$$\therefore \quad S = \underbrace{\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}}_{P_0} + \underbrace{c_1 \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{Special cases}}$$

Lecture-9 $Ax = b$ (If it has solution)

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1 \rightarrow (1)$$

$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2 \rightarrow (2)$$

$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3 \rightarrow (3)$$

L.H.S eq 3 is the sum of (1) and (2)

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

Augmented $\approx [A \quad b] \rightarrow$ matlab

matrix A with vector b tacked on.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

Pivot columns

$$\therefore \quad b_3 - b_2 - b_1 = 0$$

$$\boxed{b_3 = b_1 + b_2}$$

Suppose $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$, $u = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Solvability:

"conditions on b — to solve"

$Ax = b$ solvable when b is in the $C(A)$

If a combination of rows of A gives zero row, the same components of b need to give zero.

Assume: There is a solution:

Algorithm — To find the solution

To-find the complete solution:

$$Ax = b$$

① A particular solution x_p.

one way. → $x_{particular}$ = set all free variables to zero (convenient), solve $Ax = b$ for the pivot variables

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Free

$x_2 = x_4 = 0$

$\therefore$ $x_1 + 2x_3 = 1$

$2x_3 = 3$

$2x_3 + 4x_4 = 3$

$2x_3 = 3$

$x_3 = 3/2$

$x_1 = 1 - 3$

$\boxed{x_1 = -2}$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

$\boxed{x = x_p + x_n}$

$x_0 \rightarrow x$ null space.

$x_p \rightarrow$ particular solution

Reason

$A x_p = b$

$A x_n = 0$

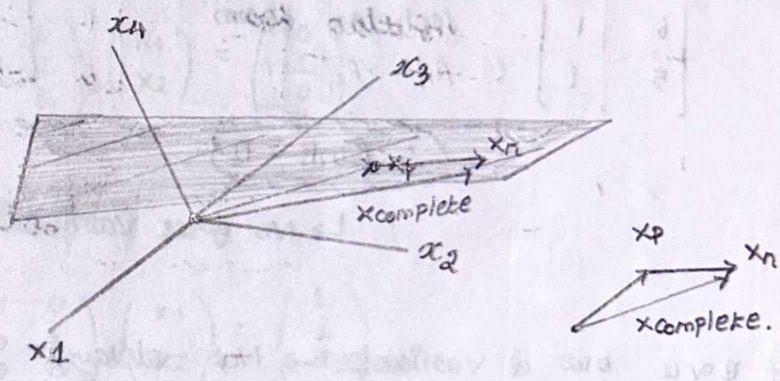
$A(x_p + x_n) = b$

'Linear'

$\therefore$ $x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Null space: Has all combinations of our special solutions.

Plot all solutions in R^4:



xcomplete $\rightarrow$ 2-d plane. (No. of independent - free constants)

$\therefore$ The plane doesn't goes through the origin. It goes through the x_p (particular).

'shifted away from the origin'

Consider $(m \times n)$ matrix A of rank r.

$$\boxed{\begin{aligned} r &\leq m \\ r &\leq n \end{aligned}}$$

Full column rank means $r = n$

<u>What does this mean?</u>

'pivot in every column'

* No free variables
* Null space has only the zero vector.

solution to $Ax=b$ is $x = x_p$ → unique solution if it exists.

$\therefore x_n = c\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_c = x_p + x_n$

$x_c = x_p$

0 or 1 solution based on solvability

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow \text{Reduced row echelon form} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank = n

↳ No free variables

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

4 eqn but 2 variables → Not always solvable.

Row reduced echelon form:

$1-6$
$1-18$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \rightarrow \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 6R_1 \\ R_4 = R_4 - 5R_1 \end{matrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & -5 \\ 0 & -17 \\ 0 & -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -17 \\ 0 & -14 \end{bmatrix} \quad R_2 \rightarrow -\frac{1}{5}R_2$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} R_3 \rightarrow R_3 + 17R_2 \\ R_4 \rightarrow R_4 + 14R_2 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad R_1 = R_1 - 3R_2$$

when it is solvable:

$Rx = b$ $\quad (4\times 2)(2\times 1)$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = b$$

From column space,

* Null space – priceless
* b must be any one of the columns itself or its combinations.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

when $x_1 = 1, x_2 = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 0, x_2 = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 1, x_2 = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 0, x_2 = 0.$

Full row rank means $r = m$

"Every row has a pivot"

"Solvability ?"

Exists

can solve $Ax = b$ for which R.H.S ?

Elimination $\longrightarrow$ free variables $= n - r$ free variables.

we can solve for every right hand side b.

$\boxed{r = m}$

$\therefore \boxed{n - r = n - m}$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

Row reduced echelon form:

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & -5 & -17 & -14 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 1 & \frac{17}{5} & \frac{14}{5} \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_2 \to -\frac{1}{5} R_2 \\ R_1 \to R_1 - 2R_2 \end{array}$$

$$A = \begin{bmatrix} 1 & 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{17}{5} & \frac{14}{5} \end{bmatrix}$$

$r = m$ → pivot variable columns.

Free variable columns

---

cases: $r = m$
$r = n$
$r = m = n$

$r = m = n$ "Full rank - square matrix" → Invertible.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \quad R_2 \to -\frac{1}{3} R_2$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad (R = I)$$

Null space of this matrix:

'zero vector only'

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $x_1 = x_2 = 0$.

For what b it's solvable?

'$r = m = n$'

For every b it's solvable - with a unique solution'

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

when $x_1 = 7$, $x_2 = 9$

---

other cases: $r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix} \rightarrow$ (0 or 1 solution)

$r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$

F could be mixed to I

'Always a solution - 1 or ∞ly many'

$\infty \rightarrow$ Because we always hall null to deal with.

so ∞ly many solutions.

$r < m$, $r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ $I, F \rightarrow$ mixed.

'No solution or ∞ly many solutions'

Rank tells you everything about the number of solution - except the exact entries in the solution.

Recitation

Find all solutions, depending on b_1, b_2, b_3

$$x - 2y - 2z = b_1$$
$$2x - 5y - 4z = b_2$$
$$4x - 9y - 8z = b_3$$

Solu:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2 - 2b_1 \\ 0 & -1 & 0 & b_3 - 4b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right] \quad R_3 \to R_3 - R_2$$

(1) $-2b_1 - b_2 + b_3 \neq 0 \rightarrow$ No solution.

(2) $-b_2 - 2b_1 + b_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & 2b_1 - b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & 2b_1 - b_2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 + 2R_2$$

$\underbrace{\uparrow \quad \uparrow}_{\text{Pivot variable}}$ $\quad \uparrow$ Free variable

* Particular solution $(Ax = b)$
* Null solution $(Ax = 0)$

$z=0$

$2b_1 - b_2 = y$

$5b_1 - 2b_2 = x$

In matrix form:

$$x_p = \begin{bmatrix} 5b_1 - 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$$

Here $z \neq 1$

∴ when $z = 1$ (0 0 0)

now can't bring $z = 1$.

Special case

$Ax = 0$

$z = 1$

$x_1 - 2z = 0$

$\boxed{x_1 = 2}$

$\boxed{y = 0}$

$$x_n = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$x = x_p + c\, x_n$$

'As many solutions are available'

Lecture-10 – Independence, Basis & dimension

A basis is a set of vectors, as few as possible, whose combinations produce all vectors in space. The number of basis vectors for a space equals the dimension of that space.

Linear independence – Bunch of vectors.

Matrix A is m by n ($m < n$). Then there are non-zero solutions to $Ax = 0$.

solu:

'More unknowns than equations', then there are non-zero solutions to $Ax = 0$.

∴ Reason: There will be some free variables (atleast one).

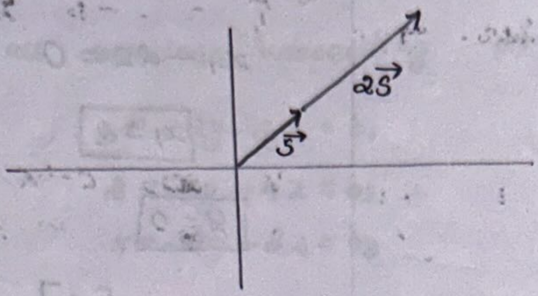
Independence

vectors $x_1, x_2, x_3, \ldots, x_n$ are independent linearly if

* If no combination gives the zero vector. (zero combination)

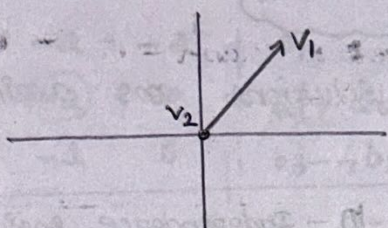
$c_1 x_1 + c_2 x_2 + \dots + c_n x_n \neq 0.$

(Except for zero combination all $c_i = 0$)



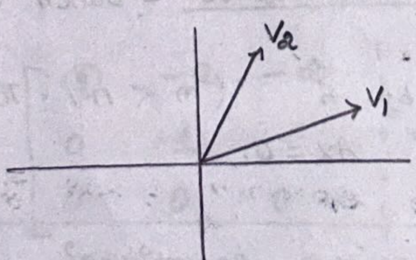
$\vec{V_2} = 2\vec{V_1}$ (Dependendent)

$\vec{V_2} - 2\vec{V_1} = 0$

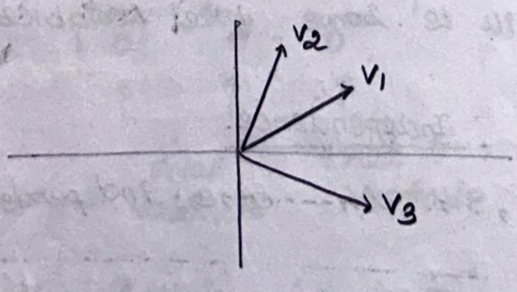


$V_2 \rightarrow$ zero vectors.

$\therefore$ $\boxed{0\vec{V_1} = \vec{V_2}}$ | $0V_1 + 6V_2 = 0$



V_1 and V_2 will be independent.



'Dependent : why?'

$$\text{Matrix} = A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

columns are dependent: If there's something in the null space.

Abstract definition: when $v_1, \ldots v_n$ are columns of A, they are independent if null space of A is only the zero vector.

★ Dependent: If something else in the null space.

$AC = 0$ for some c. [Rank $< n$]

In case of independent: Rank $= n \rightarrow$ No free variables.

The definition of independence talks about the vectors not matrices.

Spanning of space:

combination of columns: column space.

Spanning: vectors $v_1, \ldots, v_n$ span a space means: The space consists of all combination of those vectors.

Let S be the spanned space of the vectors.

we are interested in vectors

★ Span a Space.

★ Independent (correct number of vectors)

∴ less number → Don't span

More number → Not independent.

Basis

Sequence of vectors $v_1, v_2, v_3, \ldots, v_d$ in the space with two properties.

1) They are independent

2) They span the space.

Example of basis

Space is R^3

* one basis is $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ → standard

They are independent.

$ac_1 + bc_2 + cc_3 \neq 0$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = 0$$

only for zero vectors. So they are linearly independent.

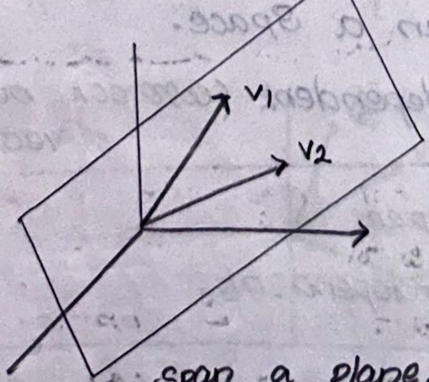
* Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\5\\8\end{bmatrix}$

$R^n \rightarrow n$ vectors give basis if the $n \times n$ matrix with those columns is invertible.

$\begin{bmatrix}1\\1\\2\end{bmatrix} \begin{bmatrix}2\\2\\5\end{bmatrix}$ → Independent

what space will they be the basis?

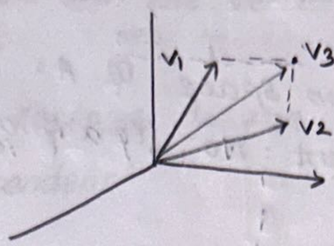
Them span.



span a plane.

If we we put another vector

$$\begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$$



They span - But they are independent
(Not a basis).

Independent columns: * Span the space (column)
* Independent.

So they are basis for column space.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$$

Not unique basic: zillions of basis (Take any invertible 3×3 matrix, its columns are basis for R^3.) columns are independent, forms a basis.

Some thing in common for all those basis:

$R^3 \to 3$ vectors

$R^n \to n$ vectors.

Every basis has the same number of vectors.

Dimension D : That number is dimension.

Number of vectors in a basis.

Independence: combinations not being zero.
Spanning: All the combinations
Basis: Combining Independence & spanning
Dimension: No. of vectors in any basis.

Example: $C(A)$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Do they span column space of A: Yes

Are they independent: No (something in null space)

eg:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Not independent

Basis for the column space:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \rightarrow \text{Natural answer (systematic)}$$

Rank = 2,

Rank(A) = # number of pivot columns

Rank (A) = Dimension of the C(A)

Rank (A) → matrix not subspace.

C(A) → column space not matrix.

Another basis:

* $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$
* c_2, c_4
* c_2, c_3
*

Another basis: For C(A)

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$$

$\dim(C(A)) = 2$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$$

* Independent
* Span

'Null space'

Dimension:

$\dim(N(A)) = ?$

$$\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \rightarrow$$ Free variables (special case) $x_3=0, x_4=1$ & $x_3=1, x_4=0$

↳ single can't span the entire null space.

Null space: Two vectors in the null space, 'Independent'

Are they basis? What's about dimension?

Dimension of Null Space = Number of free variables.

$$\dim(N(A)) = n-r$$

We can span the Null space by

$$\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \rightarrow \text{Basis.}$$

Recitation

Find the dimension of the vector space spanned by the vectors

$(1, 1, -2, 0, -1)$

$(1, 2, 0, -4, 1)$

$(0, 1, 3, -3, 2)$

$(2, 3, 0, -2, 0)$ & find a basis for that space.

Solu: Basis:

Gn: The vector space is spanned by these vectors

Are they linearly independent?

* Elimination. (As rows)

$$\begin{bmatrix}1 & 1 & -2 & 0 & -1\\1 & 2 & 0 & -4 & 1\\0 & 1 & 3 & -3 & 2\\2 & 3 & 0 & -2 & 0\end{bmatrix} \rightarrow \begin{matrix}R_2 \rightarrow R_2 - R_1\\R_4 \rightarrow R_4 - 2R_2\end{matrix}$$

$$=\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & 2 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$=\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} R_4 \to R_4 - R_3$$

$$=\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

These three rows are linearly independent. still span the same space.

Elements for basis:

(1, 1, -2, 0, -1)
(0, 1, 2, -4, 2)
(0, 0, 1, 1, 0)

we can use the first three in vectors.
∴ we don't switched.

In case of switched: we need to track it.

For safer side: we can use the answer alone.

Dimension = 3

1) Row perform elimination
2) Columns perform elimination.

∴ Transposes have same no. of pivot.

Method 2

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 0 & -4 & -3 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{Elimination}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \dim(A) = 3.$$

Problem 2: we can't use

(1 0 0 0 0)
(1 1 0 0 0)
(0 1 1 0 0) as elements

of basis. They can't span the same column space.

we need to use:

Basis

$$\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ -3 \\ 2 \end{pmatrix}$$ } Different bases.

Lecture - 10

The Four fundamental subspaces

1) Column space $C(A)$
2) Null space $N(A)$
3) Row space (Transpose the matrix)
4) Null space $N(A^T)$ [Left null space of A.]

Rows are the bases → Independent. (All combinations of rows of A).

Row space: All combinations of columns of A^T.
$C(A^T)$

Note: A is $m \times n$

* Null space $N(A)$ in R^n
* Column space $C(A)$ in R^m
* Column space $C(A^T)$ in R^n
* Null space $N(A^T)$ in R^m

R^n → maximum possible dimension space.

Null space
$Ax = 0$
possible x's → R^n
↳ Columns

$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix} \rightarrow$ Not independent

$\therefore$ Row 1, Row 2 $\rightarrow$ Identical

'Non-invertible' $\rightarrow$ Dependent.

Row space $\rightarrow$ 2-dimensional.

'Rows of the matrix tells about columns'

Column space in R^m $\rightarrow A$

$\begin{bmatrix}1 & 1 & 2\\2 & 1 & 3\\3 & 1 & 4\\4 & 1 & 5\end{bmatrix} \rightarrow$ 4 independent Rows (m components in column vector)

dimension: 4

$\hookrightarrow R^4 = R^m$

Null space in R^n: $\rightarrow A$ $\boxed{AX = 0}$

$\begin{bmatrix}1 & 1 & 2\\2 & 1 & 3\\3 & 1 & 4\\4 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$\hookrightarrow R^3 = R^n$ (n components in x)

Column space of A^T in R^n $A \rightarrow m \times n$

(n components in column vector)

$\begin{bmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\\2 & 3 & 4 & 5\end{bmatrix}$

$\hookrightarrow$ 3-dimension $R^3 = R^n$

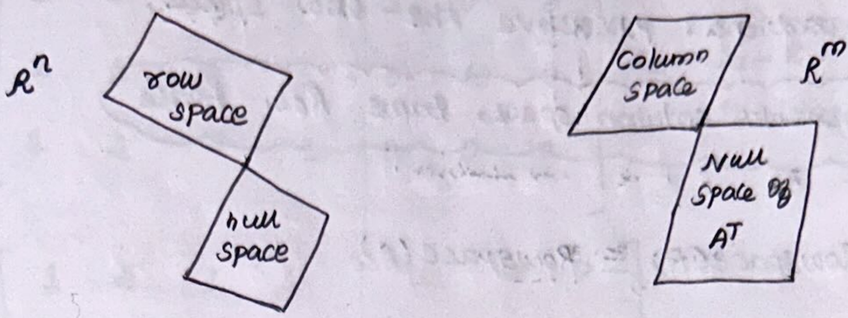
Null space of A^T in R^m:

$\begin{bmatrix}1 & 2 & 3 & 4\\1 & 1 & 1 & 1\\2 & 3 & 4 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ [m comp in X]

$\hookrightarrow$ 4-d $R^4 = R^m$

Picture:

4 Subspaces



Basis? Systematic way

1) Column space ($\star$ $\dim C(A) = \text{rank}\ (r)$
   $\star$ Basis $\rightarrow$ Pivot columns.

2) $\dim C(A^T) = \text{rank}\ (r)$) Row space
   Basis for row space is the first r rows of R.

3) $\dim N(A) = n - r$) null space.
   Basis $\rightarrow$ special solutions

No. of free variable $= n - r$

4) $\dim N(A^T) = m - r$) left null space

$$\boxed{\begin{matrix} R^m & & R^n \\ \downarrow & & \downarrow \\ m - r + r = m & & n - r + r = n \end{matrix}}$$

Basis for Row space

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

pivot

$C(R) \neq C(A) \rightarrow$ Different column space.

Row operations: preserve the row space.

Different column space, same Row space

$$\text{RowSpace}(R) \equiv \text{Rowspace}(A)$$

Basis for row space R:

Basic for row space is first r rows of R.

sitting in R in clean as possible form.

$$\begin{matrix}(1 & 0 & 1 & 1) \\ (0 & 1 & 1 & 0)\end{matrix} \rightarrow \text{All operations we did in row space (so row)}$$

$R_2 - R_1$ will give an answer that stays in the rowspace.

Dimension: r (number of pivots)

Null space of A^T

$$A^T y = 0$$

Taking transpose:

$$y^T A = 0^T$$

$$y^T A = [0 \ 0 \ 0]$$

$$[\ y^T\ ] \begin{bmatrix} & A & \end{bmatrix} = [0 \ 0 \ 0]$$

multiplying from left [so left null space]

Basis:

Gauss-Jordan method:

$$\begin{bmatrix} A_{m\times n} & I_{m\times m}\end{bmatrix} \rightarrow \text{Echelon form. } \begin{bmatrix} R_{m\times n} & E_{m\times m}\end{bmatrix}$$

$EA = R$

If $R = I$, then

$E = A^{-1}$

$\boxed{AA^{-1} = I}$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] R_2 \to -R_2$$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right] R_1 \to R_1 - 2R_2$$

R E

$\text{Rank}(A) = 2$

$$\text{Dimension of } N(A^T) = m - r = 3 - 2 = 1$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} = EA$$

→ combination gives null solution.

Left null space:

$$\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Produced by

Basis: $(-1 \quad 0 \quad 1)$

Review

New vector space:

All 3 by 3 matrices!!

$A+B$, CA → Satisfy? [vector space - eligibility]

Let our matrix space (subspace) be M:

"Space of all 3×3 matrices"

Subspaces of M:

* All upper triangular matrices
* All symmetrical matrices

Intersection of two subspaces is also a subspace.

* Intersection of upper Δ^r & symmetric matrices — Diagonal matrix.

$$\begin{bmatrix} 3 & 2 & 6 & 7 \\ 0 & 6 & 6 & 8 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} A & B & C \\ B & C & D \\ C & D & E \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

upper triangular — Symmetric matrices — Lower Δ^r matrix.

$$\begin{bmatrix} 3 & 2 & 6 \\ 2 & 5 & 4 \\ 6 & 4 & 3 \end{bmatrix} \cap \begin{bmatrix} 3 & 2 & 6 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{bmatrix} \cap \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 6 & 4 & 3 \end{bmatrix}$$

symmetric — upper triangular — Lower triangular

The intersection of Δ^r and symmetrical matrix is diagonal matrix.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Diagonal matrix: Lowest

* Diagonal matrices - Smallest subspace.

Dimension of diagonal matrix $D = 3$

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&7\end{bmatrix}$$

Are they basis? Independent? any diagonal matrix can be formed from them?

'span the subspace'

* Add, CA, (multiply two matrices - Just ignore now).

For some vectors b the equation $Ax = b$ has solutions and for others it doesn't. To understand these equations we study the

* column space
* Null space
* Row space
* Left nullspace.

} of the matrix A.

Recitation: Computing Four fundamental subspaces

Suppose
$$B = \underbrace{\begin{pmatrix}1&&\\2&1&\\-1&0&1\end{pmatrix}}_{L}\underbrace{\begin{pmatrix}5&0&3\\0&1&1\\0&0&0\end{pmatrix}}_{U}$$

Find a basis and compute the dimension of each of the four fundamental subspaces.

solu:

1) $\dim(C(B)) = 2$

$$U = \begin{pmatrix}5&0&3\\0&1&1\\0&0&0\end{pmatrix}$$

$\dim(C(B)) =$ # of pivots $= 2$.

A basis for C(B) is

1) we can do elimination & take pivot columns as basis

(or)

2) Take pivot columns in L matrix.

Basis for C(B) is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$EA = U$

$\boxed{A = LU}$

$L \to$ Inverse of A

$\to$ since for column space, the elimination doesn't preserves column operation.

Null-space:

dimension = $n - r$

$\dim N(B) = 3 - 2 = 1$

$$\left\{ \begin{pmatrix} \\ \\ 1 \end{pmatrix} \right\}$$

Plug in 1 for free variable, then back solve.

$z = 1$

$5x + 3z = 0$

$5x = -3$

$\boxed{x = -\frac{3}{5}}$

$y + z = 0$

$\boxed{y = -1}$

Nullspace $AX = 0 \to$ solution.

Basis: $\left\{ \begin{pmatrix} -3/5 \\ -1 \\ 1 \end{pmatrix} \right\}$

Rowspace:

$\dim R(B) = \dim C(B) = 2$

Basis:

"Two pivot rows"

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Left null space: $N(B^T)$

$\dim N(B^T) = m - r$

$= 3 - 2 = 1.$

$B = LU$

$\boxed{EB = U}$

Inverse of Lower triangular matrix:

Upper triangular matrix: Inverse also upper Δ^r

Lower triangular matrix: Inverse also lower Δ^r.

Inverse of L matrix = $\underset{E}{\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 1 & 0 & 1 \end{pmatrix}} \begin{pmatrix} \underset{B}{B} \end{pmatrix} = \underset{U}{\begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$

we need to have U's third row zero.

so As per basis

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

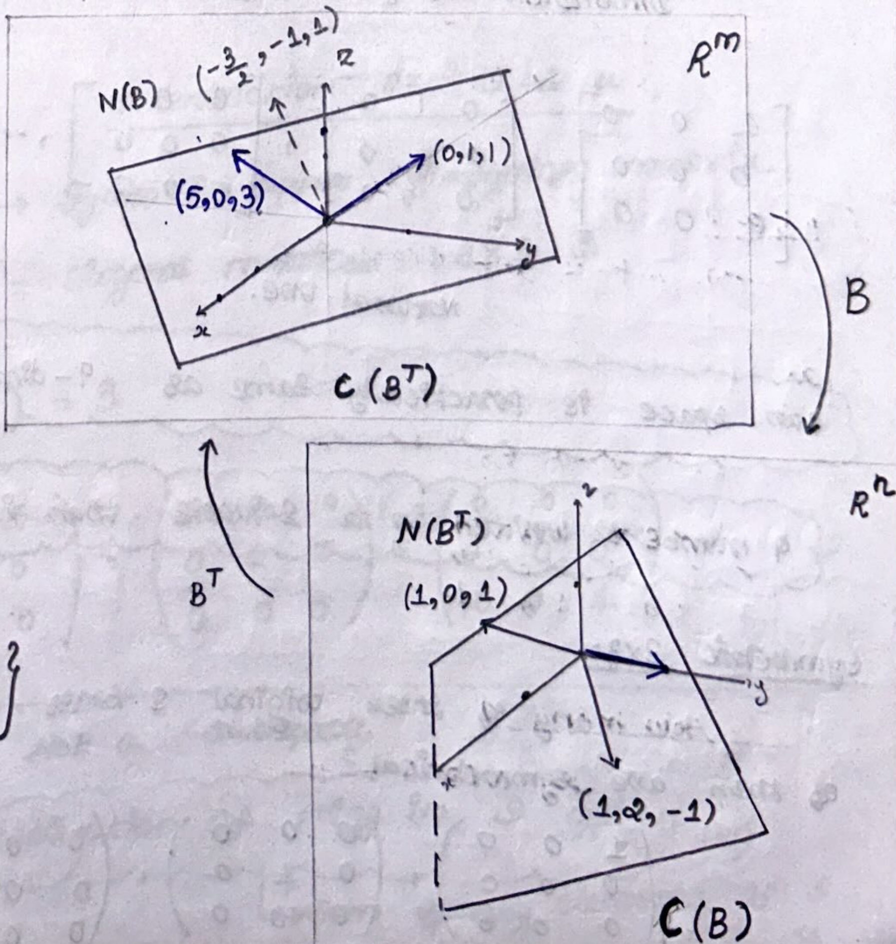
Column space: pivot columns of L matrix

Null space: U matrix

Row space: U matrix

Left null space: Invert L matrix.

Picture:



Row space:

$\left\{ \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Left null space

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Column space

$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

left Null space:

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$B^T \rightarrow$ transforms column subspace to Row.
kills $N(B^T)$

$B \rightarrow$ Transforms (maps) row subspace to column subspace. kills $N(B)$

Lecture - 12

Matrix spaces; Rank 1; Small world graphs.

Matrix space: Space of all 3x3 matrices:

* "Add them & multiply by constants"
* multiply two matrices: Not a part of vector spaces.

Subspaces:

* Symmetric
* Upper triangular matrix
* Diagonal.

Product of two symmetrical matrices is not symmetrical

Basis for M = all 3x3 's:

Dimension = 9 [3x3 = 9]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Natural one.

Our space is practically same as 9-dimensions.

9 numbers written in a square than in column.

Symmetric 3x3:

How many of these original 9 bases, how many of them are symmetrical?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & \end{pmatrix}$$

dimension of M $\dim(M) = 9$

Dimension of S (symmetric) $\dim(S) = 6$

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} \rightarrow 6 \text{ different entries}$$

Dimension of u (upper triangular) $\dim(u) = 6$

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \rightarrow 6 \text{ different entries.}$$

From original 9 bases of M:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are upper Δ^r matrices. (Accidental case) where the big bases have inner bases for the subspace.

Intersection of S & u

$S \cap u \rightarrow$ Symm & upper triangular matrix.

$=$ Diagonal matrices (3×3) s

$\dim(S \cap u) = 3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(S \cup u) \rightarrow$ Not a subspace.

$S + u \rightarrow$ combination of thing in S and u.

$\hookrightarrow$ sum of any elements of S and u.

(we have a linear combination) $\rightarrow$ subspace.

$S+U$ = All 3×3 matrices.

$\dim(S+U) = 9$

$\therefore \dim S = 6$

$\dim U = 6$

$\dim(S\cap U) = 3$

$\dim(S+U) = \dim S + \dim U - \dim(S\cap U)$

$\therefore \dim(S+U) = 9$ (we will get all 3×3 matrices)

Final subspace:

$$\frac{d^2y}{dx^2} + y = 0$$

$y = \cos x, \sin x, e^{ix} \rightarrow$ solutions.

combination

$y = a\cos x + b\sin x$

Basis: $\sin x, \cos x$

$\boxed{\dim(\text{Solution space}) = 2}$

$\sin x, \cos x \rightarrow$ don't look like vectors. (functions).

we can add them & constant multiply.

rank $\le m$

rank $\le n$

Rank 1 matrix

Example:

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \underset{2\times 1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}$$

$\dim C(A) = \text{rank} = \dim C(A^T)$ $\quad r = 1.$

$A = uv^T$ $\quad$ ($u, v \rightarrow$ column vectors) $\quad v^T \rightarrow (\ldots)$

Rank 1 matrices: Building blocks for all matrices.

M = All 5×17 matrices → Rank 4 matrices

'Subset of rank 4 matrices' → Is that a subspace.

If I add two rank 4 matrices what will be the rank of the resulting matrix?

Is it 4 → Not necessarily.

Theorem: Rank of (A+B) can't be more than Rank(A)+ Ran(B).

Clearly Rank can't be larger than 8:

How big? → could be large as 5.

Subset of rank 1 matrices? → Not a subset.

'Mostly rank will be 2'

example: In R^4, $\quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

The subspace of the vectors whose sum add to zero.

S = all v in R^4 with $v_1 + v_2 + v_3 + v_4$

↳ verify: $\left\{ \begin{array}{l} u + v = 0 + 0 = 0 \\ cu = 0 \end{array} \right\} \longrightarrow$ subspace dimension = ?

basis : ?

Connecting to null space

S = null space of A

$$AV = 0$$

$$v_1 + v_2 + v_3 + v_4 = 0$$

In matrix form:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \text{Rank} = 1$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

$r = 1$

Null Space

dimension of the null space : $n - r$

$$= 4 - 1$$

$$\dim(N(A)) = 3$$

$$A = [\,1 \quad \underbrace{1 \quad 1 \quad 1}_{\text{Free variables}}\,]$$

Basis for S:

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \text{Natural (Special) 3.}$$

$x_2 = x_3 = 0,\ x_4 = 1$

$x_2 = 1,\ x_3 = x_4 = 0$

$x_3 = 1,\ x_2 = x_4 = 0$

Column Space.

Column space is the subspace of R^1.

$$\boxed{\therefore m = 1}$$

$\therefore$ columns only have one component,