \sigma_i u_i$$

Recitation

SVD

$$C = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix}$$

soln:

$$C = U \Sigma V^T.$$

$$\boxed{\begin{aligned} C^T C &= V \Sigma^2 V^T \\ CC^T &= U \Sigma^2 U^T \end{aligned}}$$

$$C^T C = \begin{pmatrix} 5 & -1 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} 26 & 18 \\ 18 & 74 \end{pmatrix}$$

$$\det(C^TC-\lambda I) = \begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} \Rightarrow \lambda^2-100\lambda+1600=0$$

$$(\lambda-20)(\lambda-80)$$

C^TC

Eigen vectors:

$$\begin{pmatrix} 6 & 18 \\ 18 & 54 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

orthonormal

$$v_1 = \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$

$$\begin{pmatrix} -54 & 18 \\ 18 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

orthonormal

$$v_2 = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$$

$$V = \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}$$

$$\boxed{V^T = V}$$

$$\Sigma = \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{pmatrix}$$

CC^T (or)

$AV_1 = u_1$

$$\begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} -\sqrt{10} & 2\sqrt{10} \\ \sqrt{10} & 2\sqrt{10} \end{pmatrix}$$

$u\Sigma$

$$= \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{pmatrix}$$

$$u = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{pmatrix}\begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}$$

Fibonacci matrix

$$F_{n+2} = F_n + F_{n+1}$$

$$F_{K+2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{K+1} \\ F_K \end{bmatrix}$$

↳ fibonacci matrix.

Linear transformations & their matrices

When we multiply a matrix by an input vector, we get an output vector, often in a new space. We can ask what this 'linear transformation' does to all the vectors in a space. Infact, matrices were originally invented for the study of linear transformations.

Lecture - 31

Without coordinates: no matrix

With coordinates: Matrix. (for computation)

$$T(V+W) = T(V) + T(W)$$

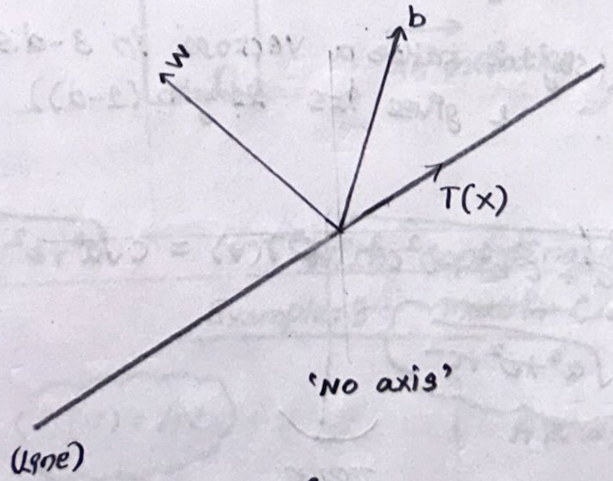
$$T(cV) = c\,T(V)$$

Example: 1: projection

$$T: R^2 \rightarrow R^2$$

linear transformation 'T' which takes every vector in R^2 into another vector in R^2.

'mapping'



$\therefore T(0) = 0$

$T(c0) = c\,T(0)$

$\boxed{T(c0) = 0}$

'project every vector on the line'

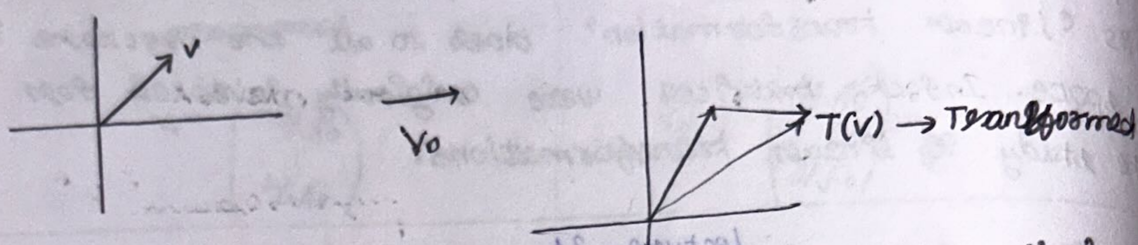
$$T(v+w) = T(v) + T(w)$$
$$T(cv) = cT(v)$$

$$T(cv+dw) = cT(v) + dT(w)$$

'If v is twice → projection will be twice'

Example: 2: Shift the whole plane by v_0

↳ Non example (Not a example of linear transformation)



'Not linear transformation'

i) Non - linear:

$$T(2v) = 2v + v_0$$
$$2T(v) = 2(v+v_0) = 2v + 2v_0$$

$$T(2v) \neq 2T(v)$$

'zero vector must be transformed to zero'

$T(0) = 0 \rightarrow T((3)\,0) = 3\,T(0) = 0$

Non example:

$T(v) = ||v||$
↳ Length

$T: R^3 \rightarrow R^1$ (say it takes a vector in 3-d space & gives its length (1-d))

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$T(cv) = \sqrt{c^2a^2 + c^2b^2 + c^2c^2}$ | $c\,T(v) = c\sqrt{a^2+b^2+c^2}$

$T(cv) = c\sqrt{a^2+b^2+c^2}$

True

when

$T(-cv) = c\sqrt{a^2+b^2+c^2} \neq cT(v) = -c\sqrt{a^2+b^2+c^2}$

$\boxed{T(cv) \neq cT(v)}$

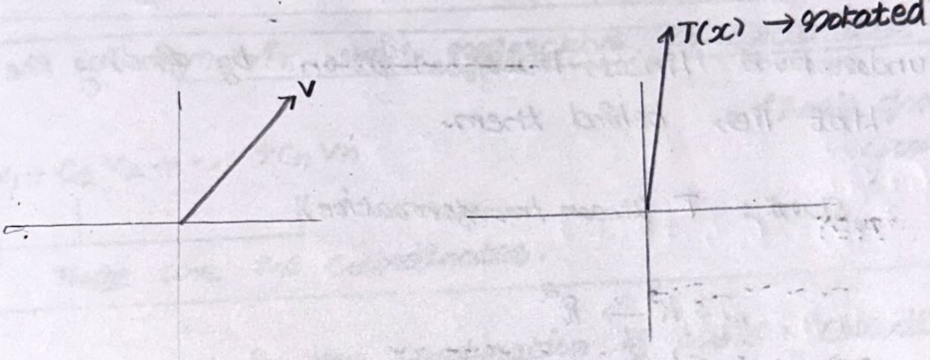
'Not a linear transformation'

Note: from here $T \rightarrow$ only for linear transformations.

Example: 2: Rotation by 45°

$T: R^2 \longrightarrow R^2$

'only rotates — not changes the matrix'.

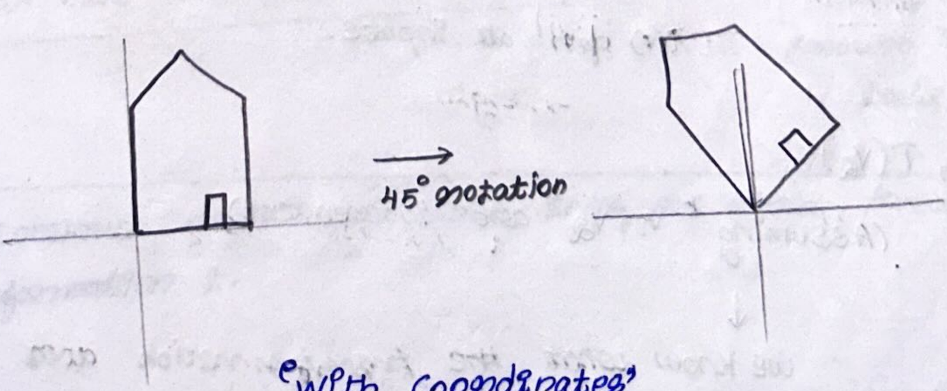


$T(cv) = c(T(v))$

$T(v+w) = T(v) + T(w)$

Rotate seperately then add up.

example: 'Take a house — full of vectors'



'With coordinates'

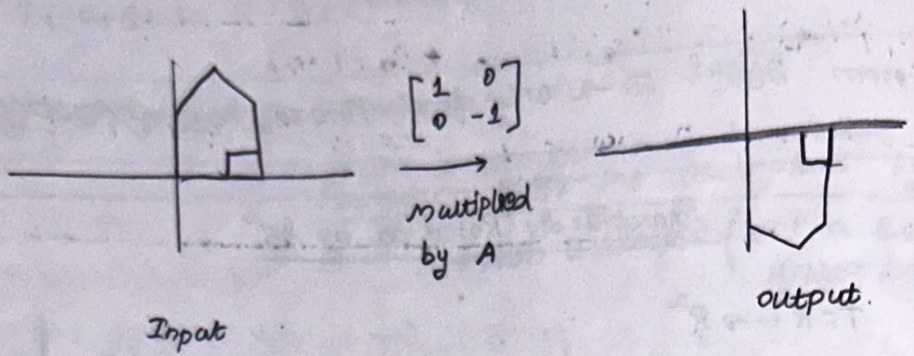
Example: 3: Matrix A

$T(v) = Av$, | $A(cw) = c(Aw)$

$A(v+w) = Av + Aw$ | $c \rightarrow$ constant

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Transforming the entire space by a matrix.



'Abstract description'

Aim: understand linear transformation by finding the matrix that lies behind them.

Start : T (linear transformation)

$T: R^3 \to R^2$

Examples:

$T(v) = Av$ → 2 by 3 matrix.

$T(v)$ ↓ output (R_2)

Av ↑ input (3-dimensions)

$(2\times3)(3\times1) = 2\times1$

$\begin{pmatrix} a \\ b \end{pmatrix} \to R^2$

Information needed to know the transformation:

$T(v)$ for all inputs.

$T(V_1), T(V_2)$

(Assuming V_1, V_2 are independent)

↓

we know what the transformation does to the basis V_1 and V_2.

If we know the transformation of basis

$T(V_1), T(V_2) \ldots T(V_n)$ for n dimensional space.

we can say what the

In n dimensional space,

every vector is a combination of

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ (c_1, c_2 – constants; v_1, v_2 – Basis Vectors)

$T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$

↳ Transformation of any vector in that n-dimension space.

Comment: If we know what T does to a basis, to each vector in a basis, then we know the linear transformation.

'Linearity'

Matrix with respective co-ordinates

'Basis for every vector'

$\underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n}$

These are the Coordinates.

'Numbers need in the combination of basis : Coordinates'

'how much each basis vectors are in v'

$$v = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

standard basis

'we can have different basis too'

Coordinates : 3, 2, 4
↓
Amount of each basis.

Construct matrix A that tells me about linear transformation t.

Soln:

$T: R^n \to R^m$

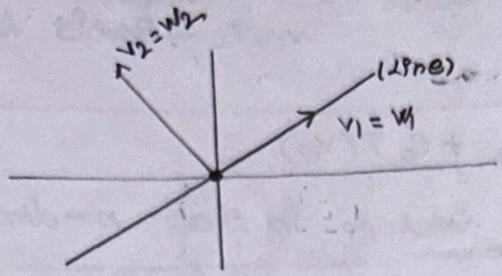
↓ Input basis (R^n) ↓ output basis (R^m)

$v_1, v_2, \ldots v_n$ → Input Basis of R^n

$w_1, w_2, \ldots w_m$ → output basis of R^m.

want matrix A: what linear transformation does:

$n = m = 2$ 'Taking projection example'



'projecting every vector on the plane to the line'

Basis:

* for simplification: Take the same basis for i/p & o/p.

1) unit vector (Right on the line)
2) unit vector ($\perp^r$ to that line)

$$V = c_1 v_1 + c_2 v_2$$

$$T(v) = c_1 v_1 + 0(\perp^r)$$

$$T(v) = c_1 v_1$$

'Already in line → so projects at the same line'

Input: (c_1, c_2) coordinates
output: $(c_1, 0)$ coordinates

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \underset{\text{Input coordinates}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\text{output coordinates.}}{\begin{bmatrix} c_1 \\ 0 \end{bmatrix}} \rightarrow c_1 v_1 \text{ (Along the line)}$$

[Combination of v_1]

$A \rightarrow$ does the job of linear transformation.

In this example:

'Input basis same as output basis'

1) Along the line
2) $\perp^r$ to the line

Eigen vector basis.
↑
'Eigen vectors of the projection' - Good basis

Eigen vectors basis $\xrightarrow{\text{leads to}}$ diagonal matrix Λ.

Aim: Transform any vector to the line.

$\therefore$

$$T(v) = c_1 v_1 + 0\, v_2$$

$$T(v) = c_1 v_1$$

In matrix form:

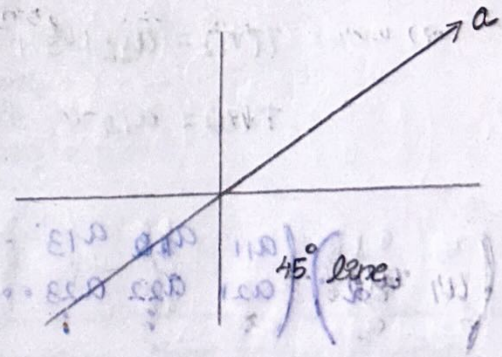
$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}\begin{bmatrix} c_1 \\ 0 \end{bmatrix} = c_1 v_1 \text{ (on the line solution)} \rightarrow 45^\circ.$$

"By $p = Pb$"

Standard basis. $T(v_1+v_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_1 , \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = w_2 .$$

Project any vector to $c\begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow$ Solution line.



$$P = \frac{aa^T}{a^Ta} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$= \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}{2} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Basis of this solution space:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{std basis.}} \rightarrow 45^\circ \text{ projecting line.}$$

Rule to find the matrix A: $\quad p = Pb$

$$(2\times 2)\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}v_1 + \frac{1}{2}v_2 \\ \frac{1}{2}v_1 + \frac{1}{2}v_2 \end{bmatrix} \rightarrow \text{on the solution line.}$$

Comment: * choosing a basis is important. Eigen value matrix $\Lambda \rightarrow$ Simplest
Diagonal
Symmetric
Easy.
) In projections.

* Standard basis $\rightarrow$ Not diagonal
Symmetric (This case)

<u>Rule to find A:</u> Given bases $v_1 \dots v_n$
$w_1 \dots w_m$

<u>1st Column of A:</u>
Write linear transformation to v_1

$T(v_1) =$ output
↓
Input

output $\rightarrow$ some combinations of the $w_1, \dots w_m$'

$$T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \dots a_{m1} w_m$$

$$T(v_2) = a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m$$

$$\begin{pmatrix} w_1 & w_2 \cdots \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ & \vdots & \vdots & \end{pmatrix} = T(V)$$

$$\underset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \underset{c}{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$$

<u>Rules to find A:</u>
'change of basis'

$$[v_1 \ v_2] = \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}, \quad [w_1 \ w_2] = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{pmatrix} 6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \left\{ \begin{aligned} v_1 &= w_1 + w_2 \\ v_2 &= 2w_1 + 3w_2 \end{aligned} \right\} \quad \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

In matrix form:

$$[v_1 \; v_2] = [w_1 \; w_2] B$$

$$= \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}$$

$\therefore$ $v_1, v_2 \rightarrow$ column vectors

$$B = W^{-1} V$$

$$= \frac{1}{6} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 3 & 8 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6 & 12 \\ 6 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$V = c_1 T(v_1) + \dots + c_n T(v_n)$

$T(v_1) = v_1 \rightarrow$ Linear

$v = c_1 v_1 + \dots + c_n v_n \rightarrow$ Input basis

$u = d_1 w_1 + \dots + d_n w_n \rightarrow$ output basis.

$T(c_1 v_1) = c_1 v_1 = d_1 w_1 \rightarrow$ Input basis = output basis.

$$\boxed{Vc = Wd}$$

$$T(v) = (w_1 \; w_2 \; \dots) \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$\downarrow$

All vectors

$T(v_1) = a_{11} w_1 + a_{21} w_2 + a_{31} w_3 + \dots$

$T(v_2) = a_{12} w_1 + a_{22} w_2 + a_{32} w_3 + \dots$

$\vdots$

Linear transformation that takes derivative.

Input space all combination $\longrightarrow$ $c_1 + c_2 x + c_3 x^2$, Basis $1, x, x^2$

output $\longrightarrow$ $c_2 + 2 c_3 x$, $1, x$

$$T : R^3 \rightarrow R^2$$

'Derivative is linear'

'If we learn derivatives of few $\rightarrow$ Through their combinations $\rightarrow$ we can do all!'

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$$

$$(2\times 3)\ (3\times 1) = (2\times 1)$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \text{will do the job.}$$

* Inverse matrix gives the inverse to the linear transformation
* product of two matrices gives the right matrix for the product of two transformation.

* matrix multiplication → comes from linear transformation

Recitation

Linear transformation

Let $T(A) = A^T$, A is 2×2.

1) why is T linear? what is T^{-1}?
2) write down the matrix of T in

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ w_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ w_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

3) Eigen values / Eigen vectors of T?

Solu:

$$\boxed{\begin{gathered} T: R^2 \rightarrow R^2 \\ R^m \rightarrow R^n \end{gathered}}$$

1) why linear?

$$T(A+B) = T(A) + T(B) = (A+B)^T$$
$$= A^T + B^T$$

↳ Linear.

Satisfies.

$$T(cA) = (cA)^T$$
$$= c\,A^T = c\,T(A) \longrightarrow \text{linear}$$

Transpose $\rightarrow$ 180° rotation)

$A^{-1} \rightarrow$ 360° rotation $\rightarrow$ 2times transpose

$$A^{-1} = (A^T)^T$$

$$\boxed{T^2 = I}$$

Two times transpose gives the same A. (Done by I)

$$\boxed{A^{-1} = A}$$

2) $T(A) = A^T$

$$T(V_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(V_1) = V_1$$

$$T(V_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T(V_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = V_2$$

$$T(V_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = V_4$$

$$M_T = \begin{array}{cccc} TV_1 & TV_2 & TV_3 & TV_4 \end{array}$$
$$M_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(In matrix form)}$$

$$\begin{tikzcd}
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \arrow[r] \arrow[d, "T"] & \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \arrow[d, bend left, "M_T"] \\
\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \arrow[r] & \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \end{pmatrix}
\end{tikzcd}$$

Basis (above the top column vector) / Basis (below the bottom column vector)

<u>for w:</u>

$T(W_1) = W_1$

$T(W_2) = W_2$

$T(W_3) = W_3$

$T(W_4) = -W_4$

$$M_T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3) Eigen values

$$TV = \lambda V$$

$$\begin{bmatrix} TW_1 = W_1 & TW_3 = W_3 \\ TW_2 = W_2 & TW_4 = -W_4 \end{bmatrix}$$

$\lambda = 1, 1, 1, -1$

Change of basis: Image Compression

video cameras record data in a poor format for broadcasting video. To transmit video efficiently, linear algebra is used to change the basis. But which basis is best for video compression is an important that has not been fully answered!

Lecture-32

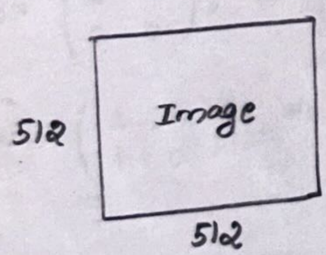
* change of basis
* compression of images
* Transformation ↔ matrix.
  Relation

matrix: Coordinate based transformation description.

---

compression:

overwhelming amount of data' - compression with out loss of data is possible.



Black & white (Greyscale) → 0 to 255

pixel: value of x_i (2^8 possibility)

$$0 \leq x_i < 255 \quad [8 \text{ bits}]$$

$\therefore x \in \text{Real}$ $n = 512^2$

pixel: vector that gives information about the image.

For colour images: we need 3 coordinates

$$n = 3(512)^2$$

Standard compression: Jpeg (Joint photographic experts group)
↳established a system of compression
(change of basis)

512^2 long

$$x = \begin{bmatrix} 121 \\ 124 \\ \\ \end{bmatrix}$$ → i th element

'Black board: Almost same (blank board) → wasting pixels'

Standard basis:

$$\begin{bmatrix}1\\0\\0\\\vdots\end{bmatrix} \begin{bmatrix}0\\1\\\vdots\end{bmatrix} \begin{bmatrix}0\\0\\1\\\vdots\end{bmatrix} \cdots \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$$

Better basis

$$\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}$$ → By itself give the complete image of our mixed solid & signal image.

'Save a lot'

Extreme vector:

$$\begin{bmatrix}1\\-1\\1\\-1\\\vdots\end{bmatrix}$$ → checker board vector (Image like a huge checker board)

Half the image darker, Half the image lighter:

$$\begin{bmatrix}1\\1\\1\\-1\\-1\\\vdots\end{bmatrix}$$

'Freedom: To choose the basis: A billion dollar decision'.

... based on the way the

signal is scanned!

movie people: Another way (A giant politics)

problem!

what basis to choose?

Best known basis : JPEG basis (Fourier basis)

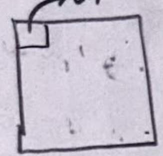
DC vector (or) constant vector.

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \begin{bmatrix}1\\w\\w^2\\ \vdots \\ w^{n-1}\end{bmatrix}, \ldots \ldots$$ "Fourier matrix"

Often 8 by 8 is a good choice

Big signal:

64 co.efficients (pixels) → changes the basis of that piece.



512

512 → reduces to 8 by 8 blocks

JPEG :

Each image → broken in to 8 by 8 blocks

↓

64 basis vectors

64 pixels (co.efficients)

we will change in 64 dimensional space (basis) using Fourier vectors.

---

signal x

↓ change basis (Better)

loss less.

coefficients c

[64 pixels come in & 64 co.efficients come out]

↓ Compression (lose information)

Setting a value (below all zero) ← 1) Thresholding (Throw away small coefficients)

↓

$\hat{c}$ (many zeros)

$\hat{C} \rightarrow$ compression came.

Smooth lecture (picture or video). $\rightarrow$ $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$ $\rightarrow$ seldom throw away
(low or zero frequency) (high Co-efficients)

Flickering (high frequency):

$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$ $\rightarrow$ Highest frequency we could have
(noise or zitter is the reason)

$\hat{C}$ (many zeros)

$\downarrow$

Reconstruct $\hat{x} = \sum \hat{c}_i v_i$

(It doesn't have 64 terms
may be two or 3 terms.

• Digitalized finger points $\rightarrow$ 512^2 by 512^2 $\rightarrow$ based
compressed $\rightarrow$ (what basis?)
& indexed

Video: Another aspect (poor):

• video as still image after another.
& compress each them, run them
make video'

$\downarrow$ problem

Video: Sequence of images (highly corrilately)
'Jumpy motion of the video'

$\downarrow$

corrilation (In time, space, things don't change instantly or smoothly changes often and we can predict one value from the previous value.)

$\downarrow$

Assume & make small corrections
(Digitize then compress).

'Pure applications of linear algebra'

wavelets: Competition for Fouriers.

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

wavelet basis (8 dimensional space)

'Sophisticated choice'

compare wavelet basis & Fourier basis:

- wavelet basis in to Fourier basis' R^8

Reconstruct $\hat{x} = \sum c_i v_i$

Extreme case:

$$\begin{bmatrix}1\\-1\\1\\-1\\1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

↳ High frequency guy.

- Every combination of R^8 is this 8 basis'

Linear algebra

① Find the coefficients.

Basis & pixel values will be given?

Job:

$$\begin{bmatrix}P_1\\P_2\\ \vdots \\P_8\end{bmatrix}$$ → Standard basis (pixels) [8 pixel values in the $P_1, P_2 \ldots$]

Combination of

$$\boxed{\text{pixel } P = c_1\begin{pmatrix}1\\1\\1\\1\\ \vdots\end{pmatrix} + c_2(\text{wavelet}_2) + c_3(w_3) + \ldots + c_8(w_8)}$$

$$P = c_1 w_1 + \cdots + c_8 w_8$$

Looking for coefficients:

Given basis $\longrightarrow$ Input signal also given

need to find co.efficients

Solve $p = c_1 w_1 + \cdots + c_8 w_8$

$\downarrow$

changing from standard basis (8-grey scale values) to the wavelet basis where the same vector is represented by eight numbers. ($\mathbb{R}^8$)

Those eight numbers are the coefficient of the basis

In matrix form

$$p = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}$$

8 columns are 8 basis vector

$W \rightarrow$ wavelet matrix.

Change of basis:

Solve $p = Wc$

$c = W^{-1} p$ $\rightarrow$ Good basis has a nice & fast inverse.

$\downarrow$

What's that?

(Still a debate updating).

"Too much time — Too much money"

Fourier basis: Fast Fourier transform.

Fast wavelet transform