$$\mathrm{var}(E[X|Q]) = \mathrm{var}(E[X_1|Q] + E[X_2|Q] + \cdots + E[X_n|Q])$$

$$= \mathrm{var}(n\,E[X_i|Q])$$

$\hookrightarrow$ Bernoulli (expect$(X_i|Q) = Q$)

$$= \mathrm{var}(nQ)$$

$$= n^2\,\mathrm{var}(Q)$$

$$= n^2\sigma^2$$

$$\boxed{\mathrm{var}(X) = n(\mu - \sigma^2 - \mu^2) + n^2\sigma^2} = n(\mu-\mu^2) + n(n-1)\sigma^2$$

verify covariance

$$\mathrm{Var}(X) = \mathrm{var}(X_1 + \cdots + X_n) =$$

$$= \sum_{i=1}^{n} \mathrm{var}(X_i) + \sum_{i \neq j} \mathrm{cov}(X_i, X_j)$$

$(j = i)$

$$\boxed{\mathrm{var}(X) = E[X^2] - (E[X])^2}$$

$$\boxed{\begin{aligned} \therefore\ \mathrm{cov}(X_i, X_j) &= E[(X - E[X])^2] \\ &= \mathrm{var}(X) \end{aligned}} \to X_i = X_j$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$\mathrm{cov}(X,Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= E[XY] - E[Y]E[X] - E[Y]E[X] + E[X]E[Y]$$

$$\boxed{\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]}$$

$E[X], E[Y] \to$ constant

Derivation

$$\mathrm{var}(X_1 + X_2) = E[((X_1 + X_2) - E[X_1 + E_2])^2]$$

$$= E[((x_1 - E[X_1]) + (X_2 - E[X_2]))^2]$$

$$= E[(x_1 - E[X_1])^2] + E[(X_2 - E[X_2])^2]$$

$$+ E[2(x_1 - E[X_1])(X_2 - E[X_2])]$$

$= \text{var}\, x_1 + \text{var}\, x_2 + 2E\left[(x_1 - E[x_1])(x_2 - E[x_2])\right]$

$= \text{var}\, x_1 + \text{var}\, x_2 + 2\,\text{Cov}(x_1, x_2)$

n variances

$\text{var}(x_1 + x_2 + \cdots x_n) = E\left[(x_1 + \cdots + x_n)^2\right]$

(Assume zero means)

$$= E\left[\sum_{i=1}^{n} x_i^2 + \sum_{\substack{i=1,2,\ldots n\\ j=1,2,\ldots n\\ i \neq j}} x_i x_j\right]$$

$$= \sum_{i}^{n} \text{var}(x_i) + \sum_{i \neq j} \text{cov}(x_i, x_j)$$

$$\text{var}(X) = \sum_{i=1}^{n} \text{var}(x_i) + \sum_{i \neq j} \text{cov}(x_i, x_j)$$

$$= n(\mu - \mu^2) + (n^2 - n)\sigma^2$$

n^2 terms (n terms $- x_i(x)x_j$)

'Exactly same as previous'

Good rule of thumb

(1 layer of randomness depends on randomness level above)

head/tails → random bias

Unit-III - Random processes

used to model the random arrivals.

* models – used to model dynamic systems (evolve probabilistically over a finite
* markov – state space. we present the general structure of markov models & study both their long-term & transient behaviour.

(science of inference)

Lecture: 13: Bernoulli process

Real world: Generate RV → Things evolve in time.

Random processes: Models capture evolution of Random phenomenon

in time → Discrete
→ continuous.

(sequence of coin flips) → Bernoulli process

Bernoulli: Family (17th Century) → of mathematicians.

* Bernoulli process: Sequence of ind. Bernoulli trial (coin flip)

$P(\text{success}) = P(X_i = 1) = P$

$P(\text{failure}) = P(X_i = 0) = 1-P$

eg:

* Sequence of lottery wins / losses
* Sequence of ups & downs of the Dow Jones. → financial models

Stock name ↑

(crude model of financial market)

* Arrivals (each second) to a bank
* Arrivals (each time slot) to servers. (Independent time slots.)

Assumption

* Independence
* P → constant.

For complete day

morning – low arrival

Afternoon – high

(Entire day – not a good appr)

10:00 to 10:15 → Probably all slots will be same.

P → constant (not vary with time)

Random process:

* Sequence of R.V.S. (Experiments associated with RVs)

$$\left.\begin{aligned} * \ & E[X_t] = P(1) + (1-P)\cdot 0 = P \\ * \ & \text{var}[X_t] = P(1-P) \end{aligned}\right] \text{ for every } t$$

Enough for description of random process?

How different R.V s relate each others?

Complete description: All possible JPdf

∴ Different R.V are independent.

$$\boxed{P_{X_2, X_5, X_7} = P_{X_2}()\, P_{X_5}()\, P_{X_7}()}$$

General Random process:

Joint distribution of X_2, X_5, X_7 ← our interest (say)

(we need to do for every collection (subset)) of JPdf

Other view:

* As one long experiment. → sample space? 1001.... (seq of 1s & 0s)

Sample space: All sequences of 0's and 1's

$\boxed{P(\text{sequence of obtaining all ones}) = P(111....)}$

Sample space: sequence of all infinite sequences.

$P(X_t = 1 \text{ for all } t) \leq P(X_t = 1 \text{ for first 10 tosses})$ (say!)

This event is contained in

$P(X_t = 1, \forall\, t = 10) = p^{10}$

$P(X_t = 1, \forall\, t = k) = p^{k}$ $\quad k = 1, 2, ...$

$\therefore P(X_t = 1, \text{ for all } t) = \lim_{k \to \infty} p^{k}$ $\quad k \to$ arbitrarily large

$\boxed{\therefore P(X_t = 1 \;\; \forall\, t) = \lim_{k \to 0} (X_t = 1, \forall\, t = k)}$

so As $k \to \infty$, $\lim_{k \to \infty} p^{k} = 0$

$\boxed{p \leq 1}$

* Particular sequence is 0 probability
* Any other infinite sequence $p^{k}(1-p)^{n-k} = 0$ (over ∞)

'counter-intuitive' → more like continuous R.V (every single point → 0 prob)

collectively → has prob

∞ sequence - more like continuous

- Bernoulli process (memoryless) → Discrete time.
- poisson process (continuous, memoryless time
* markov (memory/dependence on time)

Any real time phenomenon - evolving with time - modeled with markov models.

1) Arrivals of jobs to a facility - for n jobs - how much time for them to arrive?

Fix time slots:

number of success in (S) n time slots

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \dots, n$$

Binomial R.V'

$$E[S] = np$$

$$Var(S) = np(1-p)$$

Fix no. of arrivals - ask time

T_1: number of trials until first arrival

0 | 0 | 0 | 0 | 1

$\longleftarrow T_1 \longrightarrow$

Prob dist of T_1 ?

$\rightarrow P(T_1 = t)$, $t \rightarrow$ success, $(t-1)$ trials failures

$$\boxed{P(T_1 = t) = (1-p)^{t-1}\, p}\ , \quad t = 1, 2, \dots$$

Geometric distribution

(whatever in past is irrelevant)

memoryless property $\rightarrow$ consequence of independents.

'How long to occur 1st success doesn't affect the next success'
$\downarrow$
until you have future foresight pyshic powers

Small p $\rightarrow$ long time

$$E[T_1] = 1/p$$

$$Var[T_1] = \frac{(1-p)}{p^2}$$

Special case
you are called when 1st success.
(Based on past)

* If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days?

Solu:.

1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 ...

Losing day string

$\longleftarrow L \longrightarrow$

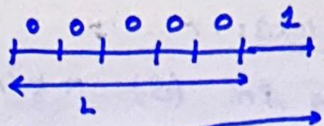
what distribution: 'past doesn't matter' 1 | 1 | 1 | 0 ...
'Time until first success'

0 0 0 0 0 1

$L+1$

Geometric R.V.

∴



L+1 = ? Geometric (P)

L = geometric (P) − 1

Geometric R.V → Takes values 1, 2, 3, ...

∴ L = geometric (P) − 1 → Takes values 0, 1, 2, 3

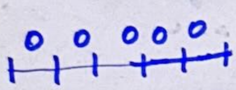
can R.V L be zero?

(minimum losing day? = 1 day (can't be zero)

$L \neq 0$

$\boxed{L+1 \neq 1}$
AS $L \neq 0$

∴ L+1 is not a geometric



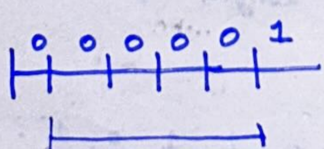
String of failures start! [Same as told → Next one is failure]

Then only you will Start watching there.

If you know what's next → No more independent (Bernoulli trial)

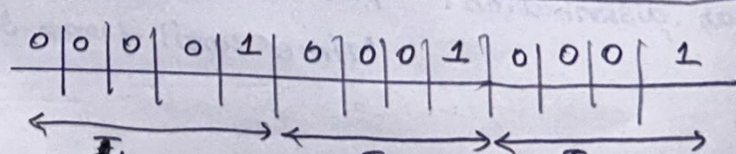
<u>Correct</u>

* Sister watched first failure (assumes: failing days Started: asks you to watch)



This will be geometric.
L (Same as L)

* L → Geometric with parameter p.



Y_3 (time for 3rd arrival) $= T_1 + T_2 + T_3$

0 0 0 0 1

$T_1 \rightarrow$ After 1 arrives you to watch

0 0 0 1

$T_2 \rightarrow$ until next success $\rightarrow$ Geometric R.V with parameter (p)

0 0 0 1

$T_3 \rightarrow$ Geometric R.V (No foresight about future - Independent Bernoulli trials).

Past doesn't affect future.

$$\boxed{Y_K = T_1 + T_2 + \cdots + T_K}$$

$T_i \approx$ geometric (p) (independent)

Distribution $(K = 100 \text{ (say)})$ $\rightarrow$ 100 customers arrive.

Convolution formula: $\rightarrow$ Extremely tedious.

[Take T_1, convolve with T_2]
T_3 with (Result of T_2 & T_1)]

'shortcut'

$$p_{Y_K}(t) = P(Y_K = t) = P\left(\begin{matrix} K-1 \text{ arrivals in time interval } [1, t-1] \end{matrix},\ \text{arrival at } t\right)$$

$$P\left(\begin{pmatrix}\text{how many success up to } t-1\end{pmatrix} \cdot \begin{pmatrix}\text{Success at } t\end{pmatrix}\right)$$

This event has to do with $[1, t-1]$ — has to do with t alone

Independent

$= P(K-1 \text{ arrival in } t-1) \cdot P(K^{th} \text{ arrival at } t)$

$$\boxed{P(Y_K = t) = \left[\binom{t-1}{K-1} p^{K-1} (1-p)^{t-K}\right] \cdot p}$$

we need atleast K time slots for K arrivals

$t \geq K$.

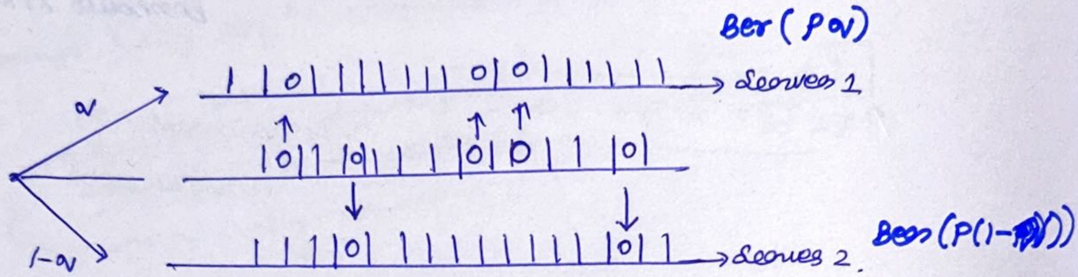
$$E[Y_K] = K\,E[Y_1]$$

$$= \frac{K}{p}$$

$$\mathrm{var}(Y_K) = K\cdot\frac{(1-p)}{p^2}$$

Several processes at time

Splitting of a Bernoulli process:



2 servers – send to a server: flipping an independent coin

coinflips: independent from arrival process itself

Process: At any time

* Any any t slot: Probability p (info) then α is the prob of sending to server 1.

∴ P(arrival saw by server 1) $= p\alpha$

P(arrival by S2) $= p(1-\alpha)$

verify?

* Independent? [different slots are independent]

∴ | | | | |

This slot has nothing to do with any other.

'Independent'

↓

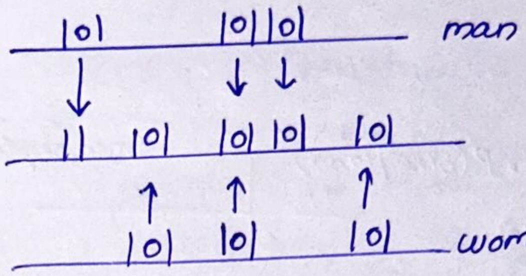
'Bernoulli process'

Also coin is also independent (flips)

functions of ind. things → remains independent

Converse

Two streams, 1 server



collisions are counted as one arrival.

Arrival: one or both has arrival.

two streams are independent.

$P(\text{arrival}) = 1 - P(\text{no arrival})$

$= 1 - (1-p)(1-q)$

$(1-p)$ → No arrival in stream1

$(1-q)$ → No arrival in stream 2

Independent.

$= p + q - pq$

Verify

Different slots are independent?

Server slot at t_i has only to with t_i of streams 1 & 2 not with any others t_{i+1} & (or) t_{i-1} s

↓

"Bernoulli process"

Bernoulli process

p: probability of getting a mosquito bite at each second

X = (time b/w successive mosquito bites)

= (time until next mosquito bite)

$P(\text{land neck}) = 0.5$

$P(\text{bite} \mid \text{land neck}) = 0.2$

$P(\text{never bitten} \mid \text{land neck}) = 0.8$

Prob distribution of getting mosquito bites are identically distributed.

solu.

$p = P(\text{lands}) \cdot P(\text{bite} \mid \text{land})$

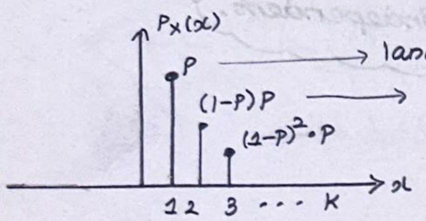
$= 0.5 \times 0.2$

$= 0.1$

"memoryless – Bernoulli"

↓

Geometric distribution



$E[X] = \frac{1}{p} = \frac{1}{0.1} = 10$

$\text{var}[X] = \frac{1-p}{p^2} = \frac{0.9}{(0.1)^2} = 90$

b) mosquito: Bernoulli $\sim p = 0.1$

Ticks lands: Prob = 0.1

Tick bites: Bernoulli $\sim q = 0.1 \times 0.7 = 0.07$

↓

Land, bite, Tick byte

Bug bites: Bernoulli $\sim r = 1 - P(\text{No mosquito \& Tick bites})$

↓ Either mosquito or tick bite

$= 1 - (1-p)(1-q)$

$= 1 - (1 - p - q + pq)$

$= p + q - pq \quad = 0.1 + 0.7 - 0.1 \times 0.7$

$= 0.163$

$Y = \{\text{time b/w successive bug bites}\}$

$E[Y] = \frac{1}{r} = \frac{1}{0.163} = 6.135$

$\text{Var}[Y] = \frac{1-r}{r^2} = \frac{1-0.163}{(0.163)^2} = 31.503.$

Lecture-14 – Poisson process-I

Random process: R.v evolve over time (∞ stages)

Continuous version of Bernoulli process

Bernoulli review:

* Discrete time: success prob p.
* No. of arrivals in n time slots: binomial pmf
* Interarrival times: geometric pmf (until first arrivals)
* Time to k arrivals: pascal pmf
* memorylessness.

$$Y_K = T_1 + T_2 + \dots + T_K$$

$$P(Y_K = t) = \left[\binom{t-1}{K-1} p^{K-1} (1-p)^{t-K}\right] \cdot p$$

pascal pmf.

'memorylessness' – $T_1, T_2, \dots T_K$ are independent geometric variables.

rule: $t = K, K+1, \dots$

Bernoulli process

$B_1, B_2, B_3, \dots$ (Trials)

$B_1, B_2, B_3, \dots, B_t, \underbrace{B_{t+1}, B_{t+2}, \dots}_{\text{watching by person 2}}$

Something interesting happens (asks to see)

↳ It is also a bernoulli trial (No foresight pyschic powers by person 1)

If the person already watched the movie (knows future)
↓
No longer a random independent Bernoulli trial

∴ T_2 is independent of past → Geometric R.V

'split → still a Bernoulli process

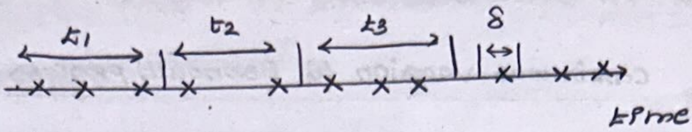
Join two Bernoulli → still a Bernoulli'

watchman → cross the (X) if came in (one second)

No one came in empty

Bank manager → millisecond slots

Idea: why ms, second time (exact)
↳ continuous.



* No need for slots

Idea: Assume intervals (same length) – behave prob in an identical fashion. Random no. of arrival at time t_i → has some prob distri)

Time homogeneity

* $P(K, \tau)$ = Prob of K arrivals in interval of duration.

* $\boxed{\sum_K P(K,\tau) = 1}$

↓ Depends on time interval length not the location of that interval in time axis.

1) Interval ↑, number of arrivals ↑ $\propto$ (proporti)

* No. of arrivals in disjoint time intervals are independent.

$\tau \to$ Fixed: Any τ slot has same prob of success

No. of arrivals described by: $P(K, \tau)$

* Different time intervals (slots) disjoint are statistically independent. (No. of arrivals in each are independent)

$P(K, \tau) \to$ Distribution?

Small interval probabilities

$P(K, \delta) = \lambda\delta$
↓
Prob density (Intensity × δ)
↓ Small interval

(for very small δ):

$$P(K,\delta) = \begin{cases} 1-\lambda\delta, & \text{if } K=0 \\ \lambda\delta, & \text{if } K=1 \\ 0, & K>1 \end{cases}$$

↳ As the time interval is very small.

$\delta \to$ length of interval
$\lambda \to$ intensity of the arrival process

Interval × intensity = $\lambda\delta$
no arrival = 1 – some arrival

As $\delta \to 0$, $O(\delta^2) \to 0$

Exactly

$$P(K,\delta) = \begin{cases} 1-\lambda\delta, & K=0 \\ \lambda\delta, & K=1 \\ 0, & K>1 \end{cases} + O(\delta^2)$$

(Approx equality)
↓
when δ is very small we ignore second order terms.

λ: arrival rate.

$$\boxed{\lim_{\delta \to 0} \frac{P(1,\delta)}{\delta} = \lambda}$$

$\therefore K=1,\ P(K,\delta) = \lambda\delta$

Double λ, prob double (Twice intense).

$E[\#\text{ of arrivals in } [0,\delta]] = (1\cdot\lambda\delta) + 0\cdot(1-\lambda\delta) + 0$ (remaining)

$E[\#\text{ of arrivals in } [0,\delta]] = \lambda\delta$ → Arrival rate (expected no)

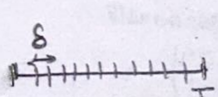
$$\boxed{\lambda = \frac{E[\#\text{ of arrivals } [0,\delta]]}{\delta}}$$

Expected no. of Arrivals per unit time.

Bernoulli: (no. of arrival during a gn interval of length δ) → Binomial Pmf

Continuous

"Big length = many intervals of small interval lengths"



* Different little intervals - indep of each other.

Bernoulli appr of Poisson process.

$\boxed{n = \frac{T}{\delta}}$ → no. of intervals.

$p = \lambda\delta$

$\boxed{p = \lambda \frac{T}{n}}$

Bernoulli process

$$P(K \text{ arrivals}) = \binom{n}{K}\left(\frac{\lambda T}{n}\right)^K \left(1 - \frac{\lambda T}{n}\right)^{n-K}$$

In continuous case:

$\delta \to 0$

$n \to \infty$

As $\delta \to 0$ (continuous)

$\therefore p = \frac{\lambda T}{n}$

$$P(K,\delta) = \begin{cases} 1-\lambda\delta + o(\delta^2) & K=0 \\ \lambda\delta + o(\delta^2) & K=1 \\ 0 + o(\delta^2) & K>1. \end{cases}$$

$\boxed{\lambda T = np}$ → no. of arrivals in Bernoulli process

App: * Deaths from horse kick in prussian army (1898)
* particle emission & radioactive decay

Bernoulli:

$$P_S(k) = \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} \qquad k = 0, 1, \dots n$$

$$\lambda = np, \; n \to \infty, \; p \to 0$$

for fixed $k = 0, 1, \dots$

$$\boxed{P_S(k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

Bernoulli in Poisson process

$$P(k \text{ arrivals}) = \binom{n}{k} \left(\frac{T\lambda}{n}\right)^k \left(1 - \frac{\lambda T}{n}\right)^{n-k} \dots$$

$\boxed{n = \frac{T}{\delta}}$, $p = \lambda\delta$

Bernoulli

* Finely discretize $[0, t]$: approximately Bernoulli
* N_t (of discrete appr): Binomial
* $\delta \to 0$ $(n \to \infty)$ gives

$$\lim_{n\to\infty} P(X=k) = \lim_{n\to\infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda T}{n}\right)^k \left(1 - \frac{\lambda T}{n}\right)^{n-k}$$

Pulling constants out

$$\left(\frac{(\lambda T)^k}{k!}\right) \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right) \left(1 - \frac{\lambda T}{n}\right)^n \left(1 - \frac{\lambda T}{n}\right)^{-k}$$

① $\lim_{n\to\infty} \frac{n!}{(n-k)!} \frac{1}{n^k}$

$$= \lim_{n\to\infty} \left(\frac{n(n-1)\dots(\cancel{n-k})(\cancel{n-k-1})}{(\cancel{n-k})(\cancel{n-k-1}) + \dots \cancel{1}}\right)\left(\frac{1}{n^k}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n(n-1)\dots(n-k+1)}{n^k}\right)$$

$$\lim_{n\to\infty} \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-K+1}{n}\right)$$

$$= \lim_{n\to\infty} (1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{K+1}{n}\right)$$

$n \ldots (n-K+1) \rightarrow K$ terms

$\downarrow$

$\therefore$ we cancelled $(n-K)!$ terms

$$= 1$$

② $$\lim_{n\to\infty} \left(1-\frac{\lambda T}{n}\right)^n$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 \quad (e = 2.718)$$

$$\lim_{n\to\infty}$$

$$x = -\frac{n}{\lambda T}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{x}\right)^{xT(-\lambda)}$$

$$= e^{-\lambda T}$$

③ $$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-K} = (1+0)^{-K} = 1$$

$$= \left(\frac{(T\lambda)^K}{K!}\right)(1)\cdot\left(e^{-\lambda T}\right)$$

$$p(K,T) = \left(\frac{(\lambda T)^K e^{-\lambda T}}{K!}\right) \quad K = 0, 1, \ldots$$

Idea: Bernoulli (time slots)

$\downarrow$

$\delta \to 0$

$\downarrow$

poisson process

$E[N_t] = \lambda t$, $\operatorname{var}(N_t) = \lambda t$

<u>In Bernoulli</u>:

$$E[N_t] = nP = \frac{T}{\delta}(\lambda\delta) = \lambda T$$

$$E[N_t] = \lambda t$$

$$\operatorname{var}(N_t) = nP(1-P) = nP(1) = \frac{T}{\delta}(\lambda\delta) = \lambda t$$

$\delta \to 0$

$p \to 0$

Double λ, double expectation

Email arrival : poisson process

whenever arrivals happen in a completely random way, without any addition of structures : poisson process

* Light source (weak): emitting photons
* Car accident.

---

email according to a poisson process, $\lambda = 5$ / hour. You check email every 30 minutes.

solu.

prob (new no messages) = ?

Prob(one new message) = ?

$\lambda t = E[N_t]$

$E[N_t] = 5 \times \frac{1}{2} = 2.5$

$$P(0, 1/2) = \frac{(t \cdot \lambda)^k \quad e^{-\lambda t}}{k!}$$

$$= \frac{1 \cdot e^{-5(1/2)}}{0!}$$

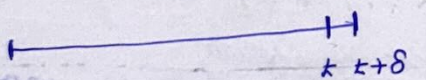
$$= e^{-2.5}$$

$$= 0.08$$

$$P(1, 1/2) = \frac{\left(\frac{1}{2} \times 5\right)^1 \quad e^{-5\left(\frac{1}{2}\right)}}{1!} = 2.5 \times e^{-2.5}$$

$$= 0.20.$$

Time until k th arrival: (Continuous R.V) → pdf



$$\boxed{f_{Y_K}(t) \cdot \delta = P(t \le Y_K \le t + \delta)}$$

For the K th arrival to happen in $t, t+\delta$

* $K-1$ → must happen b/w 0 and t
* when we take δ (very small) - Prob of having 2 arrivals in a mini slot is negligible.

∴ Disjoint slots are independ.

$$f_{Y_K}(t)\,\delta = P(t \le Y_K \le t+\delta).$$

$$= P\,(K-1 \text{ arrivals in } [0, t]) \cdot \lambda\delta$$

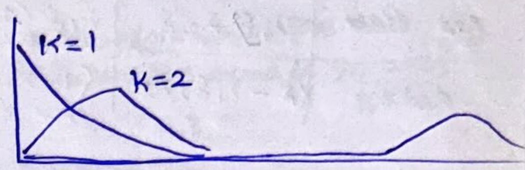
↳ 1 arrival at that (t, $t+\delta$ slot)

$$f_{Y_K}(t) \cdot \cancel{\delta} = \left(\frac{(\lambda t)^{K-1}}{(K-1)!} \cdot e^{-\lambda t}\right) \lambda \cdot \cancel{\delta}$$

$$\boxed{f_{Y_K}(t) = \frac{(\lambda t)^{K-1}}{(K-1)!} e^{-\lambda t} \cdot \lambda} \rightarrow \text{depends on number } K.$$

↓ Erlang formula. (K th arrival ment) (see pdf → graph)

K=100



K=100.

Time of first arrival:

$(K=1)$

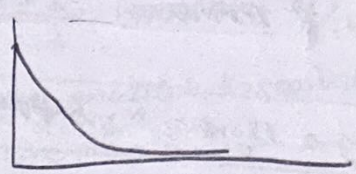
$f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$

↓ exponential distribution

memorylessness property:

(shares memorylessness prop of bernoulli process)

$\lambda \delta \rightarrow$ in every slot (no matter of past)

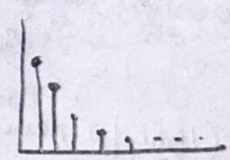


Time until 1st arrival (exponential distribution)

$Y_2 = T_1 + T_2$

Independent, exponential distribution.

In Bernoulli: Geometric distribution



Simulate: use tiny slots

↓ Random number generator

Comparison b/w poisson & Bernoulli (see pdf)

Adding & merging poisson processes

$\lambda = 1$, 0 — 2 — 5

$N[2,5] \rightarrow$ Poisson R.v (mean = 3)

$N[0,2] \rightarrow$ Poisson R.v (mean = 2)

Independent

$N[0,2) + N[2,5] = N[0,5] \rightarrow$ Poisson with mean 5.

Statement

* sum of independent poisson R.v is poisson
* merging of independent poisson processes is poisson.

Red bulb flashes (poisson) $\xrightarrow{\lambda_1}$ ○ → All flashes (poisson)

$\xrightarrow{\lambda_2}$ Green bulb flashes

	Green flash	Green doesn't
Red flash	$\lambda_1\delta \, \lambda_2\delta$	$\lambda_1\delta(1-\lambda_2\delta)$
Red off	$(1-\lambda_1\delta)\lambda_2\delta$	$(1-\lambda_1\delta)(1-\lambda_2\delta)$

$p = \lambda\delta$

$\delta \to 0$ (AS)

Keep: I orders

Throw: δ^2 terms

	Green flash	Green doesn't
Red flash	0	$\lambda_1\delta$
Red off	$\lambda_2\delta$	$1-\lambda_1\delta-\lambda_2\delta$

$\boxed{\text{Seeing a light} \approx \lambda_2\delta + \lambda_1\delta}$

* Interval of same length has same prob
* Slots are independent.

$P(\text{arrival comes from I process (Red flash)}) = \frac{\lambda_1\delta}{\lambda_1\delta+\lambda_2\delta} = \frac{\lambda_1}{\lambda_1+\lambda_2}$ (when $\lambda_1=\lambda_2$)

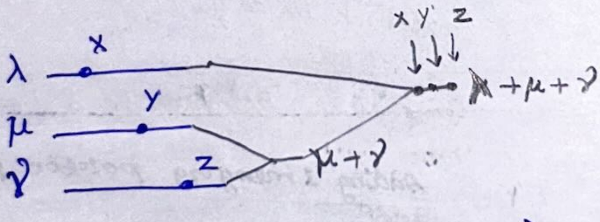
$P(\text{light} \mid \text{I process}) = \qquad = \frac{\lambda_1}{2\lambda_1} = \frac{1}{2}$

Competing exponentials

$X \sim \exp(\lambda)$

$Y \sim \exp(\mu)$

$Z \sim \exp(\gamma)$



$1 \to X$

$2 \to Y$

$P(X<Y<Z) = P(\{X<\min(Y,Z)\} \cap \{Y<Z\})$

$= P(\{X<\min(Y,Z)\}) \cdot P(\{Y<Z\})$

Poisson process: Independent of past

$= \left(\frac{\lambda}{\lambda+\mu+\gamma}\right) \cdot \frac{\mu}{\mu+\gamma} \cdot (1)$

First arrival from X

Second arrival from Y

$\left[\mu=\lambda=\gamma \text{ if } \quad \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}\right]$

* Arrivals of customers: completely random
* Independence of time slot
* Equally likely prob in each similar slots.

$$P(k, \delta) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!}, \quad k = 0, 1, 2, \dots$$

→ Duration alone matters not where the time is in axis

$$\boxed{E[N_T] = var(N_T) = \lambda \tau}$$

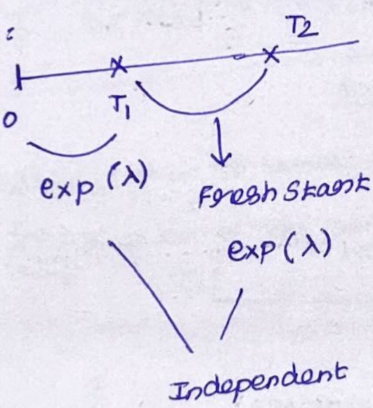
Interarrival times:

$$f_{T_1}(t) = \lambda e^{-\lambda t}, \quad t \geq 0, \quad E[T_1] = \frac{1}{\lambda} \text{ (exponential)}$$

* Find for a single interval → Extend this for multiple intervals.

Bernoulli process: no. of trial huge — P(success) tiny

Evolve:



→ Poisson has 'memorylessness as Bernoulli'

Light bulb

Not yet burnt out

Due to memorylessness (New light bulb independent of past - Future of this is same as new one) exponential distribution

'Used one is no worse no better than used one'

Time until second arrival $= \underbrace{T_1 + T_2}$ we can use convolution

$Y_2 = T_1 + T_2$ (Direct way too)

$$f_{Y_K}(y) = \frac{\lambda^K \, y^{K-1} \, e^{-\lambda y}}{(K-1)!} \qquad y \geq 0 \qquad \left[\lambda^{K-1} \cdot \lambda = \lambda^K \right]$$

↳ take previous.

$\lambda, K \rightarrow$ parameters

Poisson fishing (Anytime - little prob of caugh of fish)

* Assume : poisson $\lambda = 0.6$/hour

→ Fish for two hours

→ If no caught → Continue until 1st one.

Solu:

0 ———— 2

I caught a fish - stop & go home

* Fish 2 hours (collect as much as)
↳ left

* No fish (2 hours) → go after
Catching 1

|————|

Nothing - continue until 1

① p(more than 2 hours) = $P(0, 2) = e^{-\lambda t} = e^{-0.8 \times 2} = e^{-1.6}$

0 fish 2 hours

$\downarrow K=1$

$\int \lambda e^{-\lambda t} \cdot dt$

$\left(\lambda \frac{e^{-\lambda t}}{-\lambda} \right)_2^{\infty} = 0 + e^{-0.6 \times 2} = e^{-1.6}$

(or)

$= P(T_1 > 2) = \int_2^{\infty} f_{T_1}(t)\, dt$ → after 2 hours only he is able to catch a fish.

② p(more than 2, less than 5) = $P(0, 2) \cdot (1 - P(0, 3)) = \int_2^5 f_{T_1}(t)\, dt$

no in 2 hours; Atleast one in next 3 hours

$P(0,2) \cdot P(1,3) = e^{-1.6} \cdot \frac{(T\lambda)\, e^{-\lambda T}}{(\;)!}$

$= 0.06007$

$= e^{-1.6} \cdot (1 - e^{-1.8}) = 0.16852.$

P (atleast two fish) : Case I (within 2 hours)

$$= \sum_{K=2}^{\infty} P(K, 2) = 1 - P(0, 2) - P(1, 2)$$

0 fish, 1 fish.

③

(or)

2nd fish caught within 2 hours

$P(Y_2 \leq 2)$

E [number of fish] = Total exp theorem

= P(B/w 0 and 2) + P(after wards)

= $(2 \times 0.6) + 1 \cdot P(0, 2)$

$E[X] = (2 \text{ fishes} \times 0.6) + 1 \cdot P$

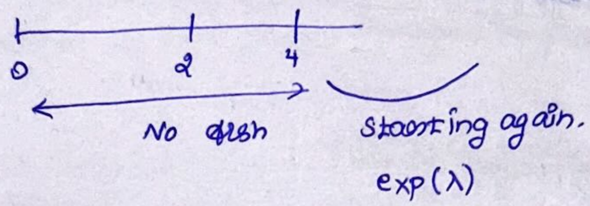
$\downarrow$ $E[2 \text{ hours}]$ $\downarrow$ 1 fish after 2 hours

$= (0.6 \times 2) + 1 \cdot P(0,2)$

$T(\text{fishing time}) = 2 + P(0,2) \cdot E[T_1]$

$= 2 + P(0,2) \cdot \frac{1}{\lambda}$

4.



$= \frac{1}{\lambda} = 0.6$ [Expected value]

Recap

i) $P(\text{more than 2 hours}) = 0$ fish in first 2 hours $= e^{-\lambda t} = e^{-1.6}$

ii) $P(\text{more than 2, less than 3}) = 0$ caught in 1st two, caught atleast 1 in next 1

$= P(0,2) \cdot (1 - P(0,3))$

iii) $P(\text{Atleast two fish}) =$ fish in 2 hours

$= \sum_{k=2}^{K} P(k,2) = 1 - P(0,2) - P(1,2)$

$\downarrow$ one $\downarrow$ Two

iv) $E[\text{no. of fish}] = E[2 \text{ hours}] + E[2 \text{ hours no fish, then 1 fish}]$

$\boxed{E[N_T] = \lambda t}$ $\downarrow$ No. of fishes.

$= \lambda t + 1 \cdot P(0,2)$ ↳ No fish in 1st 2 hours

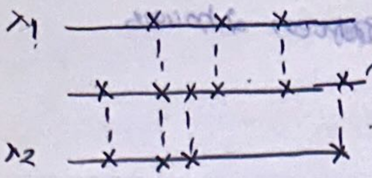
$= \lambda t + 1 \cdot P(0,2)$ ↳ No fish in 1st 2 hours

v) $E[\text{Total time}] = 2 + P(0,2) \cdot E[T_1]$

$= 2 + P(0,2) \cdot \frac{1}{\lambda}.$

more than 15 minutes)

Merging Independent Poisson Process



$\rightarrow (\lambda_1+\lambda_2)\,\delta \rightarrow$ Ignoring second order.

$\lambda_1 > \lambda_2$

$\hookrightarrow$ more likely from ①

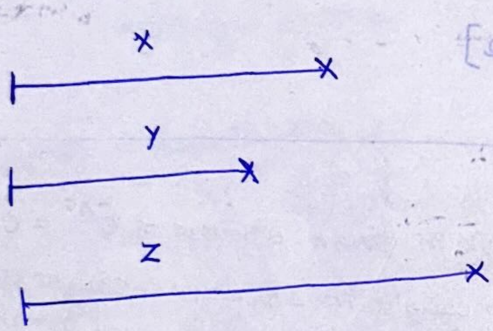
$$\frac{\lambda_1}{\lambda_1+\lambda_2} = P(\text{light} \mid ① \text{ process})$$

$$P(\text{light} \mid ② \text{ process}) = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

Install 3 bulbs - independent - die out

* exponential life time
* Install three
* Find $E[T]$ until last light bulb die out.

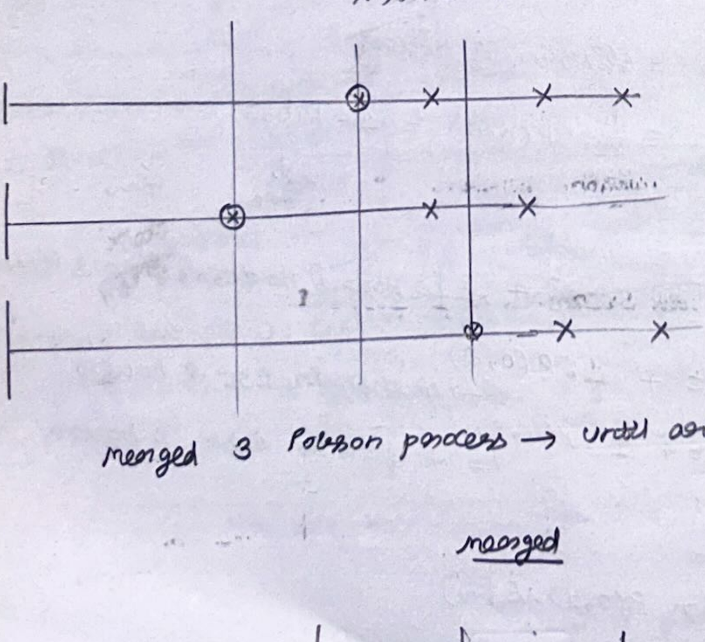
solu:



$$E\left[\max\{X, Y, Z\}\right]$$

Assume

X, Y, Z are a part of bigger poisson events



merging R.V
merging λ's
(3λ)

merged 3 Poisson process $\rightarrow$ until arrival in merged process.

merged

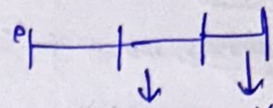
$\because$ Each has λ.

each has expected value : $\frac{1}{3\lambda}$
(Total)

1st: 1 bulb burst (New 2 bulbs → one bulb burst

↳ New 1 bulb →)

exp(λ) exp(λ) exp(λ)



$$E[\text{Time}] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}$$

↓

3λ (total) ; out of it 1

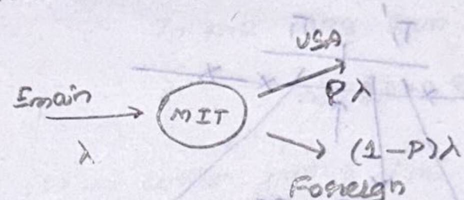
$$\boxed{E[T_1] = \frac{1}{\lambda}}$$

Exp [Time] = (1 bulb burnt) + (2nd bulb) + (3rd bulb)

After 1 bulb burnt → Starting fresh.

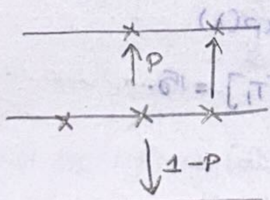
Splitting of poisson processes

* Email traffic through a server is a poisson process
* Destinations of different messages are independent.



* Each o/p is poisson

Solu:

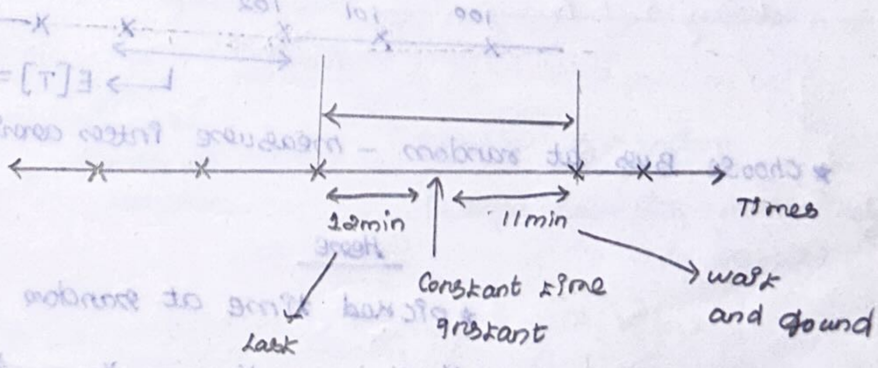


Random incidence for poisson

λ = 4 buses / hours

E [T] = 15 minutes

* poisson process that has been running forever
* Show up at some 'Random time' – really means 'arbitrary time'.



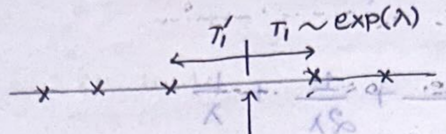
Previous to next: 23 minutes (Told: 10t more than 15 minutes)

Is that agent correct:

Interarrival time

Usual: we choose bus at random

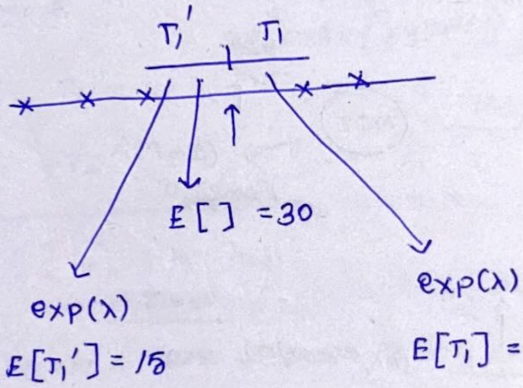
Here: we are choosing the time at random



* This interarrival interval has two intervals
* $T_1 \to$ starting from const time until a bus arrives (starting fresh - poisson process) $T_1 \sim exp(\lambda)$
* $T_1' \to$ Backwards in time (Backward sequence of cointoss (random) $\to$ still same)

Running Bernoulli backwards - still a Bernoulli.
Same with poisson.

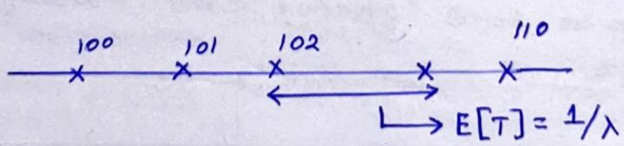
$T_1 \sim exp(\lambda)$



Doesn't contradict the claim?

what's actually happening

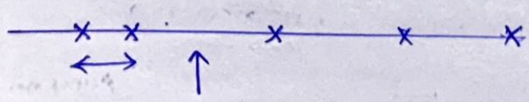
* How long is the interarrival time. (we are choosing time at random)



* Choose Bus at random - measure interarrival time

Here

* picked time at random

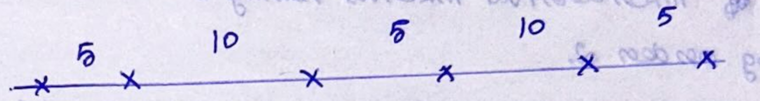


If I am picking this - more chance of picking this than the small interval

* All time are equally likely
* Large intervals - more likely.

Renewal process

* Series of successive arrive
  - Independent & identical distribution of interarrival times (Not necessarily exponential)
* Example: Bus interarrivals likely to be 5 or 10 minutes.
* Arrive: At a random time - what's the prob of selecting a time of 5 minute interval.
* E [value] of next arrival? $E[T] = 7.5$



In the long run we have as many as 5 minutes interval as 10 interval.

$\therefore$ Every 10 $\rightarrow$ covers twice the time as 5.

$\frac{2}{3}$ prob (size), $\frac{1}{3}$ prob by 5 (smaller intervals)

Prob (Selecting 10 interval) = 2/3

Prob (Selecting 5 interval) = 1/3

$$E[\text{value}] = \left(10 \times \frac{2}{3}\right) + \left(5 \times \frac{1}{3}\right) = 8.333 \neq 7.5 \text{ (given)}$$

$$= \frac{20}{3} + \frac{5}{3} = \frac{25}{3}$$

(Favours longer time)

Typical (random)

example

Family at random

person at random (Large families - more likely to occur) (Biased)

Ask a person

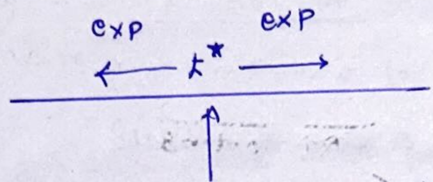
Bus — Biased (less chance of people from uncrowded buses)

Person in Bus (more likely from crowded bus)

ASKing: chocolate cookies - how many chips do you have!
↓
No use (we need to give importance to persons from uncrowded bus too)

Erlang arrival process: Erlang → Interarrival process in which the interarrival times are independent Erlang R.V of order 2, with mean $2/\lambda$. Assume that the arrival process has been ongoing for a very long time. An external observer arrives at a time. Find the pdf of the length of the interarrival interval that has t.

Solu:



* L = Length of interarrival interval having t^*
* ~ Erlang order 2.
* Erlang order K: Sum of K independent exponential r.vs.

if $T_i \sim \exp(\lambda)$

$Y_K = T_1 + \dots + T_K$ is Erlang order K

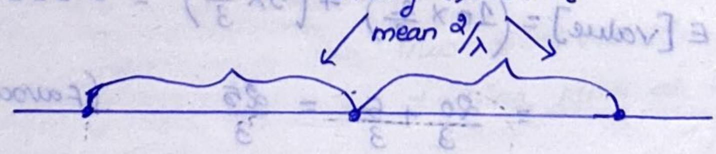
$$E[Y_K] = K\left(\frac{1}{\lambda}\right)$$

↳ Typical mean.

Erlang order of 2 with mean $2/\lambda$:

Erlang: up to K th event

$T_1 + T_2$, $T_i \sim \exp(\lambda)$



we are dealing with Erlang process:
→ No longer poisson process (No longer exponential distribution)

↓

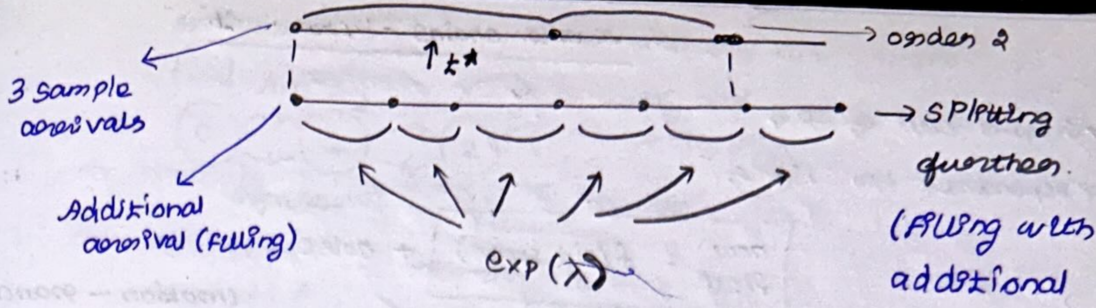
Erlang order 2

Distribution

$T_1 + T_2 \sim \exp(\lambda)$ [Sum of two exponential with parameter λ]

↓

Interpreted.



Poisson : Split in to fine intervals.

↓

Splitting the order 2 in to fine pieces.

↓

"Poisson process" [Exponential R.v]

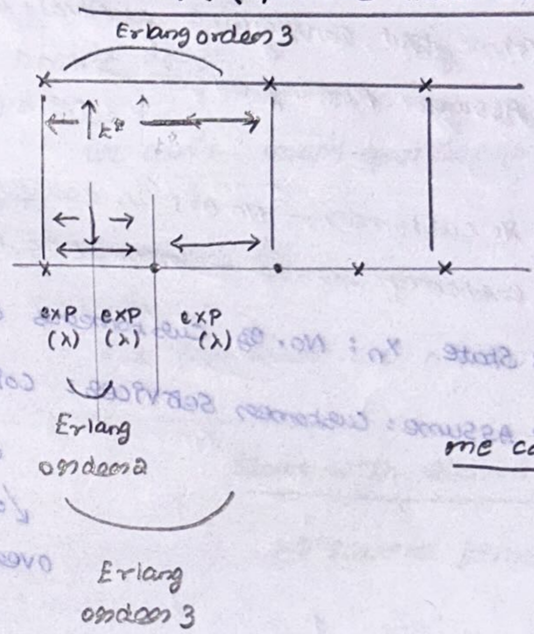
Odd & Even ↖ Two cases ↗ Even & odd.

we know :

Erlang process order: 2

For every 2 arrival in poisson process Erlang registers 1 arrival

t^* → ? (Specific interarrival interval)



Key : Choosing t at Erlang then relating with poisson

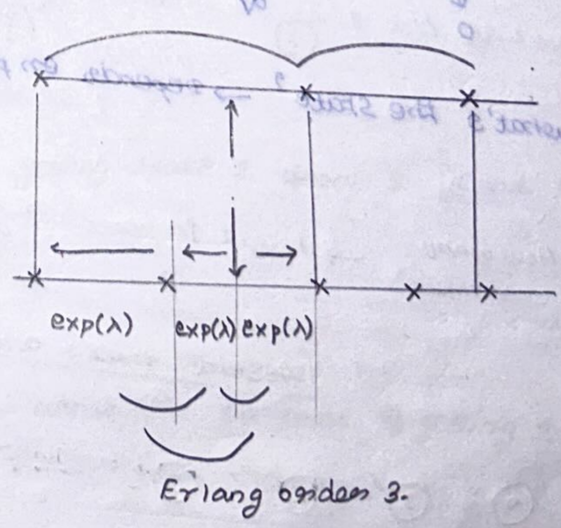
Even numbers : Actual arrivals

★ B/w odd & even
↓
t^* occured

one case

Assume : Some other case

B/w even & odd t^* occured



Erlang order 3.

Markov chains - Lecture 16

* General class of R.V
* Dependence b/w times.

$$\boxed{\text{new State} = f(\text{old State})} + \text{noise}$$

(motion - random)

markov

Example:

Checkout counter, (Stand & watch) → customers come, Stand in queue [- serve at one at time.

Assumptions

* Customers occur as Bernoulli process with parameters p.
* (Time b/w consecutive arrival: Geometric)
* Assume: Flipping a coin → Head (customer arrive) → Tail (No arrival)
* No customer - No one is departing
* Customer served - Geometric distribution (q) → Random
* State X_n: No. of customers at time n.
* Assume: Customer services: Coin flip → clerk (flips) → q: over; 1 - q: Continue service.

Time is 7:00pm - prob (departing):
- empty → 0
- Not empty → q

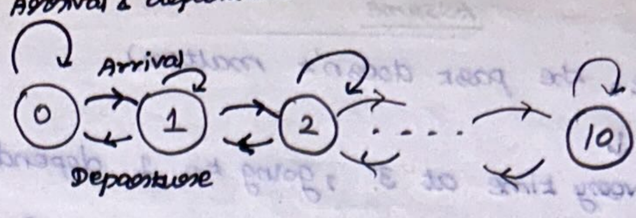
@ 7:10 pm - what's the state? → Depends on past (Large queue / Small queue)

* How many customers? → useful in predicting future

Assume max: queue of 10

0 1 2 3 10

* Customer arrival (move the state by 1 higher)
* Departure - decrease by 1
* Arrival & departure - constant



Arrival : P

Departure : α

No Arrival, departure = $(1-P)\alpha$

Arrival, No departure = $P(1-\alpha)$

Arrival, departure = $P\alpha$

No arrival, No departure = $(1-P)(1-\alpha)$

Boundary

At state 0

↓

we can't room for departure (No one)

At state 10

↓

Full (No room for others) → may have a departure

General definition

Finite state Markov chains

X_n = state after n transitions

(R.V)

↓

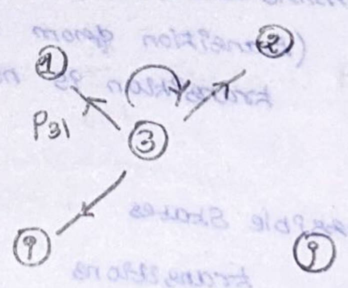
Starting X_0

↓ n transitions

X_n

Start with simple (Finite state)

*Discrete time.



* How likely am I getting state 1 from 3 [cond prob]

↓

By now 3 what's will prob of 1

P_{31} → I am in 3 what will be prob of getting 1

P_{39} → " " 9

Assume

(In current state, the past doesn't matter)

↓

Every time at 3, going to 1 depends on P_{31} alone

not how I got to 3. (past is not important)

$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

$$= P(X_{n+1} = j \mid \underbrace{X_n = i, X_{n-1}, \ldots, X_0}_{\text{Additional info.}})$$

(everything happened in past)

↓

"No bearing in future by the past, Ignore extra info,

Consider where you are now?" → choose the state with care (info relevant to future is necessary)

---

model specification

Ball in air → Future position? → Position at present

→ velocity

↓

if don't know

(use old points to find velocity).

(Don't ignore info relevant for future)

1) we will show possible transitions alone

In queue (transition from 2 to 5 in a single transition is not possible).

→ Identify the possible States

→ " transitions

→ " probabilities

we can't say exact position: provide probabilities

---

① calculate

n-step transition probabilities

State occupancy prob, given initial state i

$$r_{ij}(n) = P(X_n = j \mid X_0 = i)$$

↱ Started.

↳ After n steps j State