1) $\begin{bmatrix} x & x & x & x \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{bmatrix}$

Column 1 & Row 1 → same entry x.
other 9 numbers can be like
1, 5, 7, 2, 3, 99, π, e, 4.
(anything)

For what x → Det A will be zero

Solu:

$$x\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - x\begin{vmatrix} x & 0 & 0 \\ x & 1 & 0 \\ x & 0 & 1 \end{vmatrix} + x\begin{vmatrix} x & 1 & 0 \\ x & 0 & 0 \\ x & 0 & 1 \end{vmatrix} - x\begin{vmatrix} x & 1 & 0 \\ x & 0 & 1 \\ x & 0 & 0 \end{vmatrix}$$

$$x(1(1)) - x(x(1)) + x(x(0) - 1(x)) - x(x(-x)) = 0$$

$$x - x^2 - x^2 + x^3 = 0$$

$$x^3 - 2x^2 + x = 0$$

$$x(x^2 - 2x + 1) = 0$$

1, −1, −1, −2

$$x = 0, 1.$$

det A = 0 when x = 0, 1 → But 4×4 → won't be an identity matrix.

4×4 → Identity (9 numbers forms I)

$$\det A = \begin{vmatrix} x & x & x & x \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & 0 & 0 & 1 \end{vmatrix} = x\begin{vmatrix} 1 & x & x & x \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = x\begin{vmatrix} 1-3x & x & x & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= x(1-3x)\begin{vmatrix} 1 & x & x & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= x(1-3x)(1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= x(1-3x) = 0$$

$$x = 0, \; x = 1/3$$

a) The determinant of A is a polynomial of x. What's the largest possible degree of that polynomial?

$x(1-3x) = x - 3x^2$ → Largest degree 2

Unit - III

Positive definite matrices & applications

Special properties: symmetric, possibly complex, and positive definite. The central topic: converting matrices to nice form (diagonal or nearly diagonal) through multiplication by other matrices.

Special matrices have Eigen vectors & values

↙ Symmetric ↘ positive definite

(Extremely good properties)

Lecture - 26

what's special about Eigen vectors & values?

$A = A^T$ (symmetric)

① The Eigen values are real for a real symmetric mat.

② The Eigen vectors are perpendicular.

↓

can be chosen.

Eigen values → different → Lines of Eigen vectors

Repeated → space of Eigen vectors

(we can choose $\perp^r$ one)

Different: Lines of Eigen vectors ($\perp^r$) → 1d (vector spaces)

---

why can be chosen?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{symmetric}$$ [for this every vector is an Eigen vector]

↓

we can chose $\perp^r$ ones.

Having a complete set of Perpendicular Eigen vectors

usually, $A = S \Lambda S^{-1}$ → usual case

Symmetric case: $Q \Lambda Q^{-1}$

↳ As Eigen vectors are $\perp^r$, $S = Q$ [orthonormal matrix]

$$Q^{-1} = Q^T$$

$$\boxed{\therefore \text{ Symmetric case: } A = Q \Lambda Q^T}$$

↳ Beauty: Shows Eigen values
Eigen vectors
symmetry.

$A^T = Q^T \Lambda Q$ → spectral theorem!
(mechanics)
↳ principal axis theorem.

Spectrum: Set of Eigen values of a matrix.

Light: Spectrum of pure things

If we look at a thing at correct axis – It's diagonal
"Directions don't couple each other"

<u>Why real Eigen values?</u>

$$Ax = \lambda x$$

① <u>Always</u>:

$\bar{A}\bar{x} = \bar{\lambda}\bar{x}$ [conjugate]

$A\bar{x} = \bar{\lambda}\bar{x}$

∴ $A \to$ Real
(symmetrical)

$\boxed{A^T = A}$

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda}^T$$

$$\bar{x}^T A = \bar{x}^T \bar{\lambda}^T$$

<u>Take inner products:</u>

"Both sides multiply with x"

$$\bar{x}^T A x = \bar{x}^T \bar{\lambda}^T x$$

↓

①

$A\cancel{x} = \cancel{\lambda x}$

$\bar{x}^T A x = \lambda$

Take: $Ax = \lambda x$

multiply by $\bar{x}^T$

$$\bar{x}^T A x = \bar{x}^T \lambda x$$

Aim: To prove real Eigen values!

Prove: complex conjugate → Also real.

$Ax = \lambda x$	$Ax = \lambda x$
$A\bar{x} = \bar{\lambda}\bar{x}$ 'A-Real'	'mul by $\bar{x}^T$'
$\bar{x}^T A = \bar{x}^T \bar{\lambda}^T$	$\bar{x}^T A x = \bar{x}^T \lambda x \rightarrow ②$
$\bar{x}^T A x = \bar{x}^T \bar{\lambda}^T x \rightarrow ①$	

From ① and ②

$\bar{x}^T \bar{\lambda}^T x = \bar{x}^T \lambda x \rightarrow$ True only when

$\lambda = \bar{\lambda}^T$

'possible only when it's real' ← $\lambda = \bar{\lambda}$

Caution: $\bar{x}^T x \rightarrow$ can't be zero.

$\bar{x}^T \bar{\lambda}^T x = \bar{x}^T \lambda x$

$$\bar{x}^T x = \begin{bmatrix} \bar{x_1} & \bar{x_2} & \bar{x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix}$$

$= \bar{x_1} x_1 + \bar{x_2} x_2$

$= (a+ib)(a-ib)$

$= (a^2+b^2)$

Only real, Matrix A → also real.

$\bar{x} x \rightarrow \text{Length}^2$

99.9 % → Applications

Real matrices.

Good matrices

Real λ's

perpendicular x's

$A = A^T \rightarrow$ Real

$A = \bar{A}^T \rightarrow$ complex

$A = Q \Lambda Q^T \quad [A = A^T]$

'Even though $A \to$ Complex matrix $\to$ our proof works'

Condition: $\boxed{A = \bar{A}^T}$ Eigen value $\to$ Real., Eigen vectors $\perp^r$

$$A = Q \Lambda Q^T$$

$$= \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

'column times Row'

$$= \begin{bmatrix} q_1 \lambda_1 & q_2 \lambda_2 & \cdots & q_n \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$\boxed{A = q_1 \lambda_1 q_1^T + q_2 \lambda_2 q_2^T + \cdots + q_n \lambda_n q_n^T}$$

A breaks up to in to orthonormal Eigen vectors & real Eigen values.

$q_k q_k^T \to$ projection matrix

'Every symmetrical matrix is a combination of mutually perpendicular projection matrix'

$$\boxed{P = \frac{a a^T}{a^T a}}$$

$aa^T \to$ matrix

$a^T a \to$ number.

Here $a \to$ orthogonal vector.

Real Eigen values: Are they +ve or (-)ve.

signs of the pivots are same as the signs of the Eigen values of a Real symmetric matrix.

(numbers)

$$\boxed{\#\ \text{positive pivots} = \#\ \text{positive } \lambda\text{'s}}$$

↳ we can shift the matrix by 7

(Narrowing down) we are able to know, how many Eigen values are below 7 & above 7.

Product of the pivots (determinant) = Product of the Eigen values

Positive definite matrix

1) Symmetric

'symmetric matrices - Best one out of all' - 'simplicity'

2) All the Eigen values are positive

3) All the pivots are positive

eg: $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

pivots

$5, \frac{11}{5}$

$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$

$\therefore$ product of pivots = Determinant

$(5-\lambda)(3-\lambda) - 4 = 0$

$15 + \lambda^2 - 3\lambda - 5\lambda - 4 = 0$

$\lambda^2 - 8\lambda + 11 = 0$

$\boxed{\lambda = \frac{4 \pm \sqrt{5}}{2}}$ → Both positive

'family of matrices used in differential equ'

↳ 'we know signs → so stability'

Determinants:

$15 - 4 = 11$ (It's positive too)

$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$ → pivots → (-)ve

Eigen values → (-)ve (same pivots)

determinant → positive.

$A_1 = -1, A_2 = 3$

(Fails)

So we can't simply say det is +ve

* All sub determinants are positive $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

$A_1 = 5, A_2 = 15 - 4 = 11$

Positive definite matrix:

1) Symmetric
2) Positive pivots
3) Eigen values → +ve
4) Sub determinants → +ve.

"we are going to: Combine all pivot, Eigen, det all in a point"
until this minute: we have seen it for symmetric mat.

Aim: Expand for m×n matrices (Not necessarily Symmetric)

Recitation

Explain why the following are true?

a) Every positive definite matrix is invertible
b) The only positive definite projection matrix is $P=I$
c) D is diagonal with positive entries is positive definite.
d) S symmetric with det $S>0$ might not be positive definite.

Solu:

a) $A \to$ Invertible $\to$ det $A \to$ Non (zero)

$\therefore$ det $A = \lambda_1, \lambda_2, \ldots$ (Eigen values of A)

If A is positive definite:

$\lambda_1, \lambda_2, \ldots, \lambda_n$ must be bigger than zero

$\therefore \det(A) > 0 \to$ Invertible.

b) only +ve definite P is the identity matrix

P is the projection.

Eigen values of $P = 0$ (or) 1.

If P is positive definite matrix

↳ Eigen values > 0.

$\therefore$ Eigen values of $P = 1$

↳ Identity matrix is the only symmetrical matrix that satisfies $P=I$

If P is diagonalizable:

$\{u = S$ (Eigen vector matrix)$\}$

$$P = u\, I\, u^{-1}$$

$$\boxed{P = u u^{-1} = I}$$

[Every symmetrical matrix is diagonalizable].

$\therefore$ All projection matrices except I are singular.

→ Eigen value of P must be 1 when $P = I$ else it will be 0.

---

c) $D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 \ddots & \\ & & & d_n \end{pmatrix}$

1) symmetrical — yes
2) pivots $\longrightarrow$ +ve
3) Eigen values (pivots) → +ve
4) sub determinants ⟶ +ve.

↳ positive definite matrix

For any x (vector): (other way)

$$x^T D x > 0$$

$(1 \times 3)(3 \times 3)(3 \times 1) = (1 \times 3)(3 \times 1) = (1 \times 1) \to$ one way.

$$x^T = (x_1\ x_2\ x_3 \dots x_n)$$

$$x^T D x = d_1 x_1^2 + d_2 x_2^2 + d_3 x_3^2 + \dots + d_n x_n^2 > 0$$

---

d) $\det S > 0 \to$ might not be positive definite.

$$S = \begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}$$

'$\det S > 0 \to$ doesn't mean positive definite matrix'

↓

All sub determinant must be +ve

1) $\det S = 6 > 0$
2) pivots $< 0 \longrightarrow$ Failed
3) symmetrical ⟶ yes

$x = (1\ \ 0)^T$

$$x^T S x = (1\ \ 0)\begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -3 < 0 \quad \text{(Failed)}$$

Lecture - 27

Complex matrices; Fast Fourier transform (FFT)

The Fourier matrices have complex valued entries and many nice properties. This session Covers the basics of working with complex matrices & describes FFT.

* Even a real matrix has complex Eigen value.

Complex matrix: Fourier matrix (The most important)

FFT - In all computers

matrix with orthogonal columns — $(n \times n) \rightarrow n^2$ (multiplication will be - no. of steps)

FFT $\rightarrow$ Reducing n^2 to $n \log n$
Idea

Example:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \text{ in } C^n$$

(complex vector) [n-dimensional Complex space]

---

point: $z^T z \rightarrow$ No good.

$$\begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = z_1^2 + z_2^2 + \dots + z_n^2$$

$z_1^2 = (a+ib)(a+ib)$
$= a^2 + b^2 + 2aib$

$$\bar{z}^T z = \begin{pmatrix} \bar{z_1} & \bar{z_2} & \bar{z_3} & \dots \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$= \bar{z_1} z_1 + \bar{z_2} z_2 + \dots + \bar{z_n} z_n$$

$\bar{z_1} z_1 = (a-ib)(a+ib)$

$|z_1|^2 = a^2 + b^2 \rightarrow \text{Length}^2 \longrightarrow$ Good.

Hint: whenever we transpose $\rightarrow$ we complex conjugate.

$$\bar{z}^T z = z^H z$$

$H \to$ Hermetian (For transpose & complex conjugate)

Inner product:

$$\bar{y}^T x = y^H x$$ $\to$ Not real until if y and x are same (Real)

Inner product of it's own vector = It's length

$$z^H z = |z_1|^2 + |z_2|^2 + \dots + |z_n|^2$$

Symmetric matrix:

$A^T = A \to$ Real

$\bar{A}^T = A \to$ complex

example:

$$\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} \longrightarrow$$ Hermitian matrices $\bar{A}^T = A$

$\to$ Real Eigen values

$\perp^r$ Eigen vectors.

Perpendicular:

$q_1, q_2, \dots, q_n$

Inner product $\bar{q_i}^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ "unit orthonormal vectors"

$$Q = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{bmatrix}$$

$$\bar{Q}^T = Q = Q^H = Q$$ $\to$ orthogonal (Unitary matrix) $\to$ square $n \times n$

$\perp^r$ columns

(unit vectors)

Unitary $\to$ orthogonal complex vectors as columns

$\bar{Q}^T = Q$

'Discrete Fourier transform'

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \vdots & w & w^2 & \cdots & w^{n-1} \\ \vdots & w^2 & w^4 & \cdots & w^{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{n-2} & \cdots & w^{(n-1)^2} \end{bmatrix} \rightarrow \text{symmetric}$$

In EE → Count from 0 → $C_0, C_1, \ldots$

→ $R_0, R_1, \ldots$

$\therefore W^n = 1$

$(F_n)_{ij} = W^{ij}$

$i, j = 0, \ldots, n-1$

<u>There are lot of W:</u>

our need: Angle $\frac{2\pi}{n}$

$$\therefore W = e^{i2\pi/n} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

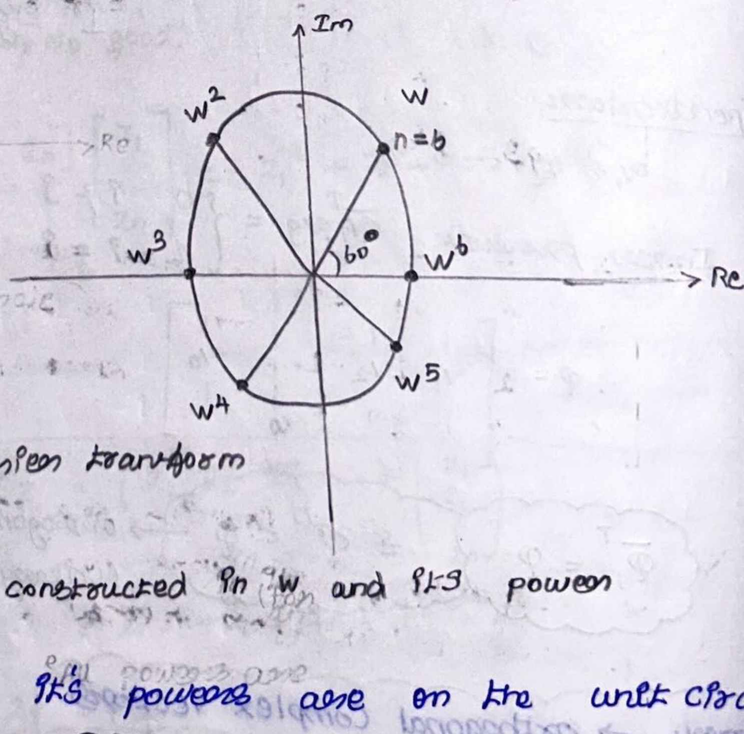
$$W^n = e^{i2\pi n/n} = \cos(2\pi) + i\sin(2\pi)$$

$W^n = 1$

$n = 6$

$\frac{2\pi}{6} = 60°$

$e^{\frac{2\pi i}{6}}$



For $\frac{6\pi}{6}$ Fourier transform

↓

It's totally constructed in W and its powers

This $e^{2\pi i/6}$ and it's powers are on the unit circle.

$W = e^{\frac{2\pi i}{6}}$ → 6/6 Fourier transform totally constructed out of W and it's powers.

∴ $e^{\frac{i2\pi}{n}}$ → absolute value are on the unit circle.

$\left(abs\left(\sqrt{\sin^2 x+\cos^2 x}\right)\right)^2 = 1.$ → All powers are on the unit circle

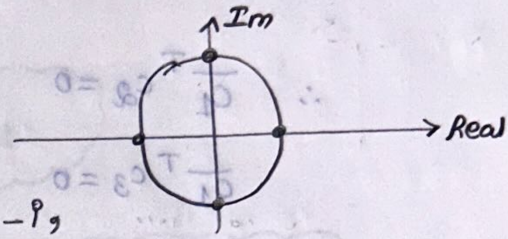
<u>when $n=6$</u>

'six roots of unity'

---

when $n=4$ 'Fourth root of unity'

$$W = e^{2\pi i/4}$$



The powers are

$$W=i,\ W^2=-1,\ W^3=-i,$$
$$W^4=1$$

∴ $i,\ i^2=-1,\ i^3=-i,\ i^4=1$

<u>Fourier matrix for 4 by 4 case</u>

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

Exponent → Row number × column number (Starting from zero)

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

→ 4×4 matrix that comes in to the four point Fourier transform.

∴ If we want to find Fourier transform of a vector with four component, we want to multiply by this 4 by 4 Fourier matrix. For inverse transform multiply by it's inverse.

'Inverse of this matrix — Also nice'

Fact: * columns are orthogonal.

'Break up into matrix with lots of zeros'

Inner product:

$$\left.\begin{matrix} c_1^T c_2 = 0 \\ c_1^T c_3 = 0 \end{matrix}\right\} \text{Real (Incase)}$$

$$(1 \quad i \quad -1 \quad -i)\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = 1 - i^2 + 1 - i^2$$

$$= 1+1+1+1 = 4$$

↳ Not zero?

Because: our matrix is a Complex matrix.

$$\therefore \overline{c_1}^T c_2 = 0$$

$$\overline{c_1}^T c_3 = 0$$

∴ Taking column 2 and 4:

$$(1 \quad -i \quad -1 \quad +i)\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix} = 0 \quad [\text{orthogonal}]$$

Are they orthonormal?

$$\overline{c_1}^T c_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}(1 \quad 1 \quad 1 \quad 1)$$

$$= 1+1+1+1 = 4$$

(need to normalize)

$$\therefore F_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -i & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \rightarrow \text{orthonormal columns.}$$

$$\therefore Q^{-1} = Q^T \rightarrow \text{Real}$$

$$\boxed{\overline{F_4}^T F_4 = I}$$

$$Q^{-1} = \overline{Q}^T$$

↳ complex.

$$\boxed{F_4^H F_4 = I}$$

Fast Fourier transform: (Idea)

* F_6 has connection (neat) with F_3
* F_8 〃 〃 F_4

(Half as big)

64 by 64 matrix:

$w \to$ 64th root

$F_{32} \to 32 \times 32$ matrix

$w \to$ 32 nd root of unity

Square

$$\boxed{(W_{64})^2 = W_{32}}$$

F_{64} is connected to two copies of F_{32}

$$F_{64} \approx \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix}$$

F_{64} is not half zero but this has zeros (half)

For

$$F_{64} = \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} \; \end{bmatrix}$$

Beauty: Almost zero.

$F_{64} \to 64^2$ calculations. $\left(\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} \to 32^2 + \text{fix up} \right)$

P $\to$ permutation matrix

$$\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \end{bmatrix}$$

$\hookrightarrow$ permutation matrix

P → multiplies with a vector (Even then odd)

$x_2, x_4, x_6, \dots$ & $x_1, x_3, \dots$

P → odds & even permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & & & \\ 0 & 1 & \cdots & & & \\ 0 & 0 & 0 & 1 & \cdots & \\ \cdots & & & & & 1 \end{bmatrix} \rightarrow \text{Right}$$

permutation matrix.

Seperates even & odd numbered components of a vector'

64 × 64 → seperated into the even & odd components then do a 32 size Fourier transform seperately & put them together.

$$F_{64} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$$

Re write all

$$\begin{bmatrix} I & D \\ I & -D \end{bmatrix}$$

P → Seperate odd & even.

∴ D → Diagonal

$= 2(32)^2 + fix$

$= 2(32)^2 + 32$ multiplication

$$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{n-1} \end{bmatrix} \rightarrow w^{31} \text{ (Here)}$$

Next idea: Break the 32 to F_{16}, F_{16}

$$F_{32} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} F_{16} & 0 \\ 0 & F_{16} \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

'Recursion'

$$F_{64} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} F_{16} & \\ & F_{16} \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$$

↓ Seperates Evens in to even even (0, 4, 8, 16) even odds (2, 6, 10, 14) odd evens odd odds.

↓ Seperates odd & even

$(32)^2$ will be reduced to

$$2\left[2(16)^2 + 16\right] + 32$$

↓

continuing

↓

simpler & simpler factors.

P P

$$\begin{bmatrix} x_{ee} \\ x_{eo} \\ x_{oe} \\ x_{oo} \end{bmatrix} \begin{matrix} \to \text{even even} \\ \to \text{even odd} \\ \to \text{odd even} \\ \to \text{odd odd.} \end{matrix}$$

Left – $\begin{bmatrix} I & D \\ I & -D \end{bmatrix}$ terms

Right – $\begin{bmatrix} & \\ & \end{bmatrix}_P$ matrix terms.

Fixups → $\log_2 n$

(32, 16, 8, 4, 2)

$$2(32)^2$$
$$\hookrightarrow 2(2(16)^2) \to 8(8)^2 \to 16(4)^2 \to 32(2)^2 \to 64(1)^2$$

$$\therefore n \log_2 n$$

$$(2(32)^2 + 32) \to 2(2(16)^2 + 16) + 32 \to 2(2(2(8)^2 + 8) + 16) + 32 \to$$
$$2(2(2(2(4)^2 + 4) + 8) + 16) + 32 +$$

$= 2(2(2(2(2 (2)^2+2)+4)+8)+16)+32$

$= 2(2(2(2(2(2(1)^2+1)+2)+4)+8)+16)+32$

$= 256. \approx \frac{n}{2} \log_2 (n)$

$$\boxed{\text{No. of steps} = \frac{n}{2} \log_2 (n)}$$

$F_n \rightarrow n^2$ steps

$FFT \rightarrow \frac{n}{2} \log_2 (n)$

If $n = 1024$ $\quad n^2 = 1048576$ (million steps)

$FFT \rightarrow \frac{n}{2} \log_2 (n) = 5(1024) = 5120$

$= \frac{1024}{2} \left(\log_2 2^{10} \right)$

$(1024 = 2^{10})$

$= \frac{1024}{2} (10 \log_2 (2)) = 1024 \left(\frac{10}{2}\right) = 5(1024)$

'we reduced the process by a process of factoring by a factor of 200'

↳ By proper factoring of matrix.

'FFT — In modern Scientific Calculations'

<u>Recitation — complex matrices</u>

Diagonalize A by constructing its Eigen value matrix Λ & Eigenvector matrix S

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix} = \bar{A}^T = A^H$$

solu:

$$\det \begin{pmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - (1-i)(1+i)$$

$$= 6 - 3\lambda - 2\lambda + \lambda^2 - (1 - i^2)$$

$$= \lambda^2 - 5\lambda + 6 - 2$$

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0$

$\boxed{\lambda = 4, 1} \rightarrow$ Real Eigen values

when $\lambda = 4$

$$\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix} x_1 = 0 \qquad \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix} x_2 = 0$$

Hermitian matrices $\rightarrow$ Have real Eigen values?

$$x_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} \qquad x_2 = \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$$

x_1 and x_2 are orthogonal

$A = S \Lambda S^{-1}$

$\hookrightarrow$ orthonormal columns

Are they orthonormal?

$$\begin{pmatrix} 1 & 1-i \end{pmatrix} \begin{pmatrix} 1-i \\ -1 \end{pmatrix} = 1-i-1+i = 0$$

$$\begin{pmatrix} 1 & 1+i \end{pmatrix} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = 1+(1-i)^2 = 1+1-i^2 = 3$$

Length $= \sqrt{3}$

$\therefore \overline{v}^T u = 0 \rightarrow v \neq u$

$= 1 \rightarrow v = u$

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i & 1 \\ -1 & 1+i \end{pmatrix}$$

$S^{-1} = \overline{S}^T \rightarrow$ orthonormal complex

$A = S \Lambda \overline{S}^T$

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i & 1 \\ -i & 1+i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1+i & -1 \\ 1 & 1-i \end{pmatrix} \left(\frac{1}{\sqrt{3}}\right)$$

$$A = \frac{1}{3}\begin{pmatrix} 1-i & 4 \\ -1 & 4(1+i) \end{pmatrix}\begin{pmatrix} 1+i & -i \\ 1 & 1-i \end{pmatrix}$$

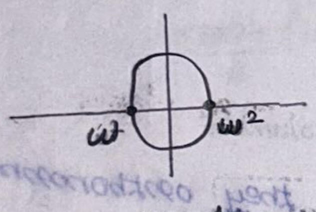
$$A = \frac{1}{3}\begin{pmatrix} 1+1+4 & -i+i^2+4-4i \\ -1-i+4+4i & i^2+(4+4i)(1-i) \end{pmatrix}$$

$$A = \frac{1}{3}\begin{pmatrix} 6 & 3-3i \\ 3+3i & 9 \end{pmatrix} = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}$$

$$\underbrace{A = (\quad)(\quad)(\quad)}_{\text{Diagonalized form}}$$

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & \omega \end{bmatrix}, \quad \omega = e^{i2\pi/2} = -1$$

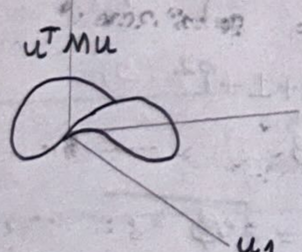
what about F_2?



$\omega = -1$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Positive definite matrices & minima



In calculus, the second derivative decides whether a critical point of $y(x)$ is a minimum. For functions of multiple variables the test is whether a matrix of II derivates is positive definite.

(several ways to test positive definiteness)

How the shape of $f(x) = x^T A x$ is determined by the entries of A?

Lecture - 28

* +ve definite matrices (Tests)
* Test for minimum ($x^T A x > 0$)
* Ellipsoids in R^n

'Ellipsoids are connected with positive definite but not parabolas are not'

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightarrow$ Symmetric (Is it positive definite)

① $\lambda_1 > 0, \lambda_2 > 0 \longrightarrow$ Eigen value test

② $a > 0$ $\boxed{ac - b^2 > 0} \rightarrow$ Determinant test.

③ pivots $a > 0$, (Product of pivots = determinant)

$\frac{ac-b^2}{a} > 0$ [other pivot]

④ $x^T A x$ being positive

'How Eigen value & Determinant test picks the same matrices'

Example:

$\begin{bmatrix} 2 & 6 \\ 6 & \end{bmatrix} \rightarrow ?$ (Be a +ve definite)

$\boxed{\text{Answer: } 19 \text{ (or) above } 19}$

If

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \Rightarrow \det(A) = 36 - 36 = 0$ [Positive - semi definite]

$\downarrow$

singular matrix

$\lambda = 0, \lambda_2 = \text{Trace} - \lambda_1$

Pivots: 2 (Singular)

Eigen values: 0, 20.

$\boxed{\lambda_2 = 20}$

$$x^T A x = (x_1 \; x_2) \begin{pmatrix} 2 & 6 \\ 6 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x \rightarrow$ Any vector

$$= \begin{pmatrix} 2x_1 + 6x_2 & 6x_1 + 18x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 2x_1^2 + 6x_1x_2 + 6x_1x_2 + 18x_2^2$$

$$= 2x_1^2 + 12x_1x_2 + 18x_2^2$$

$= 2x_1^2 + 12x_1x_2 + 18x_2^2$ (quadratic) degree 2.

$\uparrow \quad \downarrow \quad \uparrow$

$ax^2 \quad 2bx_1x_2 \quad cy^2$

(for every x_1 and x_2 is this x^TAx positive)

our Eigen value & pivot: Barely failed $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$

our Eigen " test: Totally failed $\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$

In failed case:

$= 2x_1^2 + 12x_1x_2 + 7x_2^2 \rightarrow$ for some x_1, x_2 it will be negative!

when $x_1 = 1, x_2 = -1$

$= 2(1) + 12(-1) + 7$

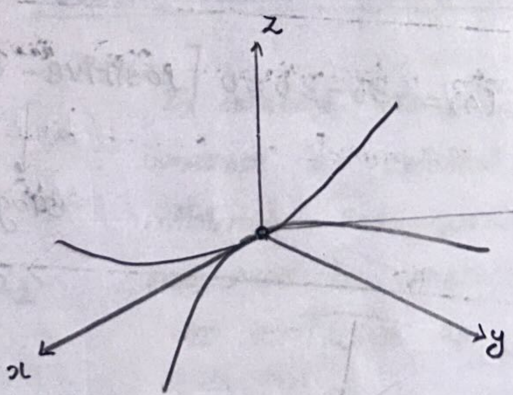
$= -3$ (So not positive definite)!

Graphs:

$f(x, y) = \vec{x}^T A \vec{x}$

$= ax^2 + 2bxy + cy^2$

$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$



saddle point (max at some direction min at some)

Better: go with the direction of Eigen vector

$2x^2 + 12xy + 7y^2 \rightarrow$ up in some directions, Down in some directions

$x = 1, y = 0 \rightarrow$ going upwards

Case with 20

$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$

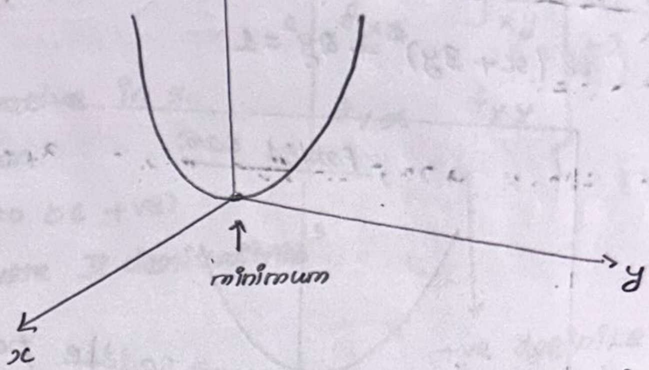
$\lambda_1 \lambda_2 \to +ve$ (determinant $\to$ +ve)

$\lambda_1 + \lambda_2 =$ Trace (+ve) only when $\lambda_1, \lambda_2 > 0$

$\boxed{\text{Exception: when both are zero}}$ $\to$ (zero vector)

Graph: 'Doesn't have a saddle point.'

$f(x,y) = 2x^2 + 12xy + 20y^2$ (pure quadratic ↳ no constants)



* 1st derivative $= 0$ (Not enough — Because saddle point too)
* 2nd derivative = +ve (upwards)
  'matrix of second derivative → +ve'

calculus: Min ~ $\frac{d^2u}{dx^2} > 0$ [1st der = 0]

Linear algebra: min $f(x_1, x_2, \ldots, x_n)$ when the matrix of second derivative is +ve definite.

Question:

$f(x,y) = 2x^2 + 12xy + 20y^2 \to$ why it's always +ve.

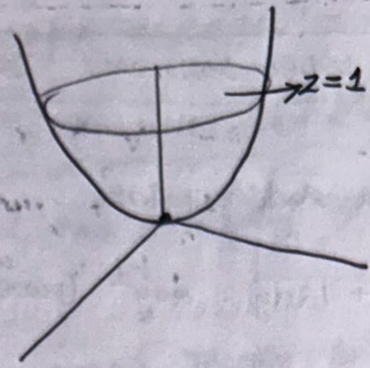
'Is it can be written in square format?

$= 2(x^2 + 6xy + 10y^2)$

$= \underbrace{2(x+3y)^2 + 2y^2}_{}$

never gonna be (-)ve [Real]

Previous case: $2(x+3y)^2 - 11y^2 \to$ (Failed case) marginal case : $= 2(x+3y)^2$

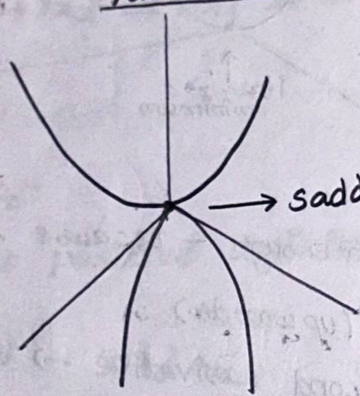


$\rightarrow z=1$ plane (elliptical curve)

Equation of this elliptical curve (surface):

$$2(x+3y)^2 + 2y^2 = 1$$

Failed case



$\rightarrow$ saddle point

'Hyperbola'

'completing a square - Good in Gaussian elimination'

$$2(x+3y)^2 + 2y^2 = 1$$

$\hookrightarrow$ Not accident

$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ 2 - pivot outside
3 $\rightarrow$ multiplier inside $(2 \times 3 = 6)$

$\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

$$A = L\,U$$

$$\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

$\downarrow$ multiplier.

Completing the square : 'Sum of squares'

Elimination (we know it for $n \times n$)

$$2(x+3y)^2 + 2y^2 = 1$$

pivots

(+ve positive pivots gives sum of squares) 'graph goes up minimum at origin'

matrix of second derivative : ?

In 2-d

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \rightarrow \text{Hessian matrix.}$$

f_{xx} - second derivative in x direction

(They have to be +ve)

↳ 'pure II derivatives'

+ve definite in order to have a minimum.

when we have minimum:

(Function of 2 variables)

1) I derivative = 0

2) II derivative = +ve

'we know n by n'

'Elimination'

Example: 3 by 3

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \text{Is this +ve definite}$$

Solu:

* Is this have ~~maximum~~ minimum?

$$\det(A) = 2(4-1) + 1(-2)$$

$$= 6 - 2 = 4$$

$$\boxed{\det(A) = 4}$$

determinants

$A_1 = 2, A_3 = 3, A_4 = 4$

Pivots

$2, \frac{3}{2}, \frac{4}{3}$

Eigen values: $2-\sqrt{2},\ 2,\ 2+\sqrt{2}$

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^3 - 2x_1x_2 - 2x_2x_3 > 0$$

'Graph: like a bowl - paraboloid'

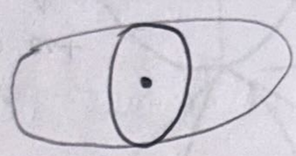
'we can write them as sum of squares'

'slice the bowl like structure'

$\downarrow$ 4d (3 variables + upward direction)

cutting at height 1:

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^3 - 2x_1x_2 - 2x_2x_3 = 1$$



3 principal direction $\rightarrow$ 1 long, 2 same (equal)

matrix $\rightarrow$ 1 equal repeated, 1 different.

sphere: All 3 directions are same

Identity matrix: 3 repeated Eigen values. (same)

typical case:

3 different Eigen values

major, minor, middle } Axis, $\rightarrow$ In the direction of Eigen vectors.

Length $\rightarrow$ by Eigen values.

---

matrix factorization: $Q \Lambda Q^T$ $\rightarrow$ principal axis theorem

Eigen vectors - Tells us the directions of principal axes

Eigen vectors - length of those principal axes.

Recitation

1) For which values of c is $B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+c \end{bmatrix}$

positive definite?

positive semi definite?

$A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Solu:

determinant test: $A_1 = 2, \quad A_2 = 3,$

$$A_3 = 2\begin{vmatrix} 2 & -1 \\ -1 & 2+c \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ -1 & 2+c \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 2(4+2c-1) + 1(-2-c-1) - 1(1+2)$$

$$= (6+4c) - 3 - c - 3$$

$\boxed{A_3 = 3c}$ If $c > 0$ positive definite

$c \geq 0$ positive semi definite

Pivot test:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2+c \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & \frac{3}{2}+c \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & c \end{bmatrix}$$

pivots: $3, 3/2, c \rightarrow c > 0$ positive def

$c \geq 0$ semi positive def.

$2 \times \frac{3}{2} \times c = 3c$ (det)

$\boxed{3c = 3c}$

Energy-test / completing the square:

$[x \ y \ z] \, B \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ≥ 0 (positive semi def)

> 0 (positive definite)

$$= 2x^2 + 2y^2 + (2+c)z^2 - 2xy - 2xz - 2yz$$

$$= \underbrace{2\left(x - \tfrac{1}{2}y - \tfrac{1}{2}z\right)^2}_{+ve} + \underbrace{\tfrac{3}{2}(y-z)^2}_{+ve} + \underbrace{cz^2}_{\text{when } c>0 \ (+ve\ def)}$$

$c \geq 0$ (+ve semi def)

case i) $c > 0$

$$\begin{bmatrix} 2 & -\frac{1}{2} & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & c \end{bmatrix} \rightarrow \frac{R_1}{2}, \frac{R_2 \times 2}{3}, \frac{R_3}{c} \rightarrow$$

$$= \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} x = 0 \quad \text{when} \quad x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

our eqn will be zero when $x = y = z = 0$

case ii) $c \geq 0$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = 0 \quad \text{when} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$1 \rightarrow$ Free variable

↑ Free variable

when $c=0$, columns of our matrix sum up to zero.

when $c=0 \rightarrow$ Even for non zero $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we will have zero

"semi +ve definite"

Lecture - 29

Similar matrices & Jordan form

Not changed by M:

* Eigen values
* Trace & determinants
* Rank
* No. of Independent Eigen vectors
* Jordan form.

Similar matrices : $B = M^{-1} A M$ for some invertible matrix M

square matrices can be grouped by similarity

(They have nice Jordan form)

$A^TA \rightarrow$ positive definite!

$x^TAx > 0$ (except for $x=0$)

$\boxed{\text{'Positive definite matrices: Come from least squares'}}$

If A is positive definite! what about A^{-1}?

Eigen values of $A^{-1} = \dfrac{1}{\text{Eigen values of A}}$

'So they are also +ve'

'A^{-1} is also positive definite'

If A and B are positive definite:

what about $A+B$ $\qquad x^TAx > 0$

$x^TBx > 0$

$x^T(A+B)x > 0 \rightarrow$ Also positive definite.

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 3 & 4 \end{bmatrix} \rightarrow \text{symmetric}$$

'positive definite'

$A^TA \rightarrow$ In least squares

Now $A_{m\times n}$ (rectangular) $\rightarrow$ Not symmetric (Not square)

$A^TA \rightarrow$ Symmetric, Square, positive definite

$x^T(A^TA)x > 0 \rightarrow$ why can't be negative!

$(Ax)^T(Ax) = \|Ax\|^2 > 0$

$(\text{Length})^2$

when will be $\text{length}^2 = 0$, only if the vector is a zero vector!

Rank $\rightarrow$ (No null space) $\rightarrow$ only zero vector.

what about rank!

Independent columns → Rank n

→ Then A^TA → Inveertible.

With a positive definite matrix - we never had to do zero pivots or row exchanges. (we never run in to unsuitably small pivots) → Great matrices.

Similar matrices

A and B are similar (n by n matrices):

* connected in the way : For some matrix M

(invertible)

$$\boxed{B = M^{-1} A M}$$

Example:

$$\Lambda = S^{-1} A S$$

'A is similar to lambda'

'Families'

$\Lambda \rightarrow$ diagonal matrix

$S \rightarrow$ Eigenvector matrix

Each one in the family is connected each other by some matrix like diagonal (outstanding - best one).

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{M^{-1}}{\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \underset{M}{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 9 \\ 1 & 6 \end{bmatrix}$$

M → some matrix

$$= \begin{bmatrix} 2-4 & 9-24 \\ 1 & 6 \end{bmatrix} = \underset{B}{\begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}}$$

A and B (they have something in common)!

'Same Eigen values'

$\lambda = 3, 1.$

Trace(B) = 4

det(B) = 3

(same as A)

Similar matrices have same λ's ! → Some M connects the members of this family.

$$\begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ 0 & 3 \end{bmatrix}$$

'They all share the same Eigen value'

$Ax = \lambda x$ $\qquad\qquad (B = M^{-1}AM)$

for any invertible matrix M

$AMM^{-1}x = \lambda x$

$M^{-1}AMM^{-1}x = \lambda M^{-1}x$

$BM^{-1}x = \lambda M^{-1}x$

'B has the same Eigen value λ'

Note: Eigen vectors don't need to be the same.

Eigen vector of B is M^{-1} (Eigen vector A)

for diagonal matrix, the Eigen values are same and the Eigen vectors may be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Nice: Different Eigen values $(3, 1)$

<u>Repeated Eigen values</u> - Bad case.

when two Eigen values are same → Might not enough be a full set of Eigen vector

$\lambda_1 = \lambda_2$ → Not diagonalizable
might

$\lambda_1 = \lambda_2 = 4$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix},$$

↓

one small family.

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= 4I$