Small angle approximation.

$$\boxed{\sin\theta \approx \theta.}$$

For cosine:

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$= \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

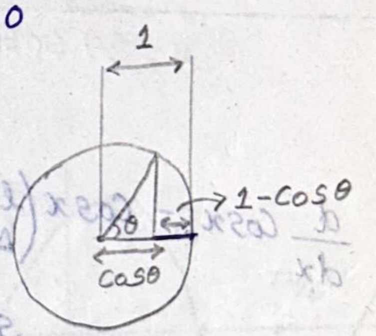
$$= \lim_{\Delta x\to 0} \frac{\cos x(\cos\Delta x - 1)}{\Delta x} - \frac{\sin x\sin\Delta x}{\Delta x}$$

$x = 0$

$$= \lim_{\Delta x\to 0} \frac{\cos\Delta x - 1}{\Delta x}$$

Geometric proof: $\Delta x \to \theta$, $\theta \to 0$

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$$



As θ is getting smaller & smaller $(\theta\to 0)$

Small angle approximation

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2$$

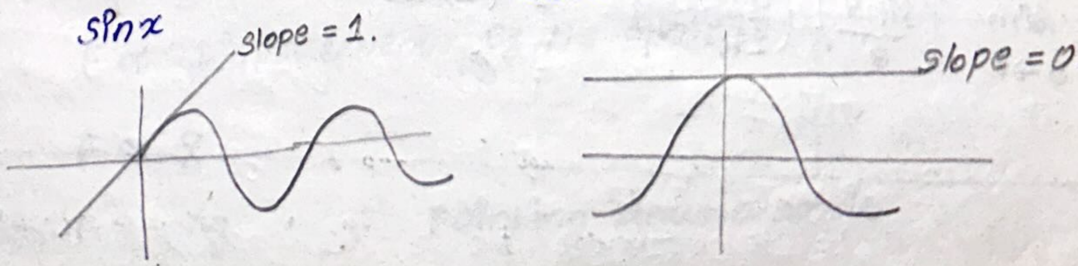
$$\lim_{\theta\to 0} \frac{1 - \frac{1}{2}\theta^2 - 1}{\theta}$$

$$\lim_{\theta\to 0} -\frac{1}{2}\theta = 0.$$

By geometry, As $\theta\to 0$

$1 - \cos\theta \approx 0$

∴ Due to -1 factor, numerator goes to zero fast than the denominator.



$$\frac{d}{dx}(\sin x) = \cos(x)\cdot \underbrace{\sin'(0)}_{1} + \sin(x)\cdot \underbrace{\cos'(0)}_{0}$$

$$= \cos x.$$

$$\frac{d}{dx}(\cos x) = -\sin x.$$

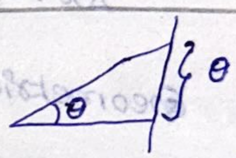
Because

$$\frac{d \sin x}{dx} = \lim_{\Delta x \to 0}\left(\sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \cos x\left(\frac{\sin \Delta x}{\Delta x}\right)\right)$$

$$= \lim_{\Delta x \to 0} \sin x\left(\frac{\cos \Delta x - 1}{\Delta x}\right) + \cos x\left(\lim_{\Delta x \to 0}\frac{\sin \Delta x}{\Delta x}\right)$$

$$= \sin x \cos'(0) + \cos x \sin'(0)$$

Analysing cosine



$$\frac{d}{dx}\cos x = \cos x\left(\lim_{\Delta x \to 0}\frac{\cos \Delta x - 1}{\Delta x}\right)$$

$$\sin x\left(\lim_{\Delta x \to 0}\frac{\sin \Delta x}{\Delta x}\right)$$

when $r = 1$ radian

$$\boxed{\sin\theta \approx \theta \\ \approx \theta}$$

$$= \cos x \cos'(0) - \sin x(\sin'(0))$$

$$= -\sin x.$$

By algebra:

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

$$= \frac{\sin\theta}{\theta}\;\frac{\sin\theta}{(1+\cos\theta)}$$

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{1+\cos\theta}\right) = (1)(0)$$

$$= 0.$$

Let $h(x) = \sin x + \sqrt{3}\cos x$. For which values of x does $h'(x) = 0$

Solu:

$h'(x) = \cos x - \sqrt{3}\sin x$

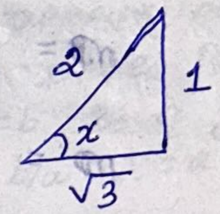
$0 = \cos x - \sqrt{3}\sin x$

$\cos x = \sqrt{3}\sin x.$

$1 = \frac{\sqrt{3}\sin x}{\cos x}.$

$\boxed{\frac{1}{\sqrt{3}} = \tan x}$

$\boxed{x = \pi/6}$, $\frac{7\pi}{6}$, $\frac{-5\pi}{6}$, ...

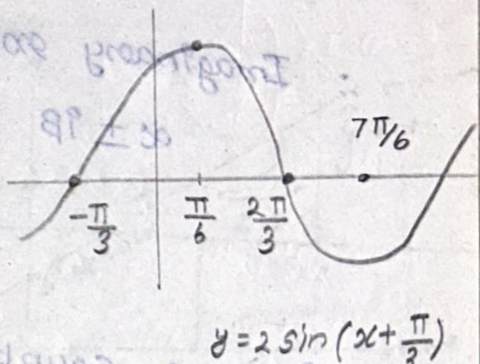


$h(x) = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$h(x) = 2\left(\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right)$

$h(x) = 2\left(\sin\left(x + \frac{\pi}{3}\right)\right)$

$h(x) = 2\left(\sin\left(x + \frac{\pi}{3}\right)\right)$



$y = 2\sin\left(x + \frac{\pi}{3}\right)$

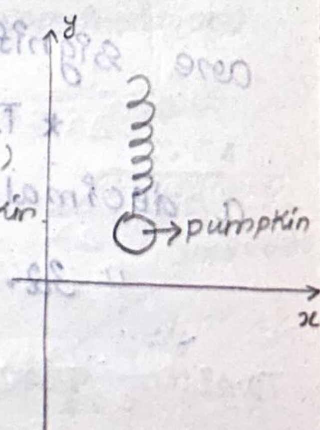
Shifting $2\sin x$ by $\pi/3$ towards left.

---

modelling oscilation

$F = ma$

$F = m(y''(t))$ (motion is in vertical direction)



$F \to$ Restoring force acting on the pumpkin

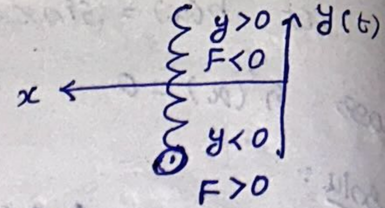
$F \propto$ Displacement of the spring.

$F \propto y$

$F = -ky$: pointing downwards.

$$m\, y''(t) = -K y(t)$$

(Differential equation).



$$y''(t) = -\frac{K}{m}\, y(t)$$

$$D^2 y + \frac{K}{m} y = 0$$

$$\boxed{\left(D^2 + \frac{K}{m}\right) y = 0}$$

$$m = \sqrt{-\frac{K}{m}}$$

$$m = \pm i\sqrt{\frac{K}{m}} \qquad \therefore \alpha = 0,\ \beta = \sqrt{\frac{K}{m}}.$$

$$y = A\cos\left(\sqrt{\frac{K}{m}}\, t\right) + B\sin\left(\sqrt{\frac{K}{m}}\, t\right)$$

$\therefore$ Imaginary roots $\Rightarrow$ solution $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\alpha \pm i\beta$

Significant figures

It is the count of those digits that carry meaning with regards to precision.

* All non-zero digits are significant
* Zeros appearing b/w non zero digits are significant
* Trailing zeros in a number containing a decimal are significant.

32.000 has 5 significant figures.

Trailing zeros in a number with no decimal are not significant - 5400 $\rightarrow$ 2 significant figures.

Leading zeros in a decimal number are

not significant - 0.0003 has 1 significant figure

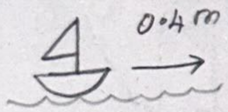
* Extraneous digits introduced in a computation to greater precision than measured data are not significant.

- If .25 and .50 are measurements accurate to ±.01, then in the product (.25) (.50) = (0.125) → the last 5 is not significant

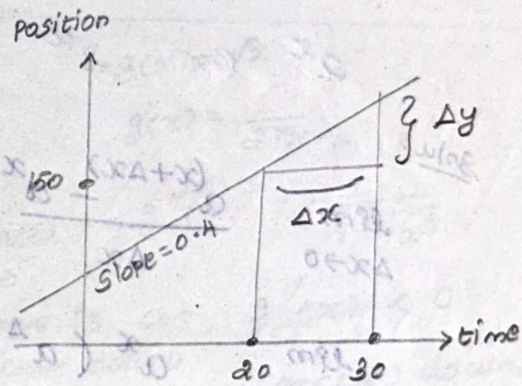
Linear approximation.

The tangent line at $x=a$ is a good approximation for the function near $x=a$. The slope of the secant line b/w $x=a$ & $x=b$ is a good approximation for the derivative b/w $x=a$ & $x=b$.

Approximation & tangent lines



$\boxed{t = 30 = ?}$



$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

$$0.4 = \frac{\Delta y}{10}$$

$$\boxed{\Delta y = 4m}$$

At $t = 30$, $d = 154m$
$(y + \Delta y)$.

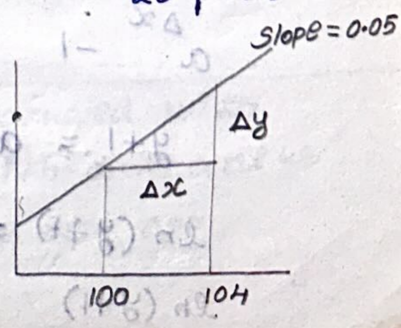
$g(x) = \sqrt{x}$, at $x = 100$. (Linear approximation)

$g(100) = 10$.

$$g'(100) = \left.\frac{1}{2\sqrt{x}}\right|_{x=100}$$

$$= \frac{1}{20} = 0.05$$

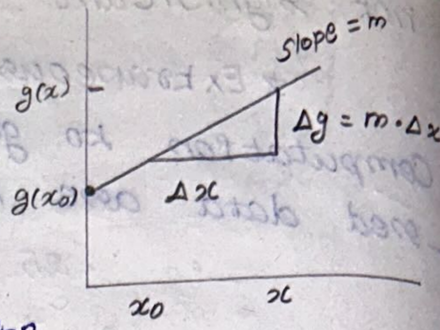
$$20 \overline{)100}\ 0.05$$



$0.05 = \frac{\Delta y}{4}$, $\Delta y = 0.2$

$\therefore \boxed{10.2 = y}$

$\Delta g \approx m \cdot \Delta x$



$$\begin{cases} g(x) - g(x_0) \approx g'(x_0)(x - x_0) \\ g(x) \approx g'(x_0)(x - x_0) + g(x_0) \end{cases}$$

True when x is near x_0.

$$\boxed{y = mx + c}$$

Tangent line.

$$m = \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} \qquad m \approx \frac{\Delta g}{\Delta x} \text{ when } \Delta x \text{ is small}$$

$$\boxed{m \Delta x \approx \Delta g}$$

linear approximation

when slope of secant, Δx is very small, it will app equal to slope of the tangent.

---

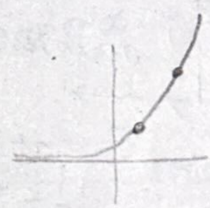
simply,

$$\Delta f = \left.\frac{df}{dx}\right|_{x=a} \cdot \Delta x \quad \text{for } \Delta x \text{ near } 0.$$

$$f(x) \approx f'(a)(x-a) + f(a) \quad \text{for } x \text{ near } a.$$

---

a^x (or) 2^x



Solu:

$$\lim_{\Delta x \to 0} \frac{a^{(x+\Delta x)} - a^x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{a^x (a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x \lim_{\Delta x \to 0} \frac{(a^{\Delta x} - 1)}{\Delta x}$$

$$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$a^{\Delta x} - 1 = y$$

$$y + 1 = a^{\Delta x}$$

$$\ln(y+1) = \ln a^{\Delta x}$$

$$\ln(y+1) = \Delta x \ln a$$

$$\frac{d}{dx} a^x = a^x \ln a$$

$$y = a^x$$

$$\ln y = \ln a^x$$

$$e^{\ln y} = e^{\ln a^x}$$

Differenciating.

$$\frac{d}{dx}\left[e^{x \ln a}\right] =$$

$$\ln a \, (e^{x \ln a})$$

$$= \boxed{\ln a \cdot a^x}$$

$$\Delta x = \frac{\ln(y+1)}{\ln a}$$

Substituting,

$$= a^x \lim_{y \to 0} \frac{a^{\frac{\ln(y+1)}{\ln a}} - 1}{\frac{\ln(y+1)}{\ln(a)}}$$

$y = 5^x$

$\ln y = x \ln 5$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 5$

$y' = y \ln 5.$

$\boxed{y' = 5^x \ln 5}$

$g(x) = \sqrt{x}$

$g'(x) = \frac{1}{2\sqrt{x}}$

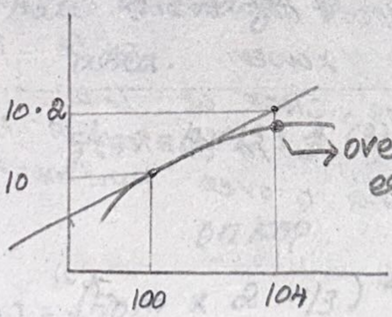
$g''(x) = \frac{1}{2} \cdot x^{-\frac{1}{2} - 1} \cdot -\frac{1}{2}$

$g''(x) = -\frac{1}{4} \cdot x^{-3/2}$

$g''(100) = -\frac{1}{4} (100)^{-3/2}$

concave down $= -\frac{1}{4} \cdot \frac{1}{1000} = -\frac{1}{4000}$

Concavity + Linear approximation.



→ over estimated by us. (curve is concave down)

$g(x) = \sqrt{x}$

$g'(x) = \frac{1}{2\sqrt{x}}$

$g''(x) = -\frac{1}{4\sqrt[2]{x^3}}$

$g''(x) < 0$

(concave down)

$\sqrt{104} = 10.198$

linear approximation is a good approximation only near the point of tangency.

So curve won't be so curvy.

(Hence, Larger the mass → easier to predict.) → More accurate for larger period of time.

$F = ma$

when m is large hard to accelerat

4151.204077

A ball is supposed to be manufactured with radius 10cm. If it gets made with a radius of 9.97cm instead. It would take up app how much less volume.

$\frac{dv}{dr} = 4\pi r^2 \; cm^3/cm$

when $r = 10$ cm $\qquad \frac{dv}{dr} = 400\pi \; cm^2$

$\boxed{\Delta r = -0.03 \text{ cm}}$

$\Delta V = 400\pi \times -0.03$

$\boxed{\Delta V = -12\pi \text{ cubic cm.}}$

$\therefore \frac{\Delta V}{V} = \frac{\frac{1}{4000\pi}}{12\pi} = \frac{12}{4000}$

$\frac{9.97}{10}$ = The radius is decreased by 0.3%

$V = \frac{4}{3}\pi r^3 \qquad = 4000\pi - 12\pi$

$\qquad = 3988\pi \qquad$ New $V = 4151.091$

$V = 4188.790$

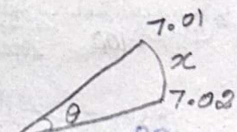
$dv = -12\pi$

$\frac{4151.091}{4188.790} = 0.991$

Difference is 0.9%
(Decrease)

The minute hand extends 14 feet out from the centre of the clock face on Elizabeth tower. It is pointed at the first minute mark of the clock at 7:01. Use linear app to estimate how much its tip of the minute hand moves horizontally b/w 7:01 and 7:02.

Solu:

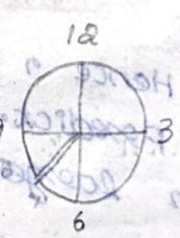


$x = 14\cos\theta$.

(since hand moves horizontally)

one minute around the clock $= \frac{\pi}{30} \left(\frac{2\pi}{1 \text{hour}} * \frac{1 \text{hour}}{60 \text{min}} \right.$

From 7:01 to 7:02

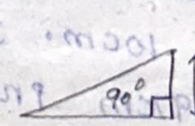


θ changes from

$\frac{\pi}{30}$ to $\frac{\pi}{15}$

$\Delta\theta = \frac{\pi}{15} - \frac{\pi}{30} \qquad \frac{dx}{d\theta} = -14\sin\theta$

$\boxed{\Delta\theta = \pi/30}$ $\qquad$ (linear approxi)



$$\frac{dx}{d\theta} = -14\sin\theta$$

$$= -14 \sin\left(\frac{\pi}{2}\right)$$

$$= -14 \text{ feet / radian}$$

So our estimation of Δx is -14.

$-\frac{\pi}{30}$ (or) $\frac{7\pi}{15}$.

---

why

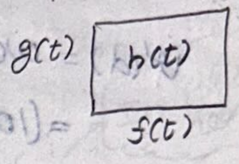
$$h(x) = f(x) \cdot g(x)$$

$$h'(x) \neq f'(x) \cdot g'(x)$$

$t \rightarrow$ time (s)

$f(t), g(t) \rightarrow$ distance (m)

$f'(t), g'(t) \rightarrow$ velocity (m/s)



$h'(t) = m^2/s$ $\quad (h(t) = m^2)$

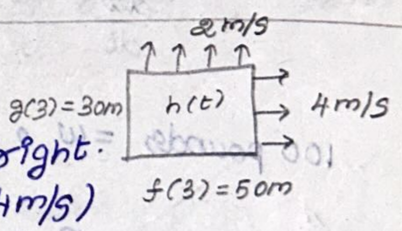
In case:

$$h'(t) \neq f'(t) \cdot g'(t)$$

$$\boxed{\frac{m^2}{s} \neq \frac{m^2}{s^2}}$$

$h'(t) \rightarrow$ Rate of change of area.

$g'(3) = 4$ m/s, $f'(3) = 2$ m/s



on top $\quad$ on right.

$$h'(t) = (50m \times 2\ m/s) + (30m \times 4m/s)$$

$$\boxed{h'(t) = 220\ m^2/s}$$

$\therefore f(3)\, g'(3) + g(3)\, f'(3)$.

[37×52]

$h'(3) \rightarrow$ Rate of change of area.

Old McDonald has a farm. This summer he's been growing a watermelon (very large size). Right now it weighs 100 pounds & it is continuing to grow at a rate of 3 pounds per day. Market price of it right now is 0.40 dollars per pound and is decreasing at a rate of 0.01 dollars per pound per day.

Solu:

Right now : price $= 100 \times 0.40$

$= 400$ dollars.

Let $w(d) \to$ weight of the watermelon on day d.
$p(d) \to$ market price per pound on day d.

If $v(d)$ is the value of old Mc Donalds old watermelon on day d, express $v(d)$ in terms of $w(d)$ and $p(d)$

solu:

$$v(d) = w(d) \times p(d)$$

In dollars per day, at what rate is the market value of old mc Donalds watermelon currently changing.

solu:

$$v'(d) = w(d)\, p'(d) + p(d)\, w'(d)$$

$$= \cancel{(100 \times 0.01) + (0.40 \times 3)}$$

$$= \cancel{1 + 1.2}$$

$$= 2.2$$

Today $= 100 \times 0.40$
$= 40$ \$

Tomorrow $= 103 \times (0.39)$
$= 40.17$

~~0.01 – per pound per day (price change)~~

$$\cancel{\frac{dP}{\frac{dw}{dt}} = 0.01} \quad \frac{dw}{dt}$$

~~100 pounds $= w$, $\frac{dw}{dt} = 3$ pounds.~~

~~40 dollars $= P$, $\frac{dP}{dt} = 0.4$ dollars.~~

~~rate of change of~~

~~$v'(d) =$ value $=$~~ $\boxed{w(d)\, p'(d) + p(d)\, w'(d)}$

~~$= (100 \times 0.4) + (40 \times 3)$~~

$$= \cancel{(100 \times 0.01) + (0.4 \times 3)}$$

$\therefore$ $w = 100$ pounds, $\frac{dw}{dt} = 3$ pounds (↑ increasing)

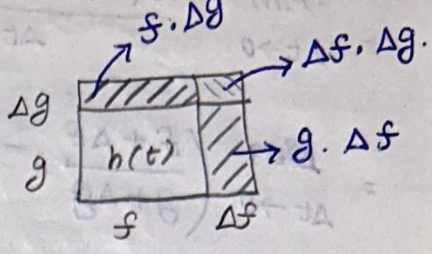
$p = 0.40$ dollars, $\frac{dp}{dt} = -0.01$ dollars (Decreasing)

$$v' = 100(-0.01) + (0.40 \times 3)$$

$$= -1 + 1.2 = 0.2$$

The product rule.

$h(t) = f(t) \cdot g(t)$



$h'(t) \approx \dfrac{\Delta h}{\Delta t}$ (Near tangency)

$$\Delta h = \underbrace{(f+\Delta f)(g+\Delta g)}_{\text{New}} - \underbrace{f \cdot g}_{\text{old}}$$

$$= fg + f\Delta g + g\Delta f + \Delta f \Delta g - fg$$

$$= f\Delta g + g\Delta f + \Delta f\Delta g$$

$$\lim_{\Delta t\to 0}\frac{\Delta h}{\Delta t} = \lim_{\Delta t\to 0}\left(f\frac{\Delta g}{\Delta t} + g\frac{\Delta f}{\Delta t} + \frac{\Delta f}{\Delta t}\cdot\frac{\Delta g}{\Delta t}\cdot\Delta t\right)$$

$$= \lim_{\Delta t\to 0}\left(fg' + gf' + f'\cdot g'\cdot\Delta t\right)$$

In previous derivation

$$= f'g + gf'$$

→ So no problem even when we missed $\Delta f\Delta g$ part

Product rule with 3 functions.

$$\frac{d}{dt}(uv) = u'v + v'u$$

$$\frac{d}{dt}(uvw) = u'vw + (vw)'u$$ (using uv product rule)

$$= u'vw + v'wu + w'uv$$

$$= u'vw + uv'w + uvw'$$

$$f(x) = x^2 \sin x \cos x$$

$$f'(x) = 2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x.$$

$$= x(2\sin x\cos x + x\cos^2 x - x\sin^2 x)$$

$$= x(\sin 2x + x\cos^2 x - x\sin^2 x)$$

$$= x(\sin 2x + x(\cos^2 x - \sin^2 x))$$

Quotient rule.

$$h(t) = \frac{f(t)}{g(t)} \qquad h'(t) \neq \frac{f'(t)}{g'(t)}$$

$$h'(t) \approx \frac{\Delta h}{\Delta t}$$ (when Δt is small)

↳ tangency.

$$= \lim_{\Delta t \to 0} \frac{h(x+\Delta t) - h}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left(\frac{f+\Delta f}{g+\Delta g} - \frac{f}{g} \right) / \Delta t$$

$$= \lim_{\Delta t \to 0} \left(\frac{g(f+\Delta f)}{g(g+\Delta g)} - \frac{f(g+\Delta g)}{g(g+\Delta g)} \right) / \Delta t$$

$$= \lim_{\Delta t \to 0} \left(\frac{gf + g\Delta f - fg - f\Delta g}{g(g+\Delta g)} \right) / \Delta t$$

$$= \lim_{\Delta t \to 0} \left(\frac{gf' - fg'}{g^2 + g\Delta g} \right) \quad \text{As } \Delta t \to 0, \ \Delta g \to 0.$$

$$h'(t) = \frac{gf' - fg'}{g^2}$$

The quotient rule.

$$h'(t) = \frac{f'(t)\,g(t) - f(t)\,g'(t)}{(g(t))^2}$$ whenever $f'(t)$ & $g'(t)$ exist And $g(t) \neq 0$.

$$\frac{d}{dx} \tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

---

power rule

$$h(x) = x^{-n} = \frac{1}{x^n}$$

$$f(x) = 1, \ g(x) = x^n.$$

$$h'(x) = \frac{x^n(0) - 1(nx^{n-1})}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}}$$

$$= -n x^{n-1-2n}$$

$\frac{d}{dx} x^{-n} = -n x^{-n-1}$, $n \to$ positive integer.

$$\boxed{\frac{d}{dx} x^n = n x^{n-1}} \quad , n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Chain rule.

time $\to$ distance $\to$ temperature.

x - metres, u - seconds, t - minutes

$x = f(u) = 2u + 200$

$x' = f'(u) = 2$ m/s

$x = g(t) = 120t + 100$

$x' = g'(t) = 120$ m/min

$\therefore g'(t) = \frac{dx}{dt} \neq \frac{dx}{du} (f'(u))$ $\times \frac{60s}{min}$

$120 \frac{m}{min} = 2$ m/s

$120 \frac{m}{min} = 120 \frac{m}{min}$.

$h(x) = f(g(x))$, $g(2) = 9m$, $f(9) = 5kg$,

$h(2) = 5$ Kg

↳ seconds.

$g'(2) = 3$ m/s ,

$\cancel{g(x) = 3x}$

solu:

when $g(2) = 9m$

$g'(2) = 3$ m/s

$g(2.01) \approx 9.03$ metres.

$\therefore$ rate $= 3$ m/s

$= 0.01 \times 3$

$= 0.03$

ii) $f'(9) = 4$ Kg

$f(9.03) \approx 5 + (0.03 \times 4)$

≈ 5.12.

iii) $f(g(2.01)) \approx h(2.01)$

$\therefore h(2) = 5$

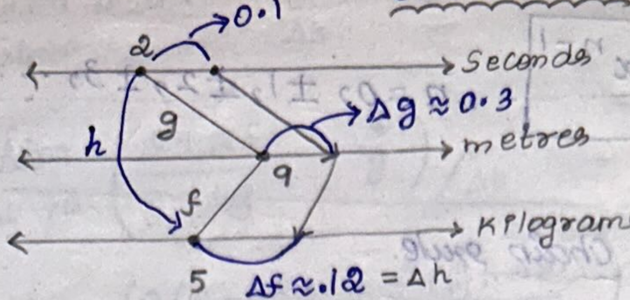
$h(2.01) \approx 5.12$

$h'(2) = 12$. Kg/s

$h'(2) = \frac{0.12}{0.01}$

$h'(2) = 12$

Chain rule.



$h(x) = f(g(x))$ | $g(2) = 9$ | $f(9) = 5$

$h'(2) = 12$ | $g'(2) = 3\,m/s$ | $f'(9) = 4\,kg/m$

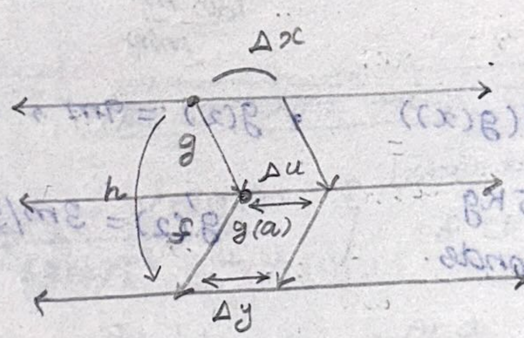
$$h'(2) \approx \frac{\Delta h}{\Delta x} \approx \frac{0.12\,kg}{0.01\,s} = 12\,kg/s$$

$$\approx f'(9) \times g'(2).$$

↓

our input is $f'(9)$.

when it is a composition of three functions.



$$h(x) = f(g(x)) = f(u)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=g(a)} \cdot \left.\frac{du}{dx}\right|_{x=a}$$

$h'(a) = f'(g(a)) \cdot g'(a)$ → Combination derivative.

$P(x) = \cos^3 x$ $\quad$ $P(x)$ as a composition of $f(g(x))$, $g(x) = ?$

$u = g(x)$ $\quad\therefore\; p(x) = (\cos x)^3$

$f(x) = x^3,\; g(x) = \cos x.$

$h(x) = f(g(x))$

$h'(x) = f'(g(x))\, g'(x).$

$p(x) = \cos^3 x$

$p'(x) = 3\cos^2 x \cdot (-\sin x)$

$= -3\sin x \cdot \cos^2 x$

$v(x) = \cos(x^3)$ $\quad u = x^3.$

$v'(x) = -\sin x^3 \cdot 3x^2$ $\quad \boxed{v'(x) = \cos' u \cdot u'}$

$v'(x) = -3x^2 \cdot \sin x^3$

$f(x) = \sqrt{x^3+2x+1}$ $\quad u = x^3+2x+1$

$f'(x) = \frac{1}{2\sqrt{u}} \cdot u'$

$f'(x) = \frac{1}{2\sqrt{x^3+2x+1}} \cdot (3x^2+2)$

$f'(1) = \frac{1}{2\sqrt{4}} (5) = \frac{5}{4}$

$f(x) = \sqrt{3}\tan x^2$

$f'(x) = \sqrt{3}\sec x^2 \cdot 2x$

$= 2\sqrt{3}\, x \cdot \sec x^2$

$f(x) = \sqrt{x}\tan x^2$

$f'(x) = \frac{1}{2} x^{-1/2} \tan x^2 + \sqrt{x}\sec^2 x^2 \cdot 2x$

$= \frac{\sqrt[2]{x^3}}{2} \tan x^2 + 2\sqrt[2]{x^3} \sec^2 x^2$

$f'(x) = \frac{1}{2\sqrt{x}} \tan x^2 + 2\sqrt{x^3} \sec^2 x^2$

$g(x) = \sqrt{x\tan x}$

$g'(x) = \frac{1}{2\sqrt{x\tan x}} (\tan x + x\sec^2 x)$

$= \frac{\tan x + x\sec^2 x}{2\sqrt{x\tan x}}$

Chain (vs) quotient rule.

$h(x) = \frac{1}{x^2+x}$, $\quad u = x^2+x$

$= u^{-1}$

$h'(x) = -u^{-2} \cdot u'$

$= -\frac{1}{(x^2+x)^2} (2x+1)$

$\boxed{= -(2x+1)/(x^2+x)^2}$

$h(x) = \frac{f(x)}{g(x)}$ $\quad r(x) = \frac{1}{g(x)}$

$h'(x) = f(x)\, r'(x) + r(x)\, f'(x)$

Let $r(x) = \frac{1}{x}$

$r(g(x)) = \frac{1}{g(x)}$

$h'(x) = f(x) \cdot (r(g(x)))' + r(g(x))\, f'(x)$

$= f(x) \cdot \frac{-1}{g(x)^2}\, g'(x) + \frac{1}{g(x)} \cdot f'(x)$

$= \frac{-f(x)\, g'(x) + f'(x)\, g(x)}{(g(x))^2}$

Find two values for θ, so that $\frac{d}{d\theta}(\cos^2(\theta^4)) = 0$

Solu:

$\frac{d}{d\theta} y = 2\cos^2(\theta^4) \cdot (-\sin\theta^4) \cdot 4\theta^3$

$0 = -8\theta^3 \cos^2(\theta^4) \sin(\theta^4)$

$\theta = 0, 90^0$

$\therefore \sin 0 = 0, \cos\theta^4 = 0$

$y = x^2$ | $\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{dw} \cdot \frac{dw}{d\theta}$

$x = \cos w$

$w = \theta^4$

$\cos\theta^4 = 0$

$\theta^4 = \cos^{-1} 0$

$\theta = \sqrt[4]{\cos^{-1} 0}$

$\theta = \left(\frac{\pi}{2}\right)^{1/4}, \ldots$ $\boxed{\theta^4 = \pi/2}$

(By adding π)

Implicit differenciation

↳ மறைமுகமான

Implicit function is a function that is defined implicitly by implicit equation.

eg: $x^2 + y^2 = 1$.

$(x^2 + y^2 = 25)$

↳ you will have two expressions. one for +ve exp, -ve exp.

$y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$

Top half:

$y = \sqrt{25 - x^2}$

Bottom half:

$y = -\sqrt{25 - x^2}$

Implicit differenciation

$x^2 + y^2 = 25.$

$\frac{d}{dx}(x^2 + (y(x))^2) = \frac{d}{dx}(25)$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$

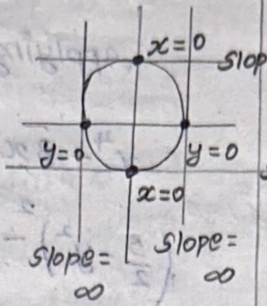
$x^2 + y^2 = 25$

↳ It has two expressions

(slope is for any point on the circle → edge.

If $y=0$, $\frac{dy}{dx}$ = undefined.

If $x=0$,
slope = 0



At fourth quadrant (+, −)

slope = +ve

At third quadrant (−, −)

slope = −ve

At I quadrant (+, +)

slope = −ve

At II quadrant (−, +)

slope = +ve

Power rule for rational numbers.

r = rational

$r = \frac{m}{n}$ (m, n → Integers)

$n \neq 0$.

$y = x^{(m/n)}$

(multiply by n (raise) power)

$y^n = x^m$

$n y^{n-1} \frac{dy}{dx} = m x^{m-1}$

$\frac{dy}{dx} = \frac{m}{n} \frac{x^{m-1}}{y^{n-1}}$

$= \frac{m}{n} \frac{x^{m-1}}{(x^{m/n})^{n-1}}$

$= \frac{m}{n} \frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}}$

$= \frac{m}{n} x^{m-1-\frac{m}{n}(n-1)}$

$= \frac{m}{n} x^{m-1-m+\frac{m}{n}}$

$= \frac{m}{n} x^{\frac{m}{n}-1}$

$n \neq 0$

$y^3 = x^2 \rightarrow$ Implicit

$y = x^{2/3} \rightarrow$ Explicit

$y^2 = x \rightarrow$ Implicit.

$\therefore y = \pm\sqrt{x}$

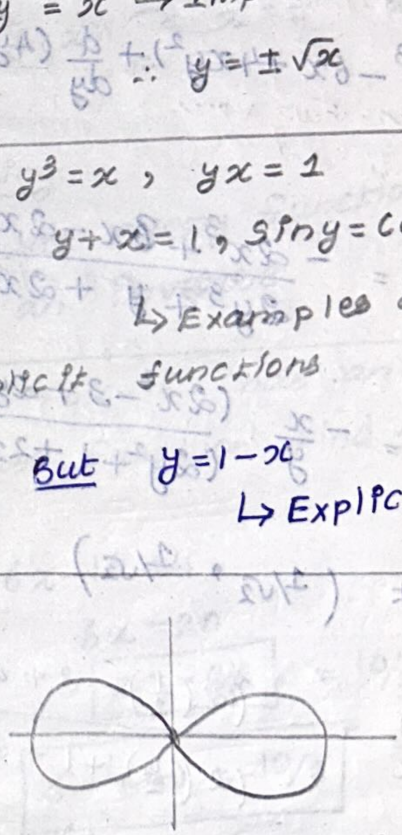
$y^3 = x$, $yx = 1$

$y + x = 1$, $\sin y = \cos y$

↳ Examples of Implicit functions

But $y = 1 - x$

↳ Explicit.



$x^4 - 3x^2 + y^4 + y^2 + 2x^2y^2 = 0$.

1) what's the slope of the tangent line at $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$?

Soln.

At what points does this function have a horizontal tangent line?

Solu:

$$\frac{d}{dx}\left(x^4 - 3x^2 + y^4 + y^2 + 2x^2y^2\right) = 0$$

$$4x^3 - 6x + 4y^3 \cdot \frac{dy}{dx} + 2y\frac{dy}{dx} + 2 \times 2x \times y^2 + 4x^2y\frac{dy}{dx} = 0$$

$$4x^3 - 6x + 4xy^2 + \frac{dy}{dx}\left(4y^3 + 2y + 4x^2y\right) = 0$$

$$(4x^3 - 6x + 4xy^2) + \frac{d}{dy}\left(4y^3 + 2y + 4x^2y\right) = 0$$

$$\frac{dy}{dx} = \frac{-2x^3 + 3x - 2xy^2}{2y^3 + y + 2x^2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}\frac{(2x^2 - 3 + 2y^2)}{(2y^2 + 1 + 2x^2)}$$

At $(1/\sqrt{2}, 1/\sqrt{2})$

$$\frac{dy}{dx} = -\frac{\left(2\left(\frac{1}{2}\right) - 3 + 2\left(\frac{1}{2}\right)\right)}{\left(2\left(\frac{1}{2}\right) + 1 + 2\left(\frac{1}{2}\right)\right)}$$

$$= -\frac{(1 - 3 + 1)}{(1 + 1 + 1)} = \frac{1}{3}$$

i) $\frac{dy}{dx} = 0$ (when $x = 0$)

(or)

$$2x^2 - 3 + 2y^2 = 0.$$

$$x^2 + y^2 = \frac{3}{2}.$$

$$x^2 = \frac{3}{2} - y^2$$

Applying x^2 in

$$x^4 - 3x^2 + y^4 + y^2 + 2x^2y^2 = 0$$

$$\left(\frac{3}{2} - y^2\right)^2 - 3\left(\frac{3}{2} - y^2\right) + y^4 + y^2 + 2\left(\frac{3}{2} - y^2\right)y^2 = 0$$

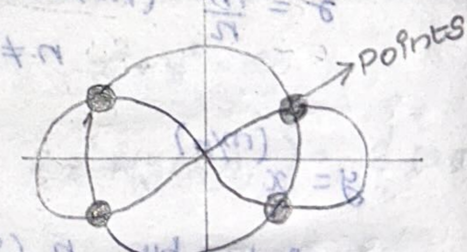
$$y^4 - 3y^2 + \frac{9}{4} - \frac{9}{2} + 3y^2 + y^4 + y^2 + 3y^2 - 2y^4 = 0$$

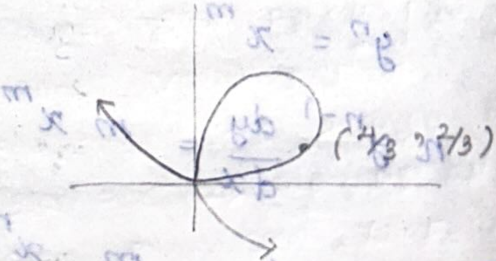
$$4y^2 = \frac{9}{4}$$

$$y^2 = \frac{9}{16} \Rightarrow \boxed{y = \pm\frac{3}{4}}$$

$$\therefore \boxed{x = \pm\frac{\sqrt{15}}{4}}$$

(The four points will be)





Slope of the tangent line to the curve $y^3 + x^3 = 3xy$ at the point $\left(\frac{4}{3}, \frac{2}{3}\right)$

Solu:

$y = \pm\sqrt{r^2 - x^2}$

↳ explicit form

$x^2 + y^2 = r^2 \rightarrow$ Implicit form

Implicit: some function of y and x equals something else. Knowing x doesn't lead directly to y.

Solu:

$$\frac{d}{dx}(y^3 + x^3 = 3xy)$$

$$3y^2\frac{dy}{dx} + 3x^2 = 3x\frac{dy}{dx} + 3y$$

$$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

At $\left(\frac{4}{3}, \frac{2}{3}\right)$

$$\frac{dy}{dx} = \frac{3\left(\frac{2}{3}\right) - 3\left(\frac{16}{9}\right)}{3\left(\frac{4}{9}\right) - 3\left(\frac{4}{3}\right)}$$

$$= \frac{2 - \frac{16}{3}}{\frac{4}{3} - 4}$$

$$= \frac{\frac{6-16}{3}}{\frac{4-12}{3}}$$

$$= \frac{-10}{3} \times \frac{3}{-8}$$

$$= \frac{5}{4}$$

Inverse functions

$$y = x^3 \rightleftarrows x = \sqrt[3]{y}$$

$$y = f(x) \rightleftarrows x = g(y)$$

∴ f undoes what g did.

$$f(g(y)) = y, \quad g(f(x)) = x$$

g is the inverse of f.

∴ $f(x) = x^3$

$$g(y) = \sqrt[3]{y}$$

The number x whose cube root is y.

$f^{-1}(y)$ = The number x such that $f(x) = y$.

$$y = f(x) \rightleftarrows x = f^{-1}(y)$$

Warning:

Not every function has an inverse.

$f(x) = 6x - 16$. has an inverse function. Find $f^{-1}(4)$

Solu:

$$6x - 16 = 4$$

$$6x = 20$$

$$\boxed{x = 20/6} = 10/3$$

$$\therefore \boxed{f^{-1}(4) = 10/3}$$

$$\therefore f(10/3) = 4.$$

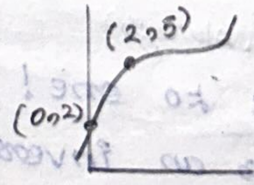
Graph of the inverse.

$f(2) = 5$

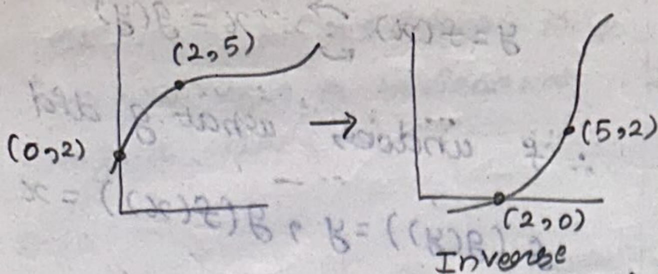
$f^{-1}(5) = 2$

$f(0) = 2$

$f^{-1}(2) = 0$



f(x)



(Reflection)

$h(x) = x^2$, $g(x) = \sqrt{x}$

$h(-2) = 4$ $g(4) = \pm 2$

$g(h(-2)) = -2$

g and h are not inverses of one other.

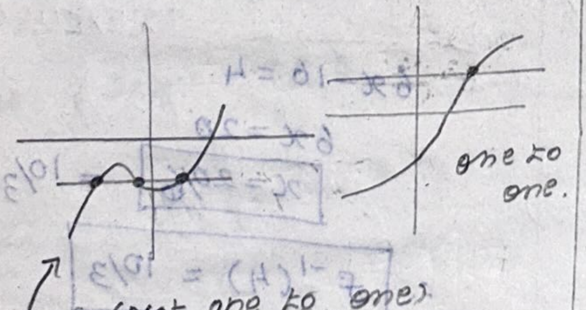
<u>Horizontal test</u>

If the function intersects more than at one point during horizontal test, it doesn't have an inverse.

Definition:

f is one-to one if, whether $a \neq b$,

$f(a) \neq f(b)$



$g(x) = 2x^3 + 3x - 1$

$g'(x) = 6x^2 + 3 > 0$

Increasing!

(Always)

one to one!

Has an inverse.

A function is one to one, if $f(a) \neq f(b)$ whenever $a \neq b$. It is one-to one if and only if its graph satisfies the horizontal line test (no horizontal line intersects its graph at more than one place.)

If $h(x) = 3 - \frac{2}{x}$

$h^{-1}(y) = ?$

$3 - \frac{2}{x} = y$

$-y + 3 = \frac{2}{x}$

$x = \frac{2}{3-y}$

$h^{-1}(y) = \frac{2}{3-y}$

<u>Domain and range, interval notation:</u>

Domain – set of allowable input values. For instance, the domain of the function $f(x) = \frac{1}{x}$ is the set of all non-zero real numbers.

Range: set of all possible output values. For instance the range of $g(x) = x^2 \rightarrow$ set of all real numbers that are non-negative.

we often use interval notation to express sets of numbers like domains and ranges. A closed interval, denoted $[a,b]$, is the set of numbers x such that

$$a \leq x \leq b.$$

open interval

denoted (a,b) is the set of numbers x such that

$$a < x < b$$

One can have a half-open half closed interval.

$[-1,3)$ is the set of numbers x such that $-1 \leq x < 3$. One can also use $\pm\infty$ as end points: $(-\infty,0)$ is the set of x

$$-\infty < x < 0.$$

(set of negative numbers)

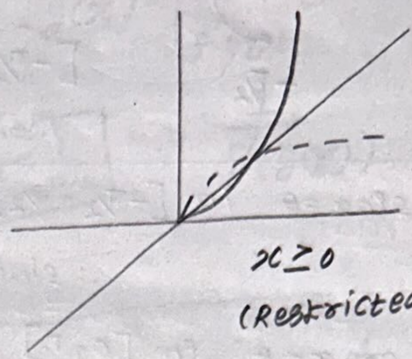
This notation using round parenthesis is not universal; many mathematicians use reversed brackets instead. For instance, they would denote the interval $3 < x < 7$ as $]3,7[$ rather than $(3,7)$.

partial inverses + Inverse Trig Functions

$$f(x) = x^2$$

(output restricted)

$$x \geq 0$$

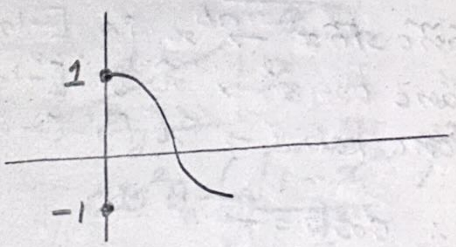


$x \geq 0$

(Restricted domain)

$\therefore \sqrt{4} = \pm 2$

Trignometric functions

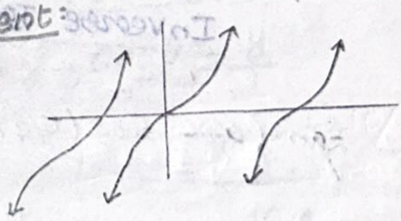


$$0 \leq \theta \leq \pi$$

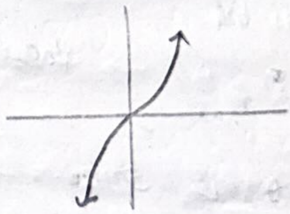
$\cos^{-1}x$ (or) arc cos x.

$$\boxed{\cos\theta = x}$$

Tangent:



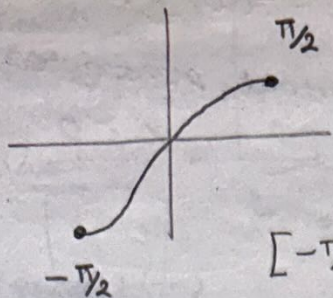
Restricted:



$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$x = \tan\theta.$$

$\sin\theta$:



$[-\pi/2, \pi/2]$

Restricted.

$\arcsin x = \theta$ in $[-\pi/2, \pi/2]$

$\therefore \sin\theta = x$

$\arccos x = \theta$ in $[0, \pi]$

$\therefore \cos\theta = x$

$\arctan x = \theta$ in $(-\pi/2, \pi/2)$

$\tan\theta = x.$

Domains

$\arcsin x \rightarrow x$ in $[-1, 1]$

$\arccos x \rightarrow x$ in $[-1, 1]$

$\arctan x \rightarrow x$ in $(-\infty, +\infty)$

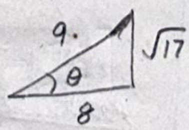
$\therefore \cos\pi = -1$

$\sin\left(-\frac{\pi}{2}\right) = -1$

$\tan\left(-\frac{\pi}{4}\right) = -1$

Combining Trig + Inverse Trig.

$\therefore \tan(\arccos(8/9))$



$81 = 64 + x^2$

$x^2 = 81 - 64$

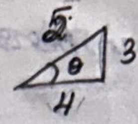
$x^2 = 17.$

$\tan\theta = \frac{\sqrt{17}}{8}$

$\therefore \theta$ is same.

$\tan\theta = \frac{\text{opp. side}}{\text{adj. side}}$

$\sin(\arctan(\frac{3}{4}))$



$\sin\theta = \frac{3}{5}$

$y = \frac{5}{2}x - 3$, slope $= \frac{5}{2}$

$x = \frac{2}{5}(y+3)$

$\boxed{\text{slope} = \frac{2}{5}}$

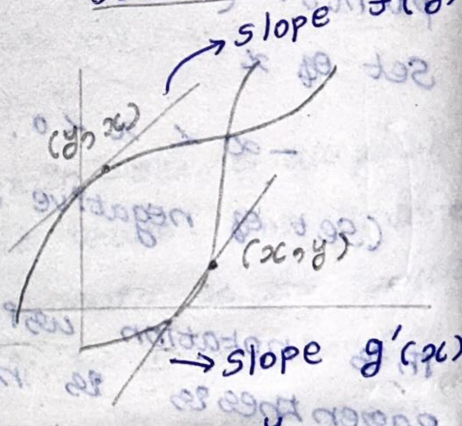
$f(m) = L$

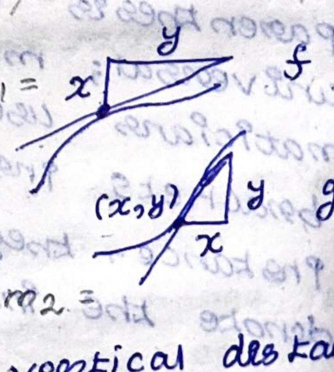
$g(L) = m$

$f' \rightarrow$ unit $\rightarrow L/m$ (Litres/metre)

$g' \rightarrow$ unit $\rightarrow m/L$

Derivatives of Inverse functions.





$\therefore$ vertical distance of f is equal to horizontal distance of g & vice versa.

Reflection $\boxed{m_1 = \frac{1}{m_2}}$

$\frac{d}{dx} f(g(x))$ If $f'(g(x))$ exists, $\neq 0$.

$\therefore f'(g(x)) \cdot g'(x) = 1.$

$g'(x) = \frac{1}{f'(g(x))}$ $\Rightarrow$

$\boxed{y = g(x)}$

$\boxed{\therefore f^{-1}(y) = x}$

$y = g(x)$

$g'(x) = \frac{1}{f'(y)}$

$\boxed{m_1 = \frac{1}{m_2}}$ $= \frac{1}{f'(g(x))}$

If g is a (full or partial) inverse of a function f, then

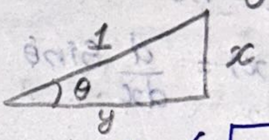
$g'(x) = \frac{1}{f'(g(x))}$

at all x where $f'(g(x))$ exists & it is non-zero.

$\frac{d}{dx}(\text{arc} \sin(x)) = \frac{1}{\cos(\text{arc} \sin x)}$

$= \frac{1}{\cos(\text{arc} \sin x)}$

θ



$g = f^{-1}$

$g'(x) = \frac{1}{f'(g(x))}$

$\frac{d}{dx} \text{arc} \sin(x) = \frac{1}{\sin'(\text{arc} \sin x)}$

$\therefore$

$\sin = f$

$g = \text{arc} \sin$

$\boxed{g = f^{-1}}$

$g'(x) = \frac{1}{f'(g(x))}$

$\frac{d}{dx} \text{arc} \sin(x) = \frac{1}{\cos(\text{arc} \sin x)}$

$\cos^2\theta = \left(\frac{\sqrt{1-x^2}}{1}\right)^2$

$\boxed{\cos\theta = \sqrt{1-x^2}}$

$y^2 = 1 - x^2$

$\boxed{y = \sqrt{1-x^2}}$

$\cos\theta = \frac{y}{1}$

$= \sqrt{1-x^2}$

$\frac{d}{dx}(\text{arc} \sin x) = \frac{1}{\sqrt{1-x^2}}$

$x < 0$

$-\frac{\pi}{2} < \theta < 0$

$\therefore$ domain of arc $\sin x$ $[-1, 1]$

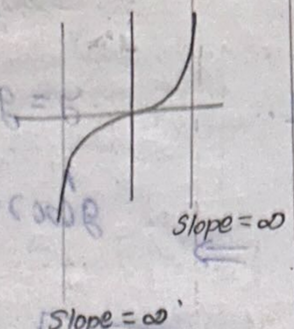
$\therefore$

$-1 < x < 1.$

$\hookrightarrow$ open interval

∴ Domain

$(-1, 1)$



Note:

$\sin'(\arcsin x) = \cos(\arcsin x)$

$\sin'(\arcsin x) \neq (\sin(\arcsin x))'$

↓ Needs chain rule.

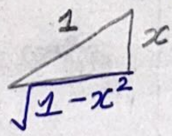
$x = \sin\theta$

$\frac{d}{dx}(x) = \frac{d}{dx}\sin\theta$

$\cos\theta \frac{d\theta}{dx} = 1$

$\frac{d\theta}{dx} = \frac{1}{\cos\theta}$

∴ $\cos^2\theta + \sin^2\theta = 1$



$\cos\theta = \pm\sqrt{1-x^2}$

$[-\pi/2, \pi/2]$

∴ $\cos\theta \to +ve.$

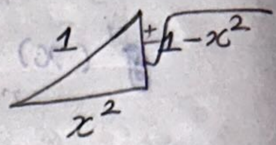
∴ $\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$

$\theta = \arccos x.$

∴ $\cos\theta = x$

$-\sin\theta \frac{d\theta}{dx} = 1$

$\frac{d\theta}{dx} = -\frac{1}{\sin\theta}$



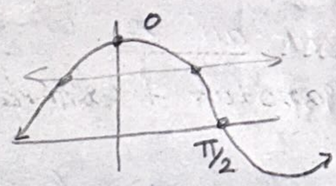
$\sin\theta = \pm\sqrt{1-x^2}$

$[0, \pi]$

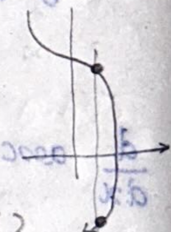
↳ +ve.

$\frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$y = \cos x$



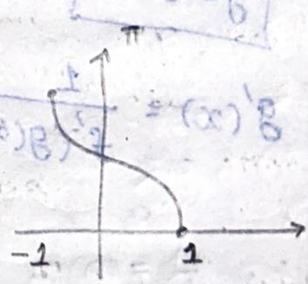
(periodic with 2π)



(Reflected) Inverse curve

vertical line test (Not a function)

Filtering.



$y = \arccos x$

Domain

$\arccos : [-1, 1]$

↳ $[0, \pi]$ range

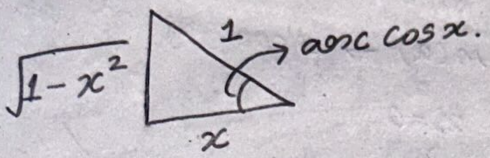
$y = \arccos x$

$\cos y = x.$

$\frac{dy}{dx} = -\frac{1}{\sin y}.$

(In terms of x)

$\frac{dy}{dx} = -\frac{1}{\sin(\arccos x)}$



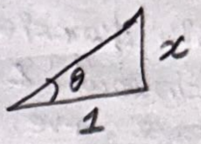
$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{\tan'(\theta)}$

$= \cos^2\theta$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

$\boxed{\frac{d}{dx} \arctan x = \frac{1}{1+x^2}}$



$\therefore \boxed{\cos^2\theta = \frac{1}{hyp} = \frac{1}{1+x^2}}$

arctan → differenciable everywhere

$g'(x) = \frac{1}{f'(g(x))}$

$\frac{d}{dx} \arctan x = \frac{1}{\sec^2(\arctan(x))}$

$\sec^2(\arc(\tan(x)))$ is non zero, since it appears in the denominator.

range of $\tan\theta$ is all real numbers, and so the domain of $\arctan x$ is also all real numbers.

In addition, $\sec^2\theta$ is non-zero at every point of the restricted domain of $\tan\theta$, $(-\pi/2, \pi/2)$

$\therefore \arctan x \rightarrow$ differenciable everywhere.

$f(x) = \arctan(3x)$, $f'(-1)$

$\therefore f'(x) = \frac{1}{1+(3x)^2} \times 3$

$= \frac{3}{10} = 0.3$

$g(x) = x^2 \arccos x$

$g'(\frac{1}{2}) = 2x\left(\frac{-1}{\sqrt{1-x^2}}\right) + 2x \arccos x$

$= (\frac{1}{2})^2\left(\frac{-1}{\sqrt{1-\frac{1}{4}}}\right) + 2(\frac{1}{2}) \arccos(\frac{1}{2})$

$= \frac{\pi}{3} - \frac{1}{4} \cdot \frac{1}{\sqrt{3}/2}$

$= \frac{\pi}{3} - \frac{1}{2\sqrt{3}}$

Exponential functions

ubiquitous - எங்கும் நிறைந்துள்ள

$a^0 = 1, \ a^1 = a, \ a^m a^n = a^{m+n}$

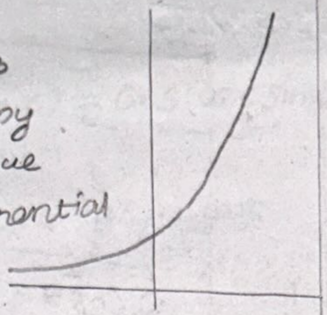
$(a^m)^n = a^{mn}, \ a^{m/n} = \sqrt[n]{a^m}$

$f(x) = 2^x$

Solu:

This function is continuous & everywhere +ve. (True for any exponential function,

$a > 0$
(Base)



$\boxed{y = 2^x}$

Properties:

The function $f(x) = a^x$ has base a for a +ve real number a.

* The function a^x is a continuous function.
* The domain of a^x is all real numbers.
* The range of a^x is all +ve real numbers.

Finding the derivative

$\frac{d}{dx} a^x = ?$

$= \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$

$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$.

$\therefore a^x \to$ constant

$= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

Let, Mystery number

$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$

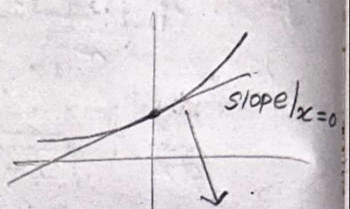
$\boxed{\frac{d}{dx} a^x = M(a)\, a^x}$

$\left.\frac{d}{dx} a^x \right|_{x=0} = M(a) \cdot a^0$

$= M(a)$

$M(a) \to$ slope of a^x at $x = 0$

$\therefore \boxed{a = 2}$ Here.



Slope $M(2)$
≈ 0.695

For

$4^x \to 1.390 \approx$ (slope)

$M(2) = \left.\frac{d}{dx} 2^x \right|_{x=0} \approx 0.695$

$\left.\frac{d}{dx} 4^x \right|_{x=0} \approx 1.390$

" $M(4)$

We need to define it:

Defining the base e to be unique, real number so that:

$M(e) = 1$.

$\frac{de^x}{dx} = M(e)\, e^x$

$= e^x$.

$\therefore \quad M(e) = \frac{d}{dx} e^x \Big|_{x=0} = 1$

Then $\frac{d}{dx} e^x = e^x$

Why e exist:

$f(x) = 2^x, \quad f'(0) = M(2)$

stretch by K

$f(Kx) = 2^{Kx} = (2^K)^x = b^x$

Let $\boxed{b = 2^K}$

when K is increased, slope gets steeper & steeper

$\frac{d}{dx} b^x = \frac{d}{dx} f(Kx)$

$= K f'(Kx)$

$\frac{d}{dx} b^x \Big|_{x=0} = K f'(0)$

$= K\, M(2)$

when $b = e$

$e^0 = K\, M(2)$

$\boxed{K = \frac{1}{M(2)}}$

If you calculate $e^{4.5} \approx 90$

$e^{4.55} = ?$

$f(x) = e^x$

$f(x) \approx f(4.5) + f'(4.5)(x - 4.5)$

$x = 4.55$

Rate of change

$f(4.55) \approx f(4.5) + \underline{f'(4.5)}(4.55 - 4.5)$

difference.

$= 90 + 90(0.5) = 94.5$

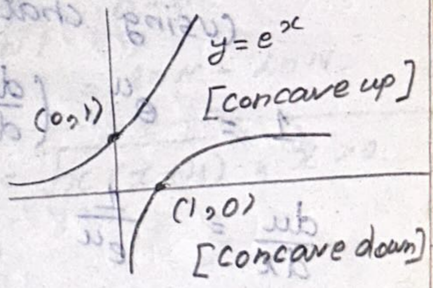
Logarithm

$\log_a(y) =$ Number x such that $a^x = y$

[Inverse of a^x]

eg:

* $\log_2(x)$ is the inverse of 2^x
* $\log_{10}(x) \rightarrow$ inverse of 10^x.
* $\log_e$ (or) $\ln(x)$ is the inverse of e^x.



$y = \ln x$

properties of $\ln(x)$ & $\log_a(x)$:

$\ln(a_1 a_2) = \ln(a_1) + \ln(a_2)$

$e^{b_1} \cdot e^{b_2} = e^{b_1 + b_2}$

$\ln(a^n) = n \ln(a)$

$(e^b)^n = e^{bn}$

$\log_{10} x \approx 8.23$

how many digits would x have.

$\therefore x \approx 10^{8.23}$ $10^8 \Rightarrow 9$ digits

Natural logarithm & derivative

$\frac{d}{dx} \ln(x) = ?$

$\therefore e^{\ln x} = x.$

$e^u = x$ $u = \ln(x)$

$$\boxed{\frac{d}{du} e^u = e^u}$$

$\frac{d}{dx} e^u = \frac{d}{dx}(x) = 1.$

$\frac{d}{dx} e^u = \frac{de^u}{du} \cdot \frac{du}{dx}$

(using chain rule)

$1 = e^u \cdot \left(\frac{du}{dx}\right)$

$\frac{du}{dx} = \frac{1}{e^u}$

$\frac{d}{dx} \ln(x) = \frac{1}{x}$

$\therefore \frac{d}{dx} a^x = M(a) \cdot a^x.$

mystery number $= M(a)$

method: 1:

changing to base e.

$\frac{d}{dx} a^x = ?$

changing to base e:

$a^x = e^{\ln a^x}$

$a^x = e^{x \ln(a)}$

$\frac{d}{dx}(a^x) = \frac{d}{dx} e^{x \ln(a)}$

$= \ln(a) \cdot e^{x \ln(a)}$

$= a^x \cdot \ln(a).$

method: 2

$$\boxed{M(a) = \ln(a)}$$

$\frac{d}{dx} 2^x = (\ln 2)\, 2^x$

$\frac{d}{dx} 10^x = (\ln 10)\, 10^x.$

method: 2:

Logarithmic differenciation

$\boxed{\cancel{u = a^x}}$

log

$\frac{du}{dx} = ??,\ \frac{d}{dx} \ln u$

Some times, it is easier to differenciate ln ~~of~~ the function than differenciating it.

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \cdot \frac{du}{dx}}$$

By chain rule:

$\frac{d}{dx} \ln u = \frac{d}{du}(\ln u) \cdot \frac{du}{dx}$

Let,

$u = a^x$

$\ln u = x \ln a$

$\frac{d}{dx}(\ln u) = \ln(a).$

$$\therefore \frac{d}{dx}\ln(u) = \frac{1}{u}\cdot\frac{du}{dx}$$
$$= \frac{1}{u}\cdot u'$$
$$= \frac{u'}{u}.$$

$$\frac{d}{dx}a^x = ?$$

$$\therefore \frac{u'}{u} = (\ln u)' = \ln(a)$$

$$\boxed{u' = \ln(a)\cdot u}$$

$$\frac{du}{dx} = \ln(a)\cdot u$$

$$\therefore u = a^x$$

$$\frac{du}{dx} = a^x\cdot\ln(a).$$

<u>Moving exponents.</u>

$V = x^x$, $v' = ?$

<u>solu:</u>

$$\ln v = x \log x.$$

$$\frac{d}{dx}(\ln v) = \frac{x}{x} + \log x$$

$$\frac{d}{dx}(\ln v) = 1 + \ln(x)$$

$$\frac{1}{v}\cdot\frac{dv}{dx} = 1 + \ln(x)$$

$$\frac{dv}{dx} = v(1 + \ln(x))$$
$$= x^x(1 + \ln(x))$$

<u>Rules ~~of~~ log</u>

1) $\ln(MN) = \ln M + \ln N$
2) $\ln\left(\frac{M}{N}\right) = \ln M - \ln N$ } Any base.
3) $\ln(M^K) = K\ln M$
4) $\ln(M^K) \neq (\ln M)^K$
5) $\log_b M = \frac{\ln M}{\ln b}$

$M, b = +ve.$

<u>proving:</u>

$$e^a = M,\ e^b = N$$

$$\boxed{e^a\cdot e^b = MN}$$

$$\ln M = a,\ \ln N = b$$

$$\ln(e^{a+b}) = \ln(MN)$$

$$a+b = \ln(MN)$$

$$\ln(MN) = \ln M + \ln N$$

$$\therefore a = \ln M,\ b = \ln N$$

$$\ln\left(\frac{M}{N}\right) = \ln\left(\frac{e^a}{e^b}\right)$$

$$\ln\left(\frac{M}{N}\right) = a - b$$
$$= \ln M - \ln N.$$

$$y = \sqrt{x(x+4)},\ x > 0.$$

$$y' = ?$$

$$\ln y = \frac{1}{2}\ln(x^2+4x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x^2+4x}(2x+4)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{x+2}{x^2+4x}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+4x}\,(x+2)}{x^2+4x}$$

$$\frac{dy}{dx} = \frac{x+2}{\sqrt{x^2+4x}}$$

$$\therefore y' = \sqrt{x(x+4)}\left(\frac{1}{2x} + \frac{1}{2(x+4)}\right)$$

$$y' = \sqrt{x^2+4x}\left(\frac{x+4+x}{2(x^2+4x)}\right)$$

$$= \frac{x+2}{\sqrt{x^2+4x}}$$

$y = x^r$, $y' = ?$

$$x^r = e^{(\ln x)^r} = e^{r \ln x}$$

$u = r \ln x$

$$\frac{d}{dx} e^u = \frac{de^u}{dx} \cdot \frac{du}{dx}$$

$$= e^u \cdot \frac{du}{dr}$$

$$= e^{r \ln x} \cdot \frac{r}{x}$$

$$= x^r \cdot \frac{r}{x}$$

$$= x^{r-1} \cdot r$$

$$= r x^{r-1}$$

(or)

$$\ln x^r = r \ln x$$

$$\therefore \frac{u'}{u} = (\ln u)'$$

$$\frac{(x^r)'}{x^r} = \frac{r}{x}$$

$$(x^r)' = r x^{r-1}$$

---

1) $f(x) = x^\pi + \pi^x$

$f'(x) = \pi x^{\pi - 1} + \pi^x \cdot (\ln \pi)$

2) $g(x) = \ln(\cos x)$

$g'(x) = \frac{1}{\cos x} \cdot -\sin x$

$-\tan x$

3) $h(x) = \ln e^{x^2}$

$h'(x) = \frac{1}{e^{x^2}} \cdot e^{x^2} \cdot 2x$

$h'(x) = 2x$

---

another way

$$\ln e^{x^2} = x^2$$

$$\therefore \boxed{h'(x) = 2x}$$

$$= \frac{d}{dx} \ln(\sqrt{x}) \Big|_{x = 1/4}$$

$$= \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2x} = \frac{1}{2\left(\frac{1}{4}\right)}$$

$$= \frac{1}{\frac{1}{2}} = 2$$

$$\frac{d}{dx} x^{\ln x} \Big|_{x=e}$$

soln:

$$\ln u = \ln(x^{\ln x})$$

$$= \ln x \,(\ln x)$$

$$= (\ln x)^2$$

$$\frac{d}{dx}(\ln x)^2 = 2(\ln x) \cdot \frac{1}{x}$$

$$= \frac{2}{x} \cdot \ln(x)$$

$$\frac{d}{dx}(\ln u) = \frac{2}{x} \ln x$$

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{2}{x} \ln x$$

$$\frac{du}{dx} = u\left(\frac{2}{x} \ln x\right)$$

$$\frac{du}{dx} = x^{\ln x}\left(\frac{2}{x} \ln x\right)$$

At $x = e$

$$\frac{du}{dx} = e^1\left(\frac{2}{e}(1)\right)$$

$$= 2.$$

(or)

$$x^{\ln x}$$

Let: $x = e^{\ln x}$