$\dot{y} = 7y$ is ce^{7t} → solution

* The constant function $y=0$ is a solution
* There are infinitely many solutions
* For each solution $y(t)$, the function $2y(t)$ is also a solution.

Order of the ODE $(y^{(5)})^4 + 4\dot{y}\,y^{(8)} - t^8 y + t^8 = 0$

Answer : 8 ⇒ $y^{(8)}$

The diff eqn $m\ddot{x} = -mg - b\dot{x}$ is one possible model for the height x of a falling object subject to air resistance.

m, g, b → Constants, x → Function of time t.

$m \to kg$

$x \to m$

$t \to s$

unit of $b = ?$

Solu:

$m\ddot{x} = -mg - b\dot{x}$ (action of air velocity)

(F=ma) (Gravity)

$\boxed{\frac{d^2x}{dt^2}}$

$kg\,ms^{-2} = (\text{units of } b)\,\dot{x}$

$kg\,\cancel{m}s^{-2} = (\text{units of } b)\cdot \cancel{m}s^{-1}$

$\frac{kg}{s}$ = unit of b.

Linear or not:

$\cos(t+y) = \cos A\cos B - \sin A\sin B$

$A = t$

$B = y$

$\dot{y} = \cos(t+y) + t$

$= \cos t\cos y - \sin t\sin y + t$ (Non linear)

$ty = (\cos t)\,\dot{y} + \sin t$

$(\cos t)\,\dot{y} - ty = -\sin t$ (In the inhomogeneous I order ODE form)

(The coefficient functions in front of $\dot{y}$ and y are allowed to be any function of t, ~~and~~ the right hand side too is allowed to be any function of t.

3) $t^2\ddot{y} = (\dot{y} - 1)\log t$

$t^2\ddot{y} - (\log t)\,\dot{y} = -\log$ [inhomogeneous linear ODE)

4) $y^{(3)} = y\ddot{y} + 2e^t\dot{y}$ → non linear ($y\ddot{y}$ → not a function of t times some derivative of y)

1) Suppose $y(t)$ is the solution to $\dot{y} = -\frac{\pi}{2} t\,(1+y^2)$ with $y(0) = 1$ what is $y(1)$

Solu:

$$\frac{dy}{dt} = -\frac{\pi}{2} t\,(1+y^2)$$

$$\int \frac{dy}{1+y^2} = -\frac{\pi}{2} t \, dt$$

$$\arctan y = -\frac{\pi}{2}\left(\left(\frac{t^2}{2}\right)\right) + c$$

$$\arctan y = -\frac{\pi}{4} t^2 + c$$

$$y = \tan\left(-\frac{\pi}{4} t^2 + c\right)$$

$y(0) = 1$

$$1 = \tan\left(-\frac{\pi}{4} t^2 + c\right)$$

$$1 = \tan(c)$$

$$c = \tan^{-1}(1)$$

$$= \frac{\pi}{4} + n\pi \quad \text{(tan has a period of } \pi\text{)}$$

$\tan(180^\circ + x) = \tan x$

$|||^{ly}$:

$$y = \tan\left(-\frac{\pi}{4} t^2 + \frac{\pi}{4} + n\pi\right)$$

$$= \tan\left(\frac{\pi}{4}(1-t^2)\right)$$

$y(1) = \tan(0) = 0$. This formula defines a valid solution on at least the interval where $-\frac{\pi}{2} < \frac{\pi}{4}(1-t^2) < \frac{\pi}{2}$, which simplifies to $-1 < t^2 < 3$ which holds if and only if $-\sqrt{3} < t < \sqrt{3}$. In particular, this solution is defined at $t = 1$, and its value there is $y(1) = \tan 0 = 0$

What function must $r(t)$ be in order for the ODE

$$2\ddot{y} + 8\dot{y} + 7y = r(t)$$

to have $y = e^{10t}$ as a solution.

Solu:

$$= 2(e^{10t})'' + 8(e^{10t})' + 7(e^{10t})$$

$$= 2 \times 10 \times 10 \times e^{10t} + 8 \times 10\, e^{10t} + 7e^{10t}$$

$$= 200e^{10t} + 80e^{10t} + 7e^{10t}$$

$$= 287e^{10t}.$$

ii) If $y = \frac{1}{287} e^{10t}$ → solution.

$$2\ddot{y} + 8\dot{y} + 7y = 287 e^{10t} \quad (y = e^{10t})$$

$$2\ddot{y} + 8\dot{y} + 7y = e^{10t} \quad (y = \frac{1}{287} e^{10t})$$

$(t+1)^2 = (t^2+1) + 2t$ $\qquad c_1 = 2, c_2 = 1$ → linear

t → Linear comb $\qquad c_1 = 1, c_2 = 0$

0 → Linear comb $\qquad c_1 = 0 = c_2.$

$(3t^2 + 4t + 5) \Rightarrow c_2$ would have 3 & 5 simultaneously (Non linear)

Find the differential equations:

$y = a\sin(2t)$ (Try to eliminate a)

Soln:

$$\frac{dy}{dt} = 2a\cos(2t)$$

$$\frac{\dot{y}}{y} = \frac{2a\cos(2t)}{a\sin(2t)}$$

$$\frac{\dot{y}}{y} = 2\cot(2t)$$

$$\dot{y} - 2\cot 2t\,(y) = 0$$

$\cot 2t$ is undefined $2t = 0, \pi, 2\pi.$

But our open interval is $(0, \frac{\pi}{2})$. So no problem

(or)

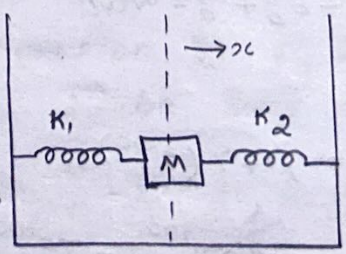
$$\sin(2t)\,\dot{y} - 2\cos(2t)\,y = 0.$$

Modelling a mass attached to two springs:

A mass m, (1kg) is attached by two spring on the left & one on the right, to the walls. The mass sits on an air table & moves horizontally with no friction

obey Hookes Law.

K_1, K_2 - Sprit



x → Displacement of the mass from the equilibrium position & choose $x > 0$ if the mass is to right of the equilibrium. (Use Newtons second law to write down a diff eqn for $x(t)$ in terms of the two spring constants K_1 & K_2

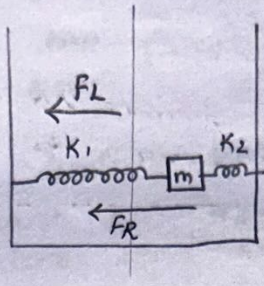
Solu:

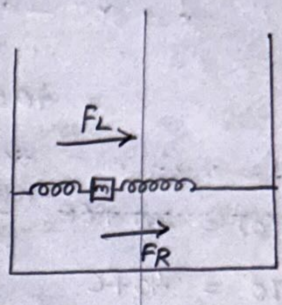
F_L = Force of left spring

F_R = Force of Right spring

$$|F_L| = K|x| \quad , |F| = K_2|x|$$

Assume :





$$F_L = -K_1 x \ , \ F_R = -K_2 x \qquad\qquad F_L = K_1 x \ , \ F_R = K_2 x$$

By II law:

(F = ma = sum of two spring forces)

$$m\ddot{x} = K_1 x + K_2 x$$

$$m\ddot{x} = (K_1 + K_2)x.$$

$$m\ddot{x} = -K_1 x - K_2 x$$

$$= -(K_1 + K_2)x$$

∴ Differential eqn for the horizontal displacement of the mass.

Heating a house

The temp of a deserted, unheated house is $T = T(t)$, in degrees Fahrenheit at time t in hours. If the outside temp is $40°$. It takes 24 hours for the temp of house to drop from $70°$ to $55°$. (use newton's law of cooling)

Solu:

$$\dot{T} = K(T_e - T)$$

$$\dot{T} = K(40 - T)$$

$$\dot{T} + KT = 40K$$

Solving homogeneous eqn

$$\dot{T} + KT = 0$$

$$\dot{T} = -KT$$

$$\frac{dT}{T} = -K\,dt$$

$$\ln T = \int -K\,dt$$

$$|T| = e^{-Kt} \cdot e^{c}$$

$$T = Ce^{-Kt}$$

$$T_h = e^{-Kt} \quad (C = 1)$$

$$y = T_h u$$

Applying:

$$\frac{d}{dt}(e^{-Kt}u) + K(e^{-Kt} \cdot u) = 40K$$

$$\dot{u}e^{-Kt} - Kue^{-Kt} + Kue^{-Kt} = 40K$$

$\dot{u}\, e^{-Kt} = 40K$

$\frac{du}{dt} = 40K \cdot e^{Kt}$

$du = 40K \cdot e^{Kt}\, dt$

$u = 40K \int e^{Kt}\, dt$

$= K40 \times \frac{e^{Kt}}{K} + C$

$= 40e^{Kt} + C$

$T = (40e^{Kt} + C)\, e^{-Kt}$

$\boxed{T = 40 + Ce^{-Kt}}$

C - Any real number

It takes 24 hours to cool from 70° to 55°.

$T(0) = 40 + Ce^{-K(0)}$

$70 = 40 + C$

$\boxed{30^\circ = C}$

$T(24) = 40 + Ce^{-K \times 24}$

$55 = 40 + 30e^{-24K}$

$15 = 30e^{-24K}$

$1 = 2e^{-24K}$

$e^{24K} = 2$

$24K = \ln 2$

$K = \ln 2/24.$

$\boxed{K = 0.0288}$

$\therefore \dot{T} = 0.0288(40 - T)$

Finding long time or steady state temperature.

A family then bought the deserted house & turned on the heater. The eqn for the temp. T(t) of the house with the furnace on is

$$\dot{T} = K(40 - T) + a$$

$\lim_{t \to \infty} T(t) \to$ as a function of K and a.

$$\dot{T} + KT = 40K + aK.$$

Sub:

$\dot{T} + KT = a$

$\frac{d}{dt}(ue^{-Kt}) + K(ue^{-Kt}) = a$

$\dot{u}e^{-Kt} - Kue^{-Kt} + Kue^{-Kt} = a$

$\dot{u} = aKe^{Kt}$

$u = a\frac{e^{Kt}}{K} + C \quad \Rightarrow \boxed{u = \frac{a}{K}e^{Kt} + C}$

$$y = \left(\frac{\alpha}{k}e^{kt} + c\right)e^{-kt}$$

$$\boxed{y = \frac{\alpha}{k} + ce^{-kt}}$$

By superposition:

$$\dot{T} = k(40 - T) + \alpha$$

$$\boxed{T = 40 + \frac{\alpha}{k} + ce^{-kt}}$$

As $\lim_{t \to \infty} \left(40 + \frac{\alpha}{k} + ce^{-kt}\right) = 40 + \frac{\alpha}{k}$.

The limiting temperature is also the value of the constant solution, the steady state solution. Since all solutions tend to the steady state solution, we say the equation is "stable"

Energy efficiency:

Find the ratio of energy (the values of α) required to maintain a temperature of $70°$ (vs) $55°$.

In other words, Find $\frac{\alpha_{70}}{\alpha_{55}}$, where $\alpha_{70} \rightarrow$ No. of degrees per hour required to maintain a temp of $70°$ & α_{55} - no. of degrees per hour req to maintain a temp of $55°$.

solu:

$$70 = 40 + \frac{\alpha_{70}}{k} \Rightarrow \alpha_{70} = 30k$$

$$55 = 40 + \frac{\alpha_{55}}{k} \Rightarrow \alpha_{55} = 15k$$

$$\frac{\alpha_{70}}{\alpha_{55}} = \frac{2k}{1k} = 2.$$

The parameter α is roughly proportional to the power consumed in heating the house, and thus to the cost per hour. So pushing the steady state temp from $55°$ to $70°$ doubles the cost per hour. This is not so surprising, since the difference in temperature from the outside is doubled. may be more surprising is that this ratio is independent of the constant k, which reflects the insulating prop of the house. But the absolute costs do depend linearly on k!

Grand mother Helena sets up a trust fund for her grand daughter Emma. Helena's fool-proof investments on the money in the trust fund make interest at a rate of I, measured in units of year^{-1}. She removes money from the trust fund to give to her grand daughter at a rate of $r(t)$ dollars per year

solu:

$x(t) \rightarrow$ In dollars $\rightarrow$ Amount in the trust fund at t years after the inception of the fund. Find $\dot{x}$ in terms of x, r and the interest rate I.

solu:

$$\dot{x} = xI - r$$

If I varies with $x \rightarrow$ eqn is nonlinear

I is constant $\rightarrow$ Linear

Grandmother decides to give a monthly allowance to her Granddaughter from the fund. The allowance is to start at \$0 at the birth of her granddaughter, and then increase by \$10 each month afterwards.

Describe the grandmother's gifts, approximately, in terms of a continuous rate $r(t)$ (in dollars per year)

solu: (use a linearly increasing function with no constant term for $r(t)$).

$$r(t) = Kt$$

At the end of the year 1: The allowance will be \$120.

~~$r(1) = 120(12)$~~

~~$= 1440$~~

Monthly	cumulative
10	10
20	30
30	60
40	100
50	150
60	210
70	280
80	360
90	450
100	550
110	660
120	780
130	910
140	1050
150	1200
160	1360
170	1530
180	1710
190	1900
200	2100
210	2310
220	2530
230	2760
240	3000

$= 780$

$= 2220$

year 0 = 780

year 1 = year 0 + 1440

year 2 = year 1 + 1440

$A_1 - A_0 = 1440$

$A_2 - A_1 = 1440$

$\therefore \quad r(t) = 780 + (1440(t-1))$

we are instructed to use a linearly ↑ function with no constant term:

$$r(t) = 1440t$$

3) Assume that the interest rate stays constant at I ($year^{-1}$) and that GM Helena carries out her plan of steadily ↑ the monthly allowance to her grand daughter as described. Find the general solution to the

DE. (simplify your exp for c) eg: instead of C+10 ↳ use c

soln:

$$\dot{x} - xI = -v$$

$$\dot{x} - xI = -1440t$$

$$\dot{x} - xI = 0$$

$$\dot{x} = xI$$

$$\frac{dx}{x} = I\,dt$$

$$\ln x = It + c$$

$$x = ce^{It}$$

$$\boxed{x_h = e^{It}}$$

$$x = u x_h$$

$$\frac{d}{dt}(ue^{It}) - I(e^{It}u) = -1440t$$

$$\dot{u}e^{It} + Iue^{It} - Iue^{It} = -1440t$$

$$\dot{u}e^{It} = -1440t$$

$$\dot{u} = -1440t \cdot e^{-It}$$

$$\int du = -1440 \int t e^{-It}\,dt$$

$$\int u\,dv = uv - \int v\,du$$

Let:

$$U = t, \quad dV = e^{-It}\,dt$$

$$dU = dt, \quad V = \frac{e^{-It}}{-I}$$

$$\int du = -1440 \int t e^{-It}\,dt$$

$$u = -1440 \left[t\frac{e^{-It}}{-I} - \int \frac{e^{-It}}{-I}\,dt \right] + c$$

$$u = -1440 \left[\frac{te^{-It}}{-I} - \frac{e^{-It}}{+I^2} \right] + c$$

$$u = 1440 \left[\frac{e^{-It}}{I^2} + \frac{te^{-It}}{I} \right] + c$$

$$= \frac{1440}{I^2} \cdot e^{-It}(1 + It) + c$$

$$x(t) = \frac{1440}{I^2}(1+It) + ce^{It}$$

Let the fund run dry:

GM plans to pay the allowance to her grand daughter as described before but also want the trust fund to decrease to 0 when she becomes an adult at 18 years of age. If the interest rate is constant at 1% per year, Find the amount $x(0)$ that she should put in the fund when her gd is born.

Solu:

$x(0) = ?$ $\quad x(18) = 0$

$I = 0.01 \left(\frac{1}{100}\right)$

$t = 18$

$$0 = \frac{1440}{I^2}(1+It) + ce^{It}$$

$$\frac{-1440\, e^{-0.01\times18}}{I^2}(1+0.01\times18) = c$$

$$c = \frac{-1440\, e^{-0.18}}{0.0001}(1.18)$$

$$\boxed{c = -14192{,}911.43}$$

$$207000 = \frac{1440}{(0.0001)}(1+0.01(0)) + ce^{0}$$

$$c = -14193{,}000.13$$

→ c is common for the entire eqn (solutions)

$$x(0) = \frac{1440}{0.0001}(1) + (-14192{,}911.43)$$

$$x(0) = 207088.57 \quad \approx \$207000$$

Initial amount of fund req so that the fund will run out at the end of the year 18 is \$207000. to the nearest thousand dollars.

Complex numbers:

1) Represent complex numbers geometrically as points on the complex plane.

2) Represent complex numbers algebraically in cartesian form & polar form as a complex exponential using Euler's Formula.

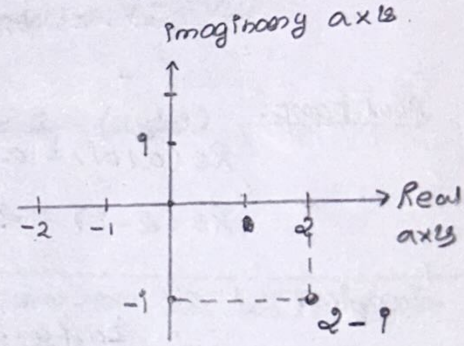
3) perform basic operations on complex numbers: conjugation, taking the real & imaginary parts, finding the absolute value & determining possible values of the argument.

4) perform arithmetic on complex numbers: add, sub, *, /, roots & powers in both cartesian & polar forms.

5) Differentiate complex-valued functions of a real value variable by breaking up such functions into their real & imaginary parts, and differenciating (& integrating) component wise.

Complex numbers and the complex plane.

Complex numbers are expressions of the form $a+bi$. a and b are real numbers, and i is a new symbol. Just as real numbers can be plotted on a line, complex numbers can be plotted in a plane: plot $a+bi$ at the point (a, b)



$0+ib \rightarrow$ purely imaginary

$a \rightarrow$ Real.

multiplication will eventually be defined so that $i^2 = -1$.

. (Electrical Engineers use j . ∴ $i \rightarrow$ current (reserved))

Set of real numbers $= \mathbb{R}$

Set of complex numbers $= \mathbb{C}$

Historical origin of complex numbers:

most people think that complex numbers arose from attempts to solve quadratic eqn, but actually they first appeared in connection with cubic equations. Everyone knew that certain quadratic eqn like

$$x^2+1=0 \quad (\text{or}) \quad x^2+2x+5=0 \quad \text{had}$$

no solutions. The problem was with certain cubic eqn,

$$x^3-6x-2=0$$

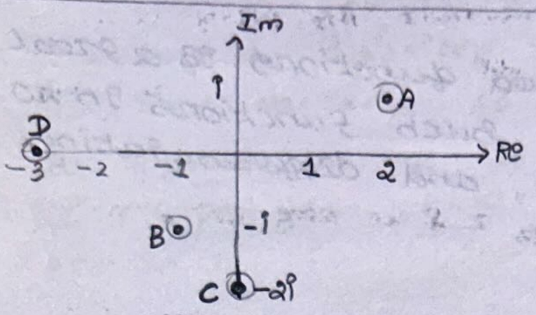
This eqn was known to have 3 real roots gn by simple combinations of the expressions.

$$A = \sqrt[3]{1+\sqrt{-7}}$$

$$B = \sqrt[3]{1-\sqrt{-7}}$$

Check that one of the roots is $A+B$; it may not look like a real number, but it turns out to be one!

Note: Complex doesn't mean complicated.
It refers to a complex of real numbers.
(அடுக்கு)



$A = 2+i$

$B = -1-i$

$C = -2i$

$D = -3$

Is 9 a real number or complex number.

Ans: Both

How? $9+0i \rightarrow$ Both real & img part (Purely real)

Operations on complex numbers

Real part:

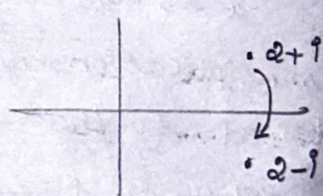
$Re(a+bi) = a$

$Re(2-i) = 2$

Imaginary part:

$Im(a+bi) = b$

$Im(2-i) = -i$



(Conjugate)

Complex Conjugate:

$\overline{a+bi} = a-bi$

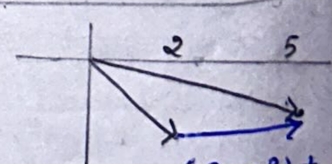
$\overline{2-i} = 2+i$

$\therefore b \rightarrow$ real number.

Complex numbers can be added, subtracted, multiplied & divided (except for division by 0). Addition, sub & mul are defined as for polynomials, except after multiplication one should simplify by using $\boxed{i^2 = -1}$

eg: 3.1 $(a+bi) + (c+di) = (a+c) + (b+d)i$

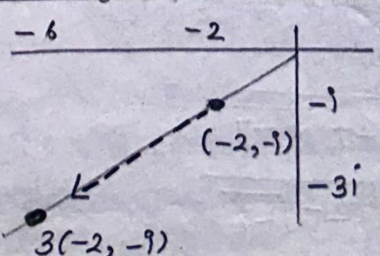
$(2-2i) + (3+i) = 5-i$



Multiplication by a real number:

$c(a+bi) = ca + (cb)i$

$3(-2-i) = -6-3i$



$(2+3i)(1-5i) = 2-7i-15i^2$
$= 2-7i+15$
$= 17-7i$

$\frac{2+3i}{1-5i} = \frac{2+3i}{1-5i} \cdot \frac{1+5i}{1+5i} = \frac{2+13i+15i^2}{1-25i^2}$
$= \frac{-13+13i}{26} = -\frac{1}{2}+\frac{1}{2}i$

Properties of arithmetic on Complex numbers:

The arithmetic operations on complex numbers satisfy the same properties as for real numbers. $(zw = wz)$ etc.. The mathematical jargon for this is that C, like R, is a field.

1) Closure under add & mul : if z & w are complex numbers so are $z+w$ and zw.

2) Commutativity of addition & mul : $\boxed{z+w = w+z \ \& \ zw = wz}$

3) Associativity of add & mul , $v+(z+w) = (v+z)+w$
$v(zw) = (vz)w$

4) Distribution of mul over add:
$v(z+w) = vz+vw$

5) Multiplicative inverses : $z\left(\frac{1}{z}\right) = 1 \quad (z \neq 0)$

6) Additive inverses: $z+(-z) = 0$

In particular, for any complex number z & integer n, the nth power z^n can be defined in the usual way
$(z \neq 0,$ if $n<0)$

$z^3 = zzz$
$z^0 = 1$
$z^{-3} = \frac{1}{z^3}$

Add, sub of complex numbers has the same geometric interpretation (zonisl) as for vectors in R^2. The same holds for scalar multiplication of a complex number by a real number.

$z = 1+3i, \quad w = 3-2i$

$\bar{z}+\bar{w} = (1-3i)+(3+2i)$
$= 4-i$

$wz = (3-2i)(1+3i)$
$= 3-6i^2-2i+9i$
$= 3+6+7i$
$= 9+7i$

$\frac{1}{z} = \frac{1}{1+3i} \times \frac{1-3i}{1-3i}$
$= \frac{1-3i}{1+9} = \left(\frac{1}{10}\right)-\left(\frac{3}{10}\right)i$

Complex conjugation:

changing i to $-i$, transforms the eqn $i^2 = -1$ in to a true solution $(-i)^2 = -1$, so it will also preserve any eqn that is deduced from $i^2 = -1$. The definitions of add & mul used only $i^2 = -1$, so add & mul respect complex conjugation. For eg. if complex numbers satisfy

$$y = z + w, \text{ then } \bar{y} = \bar{z} + \bar{w}$$

$$\overline{z+w} = \bar{z} + \bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}.$$

what does this multiplication rule say if z happens to be a real number a?

$$\overline{aw} = a\bar{w}$$

If $\bar{z} = -z$, what does that tell us about the value of $z = a+bi$?

$$\bar{z} = -z,$$

$a - bi = -a - bi$ $(a = 0, z = bi)$ $z \to$ purely imaginary. These implications are reversible, so $\bar{z} = -z$ is equivalent to z being purely imaginary.

Fundamental theorem of Algebra:

Every non zero, single-variable, degree n polynomial with complex co.eff has, counted with multiplicity, exactly n complex roots. The equivalence of the two statements can be proven through the use of successive polynomial division.

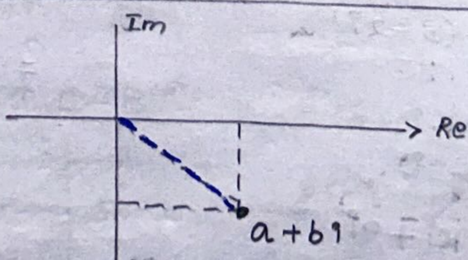
$$z = 0 + 0i$$

$$\bar{z} = 0 - 0i \quad , \quad -z = -0 - 0i = 0 - 0i$$

$$\bar{z} = -z$$

$\therefore$ $z = 0$ is actually not a real number. But purely imaginary!

Absolute value:



The absolute value (or magnitude or modulus) $|z|$ of a

Complex numbers $z=a+bi$ is its distance from the origin.

$$|a+bi| = \sqrt{a^2+b^2} \quad \text{(Real number)}$$

w.k.t

when we multiply a complex number by its complex conjugate, we get a real number. This real number is the square of the distance to the origin.

$$z\bar{z} = (a+bi)(a-bi)$$
$$= a^2 - b^2i^2$$
$$= a^2+b^2$$
$$= |z|^2$$

For a complex number z, inequalities like $z<3$ don't make any sense, but inequalities like $|z|<3$ make (do), because $|z|$ is a real number. The complex numbers satisfying $|z|<3$ are those in the open disk of radius 3, centered at 0 in the complex plane. (open disk means the disk without its boundary.)

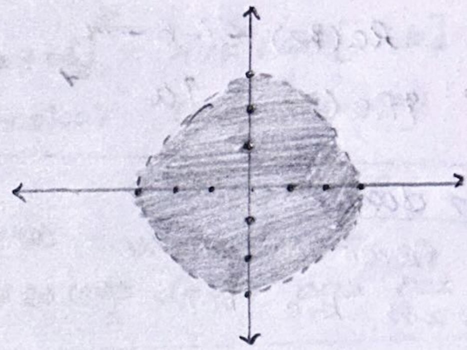
1) $z = 1+3i$

$$|z| = \sqrt{1+9}$$
$$= \sqrt{10}$$

2) $w = 3-2i$

$$|w| = \sqrt{9+4}$$
$$= \sqrt{13}$$

$$|(bi)^2| = |b^2i^2|$$
$$= |-b^2| = b^2$$



$|z|<3$

Some useful identities:

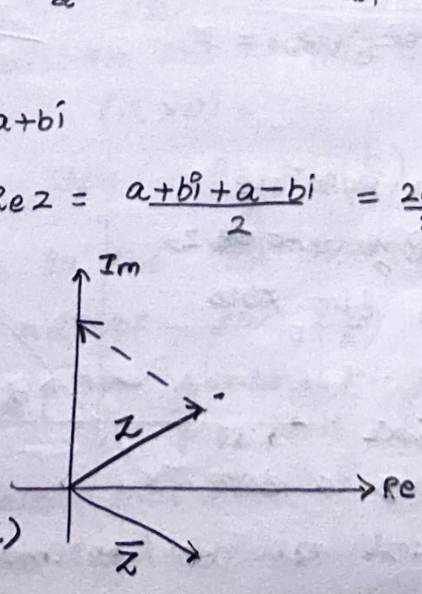
The following are true for all complex numbers z.

$$\operatorname{Re} z = \frac{z+\bar{z}}{2}, \quad \operatorname{Im} z = \frac{z-\bar{z}}{2i}, \quad \bar{\bar{z}} = z, \quad z\bar{z} = |z|^2$$

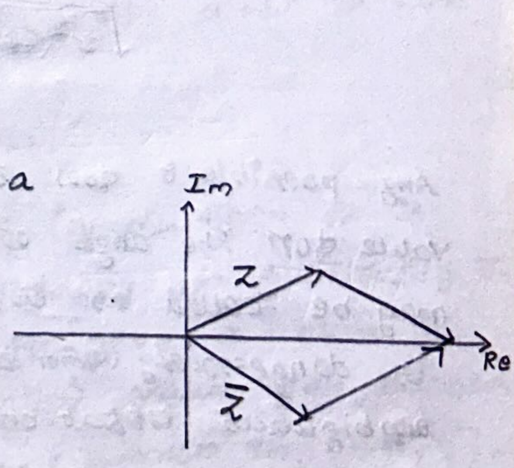
Proof.

$z = a+bi$

$$\operatorname{Re} z = \frac{a+bi+a-bi}{2} = \frac{2a}{2} = a$$



$z-\bar{z} = 2i\,\operatorname{Im}(z)$



$z+\bar{z} = 2\operatorname{Re} z$

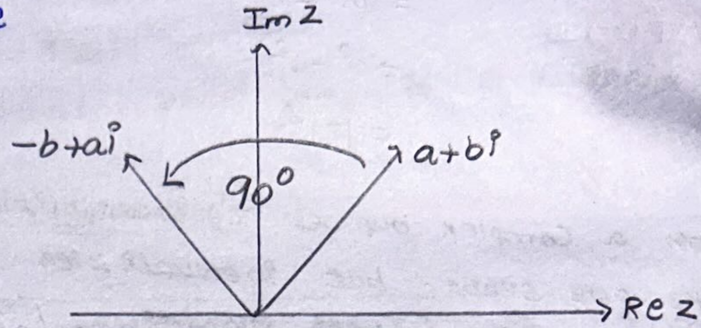
Also for any real number c and complex no. z

$$Re(cz) = c\,Re\,z, \quad Im(cz) = c\,Im(z).$$

Note: $c \rightarrow$ Real

Effect does multiplication by i have on a complex number in the complex plane?

Ans: It rotates the numbers around the origin by $90°$ counterclockwise



$$(a+bi) \times i = ai + bi^2$$
$$= -b + ai$$

$i.1 = i, \quad i.i = -1,$

$$Re(iz) \neq i\,Re(z)$$

$\therefore$ $z = a + ib$

$iz = -b + ai$

$Re(iz) = -b$

$i\,Re(z) = ia$

$\Rightarrow \quad -b \neq ia$

Polar form:

Given a non-zero complex number $z = a+bi$, we can express the point (a,b) in polar coordinates r and θ

$$a = r\cos\theta, \quad b = r\sin\theta.$$

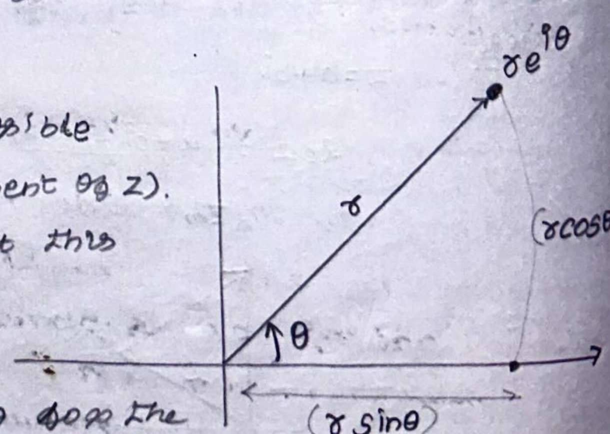
$$a+bi = r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

one has $r = |z|$; here r must be a +ve real number (assuming $z \neq 0$)

$$\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$$

$\boxed{|z| = r}$

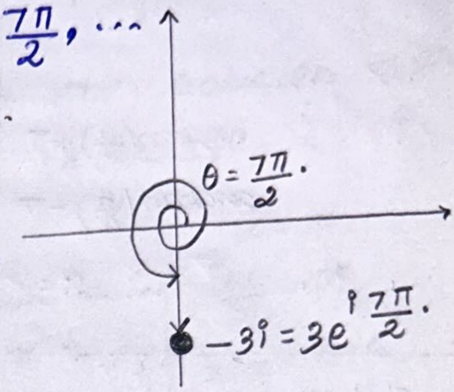
Any possible θ for z (a possible value for the angle or argument of z). may be called as $\arg z$, but this is dangerously (தெளிவற்ற) ambiguous notation. Since there are many values of θ for the same z.



ex: 7.1 Suppose $z = -3i$, z corresponds to the point $(0, -3)$ then $r = |z| = 3$, But there are infinitely many possibilities for the angle θ. One possibility is $-\frac{\pi}{2}$. All the others are obtained by adding integer multiples of 2π.

$$\arg z = \ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$$

To specify a preferred polar form, we would have to restrict the range for θ to some interval of width 2π. The most common choice is to require



$-\pi < \theta \leq \pi$. This special θ is called the principal value of the argument, and is denoted in various ways.

$$\theta = \operatorname{Arg}(a+bi) = \operatorname{Arg}[a+bi] = \operatorname{ArcTan}[a, b]$$

(Mathematica) (Mathematica)

$$= \operatorname{atan2}(b, a)$$

Matlab.

warning: In matlab, be careful to use (b,a) and not (a,b)

Given a complex number $a+bi$ with $a, b \neq 0$, how do we find
the principal value of the argument $\theta \in (-\pi, \pi]$? The
number θ satisfies $\tan\theta = \frac{b}{a}$. But since tan has
period π (there are two angles in $(-\pi, \pi]$ having
tangent equal to $\frac{b}{a}$, corresponding to opposite directions.
By definition, $\tan^{-1}\left(\frac{b}{a}\right)$ is the one in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, corresponding
to a direction in the right plane. If $a+bi$ is in the
right half plane $(a>0)$,

then $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ works.

But if $a+bi$ is in the left half plane $(a<0)$, then it is
necessary to adjust $\tan^{-1}\left(\frac{b}{a}\right)$ by adding or subtracting π.
Either adding or subtracting will give a value of
the argument, but for the principal value you should
do the operation that results in a number in $(-\pi, \pi]$.
This is summarized in the table below.

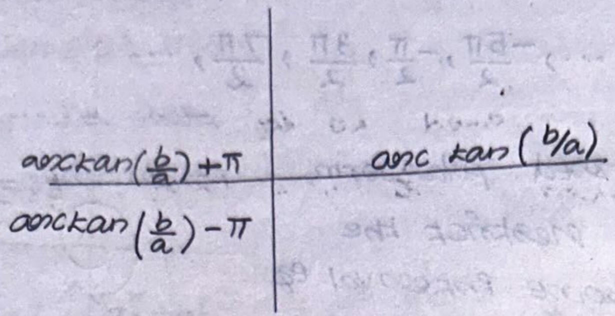
If $a > 0$ $\quad \theta = \arctan\left(\frac{b}{a}\right)$

If $a < 0$ and $b \geq 0$ $\quad \theta = \arctan\left(\frac{b}{a}\right) + \pi$

If $a < 0$ and $b < 0$ $\quad \theta = \arctan\left(\frac{b}{a}\right) - \pi$

∴ If a is −ve,
$\theta \to$ will be +ve

But If b is (−)ve
the θ will be in
III (or) IV quadrant



Find the principle argument of $-1-i$ $\quad a = -1$, $b = -1$

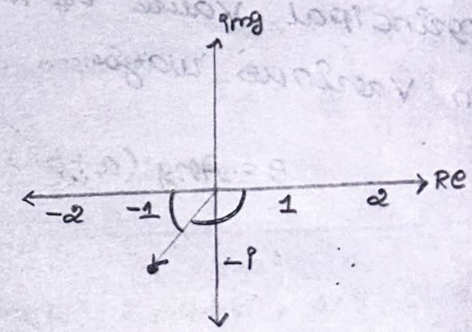
Solu:

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) - \pi$$

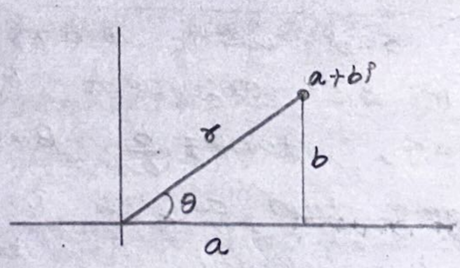
$$= \tan^{-1}(1) - \pi$$

$$= \frac{\pi}{4} - \pi$$

$$= -\frac{3\pi}{4}.$$



Polar representation:



$$a+bi = r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$\boxed{a+bi = re^{i\theta}.}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

↳ Euler's definition

Exponential:

1) $a^x \cdot a^y = a^{x+y}$ (exp function)

2) e^{at}, $\frac{dy}{dt} = a e^{at} = ay$

$e^{i\theta} = ?\ \cos\theta + i\sin\theta$. ?

$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$?

$\frac{d}{dt}(e^{it}) = ie^{it}$?

* Infinite series should work out?

For students in Science & Engg, you will often use exponential notation to abbreviate the polar form $\cos\theta + i\sin\theta$ of a complex number of modulus 1. This equ is known as Euler's formula.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

The function e^t is defined to be the solution of the Initial value problem $\dot{x} = x$, $(x(0) = 1)$

$\dot{x} = x$

$\frac{dx}{x} = dt$

$\ln x = t + c$

$x = c e^t$

when $x(0) = 1$

$\boxed{c = 1}$

$\boxed{x = e^t}$

For any constant a,

$$\boxed{e^{at} \text{ is the solution of } \dot{x} = ax, \quad x(0) = 1}$$

In addition, for Euler's formula to be a good definition, we want e^{it} to satisfy exponential prop we know from calculus, This is to justify Euler's formula, we expect to have

Exponential law: $e^{it_1} \cdot e^{it_2} = e^{i(t_1 + t_2)}$

Initial value problem: $\frac{d}{dt} e^{it} = i e^{it}$ $\quad (e^{i0} = 1)$

Taylor's formula $e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$

<u>Practice with polar form:</u>

1) $z = -8i$,

$\tan^{-1}\left(-\frac{8}{0}\right) - \pi = \frac{\pi}{2} - \pi$

$= -\frac{\pi}{2}$

$r = \sqrt{8^2}$

$\boxed{r = 8}$

$z = \left(i\sin\left(-\frac{\pi}{2}\right)\right) 8$

$z = 8 e^{i\left(-\frac{\pi}{2}\right)}$

2) $z = -2$

$\tan^{-1}\left(\frac{0}{2}\right) + \pi = \pi$

$z = 2e^{i\pi}$

3) $z = 4 + 4i$: $r = \sqrt{32} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$

$= 4\sqrt{2}\, e^{i\pi/4}$

$z = -6 - 6\sqrt{3}i$ $\quad \sqrt{36+36\times 3} = \sqrt{144} = 12$

$$\begin{array}{r} 36 \\ \times 3 \\ \hline 108 \\ +36 \\ \hline 144 \end{array}$$

$$\theta = \tan^{-1}\left(\frac{-6\sqrt{3}}{-6}\right) - \pi$$

$$= \tan^{-1}(\sqrt{3}) - \pi \qquad z = 12e^{i\left(-\frac{2\pi}{3}\right)}$$

$$= \frac{\pi}{3} - \pi$$

$$= -\frac{2\pi}{3}.$$

$2e^{-i\frac{\pi}{3}}$ Cartesian form using Euler's Formula -

$$\Rightarrow 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$\Rightarrow$

$x = r\cos\theta$	$y = r\sin\theta$
$x = 2\cos\left(-\frac{\pi}{3}\right)$	$= 2\sin\left(-\frac{\pi}{3}\right)$
$= 2\left(\frac{1}{2}\right)$	$= -2\frac{\sqrt{3}}{2}$

$$z = 1 - \sqrt{3}i$$

$-4e^{i\frac{7\pi}{3}}$ $\quad r = -4$

$x = -4\cos\left(\frac{7\pi}{3}\right)$	$y = -4\sin\left(\frac{7\pi}{3}\right)$
$x = +2\sqrt{3}$	$y = 2$

$$z = 2\sqrt{3} + 2i$$

$$e^{i\theta_1} \cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2)$$

$$= (\cos\theta_1\cos\theta_2) + (i\sin\theta_1 \times i\sin\theta_2) + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i(\sin(\theta_1+\theta_2))$$

$$= e^{i(\theta_1+\theta_2)}$$

Gives a handy way to remember the trignometric addition formulas

---

$$\frac{d}{dt} e^{it} = \frac{d}{dt}(\cos t + i \sin t)$$
$$= -\sin t + i \cos t$$
$$= i(\cos t + i \sin t)$$
$$= i e^{it}$$

$$e^{i0} = \cos 0 + i \sin 0$$
$$= 1 + 0i$$
$$= 1.$$

In order to justify Euler's formula, we want to show that e^{it} satisfies the exponential principle. To do this, we need to explain how to take the derivative of a complex-valued function of a real variable.

<u>complex valued functions of a real-variable:</u>

$y(t) \rightarrow$ complex valued function of a real variable (t).

$y(t) = f(t) + i g(t)$. for some real valued functions of t.

Here $f(t)$: $\mathrm{Re}\, y(t)$ and $g(t)$: $\mathrm{Im}\, y(t)$.

<u>Differenciation & Integration can be done component-wise:</u>

$$y'(t) = f'(t) + i g'(t)$$
$$\int y(t)\,dt = \int f(t)\,dt + i \int g(t)\,dt$$

This is exactly same as for vector-valued functions

<u>Justification:</u>

we justify the Euler's formula by showing e^{it} satisfying the ivp $\boxed{y' = iy,\ y(0) = 1}$

$$\frac{d}{dt} e^{it} = \frac{d}{dt}(\cos t + i \sin t)$$
$$= -\sin t + i \cos t$$
$$= i(\cos t + i \sin t)$$
$$= i e^{it}$$

last we need to check the ivp,

$$e^{i0} = \cos 0 + i \sin 0$$
$$= 1$$

Together with the exponential law, this proves Euler's genius in using exponential notation to simplify the polar form of a complex number with modulus 1.

$$e^{it} = \cos(t) + i\sin(t)$$

$$|e^{it}| = \sqrt{\cos^2 t + \sin^2 t}$$

$$= 1.$$

<u>Advantage of polar form:</u>

* Good for multiplication

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

multiply the moduli and add the angles (arguments).

Now we can write complex numbers $z = a+bi$ in polar form as $re^{i\theta} = r\cos\theta + ir\sin\theta$.

Test for equality of two non zero complex numbers in polar form:

$r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi K$.

for some integer K.

$r_1, r_2 \rightarrow$ Real +ve numbers and that $\theta_1, \theta_2 \rightarrow$ Real numbers

Some arithmetic operations on complex numbers are easy in polar form:

Multiplication: $(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Reciprocal: $\frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

Division: $\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

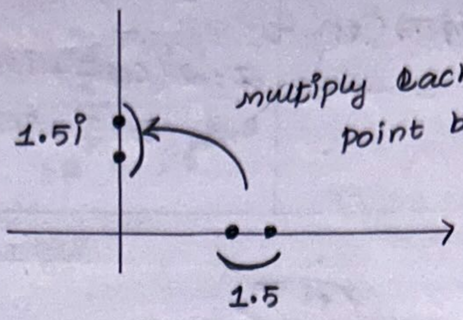
n^{th} power: $(re^{i\theta})^n = r^n e^{in\theta}$ $\quad n \rightarrow$ integer.

Complex Conjugation: $\overline{re^{i\theta}} = re^{-i\theta}$

$|z_1 z_2| = |z_1||z_2|$, $\frac{1}{|z|} = \left|\frac{1}{z}\right|$, $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$,

$|\bar{z}| = |z|$ $\quad |z^n| = |z|^n$

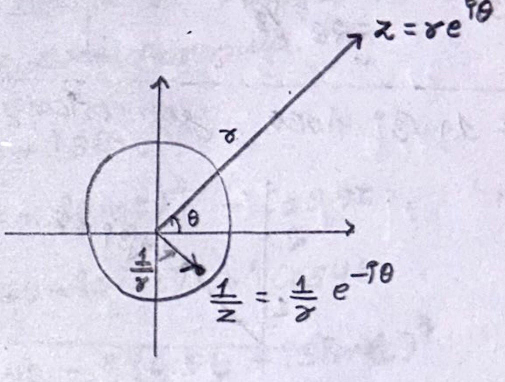
$i = e^{i\pi/2}$, multiplying i adds $\pi/2$ to the angle of each point. (Rotates $90°$ by counter clock wise)



How do you trap a lion?

Build a cage in the shape of the unit circle $|z| = 1$. Get inside it. make sure the lion is outside the cage. Apply the function $\frac{1}{z}$ to the whole plane. voilà! (There it is) The lion is now inside the cage, and you are outside it.

solu: only problem: there's a lot of other stuff inside the cage too. Also don't want to stand too close to $z = 0$ when you apply $\frac{1}{z}$.



why not always write complex numbers in polar form?

Because addition & sub are difficult in polar form.

Express $(1+i)^6$ in cartesian form.

solu:

$z = 1 + i$, $\quad x = r\cos\theta$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$y = r\sin\theta$

$r = \sqrt{2}$ $\quad \theta = \tan^{-1}(1)$

$\theta = \pi/4$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z = \sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}$

$$z^6 = (1+i)^6 = (\sqrt{2})^6 \left(e^{i\left(\frac{\pi}{4}\right)}\right)^6$$

$$= 8\, e^{i\left(\frac{3\pi}{2}\right)}$$

$$= 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 8\,(0 + (-i))$$

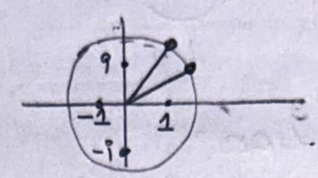
$$= -8i$$

$$(1+i)^4 = ?\ (\sqrt{2})^4 \left(e^{i\left(\frac{\pi}{4}\right)}\right)^4$$
$$= 4e^{i\pi}$$
$$= 4(\cos\pi + i\sin\pi)$$
$$= 4(-1+0)$$
$$= -4$$

Express $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in Polar form.

$r_1 = \sqrt{3+1} = 2$ | $r_2 = \sqrt{3+1} = 2.$ | $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$, $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$z = \frac{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} = e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)} = e^{i\left(\frac{\pi}{6}\right)} = \cos\frac{\pi}{6} + i\sin\pi/6$$

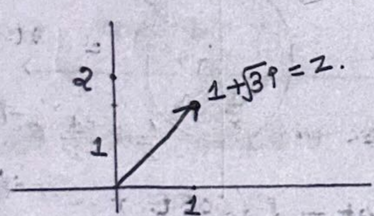


1) $(1+\sqrt{3}i)(a+bi) = a+bi+\sqrt{3}ia-\sqrt{3}b = a-\sqrt{3}b+bi+\sqrt{3}ia$
$= a-\sqrt{3}b+i(b+\sqrt{3}a)$

2) $(1+\sqrt{3}i)$.
$a=1,\ b=\sqrt{3}$
$\boxed{r=2}$
$\tan^{-1}(\sqrt{3})$
$z = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
$= 2e^{i\pi/3}$

3) $(1+\sqrt{3}i)(a+bi) = 2e^{i\pi/3}(re^{i\theta})$
$= 2re^{i\left(\frac{\pi}{3}+\theta\right)}$

what $1+\sqrt{3}i$ does geometrically when multiplying?



Adds 45° to the Component (Counterclockwise)

Does the sequence of powers of $1+\sqrt{3}i$, +ve and negative

case : +ve
$$(1+\sqrt{3}i) = 2e^{i(\pi/3)}$$
$$(1+\sqrt{3}i)^2 = 4e^{i(2\pi/3)}$$
$$(1+\sqrt{3}i)^n = 2^n e^{i(n\pi/3)}$$

Case : -ve
$$(1+\sqrt{3}i)^{-n} = \frac{1}{2^n} e^{-in\frac{\pi}{3}}$$

2) Explain why $|z^n| = |z|^n$ and $\arg(z^n) = n\arg z$ for $n \to$ a +ve integer

solu:
$$z = \cos\theta + i\sin\theta$$
$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= e^{in\theta} = \cos n\theta + i\sin n\theta$$

$|z^n| = |\cos n\theta + i\sin n\theta| = \sqrt{\cos^2 n\theta + \sin^2 n\theta}$

$= 1.$

$|z|^n = |\cos\theta + i\sin n\theta|^n = (1)^n = 1.$

$z = re^{i\theta}$, $z^n = re^{in\theta}$ $\cos n\theta + i\sin\theta$

$\arg(z^n) = n \arg z$

$\arg(z^n) = \arg(r^n e^{in\theta})$ $a = r\cos n\theta$

$a = \cos n\theta$

$= \arg\left(r^n(\cos n\theta + i\sin n\theta)\right)$

$= \tan^{-1}\left(\frac{r^n \sin n\theta}{r^n \cos n\theta}\right) = \tan^{-1}(\tan n\theta)$

$= n\theta$

$= n \arg(z)$

Find an expression for $\sin 4t$ in terms of powers of $\cos t$ & $\sin t$, using $(e^{it})^4 = e^{4it}$ & Euler's Formula.

$(\cos 4t + i\sin 4t) = (\cos t + i\sin t)^4$

$i\sin 4t = (\cos t + i\sin t)^4 - \cos 4t$

$+\sin 4t = -i(\cos t + i\sin t)^4 + i\cos 4t$

$\sin 4t = i\cos 4t - i(\cos t + i\sin t)^4$

$\sin 4t = i(\cos 4t - (\cos t + i\sin t)^4)$

$e^{i4t} = \cos 4t + i\sin 4t$

$(e^{it})^4 = (\cos t + i\sin t)^4$

$\cos 4t + i\sin 4t = (\cos^4 t - \cos^2 t\sin^2 t + \sin^4 t) + i(4\cos^3 t\sin t - 4\cos t\sin^3 t)$

$\boxed{\sin 4t = 4\cos^3 t\sin t - 4\cos t\sin^3 t}$

Complex exponential function:

1) Extend Euler's formula to define the complex exponential function e^{zt} of a real variable t for all complex numbers z.

2) Describe the trajectory of a complex exponential function depending on a real variable t.

3) Find complex roots of polynomials, including the special case of unity.

The complex exponential function:

Def: 1: For any complex number z, the complex function e^{zt} of a real variable t is defined as the solution to the initial value problem

$$\frac{d}{dt} e^{zt} = z e^{zt} \quad , \quad e^{z0} = 1.$$

Properties of e^{it}:

For every real number t,

1) $e^{it} = \cos t + i\sin t$

2) $e^{-it} = \overline{e^{it}} = \cos t - i\sin t$

3) $|e^{it}| = 1.$

$$\frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t$$
$$= i(\cos t + i\sin t)$$

Shows that the function $F(t) = \cos t + i\sin t$ is the solution to the DE with initial condition.

$$\dot{F}(t) = iF(t) \quad , \quad F(0) = 1.$$

But by definition $G(t) = e^{it}$ also satisfies

$$\dot{G}(t) = iG(t), \quad G(0) = 1$$

The existence & uniqueness theorem for I order diff eqn applies to complex-valued functions of a real variable. The uniqueness theorem tells us that

$$F(t) = G(t), \text{ or}$$

$$\boxed{e^{it} = \cos t + i\sin t} \quad \text{(or)} \quad e^{it} = \cos t + i\sin t$$

Properties of the Complex exponential e^z:

1) $e^{a+ib} = e^a(\cos b + i\sin b)$ for all real numbers a & b

2) $e^{z+w} = e^z e^w$ for all complex numbers z & w.

3) $(e^z)^n = e^{nz}$ for every complex number z and integer n.

n=0 case says $(e^z)^0 = e^{0+i0} = \cos 0 + i\sin 0 = 1$

n=−1 case says $\frac{1}{e^z} = (e^z)^{-1} = e^{-z}$

Taylors series:

$$e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

This formula can be deduced by mul z by t,

$$e^{zt} = 1+zt+\frac{(zt)^2}{2!}+\frac{(zt)^3}{3!}+\cdots$$

and showing that the rh. side satisfies the IVP

$\dot{F} = zF$, $F(0)=1$ [uniqueness guarantees that the two sides are then equal. By setting $t=1$, the original statement is shown.

Generalized Euler eqn or Taylors series for e^z as the definition of complex exponential function.

The definition we gave in terms of the IVP

$$\frac{d}{dt}e^{zt} = ze^{zt} \quad (e^{z0}=1).$$

solu.

$e^{a+ib} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b)$ by showing

$$e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt) \text{ for real numbers.}$$

$$\frac{d}{dt}\left(e^{at}(\cos bt+i\sin bt)\right) = ae^{at}(\cos bt + i\sin bt) + e^{at}(-b\sin bt + ib\cos bt)$$

$$= ae^{at}(\cos bt + i\sin bt) + e^{at}ib(\cos bt + i\sin bt)$$

$$= (a+ib)e^{at}(\cos bt + i\sin bt)$$

This shows that $e^{at}(\cos bt + i\sin bt)$ is a solution to the IVP

$$\dot{F} = (a+ib)F, \quad F(0)=1$$

But $e^{(a+ib)t}$ is a solution to the IVP by ... uniqueness theorem for diff eqn tells us that for all real values of t,

$$e^{(a+ib)t} = e^{at}(\cos bt + i\sin bt)$$ (In particular, this eqn is true when $t=1$. [But true for all real values]

---

2) $\boxed{\text{when } t=1}$

$$e^{(a+ib)} = e^{a}(\cos b + i\sin b)$$

when $t=0$

$\boxed{\text{Rough}}$

$$e^{(a+ib)0} =$$

$$e^{0+0i} = \cos 0 + i\sin 0$$

$$= 1$$

when $t=2$

$$e^{(a+ib)2} = e^{2a}\cdot e^{2ib} = e^{2a}(\cos 2b + i\sin 2b)$$

Let $z = a+ib$, $w = c+id$ then

$$e^{z}\cdot e^{w} = e^{a+ib}\cdot e^{c+id}$$

$$= e^{(a+c)+i(b+d)}$$

$$= e^{(a+c)}\cdot e^{i(b+d)}$$

$$= e^{(a+c)}\cdot e^{ib}\cdot e^{id}$$

So we need to show:

$$e^{i(b+d)} = e^{ib}\cdot e^{id}$$

$\therefore$ $\dot{F} = ibF$ where the solution is

$$\boxed{F = e^{ibt}}, \quad F(0) = 1$$

$$\therefore \frac{d}{dt}e^{ibt}e^{idt} = ib\,e^{ibt}\cdot e^{idt} + id\,e^{idt}e^{ibt}$$

$$= (ib+id)\,e^{ibt}e^{idt}$$

$\therefore$ Thus both $e^{ibt}e^{idt}$ and $e^{(ib+id)t}$ solve

$$\dot{F} = (ib+id)F, \quad F(0) = 1.$$

The uniqueness theorem for diff eqn implies that these two functions must be equal for all real values of t. In particular, they are equal when $t=1$.

If $n=0$, then this $1=1$ by definition. If $n>0$

$$(e^z)^n = \underbrace{e^z \cdot e^z \cdot e^z \cdots e^z}_{n \text{ copies}} = e^{z+z+z+\cdots+z} = e^{nz}.$$

If $n = -m < 0$ then

$$(e^z)^{-m} = \frac{1}{(e^z)^m} = \frac{1}{(e^{mz})} = e^{-mz}.$$

↳ As above

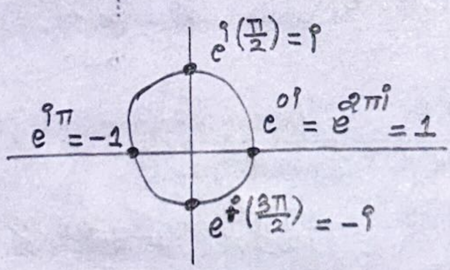
$$\therefore e^{mz} \cdot e^{-mz} = e^0 = 1.$$

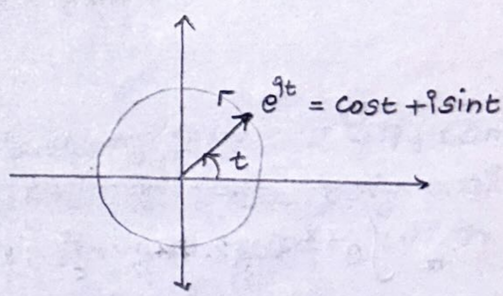
Graphing the Complex exponential function:

Graphs- Easy to understand, our aim is to understand the complex exponential function $e^{(a+bi)t}$ by plotting its trajectory as t varies.

Fundamental case:

As $t \uparrow$, the $e^{it} = \cos t + i\sin t$ travels counter clockwise around the unit cle.

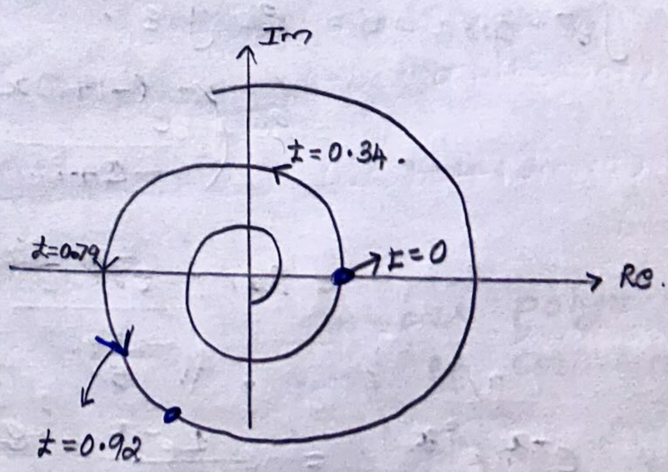




$e^{(a+bi)t}$

For a non zero complex number $a+bi$. As $t \uparrow$ $e^{(a+bi)t}$ moves along part of a line, cle, or a spiral, depending upon $a+bi$.

$a+bi = 0.50 + 4.00i$



As $t \uparrow$, the

Give the values of a and b so that the trajectory of $e^{(a+ib)t}$ is a line.

Solu: $a \neq 0$, $b = 0$ (Angle will be 0)

$e^{(-5-2i)t} = e^{-5t} \cdot e^{i(-2t)}$ as $t \to \infty$ [consider the behaviour]

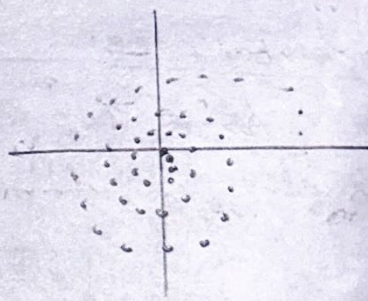
Solu:

* $\left| e^{(-5-2i)t} \right| = e^{-5t}$ tends to 0 as $t \to \infty$, so the point is moving inward.

* The argument $\operatorname{Arg}\left(e^{(-5-2i)t}\right) = -2t$ is decreasing so the point is moving clockwise.

$\therefore$ Therefore the trajectory is spiraling inwards clockwise.

[Spirals clockwise as it moves inward]



$\int e^{-x} \cos x \, dx = ?$

$e^{-x} \cos x$ is the real part of e^{-x+ix}

$e^{-x} \cos x = \operatorname{Re}\left(e^{(-1+i)x}\right)$

$\int e^{-x} \cos x \, dx = \operatorname{Re} \int e^{(-1+i)x}$

$= \operatorname{Re}\left(\dfrac{e^{(-1+i)x}}{-1+i}\right)$

$= \operatorname{Re}\left(\dfrac{e^{-x} \cdot e^{ix}}{-1+i}\right)$

$\dfrac{e^{-x} e^{ix}}{-1+i} = \dfrac{1}{-1+i} \times e^{-x} (\cos x + i \sin x)$

$$= \frac{-1-i}{1+1} e^{-x}(\cos x + i\sin x)$$

$$= -\frac{1-i}{2} e^{-x}(\cos x + i\sin x)$$

$$Re\left(\frac{e^{-x}\cdot e^{ix}}{-1+i}\right) = Re\left(-\frac{1}{2}e^{-x}\cos x - \frac{i\sin x}{2} - \frac{ie^{-x}\cos x}{2} + \frac{\sin x \cdot e^{-x}}{2}\right)$$

$$= -\frac{1}{2}e^{-x}\cos x + \frac{e^{-x}\sin x}{2}$$

$$\boxed{\int e^{-x}\cos x\, dx = \frac{e^{-x}}{2}(-\cos x + \sin x)}$$

Complex roots of polynomials:

Exam: 5.1

How many roots does the polynomial z^3-3z^2+4 have? It's factors are (It factors as) $(z-2)(z-2)(z+1)$. So it has only two distinct roots (2 and -1). But if we count twice (2), then the multiplicity is 3, equal no. of roots counted with multiplicity to the degree of polynomial.

Some polynomials with real co.eff, like z^2+9, can't be factored completely into degree 1 real polynomials, but do factor in to degree 1 polynomials with complex co.eff: $(z+3i)(z-3i)$

Real polynomial: polynomial with real co.eff

Complex polynomial: polynomial with complex co.eff

In fact, every complex polynomial factors completely into degree 1 complex polynomials - this is proved in advanced courses in complex analysis.

Fundamental theory of algebra:

Every degree n complex polynomial $f(z)$ has exactly n complex roots, if counted with multiplicity.