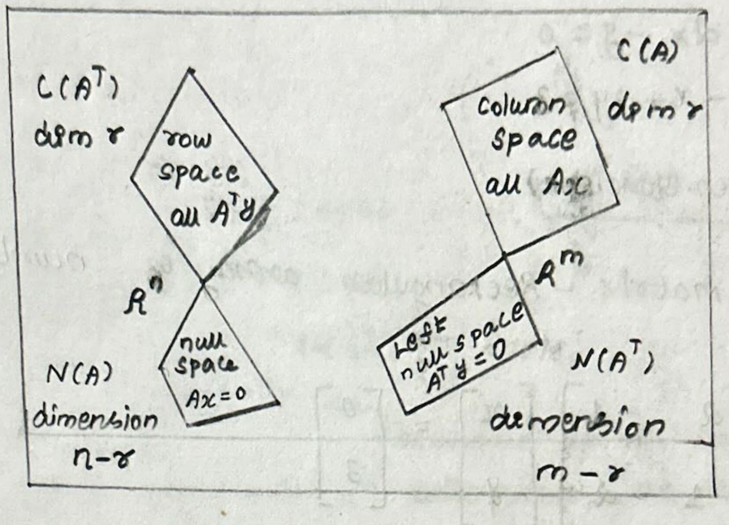
Vector spaces

Four fundamental subspaces:



Big picture of linear algebra.

* Mathematics - A tool describing the world around us
* linear equations give some of the simplest descriptions.
* System of linear equations: combining such several descriptions.

Lecture-1 Geometry of linear equations

Major application of linear algebra: Solving linear equations. Three ways of thinking.

1) Row method - focuses on the individual equ
2) Column method - combining the columns
3) Matrix method - more compact & powerful ways of describing linear equ.

Geometry of linear equations

http://web.mit.edu/18.06

Echelon forms – practically never used

ocw.mit.edu/18.06

n-linear equations with n unknowns: (Fundamental problem)

$$2x - y = 0$$
$$-x + 2y = 3$$

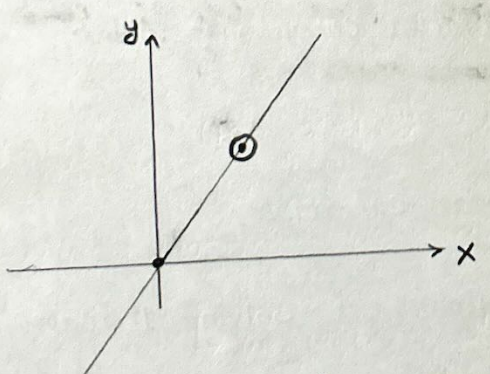
matrix (co-efficient)

matrix – Rectangular array of numbers.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \quad \underbrace{X \quad = \quad b}_{\rightarrow \text{vector}}$$

Row picture:



→ The solutions lies in a straight line → 'Linear'

That straight line is the solution.

All points satisfies equ ①

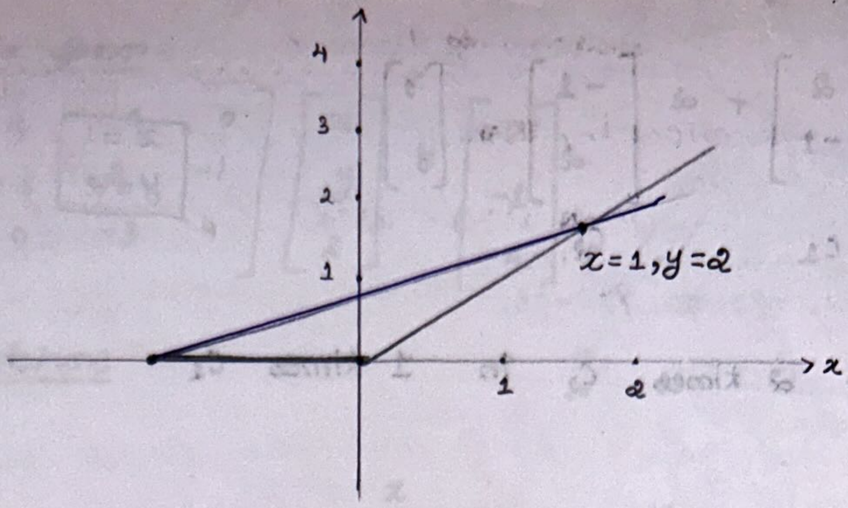
* (0,0) → satisfies (solves) that equ

$x=1, y=2$

$2(1) - 2 = 0$ → solves.

equ 1 $\quad 2x - y = 0$ → Goes through origin

equ 2 $\quad -x + 2y = 3$ → Not going through origin.



- Line 1 (equ 1)
- Line 2 (equ 2)

(0, 0)	-1, 1
(1, 2)	-3, 0
(2, 4)	

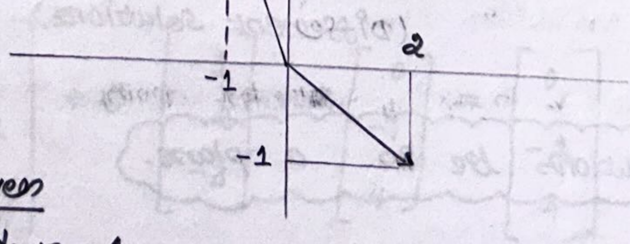
our Solution: $x=1, y=2$

(At some point, both equ will meet at a point)

Column picture:

$$x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

Linear combination of the two co.eff column matrices will give the column matrix: $\begin{bmatrix}0\\3\end{bmatrix}$.



First column over
2 down -1

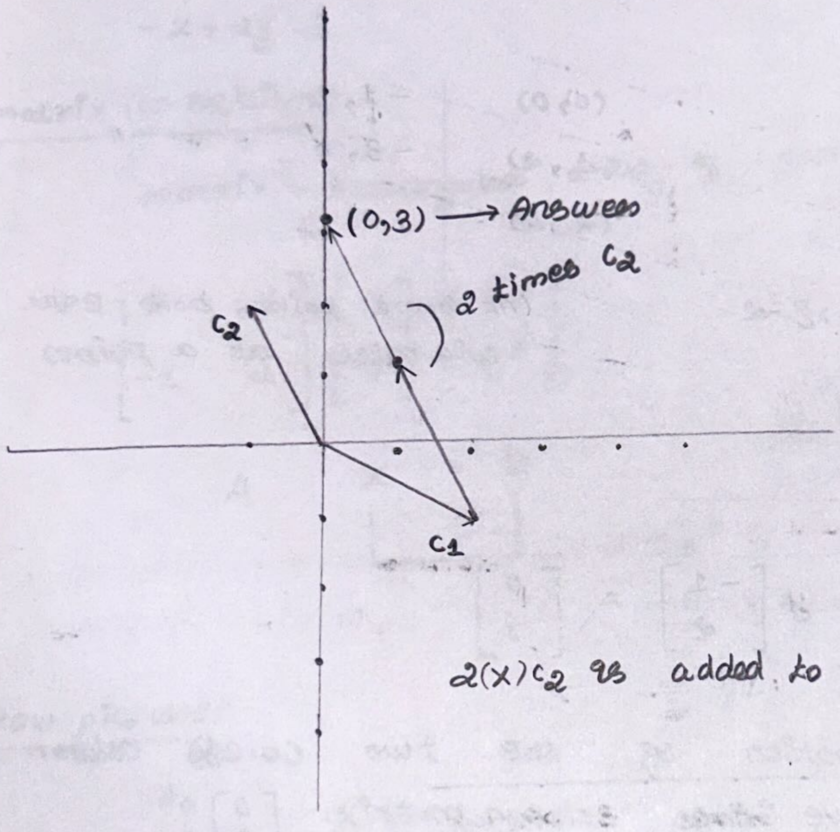
Right combination - To produce $\begin{bmatrix}0\\3\end{bmatrix}$.

$$1\begin{bmatrix}2\\-1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

c_1 c_2.

$\boxed{\begin{matrix}x=1\\y=2\end{matrix}}$

Adding 2 times c_2 in 1 times c_1



$2(\times)c_2$ is added to c_1

(for $x=1$, $y=2$)

Lot of combinations are available for different x and y.

↳ we will get different combinations
(Different solutions)

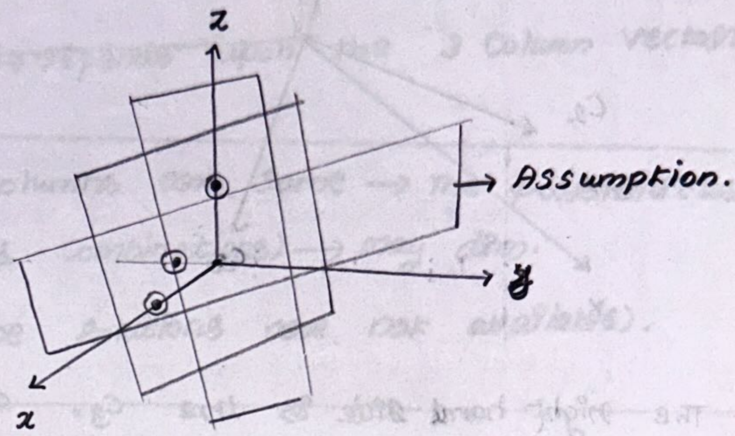
All the solutions lie in a plane.

$$\left.\begin{matrix}2x-y=0\\-x+2y-z=-1\\-3y+4z=4\end{matrix}\right\}\ 3\ \text{col (3 dimensions).}$$

Matrix form:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row picture



eqn:2

$z = 1$, $\boxed{x \text{ and } y = 0}$

$x = 1$, z and $y = 0$

$x = 0$, $z = 0$, $y = -\frac{1}{2}$.

Solutions (combination) will be a plane.

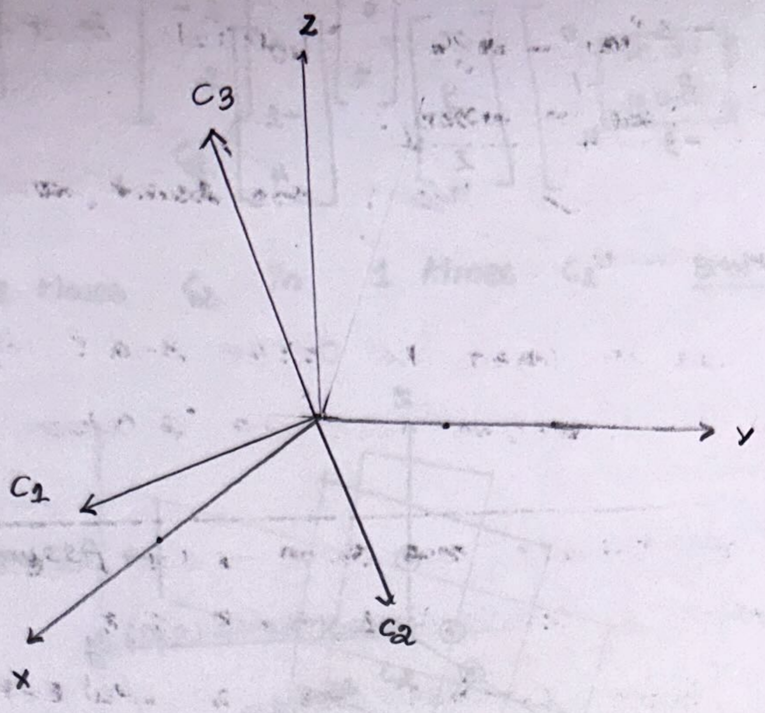
So three eqn — Gives 3 planes.

In 3d, 4d → less clear.

Column picture:

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Linear combination.



The right hand side is the c_3. one possible solution is elimination of c_1 and c_2.

$x=0, y=0, z=1$

If

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}1\\1\\-3\end{bmatrix}$$

↓ one of this ↓ one of this ↓ None of this → gives this

$x=1, y=1, z=0$

'3 planes meeting at this point'

Solving $AX=b$ for every right hand side b?

↓

Algebraic question.

In linear combination: Do the linear combination of the columns fill 3-D space'

For this particular matrix $A \rightarrow$ yes.

'Non - singular, invertible matrix'

when this could be wrong?

* If the columns lies in the same plane $\rightarrow$ Nothing new can happen.

Can we able to cover the entire 3-d? (Right handside)

↳ Depends upon the 3 column vectors.

If any two columns are same $\rightarrow$ The possibilities (permutations & combinations) $\rightarrow$ may dim.

(some solutions are not available).

Matrix form:

$$AX = b$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+10 \\ 1+6 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$AX \rightarrow$ Combination of A.

Linear independence:

$$AX = b.$$

* Can we solve for every possible vector b? the

* In other words, Do the linear combinations of the column vectors fill the x-y plane.

(Space in 3-d case)?

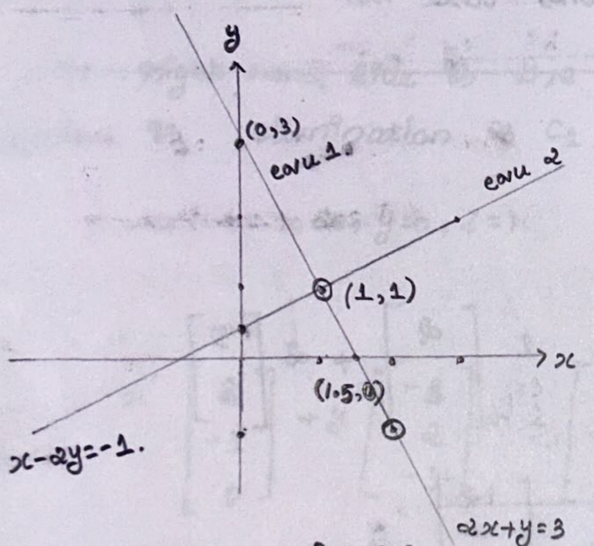
If the answer is no: Then A is a singular matrix. singular case it's column vectors are linearly dependent. The combinations don't fill the whole space.

Linear combination:

$$cv + dw = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

Solve $2x+y=3$, $x-2y=-1$ and findout row picture & column picture.

Solu:



equ 1

$x=1, y=1$

$x=2, y=-1$

eq 2

$x=1, y=1$

$x=3, y=2$

Row- picture.

$x = 2y-1$

$2(2y-1)+y=3$

$4y+y=5$

$5y=5$

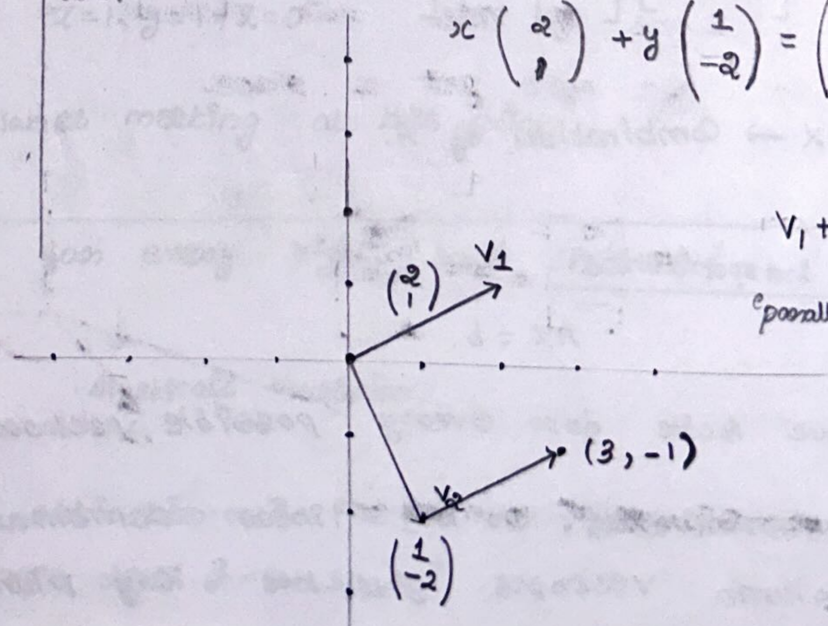
$y=1$

$x=2(1)-1$

$x=1$

Column picture

$$x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$$



$V_1 + V_2 = \begin{pmatrix}3\\-1\end{pmatrix}$

"parallelogram"

matrix representation:

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$A = [v_1 \;\; v_2]$

2x2 matrix.

Scalar eqn

$ax = b$

$\boxed{x = \frac{b}{a}}$

vector eqn

$x = A^{-1} b$

$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow$ Identity matrix.

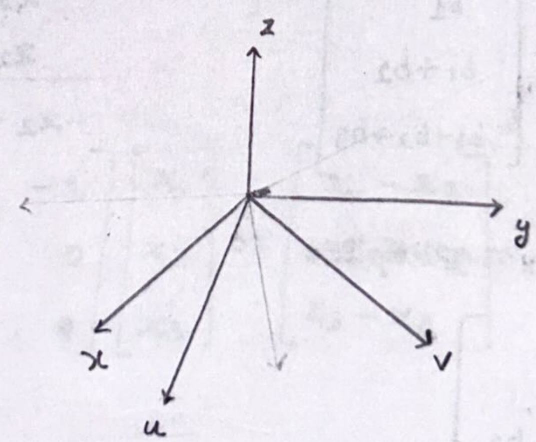
Lecture-2: An overview of key ideas

$u, v, w \rightarrow$ vectors

$x_1, x_2, x_3 \rightarrow$ scalars

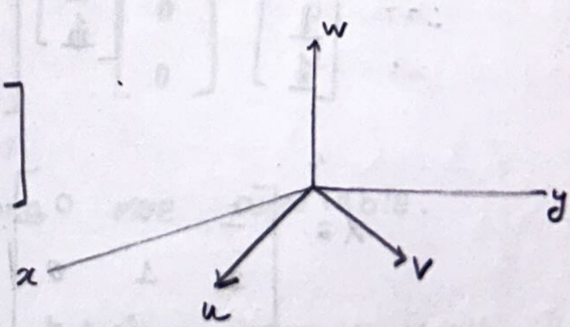
$x_1 u + x_2 v + x_3 w = b$

(Linear combination)



All combinations of this two vectors u and v, we will get a plane.

$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \; v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \; w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



vectors - Columns of a matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad x \quad = \quad b$$

$A \rightarrow$ First difference matrix $\begin{pmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{pmatrix}$

$$\begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4-1 \\ 9-4 \end{bmatrix}$$

Forward: $Ax = b$

Reverse: $x = A^{-1} b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$

$-x_1 + x_2 = b_2$

$x_2 = b_2 + x_1$

$x_2 = b_2 + b_1$

$-x_2 + x_3 = b_3$

$x_3 = b_3 + x_2$

$\Rightarrow x_3 = b_1 + b_2 + b_3$

columns of the matrix multiplies to give $\begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$

$$\therefore \; b_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} b_3 =$$

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \text{Inverse matrix.}$$

$Ax = b \longrightarrow$ difference $\begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$

$x = A^{-1}b \longrightarrow$ sum $\begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$

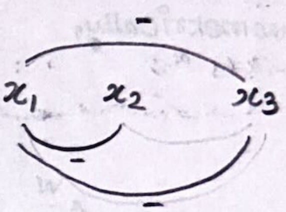
If $b_1 = 1, b_2 = 3, b_3 = 5$ then $x_1 = 1, x_2 = 4, x_3 = 9$.

Always has only one solution (Invertible solution)

'perfect solution - Transform from one to another'

Example:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$



$Cx = b$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ -x_1 + x_2 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

we will have

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A number of solutions are available.

$x_1 = x_2 = x_3 = 0$
$x_1 = x_2 = x_3 = 1$) Line of solutions are available.

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$b_1 = b_2 = b_3 = 0$ when $\left.\begin{matrix} x_1 = x_3 \\ x_2 = x_1 \\ x_3 = x_2 \end{matrix}\right\} x_1 = x_2 = x_3$

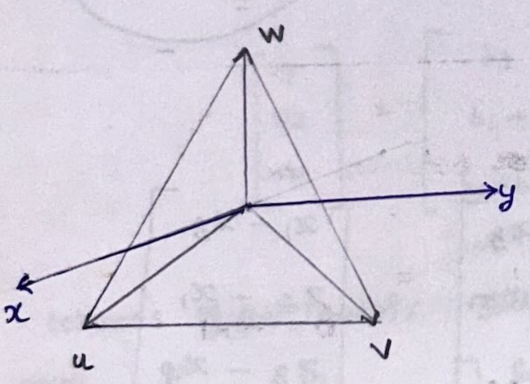
'A number of solutions are available'

Physical meaning: 'Net force = 0' when 'resultant force = 0'

$\therefore \quad CX = 0$

we can't have C^{-1}

Geometrically,



Covers the entire space

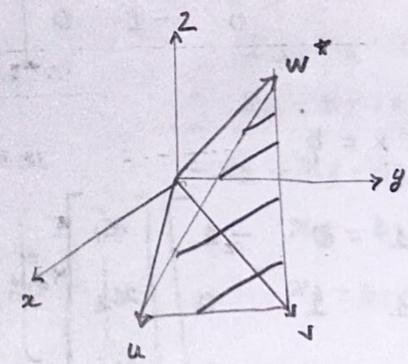
'Independent'

'Basis : For a 3-d space'

Their combination gives the entire space.

'Invertible' → matrix with basis sitting in it's column.

[In this case - the entire space is a subspace]



'Not covering all the space'

'Dependent.

↓

Their combinations are in a particular plane.

[subspace].

A plane is a subspace

All combinations of cx → covers the space (Independent)

In our example,

$$b_1 = b_2 = b_3 = 0$$

Vector space:

Bunch of vectors – (we take combinations)

Subspace:

0d – point – origin (smallest subspace)

1d – line

2d – plane

3d – whole space

$$A^T A = \text{Square matrix}$$

$$(7\times3)(3\times7) = (7\times7)$$

$$(5\times2)(2\times5) = (5\times5)$$

Recitation

A is a matrix the complete solution to

$$Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix} \text{ is } \quad x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$b = 4\times1$

What can you say about columns of A?

Soln:

$$(4\times3)(3\times1) = (4\times1)$$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \rightarrow \text{vectors}$$

we are gn with

$$x = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{x_p} + c \underbrace{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}_{x_s}$$

$$A(x_p + c x_s) = b \quad [\text{For any numbers } c]$$

If $c = 0$

$Ax_p = b$

If $c = 1$

$$Ax_p + Ax_s = b$$

$$b + Ax_s = b$$

$$\boxed{Ax_s = 0}$$

$\therefore \boxed{Ax_p = b}$

$$Ax_p = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{c_2 + c_3 = b}$$

$$-c_2 = b$$

$$Ax_s = 0$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0$$

$$2c_2 + c_3 = 0$$

$$\boxed{c_3 = -2c_2}$$

$$\boxed{c_3 = -2b}$$

$$\boxed{x = x_p + c\,x_s \rightarrow \text{special solution}}$$

$\hookrightarrow$ particular solution

$$x = x_p + c\,x_s$$

$\hookrightarrow Ax_s = 0.$

$\dim(N(A)) = 1$ [Dimension of null space]

$\text{rank}(A) = 3 - 1 = 2$

$\rightarrow$ c_1 not a multiple of b. [$\therefore c_3 = -2c_2$]

[we don't know about c_1]

Lecture:3: Elimination with matrices

* Elimination $\nearrow$ success, $\searrow$ failure.) way every software package solves equation.

* Back-Substitution
* Elimination matrices
* Multiplication matrices

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 2$$

$$\boxed{Ax = b}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

Purpose: To eliminate x.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$1 \rightarrow$ pivot

$\rightarrow$ pivot row

3 times pivot row will knock out row 2 (x) variable

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{pmatrix} \leftarrow R_2 \rightarrow R_2 - 3R_1$$

Already zero. $\leftarrow$ 0

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

$2 \rightarrow$ 2nd pivot.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$5 \rightarrow$ 3rd pivot.

$U \rightarrow$ upper triangular matrix.

Pivot's can't be zero

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix} \rightarrow \text{upper triangular matrix.}$$

Failures:

1) very first number = 0. [switch rows]

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix} R_2 \rightarrow R_2 - 3R_1 \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 4 & 1 \end{pmatrix}$$

↳ 2nd pivot zero (need to switch)

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{pmatrix} \rightarrow R_3 \rightarrow R_3 - 2R_2 \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Temporary failure: Rectified by Row switching by (Below rows)

Permanent failure: (Like above) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

↓ No further rows

• Not invertible'

Back Substitution:

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{pmatrix} \rightarrow \text{Augmented matrix}$$

(Tack something on)

A b (extra column)

$$R_2 \rightarrow R_2 - 3R_1 \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \begin{array}{c} \begin{array}{cc} u & \quad c \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{pmatrix} \end{array}$$

u → what happens to A

c → what happens to the extra row.

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10$$

$z = -2$

$2y = 6 + 2(-2)$

$y = \frac{2}{2} \Rightarrow y = 1$

$x = 2 - z - 2y$

$= 2 + 2 - 2(1)$

$= 2$

$\boxed{x = 2}$

Back Substitution: Solving eqn in reverse order as the system is triangular.

Matrices

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$$\begin{array}{c} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \\ c_1 \quad c_2 \quad c_3 \end{array} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 3c_1 + 4c_2 + 5c_3$$

(or)

$$(1 \; 2 \; 7) \begin{pmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{pmatrix}$$

$R_1 + 2R_2 + 7R_3$

∴ matrix times column = column

Row times matrix = Row

---

matrix: subtract 3 times R_1 from R_2

Reflecting

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

(Find)

(E_{21})

Reflecting

↳ As a result of $R_2 \to R_2 - 3R_1$

→ matrix needed to do $R_2 \to R_2 - 3R_1$ by multiplication.

Step: 2: Subtract $2R_2$ from R_3

$$\begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

AS R_1 & R_2 are not changed

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$\hookrightarrow E_{32} \rightarrow$ Fix E_{32} element

Upper triangular matrix

$$E_{32}\,(E_{21}\,A) = u$$

---

what matrix get the job done from A to u:
(At a single step)

$\boxed{\text{matrix} = E_{32} \times E_{21}}$ $\therefore$ Associative law.

matrix $(x)\,A = u$

other type of Elementary matrix – Exchanges two rows
(permutation matrix)

* Exchange rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \rightarrow \text{Exchange rows}$$

Since we need to teach the procedure to the computer to do so.

Exchange columns:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Row operation: Left side to the matrix

Column operation: Right side to the matrix

Can't exchange the order of the matrices in multiplication:

"Commutative law fails" $\Rightarrow (A\times B) \neq (B\times A)$

Inverse matrix:

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

↳ what matrix undoes $R_2 \rightarrow R_2 + 3R_1$

No change in R_1 & R_3

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse matrix: undoes our operation

Elimination matrix → Eliminates using some operations

Inverse matrix → undoes the operation.

Recitation

$x - y - z + u = 0$

$2x + 2z = 8$

$-y - 2z = -8$

$3x - 3y - 2z + 4u = 7$

"method of elimination"

Solu:

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{pmatrix} \longrightarrow \text{upper triangular one.}$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & -1 & -2 & 0 & -8 \\ 3 & -3 & -2 & 4 & 7 \end{pmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 1 & 7 \end{pmatrix} \quad \begin{aligned} R_4 &\to R_4 - 3R_1 \\ R_3 &\to R_3 + \tfrac{1}{2} R_2 \end{aligned}$$

pivot can't be zero.

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -2 & 8 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & -1 & -4 \end{pmatrix} \quad R_3 \rightleftharpoons R_4$$

(switched)

'upper triangular equation'

$-u = -4$	$z + u = 7$	$2y + 4z - 2u = 8$	$x = y + z - u$
$u = 4$	$z = 7 - 4$	$y = \dfrac{8 - 4(3) + 2(4)}{2}$	$x = 2 + 3 - 4$
	$z = 3$	$y = 2$	$x = 1$

$u = 4,\ x = 1,\ y = 2,\ z = 3$

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lecture: 4: Multiplication & inverse matrices

* why Gauss-Jordan method works?

Matrix multiplication (4 ways!) $\longrightarrow$ lecture 3.

Inverse of A, AB, A^T

Gauss-Jordan, find A^{-1}

---

method: 1

$$[\ \ ]\,[\ \ ] = [\ \ ]$$

$$A \times B = C$$

c_{34} = (row 3 of A) . (column 4 of B)

$$c_{34} = a_{31} b_{14} + a_{32} b_{24} + \cdots$$

$$\boxed{c_{34} = \sum_{k=1}^{n} a_{3k} \cdot b_{k4}}$$

Rule: $\underset{(m \times n)}{A} \times \underset{(n \times p)}{B} = \underset{(m \times p)}{C}$

(No. of column of A = No. of Row of B)

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columns of C are the combinations of columns of A. [column × column]

Rows of C are the combination of rows of B [Row × Row]

method: 2: $C_{(\text{column } 1)} = A \times B_{(\text{column } 1)}$

$$\underset{A_{m\times n}}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} \underset{B_{n\times p}}{\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}} = \underset{C_{m\times p}}{\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}}$$

$$A_{m\times n} \underbrace{\begin{pmatrix} | \\ c_1 \\ | \end{pmatrix}}_{\text{column 1 of matrix B}} = \underbrace{\begin{pmatrix} | \\ c_1 \\ | \end{pmatrix}}_{\text{column 1 of matrix C.}}$$

"Matrix multiplied by a vector gives vector"

Columns of C are the combination of columns of A.

---

method: 3 (By rows)

A row of A takes the combination of rows of B to get a row of C.

"Rows of C is the combination of rows of B"

---

method: 4: (Regular column × row way).

Column of A × Row of B

$$\begin{pmatrix} | \\ c_1 \\ | \end{pmatrix} \times (-R_1-) =$$

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \underset{(m\times 1)}{}(1\ 6) = \begin{pmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{pmatrix}$$

what a Column × Row yields

$(3\times1)\ (1\times2) = (3\times2)$

Fourth way: $AB = (\text{Sum of Columns of } A) \times (\text{Rows of } B)$

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{pmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{pmatrix}$$

↳ special matrix

Row space: combination of all row is just a line.

Column space: Also a line through $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ vector.

multiplication by blocks in matrix:

$$\underset{A}{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]} \underset{B}{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]} = \left[\begin{array}{c|c} C_1 & C_2 \\ \hline C_3 & C_4 \end{array}\right]$$

$C_1 = A_1 B_1 + A_2 B_3$
$C_2 = A_1 B_2 + A_2 B_4$
$C_3 = A_3 B_1 + A_4 B_3$
$C_4 = A_3 B_2 + A_4 B_4$

→ Just like ordinary Row × column multiplication

Inverses

Invertible : $\boxed{A^{-1}A = I = AA^{-1}}$

(If inverse exist)

$A^{-1}A \rightarrow$ Left inverse $\times A$

$A A^{-1} \rightarrow A \times$ Right inverse.

---

* Invertible (or) non-singular matrices
* singular (or) non invertible matrices

<u>Singular :</u>

$$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

<u>Determinant</u> : $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6-6 = 0$ (Singular)

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} & ? & \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Not exist

(Both vectors same on the same line)

↳ So we can't get $(1, 0)$ (or) $(0, 1)$

Matrix has no inverse : If $\boxed{Ax = 0}$
(Square matrix)

$$Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$Ax = 0$
↓
we can find a vector.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

---

* $A^{-1} \times 0 \neq A$ (Never) for a non-zero matrix A.

(So No inverse).

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}, \quad |A| = 7-6 = 1.$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \quad A^{-1} = I$$

$$\left(A \begin{bmatrix} a \\ b \end{bmatrix} \cup A \begin{bmatrix} c \\ d \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\hookrightarrow c_1$ $\hookrightarrow c_2$ of answers

Gauss – Jordan idea: (Solve two equations at once):

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

'Augmented'

Elimination upwards

In normal elimination we quit at this step but Jordan says keep going.

(upwards)

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] \quad \underset{I \qquad A^{-1}}{}$$

verify $\longrightarrow$

$$\begin{array}{cc} A & A^{-1} \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & (7-6) & (21-21) \\ 2 & 7 & (2-2) & (-6+7) \end{array}\right]$$

Gauss – Jordan idea.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right]$$

$$A \qquad A^{-1}$$

$A^{-1}A = I$, $AI = A$ $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&2\\3&4\end{pmatrix}$

Recitation

Find the conditions on a and b that make the matrix A invertible, & find A^{-1} when it exists.

Solu:

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$$

A is not invertible if $a = 0$
$a = b$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & \frac{b}{a} & \frac{b}{a} & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$R_1 \to R_1 - \frac{b}{a} R_2$

$R_1 \to R_1 - \frac{b}{a} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$$A^{-1} = \frac{1}{a-b}\begin{bmatrix} 1 & 0 & -b/a \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$a \neq 0, \; a-b \neq 0$

Matrix multiplication by columns

$$\begin{bmatrix} 3 & 1 & 4 \\ 8 & 2 & 5 \\ 1 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+y+4z \\ 8x+2y+5z \\ x+6y+7z \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}[x] + \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}[y] + \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}[z] = \begin{bmatrix} 3x+y+4z \\ 8x+2y+5z \\ x+6y+7z \end{bmatrix}$$

Column times row

$$\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}\begin{bmatrix} 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ 9 & 15 & 6 \\ 18 & 30 & 12 \end{bmatrix}$$ Rank = 1

$(3\times1)\,(1\times3) = (3\times3)$

Every row is a combination of other row.

columns times rows

$$\begin{bmatrix} 3 & 1 & 4 \\ 8 & 2 & 5 \\ 1 & 6 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 17 & 11 \\ 27 & 37 & 23 \\ 34 & 34 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \end{bmatrix}$$

$A = LU$ (without row exchange)

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 7 & 3 \\ 4 & 8 & 9 \end{bmatrix}, \quad LU = \begin{bmatrix} 1 & & \\ NAN & & \\ NAN & & \end{bmatrix}\begin{bmatrix} 2 & 1 & 5 \\ 0 & 4 & -12 \\ 0 & 6 & -1 \end{bmatrix}$$

$l_{11} = 1, \; l_{21} = 3, \; l_{31} = 2$

$$\begin{bmatrix} 2 & 1 & 5 \\ 0 & 4 & -12 \\ 0 & 6 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 1.5R_2} \begin{bmatrix} 2 & 1 & 5 \\ 0 & 4 & -12 \\ 0 & 0 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 0 & 4 & -12 \\ 0 & 0 & 17 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 7 & 3 \\ 4 & 8 & 9 \end{bmatrix}$$

$A = LU \rightarrow$ fails

'upper left triangular matrix $\rightarrow A = LU$ fails'

<u>PA = LU (works!)</u>

<u>Permutation P does row exchange.</u>

<u>First pivot: zero not allowed</u>

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 7 \\ 4 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}} = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 1 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -0.5 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 4 & 3 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 5.5 \end{bmatrix}}$$

$PA = LU$ (3×3 permutation)

$$\underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{P_{21}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{P_{32}\,P_{21}}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

P_{31} P_{32} P_{21} P_{32}

(Different permutations : we can use them as per our need) → (eg: shift 1 and 3)

---

lecture:5 <u>Factorization into $A=LU$</u>

"How hard a computer will work to invert a very large matrix"

<u>Lecture-4</u>

<u>Inverse of AB, A^T.</u>

Both A and B are invertible.

$(AB)(B^{-1}A^{-1}) = I$

$(ABB^{-1}A^{-1}) = I$

$(AIA^{-1}) = I$

$(AA^{-1}) = I$

$\boxed{I = I}$

$(B^{-1}A^{-1})(AB) = I$

$(B^{-1} I B) = I$

$\boxed{B^{-1}B = I}$

$\boxed{I = I}$

$AA^{-1} = I$

$\boxed{(A^{-1})^T (A^T) = I}$

$I^T = I$

$$(A^T)^{-1} = (A^{-1})^T$$

(No row exchanges)

$A = LU$ $U \rightarrow$ upper triangular matrix of A.

'Connecting A and U by a matrix L'

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}^{E_{21}} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$E_{21} \rightarrow$ making that element zero for U

Reverse process:

$A = L U$

$\hookrightarrow$ Inverse of E_{21}

- $$\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$A \quad = \quad L \quad U$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

- 'often we need to split out pivot alone'

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}}$$

Balanced.

$$\underbrace{E_{32}\ E_{31}\ E_{21}}_{\text{Elimination matrices}}\ A = U$$ [No row exchanges]

$$A = (E_{21})^{-1} (E_{31})^{-1} (E_{32})^{-1}\ U$$

$A = L U$ $L \rightarrow$ product of inverses.

E_{32} (suppose) E_{21}

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-5&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\10&-5&1\end{bmatrix}=E$$

$EA=U$

(Inverse - reverse order)

$(E_{21})^{-1}$ $(E_{32})^{-1}$

$$\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&5&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\0&5&1\end{bmatrix}=L$$

$A=LU$

$A=LU$

If no row exchange → multipliers go directly into L.

How expensive is elimination?

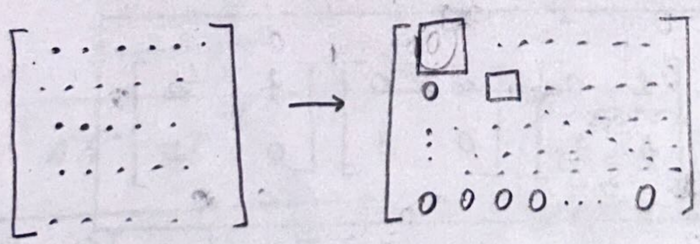
How many operations need to be done?

$n \times n$ matrix A.

Say: $n=100$:

100 elements in each row

$R_2 \to R_2 - R_1$ (has 100 subtraction operations).



Is it n (or) n^2 (or) n^3 (or) $n!$ (or) others?

'How step complexity tend to increase?'

* 100^2 → First step (tackle remaining 99 rows with row 1)
* 99^2 → Second step (Tackle rem 98 rows with R_2)

$$\therefore\ n^2+(n-1)^2+\dots+(1)^2 = n^2+\dots+1^2$$