Initial conditions

1)
$$\theta(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$$

$\theta(x,0)=1 \rightarrow$ Initial condition.

$$\theta(x,0) = \sum_{n=1}^{\infty} c_n e^{-0n^2} \sin(nx) = 1.$$

$$= \sum_{n=1}^{\infty} c_n \sin(nx) = 1$$

'Square wave'
$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0. \end{cases}$$

$$c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx$$

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases} \rightarrow \text{square wave.}$$

Summary:

we modeled an insulated rod (metal) with exposed ends held at $0°c$.

using physics, we found its temperature $\theta(x,t)$ was governed by the PDE

$$\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2} \quad 0<x<\pi \quad \text{(the heat eqn)}$$

For simplicity, we specalized the case to

$\vartheta = 1$, length $= \pi$, $\theta(x,0) = 1$ (Initial condition)

Trying $\theta = v(x) \cdot w(t)$ led to seperate ODEs for v and w, leading to solutions

$$e^{-n^2 t} \sin(nx) \quad \text{where } n=1,2,3\ldots.$$

to the PDE with boundary conditions

we took linear combinations to get the general solutions.

$$\boxed{\theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + b_3 e^{-9t}\sin 3x + \cdots}$$

to the PDE with homogeneous boundary conditions

$\theta(0,t) = 0$ and $\theta(\pi,t) = 0$

Initial conditions: As usual, we postponed imposing the initial condition, but now it's time to impose it.

---

question: 1:

which choices of $b_1, b_2 \cdots$ make the general solution above also satisfy the initial condition

$\theta(x,0) = 1$ for $x \in (0,\pi)$

set $t=0$ in

General solution: $\theta(x,t) = b_1 e^{-t}\sin x + b_2 e^{-4t}\sin 2x + \cdots\cdots$

(The general solution to the heat equation).

use the I.C to get

$$\theta(x,0) = b_1 e^{-0}\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

$$1 = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

which must be solved for $b_1, b_2, \ldots$

$\therefore$ The right hand side is odd and of base period 2π, to find such b_i, the left hand side must be extented to an odd period (2π) function namely,

$Sq(x)$. So we need to solve,

$$Sq(x) = b_1\sin x + b_2\sin 2x + b_3\sin 3x \ldots$$

for all $x \in R$.

we already know the answer

$$Sq(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots$$

In other words

$b_n = 0$ for even functions

and $b_n = \frac{4}{n\pi}$ for odd n. Substituting these b_n back into our general solution gives,

$$\theta(x,t) = \frac{4}{\pi} e^{-t} \sin x + \frac{4}{3\pi} e^{-9t} \sin 3x + \frac{4}{5\pi} e^{-25t} \sin 5x + \dots$$

Example: 5.2:

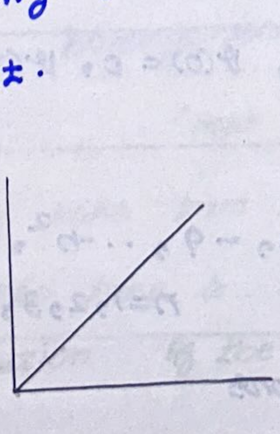
What does the temperature profile look like when t is large?

$\therefore$ All the Fourier components are decaying, so $\theta(x,t) \to 0$ as $t \to \infty$. at every position.

Thus the temperature profile approaches a horizontal segment, the graph of the zero function. But the Fourier components of higher frequency decay much faster than the first Fourier component, so when t is large, the formula

$\theta(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$ is a very good approximation. Eventually the temperature profile is indistinguishable from a sinusoid of angular frequency whose amplitude is decaying to 0. This can be observed in mathlet.

As time goes on



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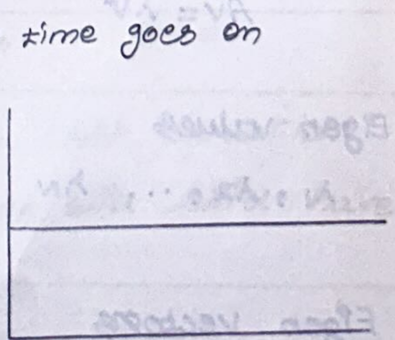


$b_3 \, e^{-16Kt} \sin(4x)$.

what's the steady state solution $\theta(x)$ defined

$$\theta(x,t) \rightarrow \theta(x) \quad \text{as} \quad t \rightarrow \infty$$

$\theta(x)$

$$\therefore \quad \theta(x) = 0.$$

$$\therefore \quad \theta(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$$

$$\text{as } t \rightarrow \infty, \quad \theta(x,t) \longrightarrow 0.$$

If you submerge the ends of a metal rod in an ice bath, eventually, the temperature every where in the bar will be 0.

Linear algebra and Fourier analogy

Analogy b/w linear algebra and fourier techniques. On the L.H.S, we present a general linear algebra example, on the R.H.S, we illustrate the analogy using the specific example we worked through on the previous pages.

System of ODE	System (Heat equation)
Vector v	Function $v(x)$
Matrix A	linear operator $\frac{d^2}{dx^2}$
$Av = f$	$\frac{d^2}{dx^2} v(x) = f(x)$ on $0 < x < \pi$; $v(0) = v(\pi) = 0$.
Eigen-value-Eigen vector problem	Eigen value-Eigen function problem
$Av = \lambda v$	$\frac{d^2}{dx^2} v = \lambda v$, $v(0) = 0$, $v(\pi) = 0$.
Eigen values $\lambda_1, \lambda_2, \ldots \lambda_N$	Eigen values $\lambda = -1, -4, -9, \ldots -n^2$, for $n = 1, 2, 3, \ldots$
Eigen vectors $V_1, V_2, \ldots V_n$	Eigen functions $v(x) = \sin nx$ for $n = 1, 2, 3, \ldots$

Linear system of ODEs	Heat equ with boundary conditions
$\dot{x} = Ax$	$\dot{\theta} = \frac{\partial^2}{\partial x^2}\theta$ on $0 < x < \pi$ $\theta(0, t) = 0, \ \theta(\pi, t) = 0.$
Normal modes: $e^{\lambda_n t} v_n$ for $n = 1, \dots, N$	normal modes: $e^{\lambda t} v(x) = e^{-n^2 t}\sin(nx)$ for $n = 1, 2, 3, \dots$
General solution: $u(t) = \sum c_n e^{\lambda_n t} v_n$	General Solution: $\theta(x, t) = \sum b_n e^{-n^2 t} \sin nx$
solve $u(0) = \sum c_n v_n$ to get the c_n	Solve $\theta(x, 0) = \sum b_n \sin(nx)$ to get the b_n

This analogy does carry through in other examples & different homogeneous boundary conditions. we used the specific example for illustration purposes only.

Solving the PDE with inhomogeneous boundary conditions.

$\theta(0, t) = 0^\circ C$ $\qquad$ $\theta(x, 0) = 1$

$\theta(\pi, t) = 20^\circ C$

Seperation of variables can only be used to an homogeneous equation.

("Heat equation $\rightarrow$ Linear")

If two take two solutions of the heat equation and then take a sum of them, we will still have a solution of the heat equation.

Now take,

$x(\theta, t)$ as the sum of two functions.

$$\theta(x,t) = \theta_p + \theta_h.$$

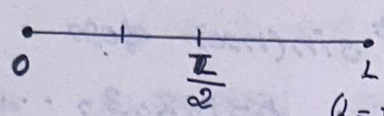
Solution:

$$\theta_p = \frac{20x}{\pi} \qquad \therefore \theta(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$$

as $t \to \infty$

$\theta(x,t) \approx \frac{4}{\pi} e^{-t} \sin x$ previous

After attaining equilibrium,



$(l = \pi)$ $\boxed{\theta_0 = 20°C}$

$$\theta_{eq}(x) = \frac{\theta_0 x}{L}$$

At $\frac{L}{2} \to \theta = \frac{\theta_0}{2}$

$\frac{L}{4} \to \frac{\theta_0}{4}$

$$= \frac{\theta_0 x}{\pi}$$

$$\boxed{\theta_p = \frac{20x}{\pi}}$$

verifying is θ_p satisfying Initial & boundary conditions?

$\theta_p(0) = 0°c$
$\theta_p(\pi) = \frac{20(\pi)}{\pi} = 20°c$ } satisfying.

$\therefore$ $\theta_p \to$ Independent of time.

sum ($\theta_p + \theta_h \to$ satisfies Initial conditions).

$\theta_h = ?$

$\theta_h(0,t) = 0$
$\theta_h(\pi,t) = 0$ } then only θ_p will be
$\theta_p(0,t) = 0$
$\theta_p(\pi,t) = \pi$

$\therefore$ Then over all solution will be

$$\boxed{\begin{aligned} \theta(0,t) &= 0 + 0 = 0 \\ \theta(\pi,t) &= \pi + 0 = \pi \end{aligned}}$$

$\therefore$ we already found that expression for $\theta(0,t) = 0$ and $\theta_h(\pi,t) = 0$.

$$\theta_h(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)$$

↳ From seperation of variable.

↓

This need to satisfy the initial condition.
$\theta(x,0) = 1.$

$$\theta_h(x,0) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin(nx)$$

our expression:

$$\theta = \theta_p + \theta_h$$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(u)\,du$$

$$\therefore \theta(x,0) = \frac{20x}{\pi} + \theta_h$$

$$\theta_h = \theta(x,0) - \frac{20x}{\pi}$$

$$\boxed{\theta_h = 1 - \frac{20x}{\pi}}$$

$$C_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{20x}{\pi}\right)\sin(nx)\,dx$$

$$\theta(x,t) = \underbrace{\frac{20x}{\pi}}_{\theta_p} + \underbrace{\sum_{n=1}^{\infty} C_n e^{-n^2 t}\sin(nx)}_{\theta_h}$$

Boundary conditions that are not zero are called inhomogeneous boundary conditions.

Steps to solve a linear PDE with inhomogeneous boundary conditions:

Find a particular solution to the PDE with the inhomogeneous boundary conditions. (but with IC). If the BC doesn't depend on t, try to find the steady-state solution $\theta_p(x)$.

(i.e - the solution doesn't depend on t).

2) Then $\theta = \theta_p + \theta_h$ is the general solution to the PDE with the inhomogeneous boundary conditions, where

* θ_h - General solution to the PDE with IC

3) If IC $\theta(x,0)$ are given, use the IC $\theta(x,0) - \theta_p$ to find the specific solution to the PDE with the inhomogeneous boundary conditions.

(This often involves finding Fourier co.efficients)

7.1:

consider the same insulated uniform metal rod as before ($\vartheta = 1$, length $= \pi$, Initial temp $1°c$). But now suppose that the left end is held at $0°c$. while the right end is at $20°c$.

Solu:

$$\theta_p = \frac{20x}{\pi}$$

$$\theta_h = b_1 e^{-t} \sin x + b_2 e^{-4t} \sin 2x + b_3 e^{-9t} \sin 3x + \cdots$$

$$\therefore \quad \theta(x,t) = \theta_p + \theta_h$$

$$\theta(x,t) = \frac{20x}{\pi} + b_1 \sin x \cdot e^{-t} + b_2 e^{-4t} \sin 2x \cdots$$

$$\theta(x,0) = 1$$

$$\boxed{1 - \frac{20x}{\pi} = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x \cdots \quad \text{for all } x \in (0,\pi)}$$

$$\therefore \theta_h = 1 - \frac{20x}{\pi}$$

Extend $\left(1 - \frac{20}{\pi}x\right)$ on $(0,\pi)$ to an odd periodic function $f(x)$ of period 2π. Then use the Fourier co.eff formulas to find the b_n such that

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

alternatively, find the F.S for the odd periodic extensions of 1 and x seperately, and take a linear combination to get

$\left(1 - \frac{20}{\pi} x\right)$. once the b_n are found, plug them back in to the general solution for the heat equ with inhomogeneous BC.

Find the steady state solution

consider the same insulated uniform metal rod as before ($\nu = 1$, length $= \pi$) but with initial temperature

$$\theta(x, 0) = x^2$$

suppose that the left end is held at $20°c$ while the right end is held at $20°c$.

Find the S.S $\theta(x)$:

$$\theta_{equ} = \frac{\theta_0 x}{L} = \frac{20(\pi)}{\pi} = 20°c$$

steady state solution occurs when the entire bar has temperature $20°c$.

Find the IC & BC

$\theta(x,t) = \theta(x) + \theta_h(x,t)$ where $\theta(x)$ is the steady state solution you found in the previous problem.

The function $\theta_h(x,t)$ satisfies

$$\frac{\partial}{\partial t} \theta_h = \frac{\partial^2}{\partial x^2} \theta_h(x,t) \qquad 0 < x < \pi.$$

what IC must $\theta_h(x,t)$ satisfy

solu..

$0 < x < \pi$, $\theta_h(x, 0) = ?$ $0 < x < \pi$

$\theta_h(0, t) = ?$ $t > 0$

$\theta_h(\pi, t) = ?$ $t>0$.

Solu:

$$\theta(x,0) = x^2$$

$$\therefore \quad \theta(x,0) = \theta_p + \theta_h$$

$$\theta_h(x,0) = \theta(x,0) - \theta_p$$

$$\boxed{\theta_h(x,0) = x^2 - 20}$$

$$\left.\begin{aligned} \theta_h(0,t) &= 0 \quad t>0 \\ \theta_h(\pi,t) &= 0 \quad t>0 \end{aligned}\right\} \rightarrow \text{Homogeneous } \theta. \ (\because BC = 0)$$

$$\theta(0,t) = \theta(0) + \theta_h(0,t)$$

$$\theta_h(0,t) = 20 - 20$$

$$\boxed{\theta_h(0,t) = 0}$$

$$\theta(\pi,t) = \theta(\pi) + \theta_h(\pi,t)$$

$$\theta_h(\pi,t) = 20 - 20$$

$$\boxed{\theta_h(\pi,t) = 0}$$

A new boundary condition: insulated ends

Differential eqn with boundary conditions:

$$v''(x) = \lambda v(x), \quad \text{for } 0 < x < \pi$$

$$v'(0) = 0, \quad v'(\pi) = 0$$

Solu:

$$\frac{\partial^2 v}{\partial x^2}(x) = \lambda v(x)$$

case: 1:

If $\lambda > 0$, then the general solution is

$$v(x) = c_1 e^{nx} + c_2 e^{-nx}. \quad \text{where } n^2 = \lambda$$

$$\therefore \quad \gamma^2 = n^2$$

$$\boxed{\gamma = \pm n} \qquad v(x) = c_1 e^{nx} + c_2 e^{-nx}.$$

when $c_1 = c_2$ $\qquad v'(0) = 0, \quad v'(\pi) = 0.$

$$\therefore \quad v'(x) = n c_1 e^{nx} + (n)(-c_2) e^{-nx}$$

$$\boxed{nc_1 - nc_2 = 0} \qquad c_1 = c_2$$

when

$\vartheta'(\pi) = nc_1 e^{\pi n} - nc_2 e^{-n\pi}$

$c_1 e^{\pi n} - c_2 e^{-n\pi} = 0.$ $\therefore\ c_1 = c_2.$

$e^{\pi n} = e^{-\pi n}$ (contradiction)

$\therefore$ The last eqn hold for any $n>0$, therefore there are no solutions in this case.

$\underline{\lambda = 0:}$

$s^2 = 0 \Rightarrow s = 0, 0$

$\therefore$ roots $\vartheta(x) = (c_1 x + c_2) e^{0x}$

$= c_1 x + c_2.$

$\vartheta'(x) = c_1$

$\vartheta'(0) = 0 = c_1$ $\therefore\ \vartheta(x) = c_2$

$\vartheta'(\pi) = 0 = c_2$ $\therefore$ many $(c_2) \rightarrow$ family of solutions are available.

$\underline{\lambda < 0:}$

$s^2 = -n^2$

$\vartheta(x) = c_1 \cos(nx) + c_2 \sin(nx)$

$\vartheta'(0) = 0,\ \vartheta'(\pi) = 0$

$-c_1 \sin(nx) \times n + c_2 \cos(nx) \times n = \vartheta'(x)$

$\vartheta'(0) = 0 = nc_2$

$\boxed{c_2 = 0}$

$\vartheta'(\pi) = 0 = + c_2 \cos(n\pi) \times n$

$0 = (-1)^n \times n c_2 + c_1 \sin(n\pi) n$

we need a non-trivial solution

$\boxed{c_1 = 0}$ (or) $\sin(n\pi) = 0$ $\boxed{c_1 \neq 0}$

The non trivial solution occurs when n is an integer. $\lambda = -n^2$ and $n = 0, 1, 2, 3, \ldots$

$\therefore \ v(x) = c_1 \cos(nx) = C \cos(nx)$ is the general solution when $\lambda = -n^2$.

By allowing $n=0$, we are covering the case $\lambda = 0$ of constant solutions.

Ans: $C\cos(nx)$ for $\lambda = -n^2$ $\quad n = 0, 1, 2, 3, \ldots$

Consider the same insulated uniform metal rod as before ($\nu = 1$, Length of the rod π) but assume now the ends are insulated too (instead of exposed and held in ice). and that the Initial temperature is gn by

$$\theta(x, 0) = x \quad \text{for } x \in (0, \pi).$$

Soln:

Insulated ends means that there is zero heat flow through the ends, so the heat flux density function $\propto -\frac{\partial \theta}{\partial x}$ is 0. when $x = 0$ or $x = \pi$. In other words, the boundary condition when 'insulated ends' mean

$$\frac{\partial \theta}{\partial x}(0, t) = 0, \quad \frac{\partial \theta}{\partial x}(\pi, t) = 0 \quad \text{for all } t > 0.$$

Instead of $\theta(0, t) = 0$ and $\theta(\pi, t) = 0$. So we need to solve the heat equation

[Heat flux density $= \dfrac{\text{Heat rate}}{\text{Area}}$.

In thermally Insulated $\rightarrow$ No heat transfer.

Heat rate $\rightarrow$ Heat transferred.

$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$ with boundary conditions for insulated ends. seperation of variables $\theta(x,t) = a(x) b(t)$. leads to

$$\frac{\partial}{\partial t}(a(x)\,b(t)) = \frac{\partial^2}{\partial x^2}(a(x)\,b(t))$$

$$a\frac{\partial b}{\partial t} = b\frac{\partial^2 a}{\partial x^2}$$

$$\boxed{\frac{1}{b}\frac{\partial b}{\partial t} = \frac{1}{a}\frac{\partial^2 a}{\partial x^2}}$$

$\frac{\partial \theta}{\partial x}(0,t) = 0$

$\frac{\partial \theta}{\partial x}(\pi,t) = 0.$

$$\frac{1}{b}\frac{\partial b}{\partial t} = \lambda = \frac{\partial^2 a}{\partial x^2}$$

$$\lambda = \frac{\partial b}{\partial t}$$

$$\therefore \frac{\partial \theta}{\partial x}(0,t) = \frac{\partial a}{\partial x}(0)\cdot b(t)$$

$$\lambda = \frac{\partial^2 a}{\partial x^2}\cdot\frac{1}{a}$$

$$\frac{\partial \theta}{\partial x}(\pi,t) = \frac{\partial a}{\partial x}(\pi)\cdot b(t)$$

where,

$$\boxed{\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}}$$

$\therefore \frac{\partial b}{\partial t} = \lambda b$

$s = \lambda$ when $\lambda \geq 0$

$s = n^2$

$\theta'(0) = 0, \; \theta'(\pi) = 0$

$b = c_1 t + c_2$

$\frac{\partial^2 a}{\partial x^2} = a\lambda$

$s^2 = \lambda$

$s^2 = n^2 \qquad s = \pm n.$

$\theta'(0) = 0, \; \theta'(\pi) = 0$

$a = c_1 e^{-nx} + c_2 e^{nx}$

$a'(0) = 0, \; a'(\pi) = 0.$

"Trivial solution"

From previous example:

we get a trivial solution for our $\partial^2 a/\partial x^2 = a\lambda$ when $\lambda = -n^2$ $(n = 0, 1, 2, \ldots)$

$$a(x) = \cos nx \text{ (times a scalar).}$$

correspondingly

$$\frac{\partial b}{\partial t} = -n^2 b$$

$$\boxed{s = -n^2}$$

$b(t) = e^{-n^2 t}$ (times a scalar), and the normal mode is:

$$\theta_n(x, t) = e^{-n^2 t} \cos(nx).$$

The case $n=0$ is the constant function 1, so the general solution will be

$$\theta(x,t) = \frac{a_0}{2} + a_1 e^{-t} \cos x + a_2 e^{-4t} \cos 2x + \cdots$$

Finally our initial condition $t=0$

$$\theta(x, 0) = x$$

$$\theta(x,0) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

for all $x \in (0, \pi)$.

$\therefore$ The right hand side is a period 2π even function, so we extend the L.H.S to a period 2π even function gives $T(x)$, a Δ^r wave.

F.S of Δ^r wave is

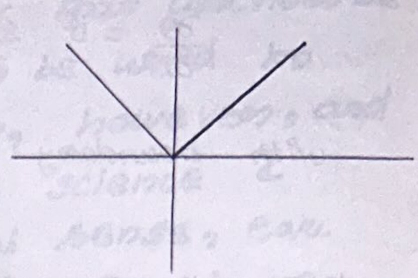
$$T(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right)$$

The solution that satisfies the I.C is

$$\theta(x,t) = \frac{\pi}{2} - \frac{4}{\pi}\left(e^{-t}\cos x + e^{-9t}\frac{\cos 3x}{9} + \cdots\right)$$

This answer makes physical sense: when the entire bar is insulated, it's temperature need to (tends) constant equal to the average of the Initial temperature.

"Even signal" – Triangular wave.

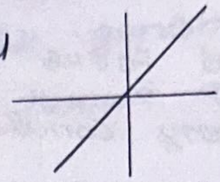


$|x| \rightarrow$ Triangular wave.

why odd signal even signal?

In case of odd signal

$f(x) = (x) \quad (-\pi < 0 < \pi)$



$$F.S = \sum_{n=1}^{\infty} b_n \sin nx.$$

But our solution from general eqn is as $\boxed{t=0}$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

So the function is $|x| \longrightarrow$ Even function

Another setup

Suppose that a metal bar of length π initially has a temperature distribution given by $\theta(x,0) = x$ for $0 \le x \le \pi$. The right end is then put into an ice reservoir (temp 0 degrees celcius) and the left end is insulated.

Solu:

$$\boxed{\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2}}$$

$\theta(\pi, t) = 0$ (Ice)

$\frac{\partial}{\partial t}\theta(0,t) = 0$ (Insulated)

Note: The boundary conditions are not consistent with the IC, because $\theta(x,t)$ is discontinuous at $t=0$.

(Jump discontinuity occurs).

Procedure Summary.

Steps for solving.

$$\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2} \quad 0 < x < L, \; t > 0$$

with boundary & I C $\boxed{\theta(x, 0) = f(x)}$

1) Ignore I C and Focus on B C

2) If any Boundary condition is non zero. Find a steady state solution $\theta(x)$ which satisfies the given B.C. Reduce problem to solving

$$\theta_h(x, t) = \theta(x, t) - \theta_p(x)$$

which has homogeneous B.C and I.C

$$\theta_h(x, 0) = f(x) - \theta_p(x)$$

3) Look up standard form of Eigen values, Eigen functions and normal modes for the homogeneous cases already computed.

Else, use seperation of variables.

That's why try $\theta_h(x, t) = \vartheta(x)\, w(t)$ (or $\theta(x, t) = \vartheta(x)\, w(t)$)

~~If~~ original problem is homogeneous. to find the family of normal modes

$$\theta_n(x, t) = \vartheta_n(x)\, w_n(t)$$

4) Take linear combinations to get the general solution

$$\boxed{\theta_h(x, t) = b_1 w_1(t)\, \vartheta_1(x) + b_2\, w_2(t)\, \vartheta_2(x) + \cdots}$$

5) Extend the I.C $f(x) - \theta_p(x)$ to have the correct base period and even / odd properties in order to be able to solve for the Fourier Coefficient.

At $t = 0$

$$f(x) - \theta_p(x) = b_1 \vartheta_1(x) + b_2 \vartheta_2(x) + b_3 \vartheta_3(x) + \cdots$$

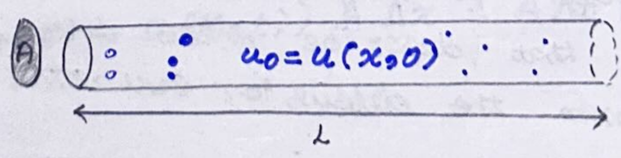
The diffusion equation.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0.$$

was first introduced as a P.D.E that describes the flow of heat in rod. This eqn can be used to model much more than heat flow, however, and appears across many fields in science & Engineering. In the most general sense, eqn of the above form are known as diffusion eqn.

Diffusion eqn are used to model the 'spreading out' of chemical species, biological cells, fluid vortices and even people in a crowd. This all comes from the fact that the assumptions made when deriving the eqn for heat are actually very simple assumptions based on the conservation of some quantity and the desire of that quantity to spread out in a simply way. We'll go through a problem that models the spreading of some chemical molecule along the length of a pipe. Such a model would be useful when designing a chemical reactor, an industrial device for synthesizing important chemicals from other more available inputs.

Physics for modelling concentrations in mixtures:



Imagine a pipe filled with carbonated water. In this fluid, the CO_2 has a concentration which varies along the length of the pipe. (Assume the length of the pipe is very large compared to its diameter; this lets us ignore the variations in concentration in the radial direction of the pipe.) The CO_2 will 'diffuse' through the pipe as it randomly move about, tending to spread out from areas of high concentration to low concentration. We'll list some of the important quantities. (similarities to the model used for heat transfer in a rod:)

L - length of the pipe

A - cross sectional area inside the pipe

u_0 - Initial conc. of CO_2 in the pipe

t - time

x - position along the length of the pipe.

u - the concentration of CO_2 at a given point in the pipe at a given time, in terms of mass.

ϕ - CO_2 mass flux at a given point in the pipe at a given time.

These all are analogous to those used in the heat equ.

Conservation of motion:

The total amount of CO_2 in the pipe is fixed - none is leaving or being created. This means that if the mass is changing in any small segment of pipe, this must occur because CO_2 is flowing in or out of the region.

Fick's law of mass transfer:

The idea is that CO_2 moves from high concentration region to regions of low concentration. The equations that describe these two laws combine together to form the diffusion equation for concentration transfer.

$$\boxed{\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}}$$

Putting it together:

1) Conservation of mass: The change in the total mass of chemical in any regions equal to the net mass flowing in or out of that region. (we assume no CO_2 being

created through reactions- like active yeast cultures which produce CO_2 in beer.)

concentration (u) is a density, and therefore the units are $\frac{\text{mass}}{\text{volume}}$, which is equivalent to $\frac{\text{mass}}{\text{length}^3}$. Consider a small slice of pipe with cross sectional area A and length Δx. The volume of this slice is $A\Delta x$, and the mass of CO_2 inside is approximately $u(x)\, A\, \Delta x$.

(mass = density (concentration) × volume)

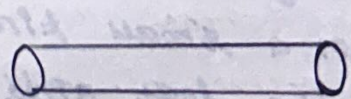
volume = Area × length.

How does this mass change in a small amount of time Δt? The only way for the mass to change in our small volume is by diffusing in through either of the ends. The diffusion through either end is known as flux. Flux, in this case CO_2 flux density, has units of mass/[time. length^2].

we can write the change in the CO_2 mass in a small volume. $A\Delta x$ that occurs in a short time Δt as

$$(u(x, t+\Delta t) - u(x, t))\, A\, \Delta x = A\, \Delta t\, (\phi(x, t) - \phi(x+\Delta x, t))$$

change in Flux = change in concentration. b/w two points



$$A \frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} = -A \frac{v(x+\Delta x, t) - v(x, t)}{\Delta x}.$$

Taking $\Delta x, \Delta t \to 0$, we can

$$\boxed{\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}}$$

mass = Density × volume

CO_2 mass flux in a given point at a given time $= v$ $\left(\text{unit} : \frac{\text{mass}}{(\text{time} \times \text{length}^2)} \right)$

Concentration $= \frac{\text{Mass}}{(\text{Length})^3} \times \text{Area} \times \Delta x = \frac{\text{Mass}}{\text{time} \times \text{length}} \times \Delta t \times A$

$$\boxed{\text{Mass flux} = \frac{\text{Rate of mass flow}}{\text{unit area}}}$$

v – mass flux in a point at a time.

So equating:

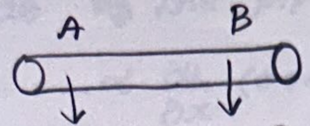
$$A\left(u(x, t+\Delta t) - u(x, t)\right) \cdot \Delta x = A \Delta t \left(v(x, t) - v(x+\Delta x, t)\right)$$

$$A \frac{u(x, t+\Delta t) - u(x, t)}{\Delta t} = -A \frac{v(x+\Delta x, t) - v(x, t)}{\Delta x}$$

The sign on the $v(x+\Delta x, t)$ term is negative because $Av(x+\Delta x, t)\Delta t$ is the CO_2 mass passing to the right, out of the small Δx length over a small time interval Δt, while the flux on the left side is bringing new mass in,

Note: Dividing by Δt and Δx we get.

$$A\,\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = -A\,\frac{\left(v(x+\Delta x,t)-v(x,t)\right)}{\Delta x}$$



$\therefore$ mass flux in A is greater than B when CO_2 is from left side.

$$\boxed{\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}}$$

Fick's law of mass transfer:

Analogous to the Fourier's law of heat transfer, Fick's law of mass transfer says that the flux b/w two points is proportional to the difference in concentration b/w the two points, with the appropriate sign so that the flow is from the higher to lower concentration. We also note that it states that the flow b/w two points is inversely proportional to the distance b/w the points. This makes intuitive sense, as a small difference in concentration is more likely to generate a greater flux when the difference is over a short distance compared to a longer one. If we imagine two points along our pipe, x and $x+\Delta x$, then the CO_2 concentration at these points will be $u(x,t)$ and $u(x+\Delta x,t)$ respectively.

We can write Fick's law of mass transfer as

$$\boxed{v \propto -\frac{\partial u}{\partial x}}$$

We can make this an equality by adding the 'Diffusion Constant' $\alpha > 0$.

$$\boxed{v = -\alpha\,\frac{\partial u}{\partial x}}$$

why is there a negative sign ... the expression above? we want v to be +ve whenever we have a flux to the right, which means

$$u(x+\Delta x, t) < u(x, t)$$

Assume $\Delta x > 0$.

write the above two laws each giving an equation in terms of v and u, we can plug the second equation into the first to get a partial differential equation for just u. as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\alpha \frac{\partial u}{\partial x}\right) \qquad \left[\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}\right]$$

$\alpha \rightarrow$ constant

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

In some cases, α may vary with x (such as a non-homogeneous material where it's easier for cells to move in one region (vs) another) and this changes the form of the equation.

Physical boundary conditions

The diffusion equation governs the evolution of the concentration profile inside of the length of CO_2. our pipe is a finite length L. How do we handle what happens at either end of the pipe?

we need to specify the initial condition. (i.e. the initial concentration distribution of CO_2).

$$u_0 = u(x, 0)$$

and constraints that tell us what happens to the boundary conditions at all +ve times.

What are boundary conditions? What can we specify? The most natural thing to prescribe in a physical sense would be the flow rate of CO_2 into / out of the ends of the pipe:

$$-\alpha \frac{\partial u}{\partial x}(0,t) \text{ and } -\alpha \frac{\partial u}{\partial x}(L,t).$$

Another, less physically intuitive example in this case, but which will be mathematically simpler, is the case where the CO_2 concentration is prescribed at each end of the pipe:

$$u(0,t) = c_1, \quad u(L,t) = c_2.$$

We can imagine this is due to the pipe connecting in to a large reservoir of constant concentration; even as CO_2 flows into and out of the pipe, the reservoir is so large that its net concentration remains constant. This could be represented as a large tank in a brewery with small pipes attached, for example.

Fluid (vs) cell flow:

It's important to note that for the model created above, we are only considering the diffusion of CO_2 in a non-flowing liquid, and not the transport of CO_2 by the flow of the liquid itself. Such a process is called advection and is modeled by a different set of PDEs. Also we can combine both of these methods of transport, and this results in a partial diff eqn called the **Advection-diffusion equation**. For now we will assume that the liquid is near-stationary so that all the transport is dominated by diffusion, so we will be fine using the diff eqn for our models.

Boundary Conditions

If we are prescribing the flow rate of CO_2 at the boundary, this is just the flux times the cross section.

$$\nu A = -A\alpha \frac{\partial u}{\partial x}$$

For a pipe of length L with boundaries at $x=0, L$ after multiplying the constant factors all together, we get the conditions for $t>0$ as

$$\frac{\partial u}{\partial x}(0,t) = a$$

$$\frac{\partial u}{\partial x}(L,t) = b$$

where a and b are constants. Note we could have a and b vary with time, but the techniques for solving such PDEs are beyond the level of this class. In general, BC that set the derivative at a boundary are known as Neumann boundary conditions.

Concentration BC

The concentration BC is similar to the above, with the difference being that we prescribe u itself of its derivative. We get an analogous set of equations to before, for $t>0$.

$$u(0,t) = a$$

$$u(L,t) = b.$$

We note that in mathematical terms these are called **Dirichlet boundary conditions**.

Other boundary conditions:

Besides the two simple BC we described above, there are a few others that can be useful. One other BC, not used as often but still important, is what's know as the Robin boundary condition. This condition has the form on the boundary

$$u + a\frac{\partial u}{\partial x} = b$$

where $a, b \rightarrow$ constants.

Such a condition is usually used to represent some sort of convective transport occurring at the boundaries. Imagine a glass of beer with the top open to the atmosphere and a wind is blowing over it. CO_2 naturally diffuses into the air above the beverage, and the wind will tend to carry it away. The above BC deals with this case.

Neumann BC $\longrightarrow$ Derivative BC

Dirichlet $\longrightarrow$ $u(0,t) = a$

Robin $\longrightarrow$ $u + a\frac{\partial u}{\partial x} = b$.

In analogous heat eqn, the case of an insulated bar with ends held at $0°C$ corresponds to

Dirichlet BC. $u(0,t) = 0$.

Insulated bar & ends?

Neumann BC $\frac{\partial u}{\partial t}(0,t) = 0$

Recitation - Flux Zero

1) a) Saline concentration u in a thin metal tube of length 1 having the solution satisfies the diffusion equ

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, \quad t > 0.$$

solu:
Assume flux of saline at the boundary is zero, that is, assume $\frac{\partial u}{\partial x} = 0$ at the boundary. Thus the Initial and boundary conditions in this situation are

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \quad t > 0.$$

$$u(x,0) = x, \quad 0 < x < 1.$$

Use seperation of variables to look for solutions of the form $v(x)\,w(t)$. Use diffusion constant 2 only in finding $w(t)$.

Determine a basis of normalized functions (Amplitude 1) that spans the space of possible solutions $u_K(x,t) = v_K(x)\, w_K(t)$

solu:

$$u_K(x,t) = v_K(x)\, w_K(t)$$

$$\frac{\partial}{\partial t}\left[v_K(x) \cdot w_K(t)\right] = 2 \frac{\partial^2}{\partial x^2}\left[v_K(x) \cdot w_K(t)\right]$$

$$v_K \frac{\partial w_K}{\partial t} = 2 w_K \frac{\partial^2 v_K}{\partial x^2}$$

we have,

$$v_K \frac{\partial w_K}{\partial t} = \lambda$$

$$\boxed{S = \lambda}$$

$$\boxed{\frac{1}{2 w_K} \cdot \frac{\partial w_K}{\partial t} = \frac{1}{\vartheta_K} \frac{\partial^2 \vartheta_K}{\partial x^2}}$$

$$\frac{1}{2 w_K} \cdot \frac{\partial w_K}{\partial t} = \lambda$$

$$\boxed{\frac{\partial w_K}{\partial t} = 2\lambda \cdot w_K}$$

$$s = 2\lambda$$

$$\boxed{c_1 e^{-2n^2 t}}$$

i) $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0$

$$\frac{1}{\vartheta_K} \frac{\partial^2 \vartheta_K}{\partial x^2} = \lambda$$

$$\frac{\partial^2 \vartheta_K}{\partial x^2} = \lambda \vartheta_K$$

$$s^2 = -n^2$$

$$s = \pm i n$$

$$\vartheta_K = c_2 \sin nx + c_3 \cos nx.$$

$$\vartheta_K' = +c_2 (\cos nx) \cdot n - n c_3 \sin(nx)$$

$$\frac{\partial v_K}{\partial x}(0) = n c_2$$

$$\boxed{c_2 = 0}$$

$$\frac{\partial \vartheta_K}{\partial x}(1) = c_2 (\cos n) \cdot n - c_3 \sin(n) \cdot (n)$$

$$0 = -c_3 \sin(n)(n)$$

$c_3 \sin(n) = 0$ $\quad \boxed{c_3 \neq 0} \rightarrow$ For a non-trivial answer.

$$\boxed{n = K\pi}$$

$$\therefore \quad u(x, t) = \left(e^{-2K^2 \pi^2 t}\right) \cdot \left(\cos(K\pi x)\right)$$

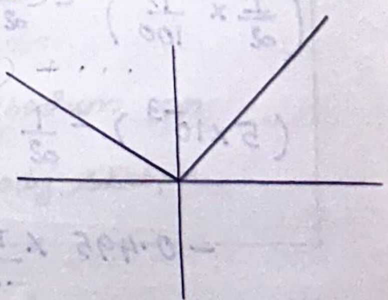
(constant multiple).

$u(x, 0) = \cos(K\pi x), a_K$ $\quad 0 < x < 1, \; t > 0$

$$u(x, 0) = \sum_{K=1}^{\infty} \cos(K\pi x) a_K.$$

$\therefore$ This is $u(x, 0) = x$ for $0 < x < 1$.

"Even function"



$$F.S = \frac{1}{2} - \frac{4}{\pi^2} \sum_{\substack{K \geq 1 \\ (\text{odd})}}^{\infty} \cos(K\pi t)$$

$$u(x,0) = \frac{1}{2} - \left(\frac{4}{\pi^2} \cos(\pi x) + \frac{4}{\pi^2} \frac{\cos(3\pi x)}{9} + \frac{4}{\pi^2} \frac{\cos(5\pi x)}{25} + \dots \right.$$

Find the solution $u(x,t)$ as a linear combination of $u_k(x,t)$.

Solu:

$$u(x,t) = \frac{1}{2} - \left(\frac{4}{\pi^2} e^{-2\pi^2 t} \cdot \cos(\pi x) + \frac{4}{\pi^2} e^{-18\pi^2 t} \frac{\cos(3\pi x)}{9} + \frac{4}{\pi^2} e^{-50\pi^2 t} \cdot \frac{\cos(5\pi x)}{25} + \dots \right)$$

Steady state solution: $u_{st}(x)$.

$u(x,t) \to u_{st}(x)$ as $t \to \infty$.

$$u_{steady}(x,t) = \frac{1}{2} - (0 + 0 + 0 \dots)$$

$$\boxed{u_{steady} = \frac{1}{2}}$$

Estimate 2 significant figures, the time T it takes to be within 1% of the steady solution.

Solu:

Dominant term $\frac{4}{\pi^2} \cos(\pi x)\, e^{-2\pi^2 t}$

Let $x = 0$

$$u(0,t) = \frac{1}{2} - \frac{4}{\pi^2} e^{-2\pi^2 t}$$

1% of steady state

$$\left(\frac{1}{2} \times \frac{1}{100} \right) = \frac{1}{2} - \frac{4}{\pi^2} e^{-2\pi^2 t}$$

$$(5 \times 10^{-3}) - \frac{1}{2} = -\frac{4}{\pi^2} e^{-2\pi^2 t}$$

$$-0.495 \times \frac{\pi^2}{-4} = e^{-2\pi^2 t}$$

$$1.221363545 = e^{-2\pi^2 t}$$

$$\ln(1.221363545) = \ln(e^{-2\pi^2 t})$$

$$0.1999678942 = -2\pi^2 t$$

$$t = 0.010130$$

Taking dominant term

$$\frac{4}{\pi^2} \cos(\pi x)\, e^{-2\pi^2 t}$$

let $x=0$

$$\frac{1}{2} \times \frac{1}{100} = \frac{4}{\pi^2} e^{-2\pi^2 t}$$

$$(5 \times 10^{-3}) \times \frac{\pi^2}{4} = e^{-2\pi^2 t}$$

$$\boxed{t = 0.22266\ s}$$

Non-zero concentration specified at endpoints

1) Consider a thin rod of length 1 having a saline solution. The left end point is placed in a large saline bath of concentration 1. The right end is placed in a large bath of fresh water (concentration 0). The concentration u satisfies the following Initial value problem.

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1, \quad t > 0$$

$$u(x,0) = x, \qquad 0 < x < 1.$$

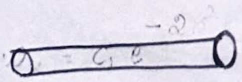
Solu:

BVC: (Boundary value conditions):

For $t > 0$

$u(0,t) = 1$

$u(1,t) = 0$



$x = 0$ $\qquad$ $x = 1$

$v_K = C_2 \sin nx + C_3 \cos(nx)$

Conc = 1 $\qquad$ Conc = 0

$v_K(0,t) \rightarrow C_3 \cos(0)$

$\boxed{C_3 = 1}$

$v_K \in C_2 \sin(n) + C_3 \cos(n)$

$0 = C_2 \sin(n) + \cos(n)$

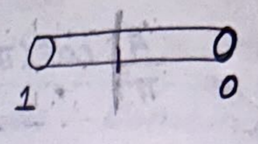
Steady state solution:

Our problem is a non-homogeneous problem.

$$u(x,t) = u_{st}(x) + u_t(x,t)$$

$$\boxed{\begin{gathered}\lim_{t \to \infty} u(x,t) = u_E(x) \\ \text{Equilibrium or} \\ \text{Steady state}\end{gathered}}$$

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1.$$



<u>Concentration at x:</u> (After equilibrium)

$$= 1 - x$$

In heat eqn

$$\theta_{eqn} = \frac{\theta_0 \cdot x}{L}$$

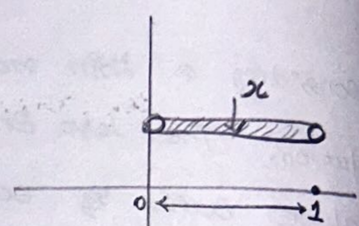
As $t \to \infty$

$$\boxed{\frac{\partial u}{\partial t} \to 0}$$

$$0 = \alpha \frac{\partial^2 u}{\partial x^2}$$

$u(x) =$? (At equilibrium)

$$\frac{\partial^2 u}{\partial x^2} = 0$$



$$\frac{\partial u}{\partial x} = c_1$$

$$\boxed{u(x) = c_1 x + c_2}$$

$u(x)$ $1 - x$

$x \to$ Initial conc

Incoming - outgoing conc. conc.

2) b)

$$u(x,t) = u_{st}(x) + u_h(x,t)$$

$$u_{h,k}(x,t) = \vartheta_k(x)\, w_k(t)$$

Solu:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$$S = \alpha\lambda$$

$$\vartheta_k \frac{\partial w_k}{\partial t} = \alpha w_k \cdot \frac{\partial^2 \vartheta_k}{\partial x^2}$$

$$\boxed{\frac{1}{\alpha w_k} \cdot \frac{\partial w_k}{\partial t} = \frac{1}{\vartheta_k} \frac{\partial^2 \vartheta_k}{\partial x^2}}$$

use Superposition $u(x,t) = u_{st}(x) + u_h(x,t)$

Solu: **Find the Eigen values of $V_k(x) \to \lambda_k$.**

Solu: $u_h(x,t) = u(x,t) - u_{st}(x)$