$u = t$
$du = dt$

$dv = \lambda e^{-\lambda t}\, dt$

$v = \lambda \dfrac{e^{-\lambda t}}{-\lambda}$

$v = -e^{-\lambda t}$

$$E[X] = \int_0^{\infty} t \lambda e^{-\lambda t}\, dt = t e^{-\lambda t} - \int_0^{\infty} -e^{-\lambda t} \cdot dt$$

$$= -t e^{-\lambda t} + \left(\frac{e^{-\lambda t}}{-\lambda}\right)_0^{\infty}$$

$$= \lim_{x \to \infty} \left(-t e^{-\lambda t} - \left(\frac{e^{-\lambda t}}{\lambda}\right)_0^{x} \right)$$

$$= \lim_{x \to \infty} \left(-x e^{-\lambda x} + \left(-\frac{1}{\lambda} e^{-\lambda t}\right)_0^{x} \right)$$

$= 0$ As $x \to \infty$

Exponentials $\to 0$ $\to$ fastes than x (wins)
$x \to \infty$

$$= \lim_{x \to \infty} \left(-x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda t} \right) + \frac{1}{\lambda} e^{0} = 0 + \frac{1}{\lambda} = \frac{1}{\lambda}$$

$E[X] = 1/\lambda$

(iii) $\mathrm{Var}(X) = E[X^2] - (E[X])^2$

$u = t^2$, $du = 2t\,dt$ | v = same, dv = same

$$E[X^2] = \int_0^{\infty} t^2 f_X(t)\, dt$$

$$= \int_0^{\infty} 2 t^2 \lambda e^{-\lambda t} \frac{dt}{2}$$

$= \dfrac{2}{\lambda^2}$ (solving)

$$\mathrm{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = 1/\lambda^2$$

$X_1, X_2, X_3 \sim exp(\lambda)$

X_1, X_2, X_3 independent

PDF of $Z = max(X_1, X_2, X_3)$

Answer of $max(X_1, X_2, X_3)$ say Z must be larger than X_1, X_2, X_3

Solu:

$F_Z(z) = \mathbb{P}(Z \leq z)$

$= \mathbb{P}(max(X_1, X_2, X_3) \leq z)$

$$CDF(z) = \begin{cases} 0, & x < 0 \\ & x > 0 \end{cases}$$

$\mathbb{P}(X_1 \leq z, X_2 \leq z, X_3 \leq z)$ = why?

X_1, X_2, X_3 are independent.

If $max(X_1, X_2, X_3) \leq z$ then X_1, X_2, X_3 must be $\leq z$

$= P(X_1 \leq z)\, P(X_2 \leq z)\, P(X_3 \leq z)$

$= (1 - e^{-\lambda z})^3 \longrightarrow$ From before

$pdf = \frac{d}{dx} F_Z(z)$

$= 3(1 - e^{-\lambda z})^2 \cdot (-(-\lambda \cdot e^{-\lambda z}))$

$= 3(1 - e^{-\lambda z})^2 \lambda e^{-\lambda z}$

$$C.D.F = \begin{cases} (1 - e^{-\lambda z})^3, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$P.d.f = \begin{cases} 3(1 - e^{-\lambda z})^2 \lambda e^{-\lambda z}, & z > 0 \\ 0, & \text{otherwise.} \end{cases}$$

p.d.f of $W = \min(X_1, X_2, X_3)$

$F_W(\omega) = \mathbb{P}(W \leq \omega)$

$= \mathbb{P}(\min(X_1, X_2) \leq \omega)$ → X_2 can be large, X_1 can be $\leq$

$= 1 - \mathbb{P}(\min(X_1, X_2) > \omega)$ → X_1 can be large, X_2 can be $\leq$

∴ X_1 large, X_2 small
X_1 small, X_2 large
Both small
Both big

$= 1 - P(X_1 > \omega) \cdot P(X_2 > \omega)$

$= 1 - \ ?$

w.r.t

$1 - e^{-\lambda x}, \ x \geq 0$

what about (starting from 0)

$x > 0 \Rightarrow \boxed{(x \geq 0) - (x = 0)}$

∴ $x = 0 \Rightarrow 1$

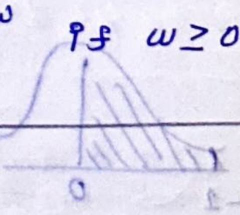
$x > 0 \Rightarrow -e^{-\lambda x}$

$F_W(\omega) = 1 - e^{-\lambda\omega} \cdot e^{-\lambda\omega}$

$= 1 - e^{-2\lambda\omega}$

$$= \begin{cases} 0 & \text{if } \omega < 0 \\ 1 - e^{-2\lambda\omega} & \text{if } \omega \geq 0. \end{cases}$$

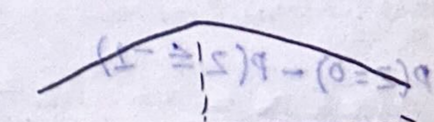
Just rate is different from normal cdf. ←



Standard normal

$X \sim N(0, 1)$

$Y \sim N(1, 4)$

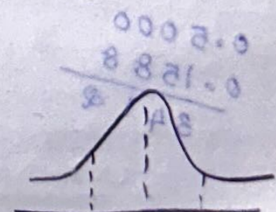


i) $P(X \leq 1.5) = ?$

∴ X is normalized

$P(X \leq 1.5) = \Phi(1.5)$

$= 0.9332$ (Table)



$P(X \le -2.5) \rightarrow$ negative values are not in table

Symmetric?

$P(X \le -1) = P(X \ge 1)$

$= 1 - P(X \le 1) = 1 - \Phi(1)$

$= 1 - 0.8412$

$= 0.1588$

$$\begin{array}{r} 1.0000 \\ 0.8412 \\ \hline 1588 \end{array}$$

B) Distribution of $\frac{y-1}{2}$

?

$\therefore$ Any linear function of Normal dist is also normal

$E\left[\frac{Y-1}{2}\right] = \frac{1}{2}E[Y] - \frac{1}{2}$

$= \frac{1}{2}(1) - \frac{1}{2} = 0$ [mean]

$\downarrow$

Given

$Y \sim (1, 4)$

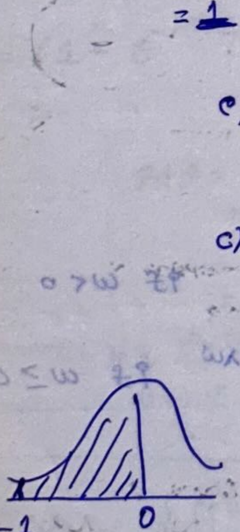
$var\left(\frac{Y}{2} - \frac{1}{2}\right) = \frac{1}{4} var(Y) = \frac{1}{4}(4)$

$= 1$

Standard Normalized one of y is $\frac{Y-1}{2}$?

c) $P(-1 \le Y \le 1) = P(-1 \le Y \le 0) + P(0 \le Y \le 1)$

$= P\left(\frac{-1-1}{2} \le \frac{Y-1}{2} \le \frac{1-1}{2}\right)$



$= P\left(-1 \le \frac{Y-1}{2} \le 0\right)$

$= P(-1 \le Z \le 0)$

$P(Z=0) - P(Z \le -1)$

$= P(Z=0) - P(Z \le -1)$

$= 0.5 - (1 - 0.8413)$

$$\begin{array}{r} 0.5000 \\ 0.1588 \\ \hline 3412 \end{array}$$

$= 0.3412.$

Lecture : 9

multiple continuous Random variable (see pdf s)

* Joint density functions
* condition d.f
* conditional p.d.f

* $f_X \geq 0$
* $\sum_{-\infty}^{\infty} f_X(x)\, dx = 1.$
* $P(X=a)=0$ (particular point)

'Think of little intervals'

* $P(x \leq X \leq x+\delta) \approx f_X(x)\cdot\delta$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

↓

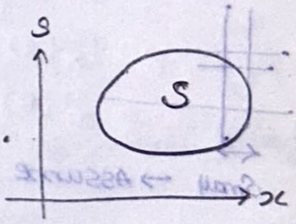
Average how much $g(x)$ is going tobe?

g(x) has input x to give a o/p → occuring of x has some prob.

multiple R.v (Jointly continuous)

$$P((X,Y)\in S) = \iint_S f_{X,Y}(x,y)\, dx\, dy.$$

Jointly continuous: If we can calculate probabilities by integrating a certain function that we call the joint density function over the set of interest (2-d plane)



The probability of occuring inside the set S.

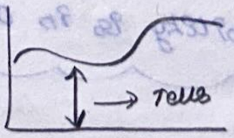
So the surface underneath it has a certain total volume. (Total volume = 1)

→ Total plane (x y)

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1}$$ → 1 pound of prob - spread all over the space.

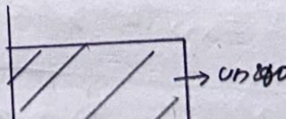
$f_{X,Y} \geq 0$

* Height of JPdf



→ Tells how much prob accumulated (tends to be) in certain space.

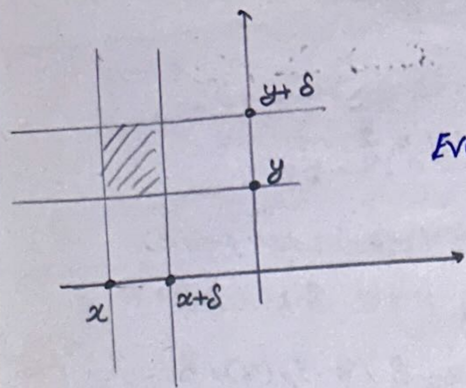
Unit square



→ uniform

'But in some cases → not uniform' (likelihoods)

x & $x+\delta$



Event: fall under rectangle.

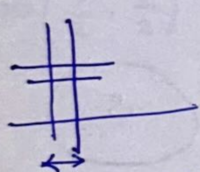
event x falls b/w x & $x+\delta$
y falls b/w y & $y+\delta$

Prob of rect going to be = integral of the density over that rectangle (volume in that rectangle)

Larger density (volume) (or) larger prob in that rect = It affects neighbourhood.

volume = height × area of the base.

(smaller area → small rectangle (distance) → The density not going to change a lot.



small → Assume density not changing too much.

volume = height × base area

$P(x \leq X \leq x+\delta,$
$y \leq Y \leq y+\delta)$ $= f_{X,Y}(x,y) \cdot \delta^2$

∴ In continuous probability is in density. → how much volume.

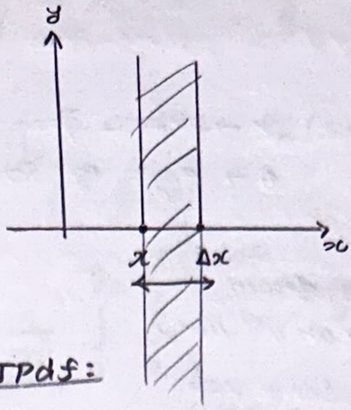
$$P(x \leq X \leq x+\delta, y \leq Y \leq y+\delta) \approx f_{X,Y}(x,y) \cdot \delta^2$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$\boxed{f_X(x) \cdot \delta \approx P(x \leq X \leq x+\delta) = ?}$$

→ marginal case.

marginal



$f_X(x) \cdot \delta \approx P(x \le X \le x + \delta x)$

In terms of JPdf:

$$P(x \le X \le x+\delta) = \int_{-\infty}^{\infty} \int_{x}^{x+\delta} f_{X,Y}(\bar{x}, y)\, d\bar{x}\, dy$$

$\delta \rightarrow$ small distance

As x varies very little in $x+\delta$

$f_{X,Y}(x,y)$ is appr. a constant

$$\int_{x}^{\delta + x} dx = \delta + x - x = \delta$$

$$= \int_{-\infty}^{\infty} \delta \cdot f_{X,Y}(\bar{x}, y)\, dy$$

$$f_X(x) \cdot \delta \approx \int_{-\infty}^{\infty} \delta\, f_{X,Y}(\bar{x}, y)\, dy$$

$$\boxed{\therefore f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}$$

* x & y are independent if

$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$ for all x & y.

$\boxed{\star\ P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)}$

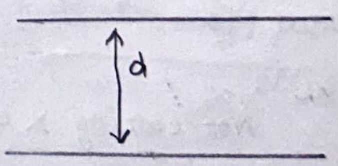
$\therefore$ ~~X~~ & ~~Y~~ are independent.

Famous needle ~~of~~ Buffon

Buffon's needle

* Two parallel lines – seperated by the distance.
* Needle (Shorter than d)

(Don't intersect both lines)



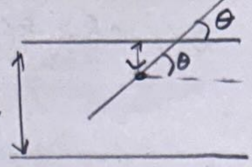
Intersect anyone

Fall b/w the two.

1) Sample space : { ? }

* P (intersects one of the lines)

Needle → small than d



x → Distance from the center

θ → Angle of intersecting

$\boxed{0 \le x \le d}$ → centre of needle from any end (near $\parallel^{el}$ line)

θ → Assume acute angle $(0 \le \theta \le \pi/2)$

2. model : (X, θ)

* Needle → randomly thrown → may at any x. (uniform distribution)

* Any θ may happen (uniform distribution)

$$f_{X,\theta}(x,\theta) = \ ? \qquad 0 \le x \le d/2,\ 0 \le \theta \le \pi/2$$

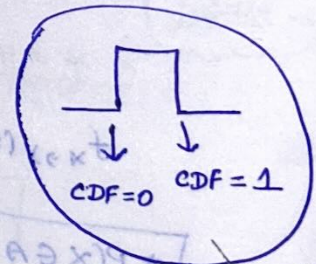
∴ Independent :

$$f_{X,\theta}(x,\theta) = f_X(x) \cdot f_\theta(\theta)$$

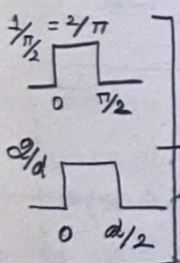
Integrating $f_X(x)$ at $d/2$ gives 1

Integrating $f_\theta(\theta)$ at $\pi/2$ gives 1

[uniform prob distribution]



∴ Same for both θ, X (Both uniform).



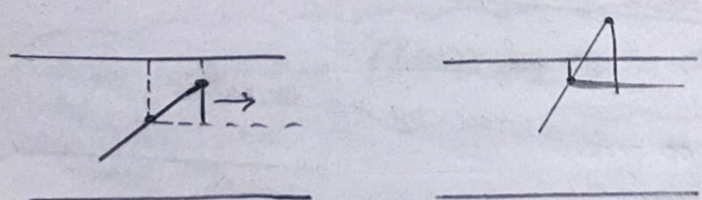
→ (uniform)

$$f_X(x) = \frac{1}{b-a}$$

$$f_{X,\theta}(x,\theta) = \left(\frac{2}{d}\right) \cdot \left(\frac{2}{\pi}\right)$$

3. Identify event of interest.

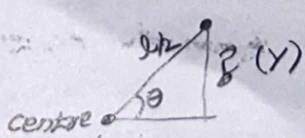
Intersect | Not intersect



Not cutting x will be greater

cutting : distance from centre to the edge will be greater.

l → length of needle



$$\sin\theta = \frac{opp}{hyp} = \frac{y}{l/2}$$

$$\boxed{y = \frac{l}{2}\sin\theta}$$

$P\left(x \le \frac{l}{2}\sin\theta\right) \rightarrow$ intersecting.

$$P\left(x \le \frac{l}{2}\sin\theta\right) = \iint_{x \le \frac{l}{2}\sin\theta} f_X(x)\, f_\theta(\theta)\, dx\, d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} \int_0^{l/2 \sin\theta} dx\, d\theta$$

$$= \frac{4}{\pi d} \int_0^{\pi/2} l/2 \sin\theta\, d\theta$$

$$= \frac{2l}{\pi d} \int_0^{\pi/2} \sin\theta\, d\theta \qquad = \frac{2l}{\pi d}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{2l}{\pi d}(1)$$

$$= \frac{2l}{\pi d}. \qquad \text{(Fix } l \text{ and } d\text{)}$$

4) verify by experiment

* Fix l & d $\rightarrow$ Throw million times.

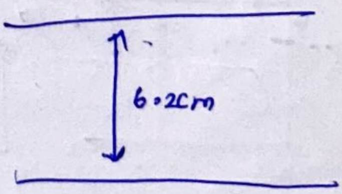
"verify the inference".

Experimental measurement:

$$P = \frac{2l}{\pi d} \qquad (D = 6.2\,cm,\ L = 3.1\,cm)$$

$$3.14 \overline{)1 0} \quad 0.3$$

$$P = \frac{2\,(3.1)\,cm}{6.2 \times \pi} = \frac{1}{\pi}$$



"Out of 8 – 4 intersected the line"

According to this is $\pi = 2$ $\rightarrow$ NO

"Throwing paper clips"

why?

This kind of methods – used by physicians & statisticians. (100 RV – No way to do 100 integrals in computers) – Generating Random samples – By simulation to estimate. $\longrightarrow$ "monte-carlo" method of evaluate Integrals.

Conditioning

$P(x \le X \le x+\delta) \approx f_X(x)\cdot\delta$ → Density gives prob of little intervals

By analogy

$P(x \le X \le x+\delta \mid Y \approx y) \approx f_{X|Y}(x|y)\cdot\delta$

↳ (approx) → (we can't condition a zero prob event)
This leads us (we will say very close)
Infinitesimally close

y is infinitesimally close to some value.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0$$

(Take discrete formulas, replace pmf by pdf s)

y is fixed
(function of x)

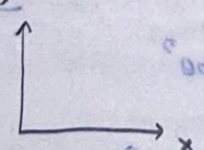
$f_{X,Y}(x,y)$ → (fixed y) → Joint density
↕
See how it varies with y

Independent

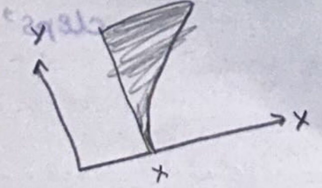
$$f_{X,Y} = f_X f_Y$$

$$f_{X|Y}(x|y) = \frac{f_X(x)\,f_Y(y)}{f_Y(y)} = f_X(x)$$

∴ knowing y, doesn't change our beliefs on x.

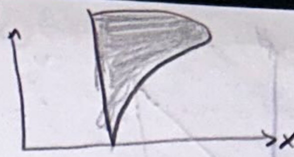


3-d plot x & y (vs) prob density
(how much mass sitting on it)

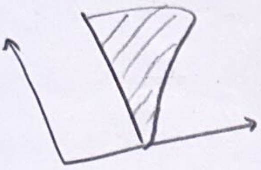


Fix x, go through y → marginal density of x.

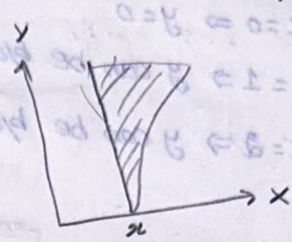
$f_{Y|X}(y|x) =$



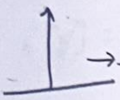
In conditional → we are rescaling our world. our new world is the conditional world.



marginal — Total area ≠ 1
prob of
x



Total area = 1
(Normalize it)
'conditional world'

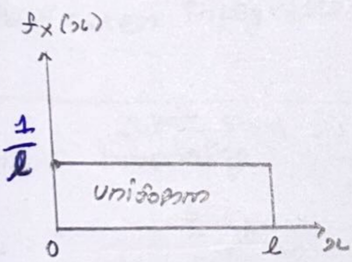


→ Impulse (Area = 1)

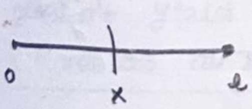
Stick-breaking problem

break a stick of length l twice: break at X: uniform in $[0, l]$; break again at Y: uniform in $[0, x]$

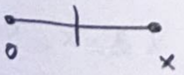
solu:



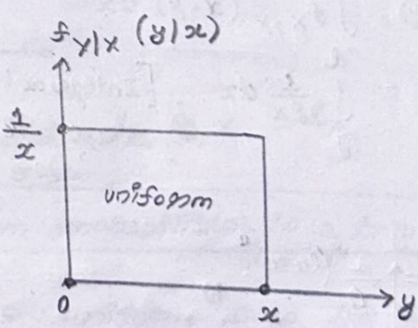
∴ uniform: $\frac{1}{b-a} = \frac{1}{l}$



(Break)



Again break (Y)



what's the JPdf

$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_{X|Y}(y|x)}$ → multiplication rule.

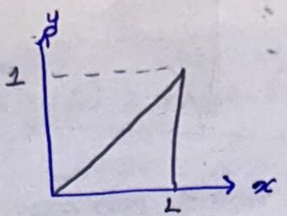
$f_{X,Y}(x,y) = \frac{1}{l} \cdot \frac{1}{x} = \frac{1}{lx}$ → valid for only x & y are possible

↓ (uniform through out)

$0 \le Y \le X \le l$

∴ x ranges from 0 to l

Y always less than X



when $x=0 \Rightarrow y=0$

$x=1 \Rightarrow$ y can be b/w 0 to 1

$x=2 \Rightarrow$ y can be b/w 0 to 2

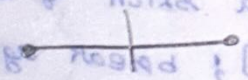
Conditional Expectation

$$E[Y|X=x] = \int y\, f_{Y|X}(y|X=x)\, dy$$

$$= \int_0^x y \cdot \frac{1}{x}\, dy$$

$$= \frac{1}{x}\left(\frac{y^2}{2}\right)_0^x$$

$$= \frac{x^2}{2x} = \frac{x}{2} \rightarrow \text{Average (midpoint)}$$



uniform b/w 0 and x

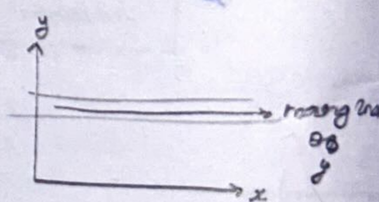
---

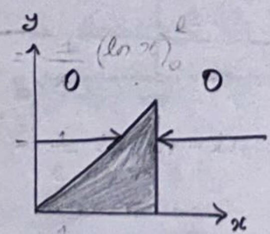
marginal (since we have JPdf)

$$f_Y(y) = \int f_{X,Y}(x,y)\, dx$$

$$= \int_y^l \frac{1}{lx}\, dx$$

[Integrating $f_{X,Y}$ through out x will be the marginal of x]





In free space $\rightarrow$ prob $= P_{X,Y} = 0$

Non zero in our Δ^{le} space

$\therefore$ So x starts from y to $l \longrightarrow$ "Not from zero to l"

$\boxed{\therefore y \text{ ranges from } 0 \text{ to } x}$

"This point x doesn't starting with zero"

$$= \int_y^l \frac{1}{lx}\,dx = \frac{1}{l}\left(\log x\right)_y^l$$

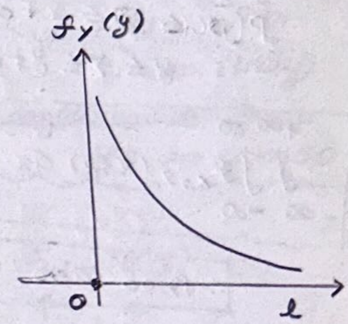
'Exclude zero region from Integration'

$$\frac{1}{lx}, \quad 0 \le x \le l, \quad 0 \le y \le x$$

$$\boxed{0 \le y \le x \le l}$$

$$= \frac{1}{l}\left(\log l - \log y\right)$$

$$= \frac{1}{l}\log\left(\frac{l}{y}\right) \qquad 0 \le y \le l$$



$$E[Y] = \int_0^l y\, f_Y(y)\, dy$$

$$= \int_0^l y \cdot \frac{1}{l} \log \frac{l}{y}\, dy$$

$$= \frac{1}{l}\int_0^l y \log \frac{l}{y}\, dy$$

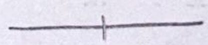
when $y = 1 \Rightarrow \log 1 = 0$

$y = 0, \log 0 = 1.$

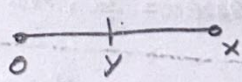
$= \frac{l}{4}$ (After Integration)

↓

1 time breaking



x (expected: 1/2



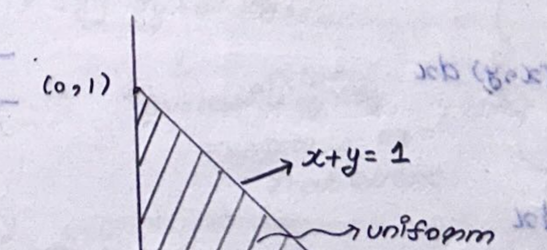
'Reasoning over average doesn't yield Correct result all the time'

Expected : 1/2 of x $= l/4$

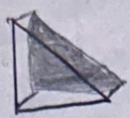
Recitation: Uniform probabilities in a triangle

R.V X & Y have a JPdf which is uniform over the Δ^{le} (0,0), (0,1) & (1,0)

Solu.



Jpdf: (uniform distribution)



→ Area.

mass coming out of it uniformly)

a) $\boxed{\text{Area} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1}$

∴ uniform → Same density everywhere.

$$P\left(\begin{matrix} x \le X \le x+\delta \\ y \le Y \le y+\delta \end{matrix}\right) = f_{X,Y}(x,y) \cdot \delta^2$$

$$\boxed{\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1}$$

$\boxed{\therefore \text{Prob also one}}$

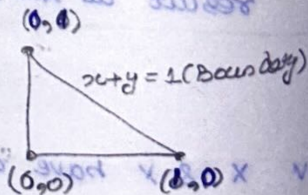
$$\iint f_{X,Y}(x,y)\, dx\, dy = \frac{\iint P\left(\begin{matrix} x \le X \le x+\delta \\ y \le Y \le y+\delta \end{matrix}\right) \cdot dx\, dy}{\delta^2}$$

$$\therefore \frac{1}{\frac{1}{2} \times b \times h} = \frac{1}{\frac{1}{2} \times 1 \times 1} = 2$$

Other way: ∴ Area $= \frac{1}{2}$, For $\iint f_{X,Y}(x,y)\, dx\, dy = 1$

$\boxed{\text{we need } f_{X,Y}(x,y) = 2}$

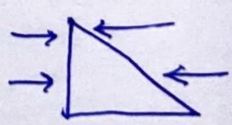
a) $f_{X,Y}(x,y) = \begin{cases} 2, & x, y \ge 0,\ x+y \le 1 \\ 0, & \text{otherwise} \end{cases}$



b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$

$= \int_0^{1-y} 2 \cdot dx$

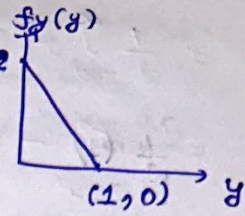
$= \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$



x always starts at zero

ends at $(1-y)$

c) $f_{X|Y}(x|y) = ?$



c) $f_{X|Y}(x,y) = \dfrac{f_{X,Y}(x,y)}{f(Y=y)}$

$= \dfrac{2}{2(1-y)}$

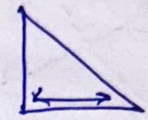
$= \begin{cases} \dfrac{1}{1-y} & \rightarrow 0 \le y < 1 \\ & \quad 0 \le x < 1-y \\ 0 & \text{otherwise} \end{cases}$

when $y \neq 1$ (Not defined)

'High density for smaller values'

Smaller values →

∴



For small values more width (more likely)

Larger values: small y's to choose.

when $y = 1$ nothing to choose

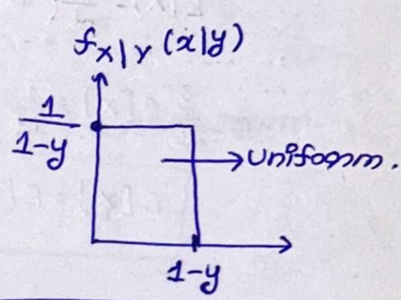
d) $E[X|Y=y] = \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y) \, dx$

$= \int_{-\infty}^{\infty} x \cdot \dfrac{1}{1-y} \, dx$

$= \int_{0}^{1-y} x \cdot \dfrac{1}{1-y} \, dx$

$= \dfrac{1}{1-y}\left(\dfrac{x^2}{2}\right)$

$= \dfrac{1-y}{2} \longrightarrow$ 'uniform' (midpoint of $\dfrac{1-y}{2}$)



$\therefore \int_{-\infty}^{\infty} f_{X|Y}(x|y) = 1$

(New world)

E) $E[X] =$ So as we have done in discrete cases

$E[X] = \int_{-\infty}^{\infty} E[X|Y=y] \, f_Y(y) \, dy.$

↓

probability of occurrence of that event.

$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) \; \vdots$

$= \int_{-\infty}^{\infty} \dfrac{1-y}{2} \, f_Y(y) \, dy$

$[E[Y] = \int y f_Y dy]$

$= \int_{-\infty}^{\infty} \dfrac{1}{2} f_Y(y) - \dfrac{1}{2} \int_{-\infty}^{\infty} y f_Y(y) \, dy$

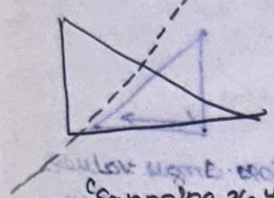
$\longrightarrow$ Integration of marginal prob

$$= \frac{1}{2}\int_{-\infty}^{\infty} f_Y(y)\,dy - \frac{E[Y]}{2}$$

$$= \frac{1}{2}(1) - \frac{E[Y]}{2}$$

$$= \frac{1}{2}(1 - E[Y])$$

$\therefore$ Symmetry in this problem



'Swapping x, y — same JPdf'

$$\therefore E[X] = E[Y]$$

$\therefore$ we need check for ranges where our work is valid.

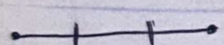
$$E[X] = \frac{1}{2}(1 - E[X])$$

$$\frac{3}{2}E[X] = \frac{1}{2}$$

$$\boxed{E[X] = E[Y] = 1/3}$$

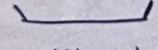
Recitation: 2

Prob that three pieces form a Δ^{le}



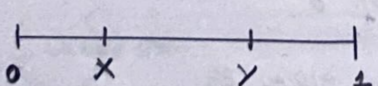
Stick — 3 pieces (Is it form a Δ^{le})

Condition: (forming Δ^{le})



Not a triange

, Any two of pieces > other side sum



Assume: $X < Y$

I lengths: X

II : $Y - X$

III : $1 - Y$

$\therefore$ condition: Sum of any two > other one

① $x + (y-x) > 1-y \rightarrow 2y > 1 \Rightarrow y > 1/2$

② $x + (1-y) > (y-x) \longrightarrow 2x + 1 > 2y$

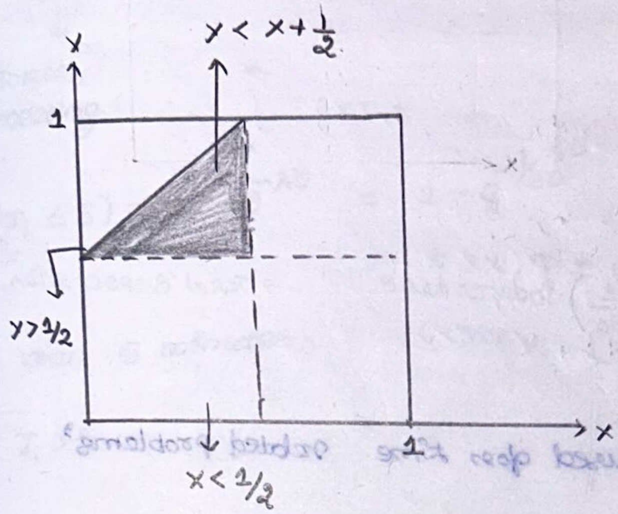
$y < x + 1/2$

③ $(y-x) + (1-y) > x$

$\hookrightarrow 2x < 1$

$x < 1/2$

'uniformly random'



∴ Both x and y are uniformly random

* $y > 1/2$
* $x < 1/2$
* $y < x + 1/2$

∴ x, y are uniform.

$y < x + \frac{1}{2}$

At $x = 0$ | At $x = 1/2$
$y < 1/2$ | $y < 1$

$P(\Delta^{le})$ = That triangle (it must) $= 1/8$ (By Geometry)

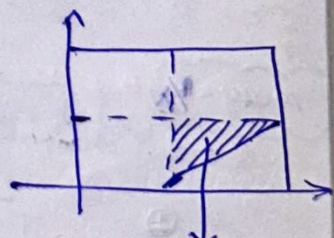
If $0 \quad y \quad x \quad 1$

we don't know which order we are cutting:

1) cut 0 to 1 then 0 to y

2) cut 0 to 1 then y to 1.

$$\boxed{\therefore x > 1/2 \quad \therefore y < 1/2 \quad \therefore x < y + \frac{1}{2}}$$



For this case this is our Δ^{le}

$\therefore 1/8$.

$$\boxed{\therefore P(\Delta^{le}) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}}$$

Question: Break at two points → Didn't gave where (precedence)

The absent minded professor

* made two appointments at the same time.
* Appointment durations are independent & exponentially distributed with mean 30 minutes.
* 1 student (1st) arrives on time
* 2nd done 5 minutes late.
* what's the expected value of the time b/w the arrival of the first student & the departure of the second one.

Solu:

$$T \sim \exp(t)$$

$\boxed{P(T \le t) = 1 - e^{-\lambda t}}$, $t \ge 0$

$E[T] = 1/\lambda$

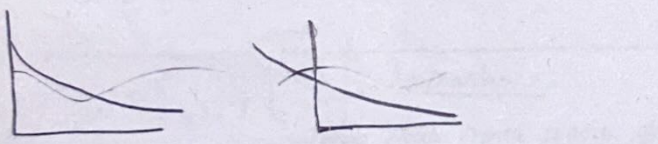
memoryless property:

T_1 & T_2 $\sim \exp\left(\frac{1}{30}\right)$ independent. → Total expectation.

'Past doesn't matter'

'exponential distribution is used for time related problems'

Distribution for going forward will be same.

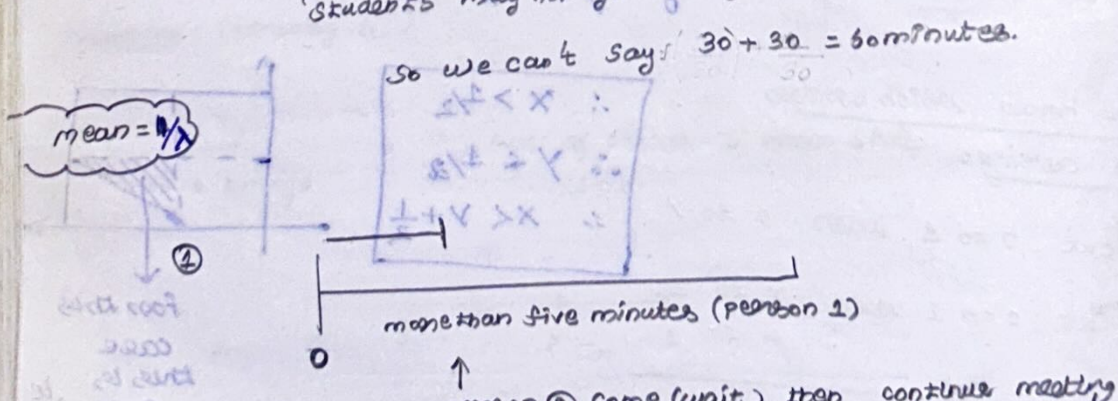


'started earlier'

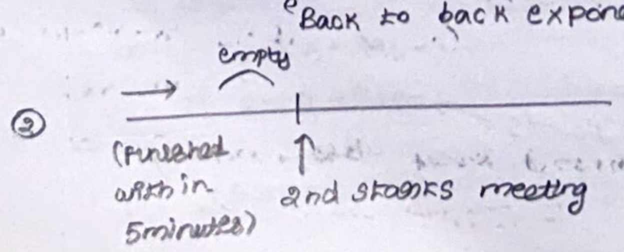
Expected time = ?

'students may not go exactly back to back'

So we can't say $\frac{30 + 30}{30} = 60$ minutes.



'Back to back exponentials'

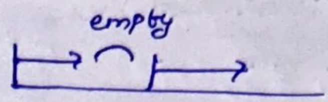


① Back to back

② 1st leaves early.

① within 5 minutes → 1st Student left

$T_1 \le 5$



$E[X \mid T_1 \le 5]$ $\cancel{E[X]} = 5 + E[T_2]$

$\cancel{E[X]} = 5 + 30 = 35$ minutes (Average)

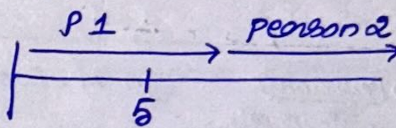
↓

Total meeting

$P(T_1 \le 5) = 1 - e^{-\lambda t} = 1 - e^{-1/30 \times 5}$

② Longer than 5 minutes

$T_1 > 5$



$\cancel{E[X \mid T_1 \le 5]}$ = 5 + 'memoryless continuation of 1st Student' + Second student

$E[X \mid T_1 > 5] = 5 + 30 + 30$
$= 65$

$P(T_1 > 5) = 1 - P(T_1 \le 5)$
$= 1 - (1 - e^{-5/30})$

$\boxed{P(T_1 > 5) = e^{-5/30}}$

$E[X] = P(T_1 \le 5)\, E[X \mid T_1 \le 5] + P(T_1 > 5)\, E[X \mid T_1 < 5]$

$= 35(1 - e^{-5/30}) + 65e^{-5/50}$

$\boxed{E[X] = 60.394}$

Even though prof is idle ↓ That time also in meeting

case: → person 1 takes longer than 5 minutes (No gap - 2 can continue after 1 finishes)

→ person 1 takes less than 5 minutes (Empty gap - until 2 arrives)

Professor idle.

Lecture: 10: Continuous Bayes Rule; Derived distributions

* Four variants of the Bayes' rule
* Derive distributions.

$P_X(x)$ | $f_X(x)$

$P_{X,Y}(x,y)$ | $f_{X,Y}(x,y)$

$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$ | $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

$P_X(x) = \sum_y P_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$

$F_X(x) = P(X \le x)$

$E[X]$, $\mathrm{var}(X)$

$f \rightarrow$ Probability / unit length | In JPDF $\rightarrow$ Prob / Area (density)

Probability: we use it to make inferences.

Bayes variations

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)} = \frac{P_X(x)\, P_{Y|X}(y|x)}{P_Y(y)}$$

$$\boxed{P_Y(y) = \sum_x P_X(x)\, P_{Y|X}(y|x)}$$

eg:

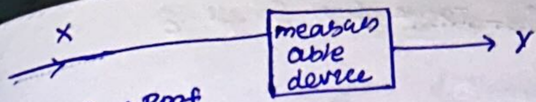
$X = 1, 0$: aeroplane present / not present

$Y = 1, 0$: something did / not registers on radar.

Counterpart

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) \cdot f_{Y|X}(y|x)}{f_Y(y)}$$

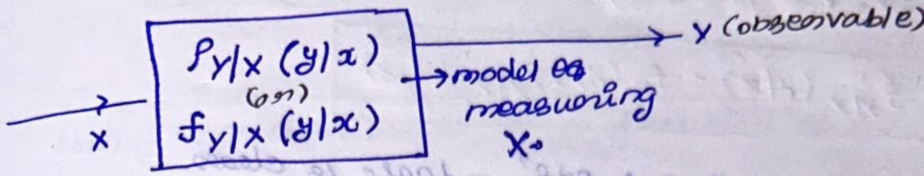
$$\boxed{f_Y(y) = \int_x f_X(x)\, f_{Y|X}(y|x)\, dx}$$



$f_X(x) \rightarrow$ pmf

$f_X(x) \rightarrow$ pdf

"Noisy phenomenon"



"make inferences:"

* "Tell the prob distribution of unknown"

"we don't know about X?": Give inference about X

$$\boxed{\begin{aligned} P_X(x) \cdot P_{Y|X}(y|x) &= P_{X,Y}(x,y) \\ P_Y(y) \cdot P_{X|Y}(x|y) &= P_{X,Y}(x,y) \end{aligned}}$$

Application: * X: current through a resistor

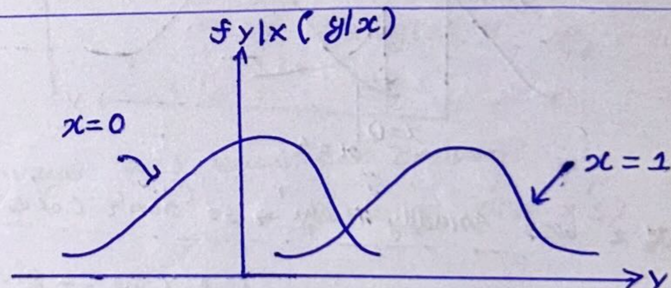
* (Noise: Gaussian noise)
* Analog equipments: continuous.

what will be X for a particular value of Y?

$X \rightarrow$ Discrete 0, 1

$Y = X + W$

↑ Gaussian noise.



we need a formula dealing with discrete & analog case

$$P(X=x, y \le Y \le y+\delta) = P(X=x) \cdot P(y \le Y \le y+\delta \mid X=x)$$

$$= P(y \le Y \le y+\delta)\, P(X=x \mid y \le Y \le y+\delta)$$

∴ Prob of little intervals → density

$$= P_X(x) \cdot f_{Y|X}(y|x) \cdot \delta$$

(or)

$$= f_Y(y) \cdot \delta \cdot P_{X|Y}(x|y)$$

"The above conv is app" — True for small δ

equating

$$\Rightarrow P(X=x,\ y \le Y \le y+\delta) = P_X(x)\, f_{Y|X}(y|x)\cdot\delta = f_Y(y)\,\delta\, P_{X|Y}(x|y)$$

$$\boxed{P_X(x)\, f_{Y|X}(y|x) = f_Y(y)\cdot P_{X|Y}(x|y)}$$

* Formula links pmf & pdf" — logic is clear.

discrete – pmf

continuous – pdf.

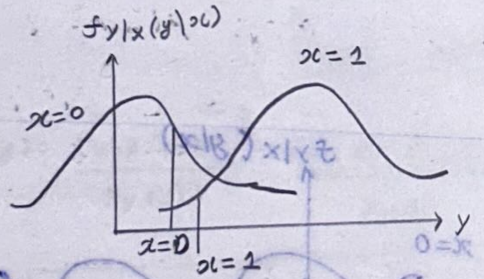
summary

$$P_{X|Y}(x|y) = \frac{P_X(x)\cdot f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_Y(y) = \sum_x P_X(x)\, f_{Y|X}(y|x) \rightarrow \text{Total prob theory.}$$

Y: Noisy version of X.

$\therefore P_{X|Y}(x|y)$ prop to $f_{Y|X}(y|x)$



$\therefore$ If x are equally likely" → so don't care about $P_X(x)$

ratio b/w $x=1$ & $x=0$ → gives the relative odds of x's in the y (observed)

Continuous X, Discrete Y

eg: Emits light (driven by current) → measure how many photons.

"unknown : x"

$$f_{X|Y}(x|y) = \frac{f_X(x)\, P_{Y|X}(y|x)}{P_Y(y)} \rightarrow \text{(Discrete → cont?) (cont → desc?) In prev case.}$$

$$\boxed{P_Y(y) = \int_x f_X(x)\, P_{Y|X}(y|x)\, dx}$$

'Bayesian inference'

Details | exact f's | model | Calculations

Real world problem.

Finding the distribution of a function of R.V

Derived distribution (Based on previous distri)

* See pdf: $g(x, y) = y/x$ (ratio of two R.V)

↳ Find dist of these ratio.

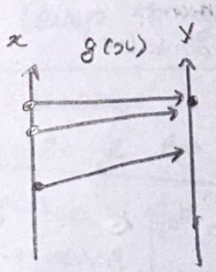
When not to find them

Don't need PDF for $g(x, y)$ if only want to compute $E[\ ]$

$$\boxed{E[g(x, y)] = \iint g(x, y)\, f_{x,y}(x, y)\, dx\, dy}$$

How to find them?

Discrete case



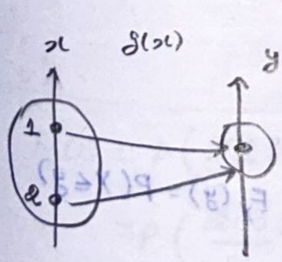
PMF of Y

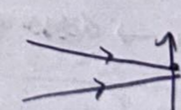
$$P_Y(y) = P(g(x) = y)$$

$$= \sum_{x : g(x) = y} P_X(x)$$

how many ways a y value can happen?

(Through diff x's)





Takes takes this or that value.

Prob of y falling in that set = x falling in that set

↓

Sum of individual prob of x's

P(1) + P(2)

'Nice & simple' - In discrete

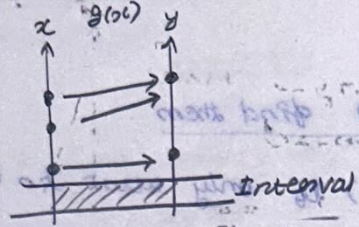
Continuous case
↓
'useless'

'Individual points has zero probability'
(single value)

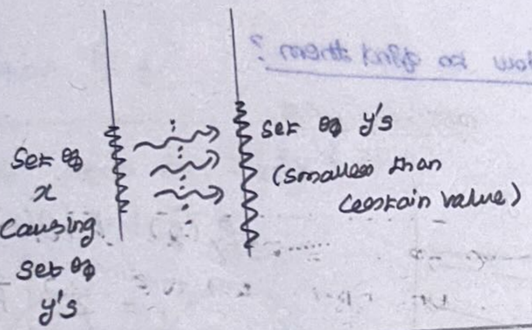
we want more!

Find density of Y:

* 'Take interval' (y's) → prob of falling in that interval.
* Visit x's causing (maps) in the interval.



CDF - Deal with large intervals



Prob of Y falls in the integral = Prob of X falling in that interval.
↓
CDF can be found.

Density → differenciating CDF

Procedure

* Two-step: * Get CDF of Y: $F_Y(y) = P(Y \leq y)$

* Differenciate to get

$$f_Y(y) = \frac{dF_Y}{dy}(y)$$

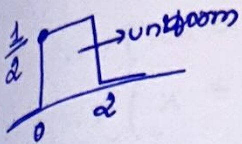
Example

* X uniform [0, 2]
* Find PDF of $Y = X^3$

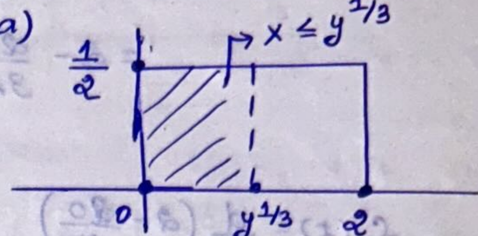
$Y = [0, 8]$ → All x's are equally likely
→ All Y's are equally likely

C.D.F $F_Y(y) = P(Y \le y) = P(X^3 \le y)$

$= P(X \le y^{1/3})$



$= \frac{1}{2} y^{1/3}$ (Area)



$f_Y(y) = \frac{d}{dy}\left(\frac{1}{2} y^{1/3}\right)$

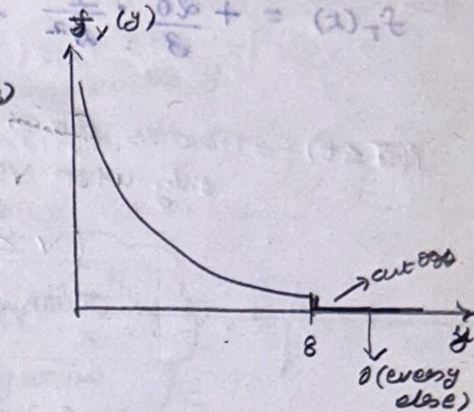
$= \frac{1}{2} \times \frac{1}{3} y^{-2/3}$

$= \frac{1}{6 y^{2/3}}$

(when $y = 0 \to$ blows)

when $y = 8$

$\frac{1}{6 \times 2} = \frac{1}{12}$



$f_Y(y) = \begin{cases} \frac{1}{6 y^{2/3}} & 0 \le y \le 8 \\ 0 & \text{otherwise} \end{cases}$

Inference: $y \to$ not uniform

(even though we started with uniform x)

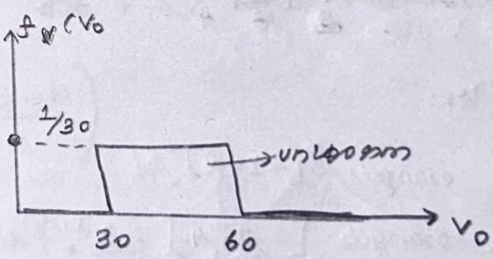
Joan is driving from Boston to Newyork. Her speed is uniformly distributed between 30 & 60 mph. What's the E [duration of trip]

solu:

$T(V) = \frac{200}{V}$

$f_T(t) = ?$

$= \frac{1}{60-30} = 1/30$



solu:

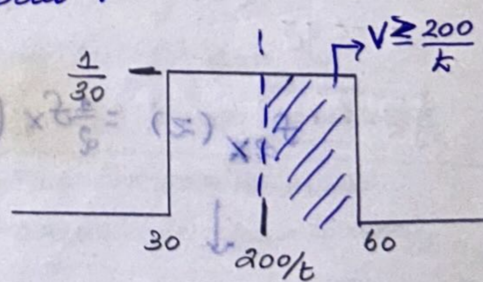
we know about V

$F_T(t) = P(T \le t)$

$= P\left(\frac{200}{V} \le t\right)$

$= P\left(V \ge \frac{1}{t} \times 200\right)$

$= P\left(V \ge \frac{200}{t}\right)$



$\therefore V \ge \frac{200}{t}$

$t = 6.667$ ($V = 30$) $= \frac{200}{30}$

$t = 3.333$ ($V = 60$) $= \frac{200}{60}$

$(30 \le V \le 60)$

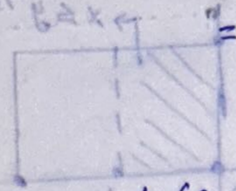
only when

$t =$

$\boxed{\frac{200}{60} \le t \le \frac{200}{30}}$

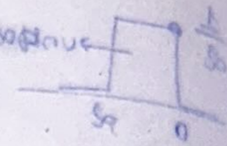
$$P\left(V \geq \frac{200}{t}\right) = \frac{1}{30} \times \left(-\frac{200}{t} + 60\right)$$

$$= 2 - \frac{200}{30t}$$



$$= 2 - \frac{20}{3t}$$

$$\boxed{\frac{200}{60} \leq t \leq \frac{200}{30}}$$



$$f_T(t) = \frac{d}{dt}\left(2 - \frac{20}{3t}\right)$$

$$f_T(t) = +\frac{20}{3} \cdot \frac{1}{t^2}$$ → monotonically decreasing.

$P(T \leq t)$ → Time is lesser than something only when velocity is bigger than something

$\left(V \geq \underline{\text{something}}\right)$ → Some inequalities must be reversed

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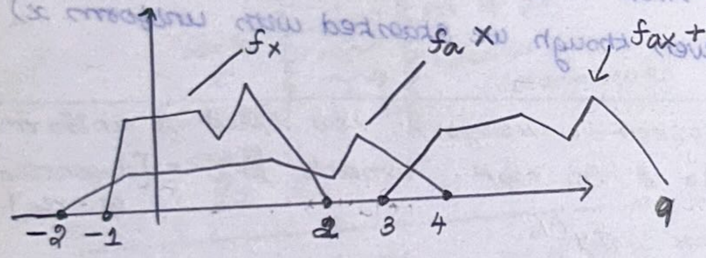
Linear function of R.V

$Y = 2X+5$ $\quad a=2, b=5$



Distribution of 2X:

X ranges $[-1, 2]$

2X ranges $[-2, 4]$ → Just changing the scale.

$2X+5$ → shifted.

$$f_{2X}(z) = \frac{1}{2} f_X\left(\frac{z}{2}\right)$$

↳ 'widening stretching'

↓

$\int$ must be $= 1$ (scaling by $1/2$)

'Shifting the function to the right'

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$ → shift

↳ scale to 1 $\boxed{\int = 1}$

$a>0$:

$F_Y(y) = P(Y \le y)$

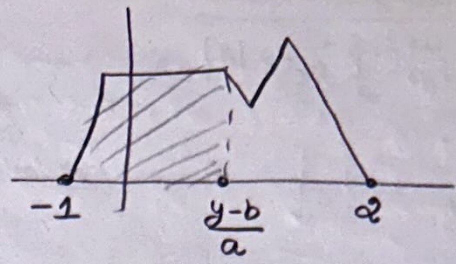
$= P(aX+b \le y)$

$= P(aX \le y-b)$

$= P\left(X \le \frac{y-b}{a}\right)$

$= F_X\left(\frac{y-b}{a}\right)$

↳ CDF up to $\frac{y-b}{a}$ is $P\left(X \le \frac{y-b}{a}\right)$



$\cancel{f_Y(y) = \frac{d}{d}}$

Derivative

→ function of y.

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

'Densities can't be negative'

So a can't be negative

when $a \le 0$:

$$F_Y(y) = P(-aX \le y-b)$$

÷ by −ve changes the sign

$$= P\left(X \ge \frac{y-b}{a}\right)$$

∴ negative is took → sign reversed

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = + f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{+a}$$

$$= f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

→ suits for all linear equations.

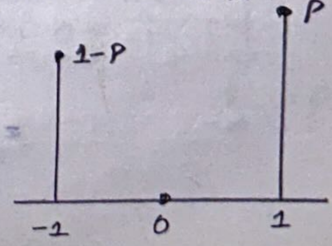
∴ linear. (linear func of linear Normal variable is also linear).

Recitation

Infer a discrete r.v from a continuous measurement.

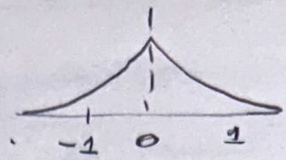
X: discrete

$$P_X(x) = \begin{cases} P & x = 1 \\ 1-P & x = -1 \\ 0 & \text{otherwise} \end{cases}$$



Y: continuous

$f_Y(y) = \frac{1}{2}\lambda e^{-\lambda|y|}$



$Z = X + Y$

solu:

$P(X=1|Z=2) = ?$ check that the expression obtained makes sense as $p \to 0^+$, $p \to 1^-$, $\lambda \to 0^+$, and $\lambda \to \infty$

solu:

$Z = \text{cont} + \text{Disc}$

$\boxed{Z = \text{cont}}$

$$P(X=1|Z=z) = p_{X|Z}(1|z)$$

$$= \frac{p_X(1) \cdot f_{Z|X}(z|1)}{f_Z(z)}$$

$$\boxed{f_Y(y) = \sum_x p_X(x)\, f_{Y|X}(y|x)}$$

∴ Y is based on the probability of x

$$= \frac{p_X(1) \cdot f_{Z|X}(z|1)}{p_X(1)\, f_{Z|X}(z|1) + p_X(-1)\, f_{Z|X}(z|-1)}$$

$Z = X + Y$

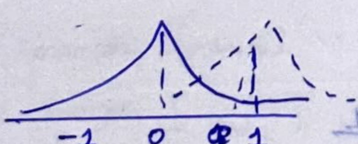
If $x = 1$

$Z = 1 + Y$

If $x = -1$

$Z = -1 + y$.

$\boxed{y = z + 1}$



"shifted by 1"

→ shifted right by 1

$$= \frac{p \cdot f_{Z|X}(1+y|x=1)}{p \cdot f_{Z|X}(1+y|1) + (1-p) \times f_{Z|X}(1-y|-1)}$$

↳ shifted left.

$$= \frac{p \cdot \frac{1}{2}\lambda e^{-\lambda|z-1|}}{p \cdot \frac{1}{2}\lambda e^{-\lambda|z-1|} + (1-p) \cdot \frac{1}{2} e^{-\lambda|1+z|}}$$

$$= \frac{p\, e^{-\lambda|z-1|}}{p\, e^{-\lambda|z-1|} + (1-p)\, e^{-\lambda|1+z|}}$$

$$= \frac{p}{p + (1-p)\,\dfrac{e^{-\lambda|1+z|}}{e^{-\lambda|z-1|}}} = \frac{p}{p+(1-p)\,e^{-\lambda|z+1|}\cdot e^{\lambda|z+1|}}$$

$$= \frac{p}{p+(1-p)\,e^{-\lambda(|z+1|-|z-1|)}}$$

verifying by limits: $p \to 0^+ : 0$

$\boxed{p_x(1) \approx 0}$

a) $p(x=1|z=2)\Big|_{p\to 0^+} = 0$

$\therefore p(x=1)$ is not possible

b) $p \to 1^-$: $\dfrac{1}{1+(1-1)\cdots} = \dfrac{1}{1} = 1$

$\longrightarrow 1 \longleftarrow$

1^- $\quad$ 1^+

c) $\lambda \to 0^+$

As $\lambda \to 0^+$

$$= \frac{p}{p+(1-p)e^0}$$

$$= \frac{p}{1} = p \longrightarrow$$ Not informative (Conditional)

d) $\lambda \to \infty$

$\boxed{\begin{aligned} e^{-\infty} &= 0 \\ e^{\infty} &= \infty \end{aligned}}$

depends upon

i) $|z+1|-|z-1|>0$

ii) $|z+1|-|z-1|<0$

when $|z+1|-|z-1|>0$

when $z > 0$

when $|z+1|-|z-1|<0$

when $z<0$

$$\frac{p}{p+0} = 1$$

$$\frac{p}{\infty} = 0$$

In all 4 cases — it makes sense

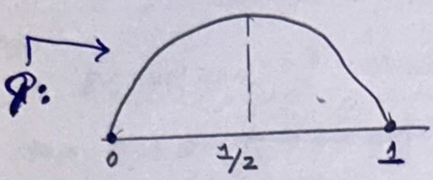
Q. Inferring a Cont. R.V from a discrete R.V

Q = Continuous

$$f_Q(q) = \begin{cases} 6q(1-q), & 0 \le q \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Q: represents the P(success) of a Bernoulli R.V X,

$$P(X=1 \mid Q=q) = q$$



X: Discrete Bernoulli

solu:

$P(X=1 \mid Q=q) = q$

$P(X=0 \mid Q=q) = 1-q$

$$P_{X|Q}(x|q) = \begin{cases} q & x=1 \\ 1-q & x=0 \\ 0 & \text{otherwise} \end{cases}$$

Solu:

$f_{Q|X}(q, x)$ for $x \in \{0, 1\}$ and all q

$$f_{Q|X}(q|x) = \frac{f_Q(q)\, P_{X|Q}(x|q)}{P_X(x)}$$

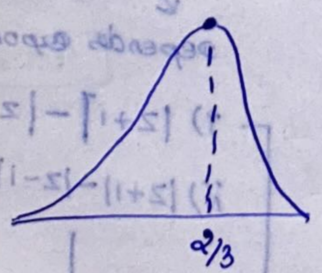
$$= \frac{f_Q(q)\, P_{X|Q}(x|q)}{\int_0^1 f_Q(q)\, P_{X|Q}(x|q)\, dq}$$

$\therefore Q$ is in range $\{0, 1\}$

(1) $\boxed{f_{Q|X}(q|1)} = \dfrac{6q(1-q)\cdot q}{\int_0^1 6q(1-q)\, q\, dq} = \dfrac{6q^2(1-q)}{\int_0^1 (6q^2 - 6q^3)\, dq}$

$$= \frac{6q^2(1-q)}{\left(\frac{6q^3}{3} - \frac{6q^4}{4}\right)_0^1}$$

$$= 12q^2(1-q)$$



(peak position changed)

Inference: Biased coin more likely be towards bias.

$f_{Q|X}(q|1)$

$\downarrow$

head $\rightarrow$ so towards head.