Probabilistic systems analysis & applied probability

1. Probability models and axioms

'Noise is random'

'manager - customer demand, stock market'

Probability:

'framework for reasoning about uncertainty'

Probabilistic model: → Probability axioms → Sample space

* list of (set) all possible outcomes (H, T)

* mutually exclusive: If H happens, T - don't

* collectively exhaustive: o/p are from set alone.

Legitimate

H
T + raining
T + not raining

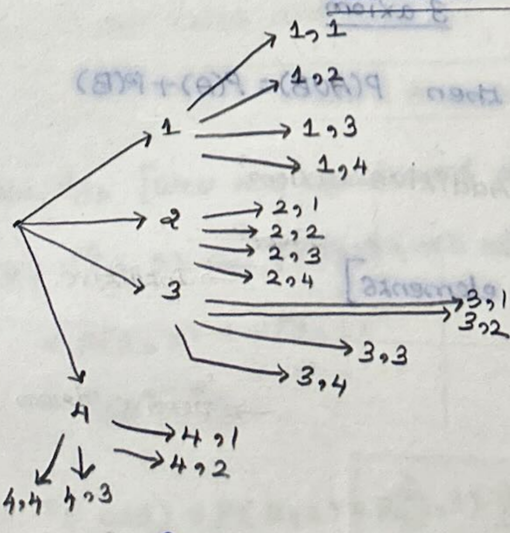
↓

If you have superstition belief rain has effect on coins go with it (Normally we don't)

(H, T) ↳ usually.

'choose details relevant, non-negligible' - Judgement (select correct sample space).

Tetrahedral die (2 rolls) (1 experiment)



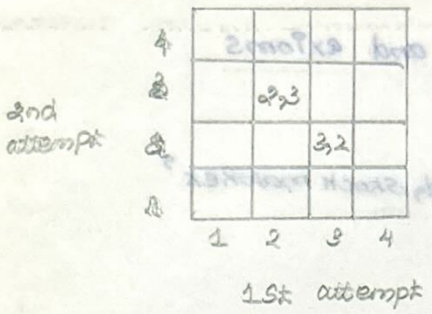
2,3 - different from 3,2

(Discrete)

16 elements

Finite space.

'Visually - sequential (tree) description'



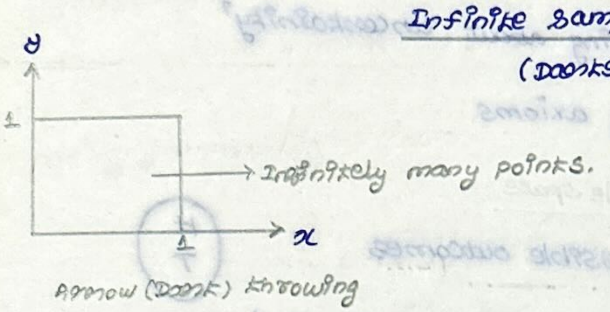
result - some result (Intermediate)

outcome - overall.

'finite'

Infinite sample space

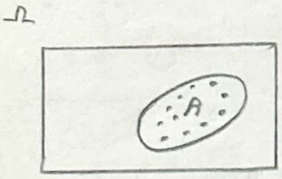
(Darts)



∞ outcomes $\rightarrow$ Sample Space

one point (probability) $= \frac{1}{\infty} = 0$

Subsets



Event A: A point (random) in A occurs.
(single outcome at a time)

Ground rules:

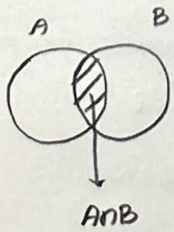
1-axiom ★ $0 \leq P(A) \leq 1$ [convention]

★ $P(A) \geq 1$ [non-negative] - Less than 1. (or equal)

2-axiom ★ $P(\Omega) = 1$ [entire axiom - sample space]

∴ collectively exhaustive - outcome must be from sample space.

3 axiom



$A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$

↓ 'Additive axiom'

'Disjoint'

[No common elements] → Intuitive.

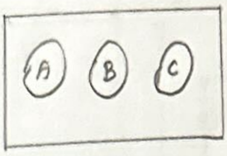
$P(\Omega) = 1$

$P(A \cup A^c) = 1$ → 'Derive from axiom 3'

'Disjoint' ← $P(A) + P(A^c) = 1$

$$\boxed{P(A) = 1 - P(A^c)}$$

$\therefore P(A) = 1 - P(A^c) \leq 1 \rightarrow$ Non-negative.



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$\therefore P((A \cup B) \cup C) = P(A \cup B) + P(C)$$
$$= P(A) + P(B) + P(C)$$

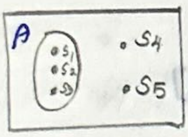
$\therefore$ $A \cup B$ and C are disjoint

A and B are disjoint.

$A_1, \ldots, A_n$ are disjoint

$$P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n)$$

<u>Special case</u> (finite set)



$$P(\{s_1, s_2, s_3, \ldots s_k\}) = P(\{s_1\}) + \ldots + P\{s_k\})$$

'Probability = union of single element sets'

$$P(A) = P(\{s_1\}) + P(\{s_2\}) + P(\{s_3\})$$
$$= P(s_1) + P(s_2) + P(s_3)$$

$\therefore$ single element set.

<u>Conclusion:</u>

'The total prob of a finite collection of possible outcomes, is equal to the prob of individual elements'

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we don't do assign probabilities for every subset.

↓

* Square (more cuts – more subsets)
* weird cuts – hard to imagine – doesn't satisfy

prob axiom – are also available.

<u>Die – example (4 sides, rolled twice)</u>

* Same prob: 1/16 [well manufactured die]

$$P(x, y) = P((1,1) \text{ or } (1,2))$$
$$= P(1,1) + P(1,1)$$
$$= 2/16 = 1/8$$

$$P(x = 1) = P(1,1) + P(1,2) + P(1,3) + P(1,4)$$
$$= \frac{4}{16} = \frac{1}{4}$$

$$P(x + y \text{ is odd}) = P(1,2) + P(2,1) + P(2,3) + P(3,2) + P(4,1) + P(1,4) + P(4,3) + P(3,4)$$
$$= \frac{8}{16} = \frac{1}{2}$$

$P(\min(x,y)=2) = P(2,2)+P(2,3)+P(2,4)+P(3,2)+P(4,2)$

$= 5/16$

Discrete uniform law

(Equally likely outcomes)

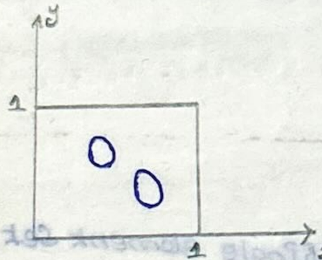
Then,

$$P(A) = \frac{\text{no. of elements of } A}{\text{Total number of sample points}} = \frac{m}{n}$$

computing probability $\equiv$ count.

Continuous uniform law

Area of the subset : probability law



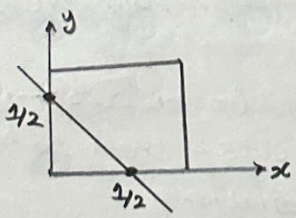
Two random numbers $[0,1]$

uniform numbers (law) : probability = Area

Assume : Equal area : Equal probability.

$P(x,y) = P(0.5, 0.3) = \frac{1}{\infty} = 0$

$P(x+y \geq 1/2)$



$x=0$, $y = 1/2$

$y=0$, $x = 1/2$

$P(x+y \geq 1/2) = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

'Countably infinite sample space'

Toss - until head occurs

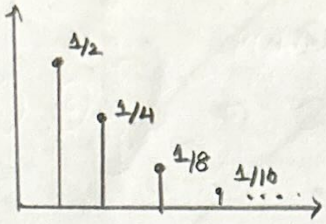
$\Omega = \{1, 2, \ldots\}$

$P(1) = 1/2$

$P(2) = 1/2 \cdot 1/2$ $\boxed{P(n) = \frac{1}{2^n}}$

↓ Tail ↓ Head

"No bound – integer" – $\{1, 2, \ldots\}$



$$P(\{2, 4, 6, 8, \ldots\}) = \text{even attempt} = P(2) + P(4) + \cdots + \ldots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \cdots$$

$$= \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right)$$

$$= \frac{1}{4} \sum_{i=0} \left(\frac{1}{4}\right)^n = \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right)$$

$$= 1/3$$

Countable additivity axiom

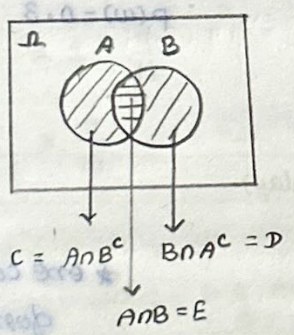
"Countable additivity axiom": If

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots \quad [\text{Disjoint}]$$

"Sequence – ordered collection"

Recitation

1)



$$P(C \cup D) = P(A) + P(B) - 2P(E)$$

$P(A) \geq 0$

$P(\Omega) = 1$

If $A \cap B = \phi$

$P(A \cup B) = P(A) + P(B)$

$P(C \cup D) = P(A) + P(B) - 0$ $\quad [P(A \cap B) = 0]$

$\boxed{P(C \cup D) = P(A) + P(B)}$

$P(C \cup D) = P(C) + P(D)$

Also

$$P(C \cup D) = P(A) - P(E) + P(B) - P(E)$$

$$\therefore P(A) = P(C) + P(E)$$

$$P(B) = P(D) + P(E)$$

$\therefore$ If A and B are disjoint

$$P(A+B) = P(A) + P(B)$$

$\boxed{P(C+D) = P(C) + P(D) \quad \therefore \text{C and D are disjoint}}$

Geniuses & Chocolates

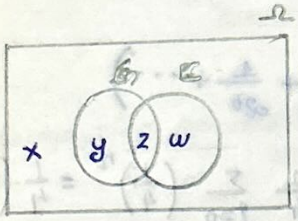
* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [Not disjoint]

$P(\text{Genius}) = 0.6$

$P(\text{Chocolate}) = 0.7$

$P(G \cap C) = 0.4$

'Neither a genius - nor a chocolator'



$P(A^c \cap B^c) = 1 - P(A \cup B)$

(outside the two circles)

$x + y + z + w = \Omega$

$P(x) + P(y) + P(z) + P(w) = 1$

$P(x) = 1 - (P(y) + P(z) + P(w))$

$$P(A^c \cap B^c) = 1 - (P(A) + P(B) - P(A \cap B))$$
$$= 1 - (0.6 + 0.7 - 0.4)$$
$$= 1 - (0.6 + 0.3)$$
$$= 0.1$$

$P(G) = P(y+z) = P(y) + P(z) = 0.6$

$P(C) = P(z+w) = P(z) + P(w) = 0.7$

$P(z) = 0.4$

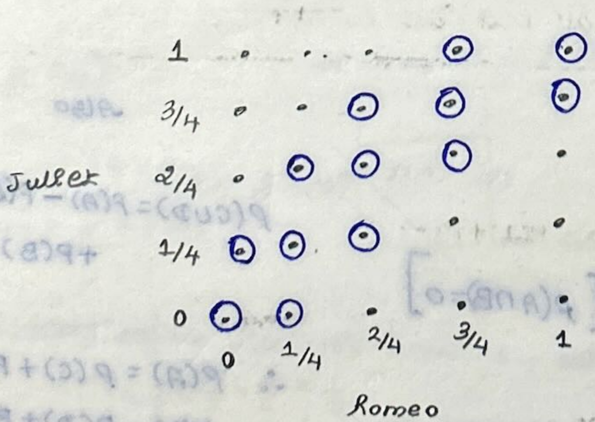
$P(y) = 0.2$

$P(w) = 0.3$

$P(x) = 1 - 0.9$

$\boxed{P(x) = 0.1}$

Arrive (0-1 hour delay)



* one can wait for 15 minutes

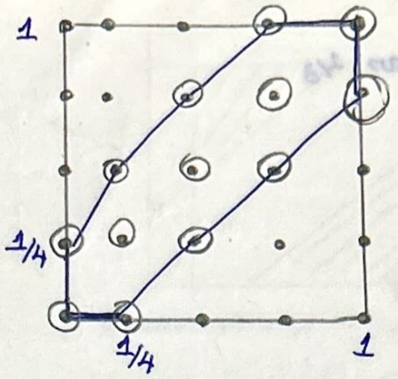
$P(\text{event}) = \frac{13}{25}$

→ 'All pairs of equally likely'

→ will wait for 15 minutes (when one came — other came before upto 15 minutes)

Romeo comes 1/4, Juliet 2/4

No wait (Both at time)



'Square - our new samplespace' — Any point.

Romeocomes at 0 — Juliet at 1/4 (If Juliet is even 1 sec late they can't meet)

'A Shape'

Area = 1 − Area of 2 Δ^s

$= 1 - 2\left(\frac{1}{2} \times b \times h\right)$ → 'continuous'

$= 1 - 2\left(\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4}\right)$

$= 1 - \frac{9}{16}$

P(meet) $= \frac{7}{16}$

In discrete – we neglect in b/w points (except $0, 1/4, 1/2, 3/4, 1$)

continuous – we take all possible points

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1) P(even face) = 2P

P(odd face) = P

P(1 or 2 or 3) = P(1) + P(2) + P(3)

= P + 2P + P

= 4P = 4 (1/9) = 4/9

P(1)+P(2)+P(3)+P(4)+P(5)+P(6) = 1

3P + 2(3P) = 1

P = 1/9

Alice & Bob each choose at random a number in the interval [0, 2]. we assume a uniform probability law under which the prob of an event is proportional to its area. Consider

i) A: magnitude of the difference of the two numbers is greater than 1/3.

B : Atleast one of the numbers is greater than 1/3

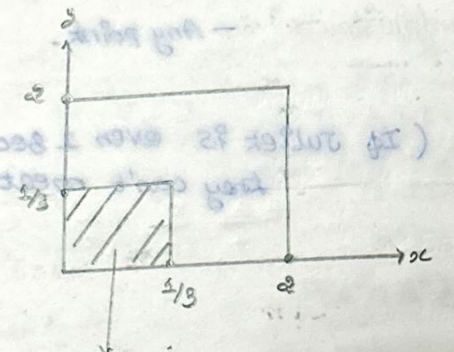
C : The two numbers are equal

D : Alice's number is greater than 1/3

$P(B)$, $P(C)$, $P(A \cap D)$

Solu :-

$P(B) = 1 - (\text{Both are lesser than } \frac{1}{3})$



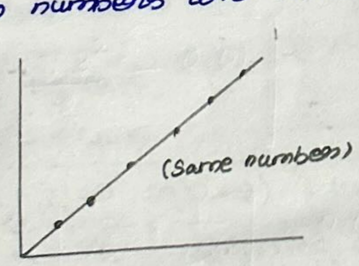
Both are below 1/3 b/w this interval.

$= 1 - \dfrac{(\frac{1}{3})(\frac{1}{3}) \rightarrow \text{our position}}{2 \times 2 \rightarrow \text{Total Square}}$

$= 1 - \dfrac{1/9}{4}$

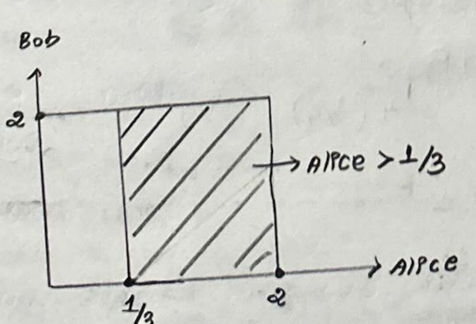
$= 35/36$

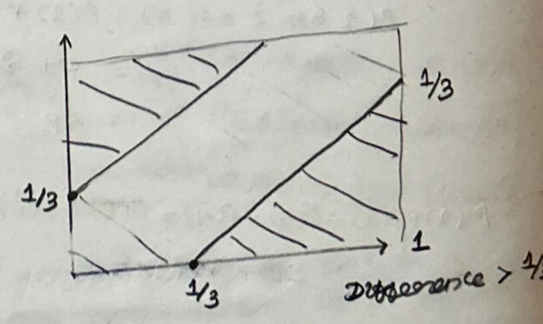
ii) $P(C)$ = Two numbers are equal



Diagonalline → Area = 0 $\quad P(C) = \frac{0}{4} = 0$

$P(A \cap D)$ = Alice number greater than 1/3 and magnitude of difference of 1/3 b/w two.



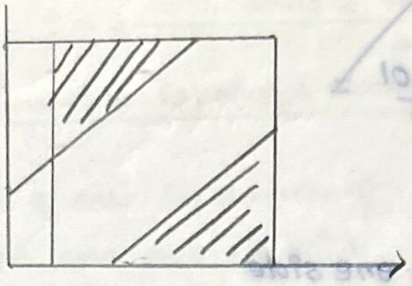


↓

Alice 1/3, magnitude diff 1/3 (Area)

Alice → 0 , BoB > 1/3

BoB → 0 , Alice > 1/3



$$P(A \cap D) = \frac{\left(\frac{1}{2} \times \frac{5}{3} \times \frac{5}{3}\right) + \left(\frac{1}{2} \times \frac{4}{3} \times \frac{4}{3}\right)}{4} = \frac{41}{72}$$

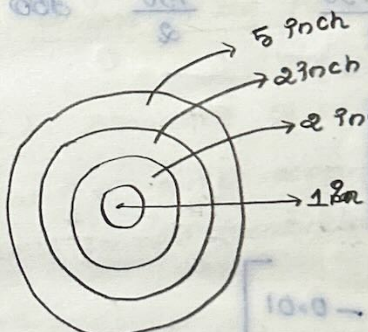
Mike & John — Dart - radius : 10in

1in of the centre — 50 pts
1 to 3 → 30
3 to 5 → 20
further five → 10

→ Able to attack dart.

(The prob of the dart falling in a gn reign is α to its area)

a) i) 50 points on 1 throw (mike)



Total board prob = 1

$$\alpha \pi R^2 = 1$$

$$\alpha \pi R^2 = 1$$

$$\alpha = \frac{1}{\pi \times 100}$$

$$\alpha = \frac{1}{100\pi}$$

i) Mike throws - 50 points

$$\alpha \pi r^2 = P(A)$$

$$P(A) = \frac{1}{100\pi} \pi (1)$$

$$= \frac{1}{100} = 0.01$$

B) 30 points on one throw.

$$\alpha \pi 3^2 - \alpha \pi 1^2 = P(B)$$

$$P(B) = \frac{1}{100}(9-1)$$

$$= 0.08$$

c) John - Right handed, twice more likely throw in the right half of the board than left.

Across each half, the dart falls uniformly in that reign. Answer A) and B) for John's.

A) For John $\Rightarrow$ $\cancel{\dfrac{\text{Right} + \text{left}}{2}}$ Finding α

$P(A) \Rightarrow \cancel{\dfrac{2(0.01) + 0.01}{2}} =$

~~For Right side~~:

$$\frac{\alpha R^2 \pi}{2} = \alpha 50\pi \rightarrow \text{one side}$$

~~Right~~ side:

Right side $\rightarrow$ Twice $\cancel{\left(\frac{2}{3}\right)}$ $\alpha 50\pi = \frac{2}{3}$

$\alpha = \frac{1}{75\pi}$

left side $\rightarrow$ $1/3$

~~$\alpha 50\pi = 1$~~

$$\alpha 50\pi = \frac{1}{3}$$

$$\alpha = \frac{1}{150\pi}$$

$\therefore \dfrac{\text{Right} + \text{left}}{2}$

$$\therefore P(A) = \frac{\alpha \pi R^2}{2} = \frac{\frac{1}{75} + \frac{1}{150}}{2} = \frac{\frac{3}{150}}{2} = \frac{3}{300} = \frac{2}{300} = 0.01$$

$$P(B) = \left[\frac{\alpha \pi 3^2}{2} - 0.01\right]$$

$$= \left[\frac{9}{2}\left[\frac{1}{75} + \frac{1}{150}\right] - 0.01\right]$$

$$= \left[\frac{9 \times 3}{300} - 0.01\right]$$

$$= \frac{9}{100} - 0.01$$

$$= 0.08$$

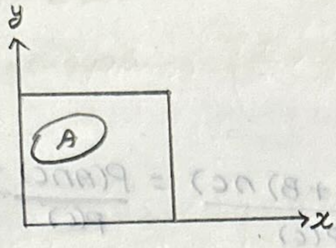
His habit doesn't affects his possibility of winning, Equal potential candidates.

Lecture-2

Conditioning & Bayes Rule

'Information is always partial'

If we take sequence of sets (or sequence), the probability of the outcome assuming the sets are disjoint.



$P(A) = area(A)$

$P(A) = \bigcup_{x,y} \{(x,y)\}$

$$1 = P[\square] : P\left(\bigcup_{x,y}\{x,y\}\right) = \sum_{x,y} P(\{(x,y)\}) \neq \bigcup_{i=1}^{\infty} A_i$$

(sequence only applicable)

one element set: area $= 0$

$\boxed{P(\text{one element set}) = 0}$

$\Sigma 0 \neq 0 \longrightarrow$ Not a sequence.

?

'where is the problem'

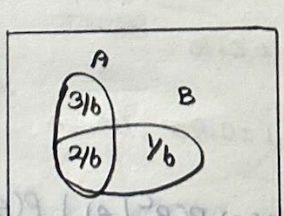
Additivity axiom: Applies to sequence of sets.

∞-sets are bigger $\rightarrow$ integers can be arranged in a sequence
$\rightarrow$ unit square (continuous) – uncountable.

$0 \rightarrow$ possibility is extremely low – don't mean doesn't.

'Based on info we built a model' $\rightarrow$ New info $\rightarrow$ update model

Conditional probability



'Event: B has occured'

↳ 'B is our new probability'

$P(A|B) = \frac{2}{3}$

$P(B|B) = 1$

Assuming $P(B) \neq 0$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\rightarrow$ what fraction of A occured in B $= \frac{2/6}{3/6} = 2/3$

$P(A|B)$ undefined if $P(B) = 0$

$$\boxed{P(A\cap B) = P(B)\,P(A|B)}$$

$$= P(A)\,P(B|A)$$

↳ 'what frequency $P(A)$ or $P(B)$ occurs × their conditional prob.

$$A\cap B = \phi$$

$$P(A\cup B) = P(A) + P(B)$$

'Event c occured'

$$P(A\cup B|C) = P(A|C) + P(B|C)$$

$$= \frac{P(A\cup B\cap C)}{P(C)} = \frac{P((A+B)\cap C)}{P(C)} = \frac{P(A\cap C)+P(B\cap C)}{P(C)}$$

$$= \frac{P(A\cap C)}{P(C)} + \frac{P(B\cap C)}{P(C)}$$

$$= P(A|C) + P(B|C)$$

Die model → uniform distribution

'$\min(x,y) = 2$ = $(2,2), (2,3), (2,4), (3,2), (4,2)$

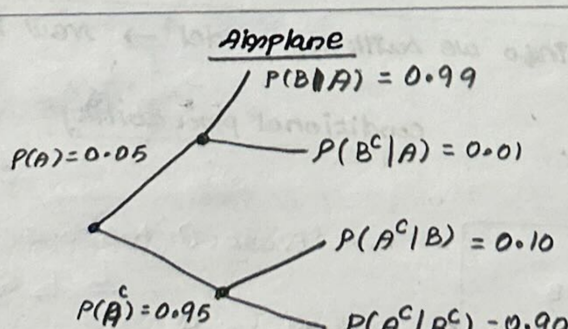
$M = \max(x,y) = 1$ = $(1,1)$

$$P(M=1 \mid \min(x,y)=2) = \frac{0}{5/16} = 0$$

$$P(M=2 \mid \min(x,y)=2) = \frac{1/16}{5/16} = 1/5$$

(still the model is uniform but in the new universe)

Airplane



$$P(A\cap B) = P(A)\cdot P(B|A) = (0.05)(0.99) = 0.0495$$

$$P(A\cap B^c) = P(A)\,P(B^c|A) = (0.05)(0.01) = 0.0005$$

$$P(A^c\cap B) = (0.95)(0.10) = 0.095$$

$$P(A^c\cap B^c) = (0.95)(0.90) = 0.855$$

$P(A \cap B) \rightarrow$ plane there, Radar detects.

$P(\text{good})$ = Plane & radar find + No plane radar don't

$= 0.0495 + 0.855$

$= 0.9045$

$P(\text{Bad})$ = Noplane & detects + plane & no detects

$= 0.095 + 0.0005$

$= 0.0955$

$P(B) = (0.05)(0.99) + (0.95)(0.10)$

$= 0.0495 + 0.095$

$= 0.1445$

$P(\overline{B}) = 0.0005 + 0.855$

$= 0.8555$

$P(A|B)$ = Some thing fly, radar shows

$= \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.0495}{0.1445} = 0.3425 \rightarrow$ only 30% $\rightarrow$ we can't rely on that

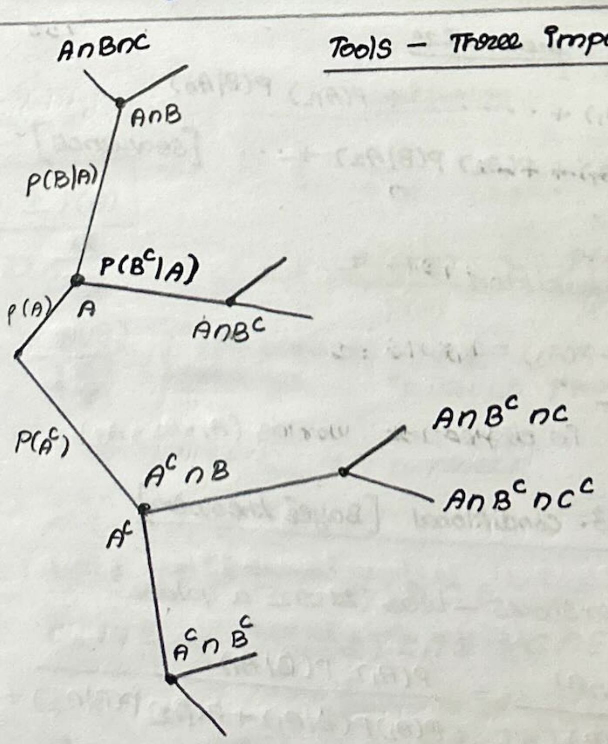
$P(A|B^c) = \dfrac{P(A \cap B^c)}{P(B^c)} = \dfrac{0.0005}{0.8555} = 0.000584$

$P(A^c|B) = \dfrac{0.095}{0.1445} = 0.6574 \rightarrow$ False alarm. [more likely]

"Mostly False alarm"

Tools - Three Important (Divide & Conquer)

1)



"Go through the branch sequences."

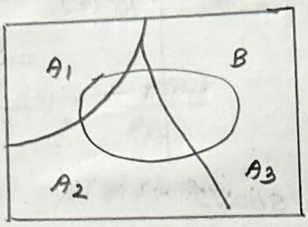
- how often A occurs $\rightarrow$ out of the times A occur how many times B occurs $\rightarrow$ " A & B " how.m. C occurs"?

$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B) \cdot P(C \mid A \cap B)$

$= P(A) \cdot P(B \mid A) \cdot P(C \mid A \cap B)$

2. Total probability

* B → Result of certain probabilities
* Divide & conquer.
* Partition of sample space A_1, A_2, A_3
* Have $P(B/A_i)$ for every i



$P(B) = P(A_1) P(B/A_1) + P(A_2) P(B/A_2) + P(A_3) P(B/A_3)$

A_1 — $A_1 \cap B$

A_2 — $A_2 \cap B$

A_3 — $A_3 \cap B$

$\boxed{P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)}$

$\therefore P(B) = P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2) + P(A_3) \cdot P(B/A_3)$

Generalize

$P(B) = P(A_1) P(B/A_1) + \dots + P(A_n) P(B/A_n)$

$P(B) = P(A_1) P(B/A_1) + P(A_2) P(B/A_2) + \dots$ [sequence]

$\therefore A_1, A_2, A_3$ → Form the partition

$\therefore P(A_1) + P(A_2) + P(A_3) = 1$

$\therefore$ weighted average of B in different worlds (A_1, A_2, A_3)

3. Conditional [Bayes theorem]

"Reverse - Radar shows - was there a plane"

$$P(A_1/B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1) P(B/A_1)}{P(A_1) P(B/A_1) + P(A_2) P(B/A_2) + P(A_3) P(B/A_3)}$$

∴ Gun Radar records → change our beliefs → update.

'Reverses the order of conditioning'

'model of cause & effect'

model

$A \longrightarrow B$ (Infer) $A_i \xleftrightarrow{\text{Inference}} B$

$P(B|A_i)$ $P(A_i|B)$ 'Learn from experience'

'Interpret new knowledge from previous knowledge'

Recitation

A: 1st toss head

B: 2nd toss head

$\boxed{A \cap B = ?}$

i) $P(A \cap B | A)$ ii) $P(A \cap B \mid A \cup B)$ ↳ 'Atleast one is head'

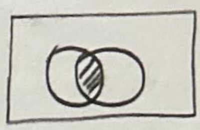
Solu:

∴ Independent

$P(A \cap B) = P(A) \cdot P(B)$

$$P(A\cap B|A) = \frac{P((A\cap B)\cap A)}{P(A)} = \frac{P((A\cap B)\cap A)}{P(A)}$$

$$= \frac{P(A\cap B)}{P(A)} = \frac{1/2 \cdot 1/2}{1/2} = 1/2$$



$$P(A\cap B|A\cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)} = \frac{1/2 \cdot 1/2}{P(A\cup B)}$$

$P(A\cap B)\cap A = P(A\cap B)$

$$= \frac{1/4}{\frac{1}{2}+\frac{1}{2}-\frac{1}{4}} = \frac{1/4}{3/4} = 1/3$$

Conclusion

$\boxed{P(A\cup B) \geq P(A)}$

$A\cup B \supset A$

$P(A) \subset P(A\cup B)$

$$\therefore \frac{P(A\cap B)}{P(A)} \geq \frac{P(A\cap B)}{P(A\cup B)}$$

$\boxed{\frac{1}{2} > \frac{1}{3}}$

'without probability of coin'

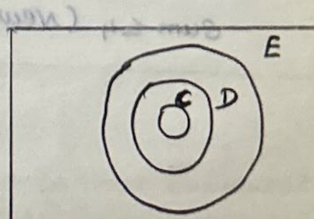
'Alice is correct'

Generalize:

C, D, E

1) $D \subset E$

2) $C\cap D = C\cap E$



$$P(C|D) \geq P(C|E)$$

∴ Knowing D occurs, Prob of C is more than knowing E occurs.

$C = A \cap B$

$D = A$

$E = A \cup B$

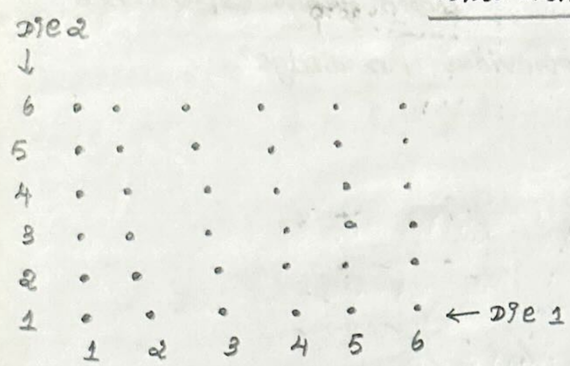
$C \subset D$

$\boxed{C \cap D = C \cap E}$

$$P(C|D) \geq P(C|E)$$

$$P(A \cap B \mid A) \geq P(A \cap B \mid A \cup B)$$

Conditioning example



Discrete uniform law:

$$P(A) = \frac{|A|}{|\Omega|}$$

Conditional example

$$P(D) = \frac{|D|}{36}$$

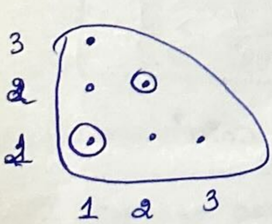
'Double - matches'

a) $P(D) = \frac{6}{36} = \frac{1}{36}$

b) $P(\text{[illegible]} \mid \text{Sum} \leq 4) = \frac{P(\text{[illegible]} \cap \text{Sum} \leq 4)}{P(\text{Sum} \leq 4)} = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}$

(or)

$$P(D \mid \text{Sum} \leq 4) = \frac{2}{6} = \frac{1}{3}$$



Sum ≤ 4 (New sample space)

c) $P(S = \text{one die is } 6) = \frac{11}{36}$

d) $P(S \mid K = \text{two diff numbers}) = \frac{10}{30} = \frac{1}{3}$

Discrete uniform law → Likely prob

(Simple over problem)

'Monty - Hall problem'

'what we are working is really important'

3 doors - Behind one of the doors (one of the doors - there is a prize)

* ★ choose a door (don't show you)
* ★ Friend: (knows prize) → look at remaining doors
* ★ open one of the remaining one (prize is not there).
* ★ Now: stay or swap

'which is the best strategy'

choose	prize	open	stay	switch
1	1	2/3	✓	
1	2	3	×	✓
1	3	2	×	✓
2	1	3	×	✓
2	2	1/3	✓	
2	3	1	×	✓
3	1	2	×	✓
3	2	1	×	✓
3	3	1/2	✓	

Switch

choose a door, stay in it.
(stick to initial)

'winning perspective'

sticking:

prize → 2nd door

Friend → can't open my door, 2nd door

(3rd door)

'wrong'

1/3 prob of winning.

prize → 3rd door

2nd door — open

'wrong'

'$\frac{2}{3}$ - Better to switch'

from this point.

Lets have a try!

Stay

1) when initial
Choice is correct

$1/3$

2) swap

Initial choice is not correct

$2/3$

'Friend can't open the door having prize'

Choosing 2 (2 at cost of 1):

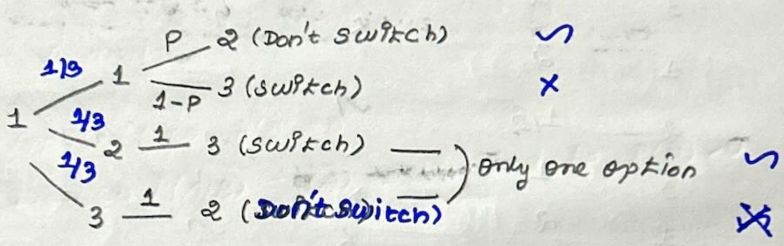
* picking group of doors.

1 or 2 and 3

Switching: 2 doors at the cost of 1. (Two opportunities)

Important assumption: Friend can't reveal the prize.

* choose 1 * If friend open 2 – Don't switch [Some idea]
* If friend open 3 – switch.



You chose 1, prize in 2 ⟶ open 3

You chose 1, prize in 3 ⟶ open 2

When you will win?

Switch = $\left(\frac{1}{3}\right)p + \frac{1}{3}(1) = \frac{1}{3}(p+1)$ = Stay.

Not switch = $\left(1/3\right)(1-p) + \left(1/3\right)(1)$.

$= \left(\frac{1}{3}\right)(1-p+1)$ ⟶ 'Depends how he is going to choose when the prize is in 1'

$= \left(\frac{1}{3}\right)(2-p)$

Assume: Choose 2 or 3 with $p = 1/2$

$\boxed{P(\text{win} \mid C_1) = 1/2}$

$\therefore \quad 0 \le p \le 1$

when $p = 0$

$\frac{2}{3}$ = Not switch, $\frac{1}{3}$ (stay) ⟶ can't be good as switching.

when $p = 1$

$\frac{1}{3}$ = not switch, $\frac{2}{3}$ (stay)

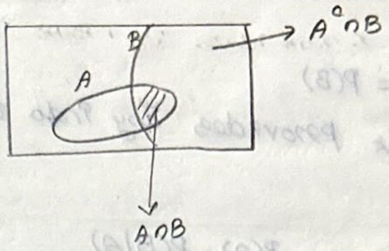
'with strategies – the probability changes'

Lecture-3 - Independence

1) $P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$

2) $P(B) = P(A_1)\, P(B|A_1) + \cdots$

3) $P(A_1|B) = \dfrac{P(A_1)\, P(B|A_1)}{P(A_1)\, P(B|A_1) + \cdots}$



$$P(B) = P(A)\,P(B|A) + P(A^c)\,P(B|A^c)$$

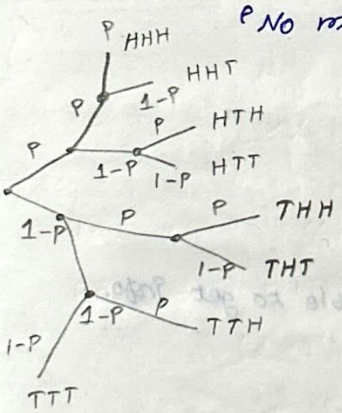
'Inference - measure - which branch occurred' → Bayes rule.

3 tosses of a biased coin:

'Sequence diagram - Notes'

$P(THT) = P(T_1) \cdot P(H_2|T_1) \cdot P(T_3|H_2 \cap T_1)$

'No matter of previous probabilities' - tosses.



$= (1-p)\,p\,(1-p)$

$= (1-p)^2 p$

$P(1\text{ head}) = TTH + HTT + THT$

$= 3(1-p)^2 p$

Bayes rule

$P(\text{First Head} \mid 1\text{ head}) = \dfrac{P(\text{First head})\; P(1\text{head} \mid 1\text{ head})}{P(H_1)\,P(1H|H_1) + \cdots +}$

$= \dfrac{P(\text{First head} \cap 1\text{ head})}{P(1\text{ head})}$ [Normal]

$= \dfrac{(1-p)^2 p}{3(1-p)^2 p} = \dfrac{1}{3}$

3 tosses

Head in 1 toss $= \dfrac{1}{3}$

(one of 3 is head)

↓

only one head.

From the image,

$T_1 \rightarrow$ Prop of second head $= P$] Not making difference.
$H_1 \rightarrow$ prob of second head $= P$

Independence

* 'Doesn't changing intuition'

$$P(B|A) = P(B)$$

'occurance of A doesn't provides any info on B's occurance'

$$\therefore P(A \cap B) = P(A) \cdot P(B|A)$$

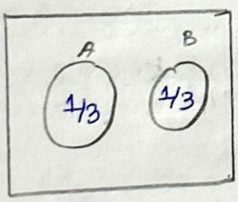
$$\boxed{P(A \cap B) = P(A) \cdot P(B)} \rightarrow \text{Not limited.}$$

Symmetric w.r.t A and B:

* Applies even $P(A) = 0$
* Implies $P(A|B) = P(A) \rightarrow$ Limited to $P(B) > 0$

'Distinct physical phenomenon'

'Physically related - But may be independent'



* If A occurs - B doesn't occur - Able to get info.

$$P(A \cap B) = P(A) \cdot P(B)$$

$$0 = 1/3 \cdot 1/3$$

$$\boxed{0 \neq 1/9} \rightarrow \text{No independence.}$$

$$\boxed{P(A|B) = 0 \neq P(A)}$$ $\therefore$ A conveys information.

'venn- never enough to say whether independent or not'

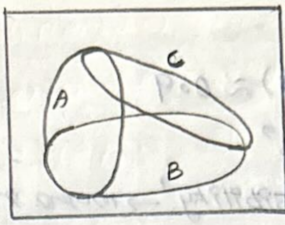
'Conditional prob are also like normal probabilities'

'Conditional independence'

$$P(A \cap B | C) = P(A|C) \cdot P(B|C)$$

$\downarrow$

'Conditional independence'



A and B are independent

★ If we are told that C occurred, are A & B independent

Solu:

$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)}$ = Is still A and B are independent?

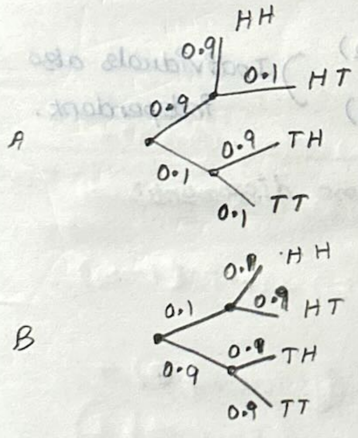
C - occurred [If we know A - occurred → B doesn't]

↳ Not independent

∴ In the new world - A & B are disjoint.

Having independence in original model doesn't implies that they are independent in the conditional world.

Two unfair coins: (0.9 - Heads; 0.1 - Tails)



2 coins - 2 flips each

↓ H - 0.9 ↘ H = 0.1

Are the coins independent?

★ 2nd & 1st toss are independent

★ one we know A is the coin! → So two tosses are independent

2) If we don't know which coin is, are tosses independent.

Compare $P(\text{toss } 11) = \text{Head})$

$P(\text{toss } 11 \mid \text{First } 10 \text{ tosses are heads})$.

Solu:

we can choose A ($\to H = 0.9$) or B ($\to H = 0.1$)

$P(\text{toss } 11 = \text{Head}) = P(A)\cdot P(\text{head}) + P(B)\cdot P(\text{head}) = \frac{1}{2}(0.9) + \frac{1}{2}(0.1)$

$= 1/2$

$P(\text{toss } 11 \mid 10 \text{ previous heads}) = P(\text{toss } 11 \mid B) \approx 0.9$

↳ 'Extremely low in B'

'A has more possibility' → more reasonable (likely)

'It is not independent anymore'

'unconditional & conditional independent are different'

'multiple independence'

The previous coin tosses doesn't change your belief on the present toss

$P(A_1 \cap (A_2^c \cup A_3) \mid A_5 \cap A_6^c) = P(A_1 \cap (A_2^c \cup A_3))$

'5 th flip - knowing' - Not going to impact 1, 2, 3, 4 flips

$P(A_1 \cap A_2 \dots \cap A_n) = P(A_1) P(A_2) \dots P(A_n)$

For any distinct indices

we also require

pairwise independent. $\left(\begin{array}{l} P(A_1 \cap A_2) = P(A_1) P(A_2) \\ P(A_2 \cap A_3) = P(A_2) P(A_3) \end{array} \right)$ Individuals also independent.

'Independence & pairwise independence are different'

Two independent fair coin tosses:

A: First toss is heads

B: second toss is heads

$P(A) = P(B) = 1/2$

$P(A \cap B) = \frac{1}{4} = P(A) P(B) = \frac{1}{4}$

C: First & second toss gn same result

$P(C) = \frac{2}{4} = \frac{1}{2}$

$P(C \cap A) = \frac{1}{4} = P(A) P(C) = 1/4$

$P(A \cap B \cap C) = 1/4 \neq P(A) P(B) P(C) = 1/8$

$P(C|A\cap B) = 1$ [certain]

* $A\cap B$ → we have one element universe (certain)
* So C $(A\cap B)$ → gives C is certain → Not independent.

$$P(C|A\cap B) \neq P(C)$$

(A, B and C) are pairwise independent.

King comes from a family of two children (Boy). What's the prob that his sibling is female?

Solu:

$$P(\text{Female}|\text{King}) = \frac{P(\text{Female} \cap \text{King})}{P(\text{King})}$$

∴ P(Female) & P(King) are independent.

$P(\text{King}) = 1/2$

$P(\text{Female}) = 1/2$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{1/2} = \frac{2/4}{3/4} = 2/3$$

↓

"only two families has both King & sibling"

Assumption

BB	BG
GB	GG

→ New universe

1 boy, 1 Girl

* Exactly ~~two~~ children

Caution:

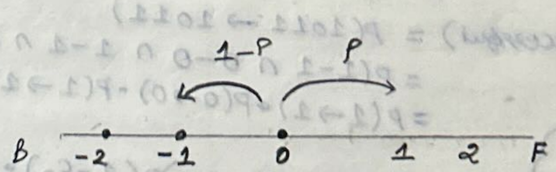
* King has precedence
* Atleast one King
* Continue until a King →

B	B
GB	

P(sibling) = 1 → depends on assumption

P(sibling) changes.

Recitation - A random walker



i) FBF
↓ ↳ Backward.
Forward

1) $\{FB, BF\} = A$

$$P(A) = P(\{FB, BF\}) = P(\{FB\}) + P(\{BF\})$$

∴ Forward & Backward are independent

$$= P\cdot(1-P) + (1-P)P$$

$$= 2P(1-P)$$

At position 1 after 3 steps:

$P(B) = \{FFB, FBF, BFF\}$

$$\boxed{P(B) = 3P^2(1-P)}$$

Probability: First step is forward

$D = \{FFB, FBF\}$

$P(D) = 2P(1-P)P$

$= 2P^2(1-P)$

$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{2P^2(1-P)}{3P^2(1-P)} = \frac{2}{3}$$

---

2. Communication over a noisy channel

1. multiplication rule
2. Total probability
3. Independence
4. Bayes rule.

Communication channel (Binary)

$0 \to 0$

$1 \to 1$

$0 \to 1$, $1 \to 0$ [Some noise is there]

P: 0 sent: $\xrightarrow{1-\varepsilon_0}$ 0 received; $\xrightarrow{\varepsilon_0}$ 1 received

$1-P$: 1 sent: $\xrightarrow{\varepsilon_1}$ 0 received; $\xrightarrow{1-\varepsilon_1}$ 1 received.

a) P(successful transmission) =

P(0 sent) . P(0 received) +

P(1 sent) . P(1 received)

$= P(1-\varepsilon_0) + (1-P)(1-\varepsilon_1)$

$P(\text{success}) = P(0) \cdot P(\text{success} \mid 0) + P(1) \cdot P(\text{success} \mid 1)$

b) P(4 bits in a row is successful) $= P(1011 \to 1011)$

$= P(1-1 \cap 0-0 \cap 1-1 \cap 1-1)$

$= P(1\to 1) \cdot P(0 \to 0) \cdot P(1 \to 1) \cdot P(1-1)$

$= (1-P)(1-\varepsilon_1) \cdot P(1-\varepsilon_0) \cdot (1-P)(1-\varepsilon_1)$

$(1-P)(1-\varepsilon_1)$

$= (1-P)^3 (1-\varepsilon_1)^3 \cdot P(1-\varepsilon_0)$

∴ Each transmission has prob structure.

= 'chosen 1' → No prob in choosing

$= (1-\varepsilon_1)^3 (1-\varepsilon_0)$

c) each symbol transmitted thrice

Outcomes

$0 \rightarrow$ Atleast 2 0's

$1 \rightarrow$ Atleast 2 1's

000	000	(0)
	001	(0)
	010	(0)
	011	(1)
	100	(0)
	101	(1)
	110	(1)
	111	(1)

Decoded.

P(0 correctly decoded) = ?

P(getting a zero decoded by passing zero) $= P(000) + P(001) + P(100) + P(010)$

$$= (1-\varepsilon_0)^3 + 3P(1-\varepsilon_0)^2$$

111 — 000 ⋮ 111

d) $P(0 \mid 101) =$ 'Inference' $= \dfrac{P(0)\, P(101 \mid \text{'0'})}{P(101)}$

↓ Being sent

$$= \frac{P\{P(101 / \text{'0'})\}}{P(101)}$$

$$P(101) = P(0)\, P(101 \mid 0) + P(1)\, P(101 \mid 1)$$

$$= P \cdot P(101/0) + (1-P)\, P(101 \mid 1)$$

$$P(0 \mid 101) = \frac{P \cdot P(101 \mid 0)}{P(0)\, P(101 \mid 0) + P(1)\, P(101 \mid 1)}$$

$$= \frac{P \cdot (1-\varepsilon_0)\,\varepsilon_0^2}{P(1-\varepsilon_0)\,\varepsilon_0^2 + (1-P)(1-\varepsilon_1)^2\,\varepsilon_1}$$

$0 \rightarrow 101$

$(000) \rightarrow 101$

Two unsuccessful & 1 successful transmission

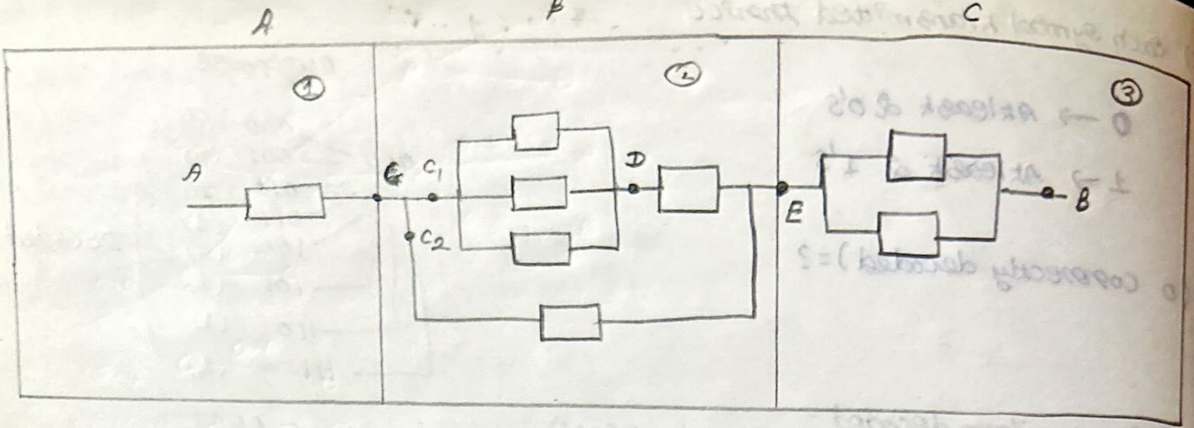
$1 \rightarrow 101$

2 success 1 failure

'why not 1/8 → ∴ No fair one'

'cost - overweighted by correctness'

Network reliability - Tutorial



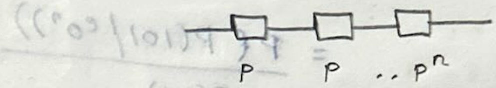
"Divide & Conquer"

* operational if a path is there (From A and B)

$P(A \to B)$: operational

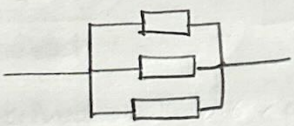
If working of a stage is P

$\therefore P(A \to B) = P \times P \times .. P^n = P^n$ [series]



Parallel:

Atleast one = 1 − No one works



$P(a \to b) = 1 - (1-P)^n$

3 collections:

Assume: Each component in both collection: independent.

$P(A \to B) = P(A \to C) \cdot P(C \to E) \cdot P(E \to B)$

$= P \cdot P(C \to E) \cdot (1-(1-P)^2)$

Atleast one of working

$P(C \to E) = \left(P(C_1 \to E) + P(C_2 \to E)\right)$

$= 1 - \left(P(C_1 \to E) \text{ Fail} \cdot P(C_2 \to E) \text{ Fail}\right)$

$= 1 - \left(1 - P(C_1 \to E)\right)\left(1 - P(C_2 \to E)\right)$

↓ ↓

conjugate of success

$P(C_1 \to E) = P \cdot (1 - \text{All parallel fail})$

$= P \cdot (1-(1-P)^3)$

$P(C_2 - E) = P$

$$P(C \to E) = 1 - (1 - P(1-(1-P)^3))(1-P)$$
$$= 1 - (1 - P + P(1-P)^3)(1-P)$$

$$P(A \to B) = P \cdot P(C \to E)(1-(1-P)^2)$$
$$= P[1-(1-P)^2]\left[1-(1-P)\left[1-(1-P)^3\right]P\right]$$

---

① one element series → must work

② parallel → c_1 → parallel + series → Atleast one of 3 blocks work.
→ c_2 → series

$\left(\begin{matrix}\text{Atleast one} \\ c_1 \text{ or } c_2 \text{ work}\end{matrix}\right)$ = 1 − All fail (1 − All fail)

A chess problem

"Belmont chess Champion" – Procedure:

* Bo & ci (Leading) → play a two game match
* Both game winning person → plays 2 games with AI (current champion)
* AI – champ → until → second round & challenger beats in both games
* AI – wins initial game → No need for other game.

i) $P(B_o \text{ beat } C_i) = 0.6$ [Any particular game]

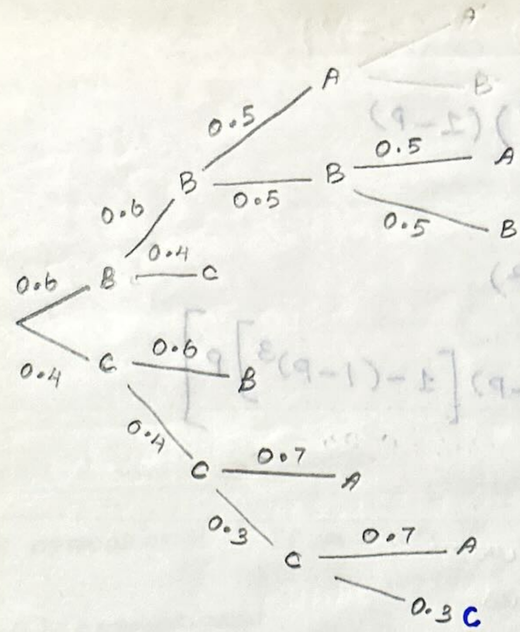
ii) $P(AI \text{ beat } B_o) = 0.5$

iii) $P(AI \text{ beat } C_i) = 0.7$

Assume: no ties & all games are independant.

a) Det

i) Second round will be required:



Second round : Play with AI (when any player win both games)

$$P(R_2) = (0.6)(0.6) + (0.4)(0.4)$$

$$= 0.6^2 + 0.4^2$$ [Bow won two games or CP won two games)

'Second round' : with AI

$$P(R_2) = 0.36 + 0.16$$

$$\boxed{P(R_2) = 0.52}$$

ii) P(Bow wins 1 round) = $0.6 \times 0.6 = 0.36$

iii) P(AI retains championship) = P(Any one win by one) + P(Lost at second round with AI)

only one win

AI wins

$$= (0.6)(0.4) + (0.4)(0.6) + [(0.6)(0.6)(0.5) + (0.6)(0.6)(0.5)(0.5)]$$

AI wins

$$+ [(0.4)(0.4)(0.7) + (0.4)(0.4)(0.3)(0.7)]$$

AI wins at 1 & 2 game resp

$$= (0.24 + 0.24) + [0.12 + 0.09] + (0.112 + 0.0336)$$

$$= 0.8956$$

or

$P(A) = 1 - P(A)^c$ [Lose deducted from 1]

$$= 1 - [(0.6)(0.6)(0.5)(0.5) + (0.4)(0.4)(0.3)(0.3)]$$

$$= 0.8956.$$

b) Given, second round is required →

i) Bo is the surviving challenger

ii) AI retains his championship

Solu:

$$P(B_0 \text{ is the surviving challenger} \mid \text{second round}) = \frac{P(B \cap R_2)}{P(R_2)}$$

$P(B \cap R_2) = P(B \text{ wins both}) = 0.6 \times 0.6 = 0.36$

Two ways → Bo wins both
→ CP wins both

$$= \frac{0.36}{0.52}$$

$$= 0.6923$$

ii) $P(AI \text{ retains} \mid \text{second round}) = \dfrac{P(AI \cap \text{second round})}{P(\text{second round})}$

AI wins in 6 ways
↓
After round 2
4 ways → win at game 1
→ win at game 2.

$$= \frac{1 - \text{lose}(AI) \text{ in 2 round}}{P(\text{second round})}$$

$$= 1 - \frac{\left[(0.6)^2(0.5)^2 + (0.4)^2(0.3)^2\right]}{0.52}$$

$= \cancel{0.8956}$

'Since AI is able to win in first round too'
↓
'Not possible'

$$P(AI \mid \text{second round}) = \frac{(0.6)^2(0.5)^2 + (0.6)^2(0.5) + (0.4)^2(0.7) + (0.4)^2(0.3)(0.7)}{0.52} = \frac{0.4481}{0.52} = 0.7992$$

c) Second round was required & that it has only one game. What's the cond. prob that it was Bo who won the first round.

I = {Second round with only 1 game}

B = {Bo won 1st round)

$P(B|I)$ = reverse order = Bayes theorem

$$= \frac{P(B) \cdot P(I|B)}{P(I)}$$

∴ Second round played : 'gn 1 round only.

∴ $I = (R_2 \cap 1 \text{ game})$

$$P(B|R_2 \cap 1 \text{ game}) = P(B|I) = \frac{P(B \cap I)}{P(I)} = \frac{P(B \cap R_2 \cap 1 \text{ game})}{P(R_2 \cap 1 \text{ game})}$$

∴ Gn: Second round with only one game $[R_2 \cap 1 \text{ game}]$

$P(B \cap I)$ = Bow wins 1 round gn 2nd round with only 1 game.

$$= \frac{(0.6)(0.6)(0.5)}{(0.6)(0.6)(0.5) + (0.4)(0.4)(0.7)}$$

$$= \frac{(0.6)^2(0.5)}{\underbrace{\underset{Bo}{(0.6)^2(0.5)} + \underset{CP \text{ wins both games}}{(0.4)^2(0.7)}}_{\text{Second round with 1 round}}}$$

$$= \frac{0.18}{0.2920} = 0.6164$$

"Building tree : the key"