$$f(0^+) = \frac{0^+}{(-\infty)^2}$$

$$= 0$$

$$f(\infty) = \frac{\infty}{(-\infty)^2} = \frac{\infty}{\infty}$$ (Indeterminate form).

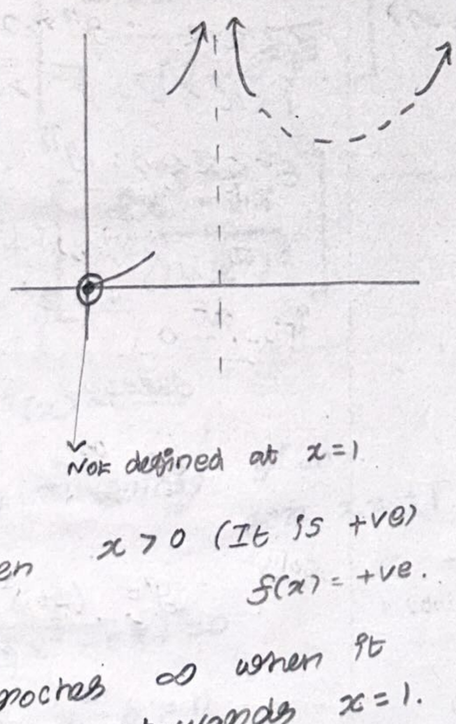
Sub high numbers:

$$f(10^{10}) = \frac{10^{10}}{(\ln 10^{10})^2}$$

$$= \frac{10^{10}}{23^2} > 10^7$$

(Big).

Graph



Not defined at $x=1$

when $x > 0$ (It is +ve)
$f(x) = +ve$.

It approaches ∞ when it approaches towards $x=1$. both from right & left.

as $x \to \infty$, $f(x) \to \infty$

$$f(x) = \frac{x}{(\ln x)^2}, \cancel{f'(x) = \frac{(\ln x)^2 - x \times 2 \times \frac{1}{x}}{(\ln x)^4}}$$

$$\cancel{f'(x) = \frac{2\ln x - 2}{4 \ln x}}$$

$$\cancel{= \frac{\ln x - 1}{2 \ln x}}$$

$$f'(x) = \frac{(\ln x)^2 - x \times 2 \times \ln x \times \frac{1}{x}}{(\ln x)^4}$$

$$= \frac{(\ln x) - 2}{(\ln x)^3}$$

$$f'(0) = \frac{\ln x - 2}{(\ln x)^3}$$

$$0 = \frac{\ln x - 2}{(\ln x)^3}$$

when,

$$\ln x = 2$$

$$\boxed{x = e^2}$$

Intervals:

$$f'(x) = \frac{\ln x - 2}{(\ln x)^3}$$

$$f'(e^1) = \frac{1-2}{1}$$

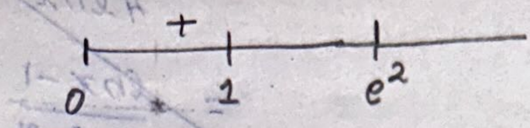
$$= -1.$$

$$f'(e^2) = 0$$

$$f'(e^3) = \frac{1}{1}$$

$$= 1.$$

$f' > 0$ on intervals?



* $0 < x < 1$ → The numerator, and the denominator is -ve. Function is ↑, ∴ So the derivative is +ve.

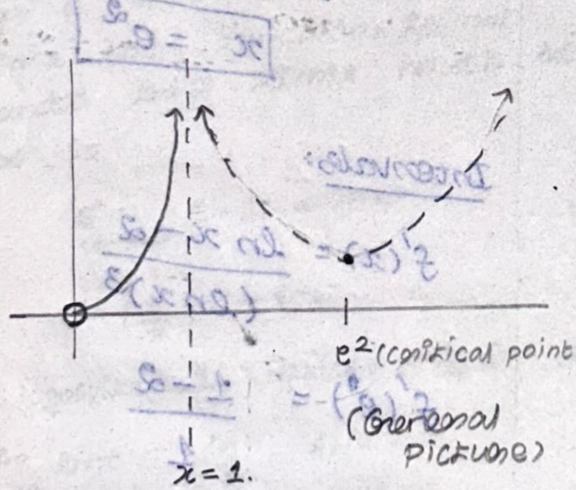
* $1 < x < e^2$ – The num is (-)ve & denominator is +ve, derivative is -ve. Function is ↓.

* $e^2 < x < \infty$, The num is +ve, den is +ve. Function is ↑.

$f' > 0$ on the intervals $(0, 1)$ and (e^2, ∞)

$f' < 0$ on $(1, e^2)$

Graph



$f' > 0$ $f' < 0$ $f' > 0$

For much accuracy

Second derivative test.

$$f''(x) = \frac{-2\ln x + 6}{x(\ln x)^4}$$

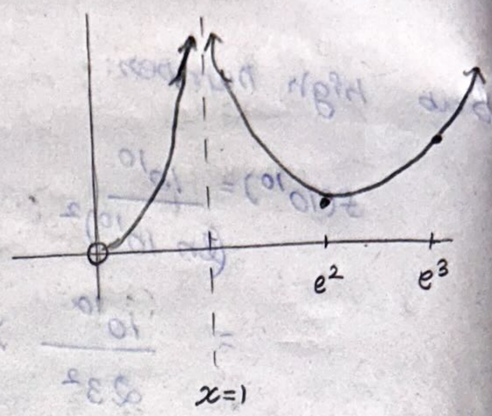
$f''(x) = 0$

when

$2\ln x = 6$

$\ln x = 3$

$\boxed{x = e^3}$



∴ $f(e^3) > f(e^2)$

$0 < x < 1 : f'' > 0$

$1 < x < e^3 : f'' > 0$

$e^3 < x < \infty : f'' < 0$

∴ The inflection point is at $x = e^3$.

Sketch

$$y = \frac{x}{1+x^2}$$

Solu:

$$y' = \frac{(1+x^2) - x(2x)}{(1+x^2)^2}$$

$$y' = \frac{1 - x^2}{(1+x^2)^2}$$

$0 \times (1+x^2)^2 = (1-x^2)$

$x^2 = 1$

$\boxed{x = \pm 1}$

This function is defined everywhere (Generally).

Long term behaviours

when $x>0$, $y>0$

$x<0$, $y<0$

$$f''(x) = \frac{(-2x)(1+x^2)^2 - (1-x^2)\, 2(1+x^2) \cdot 2x}{(1+x^2)^4}$$

$$f''(x) = \frac{(1+x^2)\left[-2x(1+x^2) - (1-x^2)4x\right]}{(1+x^2)^4}$$

$$f''(x) = \left[\frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3}\right]$$

$$= \left[\frac{-6x + 2x^3}{(1+x^2)^3}\right]$$

$$f''(x) = \left[\frac{2x^3 - 6x}{(1+x^2)^3}\right]$$

$f''(x) = 0$

critical points $f'=0$ are $x = \pm 1$

$(2x^3 - 6x) = 0$

$x^2 - 6 = 0$

$x = \pm\sqrt{3}$

$x = 0$. $\boxed{x = \pm\sqrt{3}}$

Graphing

when $x=0$, $y=0$.

when $x=1$

$y = \frac{1}{2}$, when $x=-1$

$y = -\frac{1}{2}$.

Local max Local min

$(1, 0.5)$ $(-1, -0.5)$

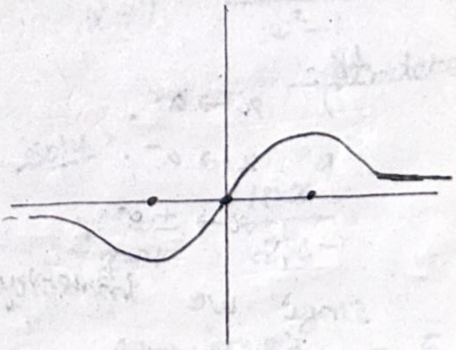
Local Extremum points

when

∴ The only two critical points are $x=1$ and $x=-1$

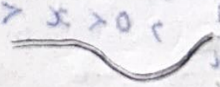
By II derivative test

Inflection points are $x=0$, $x = \pm\sqrt{3}$



(symmetric).

(odd function)



Rotated.

L' Hopitals rules

why?

uses derivatives to evaluate limits of indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

version 1: $\frac{0}{0}$.

If $f, g \to 0$

$x \to a$

f, g differential near $x=a$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

provided that the right hand side exists (or is $\pm\infty$)

version : 2:

$f, g \to \pm\infty$

as $x \to a$.

Note:

we can replace the two side limits

$x \to a$

by

$x \to a^+$

$x \to a^-$.

$x \to \pm\infty$

since we implicity assume $x \to a$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

sketch $\frac{x-1}{\ln x}$, $0 < x < 1 < x < \infty$

$$\lim_{x\to 1} \frac{x-1}{\ln x} = \lim_{x\to 1} \frac{1}{\frac{1}{x}}$$

$= 1$

Endpoints :

$$\lim_{x\to\infty} \frac{x-1}{\ln x} = \lim_{x\to\infty} x$$

$= \infty$.

. $x-1$ grows faster than $\ln x$ as $x \to \infty$.

Left end point & rh.

$$\lim_{x\to 0^+} \frac{x-1}{\ln x} = \frac{-1}{-\infty} = 0.$$

Don't use L'Hôpitals rule.

$\therefore$ we need to verify that a limit is of indeterminate form 0 over 0 or ∞/∞ before you try to use the rule.

$$\lim_{x\to 0^+} \frac{e^x - 1}{x - \sin x}$$

solu:

$\frac{+}{+} = +\infty$

$\therefore \boxed{e^x > 1, \; x > \sin x}$

(or)

$$\lim_{x\to 0^+} \frac{e^x}{1-\cos x} = \frac{1}{0}.$$

$$\lim_{x\to\infty} \frac{\ln(x^2) + 4x + 30}{8x + \sqrt{x}}.$$

$$\lim_{x\to\infty} \frac{\frac{2}{x} + 4}{8 + \frac{1}{2\sqrt{x}}}.$$

$\lim_{x\to\infty} \frac{4}{8} = \frac{1}{2}.$

$$\lim_{x\to\pi} \frac{x\sin x}{\cos^2 x+1}$$

Soln:

$$\lim_{x\to\pi} \frac{\sin x \cdot x}{\cos^2 x+1}$$

$$= -\frac{0}{2}$$

(L'Hopitals cant be app).

$$\lim_{x\to\infty} \frac{\ln(e^x+5x)}{4x+3}$$

Solu:

$$\lim_{x\to\infty} \frac{\frac{1}{e^x+5x}(e^x+5)}{4}$$

$$\frac{1}{4}\lim_{x\to\infty} \frac{e^x+5}{e^x+5x}$$

$$\frac{1}{4}\lim_{x\to\infty} \frac{e^x}{e^x+5}$$

$$\frac{1}{4}\lim_{x\to\infty} \frac{e^x}{e^x}$$

$$= \frac{1}{4}.$$

(or)

$$\lim_{x\to\infty} \frac{e^x}{e^x+5} = \lim_{x\to\infty} \frac{1}{1+\frac{5}{e^x}}$$

$$= \lim_{x\to\infty} \frac{1}{1}$$

$$= 1.$$

$$\therefore \frac{1}{4}\lim \frac{e^x}{e^x+5} = \frac{1}{4}$$

Justification

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Special case : when $f(a)=g(a)=0$

and rh. limit exists

$\therefore$ x near a, we can do linear approximation

$$\frac{f(x)}{g(x)} \approx \frac{f(a)+f'(a)\Delta x}{g(a)+g'(a)\Delta x}$$

$$\approx \frac{0+f'(a)\Delta x}{0+g'(a)\Delta x}$$

$$= \frac{f'(a)}{g'(a)}$$

both $f(a)=g(a)=0$

$f'(a)=g'(a)=0.$

use quadratic app.

$$\lim_{x\to 1} \frac{x^{10}-1}{x^2-1}$$

(Indeterminate)

Solu:

$$\lim_{x\to 1} \frac{10x^9}{2x}$$

$$\lim_{x\to 1} 5x^8 = 5.$$

How: $$\lim_{x\to 1} \frac{x^{10}-1/(x-1)}{x^2-1/(x-1)}$$

$\therefore f(x) = x^{10}-1$

$f(1) = 0$

$$\frac{f(x)-f(1)}{x-1} = f'(1)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)/x-a}{g(x)/x-a}$$

$$= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

$\therefore f(a) =$
$g(a) = 0$

$$= \frac{f'(a)}{g'(a)}.$$

when $g'(a) \neq 0$

Applying L Hopitals repeatedly

$$\frac{\cos x - 1}{x^2}, \quad x \to 0$$

solu:

$$\lim_{x\to 0} -\frac{\sin x}{2x}.$$

still $\left(\frac{0}{0}\right)$

Again

$$\lim_{x\to 0} -\frac{\cos x}{2} = -\frac{1}{2}.$$

If $-\frac{1}{2} \left(-\frac{\cos a}{2}\right)$ exists

then $-\frac{\sin x}{2x}$ exists, then in

turn $\frac{\cos x - 1}{x^2}$ exists.

$$\lim_{x\to 1} \frac{e^x - ex}{x^3 - x^2 - x + 1}$$

solu:

$$\frac{e - e}{1-1-1+1} = \frac{0}{0}$$

$$\lim_{x\to 1} \frac{e^x - e}{3x^2 - 2x - 1}$$

$$\lim \frac{0}{0}$$

$$\lim_{x\to 1} \frac{e^x}{6x-2} = \frac{e}{4}$$

$$\lim_{x\to\infty} \frac{300x^3 + 17x}{e^x}$$

solu:

$$\frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{900x^2 + 17}{e^x}$$

$$\lim_{x\to\infty} \frac{1800x}{e^x}$$

$$\lim_{x\to\infty} 1800\, e^{-x}$$

$$0$$

$$\lim_{x\to\infty} \frac{x^{1000000}}{e^x}$$

solu:

$$\frac{x^{1000000}}{e^x} \sim \frac{1000000\, x^{99999}}{e^x}$$

$$\sim \frac{1000000 \cdot 99999\, x^{99998}}{e^x}$$

$\vdots$

$$\sim \frac{1000000!\, x^0}{e^x}$$

$$\sim \frac{10^6!}{e^x}$$

$$\lim_{x\to 1} 10^6 \cdot e^{-x} = 0$$

$$\frac{x^n}{e^x} \sim \frac{n!}{e^x} \xrightarrow{x\to\infty} 0$$

$\therefore$ e^x grows faster than x^n.

$\therefore$ e^x grows faster than any other polynomial

$\ln x$ to $\sqrt[n]{x}$ where n is +ve integer. which fun grows faster as $x\to\infty$

Soln:

$$\frac{\ln x}{\sqrt[n]{x}} = \frac{\frac{1}{x}}{n x^{n-1}}$$

$$= \frac{1}{n x^n}$$

As $x\to\infty$ $= 0$.

$\therefore$ $\sqrt[n]{x} \to$ grows faster

$\lim_{x\to a} f(x) = 0$

$\lim_{x\to a} g(x) = +\infty$

$\lim_{x\to a} f(x) g(x) = ?$

we can't tell.

$\therefore$ product of a small & large number may be small or large or anything in between

---

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = +\infty$

$\lim_{x\to a} f(x) g(x)$?

very large & +ve.

(product of two large numbers)

---

other Indeterminate forms.

Solu:

$0\cdot\infty$, $\infty-\infty \rightsquigarrow \frac{0}{0}, \frac{\infty}{\infty}$

Convert to $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$

Apply L Hopital's rule.

$$\boxed{\lim_{x\to 0^+} \ln x = -\infty}$$

$$x \ln x = \frac{\ln x}{\frac{1}{x}}$$

$$= \frac{-\infty}{\infty}$$

$$\sim \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -x \to 0$$

as $x \to 0^+$

0^0, ∞^0, 1^∞ are all indeterminate forms. To apply L Hopital's rule. we must rearrange these to $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$\lim_{x\to 0^+} x^x$$

Solu:

Since we have a moving exponent.

use base e.

$$x^x = e^{x \ln x}$$

$$x \ln x \to \frac{\ln x}{1/x} \to \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$\to -x$$

as $x \to 0$ $= 0$

$\therefore$ as

$x \to 0$

$$x^x = e^0 = 1$$

Checking

$$\frac{\sin x}{x^2} \xrightarrow{LH} \frac{\cos x}{2x} \to \frac{-\sin x}{2}$$

As $x \to 0$

$= 0$

By Linear app

$\sin x \approx x$

$$\therefore \frac{x}{x^2} = \frac{1}{x} = \infty$$

$+\infty$ (near $x \to 0^+$)

$-\infty$ (near $x \to 0^-$)

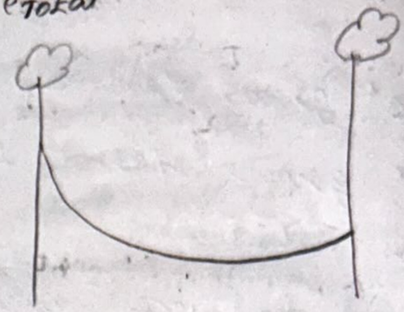
Something fishy.

$$\therefore \frac{\cos x}{2x} = \frac{1}{0}$$

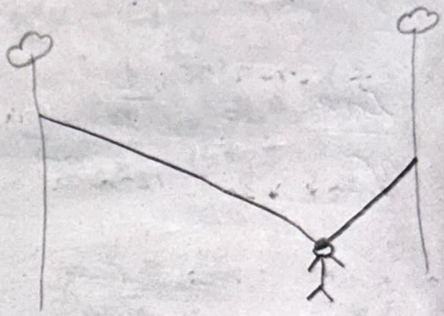
L Hopital can't be applied further.

Applications

e Total vertical drop'



zip line



(path)

(minima)

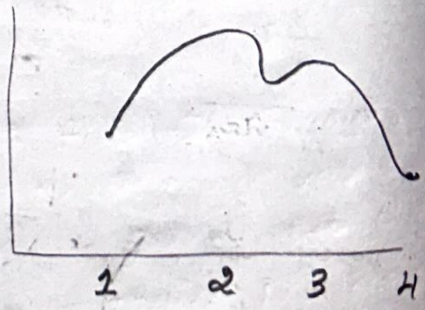
Assuming

Rope – zero mass

cable doesn't sag.

(தொய்வு)

Extrema on an interval.



Two local maximum.

But only one maximum.

f attains its maximum on the interval I at $x = c$ is

$f(c) \geq f(x)$ for all x in I.

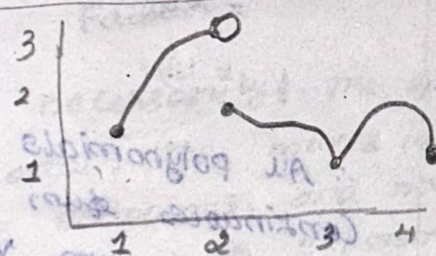
$f(x) \rightarrow$ The biggest

$f(4) = 1.$ $f(2) = 3$

$\therefore$ f attains its max on I at $x=2$

max value on I is 3.

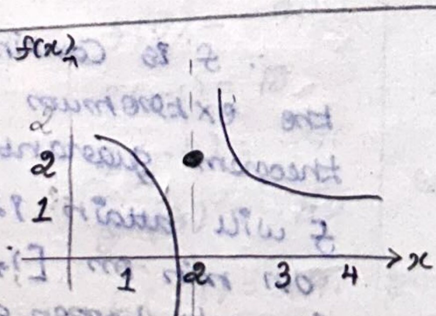
The minimum value attained by f on I is $f(4) = 1$.



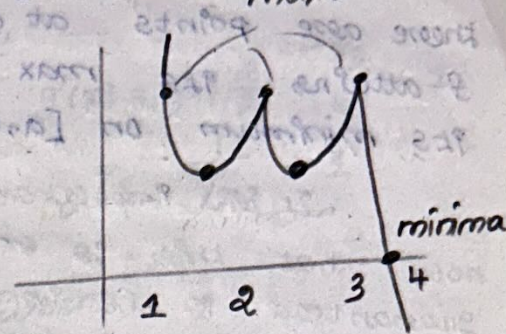
$\therefore$ f doesn't attains its maximum on I.

f attains the min values of 1 on I at $x=3$ & $x=4$.

Function can achieve max or min at more than one points (multiple) or at no points



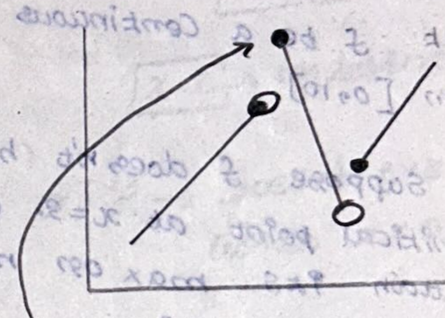
It neither has a minimum or maximum at [1,4]



Has both maxima & minima.

At 1, 2, 3 — maxima

4 — minimum



maximum

No minima.

The functions that don't attain maxima or minima are problematic.

(They were not continuous)

But differenciable at some intervals.

Extreme value theorem

If f is continuous on a closed interval [a, b], then there are points at which f attains its max and its minimum on [a, b]

It is important to note that this is only guaranteed if the interval is finite & closed. we will talk about infinite intervals and open intervals a little later.

critical points: where f is not differenciable or zero.

Let f be a continuous fun on [0, 10].

1) Suppose f doesn't have a critical point at $x = 3$. can f attain its max or min on [0, 10] at $x = 3$?

No.

$\therefore$ $f'(3)$ exists (f doesn't have a critical point at $x = 3$)

$f'(3) \neq 0$.

Tangent line is not horizontal

At $x = 10$:

we need to plug in values to determine if it has a max or min there. (end points)

we want to find the max or min value of a continuous function on a closed interval. The Extreme value theorem guarantees that they will be attained. The max & min can only be attained at critical points or endpoints, so we just need run through all of those candidates to find the larger & smallest values of f.

$f(x) = x^3 - 6x^2 - 15x + 10$

$[-4, 6]$

$f'(x) = 3x^2 - 12x - 15$

when $f'(x)$

$x^2 - 4x - 5 = 0$

$x = 5, -1$

$f(5) = 125 - 6(25) - 15(5) + 10$

$= 125 - 150 - 75 + 10$

$= 135 - 225$

$= -90$

$f(-1) = -1 - 6 + 15 + 10$

$= 25 - 7$

$= 18$

$\therefore$ All polynomials are continuous fun, so the extremum value theorem states that it will attain its min or max on any interval of the form [a, b].

No critical points

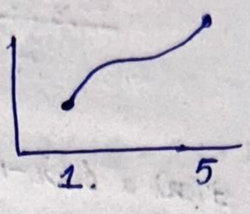
If f is a continuous function on the interval [1, 5] but has no critical points in the interval, what about f

sol

$\therefore$ f is continuous, the extremum value theorem guarantees that f will attain its max or min on [1, 5]. These can only happen at critical points or endpoints.

∴ No critical points mean

It attains max at one endpoint & min at other.

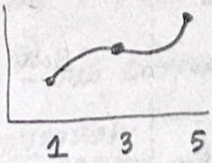


one critical point.

If f is a continuous fun on $[1, 5]$ and the only critical point for f within the interval is at $x = 3$, then f must attain either its max or its min on the interval at $x = 3$.

False.

Not necessarily! The fun could attain its max & min at the endpoints, and not have any sort of extremum at $x = 3$



The maxima or minimum can be achieved either at the critical points or at end points.

$\therefore f(5) = -90$

$f(-1) = 18$

$f(-4) = -90$

$f(6) = -80.$

$f(-1) = 18.$

$$g(x) = \frac{3}{2} x^{2/3} + x + 1$$

$[-2, 1]$

Solu:

$$g'(x) = \frac{3}{2} \times \frac{2}{3} x^{-1/3} + 1$$

$$0 = \frac{1}{\sqrt[3]{x}} + 1$$

when

$$\frac{1}{\sqrt[3]{x}} = -1$$

$$\sqrt[3]{x} = -1$$

when, $\boxed{x = -1}$

$\boxed{x = 0} \rightarrow$ undefined.

$\therefore x^{-1/3} \rightarrow$ Doesn't exist (when $x = 0$)

$g(0) = 1$

$$g(-1) = \frac{3}{2}(-1)^{2/3} + (-1) + 1$$

$$= \frac{3}{2}(-1)^{2/3}$$

$$= \frac{3}{2} = 1.5$$

$$g(-2) = \frac{3}{2}(-2)^{2/3} - 2 + 1$$

$$= \frac{3}{2}(-2)^{2/3} - 1$$

$$= 1.3811$$

$$g(1) = \frac{3}{2} + 2$$

$$= 7/2$$

$$= 3.5$$

$f(x) = \frac{1}{x} + \sin x$

achives its maximum & minimum as guaranteed by mean extremum value theorem in the intervals

$[-12, -3]$

$[\pi, 2\pi]$

Solu:

$f'(x) = -\frac{1}{x^2} + \cos x$

$\frac{1}{x^2} = \cos x.$

under these intervals.

This Extreme value theorem applies when dealing with a continuous function on a closed interval $[a, b]$. It doesn't apply when the interval is $(-2, -1)$ or $[5, \infty)$

It doesn't apply is $I = [-\pi, \pi]$ since f is not continuous at $x = 0$.

Infinite + open intervals

"Not Guaranteed to attain extrema".

canditates: Critical pts, endpoints.

$\therefore f(x) = \frac{2x^2 - x + 8}{x^2+4}$ on $[1, \infty)$

Solu:

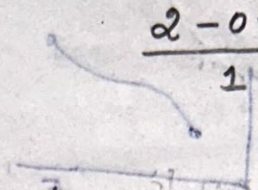
$f(1) = \frac{9}{5} = 1.8$

other end point:

$\lim_{x\to\infty} \frac{2x^2 - x + 8}{x^2+4}$.

$\lim_{x\to\infty} \frac{2 - 1/x + 8/x^2}{1 + 4/x^2}$

$\frac{2 - 0 + 0}{1} = 2$



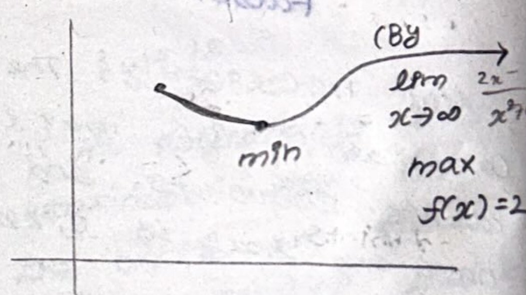
$f'(x) = \frac{(4x-1)(x^2+4) - 2x(2x^2-x+8)}{(x^2+4)^2}$

$0 = \frac{x^2-4}{(x^2+4)^2}$

when $x = +2$ (or) -2

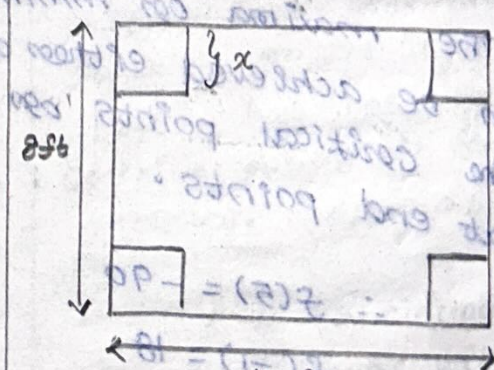
$\therefore x = +2$ (our canditate)

$f(2) = 1.75$



But f never achieves $f(x) = 2$.

Making a box



(Biggest volume when the corner squares are smaller).

maximize volume:

$$\boxed{V = lwh}$$

$$\boxed{V(x) = lwh}$$

$V(x) = (8-2x)(12-2x)x.$

$x \to$ height. $(96-16x-24x+4x^2)$

$V(x) = (8x-2x^2)(12-2x)$
$= 96x - 16x^2 - 24x^2 + 4x^3$

$= 4x^3 - 40x^2 + 96x$

$= 4(x^3 - 10x^2 + 24x)$

$V'(x) = 4(3x^2 - 20x + 24)$

$0 = 3x^2 - 20x + 24$

$\leftarrow \quad x = \frac{10 \pm \sqrt{28}}{3}$

End points:

x can't be (-)ve.

$x > 4$, the cut-out corners will start to overlap.

$\therefore$ the allowable interval is

$0 \le x \le 4.$

$x = \frac{10-\sqrt{28}}{3} \rightarrow$ lies b/w 0 and 4.

making

$x = 1.5694.$

$8 - 2x = 4.8612$

$12 - 2x = 8.8612.$

$V = 67.6$ cu ft

ft^3 ,b)

* Draw a picture
* Name the independent variable.
* write quantity we want to optimize solely in terms of that variable
* Determine the allowable interval
* check critical + end points.

Let's say we have 100 inches of wire that we want to cut into two pieces. we will use one piece to make a O^le and other to make a square.
(It is possible that to have one of the two shapes with zero size).

The perimeter of the O^le can be anywhere b/w 0 to 100 inches, and the perimeter of the square |||ly can be anywhere from 0 to 100 inches. Are these two perimeters independent of the others?

$\therefore$ once you know how much wire used by you to make O^le, you know how much wire you used to make square!
$\therefore$ The two perimeters are not independent

Let c — perimeter of $\bigcirc^{le}$

s → perimeter of square.

S in terms of C:

$$s = 100 - c$$

Total Area of two shapes:

$A(C) =$ square area + $\bigcirc^{le}$ area.

perimeter of $\bigcirc^{le}$: s

$$c = 2\pi r$$

$$c = 2\pi r$$

Area of $\bigcirc^{le}$: πr^2

$$\therefore \pi \times \left(\frac{c}{2\pi}\right)^2$$

$$\therefore \frac{\pi c^2}{4\pi^2} = \frac{c^2}{4\pi}$$

Area of square: S

$$S = 4 \times a$$

$$100 - c = 4 \times a$$

$$a = \frac{100-c}{4}$$

$$\text{Area} = \left(\frac{100-c}{4}\right)^2$$

$$\therefore \text{Total area} = \frac{c^2}{4\pi} + \left(\frac{100-c}{4}\right)^2$$

To with in 0.01 inches, what should ~~c~~ be in order to maximize the total area

Solu:

$$A'(C) = \frac{c}{2\pi} - \frac{100-c}{8}$$

$$C = \frac{100\pi}{\pi+4} \approx 43.99$$

inches.

$$A(43.99) \approx 350 \text{ sq. inches}$$

Endpoints:

$A(0) = 625$ sq. inches

$A(100) \approx \frac{2500}{\pi} \approx 796$ sq. inches

$\therefore$ A is max when

$c = 100$ inches.

(All of the wire is made in to a $\bigcirc^{le}$)

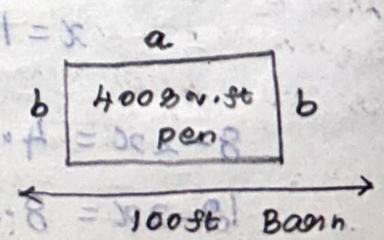
To within 0.01 inches what should c be in order to minimize the total area?

Solu:

$A(43.99) = 350$ sq. inch

(minimal area)

A farmer wants to build a rectangular pen for his sheep on the side of his barn. One side of the pen must be formed by part of the north wall of the barn, which measures 100 feet long, and the other three sides will be formed by fencing. The area of the pen should be 4000 sq. feet.



If a denotes the length of the pen and b the width (in feet) express b in terms of a.

Solu:

$$b = \frac{4000}{a}, \quad \text{Fence} = 2b + a$$

$$= \frac{8000}{a} + a$$

Interval for a

The length of the pen can't be longer than the barn, so $a \le 100$. It also can't be (-)ve. nor can it be zero (b would then be undefined).

$\therefore \; 0 < a \le 100$.

$$p(a) = \frac{8000}{a} + a$$

$$p'(a) = 8000 \times (-1)\, a^{-2} + 1$$

$$p'(a) = \frac{-8000}{a^2} + 1$$

$$8000 = a^2$$

$$a = \pm\sqrt{8000}$$

$$a = 40\sqrt{5} \quad (+ve)$$

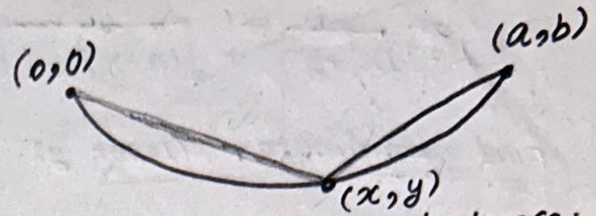
maximum fence

when $a = 40\sqrt{5}$

Fencing = 179 feet.

when $a = 100$, the total amount of fencing near is 180 feet. As $a \to 0$ from right, the amount of fencing approaches ∞! The max is not attained.

zipline

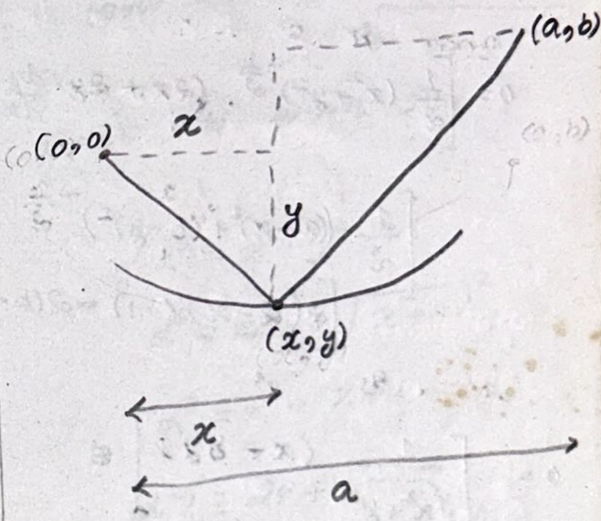


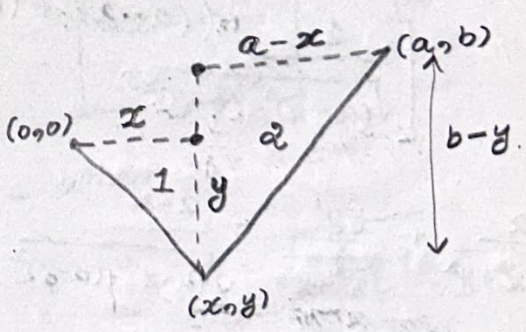
maximum vertical distance:

* Give labels (coordinate)

$(x,y) \to$ variable.

weight settles at the lowest point.
↳ lowest potential Energy.





Δle 1:

$$hyp = \sqrt{x^2 + y^2}$$

Δle 2:

$$hyp = \sqrt{(a-x)^2 + (b-y)^2}$$

Total length of rope:

$$L = \sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2}$$

Find minimum: (Least y)

From diagram, the bottom point is where the tangent is horizontal (slope = 0)

$$y' = 0.$$

(From diagram, it is observed that it's a huge ellipse). with foci $(0,0)$, (a,b) → (useless & Long way)

Implicit diff:

$$0 = \left[\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot\left(2x+2y\cdot\frac{dy}{dx}\right)\right] + \left[\frac{1}{2}\cdot\left((a-x)^2+(b-y)^2\right)^{-\frac{1}{2}}\cdot\left[2(a-x)(-1)-2(b-y)\frac{dy}{dx}\right]\right]$$

$$0 = \left[\frac{1}{\sqrt{x^2+y^2}}(x+yy')\right] - \left[\frac{1}{\sqrt{(a-x)^2+(b-y)^2}}\left((a-x)+(b-y)y'\right)\right]$$

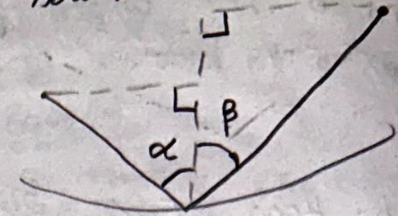
$$\frac{x}{\sqrt{x^2+y^2}} + \frac{yy'}{\sqrt{x^2+y^2}} - \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}} - \frac{(b-y)y'}{\sqrt{(a-x)^2+(b-y)^2}} = 0$$

$$\frac{x}{\sqrt{x^2+y^2}} - \frac{(a-x)}{\sqrt{(a-x)^2+(b-y)^2}} = \frac{(b-y)y'}{\sqrt{(a-x)^2+(b-y)^2}} - \frac{yy'}{\sqrt{x^2+y^2}}$$

AS $\boxed{y'=0}$

$$\therefore \frac{x}{\sqrt{x^2+y^2}} = \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}}$$

How?



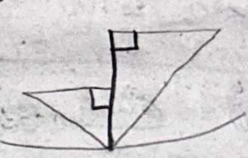
$$\sin\alpha = \frac{x}{\sqrt{x^2+y^2}}$$

$\therefore$

$$\sin\alpha = \sin\beta$$

$$\boxed{\alpha = \beta}$$

∴ This is a kind of equilibrium equation.



∴ Balanced Condition.

(No matter how the weight is tilted)

↳ The two angles are the same.

By Force diagram:

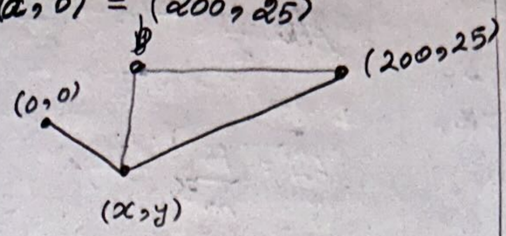
The tension of the two lines is the same.

When you build some thing which is hanging like this, it will involve the least stress.

(Both sides must carry equal weight)

↳ To be balanced.

$(a,b) = (200,25)$



* mass of rider = 50kg
* Zipline – massless
* Horizontal span b/w two fixed endpoints of the zipline ride is 200metres
* The change in y coordinate b/w the two fixed end points is 25metres.

maximum velocity = ?

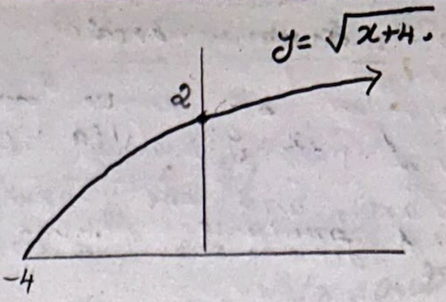
Assuming he starts with zero initial velocity.
(zero friction).

Find potential energy reduced when she reaches the min height of her trajectory.
(which equals the max amount of kinetic energy of the rider during the trip, assuming he starts with 0 initial v)

Solu:
The total vertical height $= (25-y)$

$$P.E = mgh$$
$$= 50g(25-y)$$

What point on the curve $y=\sqrt{x+4}$ comes closest to the origin?



Solu:

$$d^2 = (x-a)^2+(y-b)^2$$
$$\boxed{d^2 = x^2+y^2} \quad (a,b)=(0,0)$$

(distance problem)

$$\boxed{(d^2)' = 2dd'}$$

(minimal distance b/w (0,0) and function)

$\therefore$

Implicit diff w.r.t x.

$$d^2 = x^2+y^2$$
$$d^2 = x^2+(\sqrt{x+4})^2$$
$$d^2 = x^2+x+4$$
$$(d^2)' = 2x+1$$
$$(d^2)'=0$$
$$\boxed{\begin{aligned} 2x &= -1 \\ x &= -1/2 \end{aligned}}$$

when $x = -\frac{1}{2}$

$$y = \sqrt{-\frac{1}{2}+4}$$
$$= \sqrt{7/2}$$

$(-1/2, \sqrt{7/2})$

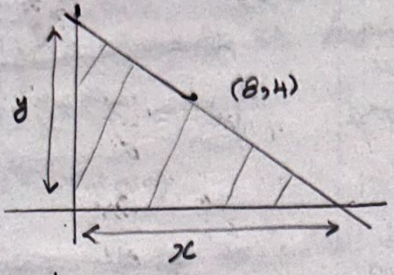
closest point to the origin.

(minimize problem)

Maximize area of a Δ^{le}

Consider Δ^{le} angles formed by lines passing through $(8,4)$, the x axis, and the y-axis. Find the dimensions that minimize area.

Solu:



$$\text{Area} = \frac{1}{2} \times b \times h$$

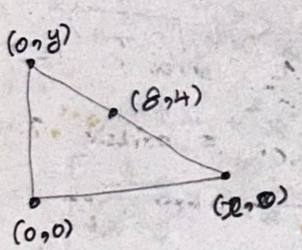
$$\boxed{\text{Area} = \frac{1}{2} x y.}$$

It has to go through $(8,4)$

$$y = mx + c$$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = m(x - 8)$$



At $(0,y)$

$$y - 4 = m(-8)$$

$$y = -8m + 4$$

At $(x,0)$

$$-4 = m(x-8)$$

$$-4 = mx - 8m$$

$$\boxed{x = -\frac{4}{m} + 8}$$

Substituting

$$A = \frac{1}{2}\left(-\frac{4}{m} + 8\right)(-8m + 4)$$

$$= \frac{1}{2}\left(32 - 64m - \frac{16}{m} + 32\right)$$

$$= \frac{1}{2}\left(32 - \frac{64m^2 - 16}{m} + 32\right)$$

$$= \frac{1}{2}\left(64 - 64m - \frac{16}{m}\right)$$

$$A' = \frac{1}{2}\left(-64 + \frac{16}{m^2}\right)$$

$$64 = \frac{16}{m^2}$$

$$64m^2 = 16$$

$$m^2 = \frac{16}{64}$$

$$= \frac{4}{16}$$

$$= \frac{1}{4}.$$

$$4\overline{)64}$$
$$16$$
$$\frac{4}{24}$$

$$\boxed{m = \pm \frac{1}{2}}$$

when $m = \frac{1}{2}$

$$A\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(64 - \frac{64}{2} - 32\right)$$

$$= \frac{1}{2}(64 - 32 - 32)$$

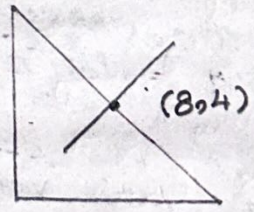
$$= \frac{1}{2}(64 - 64)$$

$$= 0$$

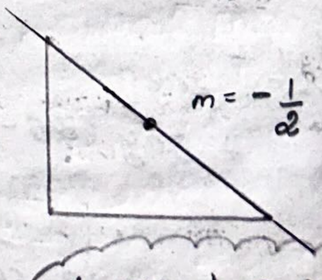
$$A\left(-\tfrac{1}{2}\right) = \frac{1}{2}\left(64 + \frac{64}{2} + 32\right)$$

$$= \frac{1}{2}(64 + 32 + 32)$$

$$= 64.$$



$m = +\frac{1}{2}$



$\therefore m = -\frac{1}{2} \rightarrow$ our candidate

End points $(x,0)$, $(0,y)$

Function $= \frac{1}{2}xy$.

$A=0$ (at $(x,0)$ & $(0,y)$)

so: The minimum occurs at $(8,4)$

Dimensions:

$m=-\frac{1}{2}$

$x=-\frac{4}{m}+8 = 8+8 = 16$ | $y=-8m+4 = 4+4 = 8$

$(16,8)$.

Second derivative test

$A'=\frac{1}{2}\left(-64+\frac{16}{m^2}\right)$

$A''=\frac{1}{2}\left(16\times -2\times\frac{1}{m^3}\right)$

$A''=\frac{1}{2}\left(-32/m^3\right)$

$A''=-16m^{-3}$

$\boxed{A''=0}$ $\boxed{m=0}$

maximum surface area

A cylinder has a fixed volume. what ratio b/w radius & height minimizes surface area?

$V=\pi r^2h$, $\boxed{S.A=2\pi r^2+2\pi rh}$

Solu: $V=\pi r^2h \rightarrow$ constraint eqn

$S.A=2\pi r^2+2\pi rh \rightarrow$ optimising eqn.

$\boxed{\frac{V}{r}=\pi rh}$

$S.A=2\pi r^2+2\left(\frac{V}{r}\right)$

$(r\neq 0,\infty)$

$A'=4\pi r - \frac{2V}{r^2}$

$4\pi r=\frac{2V}{r^2}$

$r^3=\frac{2V}{4\pi}$

$r^3=\frac{2V}{4\pi}$, $r^3=\frac{V}{2\pi}$

$r^3=\frac{\pi r^2h}{2\pi}$

$\frac{r}{h}=\frac{\pi}{2\pi}$

$\boxed{\frac{r}{h}=\frac{1}{2}}$

$\frac{r}{h}=\frac{1}{2}$ ratio for minimizmal surface area

Rates.

1) An oil tanker has ruptured and is spilling oil into the ocean at 2 million litres per hour. We can model the spill as a very thin cylinder on the surface of the ocean, the volume of which is gn by

$V=\pi r^2h$.

r – radius of the spill

h – height or thickness of the oil.

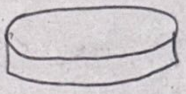
Suppose that the spilled oil has a constant thickness h of 2cm. At this moment, the spill radius is 250 meters. we would like to know at what outer perimeter of the spill is moving.

solu:

$$\boxed{V = \pi r^2 h}$$

The answer we seek is the speed of the edge. Diff w.r.t the radius didn't result in a speed, now will diff w.r.t the volume.

Gn: 2000000 L/hr

2cm 

$$\frac{dv}{dt} = 2000000.$$

h = 2cm.

$$\boxed{\text{we want: } \frac{dr}{dt}}$$

$\frac{dr}{dt}$ (r = 250m)
↳ r is not a constant

$$v = \pi r^2 h$$

$$\frac{dv}{dt} = \pi h \times 2r \cdot \frac{dr}{dt}.$$

$$\frac{dv}{dt} = 4\pi r \frac{dr}{dt} \quad (250m = r)$$

$$\frac{dv}{dt} = 1000\pi \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{1000\pi} \frac{dv}{dt}.}$$

$$\frac{dr}{dt} = \frac{2000000}{1000\pi} \quad \frac{dv}{dt} = 2000000\,L/hr$$

$$= \frac{2000}{\pi} \frac{L/hr}{m \times cm}$$

$$= \frac{2000}{\pi} \frac{L/hr}{m \times cm}$$

Litres to cubic metres.

Decimetre = 10cm.

$$1\,dm^3 = 1000\,ml$$

$$\boxed{1\,dm^3 = 1\text{ litre}}$$

100cm = 1m

1cm = 0.01m

$$1m = 10\,dm$$

$$\boxed{1\,m^3 = 1000\,dm^3}$$

$$0.001\,m^3 = 1\text{ litre}.$$

$$\frac{dr}{dt} = \frac{2000}{\pi} \times \frac{0.001\,m^3}{m \times 0.01m} \times \frac{1}{hr}$$

$$= \frac{2000}{\pi} \times 0.1 \frac{m}{hr}$$

$$= \frac{200}{\pi} \, m/hr.$$

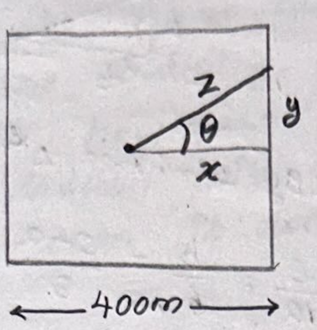
$$= 63.6619 \, m/hr.$$

Planning a prison break

Your prison's yard is a square with 400 m sides. At a centre that is a large spotlight that casts a beam which makes a bright spot on the walls of the yard, and rotates at 2 revolutions per minute. For simplicity we will treat the beam as a straight line.

Your escape plan requires you to outrun the brightspot along the wall. So you need to figure out the fastest speed at which it travels.

By symmetry, we only need to think about how the spot travels along just one of the walls.



sol^n:

Rate given in this problem is

∴ 2 revolutions per minute

∴ Rotation speed of the spot light, so

$$\frac{d\theta}{dt} = 2 \text{ rev/min}$$

we need to find $\frac{dy}{dt} = ?$

$$\frac{rev}{min} \Rightarrow \frac{rad}{min}$$

$$2 \frac{rev}{min} = 4\pi \frac{rad}{min}$$

∴ $\boxed{1 \text{ rev} = 2\pi}$

∴ $\boxed{x = 200m}$

$$\frac{y}{200} = \tan\theta$$

$$\therefore \frac{dy}{dt} = 200 \frac{d}{dt} \tan\theta.$$

$$\frac{dy}{dt} = 200 \cdot \sec^2\theta \cdot \frac{d\theta}{dt}.$$

$$\sec^2\theta = \frac{d}{d\theta} \tan\theta.$$

∴

$$\frac{dy}{dt} = 200 \sec^2\theta \cdot \frac{m}{rad} (4\pi) \frac{rad}{min}.$$

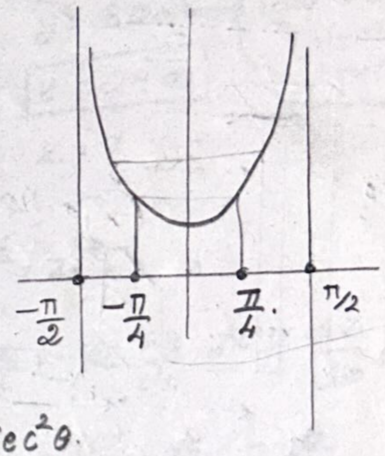
$\boxed{200m}$ Let $z = \tan\theta$

$$\frac{dz}{d\theta} = \sec^2\theta. \left(\frac{1}{rad}\right)$$

$$\frac{dy}{dt} = 800\pi \cdot \sec^2\theta \left(\frac{m}{min}\right)$$

max velocity:

$$0 = 800\pi \sec^2\theta \text{ m/min}$$



$$0 = \sec^2\theta.$$

At $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ (Interval)

$\sin 90° = 1$

$\cos 90° = 0$

$\therefore \theta = \pm \pi/4$ (max)

$$\therefore \frac{dy}{dt} = 800\pi \sec^2\left(\frac{\pi}{4}\right)$$

$$= 800 \times 2\pi$$

$$= 1600\pi \text{ m/min.}$$

$$= \frac{1600\pi}{60} \text{ m/s}$$

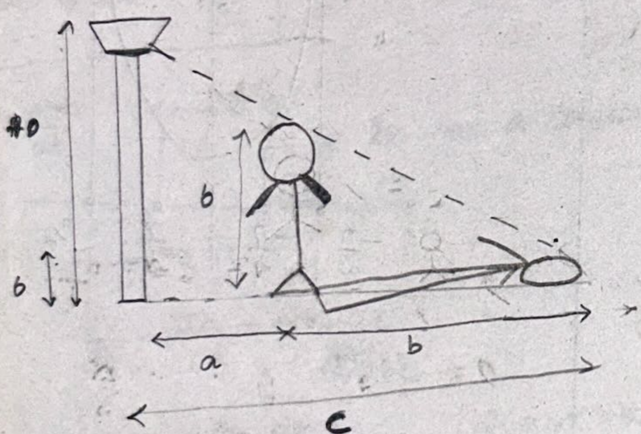
$$= 83.775 \text{ m/s.}$$

$$\boxed{\frac{dy}{dt} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dt}}$$

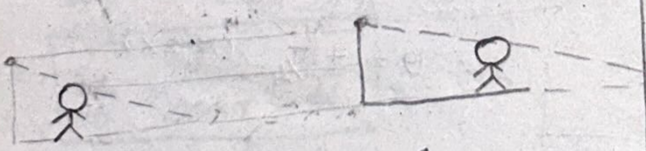
So the escape plan is not going to work.

A 6 foot tall man is walking down the street. Directly behind him is a 10-foot tall lamp post that casts a shadow onto the street. You are stationary, 40 feet from the post. When the tip of the shadow passes you, the tip is moving at 3 feet per second. We wish to determine how fast the man is moving.

Soln:



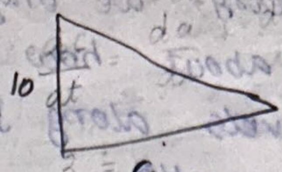
The tip is moving at 3 feet/sec

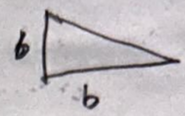


∴ It gives $\frac{dc}{dt}$.

$$\frac{dc}{dt} = 3 \text{ feet/sec}$$

How fast the man is moving $\frac{da}{dt}$.





By similar Δ^{le}.

$$\frac{c}{10} = \frac{b}{6} = \frac{c-a}{6}$$

$$\frac{6c}{10} = c-a$$

$$10a = 4c$$

$$a = \frac{2c}{5}$$

$$\frac{da}{dt} = \frac{2}{5}\frac{dc}{dt}$$

∵ 3 feet per second

$$\frac{da}{dt} = \frac{6}{5}$$

$= 1.2$ feet per second

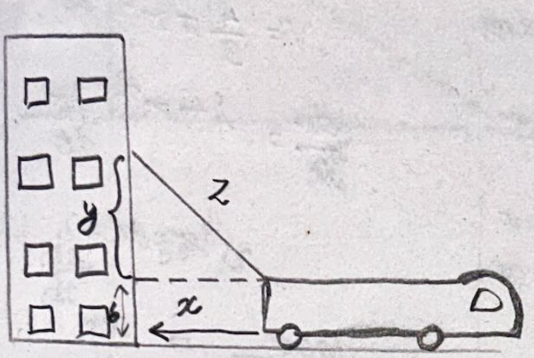
(we don't need the info about how far you are from the post - this result is true regardless of the number)

Fire truck

A fire truck is backing up to a building at 0.5 feet/second. It has its ladder against the side of the building, and the ladder is being lengthened at a rate of 0.2 feet per second. The ladder starts from the back of the truck at a height of 6 feet.

The top of the ladder is going to hit a gargoyle 30 feet up on the side of the building. When the truck is 7 feet away from the building. We want to know how fast the top of the ladder will be moving when it does that.

Solu:



$\frac{dy}{dt} \rightarrow$ we need to find

$\frac{dz}{dt} \rightarrow 0.2$ feet/sec (Ladder height ↑)

x measures the distance from the truck to the building, but is not true that

$$\frac{dx}{dt} = 0.5.$$

∴ The truck is backing up.

$$\therefore \boxed{\frac{dx}{dt} = -0.5}$$

($\frac{dx}{dt} \rightarrow$ should be (−)ve)

↗ human or animal face or figure projecting from the gutter of a building, typically acting as a spout to carry water clear of a wall. (உழகால்)

(Convey water from the roof and away from a side wall of the building)

Solu:

we need to find $\frac{dy}{dt}$ when height of building 30 feet & truck 7 feet away.

$\therefore y = 30 - 6$ — Ladder is 6m above truck

$$\boxed{y = 24\,m}$$

$$x = 7$$

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

$$7\frac{dx}{dt} + 24\frac{dy}{dt} = z\frac{dz}{dt}$$

when $y = 24$, $x = 7$

$$(24)^2 + (7)^2 = z^2$$

$$\boxed{z = 25}$$

$$\frac{dy}{dt} = \frac{25(0.2) - 7(-0.5)}{24}$$

$$= 0.354 \text{ feet/sec.}$$

Here $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
$\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}$ } are not applicable here.

Iceberg cone

we want to think about the volume of an iceberg, and we've decided to model it as a cone. Note that the volume of a cone is gn by
$V = \frac{1}{3}\pi r^2 h$, where
r - radius of the base
h - height.

∴ Right now, the radius is 300meters & the height is 60metres. we have determined that the height is ↓ by 0.10 metres per day, and the radius ↓ by 0.50 meters per day. we want to determine the rate of change of the volume.

when implicitly diff $V = \frac{1}{3}\pi r^2 h$ w.r.to the rh side of t. what rules we need to use

Solu:
$$\boxed{\frac{dh}{dt} = -0.10} \text{ m/day} \qquad \frac{dr}{dt} = -0.50 \text{ m/day}$$

$$\therefore V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$$

$r = 300m$, $h = 60m$

$$\frac{dV}{dt} = \frac{1}{3}\pi(300)^2(-0.10)$$
$$\frac{2}{3}\pi(300)(60)(0.50)$$

$$= -9000\pi \qquad = -28274.3 \text{ m}^3/\text{day}$$

"product rule"

"Also we can solve it by chain rule.

Chain rule can ... when diff r^2 with respect to t.

Filling a baloon

Air is being blown into a spherical baloon at a rate of 1000 cm³/s. How fast is the radius growing when r= 8cm? what about S.A

Solu:
$$\boxed{\frac{dV}{dt} = 1000 \text{ cm}^3/\text{s}}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$(r = 8cm)$

$$\frac{1000}{4\pi \times 64} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = 1.243 \frac{cm}{s}$$

$A = 4\pi r^2$ (Surface area)

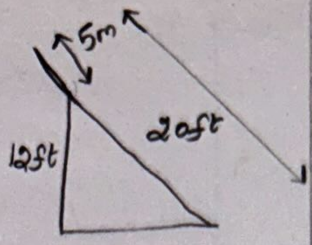
$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \qquad (r=8)$$

$$= 64\pi(1.24339799\text{3}$$

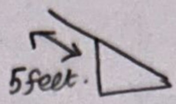
$$\boxed{\frac{dA}{dt} = 250 \text{ cm}^2/\text{s}}$$

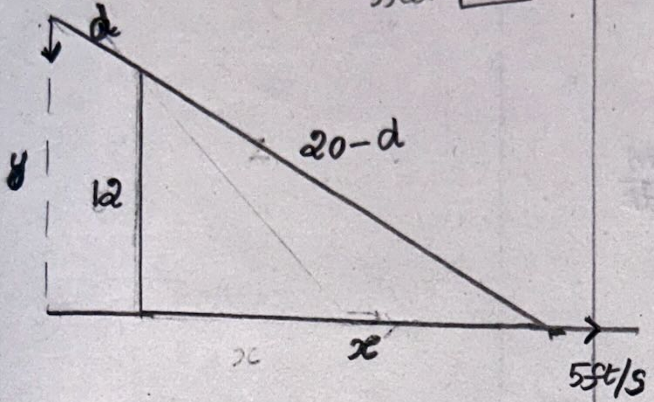
A 20ft ladder leans over a 12 ft wall so that 5 ft project over the wall. The bottom of the ladder is pulled away from the wall at 5 ft/s. How quickly is the top of the ladder approaching the ground?

Solu:



5ft → project over the wall.





$\frac{dx}{dt} = 5ft/s$

Initially,

$15^2 - 12^2 = x^2$

$x^2 = 81$

$\boxed{x = 9}$

Similar Δles:

By Pythagoras:

$144 + x^2 = (20-d)^2$

$144 + x^2 = 400 + d^2 - 40d$

$x^2 = 256 + d^2 - 40d.$

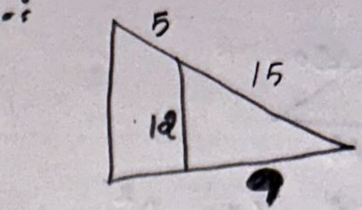
$\frac{20}{y} = \frac{20-d}{12}$

$\frac{20}{20-d} = \frac{y}{12}$

At that moment

$d = 5ft, \ \frac{dx}{dt} = 5ft/s$

$\therefore$



$x^2 + 12^2 = 15^2$

$x^2 = 225 - 144$

$\boxed{x = 9}$

$225 - 144 = 81$

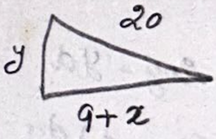
$\frac{20}{15} = \frac{y}{12}$

$y = \frac{20 \times 12}{15}$

$= \frac{20 \times 4}{5}$

$\boxed{y = 16}$

$\therefore$



$(9+x_1)^2 + y^2 = 20^2$

$81 + x_1^2 + 2 \times 9 \times x_1 + y^2 = 400$

$x_1^2 + 18x_1 = 400 - 256 - 81$

$x_1^2 + 18x_1 = 63$

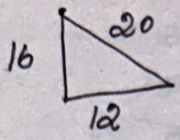
we need to find how fast the top of the ladder approaching the ground. $\left(-\frac{dy}{dt}\right)$

$x_1^2 + 18x_1 - 63 = 0$

$x_1 = 3, -21$

$\boxed{x_1 = 3}$

Totally



$$x^2 + 12^2 = (20-d)^2$$

$$2x \cdot \frac{dx}{dt} + 0 = 2(20-d)\left(-\frac{dd}{dt}\right)$$

$$2x \frac{dx}{dt} = (40-2d)\frac{-dd}{dt}$$

$$(2\times 9)(5) = (40-2(5)) - \frac{dd}{dt}$$

$$-\frac{dd}{dt} = \frac{90}{30}$$

$$\boxed{\frac{dd}{dt} = -3}$$

Also

$$\frac{20}{20-d} = \frac{y}{12}.$$

$$240 = 20y - yd.$$

$$0 = (20-d)\frac{dy}{dt} - y.\frac{dd}{dt}$$

$$y.\frac{dd}{dt} = (20-5)\frac{dy}{dt}$$

$$-3y = 15\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{-3y}{15}$$

$$= -3.2$$

$\therefore$ Top is falling down.