Since real polynomials are special cases of complex polynomials, the fundamental theorem of algebra applies to them too. For real polynomials, any complex roots will occur with their complex conjugate pairs.

5.2. Degree 3 polynomial z^3+z^2-z+15 factors as $(z+3)(z-1-2i)(z-1+2i)$. $\rightarrow$ Has 3 distinct roots: $-3, 1+2i, 1-2i$ ($-3 \rightarrow$ Real, $1+2i$ and $1-2i$ form a complex conjugate pair).

5.3 Want a fourth root of i? The Fundamental theorem of algebra guarantees that $z^4-i=0$. has a complex solution (infact, four of them).

The fundamental theorem of algebra will be useful for constructing solutions to higher order linear ODEs with constant coeff and for discussing Eigen values.

$$\sqrt[4]{i} = \left(e^{i\left(\frac{\pi}{2}\right)}\right)^{1/4} = e^{\frac{i\pi}{8}} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$$

$$\approx 0.924 + 0.383i$$

Finding roots:

$$z^5 = -32$$

Solu:

$$(re^{i\theta})^5 = (32\, e^{i\pi})$$

$$r^5 = 32$$

$$\boxed{r = 2}$$

$$r^5 e^{5i\theta} = 32\, e^{i\pi}$$

$\therefore$ $5\theta = \pi + 2\pi K$ ($\because$ periodic)

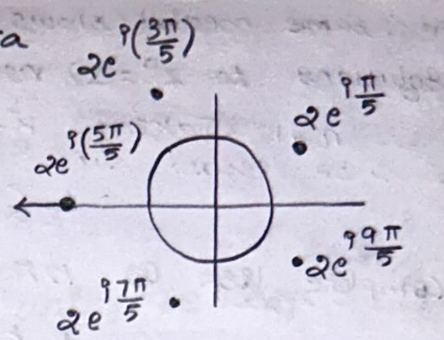
$\theta = \dfrac{\pi + 2\pi K}{5}$ for some integer K.

$z = 2e^{i\left(\frac{\pi+2\pi K}{5}\right)}$ for some integer K.

$K = 0, 1, 2, 3, 4, \ldots$

$$2e^{i(\pi/5)},\ 2e^{i(3\pi/5)},\ 2e^{i(5\pi/5)},\ 2e^{i(7\pi/5)},\ 2e^{i(9\pi/5)}.$$

The fundamental theorem of algebra predicts that the polynomial z^5+32 has 5 roots when counted with multiplicity. we found 5 roots, so each must have multiplicity 1.



Complex roots activity:

1) If a complex number z has modulus $|z| = 1$, then its nth roots also have modulus $|z^{1/n}| = 1$; If $|z| > 1$, then $|z^{1/n}| > 1$; If $|z| < 1$ then $|z^{1/n}| < 1$.

2) If a complex number z has modulus greater than 1, then the roots lie on a ○le closer to the unit circle than z.

3) If a complex number z has modulus less than 1, then the roots lie on a ○le closer to the unit ○le than z.

4) place the complex number z on the +ve real axis. For what values n does z have a +ve real root.

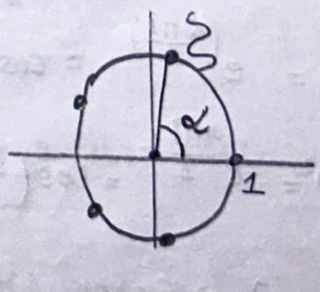
5) place the complex number z on the (−)ve real axis. For what values n does z have a (−)ve real root.

$(i)^{1/2} \rightarrow \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \; -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

Roots of unity

$\sqrt[n]{1} \rightarrow$ In complex domain, there are always n answers

Zeta $\zeta = \zeta = \zeta$ $\quad \therefore \alpha = \frac{2\pi}{5}$



$\zeta = e^{i\alpha} = e^{i\left(\frac{2\pi}{5}\right)}$

(5 equally spaced points)

$\zeta^5 = e^{i2\pi} = 1.$ ($\because 2\pi$ and 0 are same)

The same method shows that the n^{th} roots of unity (the solutions to $z^n = 1$) are the numbers $e^{i\left(\frac{2\pi k}{n}\right)}$ for $k = 0, 1, 2, \dots n-1$. Taking $k=1$, gives the number

$\zeta = e^{2\pi i/n}$ (In terms of ζ, the complete list of nth roots of unity is

$$1, \zeta, \zeta^2, \zeta^3, \dots, \zeta^{n-1}$$

$$\left[\zeta^n = 1\right]$$

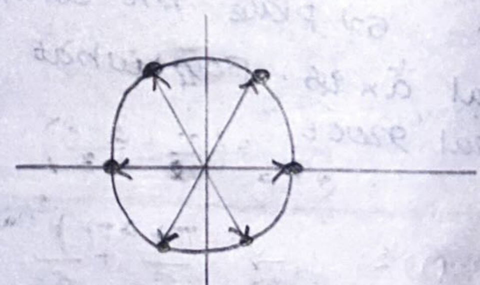
Roots of unity (De moivre numbers):

Is any complex number that yield 1 when raised to some +ve integer power n.

$$\boxed{z^n = 1}$$

$$\exp\left(\frac{2k\pi i}{n}\right) = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

This shows there are n complex nth roots of unity. Geometrically, they all lie on the unit circle in the complex plane. Each root has the form $e^{i\theta}$, which has absolute value 1. The roots are evenly spaced around a circle (unit circle), starting with the root $z=1$, and the angle b/w two consecutive roots is $\left(\frac{2\pi}{n}\right)$.



$z^6 = 1$.

Find in Cartesian form all values of $\sqrt[3]{1}$.

Solu:

cube roots are $1, \omega, \omega^2$

$$\omega = e^{\left(\frac{2\pi i}{3}\right)} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\omega^2 = e^{\frac{4\pi i}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

$\omega \rightarrow$ Greek letter (used traditionally for cube roots).

$$\boxed{\omega^2 = \overline{\omega}}$$

$$\sqrt[4]{i} = e^{i\left(\frac{\pi}{8}+\frac{2\pi K}{4}\right)} = e^{i\pi/8}\, e^{i\frac{2\pi K}{4}}.$$

where, $e^{i\left(\frac{2\pi K}{4}\right)}$ are the 4th roots of unity

$$\sqrt[4]{1} = 1, i, -1, -i$$

Let one of the 4th roots of unity be $z_0 = e^{i\pi/8}$

$$\sqrt[4]{i} = z_0, iz_0, -z_0, -iz_0.$$

Notice: $z_0 = e^{i\pi/8}$ by i is the same as 90° rotation of z_0.

$$x^6 - 2x^3 + 2 = 0$$

$$(x^3)^2 - 2(x^3) + 2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x^3 = \frac{+2 \pm \sqrt{4-4(1)(2)}}{2}$$

$$x^3 = \frac{+2 \pm \sqrt{-4}}{2}$$

$$x^3 = +1 \pm \frac{2i}{2}$$

$$x^3 = -1 \pm i$$

$$x^3 = +1+i, +1-i$$

$$1+i = re^{i\theta}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

$$r = \sqrt{2}$$

$$1+i = \sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}$$

$$(1+i)^{\frac{1}{3}} = 2^{1/6}\, e^{i\left(\frac{\pi}{4}+2\pi K\right)^{1/3}}$$

$$= 2^{1/6}\, e^{i\left(\frac{\pi}{12}+\frac{2\pi K}{3}\right)}$$

$$= 2^{1/6}\, e^{i(\pi+8\pi K)^{1/12}} \qquad = 2^{1/6}\, e^{i(1+8K)^{\pi/12}}$$

$$= 2^{1/6} \cdot e^{i\frac{\pi}{12}} \cdot e^{i\frac{2\pi K}{3}}$$

3 cube roots

$$2^{1/6} e^{i\pi/12},\ 2^{1/6} e^{i9\pi/12},\ 2^{1/6} e^{-i7\pi/12}$$

For $1-i$ $\quad 2^{1/6} e^{-i\pi/12},\ 2^{1/6} e^{-i9\pi/12},\ 2^{1/6} e^{i7\pi/12}$

we may write

$$2^{1/6}\, e^{i(\pi/4+2\pi K)/3} = 2^{1/6} e^{i\pi/12}\, e^{i2\pi K/3}$$

where $e^{i2\pi K/3} = \omega$ where ω is one of the cube root of unity, $\omega^3 = 1$.

The cube roots can also be described as

$$z_1, z_1\omega, z_1\omega^2 \text{ and } z_2, z_2\omega, z_2\omega^2$$

where $z_1 = 2^{1/6} e^{\pi i/12}$, $z_2 = \overline{z_1} = 2^{1/6} e^{-i\pi/12}$, and $\omega = e^{i\frac{2\pi}{3}}$

Doubt:

when $K = 2$

$$e^{i(1+8(2))\pi/12} = e^{17i\pi/12} = e^{i\pi\left(-\frac{5\pi}{12}\right)}$$

$$= e^{i\pi\left(\frac{2-7}{12}\right)}$$

$$\therefore \boxed{\frac{17\pi}{12} = -\frac{5\pi}{12}}$$

$$= e^{i2\pi} \cdot e^{-\frac{7i\pi}{12}}$$

$$= (1) \cdot e^{-\frac{7i\pi}{12}}$$

Complex numbers & Euler's Formula

a) $z = -2 + 3i$ in polar form

b) $3e^{i\pi/6}$ in rect form

c) Draw & lable the Δ^{le} relating rect to polar coordinates

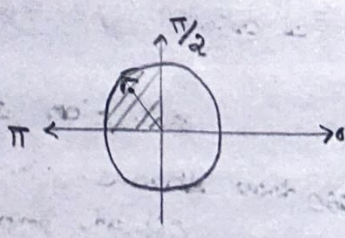
d) $\frac{1}{-2+3i}$ in polar form

e) cube root of 1

Solu:

$r = \sqrt{4+9} = \sqrt{13}$, $\theta = \tan^{-1}\left(-\frac{3}{2}\right)$ → Not an elementary angle.

$$\boxed{a + ib = r\cos\theta + i\sin\theta}$$



$\frac{\pi}{2} \le \theta \le \pi$

Answer.

$$\boxed{\sqrt{13}\, e^{i\left(\tan^{-1}\left(-\frac{3}{2}\right)+\pi\right)}} \quad \therefore r e^{i\left(\tan^{-1}\left(\frac{b}{a}\right)+\pi\right)}$$

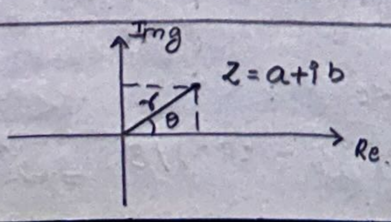
---

2) $3e^{i\pi/6} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

---

3)

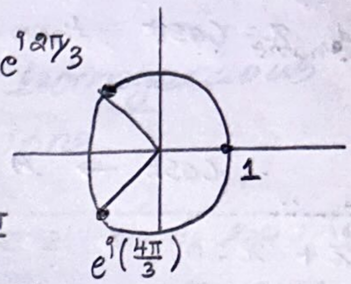


d) $\frac{1}{-2+3i} = \frac{1}{\sqrt{13}\, e^{i\theta}} = \frac{1}{\sqrt{13}} e^{-i\theta}$ $\qquad \theta = \tan^{-1}\left(\frac{-3}{2}\right) + \pi$

$$= \frac{1}{\sqrt{13}} e^{-i\left(\tan^{-1}\left(-\frac{3}{2}\right)+\pi\right)}$$

---

e) $(1)^{1/3} = \left(e^{i2n\pi}\right)^{1/3}$



when $n=0$, root 1

when $n=1$, root $e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

when $n=2$, root $e^{i4\pi/3} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$.

$\arctan\left(\frac{b}{a}\right)+\pi$	$\arctan\left(\frac{b}{a}\right)$
$\arctan\left(\frac{b}{a}\right)-\pi$	

---

1) $z^n = 1$, what $|z|$ be $\rightarrow$ possible values of $\arg z$

solu:

$\arg(z) = \frac{2\pi k}{n}$ for k any integer

Second order homogeneous linear ODE

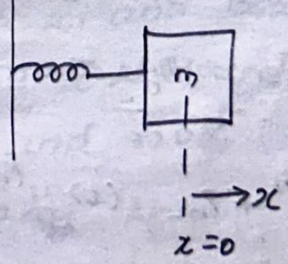
1) model spring-mass-dashpot systems using 2nd order homogeneous linear ODE's

2) Apply superposition principle to find solutions to the unforced spring mass-dashpot using 2nd order homogeneous linear ODEs.

3) Use the characteristic polynomial to find the exponential solutions for II order homogeneous linear ODE with constant co.eff

4) Find the general solution & the general real solution to the 2nd order linear homogeneous constant co.eff DEs with distinct char. roots.

Recalling spring model:



$m\ddot{x} = -Kx$

$m\ddot{x} + Kx = 0$

(For simplicity)

Let $m=1, K=1$

$$\ddot{x} + x = 0$$

we will use this example to discuss some properties of the solution.

$$\ddot{x} = -x$$

If $x = \cos t$ then, $\dot{x} = -\sin t$, $\ddot{x} = -\cos t$

$$\boxed{\ddot{x} = -x}$$

$\therefore \cos t \rightarrow$ A solution to this equ.

$\ddot{x} + x = 0$.

i) $\sin t + \cos t$, $\dot{x} = \cos t - \sin t$

$\ddot{x} = -\sin t - \cos t = -(\sin t + \cos t)$

$= -x \longrightarrow$ Solution

ii) $3\sin t \Rightarrow \dot{x} = 3\cos t \Rightarrow \ddot{x} = -3\sin t$

$= -x \rightarrow$ Solution

iii) $-2\cos t \Rightarrow \dot{x} = 2\sin t \Rightarrow \ddot{x} = 2\cos t = -x \rightarrow$ Solution.

iv) $\ddot{x} - x = 0 - 0 = 0$ (Solution but Garbage solution)

Superposition for homogeneous solutions:

The general principle that is behind the problem you just solved is that any linear combination of cosine & sine.

$$x(t) = c_1 \cos t + c_2 \sin t \qquad (c_1, c_2 \rightarrow \text{Any real number})$$

is a solution to $\ddot{x} + x = 0$. This is a consequence of the superposition principle for linear homogeneous differential equations.

Superposition for linear homogeneous differential equations:

The solution to a homogeneous linear II order ODE,

$$P_2(t)\ddot{y} + P_1(t)\dot{y} + P_0(t)\, y = 0$$

(or) more generally the solution to a linear hom. order n equ

$$P_n(t)\, y^{(n)} + P_{n-1}(t)\, y^{(n-1)} + \dots + P_0(t)\, y = 0$$ have

the following properties:

1. The zero function 0 is a solution

2. Multiplying any one solution by a scalar gives another solution.

3. Adding any two solutions gives another solution.

Summary:

All linear combinations of a homogeneous solutions are homogeneous solutions.

This is why homogeneous linear ODEs are so nice. If you know some solutions, you can form linear combinations to build new solutions, with no extra work.

All solutions to a second order linear homogeneous ODE:

$$\ddot{x} + x = 0$$

$x(t) = c_1 \cos t + c_2 \sin(t)$ is a solution for any values of the two parameters c_1 and c_2. It turns out that this 2-parameter family of solutions comprise all of the solutions to the given II order DE.

All solutions to a II order linear homogeneous DE:

The collection of all solutions to any II order linear homogeneous DE is

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

where $x_1(t)$ and $x_2(t) \rightarrow$ are solutions to the II order DE, and the two parameters c_1 and c_2 can take any constant values. This two parameter family of solutions is also called the general solution to the II order linear DE.

Important: $x_1(t)$ and $x_2(t)$ must be linearly independent! This means x_1 and x_2 can't be a constant multiple of the other. In particular this means neither x_1 and nor x_2 can be zero. In this case, we say that x_1 and x_2 form a basis for the set of all homogeneous solutions.

Linearly dependent: $c_1 x_1 + c_2 x_2$

when $\boxed{x_1 = K x_2}$ (or) $\boxed{x_2 = K x_1}$

x_1 and x_2 of a gn ~~diff~~ eqn are defined to be linearly dependent if one is a constant multiple of the other.

Remark: 1. This means that the problem of solving a II order linear ODE is reduced to finding only two linearly independent solutions, since all others can be built by taking linear combinations of these.

Remark 2: There are two parameters in the general solution to a II order DE roughly because to solve a II order DE, in one way or another (and it may be concealed), we need to integrate twice, and with each integration comes a constant of integration. Later on, we will find that in general, the number of arbitrary constants in the general solution corresponds to the order of a linear ODE.

Initial conditions:

Every II order linear ODE has a 2-parameter family of solutions. To nail down a specific solution we need to specify two pieces of data for the solution. It is a beautiful fact that this data can be provided always by two initial conditions at the same starting time, such as $y(0)$ and $\dot{y}(0)$. The starting time could also be some number a other than 0.

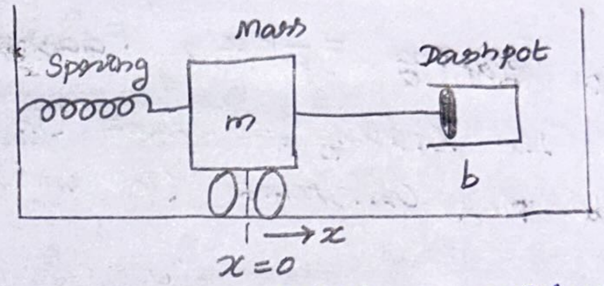
Once we have found a basis x_1 and x_2 for the solutions, the initial conditions are satisfied by choosing the values for the two parameters c_1 and c_2.

Compare this with what we have learned for first order linear eqn. Using separation of variables (in the homogeneous case) and variation of parameters (In the inhomogeneous case), we showed that every I-order ODE has a 1-parameter family of solutions. To nail down a specific solution in this family, we need only one initial condition, such as $y(0)$.

Problem: 5.1:

A mass that sits on a cart is attached to a spring attached to a wall. The mass is also attached to a dashpot, a damping device. (A dashpot could be a cylinder filled with oil that a piston moves through). Door dampers and car shock absorbers often actually work this way. Find the diff eqn for the position of the mass.

Solu:



(Equilibrium position)

$$mx'' = -kx - cx'$$

Force (Newton)   Spring (Hooke)   (Dashpot)

where

$cx' \rightarrow$ To dashpot.

∴ As m moves over time

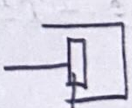
It resists the velocity of the mass

(Then opposes it)

$$mx'' + kx + cx' = 0$$

$$\boxed{x'' + \frac{k}{m}x + \frac{c}{m}x' = 0}$$

when door is pushed



(Then it comes to its original position by release of damper)

$t \rightarrow$ time (seconds s)

$x \rightarrow$ position of the mass (metres)

At $x = 0 \rightarrow$ No exert of force by spring

$m \rightarrow$ mass (kg)

$F_{spring} \rightarrow$ Force exerted by the spring on the mass (N)

$F_{dashpot} \rightarrow$ Force exerted by the dashpot on the mass (N)

$F \rightarrow$ Total force on the mass.

The independent variable is t; the position x & the forces are functions of the independent variable t.

From physics,

* F_{spring} = Function of the position x, and has the opp. sign of x.

* $F_{dashpot}$ = Function of the velocity $\dot{x}$, and again of opp. sign of $\dot{x}$.

To simplify, approximate these by linear functions (probably OK if x and $\dot{x}$ are small):

$$F_{spring} = -Kx, \quad F_{dashpot} = -b\dot{x},$$

where K is the spring constant (in units N/m) and b is the damping constant (in units Ns/m); here $K, b > 0$.

Substituting this and Newton's second law $F = m\ddot{x}$ into

$$F = F_{spring} + F_{dashpot}$$

$$m\ddot{x} = -Kx - b\dot{x}$$

A II order linear ODE, which we should usually write as

$$m\ddot{x} + b\dot{x} + Kx = 0$$

(II order linear homogeneous ODE with constant coeffs)

In the Input / system / response language:

Input signal: 0 (No external force on the mass)

System: spring, mass & dashpot

System response: $x(t)$

5.2: We've assumed that the mass m, the damping constant b, and the spring constant k don't vary with time. One can imagine this model still applies with m, b and k being functions of time, but we won't solve such here.

Std. Form $\ddot{x} + \frac{b}{m}\dot{x} + \frac{K}{m}x = 0$ (This is common for all oscillatory behaviour).

To solve: ODE Find 2 solutions

$y = c_1 y_1 + c_2 y_2$ (y_1 shouldn't be a constant multiple of y_2)

'Look different'. — Independent

solu:
Basic method:

(Traditional) → Try $y = e^{rt}$ $t \to$ independent variable)

$y' = re^{rt}$ ∴ exponential equ are easy to ∫ and diff. (That's why they are best choice)

$\boxed{y'' = r^2 e^{rt}}$

$$y'' + y' + y = 0$$

$$r^2 e^{rt} + Are^{rt} + Be^{rt} = 0 \text{ [Integrating]}$$

$\boxed{\div e^{rt}}$ $\therefore \boxed{e^{rt} \to \text{Never } 0}$

$r^2 + Ar + B = 0 \to$ characteristic equation.

What are the solutions to $m, b, K \to$ Real constants.

$$m\ddot{x} + b\dot{x} + Kx = 0$$

solu:
Try $x = e^{rt}$, where r is a constant to be determined

$$m(r^2 e^{rt}) + bre^{rt} + Ke^{rt} = 0$$

$$(mr^2 + br + K)\, e^{rt} = 0$$

This equation holds as an equality of functions if and only if $\boxed{mr^2 + br + K = 0}$

The polynomial $p(r) = mr^2 + br + K$ is known as the characteristic polynomial of the given DE, and the equ $p(r) = 0$ is the char. equ.

As usual, there are three scenarios of the roots depending on the value of $\sqrt{b^2 - 4Km}$.

In this → we will discuss the general solution to the DE in the cases when the two roots are distinct, whether real, complex, and leave the case when the roots are repeated to a later lesson

$\ddot{x} + x = 0$ (Let $x = e^{rt}$)

$r^2 e^{rt} + e^{rt} = 0$

$(1+r^2) e^{rt} = 0$ $P(r) = r^2 + 1$ (we can read

$P(r)$ directly from the DE by replacing each

diff operator with a factor of r.

$$r^2 + Ar + B = 0$$

① Real & Distinct
② pair of Complex Conjugate numbers.
③ Real & Equal.

case: i Real & Distinct (un equal).

Roots $r_1 \neq r_2$

$$\boxed{y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

Taking

$b = 4,\ K = 3$ [Damping & spring Constants]

1) $y'' + 4y' + 3y = 0 \Rightarrow (r^2+4r+3) e^{rt} = 0$ $e^{rt} \neq 0$

$(r^2+4r+3) = 0$

$(r+3)(r+1) = 0$

$r = -3,\ r = -1$

$\therefore\ y = c_1 e^{-3t} + c_2 e^{-t}$ Suppose $y(0) = 1$, (Rest) $y'(0) = 0$

$y' = -3c_1 e^{-3t} - c_2 e^{-t}$

$y'(0) = 0 = -3c_1 e^{-3t} - c_2 e^{-t}$

$3c_1 e^{-3t} + c_2 e^{-t} = 0$

$\boxed{3c_1 + c_2 = 0} \rightarrow$ ①

$\boxed{1 = c_1 + c_2} \rightarrow$ ② Solving

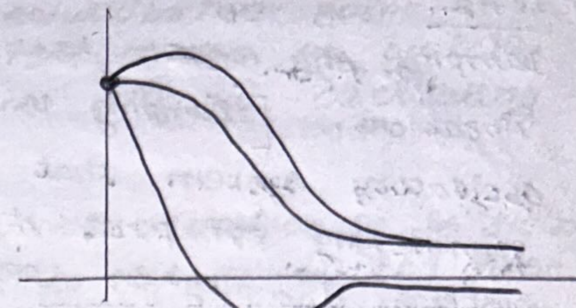
$3c_1 + c_2 = 0$
$c_1 + c_2 = 1$
$2c_1 = -1$
$c_1 = -\frac{1}{2}$

$c_1 = \frac{3}{2}$

$\therefore$

$y = -\frac{1}{2} c e^{-3t} + \frac{3}{2} c e^{-t}$

e^{-3t} → Is more negative (Due to big damping, it may outcross the equilibrium)



(possibilities)

General solution to $\ddot{y} + 5\dot{y} + 6y = 0$

solu:

$y = e^{rt}$ (Let)

$(r^2 + 5r + 6)e^{rt} = 0$

$r^2 + 5r + 6 = 0$

$r = -2, -3$

6
2 3
5

∴ The general solution to the DE is

$c_1 e^{-2t} + c_2 e^{-3t}$.

c_1, c_2 → Arbitrary constants

$y(0) = 0$, $\dot{y}(0) = 1$.

$c_1 + c_2 = 0$

$y'(0) = -2c_1 e^0 - 3c_2 e^0$

$1 = -2c_1 - 3c_2$

Solving

$2c_1 + 2c_2 = 0$

$-2c_1 - 3c_2 = 1$

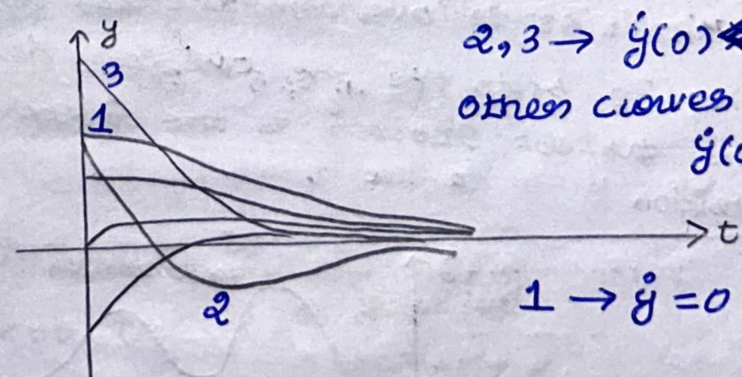
$-c_2 = 1.$

$\boxed{c_2 = -1}$

$\boxed{c_1 = 1}$

∴ $y = e^{-2t} - e^{-3t}$

other graphs (For IC)s



2, 3 → $\dot{y}(0) < 0$

other curves $\dot{y}(0) > 0$

1 → $\dot{y} = 0$.

The initial positions $y(0)$ are different for different curves.

case: ii) Complex roots:

Damping

Damping: Any effect that reduces the amplitude of vibrations. [Damping is an influence within or upon an oscillatory system that has the effect of reducing, restricting or preventing its oscillations. In physical systems, damping is produced by processes that dissipate the Energy stored in the oscillation.

$$m\ddot{x} + b\dot{x} + K = 0 \qquad (ax + by + c = 0)$$

$$\frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

$b^2 < 4mK$ (underdamping, b is small relative to m & K)

$b^2 > 4mK$ (overdamping, b is large relative to m and K)

$b^2 = 4mK$ (critical damping, b is just b/w over & under damping)

underdamping $\longrightarrow$ Roots are non-real complex roots

overdaming $\longrightarrow$ Distinct real roots

critical damping (repeated real roots)

(Ref material)

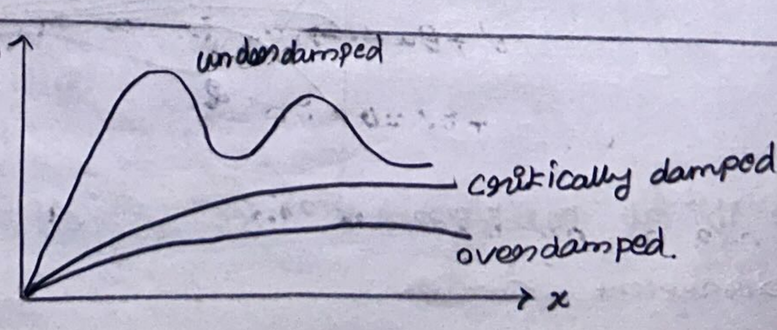
$b \rightarrow$ small (small frictional force).

↳ it will still osciuate.

(Ref material) ⟶ Ref.

overdamping: when damping is large the frictional force is so great that the system can't oscillate.

critical: As in overdamped case, this doesn't oscillate. choosing b to be the critical damping value gives the fastest return of the system to its equilibrium position.



* An overdamped system moves slowly toward equilibrium.
* An underdamped system moves quickly to equilibrium.
* But will oscillate about the equilibrium point as it does so. A critically damped system moves as quickly as possible toward equilibrium without oscillating about the equilibrium.

critical damping → Just prevents vibration or is just sufficient to allow the object to return to its rest position in the shortest period of time.

(Automobile shock absorber).

<u>underdamped</u>:

In this case, the system can't oscillate & quickly returns to equilibrium,

---

$$r = (a \pm bi)$$

$$\therefore e^{(a+bi)t} = e^{at}(e^{bit})$$

$$= e^{at}(\cos bt + i\sin bt)$$

<u>Theorem</u>:

If you have a complex solution $u + iv$, (Function of time) → to $y'' + Ay' + By = 0$, then u, v are real solutions.

<u>proof</u>:

$$(u+iv)'' + A(u+iv)' + B(u+iv) = 0$$

$$(u'' + Au' + Bu) + i(v'' + Av' + Bv) = 0$$

$\underset{0}{\parallel}$ only when $\underset{0}{\parallel}$

u and v are zero $(0 + 0i = 0)$

$\therefore$ In $u'' + Au' + Bu = 0 \rightarrow u$ is a solution

In $v'' + Av' + Bv = 0 \rightarrow v$ is a solution

$A, B \rightarrow$ Real numbers. $(\because)$ So $u, v \rightarrow$ Real solutions

Case: 2: Two distinct complex roots:

Exam: 8.1

$\ddot{y} + A\dot{y} + By = 0$ where $A, B \rightarrow$ Real, has char. roots $a \pm b$. Give general real solution

Solu: Since the roots of characteristic polynomials are $a \pm ib$, the two complex roots (exponentials) $e^{(a+ib)t}$, $e^{(a-ib)t}$ form a basis of the collection of all solutions. Now, because all the co.effs in the DE (an in char. polynomial) are real, and imaginary parts as well as real parts of either exponential solution.

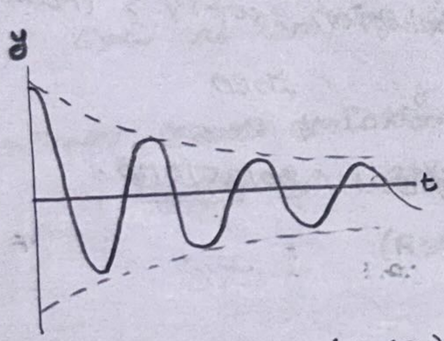
$$Re\left(e^{(a+ib)t}\right) = e^{at}(\cos bt)$$

$$Im\left(e^{(a+ib)t}\right) = e^{at}\sin bt$$

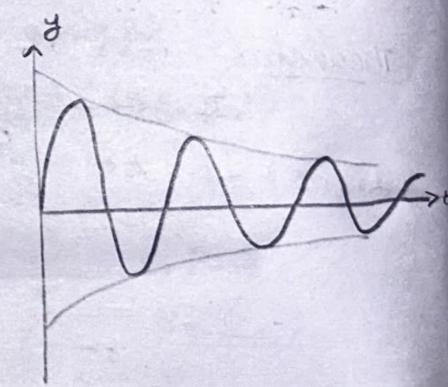
$\therefore$ These two solutions are real & linearly independents, and give the general real solution.

$$y(t) = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$$

$c_1, c_2 \rightarrow$ Arbitrary real constants.



$e^{at}\cos(bt)$ $(a<0)$



$e^{at}\sin(bt)$ $(a<0)$

In $y(t) = u(t) + i\,v(t)$ is a solution to a second order homogeneous linear DE with real co.eff.

$$\ddot{y} + A\dot{y} + By = 0 \quad \text{where } A, B \text{ real.}$$

plugging y in to the eqn,

$$(u+iv)'' + A(u+iv)' + B(u+iv) = 0$$

$$\underbrace{(\ddot{u} + A\dot{u} + Bu)}_{\text{Real}} + i\,\underbrace{(\ddot{v} + A\dot{v} + Bv)}_{\text{Real.}} = 0$$

$A, B \rightarrow$ Real.

$\therefore$ u and v are solutions to the original DE.
(All co-effs of the DE are real)

Roots are imaginary $\alpha \pm i\beta \Rightarrow e^{\alpha x}(A\cos\beta x + B\sin\beta x)$
solution

$r = a \pm bi$: $e^{a+bi} \xrightarrow{Re} e^{at}\cos bt = y_1$

$\xrightarrow{Im} e^{at}\sin bt = y_2$.

$\boxed{y = c_1 y_1 + c_2 y_2}$

Then what about $a-bi$

$e^{a-bi} \xrightarrow{Re} e^{at}\cos(-bt) = y_1$

$\xrightarrow{Im} e^{at}\sin(-bt) = -y_2$ (only change in sign $\rightarrow$ nothing new) $\rightarrow$ [Real solution]

$\therefore$ The other root doesn't give anything new.

Undamped system revisited:

Let us now use the char polynomial to solve the DE that we started with in this lecture, the one that models a spring-mass system with no dashpot.

$$\ddot{y} + y = 0$$

1 approach

$$(r^2+1)e^{rt} = 0$$

Solving, $r = \pm i$. Its roots are i and $-i$. Thus the exponential solutions

e^{it}, e^{-it} form a basis for the collection of all solutions.

$\therefore$ The co-effs are real (DE models a real life situation), the real & img parts of e^{it} (or equivalently of e^{-it})

$$Re(e^{it}) = \cos t$$

$$Im(e^{it}) = \sin t$$

are the two real solutions to the DE. These are linearly independent and so

$\cos t, \sin t$ form {another} basis for the same collection of all solutions.

The general real solution is all (real) linear combinations of these two real basis functions:

$$y(t) = c_1 \cos t + c_2 \sin t, \quad c_1, c_2 \in R.$$

(This is the same as we have guessed).

Question

The general solution gn by the exponential solutions e^{it} and e^{-it} corresponding to the roots of the char polynomial is

$$y(t) = c_1 e^{it} + c_2 e^{-it}$$

, However, this is complex since $e^{\pm it}$ are complex. Can the real solutions we found above, $c_1 \cos t + c_2 \sin t$, be obtained from this general complex solution?

Answer:

Yes.

Since the roots i and $-i$ of the char polynomial are complex conjugates of one another, the exponential solutions e^{it} and e^{-it} corresponding to these roots are also complex conjugate of one another. Now recall that the real and imaginary part of any complex function z can be obtained by linear combinations with complex co.eff, of z and its conjugate $\bar{z}$.

$$z = a + ib, \quad \bar{z} = a - ib$$

Adding real parts alone:

$$Re(z) + Re(\bar{z}) = a + a$$

$$\boxed{Re(z) + Re(\bar{z}) = 2a}$$

Also:

$$Re(z) = \frac{z + \bar{z}}{2} = \frac{a + ib + a - ib}{2} = \frac{2a}{a}.$$

$$\therefore \boxed{Re(z) = \frac{z + \bar{z}}{2}}$$

$$Im(z) = \frac{z - \bar{z}}{2i} = \frac{a + ib - a + ib}{2i} = \frac{2ib}{2i} = b$$

$$\therefore \boxed{Im(z) = \frac{z - \bar{z}}{2i}}$$

$$\cos t = Re(e^{it}) = \frac{e^{it} + e^{-it}}{2}$$

$$\sin t = Im(e^{it}) = \frac{e^{it} - e^{it}}{2}$$

(This is Euler's formula backwards). To get the general real solution, we take linear combinations of the expressions on the right above for cost and sint. Therefore the real solutions can be written as

$$c_1 e^{it} + c_2 e^{-it}$$, provided that the co.eff c_1 and c_2 can take complex values.

9.2: which basis should be used?

It depends:

* The basis e^{-it}, e^{it} is easier to calculate with, but we need to be careful which linear combinations of these functions are real valued.
* The basis cost, sint consisting of real-valued functions is useful for interpreting solutions in a physical system.

[So convert back and form as per requirement]

Explicit relation b/w the real and complex solutions:

$\boxed{2 \text{ approach}}$

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t}$$

$\boxed{y = c_1\cos\theta + c_2 \sin\theta} \rightarrow$ Real solutions where A, B are real numbers.

Approach: 2 :-

$$y = c_1 e^{(a+ib)t} + c_2 e^{(a-ib)t}$$

Real solution of this complex function:

c_1 and c_2 are allowed to be complex.

Ans — Hack (mul all out & make Imaginary part = 0)

— $u + iv \rightarrow (v = 0$ is real)

$u + iv$ — $v = 0$ (or)
— change $i \to -i$ ∴
want real

If we change $i \to -i$, the expression doesn't change the expression is real.

∴ Complement
$\bar{z} = a - ib$
↳ sign change in Img part

Steps:

$$c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} \rightarrow ①$$

change $i \to -i$ [Complex Conjugate]

$$\bar{c_1} e^{(a-bi)t} + \bar{c_2} e^{(a+bi)t} \rightarrow ②$$

expression (1) and (2) are equal when

$\bar{c_1} = c_2$ and $\bar{c_2} = c_1$ (Both $\bar{c_1} = c_2$ & $\bar{c_2} = c_1$) are the same

Real solution to this eqn:

Complex basis.

$c_1 = c + id$

$$\boxed{c_2 = c - id = \bar{c_1}}$$

$$\therefore (c+id) e^{(a+bi)t} + (c-id) e^{(a-ib)t}$$

↳ Engineers & physicists write in this way

Change this to old form

Hack method:

$$e^{at} \left(c \left(e^{bit} + e^{-ibt} \right) + id \left(e^{ibt} - e^{-ibt} \right) \right)$$

$z + \bar{z} = 2\,Re(z)$
$= 2\,Re(\cos bt + i \sin bt)$
$= 2\cos bt$

$z - \bar{z} = 2\,Im\,z$
$= 2i \sin bt$

Real Basis

$$= e^{at} \left(2c \cos bt + 2i^2 d \sin bt \right)$$

$$= 2e^{at} \left(c \cos bt - d \sin bt \right)$$

Summary on complex solutions:

we've been interchanging b/w a complex basis & a real basis for the solutions to a second order homogeneous linear DE with two distinct complex characteristic roots. The general principle, which applies to more complicated ODEs, is summarized below.

Complex basis vs real valued basis:

Let $y(t)$ be a complex-valued function of a real-valued variable t. If y and $\bar{y}$, the complex conjugate of y, are a basis of the solutions to a second order homogeneous linear ODE with real co.eff, then

Replacing $y, \bar{y}$ by $Re(y)$ and $Im(y)$ gives a new basis. Taking real linear combinations of the real basis $Re(y), Im(y)$ then gives all the real solutions to the DE.

① we can also replace $y, \bar{y}$ instead by $Re(y), iIm(y)$ [∴ Aim is to obtain a basis having a real-valued functions]

② would it be okay to replace $y, \bar{y}$ instead by $Re(y), Re(\bar{y})$

No, because if $y = f + ig$, then $\bar{y} = f - ig$. So $Re(y)$, $Re(\bar{y})$ are both f! They are linearly dependent, so they can't be part of a basis.

$$\ddot{x} + 5\dot{x} + 4x = 0$$

solu:

$$(r^2 + 5r + 4)e^{rt} = 0$$

$$(r+4)(r+1)$$

$$x(t) = c_1 e^{-t} + c_1 e^{-4t}$$

$$\ddot{x} + \dot{x} + x = 0$$

solu:

$$(r^2 + r + 1) = 0 \qquad \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

$r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$z(t) = e^{(-1+i\sqrt{3})t/2}$, $\bar{z}(t) = e^{(-1-i\sqrt{3})t/2}$

Basis of real solution $x_1(t) = Re(z(t)) = e^{-t/2} \cos\left(\frac{\sqrt{3}t}{2}\right)$,

$Im(z(t)) = e^{-t/2} \sin\left(\frac{\sqrt{3}t}{2}\right)$.

General ~~real~~ solution: $x(t) = e^{-t/2}\left(c_1 \cos\left(\frac{\sqrt{3}t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}t}{2}\right)\right)$

$\ddot{x} + 4\dot{x} + 5x = 0$

Solu:-

$(r^2 + 4r + 5) = 0$ $= \frac{-4 \pm \sqrt{16 - 4(5)(1)}}{2}$

$5 > 4$

$= -2 \pm \frac{\sqrt{-4}}{2}$

$= -2 \pm i$

$z(t) = e^{(-2+i)t}$, $z(t) = e^{(-2-i)t}$

$z(t) = e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t)$

$Re(z(t)) = e^{-2t} \cos t$, $Im(z(t)) = e^{-2t} \sin t$

$\therefore$ we now have a basis of two real valued functions, and can use superposition to find the general real valued solution,

$x(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$.

we can find the basis of solutions by taking the real & imaginary parts of $\bar{z}(t)$ instead:

$\bar{z}(t) = e^{(-2-i)t} = e^{-2t}(\cos(-t) + i \sin(-t))$

then the real valued basis solutions would be

$e^{-2t} \cos(-t) = e^{-2t} \cos t$ and $e^{-2t} \sin(-t) = -e^{-2t} \sin t$.

which is not equal to the above

exact solution. $\to$ can't be done.

---

up to a sign these are the same basic solutions as was obtained from $z(t)$, so $\bar{z}(t)$ would have work just as well.

If $\bar{z} = -z$ what does that tells us about $z = a+bi$

If $z = a+bi$, $\bar{z} = a-bi$ when $a=0$ $\boxed{z = \bar{z}}$

<u>Inference</u>: z is purely imaginary. (i.e. a real number times i).

As the +ve real numbers t tends to $+\infty$, how does the value of t^2 move in the complex plane?

Ans: Radially outward.

$\therefore$ The number t^2 is always on the real axis, and moves outwards as t increases. (Note if we are graphing the function, we would be plotting the points (t, t^2) in R^2 as t varies, which would give a parabola. But instead we are plotting the points $(t^2, 0)$ in the plane.).

How many complex numbers c satisfy $e^c = 0$

Ans: None $[\because e^c$ never become $0]$

What happens in the spring-mass-dashpot model if the cart is attached to a dashpot, but there is no spring and no external force? Assume that the initial velocity of the cart is non zero. Check all that apply.

solu: The ODE is a homogeneous linear ODE; the cart never changes direction. stops moving; the cart's speed tends to 0; the cart's acceleration tends to 0.

For a general spring-mass-dashpot system, the ODE is

$$m\ddot{x} + b\dot{x} + Kx = F_{external}$$

$\begin{bmatrix} \text{No } Kx \text{ term} \\ \text{No } F_{external} \end{bmatrix}$

$$m\ddot{x} + b\dot{x} = 0$$

This is a homogeneous linear ODE, so (The cart never changes direction). It can be solved by the general method for homogeneous linear ODEs with constant co.eff, but it is easier to set $V := \dot{x}$, which satisfies the first-order homogeneous ODE.

$$m\dot{v} + bv = 0$$

This leads to

$$\dot{v} + \frac{b}{m}v = 0$$

$$\dot{v} = -\frac{b}{m}v$$

Solving,

$$\frac{\dot{v}}{v} = -\frac{b}{m}$$

$$\frac{dv}{v} = -\frac{b}{m}dt \Rightarrow \ln|v| = -\frac{b}{m}t$$

$$v = Ce^{(-\frac{b}{m})t}$$ for

some number c, which must be non zero. Since the initial velocity is non zero. At every t, the sign of v is the same as the sign of c, so the cart never stops moving, and never changes direction; this means it can't oscillate.

As $v \to 0$, $t \to \infty$. Finally the acceleration is $\dot{v} = -(\frac{b}{m}) ce^{-(b/m)t}$, which also tends to 0 as $t \to \infty$.

I) Change in the form $z = a+bi$, $f(t) = Re\, e^{zt}$

$f(t) = \cos(2\pi t)$

solu:
$$f(t) = Re\, e^{zt}$$
$$= Re\, e^{(a+ib)t}$$
$$= e^{at}(\cos bt)$$

i) $f(t) = \cos 2\pi t$

solu:
$$f(t) = e^{i2\pi t} \qquad (\cos 2\pi t + i\sin 2\pi t) = \cos 2\pi t$$

Comparing with $e^{at}\cos bt$

$$a = 0, \; b = 2\pi$$

$$\therefore \; z = 2\pi i$$

---

ii) $f(t) = e^{-t}$ comparing $a = -1, \; b = 0$

$$z = -1$$

iii) $f(t) = e^{-t}\cos(2\pi t) = e^{-t}(e^{i2\pi t})$

$$= e^{+t(-1+i2\pi)}$$

$$z = -1 + 2\pi i$$

$f(t) \Rightarrow e^{0t} = 1 \Rightarrow a=0, b=0, z=0$

Swinging door:

once opened & let swing, our kitchen door can freely swing both directions. This seemed dangerous, so we add a swing-damper at the top of the door which satisfies the eqn

$$m\ddot{x} + b\dot{x} + Kx$$

where,

$$m = 0.5$$
$$b = 1.5$$
$$K = 0.625$$

solu:

$$(0.5r^2 + 1.5r + 0.625) = 0$$

Finding the solution that satisfies the IC $x(0) = x_0$, $\dot{x}(0) = v_0$.

$$r = \frac{-1.5 \pm \sqrt{(1.5)^2 - 4(0.5)(0.625)}}{2(0.5)}$$

$$r = -1.5 \pm 1$$

$$r = -0.5, \quad r = -2.5$$

$$x(t) = c_1 e^{-0.5t} + c_2 e^{-2.5t}$$

$x(0) = x_0$, $\dot{x}(0) = v_0$

$$x_0 = c_1 e^{-0.5(0)} + c_2 e^{0}$$

$\boxed{c_1 + c_2 = x_0}$, $\dot{x}(0) = c_1(-0.5)e^{-0.5t} + c_2(-2.5)e^{-2.5t}$

$$\dot{x}(0) = -0.5c_1 - 2.5c_2$$

$\boxed{c_2 = x_0 - c_1}$

$$v_0 = -0.5c_1 - 2.5c_2$$

$$v_0 = -0.5c_1 - 2.5(x_0 - c_1)$$

$$= -0.5c_1 + 2.5c_1 - 2.5x_0$$

$$v_0 = 2.0c_1 - 2.5x_0$$

$$c_1 = \frac{v_0 + 2.5x_0}{2}$$

$\boxed{c_1 = 0.5v_0 + 1.25x_0}$, $c_2 = x_0 - 0.5v_0 - 1.25x_0$

$\boxed{c_2 = -0.25x_0 - 0.5v_0}$

$$x(t) = c_1 e^{-0.5t} + c_2 e^{-2.5t}$$

$$x(t) = (0.5v_0 + 1.25x_0)e^{-0.5t} - (0.25x_0 + 0.5v_0)e^{-2.5t}$$

3) For which of the following IC is the solution an exponential ce^{rt} rather than a sum of such terms

Solu: when $x_0 = -2V_0$, $x_0 = -0.4V_0$

(For a single exp function, any one exp term must be zero)

$$c_1 = 0.5V_0 + 1.25(-2V_0)$$
$$= 0.5V_0 - 2.5V_0$$
$$= -2.0V_0$$
$$c_2 = 0.5V_0 + 1.25(-0.4V_0)$$
$$= 0.5V_0 - 0.5V_0 = 0.$$

---

4) For $x_0 = 0.25$, what can you say about initial velocity of the door. If once the door is let go, it swings through the closed position & then swings back from the other side.

Solu:

V_0 must be $< -\frac{5}{8}$.

when $x_0 = 0.25$ [we want to ensure $x(t)=0$] for some $t>0$.

Solu: $c_1 e^{-0.5t} + c_2 e^{-2.5t} = 0$ $\quad \div$ by $c_1 e^{-0.5t}$

$$1 + \frac{c_2}{c_1} e^{-2t} = 0$$
$$-1 = \frac{c_2}{c_1} \cdot e^{-2t}$$
$$\boxed{e^{2t} = -\frac{c_2}{c_1}}$$

(This admits a solution as long as c_2 & c_1 have opp. sign

Also, since $t>0$, we must also have

$$-\frac{c_2}{c_1} > 1.$$

$\therefore e^0 = 1$
$e^1 = 2.71828$
$e^2 = 7.38905$

This happens if either,

$$-c_2 > c_1 > 0$$
$$-c_2 < c_1 < 0$$

$$0.25x_0 + 0.5V_0 > 1.25x_0 + 0.5V_0 > 0 \Rightarrow x_0 < 0$$
$$0.25x_0 + 0.5V_0 < 1.25x_0 + 0.5V_0 < 0 \Rightarrow x_0 > 0$$

If $x_0 < 0$, then we must have $0.5V_0 > -1.25x_0$. If $x_0 > 0$ then we must have $0.5V_0 < -1.25x_0$.

$\therefore$ we have $x_0 \geq 0.25$, we must have $V_0 < -2.5 x_0 = -2.5(0.25)$
$= -0.625.$

unfortunately, the damper we purchased in the previous problem was of poor quality. It broke, and we had to quickly replace it with a damper having

$$m = 0.5$$
$$b = 0.25$$
$$K = 2.04$$

Solu:

$$m\ddot{x} + b\dot{x} + Kx = 0$$

$\therefore$ char equ $= (0.5r^2 + 0.25r + 2.04)$

roots:

$$r = \frac{-0.25 \pm \sqrt{0.25^2 - 4(0.5)(2.04)}}{2(0.5)}$$

$$= -0.25 \pm 2i$$

$\therefore$ The general solution is

$$x(t) = c_1 e^{-t/4}(\cos 2t) + c_2 e^{-t/4} \sin 2t.$$

IVP: $x(0) = 0$, $\dot{x}(0) = 1$

$0 = c_1(1) \rightarrow (1)$

$\boxed{c_1 = 0}$

$$x'(t) = c_1 e^{-t/4}\left(-\frac{1}{4}\right)\cos 2t + c_1 e^{-t/4}(2)(-\sin 2t)$$
$$+ c_2\left(-\frac{1}{4}\right)e^{-t/4}\sin 2t$$
$$+ c_2(2)e^{-t/4}\cos 2t.$$

$$1 = c_1 e^{-t/4}\left(-\frac{1}{4}\right)\cos 2t + 2c_2 e^{-t/4}\cos 2t$$

$\rightarrow (2)$

$$1 = 0 + 2c_2 e^{-0/4}$$

$$1 = 2c_2 \Rightarrow \boxed{c_2 = \frac{1}{2}}$$

IVP:

(from mathlet tool)

we measure the zeros to occur when

$t = 0, 1.57, 3.19, 4.71, 6.28.$

The difference b/w successive zero is

1.57, 1.62, 1.52, 1.52

which is fairly close to constant. This time interval represents the half period of the sine & cosine in the solution with angular functions that appear in the solution with angular frequency.

$$\omega \approx \frac{\pi}{1.57} \approx 2.$$

4) (Give a formula for this angular frequency)

$$\omega = \frac{\sqrt{4mK - b^2}}{2m}$$ [char polynomial has complex roots].

Note that in the limit as $b \to 0$, this frequency becomes $\sqrt{\frac{K}{m}}$, which is the natural frequency of the harmonic oscillator with no damping term.

Sinusoidal Functions

1) Represent a linear combination of a sine & cosine with the same frequency as a sinusoidal function.

2) Describe & graph sinusoidal functions in terms of amplitude & phase angles.

3) Write the eqn for a sinusoidal function from a graph.

4) Graph oscillatory solutions to spring-mass systems with no external force.

5. Understand how two sinusoids of close frequencies create beats.

Solutions to the spring-mass-dashpot system

(without external Force)

$$m\ddot{x} + b\dot{x} + Kx = 0$$ (Let the damping constant be small $b^2 < 4mK$ and the char polynomial has 2 distinct complex roots. (under damped).

Find the two char roots r_1 and r_2 of this system in terms of the mass m, K and b.

$$r_1 = ?\ ,\ r_2 = ?$$

$mr^2 + br + K$ is the char. polynomial.

$$r = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

$b^2 < 4mK$

$$r = \frac{-b \pm \sqrt{-(4mK - b^2)}}{2m}$$

$$r = \frac{-b \pm i\sqrt{4mK - b^2}}{2m}$$

$$r = -\frac{b}{2m} \pm i\sqrt{\frac{K}{m} - \frac{b^2}{4m^2}}$$

Consider the same spring-mass-dashpot system as above. Let the two distinct char roots be $-s \pm i\omega_d$ (s and ω_d can be expressed in terms of m, K and b)

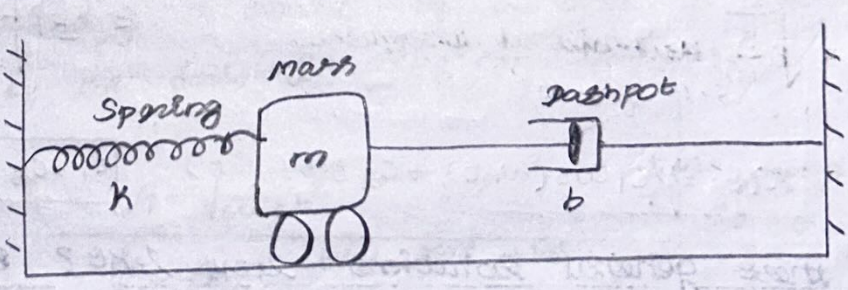
Solu: (Bases of the solution to the system).

Solution will be: $e^{(a+ib)t} = e^{(-s+i\omega_d)t}$; $e^{(-s-i\omega_d)t}$

$$e^{(-s+\omega_d i)t} = e^{-st}(\cos\omega_d t + i\sin\omega_d t)$$

$$= \underset{(y_1)}{e^{-st}\cos\omega_d t} + \underset{y_2)}{e^{-st}\sin\omega_d t}$$

Real solutions to spring-mass-dashpot system.



(with no external factors)

Solu:
$$m\ddot{x} + b\dot{x} + Kx = 0 \qquad [m, K > 0 \; ; \; b \geq 0]$$

When there are two distinct characteristic complex roots, that is, when the discriminant $b^2 - 4Km < 0$, the real basis that you found in the previous problem gives the general real solution.

Real Solution $\rightarrow$ $c_1, c_2 \in \mathbb{R}$.