Solving a homogeneous linear ODE with constant co.eff

Equipped with the linear algebra notions, we now generalize the solution method we used for II order homogeneous constant co.eff ODEs such as $\ddot{y} - \dot{y} + 6y = 0$, to solve homogeneous constant co.eff ODEs of any order.

Given,

$$a_n y^{(n)} + \dots + a_1 \dot{y} + a_0 y = 0$$

where $a_n, \dots, a_0$ are real constants.

1) write down char eqn

$$a_n r^n + \dots + a_1 r + a_0 = 0.$$

in which the co.eff of r^i is the co.eff of $y^{(i)}$ from the ODE. The LHS is called the characteristic polynomial $P(r)$. eg: $y^{(4)} - 2y^{(2)} + y = 0$ has characteristic polynomial

$$r^4 - 2r^2 + 1$$

2) Factor $P(r)$ as

$$P(r) = a_n (r - r_1)(r - r_2) \dots (r - r_n)$$

where the n roots $r_1, \dots, r_n$ are (possibly complex) numbers. These are guaranteed to be n (possible complex) roots with counted with multiplicity by the fundamental theorem of algebra. The roots may be all distinct or some roots may be repeated.

3) If $r_1, \dots, r_n$ are distinct, then the functions $e^{r_1 t}, \dots, e^{r_n t}$ form a basis for the vector

space of solutions to the constant co-eff ODE. In other words, the general solution is

$$c_1 e^{r_1 t} + \dots + c_n e^{r_n t}$$

Note: Complex roots always appear in pairs of conjugates, and if some of the roots are complex, the co-eff $c_1, \dots, c_n$ will have to be complex as well.

4) If $r_1, \dots, r_n$ are not distinct, then $e^{r_1 t}, \dots e^{r_n t}$ can't be a basis since some of these functions are redundant (definitely not linearly independent)! If a particular root r is repeated m times then,

$$\text{replace } \overbrace{e^{rt}, e^{rt}, e^{rt}, \dots, e^{rt}}^{m \text{ copies}}$$

by

$$e^{rt}, te^{rt}, t^2 e^{rt}, \dots, t^{m-1} e^{rt}.$$

Important: In all cases,

Number of roots of $p(r)$ counted with multiplicity = order of ODE = Number of functions in basis.

(By dimension theorem).

Example: 8.1:

Find the general solution to

$$y^{(6)} + 6y^{(5)} + 9y^{(4)} = 0$$

Solu:

char polynomial is $r^6 + 6r^5 + 9r^4 = 0$

$$r^4 (r^2 + 6r + 9)$$

$$r^4 (r+3)^2$$

whose roots listed with multiplicity are

$$0, 0, 0, 0, -3, -3.$$

since the roots are not distinct, the basis is not

$$e^{0t}, e^{0t}, e^{0t}, e^{0t}, e^{-3t}, e^{-3t}$$

we need to replace it with the correct basis as

$$\therefore e^{0t}, te^{0t}, t^2 e^{0t}, t^3 e^{0t}, \quad e^{-3t}, te^{-3t}$$

which simplifies to

$$1, t, t^2, t^3, e^{-3t}, te^{-3t}$$

Thus the general solution is

$$c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{-3t} + c_6 t e^{-3t}.$$

As expected the set of all solutions to this 6th order linear ODE is 6 dimensional.

Example: 8.2:

Find the simplest constant-co.effs homogeneous linear ODE having $(5t+7)e^{-t} - 9e^{2t}$ as one of its solution.

Soln:

The gn function is a linear combination of

$$e^{-t}, e^{2t}, te^{-t}$$

so the roots of the char polynomial (with multiplicity) should include $-1, 2, -1$. So the simplest char polynomial is

$$(r+1)(r+1)(r-2) = r^3 - 3r - 2$$

corresponding ODE is

$$y^{(3)} - 3\dot{y} - 2y = 0.$$

Complex roots:

$$\ddot{y}+y=0 \quad (y=c_1e^{it}+c_2e^{-it}) \text{ where } c_1, c_2 \in \mathbb{C}$$

$\therefore$ r^2+1 factors as $(r-i)(r+i)$.

$e^{it}=\cos t+i\sin t$ (or) $\cos t=\dfrac{e^{it}+e^{-it}}{2}$, $\sin t=\dfrac{e^{it}-e^{-it}}{2i}$,

General solution:

$$d_1\cos t+d_2\sin t \quad (d_1, d_2 \in C)$$

In our terminology,

$$\text{Span}(e^{it}, e^{-it})=\text{Span}(\cos t, \sin t)$$

Assuming complex numbers are allowed as co.efficients in the linear combinations.

moreover, the two pairs of functions on the left & on the right above are both bases of the set of all solutions since both pairs of functions are linearly independent.

Higher order ODES with Complex roots

Find a basis of solutions to

$$y^{(3)}+3\ddot{y}+9\dot{y}-13y=0 \quad \text{(Consisting of real valued functions)}$$

Solu:-

$$p(r)=r^3+3r^2+9r-13$$
$$=(r-1)(r^2+4r+13)$$

13

4

$$r^2+4r+13=0$$

$$r=\frac{-4\pm\sqrt{16-4(13)(1)}}{2}$$

13
4
52
16

$$=-2\pm\frac{\sqrt{-36}}{2}$$

$$=-2\pm 3i \Rightarrow -2-3i \text{ (or) } -2+3i.$$

Thus we have 3 solutions

$$e^{(t)}, \quad e^{(-2-3i)t}, \quad e^{(-2+3i)t}$$

$$e^{(-2+3i)t} = e^{-2t}\cos 3t + ie^{-2t}\sin 3t$$

$\therefore e^{t}, \; e^{-2t}\cos(3t), \; e^{-2t}\sin(3t)$

→ Real part → Imaginary part

is another basis, this time consisting of real-valued functions.

10.2

Suppose the roots – with multiplicity – of the char polynomial of a certain homogeneous constant coeff linear equ are

$$3, 4, 4, 4, 4, 5 \pm 2i, 5 \pm 2i$$

Solu: A basis for the real valued solutions is gn by

$$\{ e^{3t}, e^{4t}, te^{4t}, t^2e^{4t}, t^3e^{4t}, e^{5t}\cos 2t, e^{5t}\sin(2t), te^{5t}\cos 2t, te^{5t}\sin 2t \}$$

$$x(t) = c_1e^{3t} + c_2e^{4t} + c_3te^{4t} + c_4t^2e^{4t} + c_5t^3e^{4t} + c_6e^{5t}\cos 2t + c_7e^{5t}\sin 2t + c_8te^{5t}\cos 2t + c_9te^{5t}\sin 2t$$

(8 roots) → So the order will be 8.

we can obtain real solutions to linear ODEs with non constant real co-efficients as well:

Complex basis (vs) real basis:

Let $y(t)$ be a complex-valued function of a real-valued variable t. If y and $\bar{y}$ are part of a basis solutions to a homogeneous linear system of ODEs with real co-efficients, then replacing y and $\bar{y}$ by $Re(y)$ and $Im(y)$

gives a new basis of the homogeneous solutions.

Why is this true?

To know that the new list, which includes Re(y) and Im(y), is a basis, we need to check two things:

1. The span of the new list is the set of all solution:

This is true, because any solution is a linear combination of the old basis, and can be converted to a linear combination of the new list by substituting

$$y = Re(y) + i Im(y) \quad , \quad \bar{y} = Re(y) - i Im(y)$$

The new list linearly independent.

is not, say

$$c_1 Re(y) + c_2 Im(y) + \cdots = 0$$

then

$$Re(y) = \frac{y + \bar{y}}{2} , \quad Im(y) = \frac{y - \bar{y}}{2}$$

would show that the old basis was linearly dependent, which is impossible for a basis.

Let $y = \cos\theta + i\sin\theta$

$y = Re(y) + i Im(y)$, $\bar{y} = Re(y) - i Im y$.

$y =$

$-i\bar{y} = iRe(\cos\theta + i\sin\theta) + Im(\cos\theta + i\sin\theta)$

$= -(i\cos\theta - i\sin\theta)$

$= \cos\theta$.

y & $\bar{y}$ has same basis $(\cos t, \sin t)$.

→ so y and $\bar{y}$ can't be the basis.

Existence & uniqueness theorem

We now know how to find the general solution to a DE of the form

$$a_n y^{(n)} + \cdots + a_1 \dot{y} + a_0 y = 0.$$

but what do we need in order to pin down one specific solution? The answer is provided by the existence & uniqueness theorem below, which we have only started for I order linear ODE.

Existence & uniqueness theorem for a linear ODE:-

Let $p_{n-1}(t), \ldots, p_0(t), q(t)$ be continuous functions on an open interval I. Let $a \in I$, let $b_0, \ldots, b_{n-1}$ be given numbers. Then there exists a unique solution on I to the n^{th} order linear ODE.

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + \ldots + p_1(t)\, \dot{y} + p_0(t)\, y = q(t)$$

satisfying the n initial conditions

$$y(a) = b_0, \quad \dot{y}(a) = b_1, \ldots, y^{(n-1)}(a) = b_{n-1}$$

As before, existence means that there is atleast one solution & uniqueness means that there is only one solution.

The theorem says that, given an n^{th} order linear ODE, a single solution can be found by giving n numbers, namely the values of $y(a), \dot{y}(a), \ldots, y^{(n-1)}(a)$, the initial conditions.

Remark 11.1

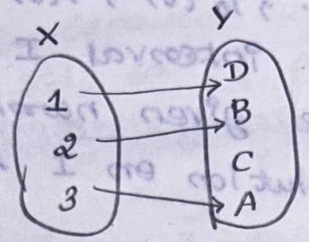
The existence & uniqueness theorem theorem in the homogeneous case explains the dimension theorem we stated previously. Each solution $y(t)$ determines a set of initial conditions, given by

$$y(t) \rightarrow (y(a), \dot{y}(a), \ldots, y^{(n-1)}(a))$$

On the other hand, the existence & uniqueness theorem says that each set of initial conditions specifies exactly one solution. The sequence of numbers $(y(a), \dot{y}(a), \ldots, y^{(n-1)}(a))$ constitutes a vector in R^n, and the 1-to-1 correspondence b/w solutions & sets of initial conditions given by existence & uniqueness theorem proves the dimension theorem.

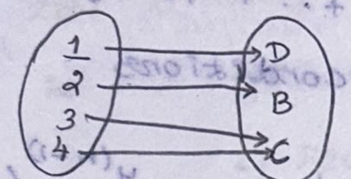
one to one function (Injective)

It is a function that maps



(Each input has unique output)

onto function:



All x has y values
one or more functions may have a common output

$y^{(4)} + 4\ddot{y} + 3y = 0$ (For which initial conditions below does the existence & uniqueness theorem guarantee a specific solution to.

soln:

$$y(0) = 0,\ \dot{y}(0) = 0,\ \ddot{y}(0) = 0,\ \dddot{y}(0) = 0$$

$$y(0) = 1,\ \dot{y}(0) = 0,\ \ddot{y}(0) = -1,\ y^{(3)}(0) = 0$$

$$y(2) = 1,\ \dot{y}(2) = 0,\ \ddot{y}(2) = -1,\ y^{(3)}(2) = 0$$

The DE is 4th order, so according to the existence & uniqueness theorem, the initial conditions $y(a)$, $\dot{y}(a)$, $\ddot{y}(a)$, $y^{(3)}(a)$, for any a will give a unique solution. Therefore, the first & last two choices are correct. (For the first choice, the unique solution is $y(t) = 0$). The others are wrong.

Problem - cylindrical buoy

(மிதவை)

A cylindrical buoy is to be placed in a pond. We begin with the bottom of the buoy touching the surface of the water, then release it. At this point, it is acted on by two forces: the downward force of gravity mg and the upward buoyancy force equal to the weight of water displaced. (Archimedes principle)

To model this buoy, we need to know its geometry. It has radius r (cm), height h (cm), and density $\rho_b = 0.5$ grams/cm^3.). Suppose further that the buoy descends symmetrically in to the water (it doesn't rock from side to side).

solu:

Find a diff eqn modelling the displacement of the bottom of the buoy $x(t)$ below the surface of the pond, as a function of time t. Choose the direction so that displacements below the surface are positive. Here $t = 0$ is the moment we release the buoy from rest with the bottom of the buoy touching the surface of the water.

(Express $\ddot{x}$ in terms of $g, h, r, \dot{x}$ and x. Type dotx for $\dot{x}$.)

solu:

$F_G \rightarrow$ Force due to gravity, F_B - Force due to Buoyancy

(Acting in opposite directions)

$F_G = mg$

$= ((\pi r^2 h) \times \rho_b) g$

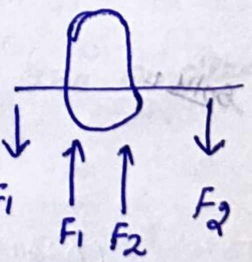
$= 0.5 \pi r^2 h g$

$m =$ volume $\times$ density

$F_B = g\, m_w$ (mass displaced in water)

$F_B = (\pi r^2 x \rho_w) g = \pi r^2 x g$

$\boxed{\text{Density of water} = 1 g/cm^3}$



By Newton's second law:

$F = ma = m\ddot{x}$

$0.5 \pi r^2 h \ddot{x} = 0.5 \pi r^2 h g - \pi r^2 x g$

(so gravity is also in F_B.)

$[F_{total} = F_G + F_B] \rightarrow$ Total force acting on the body

$$0.5\pi r^2 h \ddot{x} = 0.5\pi r^2 h g - \pi r^2 2x g$$

$$\ddot{x} = g - \frac{1}{0.5}\frac{g}{h}x$$

Period & amplitude

$$\ddot{x} + 2\frac{g}{h}x - g = 0$$

$$r^2 + 2r\frac{g}{h} - g = 0 \qquad = \frac{-\frac{2g}{h} \pm \sqrt{\frac{4g^2}{h^2} - 4(1)(-g)}}{2(1)}$$

$$= -\frac{g}{h} \pm \sqrt{\frac{4g^2}{4h^2} + \frac{4g}{4}}$$

$$= -\frac{g}{h} \pm \sqrt{\frac{g^2}{h^2} + g}$$

$$= -\frac{g}{h} \pm \sqrt{\frac{g^2 + gh^2}{h^2}}$$

The general solution to the differential eqn found above is

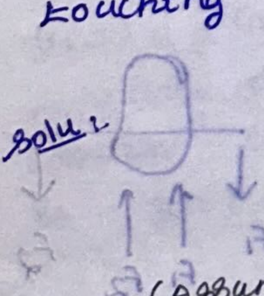
$$x(t) = h/2 + x_h(t)$$

$x_h(t) \to$ general solution to the associated homogeneous differential equation

compute T & A (amplitude) of the resulting oscillation is $h = 2$ metres. Recall at $t=0$ (we release the buoy from rest with the bottom of the buoy touching the surface of the water.

$$g = 9.8\ ms^{-2}$$

solu:



$$\ddot{x} + 2\frac{g}{h}x = g$$

(Assuming homogeneous eqn)

$$\ddot{x} + 2\frac{g}{h}x = 0 \qquad x = \frac{-0 \pm \sqrt{-4\left(2\frac{g}{h}\right)(1)}}{2}$$

$$= \pm\sqrt{-\frac{2g}{h}}$$

$$= \pm i\sqrt{2g/h}$$

$\therefore$ The general solution will be

$$x_h(t) = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t)$$

$\omega_n = \sqrt{\frac{2g}{h}}$ (Angular velocity)

$$T = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{2g}{h}}} = \frac{2\pi}{\sqrt{\frac{2\times 9.8\,m/s^2}{2m}}} \approx \frac{2\pi}{\sqrt{9.8}}\ s$$

$$\approx 2.007\ s$$

To find amplitude:

$$x(t) = x_p(t) + x_h(t)$$
$$= \frac{h}{2} + c_1 \cos \omega_n t + c_2 \sin \omega_n t$$

At $x(0) = 0$ (Buoy is released) $\rightarrow$ Rest. $\dot{x}(0) = 0$

$$0 = \frac{h}{2} + c_1 \cos \omega_n t + c_2 \sin \omega_n t \quad (\text{At } t = 0)$$

$$0 = -c_1 \omega_n \sin \omega_n t + c_2 \cdot \omega_n \cos \omega_n t$$

$$0 = c_2 \omega_n$$

$$\boxed{c_2 = 0}$$

$$0 = \frac{h}{2} + c_1 + 0$$

$$\boxed{c_1 = -\frac{h}{2}}$$

$$x(t) = \frac{h}{2} - \frac{h}{2}\cos(\omega_n t), \quad \omega_n = \sqrt{\frac{2g}{h}}.$$

and

since $x(t) = \frac{h}{2}\left(1 - \cos(\omega_n(t))\right)$

$0 \leq 1 - \cos(\omega_n t) \leq 2$, the Amplitude

$$\boxed{A = \frac{h}{2} = 1 \text{ meter}}$$

A second buoy

A second buoy with height h and a mass of 50kg floats in the water. When pressed downward slightly & released it oscillates up & down four times for every 10 second. Find the radius of the buoy in (cms)

Solu:

$F_G = ma$

$\rho_b \pi r^2 h \ddot{x} = \rho_b \pi r^2 h g - \pi r^2 x g \cdot \rho_w$

$\ddot{x} = g - \frac{x}{h \rho_b} \cdot \rho_w \cdot g$

$\ddot{x} + \frac{x}{h} \cdot g \frac{\rho_w}{\rho_b} = g$ → Taking (Homogeneous)

$x = \pm i \sqrt{\frac{g}{h} \cdot \frac{\rho_w}{\rho_b}}$.

$T = \frac{2\pi}{\sqrt{\frac{g}{h} \frac{\rho_w}{\rho_b}}}$

Turbulence - Irregular motion of air.

$T^2 \left(\frac{g}{h} \cdot \frac{\rho_w}{\rho_b} \right) = 4\pi^2$

$M = 50\,kg = \pi R^2 h \rho_b$

$h \rho_b = \frac{g \rho_w T^2}{4\pi^2}$

In this system we can use damping (resulting from [in the form of turbulence) → From Fluid mechanics)

$\frac{50}{\pi R^2} = \frac{9.8 \rho_w T^2}{4\pi^2}$

$\frac{R^2}{50\,kg} = \frac{4\pi}{9.8\,(1) T^2 \,(g/cm^3)}$

$R = \sqrt{\frac{4\pi \times 50000 g}{9.8\, ms^{-2}\, (1) g/cm^3 \cdot T^2 (s^2)}}$

Time of 1 oscillation $= \frac{10}{4}$ $= 2.5s$

$= \sqrt{\frac{4\pi \times 50000 g}{9.8\,(2.5)^2 \; mg/cm^3}}$

metres ↳ centimetres

$= \sqrt{\frac{4\pi \times 50000}{9.8\,(2.5)^2 \times 100 \; cm/cm^3}}$

≈ 10.1 cm.

Damping

we will now study the effect of damping (in mkb system)

$$\frac{\ddot{x}}{2} + b\dot{x} + \frac{x}{2} = 0 \quad (x(0) = 1, \dot{x}(0) = -1)$$

(Solve when $b < 1$, $b = 1$, $b > 1$)

Soln:

when $b < 1$, $b^2 - 1 < 0$, and the char eqn has

$$r = -b \pm \sqrt{1-b^2}\, i$$

$$x(t) = c_1 e^{-bt} \cos(\sqrt{1-b^2}\, t) + c_2 e^{-bt} \sin(\sqrt{1-b^2}\, t)$$

Using IVP:

$$x(0) = c_1 = 1$$

$$\dot{x}(0) = -b + c_2(\sqrt{1-b^2}) = -1$$

$$c_2 = \frac{b-1}{\sqrt{1-b^2}}$$

$$\therefore x(t) = e^{-bt} \cos(\sqrt{1-b^2}\, t) + \frac{b-1}{\sqrt{1-b^2}} e^{-bt} \sin(\sqrt{1-b^2}\, t)$$

Case: 2

when $b = 1$, $b^2 - 1 = 0$; char eqn $r = -1$

$$x(t) = (c_1 + c_2 t) e^{-t}$$

Using the IVP

$$x(0) = c_1 = 1$$

$$\dot{x}(0) = c_2 - 1 = -1$$

$$c_2 = 0$$

$$x(t) = e^{-t}$$

Case: 3

$b > 1$, $b^2 - 1 > 0 \Rightarrow r = -b \pm \sqrt{b^2 - 1}$

$$x(t) = c_1 e^{(-b + \sqrt{b^2-1})t} + c_2 e^{(-b - \sqrt{b^2-1})t}$$

$x(0) = c_1 + c_2 = 1$

$\dot{x}(0) = c_1(-b + \sqrt{b^2-1}) + c_2(-b - \sqrt{b^2-1}) = -1$

$= -(c_1 + c_2)b + (c_1 - c_2)\sqrt{b^2-1} = -1$

$= -b + (2c_1 - 1)\sqrt{b^2-1} = -1$

$= (2c_1 - 1)\sqrt{b^2-1} = b - 1$

$$c_1 = \frac{1}{2}\left(\frac{b-1}{\sqrt{b^2-1}}\right) + \frac{1}{2}$$

$$c_2 = \frac{1}{2} - \left(\frac{1}{2}\right)\frac{b-1}{\sqrt{b^2-1}}$$

$$\therefore x(t) = \left(\frac{1}{2} + \frac{1}{2}\frac{b-1}{\sqrt{b^2-1}}\right)e^{(-b+\sqrt{b^2-1})t} + \left(\frac{1}{2} - \frac{1}{2}\frac{b-1}{\sqrt{b^2-1}}\right)e^{(-b-\sqrt{b^2-1})t}$$

Compute the limit analytically as $b \to 1^-$ of the underdamped solution when $t = 1$.

Solu:

$t = 1$

$$= \lim_{b \to 1^-}\left(e^{-b}\cos(\sqrt{1-b^2}) + \frac{b-1}{\sqrt{1-b^2}}e^{-b}\sin(\sqrt{1-b^2})\right)$$

$$= e^{-1} + \lim_{b \to 1^-}\left(\frac{b-1}{\sqrt{1-b^2}}e^{-b}\sin(\sqrt{1-b^2})\right)$$

$$= e^{-1} + \lim_{b \to 1^-}\left(-\sqrt{\frac{1-b}{1+b}}\,e^{-b}\sin(\sqrt{1-b^2})\right)$$

$$= e^{-1}$$

Set $t = 1$ as $b \to 1^+$

$$\lim_{b \to 1^+}\left(\left(\frac{1}{2} + \frac{1}{2}\frac{b-1}{\sqrt{b^2-1}}\right)e^{(-b+\sqrt{b^2-1})} + \left(\frac{1}{2} - \frac{1}{2}\frac{b-1}{\sqrt{b^2-1}}\right)e^{(-b-\sqrt{b^2-1})}\right)$$

$$\lim_{b \to 1}\frac{b-1}{\sqrt{b^2-1}} = \lim_{b \to 1^+}\sqrt{\frac{b-1}{b+1}} = 0$$

$$\lim_{b \to 1^+} \left(\frac{1}{2} e^{(-b+\sqrt{b^2-1})} + \frac{1}{2} e^{(-b-\sqrt{b^2-1})} \right) = e^{-1}$$

choice of $t=1$ was arbitrary to simplify the algebra. This computation actually works all values of t, which explains smooth transition b/w the underdamped & overdamped cases you saw in the mathlet.

Application of sinusoids:

→ we examine the response $x(t)$ to the input $v_e(t) = \cos\omega t$

$$\dot{x} + kx = k\,v_e(t) = k\cos\omega t$$

$v_e(t) \to$ level of water in ocean (fluctuating according to the tides)

$x(t) \to$ water level in a bay connected by a channel to the ocean.

This model takes sense because the rate of change in water level in the bay $\dot{x}$ should be proportional to the difference in water level $v_e(t)$ and the water level in the bay x.

should be proportional to the difference in ocean water level $v_e(t)$ and the water level in the bay x.

∴ (Bay is connected to the ocean through a channel).

∴ It is also reasonable to model a bay in some seasons using $\cos\omega t$ since the tides shift with the pull of the moon's gravity as the earth rotates on its axis.

Bay - Coastal waterbody directly connected to some lake, ocean, sea etc.

Steady state response (After the homogeneous exponential terms decay to near 0)

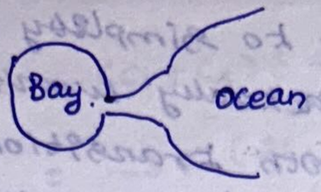
Does A ↑ or ↓ : $A\cos(\omega t - \phi)$

- when k increase, with ω fixed
- when ω ↑ with k fixed (The Earth changes frequency of rotation). (what about phase lag ϕ)

In the model of bay:

$x_e(t) \rightarrow$ water level (ocean)

$x(t) \rightarrow$ " " Channel to the ocean (Bay)



$\dot{x} \rightarrow$ change in bay level

$$\dot{x} + Kx = K\cos\omega t$$

$Kx \rightarrow$ Channel x water level

$K\cos\omega t \rightarrow$ From ocean due to waves.

The homogeneous solutions are constant multiples of a decaying exponential. Thus over time, the steady state response is the particular solution $\underline{x_p(t) = A\cos(\omega t - \phi)}$

(remove by water)

1) As conductivity $K \uparrow$, the channel has been dredged removing silt (sand, clay, ..), so the response follows the input more closely.

In other words,

As $K \uparrow$, Amplitude A also increases.

II) K constant, $\omega \uparrow$.

The oscillation is faster. So the ups and downs cancel each other. So A decreases.

<u>For more information:</u>

Taking cases $K \rightarrow 0$, $K \rightarrow \infty$

when K is very large $\rightarrow$ Nearly zero silt and a wide channel, the A amplitude of the response should be same as the input. (Near $A = 1$)

when K is nearly 0, the channel is plugged, there is hardly any response and A should be near zero.

(How changing K and ω affects ϕ)

As K increases, the channel has been dredged removing silt, so the response follow the input more closely.

So $\psi \to$ Decreases.

This is consistent with the limit as $K \to \infty$. (with zero silt, the input & output are the same (phase lag $\phi = 0$)

As $\omega \uparrow$, the oscillations are faster, but the reaction time (based on K) remains the same. Thus the phase lag ϕ increases.

In real world, there is no such thing as a perpetual motion machine. This model is missing damping! This is apparent from the fact that the general solution is a sum of sinusoids with no damping terms, meaning the masses must oscillate forever.

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perpetual motion - A state in which movement or action is or appears to be continuous and unceasing.

$\hookrightarrow$ Hypothetical machine. once started never ends. (until externally stopped).

Exponential response

1) write linear constant co.eff diff eqn using operator notation, (using Linear Time Invariant operators).
2) Use Time-shifts to solve LTI differential equations.
3) Solve inhomogeneous constant coeff ODEs with exponential inputs using the exponential response formula (ERF)
4) Recognize the ERF fails and how to apply the ERF and the generalized ERF in these situations.
5) Apply superposition to higher order, linear inhomogeneous diff eqn.

operator notation

If y_1 and y_2 are two solutions to a homogeneous linear differential equation

$$\ddot{y} + p\dot{y} + qy = 0$$

Why is $c_1 y_1 + c_2 y_2$ is a solution? → Linear Combination (Sum with Constant Co.effs)

(You know superposition for linear homogeneous diff equ) (can be higher derivative)

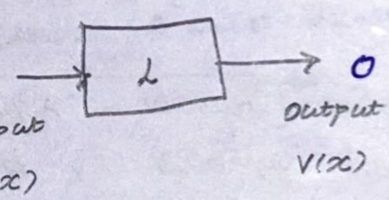
An elegant proof of superposition:

$$y'' + py' + qy = 0$$

$$D^2 y + pDy + qy = 0.$$

$$(\underbrace{D^2 + pD + q}_{\text{linear operator}})\,y = 0$$

$Ly = 0$ $\quad (L = D^2 + pD + q)$



(For a diff equ)

$L(u_1 + u_2) = Lu_1 + Lu_2$ (∴ linear operator)

$L(cu) = cL(u)$

u → function

c → constant

D → Linear why?

$$(u_1 + u_2)' = u_1' + u_2'$$

$$(cu)' = cu'$$

$\boxed{\text{ODE} : Ly = 0}$

Proof :

$$L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2)$$

$$= c_1 (Ly_1) + c_2 (Ly_2). = 0$$

only Ly_1 & Ly_2 are zero

operator notation:

The operator D:

* A function takes an input number and returns another number.

* An operator takes an input function and returns another function.

eg: Differential operator $\frac{d}{dt}$ takes an input function $y(t)$ and returns $\frac{dy}{dt}$. This operator is also called D.

For instance

$De^{4t} = 4e^{4t}$. (The operator D is linear)

$D(\alpha f) = \alpha Df$

$D(f+g) = Df + Dg$,

for any functions f and g, and any number α.

Because of this, D behaves well with respect to linear combinations, namely

$D(c_1 f_1 + \dots + c_n f_n) = c_1 Df_1 + \dots + c_n Df_n$

for any numbers $c_1, \dots, c_n$ and functions $f_1, \dots f_n$.

2.1 $Dt^3 = 3t^2$ $D8 = 0$ [$8 \rightarrow$ constant function)

when $t=2$

warning $D(8) = 3(4) = 12$ (The only way to interpret 8 in D(8) is a constant function. (D8 = 0). The point is that in order to know the function Df(t) at a particular value t, say t = a, you need to know how f(t) is changing near a as well. This is characteristic of operators. In general, we have to expect to need to know the whole function f(t) in order to know the whole function f(t) in order to evaluate an operator on it)

Def: 2.2: In general, a linear operator L is any operator that satisfies

$L(f+g) = Lf + Lg$, $L(\alpha f) = \alpha Lf$ for any functions f and g and any number α.

Exam: 2.3

The operator $L = D^2 + p(t)D + q(t)$ where $p(t)$ and $q(t)$ are any functions of t is a linear operator. This (why L is linear)?

As L is linear. |||ly D^2 is linear. To see that L is linear, verify that a linear Combination of linear operators is again a linear operator.

To apply a product of two operators, apply each operator in succession. For instance, DDy means take the derivative of y, and then take the derivative of the result; therefore

$$D^2y = \ddot{y}$$

To apply a sum of 2 operators, apply each operator to the function & add the results.

$$(D^2 + D)y = D^2y + Dy = \ddot{y} + \dot{y}$$

Any number can be viewed as the 'multiply by the number' operator: For instance, the operator 5 transforms the functions $\sin t$ into the function $5\sin t$. In particular, the 'identity' operator is the multiply by the number 1 operator.

$$1f = f \text{ for all functions } f.$$

The operator t^2 transforms the function $\sin t$ into the function $t^2 \sin t$.

Example: 3.1 The ODE

$2\ddot{y} + 3\dot{y} + 5y = 0$ (whose char polynomial)

$$2r^2 + 3r + 5 = 0$$

$$(2D^2 + 3D + 5) = 0$$

$$P(D)\, y = 0$$

The same argument shows that every constant-Coeff homogeneous linear ODE

$a_n y^{(n)} + \dots + a_0 y = 0$

can be written in the form

$P(D)y = 0.$

$P \rightarrow$ characteristic polynomial.

$$\frac{d^4x}{dt^4} + 2\frac{d^2x}{dt^2} + x = 0 \qquad [r^4 + 2r^2 + 1] = 0$$

Linear:

$4D^n + 3$, $D + t^2$, $D^2 + tD + t^2$, $mD^2 + bD + K$, $a_0 D^n + a_1 D^{n-1} + \dots + a_n$

are all linear operator having functions.

In particular, every linear diff eqn can be written in terms of a linear diff operator. The second & third choices are linear with variable coefficients. The first, fourth & fifth are linear with constant coefficients. We'll have formulas for the solutions when the operator involved have constant coeff.

Time invariance

Time invariance:

$P(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$

where all of the co-eff a_k are numbers (as opposed to functions of t). All operators of this form are linear. In addition to being linear operators, they are also time-variant operators, which means:

If $x(t)$ solves $P(D)x = f(t)$, then $y(t) = x(t - t_0)$ solves $P(D)y = f(t - t_0)$ solves $P(D)y = f(t - t_0)$

solves $P(D)y = f(t - t_0)$

In words, this says that 'delaying the input signal $f(t)$ by t_0 seconds delays the output signal $x(t)$ by t_0 seconds.

If we know that $x(t)$ is a solution to $P(D)x = f(t)$ we can solve

$P(D)y = f(t - t_0)$ by replacing t by $t - t_0$ in $x(t)$. This is a useful property because gives us the solutions to many differential eqn for free.

A system can be modeled using a linear

time invariant operator is called an LTI (Linear time invariant) system.

Exam: 4.1

$x(t) = \sin t$ solves $\dot{x} = \cos t$ what's the solution to $\dot{y} = \cos(t + \pi/2)$

solu:

$y = \sin(t + \pi/2)$ [By time invariance]

$\dot{y} = \cos(t + \pi/2)$ (1)

$\dot{y} = \cos(t + \pi/2)$

$\dot{y} = \cos t \cos \frac{\pi}{2} - \sin t \sin \frac{\pi}{t}$

$\boxed{\dot{y} = -\sin t}$

Here, Shifting input by 90° Shifts output by 90°

$y = \sin(t + \pi/2)$

$= \sin t \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \cos t$

$= \cos t.$

$\therefore \boxed{\begin{aligned} y &= \cos t \\ \dot{y} &= -\sin t \end{aligned}}$

$y \to \sin(t + 90°) = \cos t$

$\dot{y} \to \cos(t + 90°) = -\sin t$

Time invariant.

Example: 4.2 $\dot{x} + x = \cos t$ [variation of parameters or integrating factors tells us that the general solution to this differential eqn is

$$x(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t + c_1 e^{-t}$$

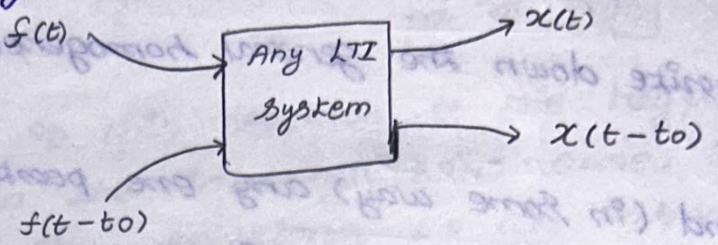
If $\dot{x} + x = \sin t$ [$\sin t = \cos(t - \frac{\pi}{2})$ delay. Time invariance tells us that

$$y(t) = x(t - \frac{\pi}{2}) = \frac{1}{2}\cos(t - \frac{\pi}{2}) + \frac{1}{2}\sin(t - \frac{\pi}{2}) + c_1 e^{+\frac{\pi}{2} - t}$$

$$= \frac{1}{2}\sin t - \frac{1}{2}\cos t + c_1 e^{\pi/2} e^{-t}$$

$$= \frac{1}{2}\sin t - \frac{1}{2}\cos t + c_2 e^{-t}.$$

should solve $\dot{y}+y=\sin t$. This agrees with the solution we'd get if we had used variation of parameters or integrating factors, we had to do almost no work to get this solution from the first.



LTI: $4D^n+3$, mD^2+bD+K, $a_0D^n+a_1D^{n-1}+\dots+a_n$

The functions $tD+2$ and D^2+tD+t^2 (are non linear))

Soln:

$$P(D)=tD+2$$

$$P(D)x=t\dot{x}+2x=t^5 \text{ (say)} \quad \text{on } (0,+\infty)$$

$$x(t)=\frac{1}{7}t^5+ct^{-2}$$

Now if the operator was time invariant, then

$$y(t)=x(t-t_0)=\frac{1}{7}(t-t_0)^5+c(t-t_0)^{-2}$$

would solve

$$t\dot{y}+2y=(t-t_0)^5$$

for any $t_0>0$. However:

$$t\dot{y}+2y=\frac{2}{7}(t-t_0)^5+\frac{5}{7}t(t-t_0)^4+2c(t-t_0)^{-2}-2ct(t-t_0)^{-3}$$

It should be clear that

$$t\dot{y}+2y\neq(t-t_0)^5 \rightarrow \text{Not time invariant.}$$

Superposition for an Inhomogeneous linear ODE

Inhomogeneous eqn: $P_n(t)y^{(n)}+\dots+P_0(t)y=q(t)$

Do the following:

1. List all the solutions to the associated homogeneous equation.

(Homogeneous eqn: $P_n(t)\, y^{(n)} + \dots + P_0(t)\, y = 0$)

i.e... write down the general homogeneous solution y_h.

2. Find (in some way) any one particular solution y_p to the inhomogeneous ODE.

3. Add y_p to y_h.

Summary:

$$\boxed{y = y_p + y_h}$$

Why does this works?

Proof: Let L be the linear operator $L = P_n(t) D^n + \dots + P_1(t) D + P_0(t)$. Then the diff eqn become

inhomogeneous eqn $Ly = q(t)$

homogeneous eqn $Ly = 0$.

Let y_p be a particular solution to the inhomogeneous solution equation

$L(y_p) = q$. Let y_h be the homogeneous solution $L(y_h) = 0$. Then

$$L(y_p + y_h) = q + 0 = q.$$

Suppose that y is also a solution to $Ly = q$ That is

$$Ly = q$$

$$L(y_p) = q.$$

Then it follows the

$L(y-y_p) = L(y) - L(y_p) = v - v = 0$.

$\therefore \boxed{y - y_p = y_h}$ In other words $\boxed{y = y_p + y_h}$

The result of this section is the key point of linearity in the inhomogeneous case. It lets us build all the solutions to an inhomogeneous DE out of one particular solution, provided that you have already solved the associated homogeneous ODE.

Super position for inhomogeneous equations (ODE)

$Ly = f(x)$ [Since $f(x) \neq 0$]
↳ In homogeneous

$\boxed{y_p + y_h = y}$ $\boxed{P \rightarrow \text{particular}}$

Here particular solution means Any one.

proof (using linear operator) $\therefore y_h = c_1 y_1 + c_2 y_2$.

$$L(y_p + c_1 y_1 + c_2 y_2) = Ly_p + c_1 Ly_1 + c_2 Ly_2.$$

$$= Ly_p + L(c_1 y_1 + c_2 y_2)$$

$$= Ly_p + 0$$

$\therefore$ The homogeneous solution y_h plugs in to an equ to give zero. $\boxed{Ly_p = f(x)}$

$\boxed{Ly = f(x)}$

$\therefore$ Satisfies the above equ

(Inhomogn)

proof:

If $u(x)$ is the solution to the ODE then $L(u) = f(x)$.

we saw that $L(y_p) = f(x)$

Solving: $L(u) = f(x)$ $\boxed{\text{where } y - y_p = y_h}$

$\boxed{L(y_p) = f(x)}$

$L(y_h) = 0$

$L(u - y_p) = 0$

$\therefore u - y_p = y_h$

$$\boxed{u - y_p = c_1 y_1 + c_2 y_2}$$

$\therefore u = y_p + c_1 y_1 + c_2 y_2$. (one of the other solutions)

Find particular solutions with exponential inputs.

$$y'' + Ay' + By = f(x)$$

soln:

Let $y'' + Ay' + By = 0$

$\boxed{y = y_p + y_h}$

$r^2 + Ar + B = 0$

$\therefore r = \dfrac{-A \pm \sqrt{A^2 - 4B}}{2}$

Our work here is to find y_p:

An arbitrary function can be built from these simple functions.

It is allowed to be complex

e^{ax} (pure osillation) $\begin{cases} \cos\omega x \\ \sin\omega x \end{cases}$ (Imposing pure vibration on the spring mass dashpot)

$(a<0)$

(1)   (2)

(or)

All ① ② & ③ are special cases when the exponent is a complex number

$\begin{cases} e^{ax} \cdot \sin\omega x \\ e^{ax} \cos\omega x \end{cases}$

(3)

Simply $e^{\alpha x}$ $(\alpha = a + i\omega)$ when $a = 0$ (case = 2)
when $\omega = 0$ (case = 1)
when $a, \omega \neq 0$ (C 3)

$(2D^2 + 3D + 5)\, e^{rt}$ (If r is a number)

Solu:

$$De^{rt} = re^{rt}, \quad D^2 e^{rt} = r^2 e^{rt}.$$

$$(2D^2 + 3D + 5) e^{rt} = 2r^2 e^{rt} + 3re^{rt} + 5e^{rt}$$

$$= (2r^2 + 3r + 5) e^{rt}$$

$$P(D) = 2D^2 + 3D + 5 \rightarrow P(r) = 2r^2 + 3r + 5$$

operator ... (char polynomial)

Theorem 7.1 : For any polynomial P and any number r

$$P(D)\, e^{rt} = P(r)\, e^{rt}$$

Shortcut

$$P(D) e^{\alpha x} = P(\alpha) e^{\alpha x} \rightarrow \text{substitutional rule.}$$

proof:

$$(D^2 + AD + B)\, e^{\alpha x}$$

$$= D^2 e^{\alpha x} + A De^{\alpha x} + Be^{\alpha x}$$

$$= (r^2 + Ar + B)\, e^{\alpha x}$$

$$= P(r)\, e^{\alpha x}$$

Exponential response formula

For any polynomial P and number r, $\frac{e^{rt}}{P(r)}$ is a Particular solution to

$$P(D)\, y = e^{rt}$$

Solu:

Super position

$$P(D)\, e^{rt} = P(r)\, e^{rt}$$

$$P(D)\left(\frac{1}{P(r)} e^{rt}\right) = e^{rt} \quad \text{(using linearity)}$$

This is called the exponential response formula (ERF)

In words, For any polynomial P and any number r such that $P(r) \neq 0$

$\frac{1}{P(r)} e^{rt}$ is a solution to $P(D)y = e^{rt}$

(Remember: This is just one particular solution. To get the general solution, we need to add the solution to the associated homogeneous equ.

<u>Exponential response formula</u>

For any polynomial P and number r, what is a P.I to $P(D)y = e^{rt}$?

<u>solu</u>: $P(D)y = e^{rt}$

By superposition

$$P(D) e^{rt} = P(r) e^{rt} \Rightarrow P(D)\left(\frac{1}{P(r)} e^{rt}\right) = e^{rt}$$

$$\ddot{y} + 7\dot{y} + 12y = -5e^{2t}$$

<u>solu</u>: $P(r) = r^2 + 7r + 12 = (r+3)(r+4)$ Roots $-3, -4$.

$$y_h = c_1 e^{-3t} + c_2 e^{-4t}$$

By ERF:

$r=2$ $\frac{1}{P(2)} e^{2t}$ is a particular solution to $P(D)y = e^{2t}$.

$\frac{1}{30} e^{2t}$ is a P. solution to $\ddot{y} + 7\dot{y} + 2y = e^{2t}$

So,

$-\frac{1}{6}e^{2t}$ is a particular solution to $\ddot{y}+7\dot{y}+12y=-5e^{2t}$.

General solution to inhomogeneous solution:

$$y = y_p + y_h$$

$$y = -\frac{1}{6}e^{2t} + c_1 e^{-3t} + c_2 e^{-4t}$$

ERF and complex roots

ERF works equally well for complex exponential inputs

e.g.: $\ddot{x} + x = e^{2it}$

solu:

$$(D^2+1)\,x = e^{2it} \qquad r = 2i$$

By ERF,

$$x_p = \frac{e^{rt}}{r^2+1} = \frac{e^{2it}}{r^2+1} = \frac{e^{2it}}{-4+1} = -\frac{1}{3}e^{2it}.$$

$(D^2+1)\,y = e^{-t}$

solu:

$$y_p = \frac{1}{r^2+1}e^{-t} = \frac{1}{2}e^{-t}$$

$(D^2+1)y = e^{-t} - 3e^{2it}$

solu:

$$y_p = \frac{1}{r^2+1}(e^{-t} - 3e^{2it}) = \frac{e^{-t}}{2} - \frac{3e^{2it}}{-4+1}$$

$$= \frac{e^{-t}}{2} + e^{2it}$$

For $v(t) = e^{at}$ does the diff eqn $\ddot{x} + x = v(t)$ not have a solution of the form Ae^{at}?

solu:

The char polynomial $P(r) = r^2+1$

$\therefore$ $r = \pm i$ (Thus we can't apply the ERF to find a solution of the form e^{at} is when $a = \pm i$.

The existence & uniqueness theorem says that

$$P(D)y = e^{rt}$$

should have a solution even if $P(r)=0$ when (ERF doesn't apply).

ERF' suppose that P is a polynomial and $P(r_0)=0$, but $p'(r_0) \neq 0$ for some number r_0

$$x_p = \frac{1}{p'(r_0)} t e^{r_0 t}$$ is a particular solution to $P(D)x = e^{r_0 t}$

Proof:

$$P(D)y = e^{r_0 t} \qquad (\text{w.k.t } P(D e^{rt} = P(r) e^{rt}) \text{ for all } r,$$

So in particular

$$P(D) e^{r_0 t} = P(r_0) e^{r_0 t}$$

. However since $P(r_0)$ is zero, we can't divide by it. Instead, let look at what happens for r near r_0 by differentiating w.r.t r, and then substituting $r = r_0$.

$$\frac{\partial}{\partial r}(P(D)e^{rt}) = \frac{\partial}{\partial r}(P(r)e^{rt}) = P'(r)e^{rt} + P(r)te^{rt}$$

we need to take the derivative of the left hand side. Recall from 18.02 calculus.

$$\frac{\partial}{\partial r}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\cdot\frac{\partial}{\partial r}$$

meaning: $$\frac{\partial}{\partial r} D = D \frac{\partial}{\partial r} \qquad \left(D = \frac{d}{dt}\right)$$

$$\therefore \frac{\partial}{\partial r} P(D) = P(D) \frac{\partial}{\partial r}$$

∴ The LHS becomes

$$\frac{\partial}{\partial r}(P(D)e^{rt}) = P(D)\left(\frac{\partial}{\partial r} e^{rt}\right)$$

$$= P(D)(te^{rt})$$