Definition 5.1:

A period function f of period $2L$ is called piecewise differentiable function if

1) At points where the derivatives exist it is bounded. (that is, there is an M such that $|f'(t)| \le M < \infty$ at all t for which $f'(t)$ exists. (M-bounded set)

2) There are at most finitely many points in $[-L, L)$ where $f'(t)$ doesn't exist, and

3) At each point T, (such) the left limit $f(T^-) := \lim_{t \to T^-} f(t)$ and right hand limit $f(T^+) := \lim_{t \to T^+} f(t)$ exist, might be unequal. (we say jump discontinuity at T)

Theorem 5.2:

If f is a piecewise differentiable periodic function, then the Fourier series of f (with the a_n and b_n defined by the Fourier coefficient formulas)

- converges to $f(t)$ at values of t where f is continuous, and
- converges to $\frac{f(t^-) + f(t^+)}{2}$ where f has a jump discontinuity.

5.3 left lim $Sq(0^-) = -1$ and right limit $Sq(0^+) = 1$. average to 0. The F.S

$$\frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$$

evaluated at 0 will give 0.

$\therefore$ The function will converge to 0 too.

Note: The examples here are for 2π - periodic functions, but this theorem on convergence holds for periodic functions of any period.

2π - periodic function

$$f(t) = \begin{cases} t & 0 < t < \pi \\ 0 & -\pi < t < 0 \end{cases}$$

Solu :

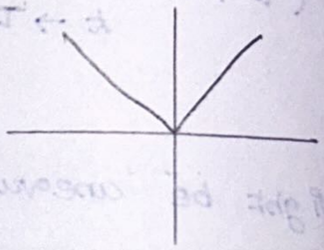
$g(t) \rightarrow$ Fourier series for $f(t)$.

$\boxed{g(\pi) = ?}$ $$g(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

$U(t) \rightarrow$ F.S of 2π - periodic Δ^r wave

$$T(t) = |t| \quad , \quad -\pi < t < \pi.$$

$U(\pi) = 0.$



$\therefore$ The function is continuous. so the F.S is the value of the Function at these points. which is π.

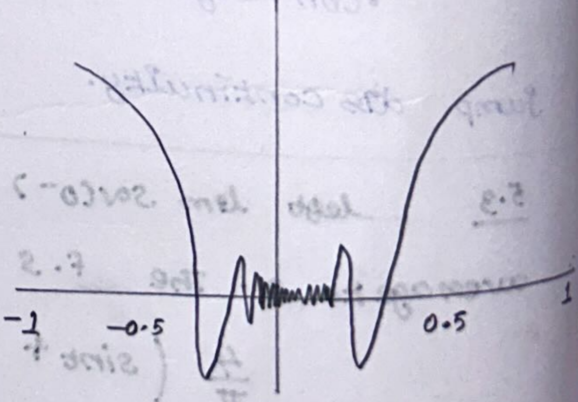
$$f(x) = x\left(\sin\left(\tfrac{1}{x}\right)\right) \quad x \neq 0 \text{ on } [-\pi, \pi]$$

The function is continuous, has a derivative everywhere in the interval, except at $x = 0$. The derivative is unbounded on that interval. (It oscillates b/w $+\infty$ & $-\infty$).

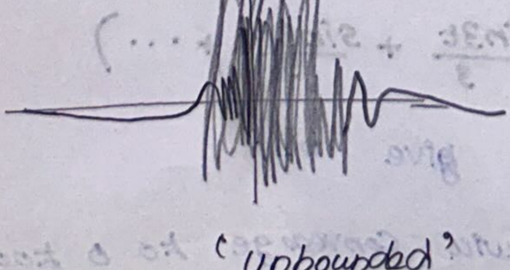
$$f: = x \rightarrow x \cdot \sin\left(\tfrac{1}{x}\right)$$

$$g: = \frac{d}{dx} f(x) = \sin\left(\tfrac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}$$

plot will be : $(f(x), x = -1, \ldots 1)$



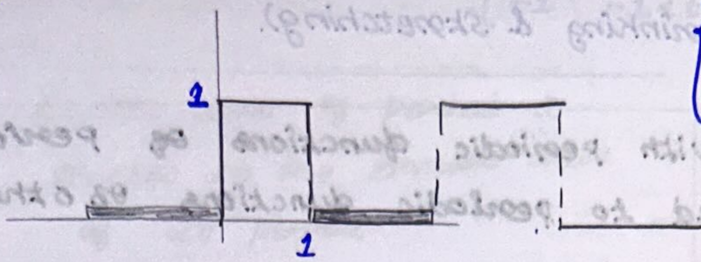
plot $(g(x), x = -1 .. 1)$



'Unbounded'

Functions of arbitrary period.

$f(t)$

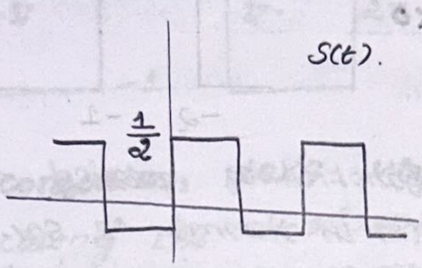


$\begin{cases} \sin n\pi t \\ \cos n\pi t \end{cases}$

$\boxed{L=1}$

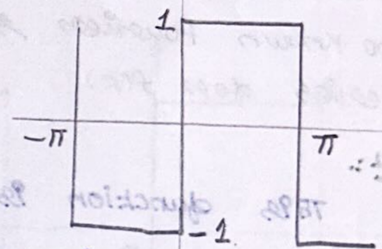
"Neither even nor odd"

$S(t) \to$ converting $f(t)$ to odd.



$S(t)$.

From salmon's function (modelled)

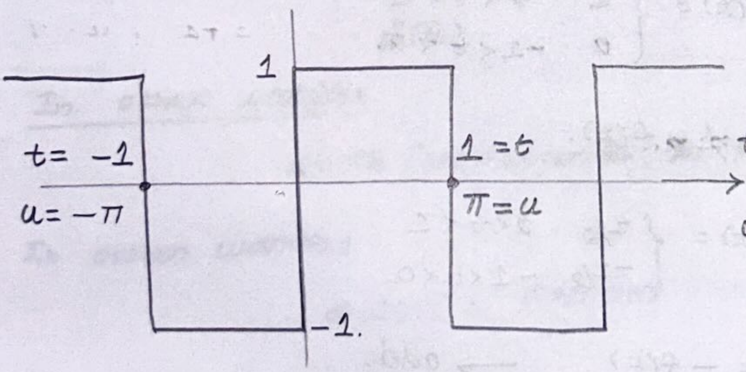


From previous answers $\leftarrow$

Consider:

$$g(u) = \frac{4}{\pi} \sum_{odd} \frac{\sin nu}{n}$$

(odd function)



} two variables (at x-axis)

when
$t = -1, \; u = -\pi$
$t = 1, \; u = \pi.$

$\therefore \boxed{u = \pi t}$

$\boxed{g(u) = g(\pi t)}$

$f(t) = S(t) + \frac{1}{2}$

$\hookrightarrow$ Lowered by $\frac{1}{2}$ (one half)

$\therefore \; S(t) = \frac{1}{2} g(u)$

$$f(t) = \frac{1}{2} g(u) + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{2} \frac{4}{\pi} \sum_{odd} \frac{\sin n\pi t}{n}$$

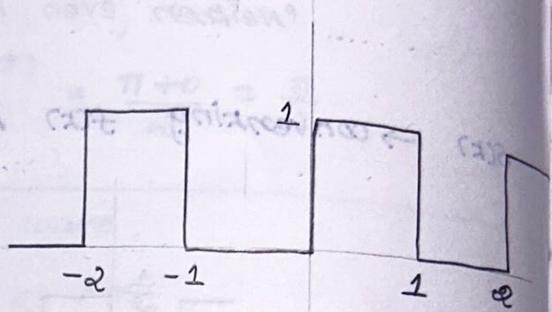
$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{odd} \frac{\sin n\pi t}{n}$$

(Found by shrinking & stretching).

Everything we did with periodic functions of period 2π can be generalized to periodic functions of other periods.

6.1:

$$f(t) = \begin{cases} 1 & \text{if } 0<t<1 \\ 0 & \text{if } -1<t<0 \end{cases}$$



solu:

It extend it to periodic function of period 2. Express this new square wave $f(t)$ in terms of sq. Then use the known Fourier series of sq to find the fourier series for $f(t)$.

solu:

This function is neither odd nor even. If we shift the function downwards by 1/2. It is an odd function.

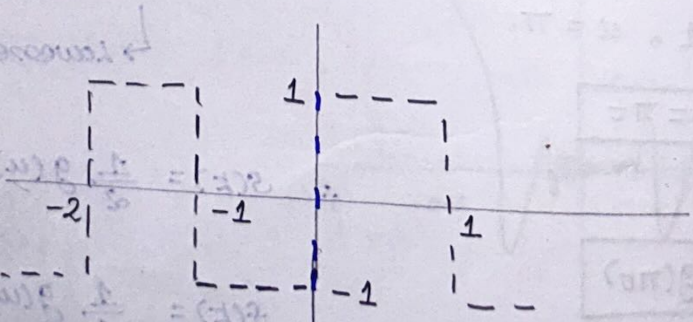
$\therefore$ For $f(t) = \begin{cases} 1 & 0<t<1 \\ 0 & -1<t<0. \end{cases}$

so $f(-t) \neq -f(t)$.

For $f(t) = \begin{cases} 1/2 & 0<t<1 \\ -1/2 & -1<t<0. \end{cases}$

so $f(-t) = -f(t) \longrightarrow$ odd.

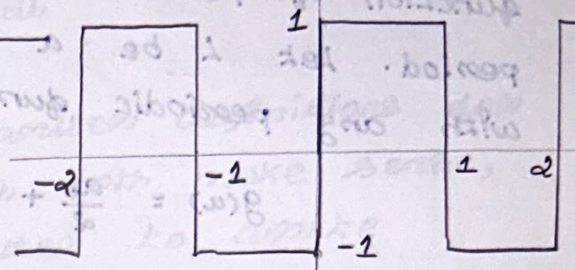
so shifting this function downwards by 1/2, it's an odd function.



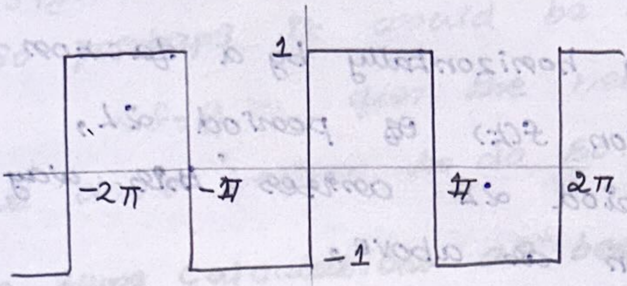
$$S(t) = f(t) - 1/2 = \begin{cases} 1/2 & 0<t<1 \\ -1/2 & -1<t<0 \end{cases}$$

If we multiply this function by 2,

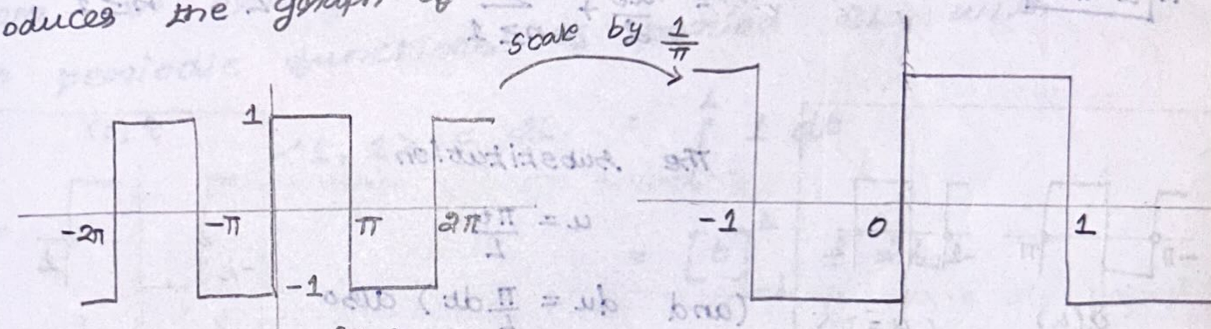
$$2S(t) = 2f(t) - 1 = \begin{cases} 1 & 0 < t < 1 \\ -1 & -1 < t < 0. \end{cases}$$



This square wave of period 2 is similar to the square wave of 2π period



To avoid confusion, let's use u as the variable for sq. Scaling the horizontal axis by factor of $1/\pi$ produces the graph of $2S(t)$



$S(u)$ $S(t)$

In other words,

$u = \pi t$ (corresponds to $t = 1$), then $2S(t) = sq(u)$.

In other words,

$$2S(t) = sq(\pi t)$$

Therefore,

$$f(t) = \frac{1}{2} + S(t) = \frac{1}{2} + \frac{1}{2} sq(\pi t)$$

From the Fourier series

$$sq(u) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nu}{n}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin n\pi t}{n}$$

Period 2L Functions:

Similarly we can scale the horizontal axis of any function of period 2π to get a function of different period. Let L be a +ve real number. Start with any periodic function.

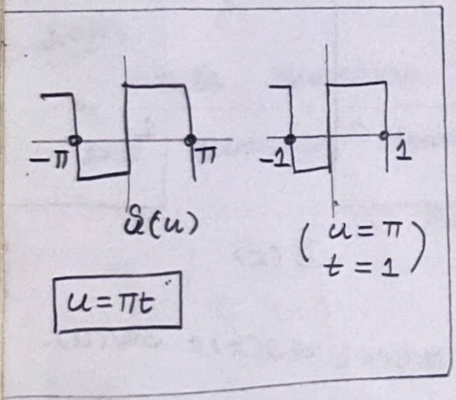
$$g(u) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nu + \sum_{n\geq 1} b_n \sin nu$$

of period 2π. Stretching horizontally by a factor $\frac{L}{\pi}$ gives a periodic function $f(t)$ of period $2L$, and 'every' f of period $2L$ arises this way. By the same calculation as above,

$$f(t) = g\left(\frac{\pi t}{L}\right)$$

$\therefore \boxed{u=\pi t}$

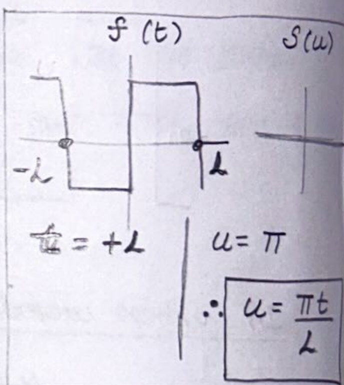
$$= \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\frac{n\pi t}{L} + \sum_{n\geq 1} b_n \sin\frac{n\pi t}{L}$$



The substitution

$$u = \frac{\pi t}{L}$$

(and $du = \frac{\pi}{L} dt$) also leads to Fourier co.eff formulas for period $2L$:



$$\therefore \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu \, du$$

$$= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi t}{L}\right)\cos\left(\frac{n\pi t}{L}\right)\frac{\pi}{L}\,dt$$

$$= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

A similar formula gives b_n in terms of f.

The inner product for periodic functions of arbitrary period.

Adapt the definition of the inner product to the case of functions f and g of period $2L$.

$$\langle f, g\rangle := \int_{-L}^{L} f(t)\, g(t)\, dt$$

This conflicts with the earlier definitions for $\langle f, g\rangle$, functions for which both make sense, so perhaps it would be better to write $\langle f, g\rangle_L$ for the new inner product, but we don't bother to do so.

The same calculations as before. Show that the functions

$$1, \cos\left(\frac{\pi t}{L}\right), \cos\left(\frac{2\pi t}{L}\right), \cos\left(\frac{3\pi t}{L}\right), \dots, \sin\left(\frac{\pi t}{L}\right), \dots$$

form an orthogonal basis for the vector space of all periodic functions of period $2L$, with

$$\langle 1, 1\rangle = 2L. = \int_{-L}^{L} 1\, dt$$

$$= [t]_{-L}^{L} = 2L.$$

$$\left\langle \cos\frac{n\pi t}{L}, \cos\frac{n\pi t}{L}\right\rangle = L \quad \left(\int_{-L}^{L} \rightarrow \text{bounds}\right)$$

$$\left\langle \sin\frac{n\pi t}{L}, \cos\frac{n\pi t}{L}\right\rangle = L$$

The average value over the whole period of $\cos^2 \omega t$ is $1/2$ for any ω, and the average value of $\sin^2 \omega t$ is $1/2$ too.

(This gives another way to derive the Fourier co.eff formulas for functions of period $2L$.)

$T(t) \rightarrow$ periodic function of period 2, such that $T(x) = |x|$ for $-1 \le x \le 1$. (Triangular wave).

$Sq(x) \rightarrow$ denote the odd square wave of period 2.

$$\langle T, Sq\rangle = \int_{-1}^{1} |x|\, sq(x)\,dx \qquad Sq(x) = -1 \quad -1<x<0$$
$$= 1 \quad 0<x<1$$

$$= \int_{-1}^{0} -(-x)\,dx + \int_{0}^{1} x\,dx$$

$$= \left(\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{0}^{1}$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

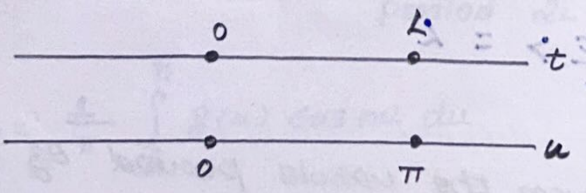
$\therefore$ Hence the Δ^{r} wave is orthogonal and square wave too. (Δ^{r} & square wave are orthogonal).

Note: This makes sense because the Δ^{r} wave is even periodic & square wave - odd periodic. So we would anticipate that every term in one fourier series is orthogonal to the terms in another.

Fourier co.effs formulas

Period 2L formulas:

L means we are just stretching our function.



when $t = 0, u = 0$
$t = L, u = \pi$ $\quad \left(u = \frac{\pi t}{L}\right) \longrightarrow$ zero at

$$\cos\left(\frac{n\pi}{L} t\right)$$

$$\sin\left(\frac{n\pi}{L} t\right) \rightarrow \text{zero at } 0, \pm L, \pm 2L$$

$$f(t) = \frac{a_0}{2} + \sum\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)$$

(Identical) $\rightarrow$ Not exactly equal.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\, dt.$$

$f(t)$ is even, period $= 2L$

$$\boxed{a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(\frac{n\pi}{L}t\right) dt.}$$

If the period is 2, 1 is the half period, because in the literature, frequently 1 is the standard normal reference, not pi.

pi – is convenient mathematically. Since it makes the sines & cosines look simple, but in actual calculation, it tends to be where $L = \pi$.

Fourier's theorem:

"Every" periodic function f of period $2L$ is a Fourier series.

$$f(t) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\frac{n\pi t}{L} + \sum_{n\geq 1} b_n \sin\frac{n\pi t}{L}$$

Given, f, the Fourier coefficients a_n and b_n can be computed using:

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt = \frac{\langle f(t), 1\rangle}{\langle 1, 1\rangle}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt = \frac{\langle f(t), \cos\left(\frac{n\pi}{L}t\right)\rangle}{\langle \cos\left(\frac{n\pi}{L}t\right), \cos\left(\frac{n\pi t}{L}\right)\rangle}$$

for all $n \geq 1$.

$$\therefore \langle \cos\left(\frac{n\pi t}{L}\right), \cos\left(\frac{n\pi t}{L}\right)\rangle = L$$

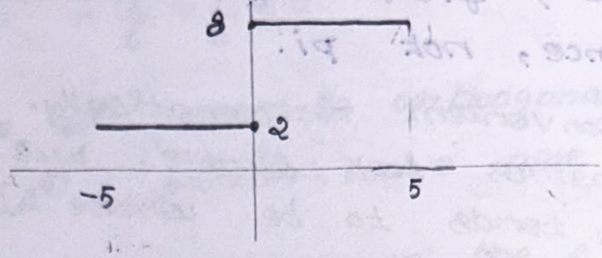
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt = \frac{\langle f(t), \sin\left(\frac{n\pi t}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi}{L}t\right), \sin\left(\frac{n\pi}{L}t\right)\rangle}$$

for all $n \geq 1$.

* If f is even, then only the cosine terms (including the $a_0/2$ term) appear.
* If f is odd, then only sine terms appear.

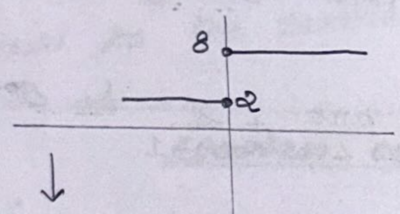
Define $S(t) = \begin{cases} 8 & \text{if } 0 < t < 5 \\ 2 & \text{if } -5 < t < 0 \end{cases}$

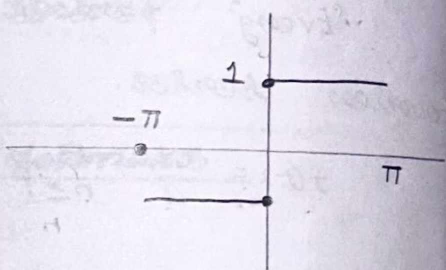
Solu:



$\boxed{\text{Period} = 10.}$

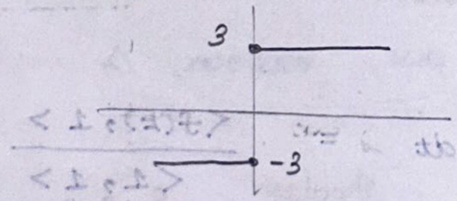
Solu: 'By stretching & shifting'.

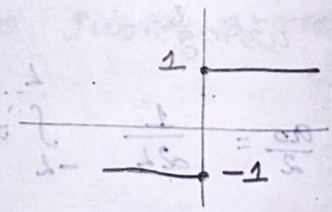




$\downarrow$

shifting downwards by 5

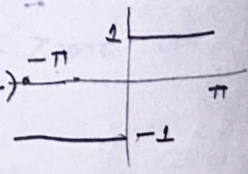




Scaling by $\frac{1}{3}$ gives

Comparing

Scaled & shifted with 5 & $\frac{1}{3}$ (&)



$$sq\left(\frac{\pi t}{5}\right) = \begin{cases} 1 & 0 < t < 5 \\ -1 & -5 < t < 0 \end{cases}$$

$$3\, sq\left(\frac{\pi t}{5}\right) = \begin{cases} 3 & 0 < t < 5 \\ -3 & -5 < t < 0 \end{cases}$$

$$\therefore\; 5 + 3\, sq\left(\frac{\pi t}{5}\right) = \begin{cases} 8 & 0 < t < 5 \\ 2 & -5 < t < 0. \end{cases}$$

$sq(t) = \frac{4}{\pi} \sum_{n \geq 1 \text{ odd}} \frac{1}{n} \sin nt.$

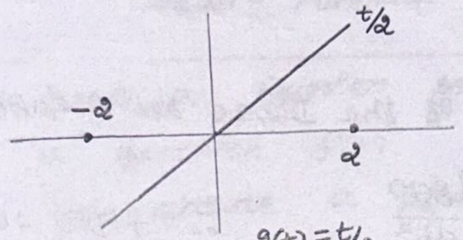
$$s(t) = 5 + 3\left[\frac{4}{\pi} \sum_{n \geq 1 \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi t}{5}\right)\right]$$

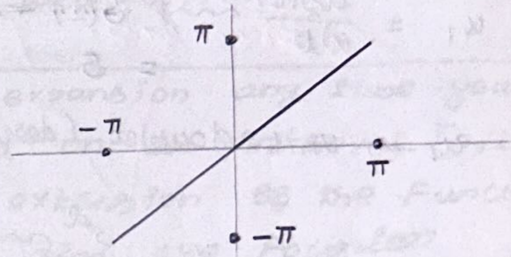
$$= 5 + \sum_{n \geq 1 \text{ odd}} \left(\frac{12}{n\pi} \sin\left(\frac{n\pi t}{5}\right)\right)$$

use [sawtooth] $f(u) = u, \; -\pi < u < \pi$ having F.S

$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nu)$ to find the F.S of the odd periodic function (of period 4). , $g(t) = \frac{t}{2} \; (-2 < t < 2)$

Solu:





At $g(1) = \frac{1}{2}$ $\quad t = 2, \; u = \pi$

$$\boxed{u = \frac{\pi t}{2}}$$

$$f(u) = \sum \frac{2(-1)^{n+1}}{n} \sin(nu)$$

$$g(t) = \frac{1}{\pi} f(u) = \frac{1}{\pi} f\left(\frac{\pi t}{2}\right)$$

$$= \sum \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(t) \sin(u) \, du$$

$$= \frac{2}{L} \int_0^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

checking,

$$b_n = \frac{2}{2} \int_0^2 \frac{t}{2} \sin\left(\frac{n\pi t}{2}\right) dt$$

$$= \frac{1}{2} \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt$$

$$= \frac{1}{2}\left[\frac{-2t}{n\pi} \cos\left(\frac{n\pi t}{2}\right)\right]_0^2 + \int_0^2 \frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) dt$$

$$= -\frac{2(-1)^n}{n\pi}$$

$$= \frac{2(-1)^{n+1}}{n\pi}$$

$\therefore$ The Fourier series is $\sum_n \frac{2(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi t}{2}\right)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\delta(t) \rightleftharpoons 1 \qquad \therefore u(t) \xrightarrow{\mathcal{L}} \frac{1}{s}.$$

$$x(t) \rightleftharpoons X(s) = s$$

$$y(t) = x(t) * u(t) \rightleftharpoons s \times \left(\frac{1}{s}\right)$$
$$= 1$$
$$= \delta(t)$$

$$f'(t) \rightsquigarrow sF(s) - f(0) \qquad \Big| \quad \mathcal{L}^{-1}\{s^n\} = \delta^{(n)}(t)$$

$$u_1 = \frac{d\delta(t)}{dt} \rightsquigarrow s(1) - f(0^-)$$
$$= s$$

$u_1 \rightarrow$ unit doublet (derivative of the Dirac delta function)

$$s^2 \underset{\mathcal{L}^{-1}}{\rightsquigarrow} \frac{d^2\delta(t)}{dt^2} \qquad \therefore f(0^-) = 0$$

(exactly defined at $x=0$)

---

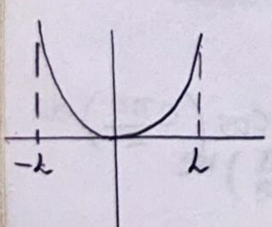
most functions are not periodic.

All this theory is not for periodic functions alone. It's about functions – really the interval on which you're interested in them is finite. **(Finite interval).**

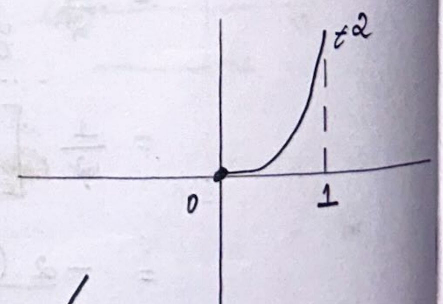
{For functions going to infinity → we need to use Fourier transforms}

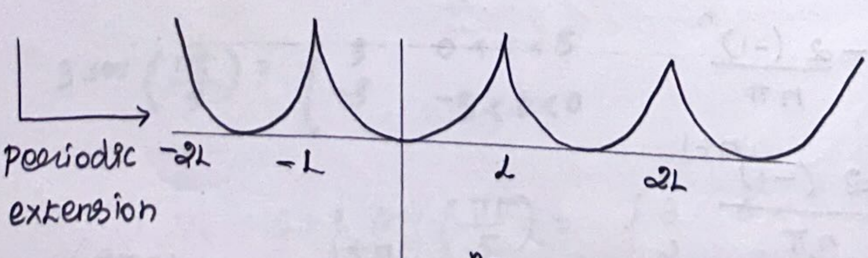
$f(t)$ **defined on** $[0, L] \rightarrow$ finite interval.

'Make a periodic extension'



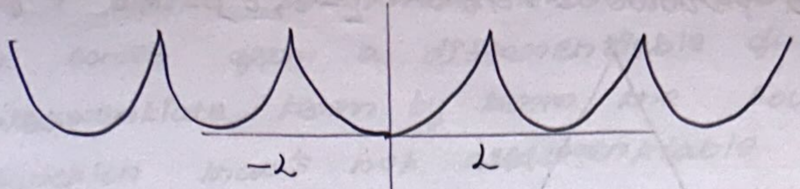
This function has even periodic extension



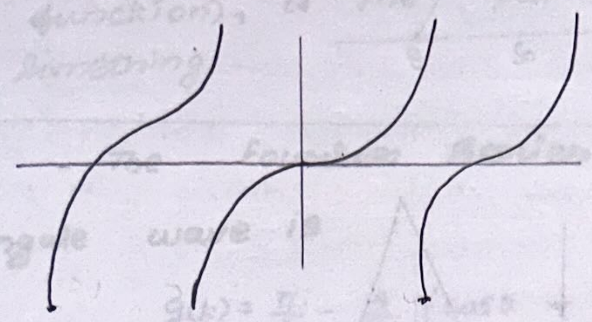


periodic extension

'Removing the other parts of t^2'



→ Even extensions.



→ odd extensions.

odd - Function will be discontinuous in this case

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we can use a Fourier series expansion any time you have a function $f(t)$ defined on an interval $[0, L]$. First you create a periodic extension of the function defined everywhere, and then find the Fourier series of the periodic function. (In order to simplify the Fourier series, we typically choose to extend our function to be either odd or even, so that we end up with a cosine or sine terms.

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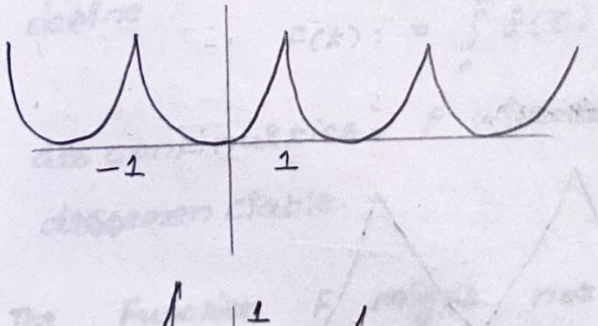
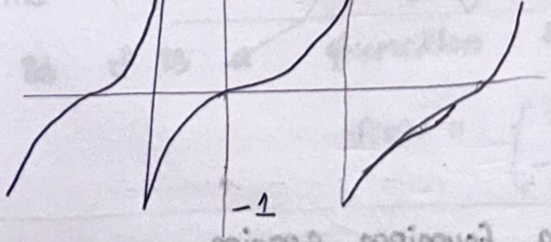
consider $y = t^2$ defined on $[0, 1]$



Even extension

$f(t) = t^2, \quad -1 \leq t \leq 1$

(period = 2)



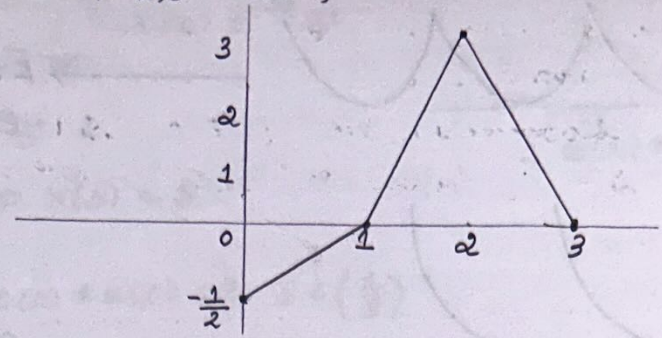
$$f(t) = \begin{cases} t^2 & 0 < t < 1 \\ -t^2 & -1 < t < 0 \end{cases}$$

Sketch the (odd) periodic function $g(x)$ defined on $[0, 3]$

$$g(x) = \begin{cases} x/2 - 1/2 & 0 < x < 1 \\ 3x - 3 & 1 < x < 2 \\ -3x + 9 & 2 < x < 3 \end{cases}$$

Solu: Sketch the odd periodic extension $[-6, 6]$

Solu:

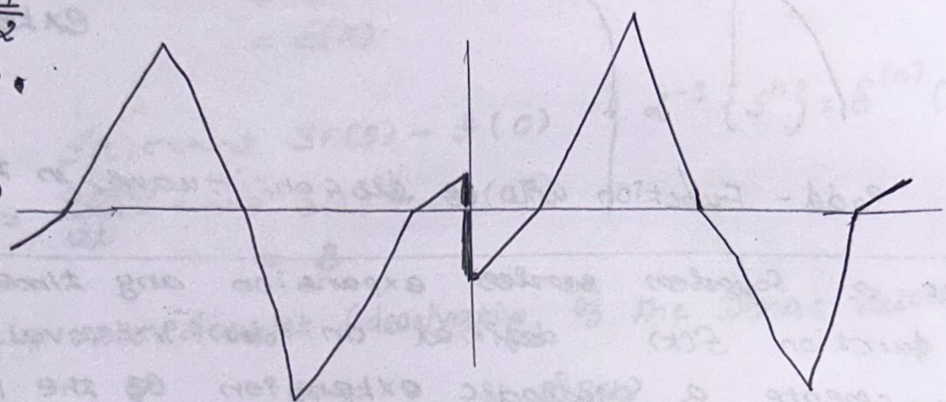


$g(0) = -\frac{1}{2}$

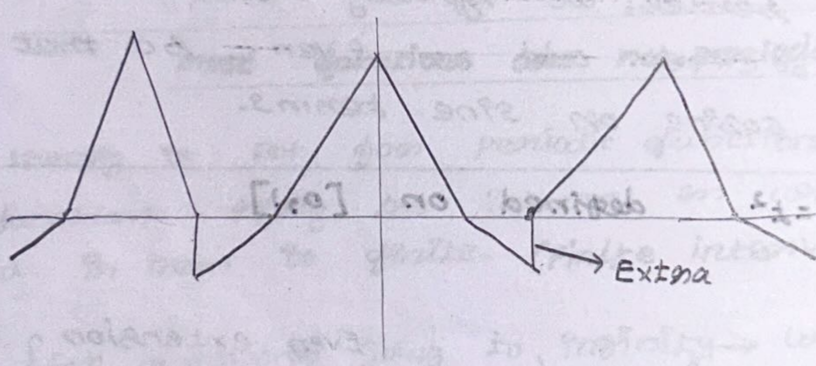
$g(1) = 0$

$g(2) = 3$

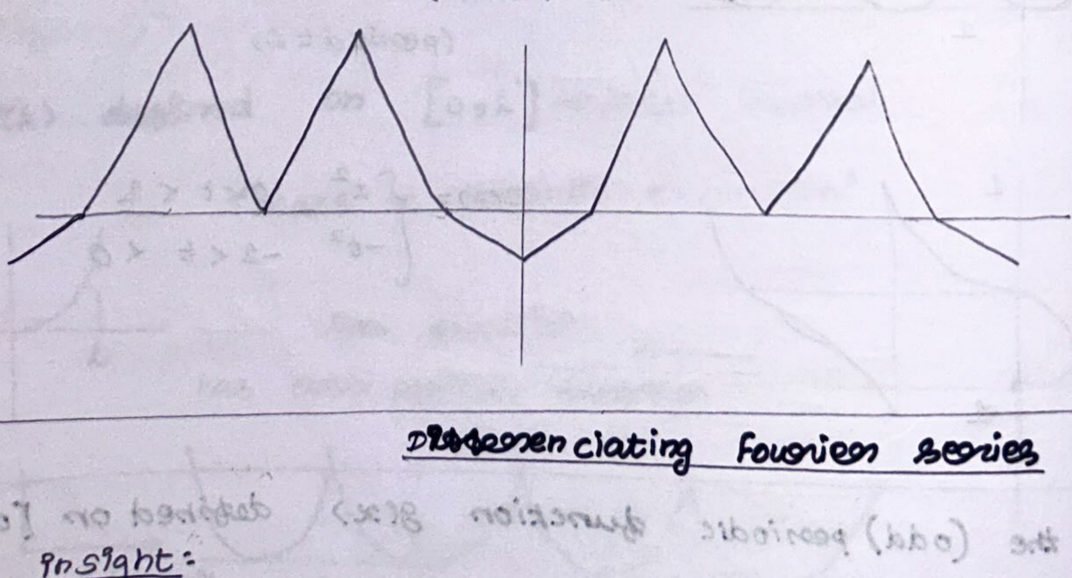
$g(3) = 0$



Even signal $[-6, 6]$



(No extra lines)



Differenciating Fourier series

Key insight:

If a function is differentiable, you can simply differenciate its Fourier series term by term to obtain the Fourier series for the derivative.

If I write arbitrary Fourier series, how do I know if it comes from a differentiable function? If I differentiate term by term the Fourier series for a function that's not differentiable (like square wave function), is the result of the fourier series for something.

10.1 The Fourier series for the period 2π triangule wave is

$$g(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos t + \frac{\cos 3t}{9} + \frac{\cos 5t}{25} + \cdots\right)$$

what's $g'(t)$?

Solu:

$$g'(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots\right)$$

(Fourier series of the 2π-periodic square wave.

Antiderivative of a Fourier series

suppose that f is a piecewise differentiable periodic function, and that F is an antiderivative of f. (If f has discontinuties (Jump), one can still define

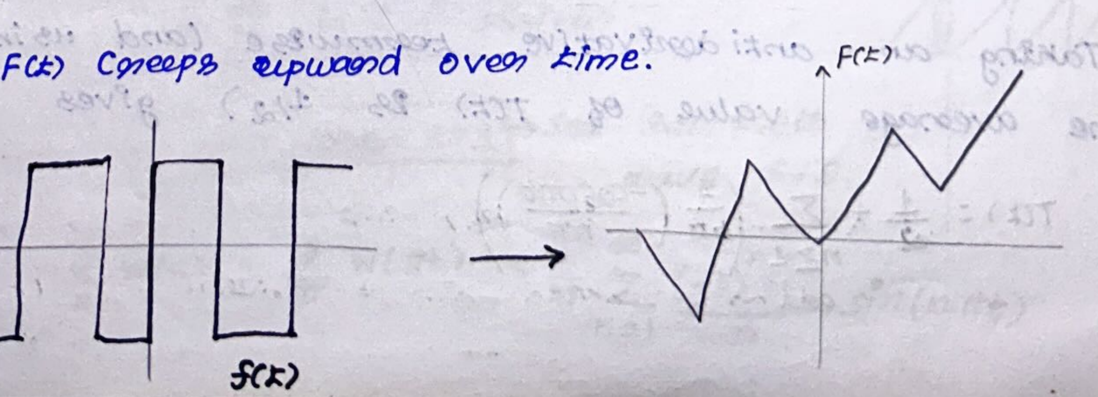
$$F(t) := \int_0^t f(\tau)\, d\tau + C,$$

but at the jump discontinuties, F will be only continuous, and not differenciable.

The Function F might not be periodic. for example, if f is a function of period 2 such that

$$f(t) := \begin{cases} 2 & \text{if } 0 < t < 1 \\ -1 & \text{if } -1 < t < 0 \end{cases}$$

the $F(t)$ creeps upward over time.



An easier example:

If $f(t) = 1$, the $F(t) = t + C$ for some C, so $F(t)$ is not periodic.

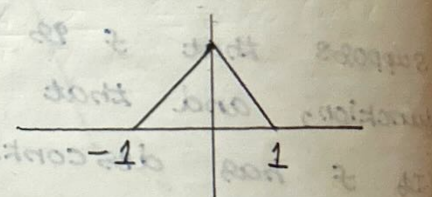
But if the constant term $\frac{a_0}{2}$ is zero (of the F.S of f) then F is periodic, and its Fourier series can be obtained by taking the simplest antiderivative of each cosine & sine term, and adding an overall $+C$, where C is the average value of F.

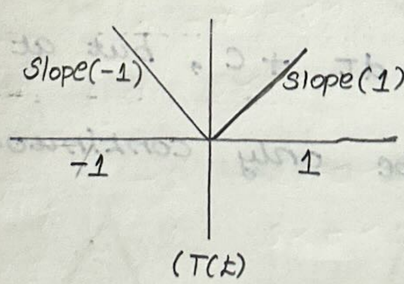
Problem 11.1

Let $T(t)$ be the triangle wave of period 2 and amplitude 1: so that $T(t) = |t|$ for $-1 \le t \le 1$. Find the Fourier series of $T(t)$.

Solu:
we would use the F.S formula. But instead, notice that $T(t)$ has slope -1 on $(-1, 0)$ and slope 1 on $(0, 1)$, so $T(t)$ is an antiderivative of the period 2 square wave.

Our Δ wave





$$\text{slope} = \frac{1-0}{0-(-1)} = 1$$

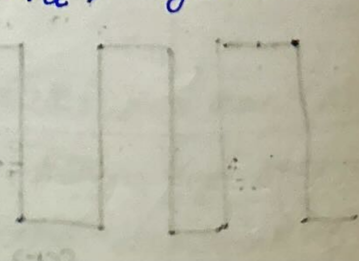
$$\text{slope} = \frac{0-1}{1-0} = -1$$

$$Sq(\pi t) = \sum_{n \ge 1 \text{ odd}} \frac{4}{n\pi}\left(\sin n\pi t\right)$$

$$= \sum_{n \ge 1 \text{ odd}} \left(\frac{4}{n\pi} \sin(n\pi t)\right)$$

Taking an antiderivative termwise (and using that the average value of $T(t)$ is $1/2$) gives

$$T(t) = \frac{1}{2} + \sum_{\substack{n \ge 1 \\ \text{odd}}} \left(\frac{4}{n\pi}\left(\frac{-\cos n\pi t}{n\pi}\right)\right)$$



$$T(t) = \frac{1}{2} - \sum_{\substack{n \geq 1 \\ (\text{odd})}} \frac{4}{n^2\pi^2} \cos(n\pi t)$$

warning: If a periodic function $f(t)$ is not continuous, it won't be an antiderivative of any piecewise differentiable function, so you can't find the fourier series of $f(t)$ by integration.

"All continuous functions are differentiable But all differentiable functions need not be continuous"

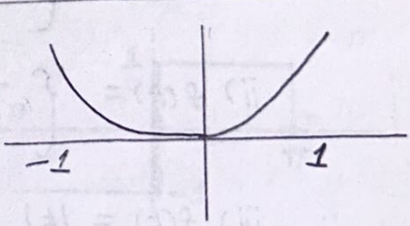
Remark: 11.2: The Fourier series for a function with discontinuities can (normally) be differentiated term by term, but the result won't converge. For example, the termwise derivative of the Fourier series for $Sq(t)$ is

$$\frac{4}{\pi} \sum_{n \text{ odd}} \cos(nt)$$

This doesn't converge anywhere (since the n^{th} term does n't even vanish as $n \to \infty$). However, note that it's possible to make sense of this series, and of the anti-derivative of the square-wave function, & the theory in terms of Dirac's delta function. of distributions - seen in more advaced courses.

Integrate to find the fourier series

$f(t) = t^2 \quad [-1, 1] \quad$ period $= 2$



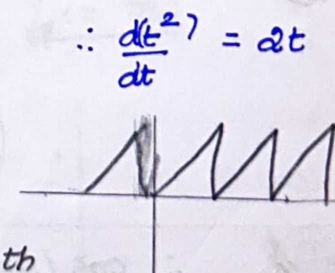
Find $\frac{a_0}{2}$

solu: The function is continuous, so it could be the antiderivative of another function

$$\therefore \frac{d(t^2)}{dt} = 2t$$

$$g(t) = 2t, \quad -1 < t < 1.$$

$$W(u) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nu).$$



$\boxed{u = \pi t, \quad \text{so } u = \pi \text{ when } t = 1} \rightarrow$ see saw tooth wave F.S

$$g(t) = \frac{2}{\pi} W(u) = \frac{2}{\pi} W(\pi t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin(n\pi t)$$

Finally, to find the F.S of $f(t)$, we integrate the F-S for $g(t)$ term-by-term.

$$\int g(t)\,dt = \frac{4}{\pi}\sum_{n=1}^{\infty}\int \frac{(-1)^{n+1}}{n}\sin(n\pi t)\,dt$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} -\frac{(-1)^{n+1}}{n}\,\frac{\cos(n\pi t)}{n\pi}$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2\pi}\cos(n\pi t) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi t)$$

To find the constant term, we find the value of $f(t)$ on interval of $f(t)$ $-1 < t < 1$.

$$\frac{a_0}{2} = \frac{\int_{-1}^{1} t^2\,dt}{1-(-1)} = \frac{1}{2}\left(\frac{1+1}{3}\right) = \frac{1}{3}.$$

$$\frac{a_0}{2} = \frac{\text{Integration of Area}}{\text{Interval}}$$

Manipulating Fourier series

1) Find the F.S for $f(t) = \cos(2t - \pi/4)$

2) Given $sq(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases} = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin(nt)$

Find the F.S for

i) $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 4 & 0 < t < \pi \end{cases}$ (period 2π)

ii) $f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$ (period 2)

iii) $f(t) = |t|$ on $-\pi < t\ \pi$ (period 2π)

Solu:

i) F.S of $f(t) = \cos(2t - \pi/4)$

$\left(\text{F.S} = \frac{a_0}{2} + \sum_n a_n\cos(nt) + \sum_n b_n\sin(nt)\right.$

$\therefore \cos(2t - \pi/4) \longrightarrow$ Amplitude form.

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$= \cos 2t\cos\left(\frac{\pi}{4}\right) + \sin 2t\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\cos 2t + \sin 2t\right)$

That's actually an F.S form.

$$\text{F.S of } f(t) = \frac{1}{\sqrt{2}}\sin 2t + \frac{1}{\sqrt{2}}\cos 2t.$$

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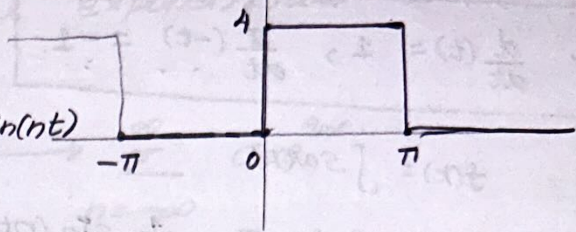
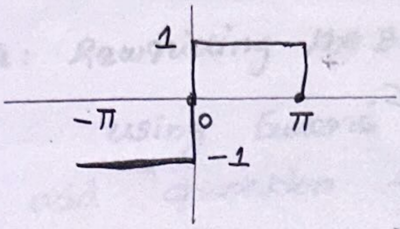
2) i) $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 4 & 0 < t < \pi \end{cases}$

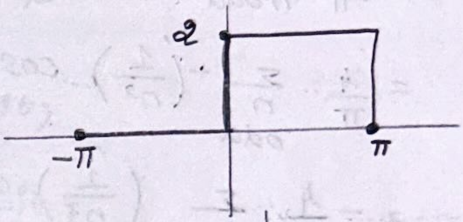


$$sq(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases} = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nt)$$

(Square wave by expanding)



shifted by 1 units up



scaling by 2 will give our new function.

$\therefore$

$$f(t) = 2\,(sq(t) + 1)$$

$$= 2\left(\frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nt)\right) + 2$$

$$= 2 + \frac{8}{\pi} \sum_{n \text{ odd}} \left[\frac{1}{n} \sin(nt)\right]$$

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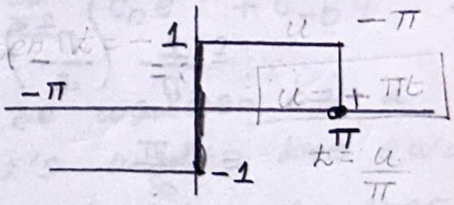
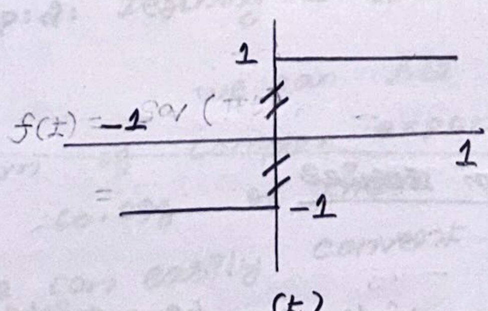
ii) $f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$

comparing $sq(u)$



$\therefore$

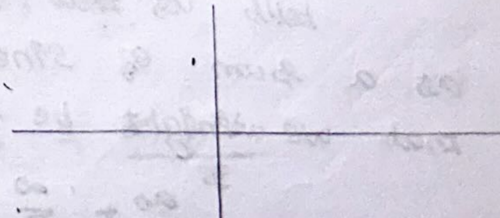


when $t = -1$, $u = -\pi$

$\boxed{u = \pi t}$

$$f(t) = sq(\pi t)$$

$$\boxed{f(1) = 1, \quad sq(\pi) = 1}$$

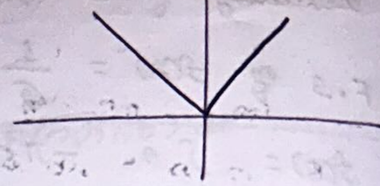


$$= \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(n\pi t)$$

(our one is periodic with 2)

The F.S too.

iii) $f(t) = |t| \qquad -\pi < t < \pi$



solu:-

Square wave fun $sq(t) = \begin{cases} 1 & 0<x<1 \\ -1 & -1<x<0 \end{cases}$

$$\boxed{\therefore \quad \frac{d}{dt}(t) = 1, \quad \frac{d}{dt}(-t) = -1.}$$

$$f(t) = \int sq(t)$$

$$= \int \frac{4}{\pi} \sum_{\substack{n \\ odd}} \frac{1}{n} \sin(nt)$$

$$= \frac{4}{\pi} \sum_{n\,odd} \left(\frac{1}{n}\right)\left(-\frac{\cos nt}{n}\right) + C$$

$$= \frac{4}{\pi} \sum_{\substack{n \\ odd}} -\left(\frac{1}{n^2}\right) \cos nt + C$$

$$= -\frac{4}{\pi} \sum_{\substack{n \\ odd}} \left(\frac{1}{n^2}\right) \cos nt + C$$

what's $C \rightarrow$ Identified by $\frac{a_0}{2}$ term of the F.S

$$C = \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |t|\, dt = \int_{-\pi}^{0} |t|\, dt + \int_{0}^{\pi} |t|\, dt$$

$$= \frac{2}{2\pi} \left(\int_0^{\pi} |t|\, dt\right)$$

$$= \frac{1}{\pi} \left(\frac{t^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right)$$

$$= \frac{\pi}{2}.$$

Complex Fourier Series

Euler's Formula: $e^{it} = \cos t + i \sin t$

tells us that complex exponentials can be written as a sum of sine & cosine functions. This suggests that we might be able to write a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

with real coeffs a_n and b_n as a series of complex exponentials $\sum_{n=-\infty}^{\infty} c_n e^{int}$, for some complex co-efficients c_n. As it turns out, this is true, that is, we can always write a Fourier series in terms of complex exponentials. Since the two series turn out to be equal. We'll also call the series in terms of complex exponentials a Fourier series.

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \longrightarrow \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Step: 1: Rewriting the sum.

using Euler's formula & the fact that $\sin t$ is an odd function & $\cos t$ is an even function,

we notice that

$$\sin t = \frac{i}{2}(e^{-it}-e^{it})$$

$$\cos t = \frac{1}{2}(e^{it}+e^{-it})$$

then we see that given any F.S f, we can write

$$f(t) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

$$= \frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n(e^{int}+e^{-int}) + \frac{ib_n}{2}(e^{-int}-e^{int})\right)$$

$$= \frac{a_0}{2}+\frac{1}{2}\sum_{n=1}^{\infty}((a_n-ib_n)e^{int}+(a_n+ib_n)e^{-int})$$

$$= c_0+\sum_{n=1}^{\infty}(c_n e^{int}+c_{-n}e^{-int})$$

Step: 2: Defining co-efficients:

we can see that f can be written as a sum of complex exponentials. Let's write the eqn nicely so that of co-eff of these exponentials b/w the two we can easily convert back and forth forms.

Define $c_0 = \frac{a_0}{2}$ $(n>0)$, define

$$c_n := \frac{a_n - ib_n}{2}, \quad c_{-n} := \overline{c_n} = \frac{a_n+ib_n}{2}$$

we can write f compactly as ($\sum_{n=-\infty}^{\infty}$ include $-n, +n, 0$)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Step:3: Relating co.eff to inner products:

Remark about inner products:

Note that for real valued 2π - periodic function f and g we define an inner product as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f \cdot g \, dt.$$

If f and g are complex valued functions, we must define the inner product as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f \cdot \bar{g} \, dt \rightarrow$$ From above

$\bar{g}$ is the complex conjugate of g.

13.1

$$\langle e^{int}, e^{int}\rangle = \int_{-\pi}^{\pi} e^{int} \cdot e^{-int} \, dt = \int_{-\pi}^{\pi} dt = 2\pi$$

If $m \neq n$

$$\langle e^{int}, e^{imt}\rangle = \int_{-\pi}^{\pi} e^{int} \cdot e^{-imt} \, dt$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)t} \, dt$$

$$= \left. \frac{e^{i(n-m)t}}{i(n-m)} \right|_{-\pi}^{\pi}$$

Expanding $\leftarrow$

$e^{it} = \cos t + i \sin t.$

$$= \frac{e^{i(n-m)\pi} - e^{-i(n-m)\pi}}{i(n-m)}$$

$$= \frac{2}{i(n-m)} \sin((n-m)\pi) = 0$$

$\therefore$ $\boxed{\sin n\pi = 0}$

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f \cdot \bar{g} \, dt$$

$$\therefore \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left((a_n - ib_n) e^{int} + (a_n + ib_n) e^{-int} \right)$$

for an operation to be inner product it must satisfy very specific conditions. It turns out that we need to take the conjugate of one of the complex functions. One of the conditions for an operation to be an inner product is that

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$

when f and g are real, $\bar{g} = g$.

An inner product is a generalization of the dot product in a vector space, it's a way to multiply vectors together, with the result of this multiplication being a scalar.

More precisely, for a vector (real) space, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following four properties. u, v, w : be vectors and α - scalar.

1) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

2) $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$

3) $\langle v, w\rangle = \langle w, v\rangle$

4) $\langle v, v\rangle \geq 0$ and equal if and only if $v=0$.

mathworld.wolfram.com/Innerproduct.html

$\therefore e^{int}$ are orthogonal.

In, particular, notice that for $n>0$, we can compute c_n by the following formula

$$c_n = \frac{\langle f, e^{int}\rangle}{\langle e^{int}, e^{int}\rangle} \begin{array}{l} \longrightarrow \text{Integral over } -\pi \text{ to } \pi. \\ \longrightarrow \text{Boundary } (2\pi) \end{array}$$

This definition is in agreement with the definition of

$$\boxed{c_n = \frac{a_n - ib_n}{2}}$$

<u>checking</u> $n>0$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt - \frac{i}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,(\cos nt - i\sin nt)\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,(\cos(-nt) + i\sin(-nt))\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt \quad \left(= \frac{1}{2\pi}\langle f, e^{int}\rangle\right.$$

$$\frac{\langle f, e^{int}\rangle}{\langle e^{int}, e^{int}\rangle}$$

An analogous computation shows that for any n,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \frac{\langle f, e^{int}\rangle}{\langle e^{int}, e^{int}\rangle}$$

$$\boxed{\langle f, e^{int}\rangle = \int_{-\pi}^{\pi} f\cdot\bar{g}\, dt}$$

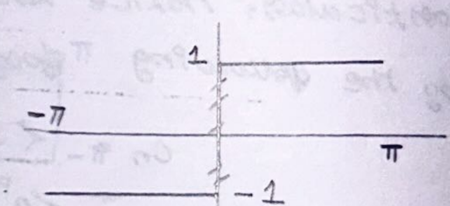
Key properties of complex Fourier series:

* often integrals involving complex exponentials are actually much easier to compute than integrals involving sines and cosines.

* we can also make sense of Fourier series of complex-valued functions more easily in this setting.

Let $sq(t)$ be the 2π-periodic square wave that's equal to 1 for $0 \le t < \pi$ and equal to -1 for $-\pi \le t \le 0$. For $n \neq 0$.

solu:



Multiple ways to approach this problem. One is to use the co.effs a_n and b_n to write c_n. Another better approach is calculating c_n directly.

$\boxed{n \neq 0}$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} sq(t)\cdot e^{-int}\, dt = \frac{\langle f, e^{int}\rangle}{\langle e^{int}, e^{int}\rangle}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-int}\, dt + \frac{1}{2\pi}\int_{0}^{\pi} e^{-int}\, dt$$

$\hookrightarrow sq(t) = -1$ $\qquad$ $\hookrightarrow sq(t) = 1$

$$= \frac{1}{2\pi}\left[\left(-\frac{e^{-int}}{-in}\right)_{-\pi}^{0} + \left(\frac{e^{int}}{-in}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{in}\right)\left[+e^{0} - e^{in\pi} + \left(-e^{-in\pi} + e^{0}\right)\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{in}\right)\left[2 - \left(e^{in\pi} + e^{-in\pi}\right)\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{in}\right)\left[2-(\cos n\pi + i\sin n\pi + \cos n\pi - i\sin n\pi)\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{in}\right)\left[2-2\cos n\pi\right]$$

$$= \frac{2}{2\pi}\left(\frac{1}{in}\right)\left[1-(-1)^n\right]$$

$$= \frac{1}{\pi i n}\left(1-(-1)^n\right) \qquad c_n = \begin{cases} 0 & n \text{ even} \\ \frac{2}{in\pi} & n \text{ odd} \end{cases}$$

when $n=0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} Sq(t)\,dt = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\,dt + \int_{0}^{\pi} 1\,dt\right]$$

$$= \frac{1}{2\pi}\left[-(t)_{-\pi}^{0} + (t)_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[-(+\pi) + \pi\right] = 0$$

Hence the F.S:

$$Sq(t) = \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{in\pi}\, e^{int} \qquad \left[\sum_{n=-\infty}^{\infty} c_n e^{int}\right]$$

Visualizing Complex Fourier Series for Sound Signals

In this course, our applications will involve real-valued signals, such as sound, rather than complex valued signals (studying signals coming from Electromagnetic waves for example won't fit in this scenario)

Complex Fourier series of a real signal:

Suppose that $2L$-periodic function $f(t)$ is real valued. Which of the following are true about the complex coefficients c_n of the complex F.S such that

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/L}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)\cdot e^{-in\pi t/L}\,dt = \frac{\langle f(t), e^{in\pi t/L}\rangle}{\langle e^{in\pi t/L}, e^{in\pi t/L}\rangle}$$

Solu:

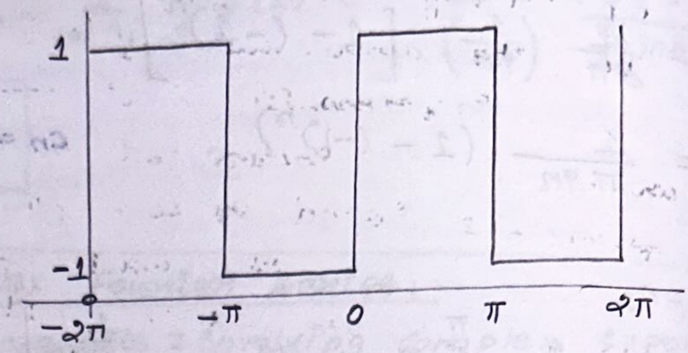
$$c_n = \frac{a_n - ib_n}{2}$$

$$c_{-n} = \overline{c_n} = \frac{a_n + ib_n}{2} \qquad \boxed{c_{-n} = \overline{c_n}}$$

$f(t)$ is real.

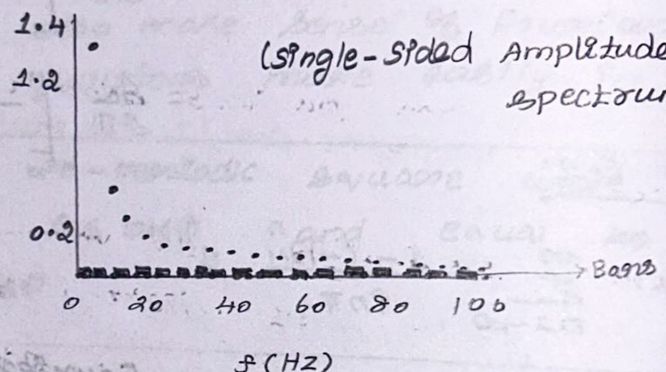
Coefficients of the complex Fourier series:

The coefficients of the complex Fourier series, c_n are complex numbers. To visualize them, we typically look at the magnitudes, c_n, which are real. For example, suppose we have a real, 2π-periodic (square wave) signal.



Square wave with period 2π.

Then we can visualize the coefficients c_n graphically as



The frequencies $\frac{\omega n}{2\pi}$ are displaced along the horizontal axis. Magnitudes of the coefficients c_n are plotted on the vertical axis.

The horizontal axis represents the frequency $\frac{\omega n}{2\pi}$, and the height of each bar represents the magnitude of each coefficient. (c_n). Because the signal is real,

$|c_n| = |c_{-n}|$, hence the graph is symmetrical about $n=0$.

To understand what this graph is telling us, remember that a sinusoid of frequency ω can be written in three equivalent ways.

$$\text{Re}(ce^{i\theta}) = \text{Re}(\bar{c}e^{-i\theta}) = A\cos(\theta-\phi)$$

$$\boxed{|c| = A, \ \text{Ang}(c) = -\phi}$$

Therefore

$$ce^{i\theta} + \bar{c}e^{-i\theta} = 2A\cos(\theta-\phi) \quad [A\cos(\theta-\phi) + A\cos(\theta-\phi)]$$

$$\sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} = c_0 + \sum_{n=1}^{\infty} (c_n e^{i\omega_n t} + \overline{c_n} e^{-i\omega_n t})$$

$$= c_0 + \sum_{n=1}^{\infty} (2A_n \cos(\omega_n t - \phi_n))$$

Therefore by knowing the magnitudes of the co-efficients c_n, we know the amplitudes of the terms A_n, but we lose the information about the phase shifts ϕ_n. When we are interested in real signals that are coming from the sound of a single note, losing the information about the phase shift is OK. The reason being that we can't actually discern phase shifts in sound signals. So this loss of information is not detrimental to our understanding of the signal.

verify to yourself that you can't hear phase shifts using the mathlet.

1. Create a real signal. set values c_n and make sure that $|c_n| = |c_{-n}|$ by clicking the box 'f(t) real'.

2. By default all of the phase shifts are zero. we can change it.

3. Warning: pure sinusoids can be damaging to your hearing. Recommendation: keep the volume level very low and don't listen for too long! (use lower frequencies and keeping the amplitude of higher frequencies small.)

4. press play. manipulate the phase shift & see if you can hear any difference in the sound signal

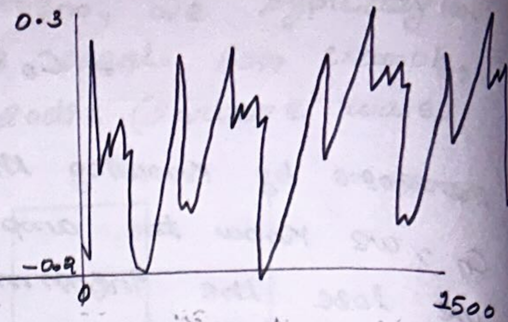
5. press pause! (Don't damage your hearing)

'phase shift' – Just a graph shifting horizontally in the time axis, so we can always set the time origin as we want.

↳ matters when both sine & cosines are in a single graph

Complex Fourier series of a Sound signal:

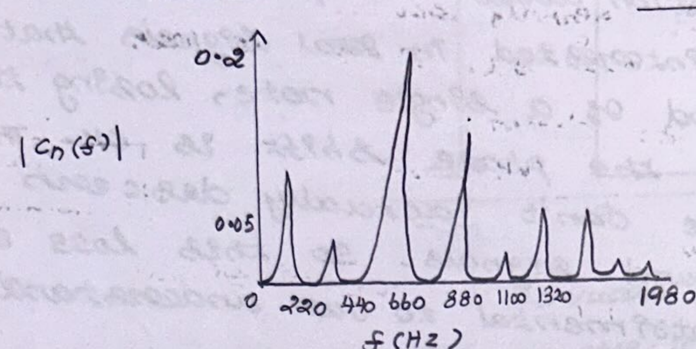
A (singing) at 220hertz.



(Pronouncing A)

The absolute value of the fast Fourier transform is

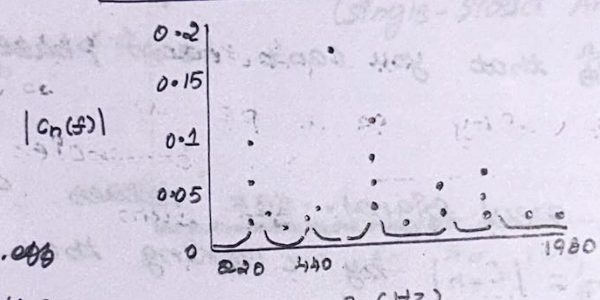
Continuous Frequency Spectrum



Discrete Frequency Spectrum

$\frac{\omega_n}{2\pi} = f_n$ are displayed along the horizontal axis.

Magnitudes of the co.eff $|c_n|$ are plotted on the



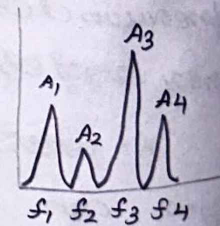
vertical axis shown continuously on the left, and descrete plot on the right.

1) First four terms of the Fourier series as

$$2A_1 \cos(2\pi f_1 t - \phi_1) + 2A_2 \cos(2\pi f_2 t - \phi_2) + 2A_3 \cos(2\pi f_3 t - \phi_3) + 2A_4 \cos(2\pi f_4 t - \phi_4)$$

$A_1 = 0.095$, $A_2 = 0.025$, $A_3 = 0.165$, $A_4 = 0.095$
(0.075) (0.02) (0.17) (0.09)

$f_1 = 220$, $f_2 = 440$, $f_3 = 660$, $f_4 = 880$

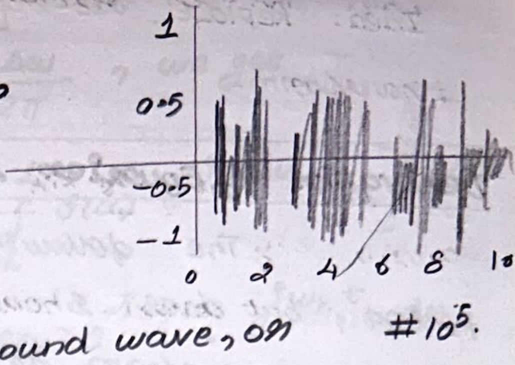


Generalizing Fourier series: The Fourier transform

So far, we have only worked with periodic functions & signals. In the real world, signals may not be periodic.

Example: 15.1: The function $f(t) = e^{-t^2}$ is a non-periodic function that's important in probability & differential equations.

Example: 15.2: Sound signals are not truly periodic as the signal exists for finite time.



Let's generalize the method of Fourier Series to non-periodic signals. This is known as the Fourier transform.

Suppose you have a radio signal, sound wave, or other signal $f(t)$ with period $2L$. Then we can write the signal in the form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{(ik\pi/L)t}$$

$$\boxed{\frac{k\pi}{L} = \omega_n}$$

where the c_k are the co-efficients determined by

$$c_k = \frac{1}{2L} \int_{-L}^{L} f(t)\, e^{-(ik\pi/L)t}\, dt$$

we can think of a non-periodic signal as the limit as L goes to infinity of a periodic signal of $2L$. As L increases, the spacing b/w the frequencies in our sum are approaching zero. This turns the sum into an integral in the limit, and we have the equations:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikt}\, dk$$

[Transforming periodic function to get our non-periodic Function]

$$\boxed{\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-ikt}\, dt}$$

we call $\hat{f}(k)$, the Fourier transform of $f(t)$.

Note: The continuous function $\hat{f}(k)$ replaces the discrete (non-Continuous) co-efficients c_k.

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikt}\, dt$$

$\longrightarrow$ Transforming periodic wave.

So now $f(t)$ can be composed of a continuous infinite sum (an integral) of complex sinusoids e^{ikt} with the weights being given by the $\hat{f}(k)$ function.

Idea: Replace discrete co-effs by $\hat{f}(t) \rightarrow$ Fourier transform.

Derivation of Fourier transform:

The following argument doesn't constitute proof, but does show you how the formula for the Fourier transform arises. We leave out some technical conditions related to the convergence of the finite fourier series & Improper integral of $f(x)$ (the function $f(x)$ satisfies the Dirichlet condition on every finite interval and

$$\int_{-\infty}^{\infty} |f(x)|\, dx \text{ is finite}$$

$f(t)$ is finite in $[-L, L]$ as $L \to \infty$

It will be only defined on that particular interval. In other places it's value is zero.

---

start with

$$f(t) = \sum_{K=-\infty}^{\infty} c_K \, e^{i \frac{K\pi}{L} t}$$

make a change of variable so that

$$\boxed{\omega_K = \frac{K\pi}{L}}, \quad \omega_{K+1} = \frac{(K+1)\pi}{L}$$

$$\boxed{\omega_{K+1} - \omega_K = \frac{\pi}{L}} = \Delta\omega$$

The Fourier series can be then written as the following where the dummy variable t has been replaced by u in the eqn for c_K to avoid confusion.

$$f(t) = \sum_{K=-\infty}^{\infty} c_K \, e^{i\omega_K t}$$

$$c_K = \frac{1}{2L} \int_{-L}^{L} f(t)\, e^{-i\omega_K t}\, dt$$

$$= \frac{1}{2L} \int_{-L}^{L} f(u)\, e^{-i\omega_K u}\, du$$

Substituting the c_K expression in to the formula of