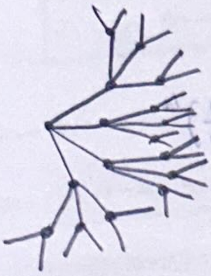
Counting - Lecture-4

'most problems result in counting'

$$P(A) = \frac{\text{num. of elements of } A}{n(\Omega)} = \frac{|A|}{|\Omega|}$$

Tricks: Counting: (proper way: sequential process)



4 choise - each having 3 - each having 2.

$4 \times 3 \times 2 = 24$ choices

3 letters - 4 digits

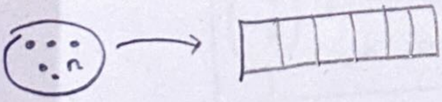
$26 \times 26 \times 26 \times 10 \times 10 \times 10$

No repitation

$26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$

Permutations

'No. of ways of ordering n elements'



$\{1, 2, 3, 4, \ldots\}$ 1 elements → n choices

2nd element → $(n-1)$ choices

$n \times (n-1) \times \cdots \times 1 = n!$ (factorial)

Subset

$\{1, 2, 3, 4, 5, \ldots\}$

'put in subset or not'

2 choices for 1 element (each element has this)

no. of subsets $= 2^n$

$\boxed{\text{No. of subsets} = 2^n}$

$n=1$ (element):

$\{1\} \rightarrow \{1\}$
$\phantom{\{1\}} \rightarrow \{\} = \phi$

Probability that six rolls of a six-sided die all give different numbers.

Solu: $\boxed{\text{Fair die}}$

* 1, 2, 3, 4, 5, 6 → one possible outcome

$$P(1,2,3,4,5,6) = \left(\frac{1}{6}\right)^6$$

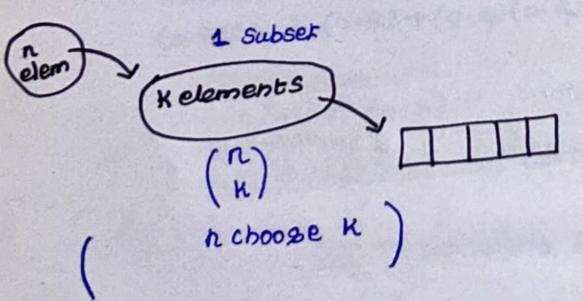
$$P(1) \cdot P(3) \cdot P(3) \cdot P(4) \cdot P(5) \cdot P(6) = \left(\frac{1}{6}\right)^6$$

∴ Independent, Fair.

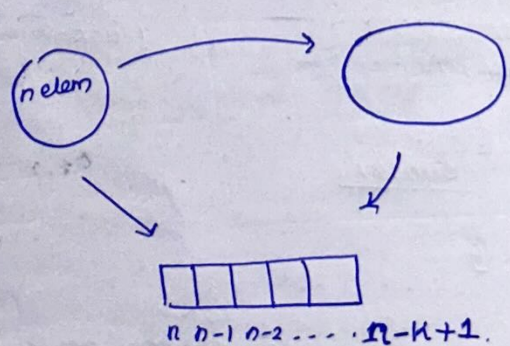
→ All give different numbers = A

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6\times5\times4\times3\times2\times1}{6^6} = \frac{6!}{6^6} = \frac{6!}{6^6} = 0.0154$$

Subset has K elements:

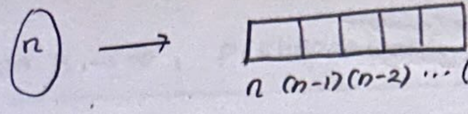


out of n elements, choose K



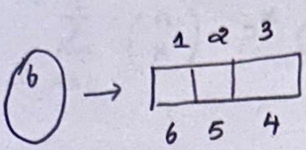
1) from n elements → order K people (out of n)

2) from n elements → choose K → Then order K

i) 

k elements.

$n(n-1)(n-2)\cdots(n-k+1) = ?$



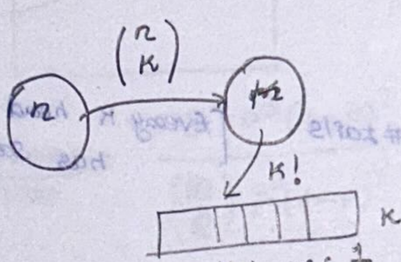
$(n-1)(n-2)(n-3)$ where $n-3 = (n-k+1) = 4$.

$$\boxed{\therefore\ n(n-1)(n-2)\cdots(n-k+1) = ?}$$

$n\cdot(n-1)(n-2)\cdots 1 = n!$

$n\cdot(n-1)(n-2)\cdots\cdots(n-k+1)(n-k)(n-k-1)(n-k-2)\cdots 1 = n!$

$$\boxed{n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}}$$

2) group 

choose k people out of n'

$$\therefore \frac{n!}{(n-k)!} = \binom{n}{k}\cdot k!$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$ → choose k people out of n

Binomial co-efficients. [how many k-element subset from n elements]

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i) $k=n$

$\binom{n}{n} = \frac{n!}{n!\,0!} = \frac{1}{0!} = 1$

∴ {} → empty set

"one ordered list is possible"

ii) $\binom{n}{n}$ → choose n from n → 1 subset (having n elements)

$k=0$ $\binom{n}{0} = \frac{n!}{n!\,k!} = 0$ → 1 zero subset can be built

Properties

$$\sum_{k=0}^{n} \binom{n}{k} = \underset{\text{subset}}{0\text{ element}} + \underset{\text{e.s}}{1} + \dots + \underset{\text{subset}}{n\text{element}}$$

$$= 2^n \text{ [Total numbers of subsets]}$$

$\{1,2,3\} \to \{1\}$
$\to \{2\}$
$\binom{3}{1} = 3 \to \{3\}$

$\binom{3}{2} = 3 \to \{1,2\}$
$\to \{2,3\}$
$\to \{3,1\}$

$\{\}$
$\{1,2,3\}$

8 subsets.

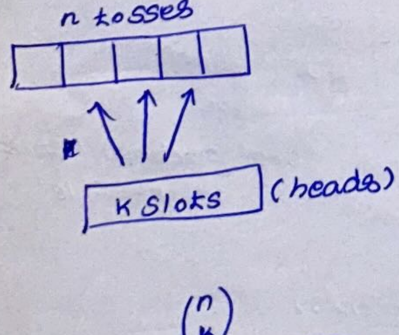
---

* n independent coin tosses
* $P(H) = P$
* $P(HTTHHH) = P(1-P)(1-P)P^3$
  $= p^4(1-P)^2$
* $P(\text{sequence}) = P^{\#\text{heads}} \cdot (1-P)^{\#\text{tails}}$ [Every k head sequence has same prob]
* $P(\text{k heads}) = \sum_{\text{k-head seq}} P(\text{seq})$

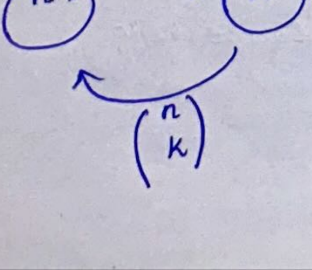
$$\boxed{[P(101) + P(011) + P(110)] = 3\,P(101)}$$

$$= (\#\text{ of k-head seq})\ P^k (1-P)^{n-k}$$

$$\boxed{P(\text{k heads}) = \binom{n}{k} p^k (1-P)^{n-k}}$$



$$\boxed{\#\text{ of k-head seq} = \#\text{ k-element subsets of } \{1,\dots,n\}}$$



$K = 0, 1, \ldots n$ $\qquad\qquad$ $K = 0, 1, \ldots, n$

when $K \to \infty$, $P(K\text{heads}) \to 0$

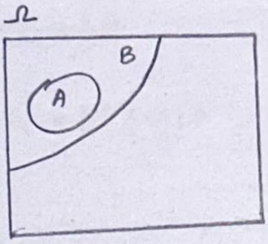
$$\sum_{K=0}^{n} \binom{n}{K} p^K (1-p)^{n-K} = \qquad \longrightarrow \text{property.}$$

$$\sum_{K=0}^{n} P(K \text{ heads}) = P(0) + P(1) + \ldots P(n)$$
$$= 1.$$

B = * 3 out of 10 tosses were heads (Independent)

* Gn B occured, prob of first two tosses are head.

solu:



→ All outcomes in set B are equally likely.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{8\ p^3(1-p)^7}{\binom{10}{3} p^3 (1-p)^7} = \frac{|A \cap B|}{|B|}.$$

$$= \frac{8}{\binom{10}{3}}$$

$$= 0.0667$$

10 tosses
↓
3 heads
↓
first 2 heads
rem head has 8 possibilities

∴ we are in B : 3 heads

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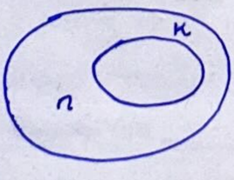
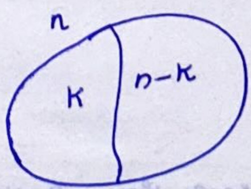
* 52 card deck - dealt to 4 players, Find P(each gets an ace)

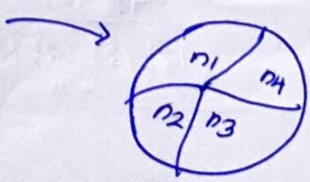
* outcome : a partition of the 52 cards : no. of outcomes?

solu:



(or)





n_1, n_2, n_3, n_4

$$\boxed{\sum_{i=0} n_i = n}$$

4 subsets of 13 subsets.

$\boxed{52 \text{ cards}}$

$\boxed{\text{4 Subsets}}$

$$\binom{52}{13} \times \binom{39}{13} \times \binom{26}{13} \times \binom{13}{13}$$

$$= \frac{52!}{13!\,39!} \times \frac{39!}{13!\,26!} \times \frac{26!}{13!\,13!} \times \frac{13!}{13!\,0!}$$

$$= \frac{52!}{13!\,13!\,13!\,13!}$$

Partitions : 4 players, 52 cards → 13 cards per head

General:

$$\frac{n!}{n_1!\,n_2!\,n_3!\,n_4!}$$

$n!$ → No. of elements.

$n_1!\,n_2!\,n_3!\,n_4!$ → 4 subsets

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5) Each player gets an ace.

Solu:

Take 4 ace → distribute each randomly.

4 choices $\left(\frac{1}{4}\right)$

4	3	2	1

$= 4!$

Remaining cards - partition

$$\frac{48!}{12!\,12!\,12!\,12!}$$

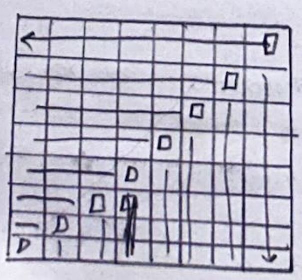
[Cardinality: no. of elements of a set]

Totally:

$$\frac{4!\,48!}{12!\,12!\,12!\,12!}$$

Recitation - Rooks

$P(A) = \frac{|A|}{|\Omega|}$



Place all 8 rooks → P(safe arrangement)
↓
Neither in same row nor in same column

solu:-

8 Ro → 64 squares

combination

$$P(\text{safe arrangement}) = \frac{|\text{safe arrangements}|}{|\text{total arrangements}|} = ?$$

Not safe yet.

- 1 rook → 64 slots
- 2 rook → 63 (normal) → safe (not on same row or column)
- 3 rook → 62
- ⋮
- 8 rook → 57 spots

Counting principle: when you have a process done in stages - in each stage we have particular number of choices. To get the total number of choices - multiply no. of choices at each stage.

- wheat
  - Ham
  - Turkey
  - S
- bread
  - H
  - T
  - S

$(2 \times 3) = 6$ choices.

$$\#\text{total arrangement} = 64 \times 63 \times \cdots \times 57 = \frac{64!}{56!}$$

$$P(\text{safe arrangement}) = \frac{|\text{safe arrangement}|}{|\text{total arrangement}|} = \frac{64 \times 49 \times 36 \times 25 \times 16 \times 9 \times 4}{\frac{64!}{56!}}$$

$= 0.0000091$

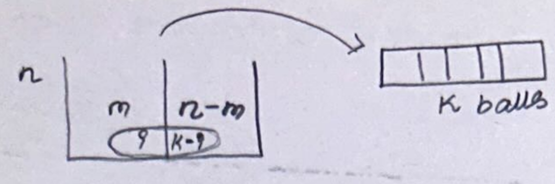
- 1 rook → 64
- 2 rook → 49
- 3 rook → 36
- 4 rook → 25
- 5 rook → 16
- 6 rook → 9
- 7 rook → 4
- 8 rook → 1

Hypergeometric problems

An urn has n balls, out of which m are exactly red. We select K of the balls at random, without replacement.

what's the prob that i of the selected balls are red?

Solu:



* $P(i \text{ red balls}) = ? = P_r$

$|\Omega| = \binom{n}{k}$ ∴ out of $\underbrace{K \text{ balls}}_{\Omega}$, $\underbrace{\text{how many are red?}}_{\text{Event.}}$

$c = \#$ ways to get i balls.

∴ A particular sequence of red balls in be obtained in many ways.

∴

∴ As we did [ace] & [other cards]

K elements

$$\underbrace{\binom{m}{i}}_{\text{Red}} \times \underbrace{\binom{n-m}{k-i}}_{\text{other balls}}$$

$\boxed{(TTH) = (1-P)^2 P}$

Reason

(sequence)

i red from m balls — $(K-i)$ other balls from $(n-m)$ balls

'stages'

$$c = \binom{m}{i} \times \binom{n-m}{K-i}$$

$$P_r = \frac{\binom{m}{i}\binom{n-m}{K-i}}{\binom{n}{K}}$$

Deck

Aces $m=4$	Remaining cards

$K = 7$ cards

∴ $\# \text{poss}(\Omega) = \binom{52}{7}$

$$P(\text{Aces}) = \binom{m}{\text{Aces}}\binom{n-m}{K-\text{Aces}}$$

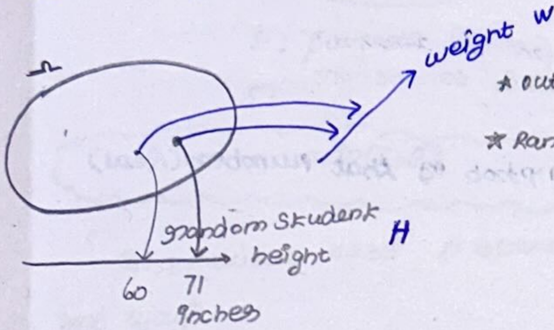
$$P(3\ aces) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} = \frac{4 \times 194580}{133784560} = 0.0058$$

Lecture-5

Discrete R.v ; probability mass function ; Expectation

Random variables:

* ways of defining a random number (value) to every possible outcome. [numerical results to outcomes].



* outcome: random student
* Random variable: Height

$H(x)$ = Height random function.

* single prob experiment may involve several random functions

$$\bar{H} = 2.5H \text{ (Height in cm)}$$

$$\begin{array}{ccc} 60 & 71 & \to H \\ \downarrow & \downarrow & \\ 150 & & \to \bar{H} \end{array}$$

$\bar{H}$ (Random variable - function of random variable)

A function not a variable.

* Random variable: Function from the sample space to real value

* $H \to$ Function takes the outcome of x and gives the boys height in inches
* $W \to$ " weight in kg
* $\bar{H} \to$ Take H as argument - gives height in $\bar{H}$.

$\bar{H}$ depends on H which in turn depends on H

$$\text{R.V} \swarrow \text{Discrete} \quad \searrow \text{Continuous}$$

Rounds: (Integer Answer) $\to$ Discrete R.v only precise → continuous random variable (61.7....... kg)

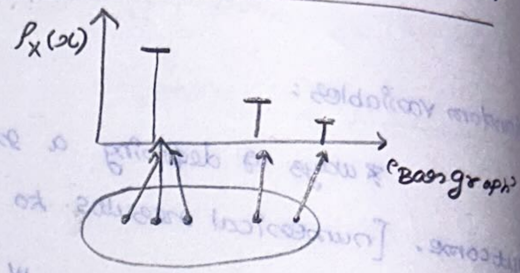
Notation:

* Random variable X (Function) [Student as i/p, height as o/p]
* x → numerical value of that function. $\in$ Real. ($\mathbb{R}$)

Example: X = number of coin tosses until first head

Assume: Independent tosses

$P(H) = p > 0$



$P_X(K) = P(X = K)$

$= P(TT....TH)$

'How likely a particular numerical value can occur'

Bar graph → prob of that number (Real)

X → function

x → o/p of function

$p_X(x)$ → probability of particular o/p of a random variable X.

$$p_X(x) = P(X = x)$$

$$= P(\{\omega \in \Omega \text{ s.t } X(\omega) = x\})$$

* $P(X = x) \geq 0$ $\boxed{\sum_x p_X(x) = 1}$

$p_X(K) = P(X = K)$ → until a head: no. of attempts. [K th attempt → head]

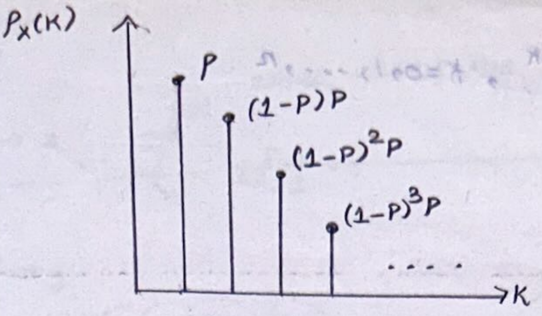
$= P(TT...TH)$ → 'Independent' $\underbrace{TTT...}_{K-1}$ $\underset{K\text{ th time}}{H}$

$\boxed{P_X(K) = (1-p)^{K-1} p}$ $K = 1, 2, 3, ...$

Note: Repeat until a head.

$P_X(K) = (1-p)^{K-1}\ p$ → PMF for this one.

why?



Geometric sequence.

$X \rightarrow$ geometric random variable (No. of tosses until a head)

Two independent rolls of a fair tetrahedral die.

F : outcome of 1) throw
S : " 2)

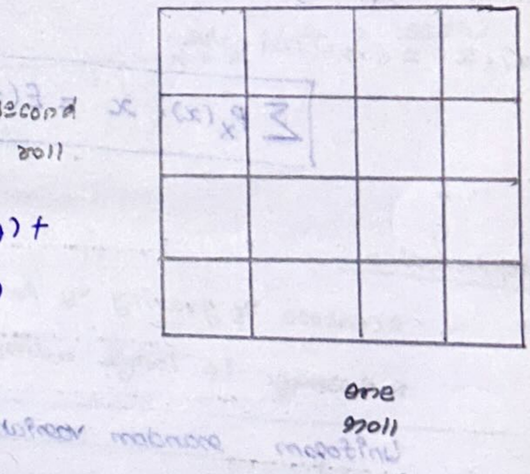
$X : \min(F, S)$

In previous case A sequence can be attained through only one way?

In case - A answer (o/p of X) - attained by many (ω) where

$\boxed{\omega \in \Omega}$ — we must calculate all.

One entry of pmf:



$$P_X(2) = P(2,2) + P(2,3) + P(2,4) + P(3,2) + P(4,2)$$

↓ minimum 2

$$= 5/16 \text{ (uniform)}$$

Output : 2 → lead by 5 possible model elements.

Binomial PMF

* X : Total number of heads. (Independent toss)

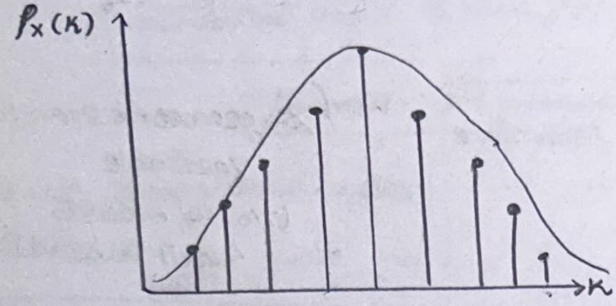
$P(H) = P$ → 4 tosses

$$P_X(2) \approx P(HHTT) + P(HTHT) + P(HTTH) + P(THHT) + P(THTH) + P(TTHH)$$

$$= 6P^2(1-P)^2$$
$$= \binom{4}{2} P^2 (1-P)^2$$

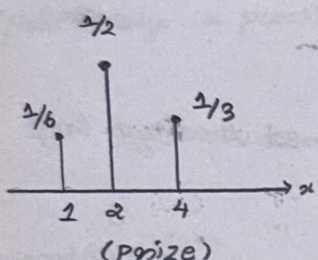
In general:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad , \; k = 0, 1, \dots, n$$



when n is big $\rightarrow$ bell curve

Expected value (loosely: Average)



$$\frac{1}{6}\cdot 1 + \frac{1}{2}\cdot 2 + \frac{1}{3}\cdot 4 = 2.5 \text{ (Average)}$$

Frequencies: with which we get different payoffs

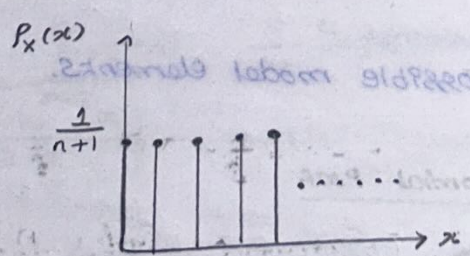
$\sum P_X(x)\cdot x$ = Expected value.

$$\boxed{\sum P_X(x)\cdot x = E(x)} \rightarrow \text{From above } (2.5)$$

Interpretation:

* Centre of gravity of PMF
* Average in large numbers of repetitions of the exp

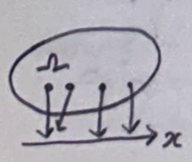
Uniform random variable

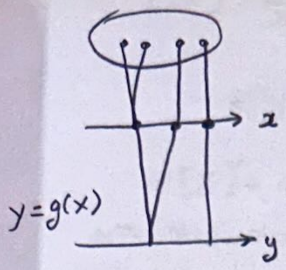


[From 0] $\rightarrow \frac{1}{n+1}$

"Symmetry" – midpoint $\rightarrow \frac{n}{2}$

Properties of Expectations





what will be the expected value of Y

$E[Y] = ?$

'Collects the outcomes of x'

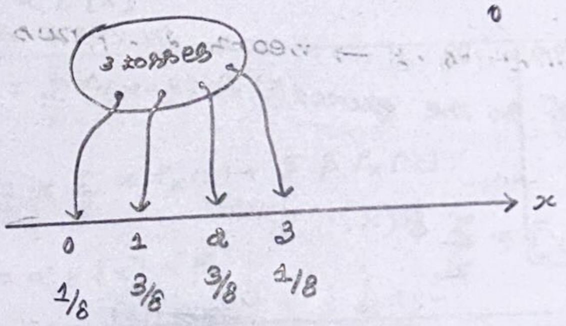
$\boxed{\text{X determines the probability of Y.}}$

$$\boxed{E[Y] = \sum_{y} y\, P_Y(y)}$$

'So every o/p of Y (function) multiplied by that o/p's probability'

Analysis

TTT	HTT, THT, TTH	HHT, THH, HTH	HHH
0	1	2	3



(uniform)

$$E(x) = 0\cdot\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = \frac{3}{2} = 1.5$$

(or)

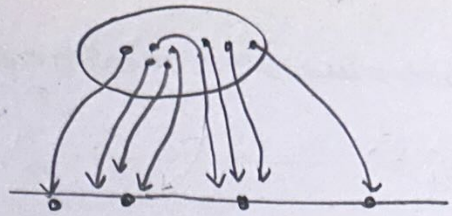
$$E(x) = \left[X(TTT)\cdot\frac{1}{8}\right] + \left[X(HTT)\cdot\frac{1}{8} + X(TTH)\,\frac{1}{8} + X(THT)\,\frac{1}{8}\right]$$

$$+ \left[X(HHT)\,\frac{1}{8} + X(HTH)\,\frac{1}{8} + X(THH)\,\frac{1}{8}\right]$$

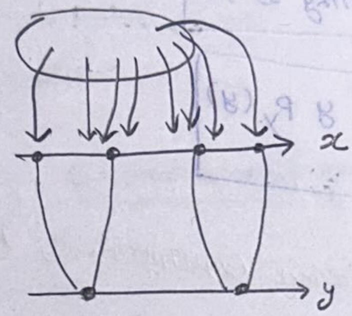
$$+ \left[X(HHH)\ 1/8\right]$$

$$= \sum X(\text{random sample})\cdot P_{\text{random}}(\text{random sample})$$

$\therefore$



we are using the previous step's probability to calculate the expectation. [$\therefore$ Probability of each x is the collection of probability of all samples which results in x]



$$E[Y] = \sum_y y\, P_Y(y)$$

$\therefore$ The probability of $y \rightarrow$ means the probabilities of all x causing y in the function Y.

$$\therefore \quad E[Y] = \sum_x g(x) \cdot P_X(x)$$

[Rearrangement]

$g(x) \downarrow$ result of Y $\qquad$ $P_X(x) \downarrow$ caused by value x

caution:

$E[g(x)] \neq g(E[X])$

Average of functions not same as function of averages.

Properties

If α, β are constant:

$$E[\alpha] = \sum_\alpha \alpha \cdot P_X(\alpha) = \alpha \sum_\alpha P_X(x) = \alpha(1) = \alpha$$

$\rightarrow \alpha$ is the only element

$\downarrow$

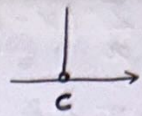
Random variable that takes always α.

[$E[2] = 2$]

i) $E[\alpha] = \alpha$

ii) $E[\alpha x] = \sum_x g(x)\, P_X(x) = ?\ \sum_x$

$E[x] = \sum_{x} x \cdot P_X(x)$



only one value

$E[x] = c \sum_{x} P_X(x)$ $\longrightarrow = 1$ (only one o/p) $\to$ sure

$E[x] = c$ $\longrightarrow$ constant function

$E[2] = 2$

---

ii) $E[2.5H]$ = Average (values of H multiplied by H)

$= 2.5$ Average (H)

$= \sum_{x} 2.5\, x \, P_X(x)$

$\boxed{g(x) = 2.5x}$

$= 2.5 \sum_{x} x P_X(x)$

$= 2.5\, E[\cancel{H}]$

$\therefore E[\alpha X] = \alpha E[X]$

$E[\alpha X + \beta] = \sum_{x} (\alpha x + \beta) P_X(x)$

$= \alpha \sum_{x} x\, P_X(x) + \sum \beta P_X(x)$

$= \alpha E[X] + \beta$
(Linear)

If I sum 2 to all o/p of x the Average $\uparrow$ by 2

$\therefore E[g(x)] = g(E[X]) \longrightarrow$ only g is linear.

Average value of x^2: $E[X^2] = \sum_{x} x^2 P_X(x)$

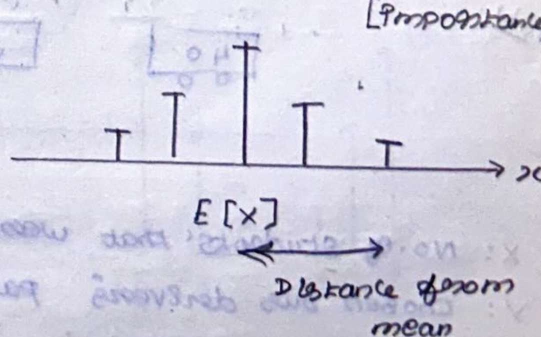
Variance

$var[X] = E[(X - E[X])^2] \longrightarrow$ R.V

X $\downarrow$ R.V, $E[X]$ $\downarrow$ R.V, together R.V

Squaring - special emphasis for large values [importance]

Variance = Average of the squared distance from the mean



$$= \sum_x (x - E[X])^2 P_X(x)$$

$$= \sum_x (x^2 - 2x(E[X]) + (E[X])^2) P_X(x)$$

$$= \sum_x x^2 P_X(x) - 2 \sum_x x E[X] . P_X(x) + \sum_x (E[X])^2 P_X(x)$$

$$= E[X^2] - 2(E[X])^2 + [E[X]]^2$$

$\therefore E[X] \rightarrow$ Real valued (constant)

$$= E[X^2] - (E[X])^2$$

Importance:

* How spread out a distribution is?
* Tightly concentrated?

(Large deviation or small deviation)

variances are always non negative.

$var[X] \geq 0$

$var(\alpha X + \beta) = \alpha^2 var(X)$

$\therefore$ Adding constant shifts right $\rightarrow$ Not disturbance the spread.

* multiply the random variable: variance goes up by their square

---

PMFs & Expectation

4 Buses carrying 148 - Job seeking MIT students arrive at a job convention. The buses carry 40, 33, 25, and 50 students. One of the students is randomly selected. If X denote the no. of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on his bus

a) $E[X]$ or $E[Y] \rightarrow$ which is large

b) Compute $E[X]$ & $E[Y]$

Sol:

40 33 25 50

Total = 148 students.

X: No. of students that were on the same bus with chosen one

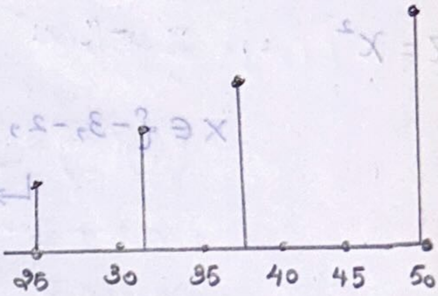
Y: Chosen bus drivers passengers

$E(x)$: may be high. As $\frac{50}{148}$ students are in the bus 4. so higher prob of getting selected. So Expectency may be higher.

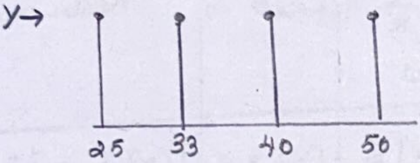
As drivers is selected: $1/4$ uniform: The answer will be some what lower than X.

$$p_X(x) = \begin{cases} 40/148, & x=40 \\ 33/148, & x=33 \\ 25/148, & x=25 \\ 50/148, & x=50 \end{cases} \qquad p_Y(y) = \begin{cases} 1/4 & y=40 \\ 1/4 & y=33 \\ 1/4 & y=25 \\ 1/4 & y=50 \end{cases}$$

Assume: Any point in A occuring



'favoured towards biggest'



$$E[X] = \sum_x x\, p_X(x)$$

$$= 40\left(\frac{40}{148}\right) + 33\left(\frac{33}{148}\right) + 25\left(\frac{25}{148}\right) + 50\left(\frac{50}{148}\right)$$

$$= 39$$

$$E[Y] = \frac{1}{4}(40) + \frac{1}{4}(33) + \frac{1}{4}(25) + \frac{1}{4}(50)$$

$$= 37$$

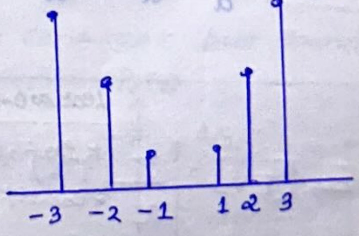
PMF of a function of a random variable

$p_X(x) = \frac{x^2}{a}$, $x \in \{-3,-2,-1,1,2,3\}$, Find a

Find PMF of $Z = X^2$

Solu:

$p_X(-3) = \frac{9}{a}$, $p_X(-2) = \frac{4}{a}$, $p_X(-1) = \frac{1}{a}$, $p_X(1) = \frac{1}{a}$, $p_X(2) = \frac{4}{a}$, $p_X(3) = \frac{9}{a}$



$$\sum_x p_X(x) = 1$$

$$2\left(\frac{9}{a} + \frac{4}{a} + \frac{1}{a}\right) = 1$$

$$\frac{14}{a} = \frac{1}{2}$$

$$\boxed{a = 28}$$

PMF of $Z = X^2$ $\qquad Z = X^2$

$P_Z[Z] = ?$

$P_Z(k) = P_k(Z = k)$

<u>what values Z takes [o/p of X]:</u>

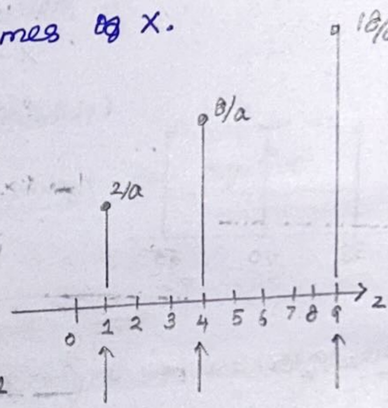
$P_X(x) = P_X(-3) = \frac{9}{28}, P_X(-2) = \frac{4}{28}, P_X(1) = \frac{1}{28}, P_X(-1) = \frac{1}{28}, P(2) = \frac{4}{28}$

$P(3) = \frac{9}{28}$

$Z = X^2$

$X \in \{-3, -2, -1, 1, 2, 3\}$

$\hookrightarrow$ outcomes of X.



$Z \in \{9, 4, 1\}$

$P_Z(K) = P(Z = K)$

Z takes 1 when X takes 1 or −1
4 when X takes 2 or −2
9 when X takes 3 or −3

∴ prob of X takes 1 is $\frac{1}{a}$, −1 is $+\frac{1}{a}$

2 is $\frac{4}{a}$, −2 is $+4/a$

3 is $9/a$, −3 is $9/a$

$\{Z = 1\} = \{X = -1\} \cup \{X = 1\}$

$P(Z=1) = \frac{1}{a} + \frac{1}{a} = \frac{2}{a} = \frac{2}{28}$

$P(Z=2) = \frac{4}{a} + \frac{4}{a} = \frac{8}{a} = \frac{8}{28}$

$P(Z=3) = \frac{9}{a} + \frac{9}{a} = \frac{18}{a} = \frac{18}{28}$

$P_Z(K) = \frac{2x^2}{a} = \frac{2x}{28}$

$$\begin{cases} \frac{2}{28} & K = 1 \\ 8/28 & K = 4 \\ 18/28 & K = 9 \\ 0 & \text{otherwise} \end{cases}$$

<u>Lecture-6 Discrete R.V - II</u>

* PMF, $E[X]$, var $[X]$
* Conditional PMF
* Geometric PMF
* Total expectation theorem
* JPMF.

* Random variable: function from Ω to the real numbers.

* PMF (For discrete variable)

$$p_X(x) = P(X = x)$$

* Expectation:

$$E[X] = \sum_x x \, p_X(x)$$

$$E[g(x)] = \sum_x g(x) \, p_X(x) \qquad \left[\Sigma[g(x)] = g(E[X]) \text{ when } g(x) \text{ is linear} \right]$$

$$E[\alpha X + \beta] = \alpha E[X] + \beta$$

* $E[X - E[X]] = \sum_x (x - E[X]) \, p_X(x)$

↓

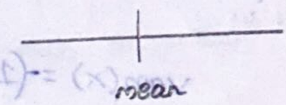
How far you will be from mean

$$= \sum_x x \, p_X(x) - \sum_x E[X] \, p_X(x)$$

$$= E[X] - \sum_x E[X] \, p_X(x)$$

↳ Real numbers.

$$\boxed{E[X - E[X]] = E[X] - E[X] = 0}$$



'Some times X - left of mean, right of mean' → Average 0

Interest

$|X - E[X]| \longrightarrow \therefore (E[X - E[X]] \rightarrow$ Not telling much)

(or)

$\text{var}[X] \longrightarrow$ more useful than $|X - E[X]|$

$$\text{var}(X) = E[(X - E[X])^2]$$

$$= E[X^2] - (E[X])^2 \rightarrow \text{units are not right}$$

If $X \rightarrow$ metres, var $\rightarrow$ metres^2

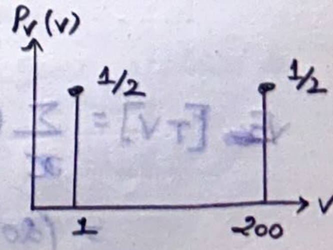
$\therefore$ Different units.

$$\sigma \text{ (standard deviation)} = \sqrt{\text{var}(X)} \rightarrow \text{Same unit}$$

---

* Traverse a 200 mile distance at constant but random speed V.

(Newyork → Fly (200 mile/hr)
→ walk (1 mile/hr)



* $d = 200$, $T = t(V) = 200/v$
(time)

$T = \frac{\text{distance}}{\text{speed}}$

$$E[V] = \frac{1}{2}(1) + \frac{1}{2}(200)$$

$$\boxed{E[V] = 100.5}$$

$$var(V) = (E[x^2] - (E[x])^2)$$

$$E[x] = \sum_x x P_x(x)$$

$$E[x^2] = \sum_x x^2 P_x(x)$$

$$= (1)\left(\frac{1}{2}\right) + (200)^2\left(\frac{1}{2}\right)$$

$$= 0.5 + 20000$$

$$= 20000.5$$

$$(E[x])^2 = (100.5)^2 = 10100.25$$

$$var(x) = -(10100.25) + (20000.5)$$

$$\boxed{Var(x) = 9900.25}$$

(or)

$$var(V) = \frac{1}{2}(1-100.5)^2 + \frac{1}{2}(200-100.5)^2$$

$$\approx 100^2$$

How far are the points (Avg)

$\boxed{\sigma_V = 99.5}$ → Large amount of spread.

∴ $V \to$ Random variable [walk ⇝ 1, Fly ⇝ 200]

$T \to$ Random variable

$$E[T] = E[t(V)] = \sum t(v) P_V(v) = \frac{1}{2}\frac{(200)}{1} + \frac{1}{2}\frac{200}{200}$$

$$= 100 + 0.5$$

$$= 100.5$$

$$[TV] = \sum_x (TV) P_x(x)$$

∴ T, V both depends on x

$$= (200)(1)\left(\frac{1}{2}\right) + (1)(200)\left(\frac{1}{2}\right) = 200.$$

As speed = 200 miles/hours, time = 1 hours

$\therefore$ As speed = 1 miles/hours, time = 200 hours.

Note: $E[TV] = 200 \neq E[T]\,E[V]$

$$\star\ E[200/V] = \left(\frac{200}{1}\right)\frac{1}{2} + \left(\frac{200}{200}\right)\frac{1}{2} = 100.5 = E[T]$$

$$200/E[V] = \frac{200}{200(\frac{1}{2})+1(\frac{1}{2})} = \frac{200}{100.5} = 1.990$$

$$\boxed{\therefore\ E\left[\frac{200}{V}\right] \neq \frac{200}{E[V]}} \rightarrow \text{Non linear.}$$

$$E[TV] = \underbrace{E[T]\,E[V]}_{\text{Non linear.}}$$

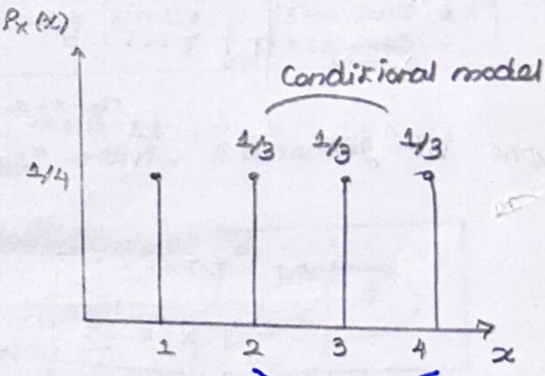
$E[g(x)] \neq g[E[x]]$

Non linear

Conditional PMFs

$\star\ p_{X|A}(x) = P(X = x|A) \longrightarrow$ Some body tells A has occurred.

$\star\ E[X|A] = \sum_x x\, p_{X|A}(x)$



Let $A = \{X \geq 2\}$

$$p_{X|A}(x) = P(X = x|A)$$

$$= 1/3 \quad [x = 2, 3, 4]$$

New model
(Three possible
↓
uniform)
↓
Equally likely.
(Rescaled)

$\star\ E[X|A] = \sum_x x\, p_{X|A}(x)$

↓

use conditional probability.

$$= \left(\frac{1}{3}\right)2 + \left(\frac{1}{3}\right)3 + \left(\frac{1}{3}\right)4$$

$$= 3 \text{ (By symmetry)}$$

'They are also like ordinary expectation just applied to conditional universe'

$$\boxed{E[g(x)|A] = \sum_x g(x)\, p_{X|A}(x)}$$

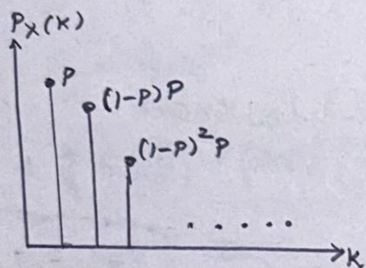
"Every formula has a conditional counterpart". Expectation got replaced by a conditional expectation & prob by a cond. prob.

Geometric PMF

"Heads for first time"

$$p_X(k) = (1-p)^{k-1} \cdot p \quad , k = 1, 2, \dots$$

$$E[X] = \sum_{k=1}^{\infty} k\, p_X(x) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p \quad , k \geq 1.$$



$Y \rightarrow$ TTT.....H
(Experiment)

Conditional world $[X > 2]$

$X \rightarrow$ flipping (Before Y is starting - be done two tosses)

$X \rightarrow$ TT | TT.....H

$Y \rightarrow$ | TTT.....H

who will get the Head first [∵ coin - Independent]

TT $\underbrace{\text{TT.....H}}_{x-2 \text{ times}}$ (x times) [first two tails]

Prove: TT | TTT.....H
TTT.....H $\rightarrow$ same prob 'memoryless'

'past doesn't affect the future'

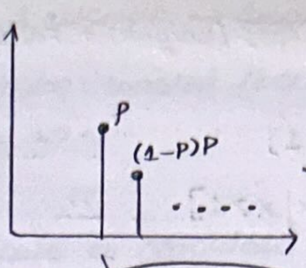
$\boxed{p_X(k)}$ we have information about person X

$$X - 2 \,|\, X > 2$$

$\downarrow$

Didn't had success in 1st 2 attempts.

$p_{X|X>2}(k)$



→ shifted version of normal

→ same as who is starting late.

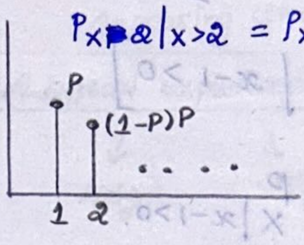
'same distribution as Y's experiment'

$p_{X-2|X>2}$ → starting fresh after 2 experiments (fail)

↓

Remaining tosses

$p_{X-2|X>2} = p_X(k)$



'starting from fresh after 2 failures - same as starting new'

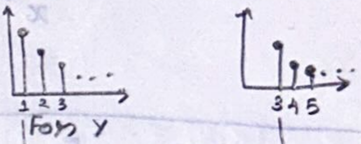
'memorylessness'

X → TT | TTT.... H

Y → TTT.... H

↓

same distribution (w.r.t time)

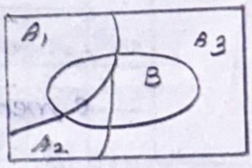


Time same
[3rd toss → X
1st toss → Y]

'same distribution'

Expected value? → Divide & conquer tool

$P(B) = P(A_1)\,P(B|A_1) + \dots + P(A_n)\,P(B|A_n)$



$p_X(x) = P(A_1)\,p_{X|A_1}(x) + \dots + P(A_n)\,p_{X|A_n}(x)$

$E[X] = P(A_1)\,E[X|A_1] + \dots + P(A_n)\,E[X|A_n]$ (In terms of R.V)

may happen in different scenarios

multiplied by x taking summation

$\sum_x x\,p_X(x) = \sum_x x\,P(A_1)\,p_{X|A_1}(x) + \dots + \sum_x x\,P(A_n)\,p_{X|A_n}(x)$

$\boxed{E[X] = P(A_1)\,E[X|A_1] + \dots + P(A_n)\,E[X|A_n]}$ → 'weighted /mean comb. of diff scenarios'

↓ weighted (based on prob)

'1 random variable - occurring w.r.t to other partitions'

Geometric example

$A_1 : [X = 1]$, $A_2 : [X_2 > 1]$

↳ first toss tail.

↳ first toss head

E[x] = heads in first toss (weight) + tails in 1 toss

$$E[x] = P(x=1)\,E[x \mid x=1] + P(x>1)\,E[x \mid x>1]$$

$\boxed{E[x \mid x>1] = ?}$

'Simplifying'

$$E[x \mid x>1] = E[x \mid x-1 > 1-1]$$

$$= E[x \mid x-1>0]$$

$$= E[x-1+1 \mid x-1>0]$$

$$E[x-1+1 \mid x-1>0] = \sum_x (x-1+1)\, P_{X \mid x-1>0}$$

$$= \sum_x (x-1)\, P_{X \mid x-1>0} + 1$$

$$\boxed{\therefore E[x \mid x>1] = E[x-1 \mid x-1>0] + 1}$$

1 st one tails doesn't tell me anything about future.

starting late [X−1]

'memoryless'

$$E[x \mid x>1] = E[x] + 1$$

$$\therefore E[x] = P[x=1]\,E[x \mid x=1] + P(x>1)\,E[x \mid x>1]$$

$$E[x] = P \cdot 1 + (1-P)[E[x]+1]$$

(only one value sure prob)

$$E[x] = P + E[x] + 1 - P - PE[x]$$

$$E[x] = E[x] - P[E[x]] + 1$$

$$P(E[x]) = +1$$

$$\boxed{E[x] = 1/p}$$

'1st toss tail' → wasted 1 attempt → started again.

Info: Not about future: wasted then start again.

A review:

$E[X \mid X > 1] \to$ Difficult to calculate

$$E[X \mid X > 1] = E[X - 1 + 1 \mid X - 1 > 0]$$

$$= E[X - 1 \mid X - 1 > 0] + 1$$

↓

starting from 1 again [after wasting 1 attempt]

A typical experiment - may have several R.V s

↓ Height ↓ weight

Individually the height & weight Pmf doesn't tell how height & weight are related each other.

Solution: Joint Pmf

'How certain X's go together with certain Y's'

$$P_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

Prop:

Pmf s ≥ 0

JPmf - 'captures association with RVs'

$$P_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

$$\boxed{\sum_x \sum_y P_{X,Y}(x,y) = 1}$$

★ $P_X(x) = \sum_y P_{X,Y}(x,y) = ?$ (say $x = 3$)

2/20
1/20
3/20
3

(Add all those 3)

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

Keep x fixed - add diff values of y.

If you know JPmf → we can know marginal pmfS (x) or (y)

Conditional Pmf

$$P_{X|Y}(x|y) = P(X=x \mid \overset{\text{conditional universe.}}{Y=y})$$

$$= \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Say: $y=2$

$$P_{X|Y}(x|y) = P(X=x \mid \overset{\text{new world (fixing)}}{Y=2}) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

→ fixing y, we can tell prob of diff y.

'function of x'

$\sum_x P_{X|Y}(x|y)$ = marginal pdf of that particular y = 1.
rescaled

New universe

$y=2$		$1/20$	$3/20$	$1/20$
	1	2	3	4

But rescaled to the new universe

$$P_{X|Y}(x|2) = P(X=x \mid Y=2) = \frac{P_{X,Y}(x,2)}{P_Y(2)}$$

→ both thing hap

→ cond. thing

Different values of x

Say $x=1$

$$= \frac{1/20}{5/20}$$

$= 1/5$ (Rescaled to new world)

Summing cond pmf all x: gives marginal distribution of y.

$$\boxed{\sum_{x} P_{X|Y}(x|y) = 1}$$

Note:

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Flipping a coin a Random no. of times?

Roll die $\Rightarrow N \in \{0, 1, 2, 3\}$ → Faces labelled (Thrown once)

As per result $N \rightarrow$ the coin is to be rolled until K number of heads in N tosses.

$N \rightarrow$ result of die

$K \rightarrow$ Total heads in N roll.

solu:

say: $N = 3 \Rightarrow HHT \Rightarrow K = 2$

i) $P_N(n) = \begin{cases} 1/4 & \text{if } n \in \{1, 2, 3, 0\} \\ 0 & \text{otherwise} \end{cases}$

ii) $P_{N,K}(n, k) = P(N = n \text{ and } K = k)$

This case after knowing result of N – tossing the coin)

$$= P(K = k \mid N = n) \cdot P(N = n)$$

$$= \frac{1}{4} P(K = k \mid N = n), \quad n \in \{0, 1, 2, 3\}$$

K \ n	0	1	2	3
3	0	0	0	1/32
2	0	0	1/16	3/2
1	0	1/8	2/16	3/32
0	1/4	1/8	1/16	1/32

0 heads, 0 rolls

$n = 0$

$$= \frac{1}{4} P(K = 0 \mid n = 0)$$

$$= 1$$

only one is possible 0 heads

$n = 1$

$$= \frac{1}{4} P(K = k \mid n = 1)$$

$$= \frac{1}{4} \cdot \frac{1}{2}$$

1 toss → head, → tail

★ $P(\text{one head} \mid \text{one toss}) = \frac{1}{4} \cdot \frac{1}{2}$, $P(\text{zero head} \mid \text{one toss}) = 1/2$

★ $P(\text{zero head} \mid \text{two toss}) = \underset{\text{2 toss}}{\frac{1}{4}} \cdot \underset{\text{Tail}}{\frac{1}{2}} \cdot \underset{\text{Tail}}{\frac{1}{2}}$

$P(1 \text{ head} \mid \text{two toss}) = \frac{1}{4} \cdot \underset{\text{Head}}{\frac{1}{2}} \cdot \underset{\text{Tail}}{\frac{1}{2}} +$

$\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}$ (T, H)

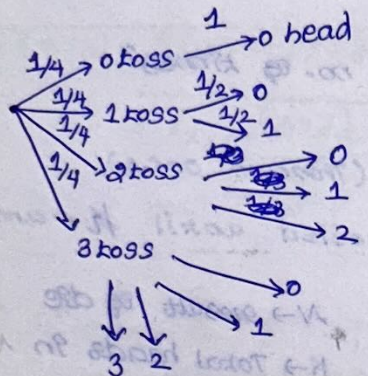
Binomial distribution:

(distribution)

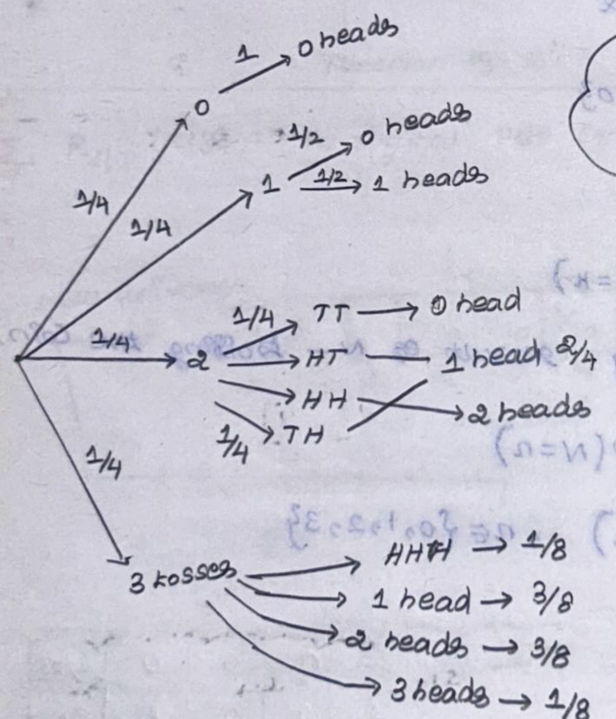
$n \in \{1,2,3\} \in \quad K = \text{Binomial}\,(n, 1/2)$

$\downarrow$ Toss $\quad \downarrow$ Fair.

$P_{N,K}(n,k) = P(K=k \mid N=n)$

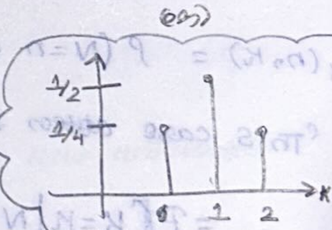


Also (Detailed)



$$P_{K|N}(k|2) = \begin{cases} 1/4, & k=0 \\ 1/2, & k=1 \\ 1/4, & k=2 \\ 0, & \text{else} \end{cases}$$

(or)



$P(3\text{ heads} \mid 3\text{ tosses}) = \frac{1}{4} \times \frac{1}{8} = \frac{1}{32}$

$\downarrow$

3 toss

"Binomial"

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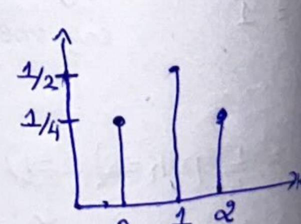
iii) $P_{K|N}(k|2) = ? \longrightarrow P_{K|N}(k|2) = \begin{cases} 1/4, & k=0 \\ 1/2, & k=1 \\ 1/4, & k=2 \\ 0, & \text{else} \end{cases}$

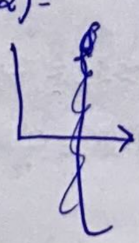


iv) $P_{N|K}(n|2) = ?$

$\downarrow$

$k=2 \rightarrow$ Lock

$P_{N|K}(n|2) =$



Knowing 2 heads what will be in die.

→ K (knowing 2 in die)

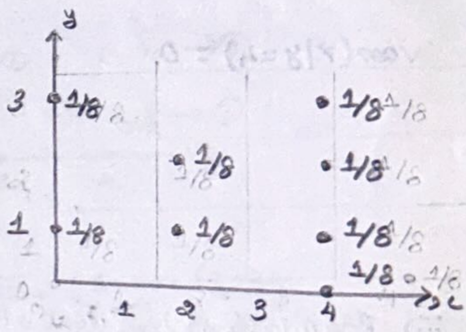
$$p_{N|K}(n|2) = \frac{p_{N,K}(N=n, K=2)}{P(K=2)}$$

$$p_{N|K}(2|2) = \frac{1/16}{5/32} = 2/5$$

$$p_{N|K}(2|3) = \frac{3/32}{5/32} = 3/5$$

$$\begin{cases} 2/5, & n=2 \\ 3/5, & n=3 \\ 0, & \text{else} \end{cases}$$

Joint pmfs



i) maximize $[y \mid x = x]$? [which x]

Solu:

$x = 0$

$E[y \mid x=0] = 2$ [mid point] → maximum at $E[y \mid x=0] = 2$.

$E[y \mid x=2] = 3/2$ [mid]

$E[y \mid x=4] = 3/2$ [mid)

ii) maximize var $(X \mid Y = y)$ → far away. [fixing y]

var $(X \mid Y=y) \Rightarrow$ var $(X \mid y=0) = 0$ [only one point – mid]

var $(X \mid y=1) =$ var $(X \mid y=1) = \text{Exp}[X^2] - [\text{Exp}[X]]^2$

↳ conditioned world

1/8 1/8 1/8

0 2 4

'occurs in same freq'

'likely: 1/8'

$$= E[X^2 \mid y=1] - E[X \mid y=1]^2$$

$$= \Big[\underbrace{0\left(\tfrac{1}{3}\right) + 2^2\left(\tfrac{1}{3}\right) + 4^2\left(\tfrac{1}{3}\right)}_{\text{conditioned universe}}\Big] - \left[2\left(\tfrac{1}{3}\right) + 4\left(\tfrac{1}{3}\right)\right]^2$$

$$= \frac{20}{3} - 2^2 = 2.666.$$

$$\cancel{= \left[\frac{4}{3} + \frac{16}{3}\right] - \left[\frac{9}{3}\right] = \frac{49}{12} = 1.833}$$

$\therefore \text{var}(x|y=0) = 0$

$\text{var}(x|y=1) = 8/3$

$$\text{var}(x|y=2) = \left[2^2\left(\tfrac{1}{2}\right) + 4^2\left(\tfrac{1}{2}\right)\right] - 3^2$$

$$= \left(\frac{4}{2} + \frac{16}{2}\right) - 9$$

$$= 1$$

$$\text{var}(x|y=3) = \left[4^2\left(\tfrac{1}{2}\right)\right] - \cancel{2.5}^2$$

$$= 8 - \cancel{6.25}$$

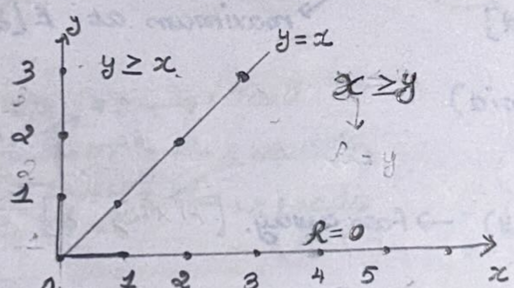
$$= \cancel{1.75}$$

$\text{var}(x|y=4) = 0$

$$\text{var}(x|Y=y) = \begin{cases} 0 & y=0 \\ 8/3 & y=1 \\ 1 & y=2 \\ 4 & y=3 \end{cases}$$

iii) $R = \min(X, Y) \rightarrow$ neat sketch of $P_R(r)$

<u>solu</u>:



$(0,1) \rightarrow 0$	$(1,1) \rightarrow 1$	$(2,1) \rightarrow 1$	$(3,1) \rightarrow 1$
$(0,2) \rightarrow 0$	$(1,2) \rightarrow 1$	$(2,2) \rightarrow 2$	$(3,2) \rightarrow 2$
$(0,3) \rightarrow 0$	$(1,3) \rightarrow 1$	$(2,3) \rightarrow 2$	$(3,3) \rightarrow 3$

<u>only points with probabilities</u>