$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

$A \to$ diagonal $\to A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

$\det A^2 = (\det A)(\det A) = (\det A)^2$

$\det 2A = \det(2I) \cdot \det(A)$ $\quad \therefore \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & \cdots & \\ 0 & \cdots & \cdots & 2 \end{vmatrix}$

$= 2^n \det(A)$.

$2 \times 2 \times 2 \begin{vmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & \cdots & 1 \end{vmatrix}$

$\hookrightarrow$ Each from one row

Like volume: double the side.

3-d $\to$ mult $2^3 \times$ volume

n-d $\to 2^n \times$ Volume.

$\det A^{-1} = \frac{1}{\det A}$ when $\det A = 0$

$\det A^{-1} \to$ doesn't exist.

---

10) $\det A^T = \det A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

- If column is all zero $\to \det A = 0$

Exchanging columns $\to$ changes sign odd $\to$ -ve

Even $\to$ +ve.

Proof:

$|A^T| = |A|$

$\downarrow \qquad \downarrow$

$|U^T L^T| = |LU|$

"Lower triangular matrix"

$|L| = 1$.

$|L^T| \to$ Also a lower triangular.

Solu:

$|U^T||L^T| = |L||U|$

$|U^T|(1) = 1|U|$

$|U^T| = |U|$

$\boxed{|U| = |A|}$

Loose end: If we got $\mapsto$ matrices row exchanges after 7 and 10 row exchanges resp $\to$

Is it means determinants equal to its negative?

↳ No (we need to have a proof).

Fact: Every permutations are either odd or even:
I could get the permutation with seven row exchanges, or 21 or twenty three or 101 - But absolutely not with even numbers of operations. → Odd one.

Even one → with only even no. of operations.

Recitation.

Find det

$$A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix}; \quad D = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

Solu:

$$A = \begin{bmatrix} 100+1 & 200+1 & 300+1 \\ 100+2 & 200+2 & 300+2 \\ 100+3 & 200+3 & 300+3 \end{bmatrix}$$

$$= \begin{vmatrix} 100 & 200 & 300 \\ 100 & 200 & 300 \\ 100 & 200 & 300 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= 0 + 0 = 0$$

$$\begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$B = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b+a)(b-a) \\ 0 & c-a & (c+a)(c-a) \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

'upper triangular'

$$= (b-a)(c-a)(c-b)$$

③ $C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (1 \ -4 \ 5) = \begin{pmatrix} 1 & -4 & 5 \\ 2 & -8 & 10 \\ 3 & -12 & 15 \end{pmatrix} \rightarrow$ Linearly dependent

(Rank 1)

det $C = 0$

④ $\begin{vmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix}$ = skew matrix

$D^T = -D$

$|D^T| = |-D|$

$= (-1)^n |D|$

$= (-1)^3 |D| = -|D|$

$\therefore |D^T| = |D| = -|D|$

only true when $|D| = 0$

$|D| = 0$

Is $|D| = 0$ for all skew matrix = ?

Not necessarily $\rightarrow$ when n = even

$|D^T| = D$

when n = odd

$|D^T| = |D| = -|D| = 0$

Lecture - 20 - Determinant formulas & cofactors

* Formula for det
* Tridiagonal matrices
* Cofactor matrices.

① det $I = 1$
② sign reverse with row exchange
③ det is linear in each row seperately.

---

① $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \rightarrow \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$

$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$

$= 0 + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + 0$

column of zero.

$= ad - \begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} \rightarrow$ Flip 3.

$= ad - bc$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & & \\ g & & \end{vmatrix} \begin{vmatrix} 0 & b & 0 \\ & & \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

we can split in to 3 ↓ That 3 → To 3 each → 9 → 27 (each 3)

'most of them are zero'

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{23} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$\left(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{11}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \right)$$

↳ To get a Identity matrix, flip R_2 & R_3

'survivor has one entry in each row & each column'

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

① $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ② $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ③ $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

④ $\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ⑤ $\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ⑥ $\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

⑦ $\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ⑧ $\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ⑨ $\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

①

$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \to 0$ $\quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \to 0$ $\quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \to 0$

②

$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \to 0$ $\quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \to 0$ $\quad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \to 1$

③

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} \to 0 \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} \to ② \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \to 0$$

④

$$\begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \to 0 \qquad \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \to ③ \qquad \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{23} \end{bmatrix} \to 0$$

⑤

$$\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

⑥ $\to 0$

⑦

$$\begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

⑧ $\to 0$

⑨

$$\begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix}$$

$$= (a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23})\, I$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \searrow \; + \qquad \swarrow \; - \; \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix}$$

For $n \times n$

$$\boxed{\det A = \sum \pm\, a_{1\alpha}\, a_{2\beta}\, a_{3\gamma} \cdots a_{n\omega}}$$

2 terms $\to 2\times 2$
6 terms $\to 3\times 3$
24 terms $\to 4\times 4$
$\Big\}$ $n!$

$(\alpha, \beta, \ldots \omega) \to$ permutations of $(1, 2, \ldots, n)$

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0 \to \text{singular.}$$

$(4, 3, 2, 1) \to +1$
$(3, 2, 1, 4) \to -1.$

4×4

$\searrow$ $(-)$ odd number of exchanges

$\swarrow$ $(+)$ even number of exchanges

Cofactor

Breaking up the big formula $n \times n$ det to determinants 1 smaller.

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

cofactors. $\begin{vmatrix} a_{22} & a_{23} \\ a_{33} & a_{32} \end{vmatrix}$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} \to +$$

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

Cofactor: $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \to -$

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} \to +$$

Cofactor of $a_{ij} = \pm \det \begin{pmatrix} n-1 \text{ matrix} \\ \text{with row } i \text{, column } j \text{ erased} \end{pmatrix}$

$\pm \to$ depending upon row exchanges.

$+ \to i+j \to$ even

$- \to i+j \to$ odd.

Pattern

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \to 3 \times 3$$

without sign: $\begin{bmatrix} a_{22} & a_{23} \\ a_{33} & a_{32} \end{bmatrix} \to$ minor.

Cofactor formula (Along row 1)

$$\det(A) = a_{11}(\text{cofactor}) + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

$$\boxed{\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}.}$$

2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$i+j \to$ odd

Applications:

'Determinants from smaller determinants'

Example: Tridiagonal matrix

$$A_4 = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1$, $|A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, $A_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

$= 1(0) - 1(1) + 0(1)$

$= \cancel{0}\,1.$

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$A_3 = -1(1) = -1.$

$\hookrightarrow i+j =$ odd

$$|A_4| = 1 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

we can use columns too. $\therefore \det(A^T) = \det(A)$

$$= 1|A_3| - 1(1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1|A_3| - 1|A_2| = -1 - 0 = -1$$

$$|A_n| = |A_{n-1}| - |A_{n-2}|$$

$$|A_4| = -1$$

$|A_5| = 0$, $|A_6| = 1$, $|A_7| = 1$. $\longrightarrow$ period $= 6$ (Repeats).

Recitation

Find the determinants of

$$A = \begin{bmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & x \end{bmatrix}, \quad B = \begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix}$$

1) Elimination 2) $\sum \pm a_{1\alpha} a_{2\beta} \dots a_{5\gamma}$ 3) By cofactor

solu:

Lower Δ^r matrix

$$\begin{bmatrix} x & y & 0 & 0 & 0 \\ 0 & x & y & 0 & 0 \\ 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ y & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

upper Δ^r matrix.

Take column → ∴ It has cofactors in the forms of U and L

$$\det A = x \begin{vmatrix} x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \end{vmatrix} + y \begin{vmatrix} y & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{vmatrix}$$

'upper triangular'

$$= x \cdot x^4 + y \cdot y^4$$

$$\det A = x^5 + y^5$$

det $R_5 \rightarrow R_5 - R_4$
$R_4 \rightarrow R_4 - R_3$
$R_3 \rightarrow R_3 - R_2$
$R_2 \rightarrow R_2 - R_1$

$$\begin{bmatrix} x & y & y & y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & y-x & x-y & 0 & 0 \\ 0 & 0 & y-x & x-y & 0 \\ 0 & 0 & 0 & y-x & x-y \end{bmatrix}$$

operations on columns:

$C_4 \rightarrow C_4 + C_5$
$C_3 \rightarrow C_3 + C_4$

$$\begin{bmatrix} x & y & 2y & 2y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & y-x & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & y-x & 0 & x-y \end{bmatrix}$$

$C_3 \rightarrow C_3 + C_5$

$$\begin{bmatrix} x & y & 3y & 2y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & y-x & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{bmatrix}$$

$$C_2 \to C_2 + C_3 + C_5 \quad \begin{bmatrix} x & 4y & 3y & 2y & y \\ y-x & x-y & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{bmatrix}$$

$$C_1 \to C_1 + C_2 \quad \begin{bmatrix} x+4y & 4y & 3y & 2y & y \\ 0 & x-y & 0 & 0 & 0 \\ 0 & 0 & x-y & 0 & 0 \\ 0 & 0 & 0 & x-y & 0 \\ 0 & 0 & 0 & 0 & x-y \end{bmatrix}$$

$\det B = (x+4y)(x-y)^4 \quad \because$ upper triangular.

$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1$, why $\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{bmatrix}$

$\therefore \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = 1.$

<u>soln</u>:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20-1 \end{bmatrix} = \underset{(n-1)}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}} + \underset{(n-2)}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & -1 \end{bmatrix}}$$

$$\det \left[\quad '' \quad \right] = 1 - 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$$

$$= 1 - 1$$

$$= 0$$

Cramer's rule, Inverse matrix, volume

Lecture - 21

$$A^{-1} = \frac{1}{(\det A)} C^T \rightarrow C \rightarrow \text{matrix of cofactors of } A.$$

Absolute value of the determinant gives the volume of a box.

* Cramer's rule for $x = A^{-1}b$
* $|\text{Det } A| = $ volume of box.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

why this works?

<u>3x3:</u>

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \text{Gauss - Jordan (Algorithm)}$$

↓

what's the algebra?

det A → product of n entries

C^T → product of $(n-1)$ entries

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$AA^{-1} = I$

$$AC^T = \det(A) \cdot I$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \vdots \\ C_{12} & C_{22} & \vdots \\ \vdots & \vdots & \\ & & C_{nn} \end{bmatrix}$$

$a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n} = \det$ of Row 1.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \vdots \\ c_{12} & c_{22} & \vdots \\ \vdots & \vdots & c_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \ \ddots \end{bmatrix}$$

question: what about off-diagonal entries.

check:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (a \ \ b) \begin{pmatrix} -b \\ a \end{pmatrix} = 0$$

Row 1 Cofactor of Row 2.

$-ab + ab$

$A_S =$ (screwed up) $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab \rightarrow$ determinant of matrix having similar rows.

cofactor of $a = b$
cofactor of $b = -a$

$\det A_S = ab - ab$
$\det A_S = 0$

why 0: determinant of taking 1st & last rows identical

$$\therefore \begin{bmatrix} a_{11} & \cdots \\ a_{21} & \cdots \\ \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots \\ \vdots & \vdots & \\ & & c_{nn} \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdots & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$AC^T = \det(A)\, I$$

$$A \cdot \frac{1}{\det(A)} C^T = AA^{-1}$$

$$\frac{1}{\det(A)} C^T = A^{-1}$$

If I move a_{11} entry what happens to A^{-1}?

$$Ax = b$$

$$x = A^{-1} b$$

$$\boxed{x = \frac{1}{\det A} C^T b}$$

$C^T b$ = Cofactor matrix (x)

Some number

$C^T b$ = determinant of something

Cramer's rule:

Formula for x:

$$x_1 = \frac{\det B_1}{\det(A)}$$

$$x_2 = \frac{\det B_2}{\det(A)}$$

Cramer realized what B_1, B_2 are?

$$B_1 = \begin{bmatrix} | & & \\ b & n-1 \text{ columns} & \\ | & \text{of } A & \end{bmatrix} = A \text{ with column 1 replaced by } b.$$

B_j = A with column j replaced by b.

$$\boxed{x_j = \frac{\det B_j}{\det A}}$$

The above one has long calculations!

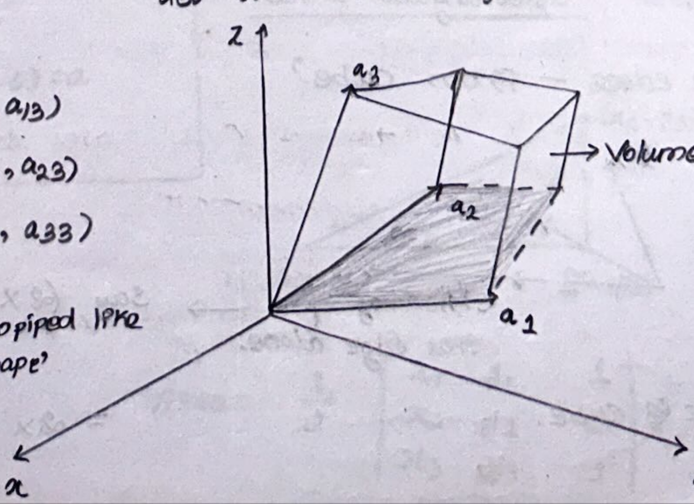
'Not efficient - disasterous way'

---

det A = volume of box.

edges
$a_1 (a_{11}, a_{12}, a_{13})$
$a_2 (a_{21}, a_{22}, a_{23})$
$a_3 (a_{31}, a_{32}, a_{33})$

'Box - Parallelopiped like shape'



→ Volume is gn by determinant

Is that be (-)ve?

↓

So we are taking it's absolute value.

If we change two rows → Changing (exchanging) two sides won't affect volume. (Right handed will be left handed)

Special case: If $A = I$

It's a unit cube

volume = 1 cu. units

$\therefore \det I = 1$

Relating to orthogonality:

If $A = Q$ (orthogonal matrix)

$Q \rightarrow$ Columns orthonormal

Shape of the box: cube (How different from Identity cube?)

↓

Rotated. (Turned in space)

$\det Q = ?$ IS $\det Q = \pm 1$

w.r.t:

$$Q^T Q = I$$

$$\det(Q^T Q) = \det I$$

$$\det(Q^T)\det(Q) = \det I$$

$$\therefore \det Q^T = \det Q$$

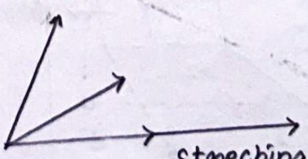
$$(\det Q)^2 = \det I$$

$$\boxed{\det Q = \pm 1}$$

Rectangular boxes:

'Stretch the edges - From cube'

Cube = a^3 cu. units



rect = $a^2 \times (2a)$

$= 2a^3$

= double × Volume of cube.

Streching that edge alone. → Say ($2 \times$ edge) (Row alone)

= $2 \times$ Volume.

$R_1 = 2R_1 \Rightarrow$ what about determinant?

determinant also doubles!

volume satisfies our 3(a) property.

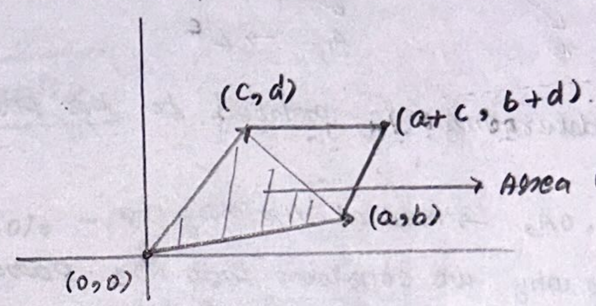
Prove 3(b) property

satisfied $\rightarrow$ prop 1 ✓

prop 2 $|\det A| =$ volume of box ✓

prop 3(a) ✓

$$3(b): \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



Area of $\Delta^{le} = \frac{1}{2} \times b \times h$

Area of the parallelogram: base × height.

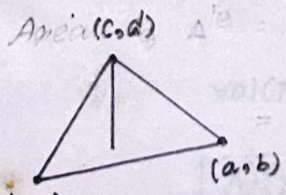
$$\text{Area}: \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \text{parallelogram}$$

$$= ad - bc$$

$$\text{Area of } \Delta^{le} = \frac{1}{2}(ad - bc)$$

[Area of Δ^{le} = $\frac{1}{2}$ Area of parallelogram]

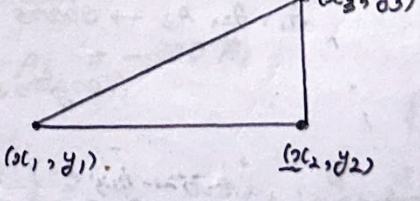
Area



Base: $(a-0) = a$

height: $d-b$.

Triangle (Not in (0, 0)



$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2 axis of parallelogram.

T is a tetrahedron with vertices

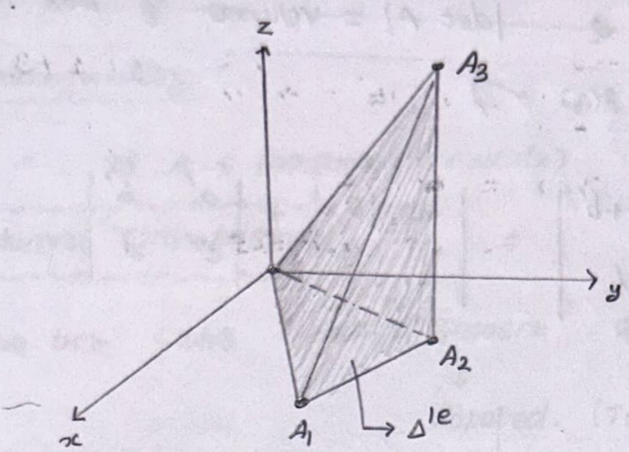
$O(0,0,0)$, $A_1(2,2,-1)$, $A_2(1,3,0)$, $A_3(-1,1,4)$

compute volume (T)

If A_1, A_2 are fixed but A_3 is moved to $A_3(-201,-199,104)$

compute volume (T)

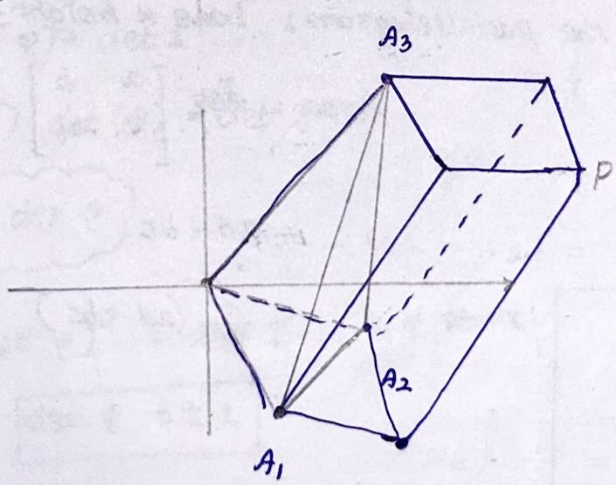
solu



Fact: w.k.t determinant is related to the parallelopiped.

$OA_1, OA_2, OA_3 \rightarrow$ meet at the origin

↳ why we consider that the parallelopiped is spanned by this same three edges?



$A_1, A_2, A_3 \rightarrow$ spans parallelopiped.

$A_1, A_2, A_3 \rightarrow$ spans three edges

(Length breadth & height).

volume of tetrahedron $= \frac{1}{3}$ Area of base × height.

Base be OA_1A_2

$vol(T) = \frac{1}{3} A(\Delta OA_1A_2) \cdot h$ $\quad$ $h = A_3$

vol (P) = Area of base × h

Base = 2× Δ^{le} area.

$vol(P) = 2A(\Delta OA_1A_2) \times h$

$vol(T) = \frac{1}{6} vol(P)$

$$vol(P) = abs\begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{vmatrix} = 2(12) - 2(4) - 1(1+3)$$

$= 24 - 8 - 4$

$= 12$ cu. units

$vol(T) = \frac{1}{6} \times 12$

$\boxed{vol(T) = 2 \text{ cu. units}}$

changing A_3 $(-201, -199, 104)$.

$$vol(P) = abs\begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -201 & -199 & 104 \end{vmatrix}$$

height will be very high
↓
shape will be like a needle

$= 2(3\times104) - 2(104) - 1(-199 + 3(201))$

$= 624 - 208 - 404$

$= 12$

$vol(T) = \frac{1}{6} . 12$

$\boxed{vol(T) = 2 \text{ cub. units}}$

Pay attention to the last row:

How much moved: $A_3' - A_3 = -100 A_1$

A_3 is the linear combination of R_1

$$\det\begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -201 & 199 & 104 \end{vmatrix} = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -200 & -200 & 100 \end{vmatrix}$$

$\therefore A_3 = -100A_1$

$= 2 + 0 = 2$

"$A_1, A_2 \rightarrow$ Fixed" $\rightarrow$ Base is fixed

↳ Changing height must not change base.

↓

Not Possible (so height not changing)

we are moving in the inverse direction - 100 times to A_1. (Height not changing)

Lecture - 22

Eigen values & Eigen vectors.

If the product Ax points in the same direction as the vector x, we say that x is an Eigen vector of A. Eigen values & Eigen vectors describe what happens when a matrix is multiplied by a vector.

* Eigen values - Eigen vectors
* $\det[A - \lambda I] = 0$

Trace $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

$\underset{x}{\longrightarrow}$ vector Ax

(Input) (output)

"Intrusted: one's that come out in the same direction"

"Ax - parallel to x" — Eigen vectors.

$AX = \lambda X$ → Eigen vector.

↳ some multiples. (Eigen value)

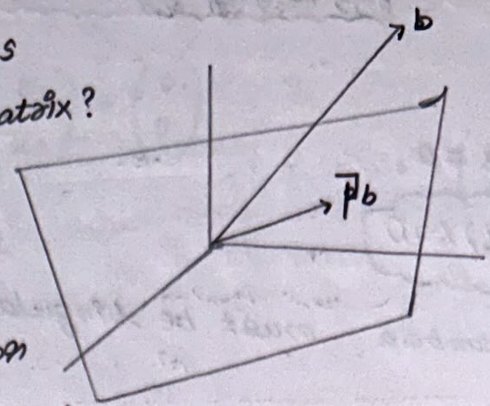
If $A \rightarrow$ singular, then $\lambda = 0$ is an Eigen value:

$X \rightarrow$ some non-zero vector.

'we can't use elimination'

'Examples - projection matrix'

what are x's and λ's
for a projection matrix?



b → Not a Eigen vector

Pb → has different direction.

what vectors are Eigen vectors of P?

Any x in the plane: projected will give me x.
(unchanged - so x is an Eigen vector.
multiplier 1 is an Eigen value).

3-d - 3-Eigen vectors: 2 inside (2-D) plane & 1 outside

'Any x perpendicular to the plane'

$$\boxed{Px = 0x}$$

'permutation matrix'

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ → Eigen value = 1
Eigen vector = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigen value = -1
Eigen vector = $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

↳ multiply by A reverses the two component]

$Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Ax = -\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\boxed{Ax = -x}$

→ In this example → This is 0.

n x n → n eigen values

'sum of the Eigen values = sum of the diagonals'
↓
Trace of the matrix

Fact

$Ax = \lambda x$

Rewrite, $Ax - \lambda x = 0$

$\boxed{(A - \lambda I)x = 0}$

A shifted by Lambda must be singular else the only solution is $\begin{pmatrix}0\\0\\0\end{pmatrix}$.

$\therefore \det(A - \lambda I) = 0$

Characteristic equ (or) Eigen value equ:

$\lambda \rightarrow$ could be repeated.

Finding $x \rightarrow$ Null space (By elemination).

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow$ symmetric

constant down the diagonal

2×2.

"Eigen values: Real for symmetric matrix"

$\rightarrow$ The Eigen vectors are $\perp^r$

eg: $(1 \; 1)\begin{pmatrix}1\\-1\end{pmatrix} = 0 \rightarrow$ previous example.

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)^2 - 1 = 0$

$(3-\lambda)^2 = 1$

$\lambda^2 - 6\lambda + 9 = 1$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda_1 = 4, \lambda_2 = 2. \rightarrow$ Real.

$\lambda^2 - 6\lambda + 8 = 0$

$\downarrow$ Trace $\quad \downarrow$ Determinant

8

-4 -2

-6

when $\lambda = 4$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x_1 = 1, \; x_2 = 1$

when $\lambda = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$x_1 = 1, \; x_2 = -1$

'we have a line of Eigen vectors – we are giving a basis'

Relation b/w

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Common Eigen vectors (Not Eigen values).

$\therefore A = B - 3I$

Observation: If I add $3I$ to a matrix, it's eigenvectors don't change' → It's Eigen values are three (sum) bigger.

$$AX = \lambda X$$

$$(A + 3I)X = \lambda X$$

$$(A+3I)X = AX + 3X$$

$$(A+3I)X = \lambda X + 3X$$

$$\boxed{(A+3I)X = (\lambda+3)X}$$

Eigen vector same X for both matrices.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$(2\times2)(2\times1) = (2\times1)$

<u>Special:</u>

$AX = \lambda X \quad (X, \lambda \rightarrow \text{Known})$

Adding some other matrix B

'Caution' B has eigen values α_1,

$Bx = \alpha x$

$(A+B)x \neq (\lambda+\alpha)x$ → False. → False case.

x → Is not the Eigen vector of B

'Eigen values → Not linear (Not multiply).

Example: 'Rotation matrix' → Rotates Every vector by 90°.

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\cos 90° = 0$, $\sin 90° = 1$ $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

↳ 'orthogonal matrix'

Solu: Eigen values & Eigen vectors!

Trace $= 0$

Sum of Eigen values $= \lambda_1 + \lambda_2 = 0$ (Trace)

$\therefore$ determinant = product of Eigen values

$\det(Q) = \lambda_1 \lambda_2$

$$\boxed{\lambda_1 \lambda_2 = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1}$$

'Rotation: what vector can come out parallel after a 90° rotation' → Trouble.

From, $\lambda_1 + \lambda_2 = 0$ $\boxed{\lambda_1 = -\lambda_2}$ | How $\lambda_1 \lambda_2 = +ve$?

$$\det \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \sqrt{-1}$$

$\lambda = \pm i$ → Complex Eigen values even though the matrix is real.

Real matrix $\xrightarrow[\text{give}]{}$ Complex Eigen values.

---

complex conjugates of each others: 'switched signs of imaginary'

'Eigen vectors'

Fact: Symmetric matrix → It won't happen. Eigen values are real.

$Q \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Q^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\boxed{Q = -Q^T}$ → Anti-symmetric. → pure imaginary Eigen values.

Good possibilities: ⊥ Eigen vectors, Real Eigen values

Bad possibilities: complex Eigen values

Even worse : partly symmetrical & non-symmetrical

('Both img + real numbers)

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ → Triangular matrices.

Solu:-

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$\boxed{\lambda = 3, 3}$

Problem: Eigen vectors:

when $\lambda = 3$

$(A - \lambda I)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

↳ singular

when

$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = 3$ (again)

∴ x_2 must be independent. (Possibility of repeated eigen values → opens the shortage of Eigen vectors)

1 → Eigen vector, instead of 2.

(x_2 = No second independant Eigen vectors.) → No Complete story given by Eigen vectors.

Recitation

Invertible

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$$, Find the Eigen values & vectors of $A^2, (A^{-1} - I)$

Solu:

$AV = \lambda V$

$A^2 V = A(AV) = A(\lambda V)$

$A^2 V = \lambda (AV)$ → scalar

$A^2 V = \lambda^2 V$

III^ly

$A^{-1} A V = A^{-1} \lambda V$

$A^{-1} V = A^{-1} \frac{AV}{\lambda}$ $(\lambda \neq 0)$

$A^{-1} V = \frac{1}{\lambda} V$

$(A^{-1} - I) V = \left(\frac{1}{\lambda} V - I V\right)$

$(A^{-1} - I) V = \left(\frac{1}{\lambda} - 1\right) V$

∴ Eigen values & Eigen vectors of $A^2, (A^{-1} - I)$:

$\det(A - \lambda I) = 0$

$$\det \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(4-\lambda) + 2\right]$$

$= (1-\lambda)\left[4 - \lambda - 4\lambda + \lambda^2 + 2\right]$

$0 = (1-\lambda)\left[\lambda^2 - 5\lambda + 6\right]$

$$0 = (1-\lambda)(\lambda^2 - 5\lambda + 6)$$

$$0 = (1-\lambda)(\lambda-2)(\lambda-3)$$

6
-2 -3
-5

$$\lambda = 1, 2, 3.$$

when $\lambda = 1$	when $\lambda = 2$	when $\lambda = 3$
$0 = (A - \lambda I)v$ $0 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} v$ $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$0 = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix} v$ $v = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$	$0 = \begin{pmatrix} -2 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} v$ $v = \begin{pmatrix} -1/2 \\ -1 \\ 1 \end{pmatrix}$ (or) $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $-2+3 = 1 - 1 = 0$

	A	A^2	$A^{-1} - I$
Eigen values	λ	λ^2	$\lambda^{-1} - 1$
Eigen vectors	v	v	v

$$A = \begin{bmatrix} 2.0 & 0 \\ 1.0 & 3.0 \end{bmatrix}$$

Eigen values : 2, 3

Eigen vectors: $\begin{pmatrix} 0.71 \\ -0.71 \end{pmatrix}$, $\begin{pmatrix} 0.00 \\ -1 \end{pmatrix}$

Trace : 5

Determinant : 6

x

Ax

'Eigen vectors'

Powers A^k have eigen values λ^k

$$A^2 x = A(\lambda x) = \lambda^2 x$$

u_0 is a combination $c_1 x_1 + \dots + c_n x_n$

Then Au_0 is $u_1 = c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n$

Dynamics in discrete time:

$u_{k+1} = Au_k$ starting from u_0

$A^K u_0$ is $u_K = c_K (\lambda_1)^K x_1 + \cdots + c_n (\lambda_n)^K x_n$

$u_0 \rightarrow$ combination $c_1 x_1 + \cdots + c_n x_n$ [combination of Eigen vectors]

$$A u_0 = u_1 = c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n.$$

Dynamics in discrete time:

$$u_{K+1} = A u_K$$

$$A^K u_0 = u_K = c_1 (\lambda_1)^K x_1 + \cdots + c_n (\lambda_n)^K x_n$$

Diagonalization

Needs n independent Eigen vectors: Automatic if the eigenvalues are all different.

Eigen vectors $x_1, \ldots, x_n$ are columns of invertible Eigen vector matrix S.

Multiply the columns x by A: $AS = [\lambda_1 x_1 \ldots \lambda_n x_n]$

This equals $\begin{bmatrix} x_1 \ldots x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda$

Then $S^{-1} A S$ = diagonal Eigenvalue matrix Λ.

In other words $A = S \Lambda S^{-1}$ $\quad$ $\Lambda \rightarrow$ diagonal matrix.

Differential equation

S = Eigen vectors, Λ = Eigen values, $A = S \Lambda S^{-1}$

$$A^K = (S \Lambda S^{-1}) \ldots (S \Lambda S^{-1})$$
$$= S \Lambda^K S^{-1}$$

$e^{At} = S e^{\Lambda t} S^{-1}$ for differential equations

$$\boxed{e^{At} u(0) \text{ is } u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n}$$

Then $u(t)$ solves $\frac{du}{dt} = Au$ starting from $u(0)$

'similar' $B = M^{-1} A M$ has the same eigen values as A

Symmetry: 1) Real Eigen values
2) perpendicular eigen vectors.

$A^T = A$ takes $A = S\Lambda S^{-1}$ into $A = Q\Lambda Q^T$

Eigen vector matrix Q has orthonormal columns

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \boxed{Q^{-1} = Q^T}$$

$A^T = A$: Real λ , $A^T = -A$: Imaginary λ

orthogonal matrix Q : All $|\lambda| = 1$.

'Stable for computing'

Positive definite

Positive definite matrices come from A^TA

Energy $u^T A^T A u = (Au)^T (Au) = \|Au\|^2 \geq 0$

Eigen values of A^TA never negative: $x^T A^T A x = x^T \lambda x$

Eigen values of A^TA never negative.

upper left determinants of A^TA never negative.

pivots of A^TA never negative.

All are positive if A has independent columns.

SVD

Single value decomposition of A (m by n) 'Factorize'

$A = U \Sigma V^T$ = (orthogonal) (diagonal) (orthogonal)

$u_1, \ldots, u_m$ in U are eigen vectors of AA^T

$v_1, \ldots, v_n$ in V are eigen vectors of A^TA.

$\sigma_1, \ldots, \sigma_r$ in Σ are positive singular values of A.

$AV = U\Sigma$ means each $Av_i = \sigma_i u_i$

'Every A is diagonalized by 2 orthonormal matrices'

'Diagonalization and powers of A'

If A has n independent eigen vectors, we can write

$A = S\Lambda S^{-1}$, where $\Lambda \to$ diagonal matrix having eigen values of A. This allows us to easily compute powers of A in which in turn allows us to solve differential equations

$u_{k+1} = A u_k$

Lecture - 23

Diagonalize a matrix $S^{-1} A S = \Lambda$

$S^{-1} \rightarrow$ Inversion: Independent Eigen vectors.

Suppose n independent eigen vectors of A, put them in columns of S.

$Ax = \lambda x$ $\rightarrow$ x - Eigen vectors

$S \rightarrow$ Eigen vector matrix

$$AS = A \begin{bmatrix} x_1 & x_2 & x_n \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column times Row multiplication

$$= S\Lambda$$

$\Lambda \rightarrow$ Diagonal matrix

$\Lambda \rightarrow$ capital lambda

n independent Eigen vectors?

$$AS = S\Lambda$$

$$S^{-1}AS = S^{-1}S\Lambda$$

$$S^{-1}AS = \Lambda$$

Also otherway:

$$ASS^{-1} = S\Lambda S^{-1}$$

$A = S\Lambda S^{-1} \rightarrow$ New factorization

Replacement of LU.

what about A^2?

$Ax = \lambda x$

$A^2 x = A(\lambda x)$

$A^2 x = \lambda (Ax)$

$A^2 x = \lambda^2 x$

'same Eigen vectors x as A'

$A^2 = (S \Lambda S^{-1})(S \Lambda S^{-1})$

$A^2 = S \Lambda^2 S^{-1}$

'Eigen vectors are the same'

diagonal matrix is squared

↳ In matrix form.

$A^k = S \Lambda^k S^{-1}$ → Eigen values are k^{th} power

Eigen vectors are the same.

In LU → multiplying 100 times with give $(LU)^{100}$ times

↳ Here Λ^k

$A^{100} = S \Lambda^{100} S^{-1}$

when to the powers of the matrix goes to zero?

'stable matrix'

$A^k \to 0$ as $k \to \infty$ if all $|\lambda_i| < 1$.

where is the information about stability:

'Not in pivots'

'In the Eigen values'

Note: we have n independent Eigen vectors: Else Diagonalization is not possible.

---

which matrices are diagonalizable?

A is sure to have n independent Eigen vectors. (If all the lambdas are different).

↳ No repeated Eigen values.

command: eig( )

Repeated Eigen values:

'May or may not have n independent Eigen Vectors'

$I_{10\times10}$ = Eigen values (Diagonal element)

$\lambda = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$

But non-repeated Eigen vectors.

If $A = I$

$S^{-1} A S = \Lambda$

$S^{-1} I S = \Lambda$

$\boxed{I = \Lambda} \rightarrow \therefore$ Already diagonalized.

Case: Triangular.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$

Eigen values $\lambda = 2, 2$.

'upper Δ^r'

$(2-\lambda)^2 = 0$

$\lambda = 2, 2$

Eigen vectors:

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$

$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ → only one independent Eigen vectors

↳ one-dim null space.

> 'As we can't find two independent Eigen vectors we can't diagonalize'

n independent Eigen vectors: Algebraic multiplicity is one.

Geometric multiplicity is one.

$(\lambda - a)(\lambda - b)$

(Independent).

Equation:

'start with u_0'

$u_{k+1} = A u_k$

$u_1 = A u_0$

$u_2 = A u_1$

$u_2 = A(A u_0)$

$u_2 = A^2 u_0 \quad [A u_1 = A(A u_0)]$

$u_k = A^k u_0$

'Solves I order differential equation'

'we are dealing with systems' – matrix

To really solve:

write u_0 as a combination of Eigen vectors.

$u_0 = c_1 x_1 + c_2 x_2 + \dots + c_n x_n \rightarrow$ Any combination of Eigenvectors

$A u_0 = A c_1 x_1 + A c_2 x_2 + \dots + A c_n x_n$

$A x_1 = \lambda_1 x_1$

$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \dots + c_n \lambda_n x_n$

$A^{100} u_0 = c_1 \lambda_1^{100} x_1 + c_2 \lambda_2^{100} x_2 + \dots + c_n \lambda_n^{100} x_n$

$\therefore$ we have pure λ

$$\boxed{u_{100} = A^{100} u_0 = \Lambda^{100} S c}$$

$A^{100} u_0 = \Lambda^{100} S c$

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, ...

F_{100} = 100th fibonacci number = ?

$$\boxed{F_{k+2} = F_{k+1} + F_k}$$

$F_{k+1} = F_{k+1}$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \rightarrow$ made it first order, inspite of II order.

$$u_{k+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

$\therefore F_{k+2} = F_{k+1} + F_k \rightarrow$ II order

converting into I order:

$u_{k+1} = A u_k$

$F_{k+2} = F_{k+1} + F_k \rightarrow (1)$

$F_{k+1} = F_{k+1} \rightarrow (2)$

$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \rightarrow$ 2x2 system (I order)

↳ Be a vector.