Resonance, frequency response, RLC circuits.

Objectives:

* Understand that resonance occurs at the input frequency at which the amplitude of the system response is largest.
* Find purely resonant solution to DE for generalized exponent resonance formula.
* Express how the amplitude (gain of the output signal changes as a function of the angular frequency of the input signal as a Bode plot.

Review

Harmonic oscillator without input

$$\ddot{x} + 9x = 0$$

$P(r) = r^2 + 9$

$PD(x) = 0$, $P(D) = D^2 + 9$

Roots : $\pm 3i$

Real valued basis : $\cos 3t, \sin 3t$

Basis of complex solutions $= e^{3it}, e^{-3it}$

General solution: $a\cos 3t + b\sin 3t$

$a, b \rightarrow$ Real numbers a, b.

These are all the sinusoids with angular frequency 3, the natural frequency.

$$A\cos(3t - \phi)$$

$(D^2+9)x = \cos\omega t \Rightarrow y_p = \dfrac{\cos\omega t}{9-\omega^2} \Rightarrow \text{Gain} = \dfrac{1}{P(D)} = \dfrac{1}{9-\omega^2}(1)$

$\because \cos^2 x + \sin^2 x = 1$

$|G(\omega)| = \left|\dfrac{1}{9-\omega^2}\right|$ (complex gain)

$= +ve$ (1)

Near resonance

Resonance:

Simple pendulum:

$$y'' + \omega_0^2 y = \cos\omega_1 t$$

For a kid need to swing in the swing, we need to give exactly the same ω as that of its ω_n.

$$(D^2 + \omega_0^2)y = 0$$

$$(D^2 + \omega_0^2)\,\tilde{y} = e^{i\omega_1 t}$$

$$\tilde{y}_p = \frac{e^{i\omega_1 t}}{-\omega_1^2 + \omega_0^2}$$

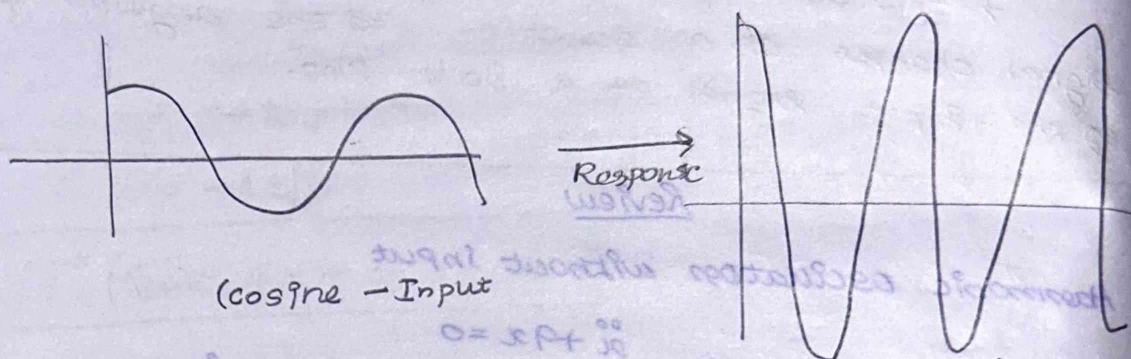
when $\boxed{\omega = \omega_0^2} \rightarrow \tilde{y}_p \rightarrow$ Blow

Real part:

$$y_p = \frac{\cos \omega_1 t}{-\omega_1^2 + \omega_0^2}$$

$\omega_1 \rightarrow$ Driving or Input frequency.

when $\omega_1 \approx \omega_0$ (Then the solution has large amplitude)



pushing something with approximately the same frequency something that wants to oscillate. (The same frequency that it would like to oscillate by itself.

3.1 We might try to model a swing on a playground as a simple harmonic oscillator. has a natural frequency ω_n. When you push your kid sister on a swing, the Key is to get in swing (sync) with the natural frequency of the swing. You keep pushing on regular intervals with the same force, but your sister swings higher & higher.

Let's model the swing & sister setup by a sinusoidally driven harmonic oscillator, $\ddot{x} + \omega_n^2 x = A\cos(\omega t)$. We find a periodic solution using the exponential response formula provided that $\omega \neq \omega_n$.

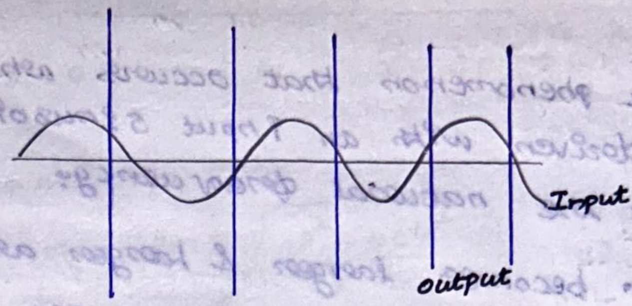
$$x_p = A\,\frac{\cos \omega t}{\omega_n^2 - \omega^2}$$

when $\omega \longrightarrow \omega_n$, the amplitude of this particular solution grows larger.

In this mathlet, the $\omega_n = 1$, and the frequency of the input signal is adjustable. Input signal $\rightarrow$ pencil, Response $\rightarrow$ output vertical axis is scaled by $\frac{1}{\sqrt{2}}$. (To fit the frame)

which is RMS — Root mean square.

For sinusoid is amplitude$/\sqrt{2}$; It has physical interpretation used for superposition of many frequencies when $\omega = \omega_n$ (1)



Pure resonance

If $\omega_1 = \omega_0$

$$(D^2 + \omega_0^2)\, y = \cos \omega_0 t.$$

Solu:

$$\ddot{z} + \omega_0^2 z = e^{i\omega_0 t}$$

$$z_p = \frac{e^{i\omega_0 t}}{-\omega_0^2 + \omega_0^2} \quad \text{(Zero)}.$$

So Taking case 2:

$$\tilde{y}_p = \frac{t e^{i\omega_0 t}}{2i(\omega_0)} = \frac{i t e^{i\omega_0 t}}{-2\omega_0}$$

$$y_p = Re(\tilde{y}_p) = -\frac{t \sin \omega_0 t}{-2\omega_0} = \frac{t \sin \omega_0 t}{2\omega_0}$$

(The resonance may collapse a building when the frequency of Earth quake is exactly the natural frequency of that building).

The case in which we drive a harmonic oscillator by the natural frequency ω_n is known as pure resonance.

$(D^2 + \omega_n^2)\, x = \cos(\omega_n t)$ → To find P.I follow the above steps.

Even though our input signal is sinusoidal, the response is not a sinusoidal. The resonance is an oscillating function whose oscillations grow linearly without bound as time increases.

Pure resonance:

It is a phenomenon that occurs when a harmonic oscillator is driven with an input sinusoid whose frequency is at the natural frequency.

* The gain becomes larger & larger as the input frequency approaches the natural frequency, and
* when the input frequency equals the natural frequency, any particular solution is unbounded.

$$\ddot{x} + 4x = 2\cos 2t$$

solu:

$$\ddot{z} + 4z = 2e^{i2t}$$

solu:

$$\frac{2e^{i2t}}{\cancel{-4} + 4} \Rightarrow (0) = z_p \qquad \Rightarrow \quad z_p = \frac{2t\,e^{i2t}}{2r} = \frac{2t\,e^{i2t}}{4i}$$

$$x_p = Re(z_p) \Rightarrow \quad z_p = \frac{2t\,i\,e^{i2t}}{-4i} \quad \Rightarrow \quad x_p = \frac{t}{2}\sin 2t$$

Resonance with damping

In a realistic physical situation, there is atleast a tiny amount of damping, and this prevents the runaway growth of the amplitude of the system response.

5.1 : what happens if $\omega = 3$ exactly. (with tiny amount of damping)

$$\ddot{x} + b\dot{x} + 9x = \cos\omega t \qquad \text{(For some } b \text{ (+ve))}$$

solu: $p(r) = r^2 + br + 9$

complexing,

$$\ddot{z} + b\dot{z} + 9 = \cancel{\cos\omega t}\; e^{i\omega t}$$

when $b = 3$

$$z_p = \frac{e^{i\omega t}}{-\omega^2 + bi\omega + 9}$$

$$= \frac{e^{i\omega t}}{bi\omega} = \frac{i\,e^{i\omega t}}{-b\omega}$$

$$x_p = \frac{\sin\omega t}{b(3)} \qquad \text{(when } b \text{ is large)}$$

$$\text{Gain} = \frac{1}{3b} \quad \text{(the system reduces to zero)}$$

So oscillations are bounded. There is a steady state solution.

$m\ddot{x} + b\dot{x} + Kx = 0$ $\quad \div m \quad$ $\ddot{x} + 2p\dot{x} + \omega_n^2 x = 0$

solving

$$r^2 + 2pr + \omega_n^2 = 0$$

$$r = \frac{-2p \pm \sqrt{4p^2 - 4\omega_n^2}}{2}$$

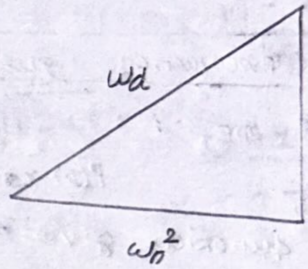
$$r = -p \pm \sqrt{p^2 - \omega_n^2}$$

$$r = -p \pm i\sqrt{\omega_n^2 - p^2}$$ → when this can is under damped.

$$\boxed{\omega_d^2 = \omega_n^2 - p^2}$$

only case when damped frequency exists when $\sqrt{p^2 - \omega_n^2}$ is imaginary. In this case $\omega_n^2 > p^2$. The damped frequency satisfies the pythagorean relationship.

$$\omega_n^2 - p^2 = \omega_d^2$$



$$m\ddot{x} + b\dot{x} + Kx = f_1(t) \qquad 2p = \frac{b}{m}, \ \omega_n^2 = \frac{K}{m}$$

$$\ddot{x} + 2p\dot{x} + \omega_n^2 x = f_2(t)$$

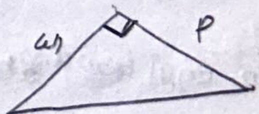
ω_n (or) ω_0 → Natural undamped frequency.

solu:

ω_1 → Natural damped frequency (or) pseudo frequency

If R.H.S = 0 (The function isn't periodic & It crosses the t-axis at regular intervals. It decays.

Bigger the damping (p), Larger the ω_1.



$$\boxed{\omega_1^2 = \omega_0^2 - p^2}$$ (From the char. roots)

ω_0 or ω_n ↓ Fixed (Spring's Frequency)

$$y'' + 2py' + \omega_0^2 y = \cos\omega t \qquad \omega - \text{undetermined frequency}$$

($p \neq 0$, In nature we always have damping)

Prob: which ω gives maximum amplitude for response.

let

$$\omega_1 = \sqrt{\omega_0^2 - p^2}$$

$$\omega_r = \sqrt{\omega_0^2 - 2p^2}$$

ω_n — Natural (undamped) frequency

ω_d — Natural (Damped) pseudo frequency

ω_r — The resonant frequency.

The resonant frequency ω_r is defined as the input frequency that leads to the maximum amplitude in the response.

($\omega_0 = \omega_n$ & $\omega_1 = \omega_d$)

(Above)

Frequency response & amplitude response

Given any ODE,

$P(D)x = Q(D)\cos(\omega t)$, the complex gain is a complex function given by

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)} = ge^{-i\phi} \quad \text{(In polar form).}$$

The gain of the response x is $g = |G(\omega)|$. which can also be thought of as a function of the input sinusoidal frequency ω. The phase lag of the response

$-\phi = \arg(G(\omega))$ can similarly be thought as a function of ω.

In Engineering, the graph of gain as a function of ω is called a Bode plot (Bode is pronounced as Boh-dee), and is usually on a log-log scale. Alternatively, a Nyquist plot shows the trajectory of the complex Gain G as ω varies. The Bode plots contain, at a glance, all of the information about the resonance response ω_r. By looking at the Bode plot of the amplitude as a function of frequency. we can observe any vertical asymptotes (pure resonant values) or local maxima (resonant values). Sometimes the gain is monotonically decreasing, this is the case of no resonance.

Graph the amplitude response for

a) $\ddot{x} + 4x = F_0 \cos\omega t$

b) $\ddot{x} + \dot{x} + 4x = F_0 \cos\omega t$

c) $\ddot{x} + 6\dot{x} + 4x = F_0 \cos\omega t$.

d) Discuss resonance

Solu:

a) $\ddot{z} + 4z = F_0 e^{i\omega t}$

$$z_p = \frac{F_0 e^{i\omega t}}{i^2\omega^2 + 4} = \frac{F_0 e^{i\omega t}}{-\omega^2+4}$$

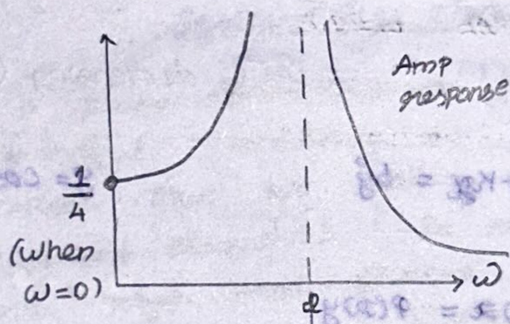
$$x_p = \frac{F_0 \cos\omega t}{|4-\omega^2|}$$

↳ Output Amplitude

Amplitude response = $\dfrac{\text{Output Amp}}{\text{Input Amp}}$ (magnitude alone)

$$= \frac{\dfrac{F_0}{|4-\omega^2|}}{F_0}$$

$$= \frac{1}{|4-\omega^2|}$$



Amp response $= \dfrac{1}{|4-\omega^2|}$

At $\omega = 2$, the function blows up.

($\therefore \omega_r = 2$)

↳ Resonant force.

b) $\ddot{z} + \dot{z} + 4z = F_0 \cos\omega t$

$$z_p = \frac{F_0 e^{i\omega t}}{-\omega^2 + i\omega + 4}$$

$$z_p = \frac{F_0 e^{i\omega t}}{(4-\omega^2) + i\omega}$$

$$x_p = Re\left(\frac{F_0 e^{i\omega t}}{(4-\omega^2)+i\omega}\right)$$

Output Amplitude $= \dfrac{F_0}{|P(i\omega)|}$, Input Amplitude $= F_0$

Amplitude response $= \dfrac{1}{|P(i\omega)|}$

$$= \frac{1}{\sqrt{(4-\omega^2)^2 + \omega^2}}$$

$|a+bi| = \sqrt{a^2+b^2}$

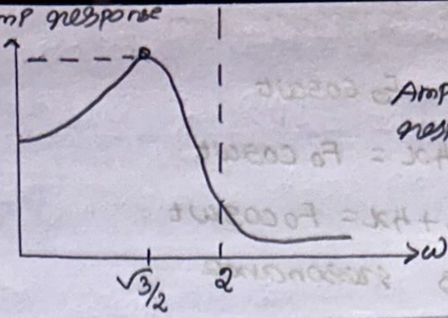
(As denominator is minimum)
↳ Amplitude max

minimize $(4-\omega^2)^2 + \omega^2$ at $\omega = \sqrt{\frac{7}{2}}$.

At $\omega=2$

(No asymptote)

At $\sqrt{7/2} \rightarrow$ max amplitude



$$\text{Amp response} = \frac{1}{\sqrt{(4-\omega^2)^2+\omega^2}}$$

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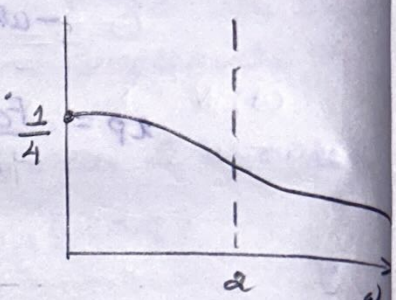
c) $\ddot{x}+6\dot{x}+4x = F_0\cos\omega t$

solu:

$$\text{Amp response} = \frac{1}{|P(i\omega)|} = \frac{1}{|(4-\omega^2)+6i\omega|} = \frac{1}{\sqrt{(4-\omega^2)^2+36\omega^2}}$$

when $\omega=0 \Rightarrow$ Am response $= \frac{1}{4}$.

As $\omega\to\infty \longrightarrow$ Function goes to 0.



d) Resonance

a) No damping At $\omega=2$

b) At $\omega=\sqrt{7/2}$

c) The function is monotonically decreasing (No peak point)

$$\ddot{x}+b\dot{x}+Kx = b\dot{y} \qquad y=\cos\omega t$$

Set $b=0.5,\ K=3$

$$P(D)x = Q(D)y$$

$$P(D) = D^2+0.5D+3$$

$$Q(D) = 0.5D$$

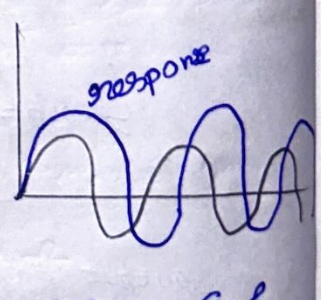
$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{0.5\omega i}{3-\omega^2+0.5\omega i}$$

when $\omega\to 0$

$$G(\omega) = \frac{0.5\omega i}{3} \to 0$$

$g=|G|$ also tends to zero as $\omega\to 0$

when ω is large



$$G(\omega) = \frac{0.5\omega i}{3-\omega^2+0.5\omega i} \approx \frac{0.5\omega i}{-\omega^2} \to 0$$

$\therefore$ $g=|G|$ also tends to zero as $\omega\to\infty$ since G & hence g is not identically zero, we expect that b/w 0 and ∞, there is some resonant freq ω_r. where g obtains a maximum value.

$$\omega_r \approx 1.75 \text{ (From mathlet)}$$

$b = 1.5,\ K = 1$

$P(D) = D^2 + 1.5D + 1,\quad Q(D) = 1,\quad G(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{1}{1-\omega^2+1.5\omega i}$

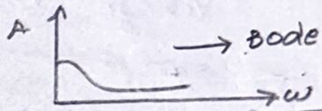
when $\omega \to 0$

$$\frac{1}{1} = 1$$

$\therefore g = |G|$ also tends to 1 as $\omega \to 0$ when ω is large

$$G(\omega) = \frac{1}{1-\omega^2+1.5\omega i} \approx \frac{1}{-\omega^2} \to 0.$$

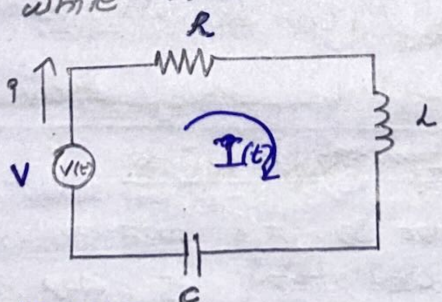
$\therefore g = |G|$ also $\to 0$ as $\omega \to \infty$. $\therefore$ It's not clear there is a resonant frequency or not. Using the bode plot we see that g is monotonically $\downarrow$ & this system has no resonance.



$\to$ Bode plot

Series RLC circuit

Resonance is a phenomenon that is very important in the design of Electrical circuits. So we will model an RLC ckt in series. Then we will explore how resonance enables different elements in the circuit act as low-pass filters, high pass filters, or mid-pass filters - selectively allowing certain driving frequencies to pass through while others diminish.



V_R: Voltage drop across the resistor

V_C: " " capacitor

V_L: " " Inductor

Step1: Draw diagram

2: Identify all variables.

3: Declare Input.

$V_R(t) = RI(t)$

$V_L(t) = L\dot{I}(t)$

$\dot{V}_C(t) = \frac{1}{C} I(t)$

} Although they aren't perfect. we can model a wide range of devices.

KVL:

$V = V_R + V_L + V_C$

$\dot{V} = \dot{V}_R + \dot{V}_L + \dot{V}_C$

$\dot{V}_R(t) = R\dot{I}(t)$

$\dot{V}_L(t) = L\ddot{I}(t)$

$\dot{V}_C(t) = \frac{1}{C} I(t)$

In case of AM radio: Input: Voltage
Output: Headphone sound

Here: we are taking voltage drop as our system response

I is common to all of the elements.

$$I = \frac{V_R(t)}{R}, \quad \dot{I} = \frac{1}{R}\dot{V}_R(t), \quad \ddot{I} = \frac{1}{R}\ddot{V}_R(t)$$

Applying

$$V_L(t) = L\dot{I}(t) = \frac{L}{R}\dot{V}_R(t)$$

$$\dot{V}_C(t) = \frac{1}{C}I(t) = \frac{1}{C}\cdot\frac{1}{R}V_R(t)$$

$$\dot{V} = R\,I(t) + \frac{L}{R}\dot{V}_R(t) + \frac{1}{CR}V_R(t)$$

$$\dot{V} = \dot{V}_R(t) + \frac{L}{R}\dot{V}_R(t) + \frac{1}{CR}V_R(t)$$

$$\boxed{R\dot{V} = L\ddot{V}_R + R\dot{V}_R + \frac{1}{C}V_R}$$ → comparing it with mkb

$$m\ddot{x} + b\dot{x} + Kx = b\dot{y}$$

$$\boxed{b = R \quad m = L \quad K = \frac{1}{C}}$$

→ RLC ckts can be modeled as some physical systems do.

AM radio receiver

The power source at left is the antenna, being driven by radio waves. The resistor at top is a speaker. The other components are a capacitor (At bottom) & Inductance coil (At right).

1) Draw dig ② Identify variable & parameters

$R \rightarrow \Omega$
L — Henry
C — Farads
V — Volts

3) Declare input signal & response
$V(t) \rightarrow$ Input
$V_R(t) \rightarrow$ Response

4) write KVL, ($F=ma$ in mechanical cases)

5) Rewrite in standard form.

Displacement → x, y — voltage drop, gain → V_R, V

mass → m — Inductance L

Damping constant → b — Resistance R

Spring constant → K — $\frac{1}{C}$ → 1/capacitance.

	Electrical
Displacement → x, y	voltage drop, gain → V_R, V
mass → m	Inductance L
Damping constant → b	Resistance R
Spring constant → K	$\frac{1}{C}$ → 1/capacitance.

|||ly a harmonic oscillator (undamped) is analogous (similarity b/w two things) to a series LC circuit. (No resistor).

DE's for voltage drops across inductor & capacitor.

DE for voltage drop across the Inductor:

$$L \ddot{V}_L + R \dot{V}_L + \frac{1}{C} V_L = L \ddot{V} \quad (m\ddot{x} + b\dot{x} + Kx = m\ddot{y})$$

DE for voltage drop across the capacitor:

$$L \ddot{V}_C + R \dot{V}_C + \frac{1}{C} V_C = \frac{1}{C} V. \quad (m\ddot{x} + b\dot{x} + Kx = Ky)$$

↳ Even though we have a single power source. we have three responses.
↳ so 3 diff equ

Gain for different system responses.

$P(D) = LD^2 + RD + L$

(for the input $V = \cos\omega t$ what are the system response as ω changes).

$$L \ddot{V}_R + R \dot{V}_R + \frac{1}{C} V_R = R\dot{V}$$
$$L \ddot{V}_L + R \dot{V}_L + \frac{1}{C} V_L = L\ddot{V}$$
$$L \ddot{V}_C + R \dot{V}_C + \frac{1}{C} V_C = \frac{1}{C} V$$

$g(\omega) = |G(\omega)|$

$$\boxed{P(D)x = Q(D)y.}$$

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)}$$

$$P(D) = LD^2 + RD + \frac{1}{C}(1)$$

$$P(i\omega) = -L\omega^2 + Ri\omega + \frac{1}{C}$$

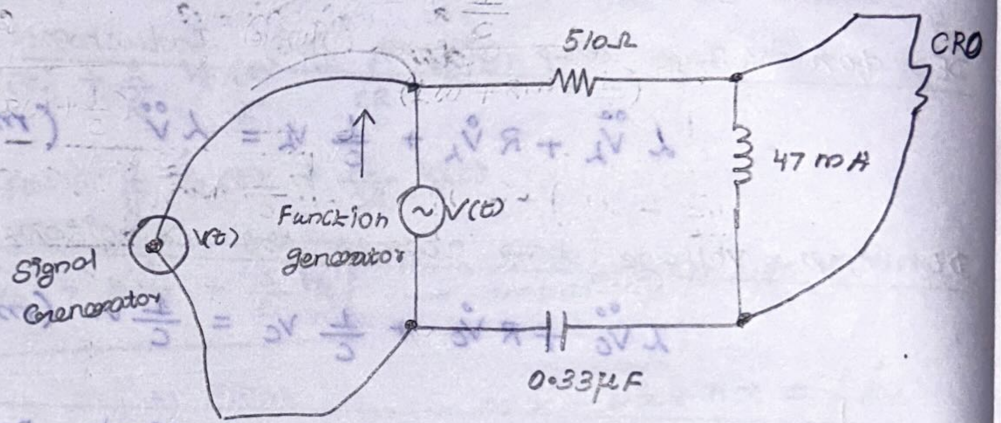
i) $\phi_R(D) = RD$, ii) $\phi_L(D) = LD^2$, iii) $\phi_C(D) = \frac{1}{C}(1)$

$\phi_R(i\omega) = i\omega R$ ii) $\phi_L(i\omega) = -L\omega^2$, $\phi_C(i\omega) = \frac{1}{C}$

$$G_R(i\omega) = \frac{iR\omega}{-L\omega^2 + iR\omega + \frac{1}{C}} = \frac{iR\omega}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}$$

$$G_L(i\omega) = \frac{-L\omega^2}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}, \quad G_C(\omega) = \frac{\frac{1}{C}}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}$$

Voltage drop across resistor as system response.



At resonance, Gain = 1 (Input = output) ≈ 1 (Due to physical imperfection)

If we input signal (very high frequencies)

↳ Response eventually becomes zero.

$z = a + ib$, $|z| = \sqrt{a^2+b^2} \Rightarrow |z|^2 = a^2+b^2$

Analyzing Gain:

$$G_R(\omega) = \frac{iR\omega}{\left(\frac{1}{C} - L\omega^2\right) - iR\omega}$$

$$g^2 = \frac{R^2\omega^2}{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}$$

What happens when we vary ω: when $\omega \to 0$

$$g^2 \to \frac{R^2\omega^2}{\frac{1}{C}} \to 0 \quad (g^2 \to 0, \ g \to 0)$$

$\omega \to \infty \Rightarrow g^2 \to 0$ (As denominator is growing faster than the numerator.

(Both cases $g \to 0$)

(In some case g may be complex)

That time we can use complex gain for our analysis

$$G_R(\omega) = \frac{iR\omega}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}$$

As $\omega \to 0$, $G_R \to 0$.

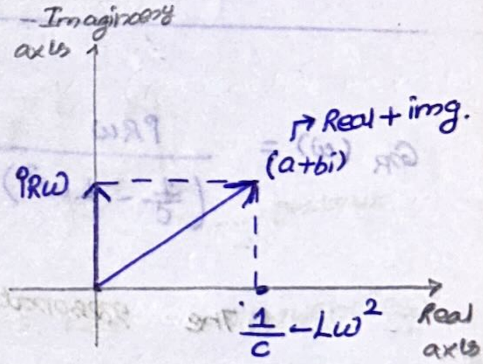
$G_R(\omega) = ?$ (when $\omega \to \infty$) $\Rightarrow G_R \to 0$ (As denomin grows faster).

what will be the maximum gain?

$$\frac{d}{d\omega}\left(g^2(\omega)\right) = 0$$

$$|G_R| = \frac{|iR\omega|}{\left|\left(\frac{1}{C} - L\omega^2\right) + iR\omega\right|} = \frac{1}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}}$$

$\therefore iR\omega \to$ purely imaginary



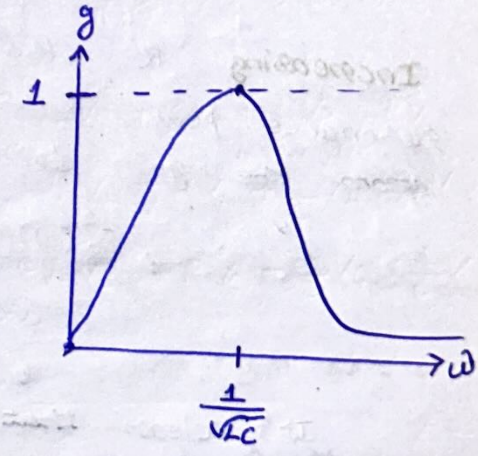
we have maximum magnitude of gain when Num = Denominator (purely imaginary).

This happens when

$$\frac{1}{C} - L\omega^2 = 0$$

$$\omega = \sqrt{\frac{1}{LC}}$$

$|G_R| \to 1$ when $\omega = \sqrt{\frac{1}{LC}}$



As $\omega \to 0$, $g \to 0$

$\omega \to \infty$, $g \to \infty$

Verifying our prediction.

$R = 510\,\Omega$, $L = 47\,mH$ and $C = 0.33\,\mu F$, Voltage drop across the resistor, shown as the red curve on the

Oscilloscope has the largest amplitude at around 1.2 KHz.

Soln:

$$G_R(\omega) = \frac{iR\omega}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}$$

$$\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{47m \times 0.33\mu}} \approx 8000 \text{ (rad/s)}$$

$$\nu_r = \frac{\omega}{2\pi} \sim 1300 HZ = 1.3 KHZ$$ (which is within 10% error)

$\omega_r = \sqrt{\frac{1}{LC}}$, we also see that decreasing C to $\frac{C}{10}$ means ↑ ω, ↑ by a factor of $\sqrt{10}$. since ν_r is proportional to ω_r, we also have

(If we use a capacitor $C = 0.033\mu F$, how will ν_r change) → Give multiplication factor.

$$\nu_r\left(\frac{C}{10}\right) = a\,\nu_r(C)$$

Increasing resistance.

$$G_R(\omega) = \frac{iR\omega}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega}$$

1) The resonat frequency will remain the same

2) The maximum gain value stay at 1.

Increasing R also has an effect on the shape of the resonance peak, even though it's not immediately obvious from the gain formula

$$g_R^2(\omega) = |G_R|^2 = \frac{R^2\omega^2}{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}$$

It's clear that $\omega_r = \sqrt{\frac{1}{LC}}$ (Doesn't depend on R)

G_R (Complex Gain)

$$G_R(\omega) = \frac{iR\omega}{\frac{1}{C} - L\omega^2 + iR\omega}$$

attains a maximum magnitude of 1 when its denominator is purely imaginary, regardless on the value of R.

Finally, in the analogy b/w the series RLC circuit with V_R chosen as response and the spring - mass - dashpot system driven by the damper, increasing R is analogous to ↑ the damping constant b. using mathlet, we can see b widens the resonance peak.

Remark : To make a AM radio, decreasing R will help filter out frequencies outside the desired band.

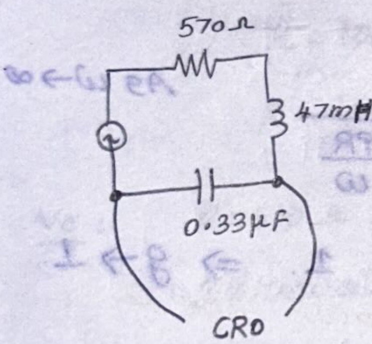
Replacing $R = 1000\Omega$ → (same resonance frequency).

(It takes long time to decrease to zero).

---

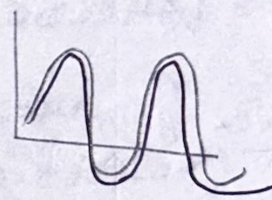
when $C = 0.033\mu F$ → $\omega_r = 28.27$ (KHz). changed from earlier.

Capacitor — Voltage drop across capacitor & Inductor.



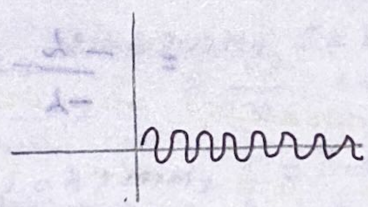
→ ~~Feest~~

At low freq: $\left(\frac{1}{\omega}\right)$ • Input



Output

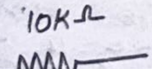
At high freq

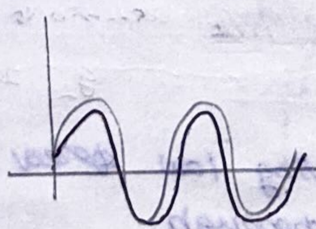


(At high frequencies)

16000KHz

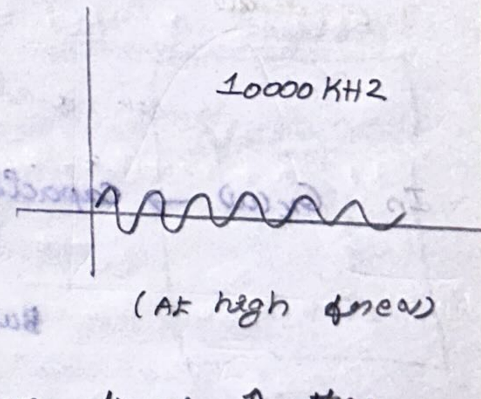
10KΩ





Low Freq (similar)

10000 KHz

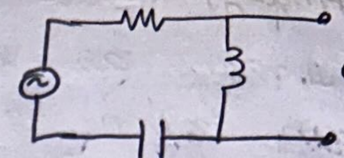


(At high freq)

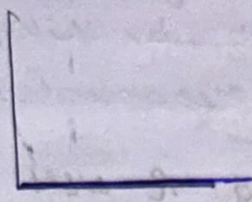
when R is small ⇒ The output curve first ↑ then ↓ to zero

R is Large → New ↑, only decreases to zero

Inductor

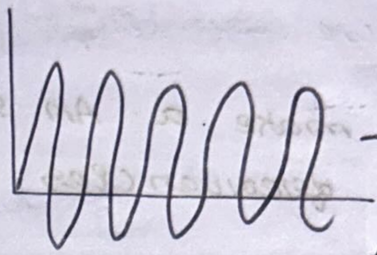


Low freq



High freq

slowly goes up



Gain → 1 ←

Analyze gain (For Inductor & capacitor)

$$G_C(\omega) = \frac{1/C}{\left(\frac{1}{C} - L\omega^2\right) + iR\omega} \xrightarrow{\omega \to 0} G_C(\omega) = 1 \Rightarrow g \to 1$$

$$\xrightarrow{\omega \to \infty} G_C(\omega) = 0 \rightarrow g \to 0.$$

$$G_L(\omega) = \frac{-L\omega^2}{\left(\frac{1}{\omega} - L\omega^2\right) + iR\omega} \xrightarrow{\omega \to 0} g \to 0$$

As $\omega \to \infty$

$$\xrightarrow{\omega \to \infty} \frac{-L}{\frac{1}{\omega^3} - L + \frac{iR}{\omega}}$$

$$= \frac{-L}{-L} = 1. \Rightarrow g \to 1.$$

midrange:

$\frac{d}{d\omega}(g_C(\omega)) = 0 \Rightarrow$ for maximum $\Rightarrow g_C$ has max for R small.

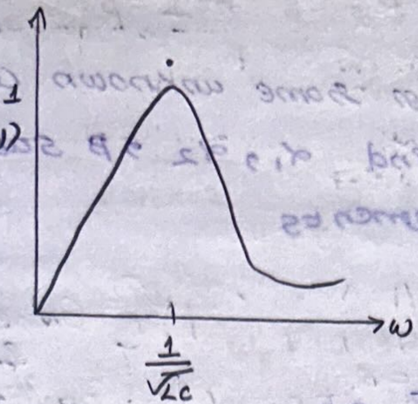
g_L always has a max.

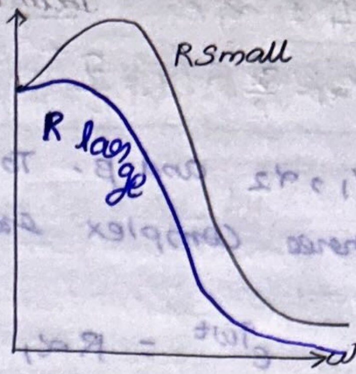
In $G_C(\omega) \rightarrow$ capacitor is allowing low freq to pass through

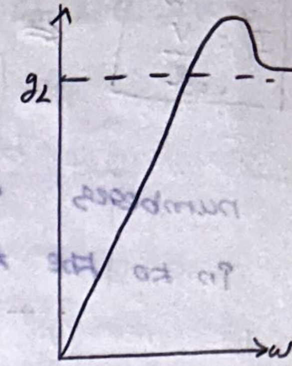
But blocks high frequencies.

In $G_L(\omega) \rightarrow$ allows high freq

blocks low freq.







__Applications__ :

* Filters. (Speakers (Allows low freq, blocks high freq) (Strong bass & Low treble).

Tweeter Speaker (Block low ω, allows high.)

__V_R:__ $R = 510\,\Omega$, $L = 47mH$, $C = 0.33\mu F$

$R = 1000\,\Omega$, $L = 47mH$, $C = 0.33\mu F$

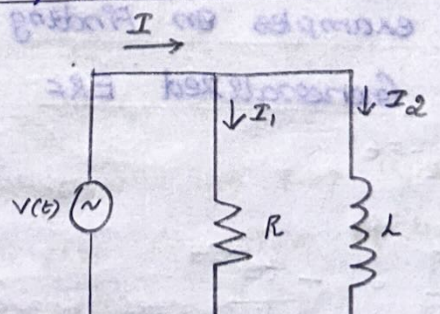
$R = 1000\,\Omega$, $L = 47mH$, $C = 0.033\mu F$

__V_C:__ $R = 510\,\Omega$, $L = 47mH$, $C = 0.033\mu F$

$R = 10000\,\Omega$, $L = 47mH$, $C = 0.033\mu F$

__V_L:__ $R = 510\,\Omega$, $L = 47mH$, $C = 0.033\mu F$.

__A parallel RL circuit.__



Physics.

$V = RI_1$

$V = L\dot{I}_2$

$I = I_1 + I_2$

__Complexing__

$\tilde{V} = R\tilde{I}_1$

$\tilde{V} = L\dot{\tilde{I}}_2$

$\tilde{I} = \tilde{I}_1 + \tilde{I}_2$

Suppose

$\tilde{V} = e^{i\omega t}$

(In general, $\tilde{V} = \gamma e^{i\omega t}$)

$\gamma \in \mathbb{C}$

$\boxed{\tilde{I}_1 = \frac{\tilde{V}}{R}}$ $\Leftarrow$ $\tilde{I}_1 = \alpha_1 e^{i\omega t}$, $\tilde{I}_2 = \alpha_2 e^{i\omega t}$, $\tilde{I} = \beta e^{i\omega t}$

$\boxed{\alpha_1 = \frac{1}{R}}$ $\boxed{\alpha_2 = \frac{1}{L}}$ for some unknown complex numbers α_1, α_2 and β. To find $\alpha_1, \alpha_2, \beta$ substitute in to the three complex replacements

$$e^{i\omega t} = R\alpha_1 e^{i\omega t}$$
$$e^{i\omega t} = L\alpha_2 e^{i\omega t}\, i\omega$$
$$\beta e^{i\omega t} = \alpha_1 e^{i\omega t} + \alpha_2 e^{i\omega t}$$

Generalized & applied in $\tilde{V} = R\tilde{I}_1$

This simplifies to

$$1 = R\alpha_1$$
$$1 = L\alpha_2 i\omega$$
$$\beta = \alpha_1 + \alpha_2$$

So $\alpha_1 = 1/R$, $\alpha_2 = \frac{1}{Li\omega}$, $\beta = \frac{1}{R} + \frac{1}{Li\omega}$

The complex gain of I_1 relative to I is the complex constant.

$$G = \frac{\alpha_1}{\beta} = \frac{1/R}{\frac{1}{R} + \frac{1}{Li\omega}}$$

. The gain is $|G|$, and the phase lag is $-\arg G$.

Review: worked examples on finding solutions using Generalized ERF

1) $\ddot{x} + 8\dot{x} + 15x = e^{-5t}$

soln:

$$x_p = \frac{te^{-5t}}{p'(-5)} = \frac{-te^{-5t}}{2}$$

2) $\ddot{x} + 2\dot{x} + 2x = e^{-t}\cos t$

soln:

$$z_p = \frac{te^{(-1+i)t}}{p'(-1+i)} = \frac{te^{(-1+i)t}}{2i} = \frac{te^{-t}(\cos t + i\sin t)}{2i}$$

$$x_p = \operatorname{Re} z_p = \frac{te^{-t}\sin t}{2}$$

For e^{it} & e^{-it} → we can't have solution of the form Ae^{at} $(r = \pm i)$

$b=1, K=2$ $\Rightarrow$ $\ddot{x} + b\dot{x} + Ky = Ky + b\dot{y}$

<u>From mathlet</u>

$$\omega_r = 1.30$$

$$g_{max} = 1.78 \longrightarrow \text{From bode plot.}$$

Effect of damping on resonance

$m\ddot{x} + b\dot{x} + Kx = A\cos\omega t$

$m \neq 1$, Fix K, vary b

<u>solu</u>:

$K = 1$ (say)

As b ↑, resonance freq ↓ (maximum)

b ↓, resonance freq ↑ (maximum height ↑)

Changing b while fixing K constant, significantly changes the height of the peak, but not its location.

No damping, $b = 0$, $G(\omega) = \dfrac{K}{K - m\omega^2}$ $(b = 0)$

attains maximum when.

$$K - m\omega^2 = 0$$

$$\omega = \sqrt{\frac{K}{m}}$$

$K=2, b=0$ & $m=1$, $\omega_r = \sqrt{2}$. This approximates the value of ω_r when $b=1$ to within 10% accuracy.

Note: Using the mathlet, we can verify that in general, even with damping, the location of the resonance is still well approximated with

$$\omega_r = \sqrt{\frac{K}{m}}.$$

<u>Effect of spring</u>

$\ddot{x} + b\dot{y} + Kx = Ky + b\dot{y}$ $\quad y = \cos\omega t$

<u>solu</u>:

Fix b, vary K → Look at resonance.

It significantly ↑ the height of the resonant peak
It significantly ↑ the frequency at which resonance happens.

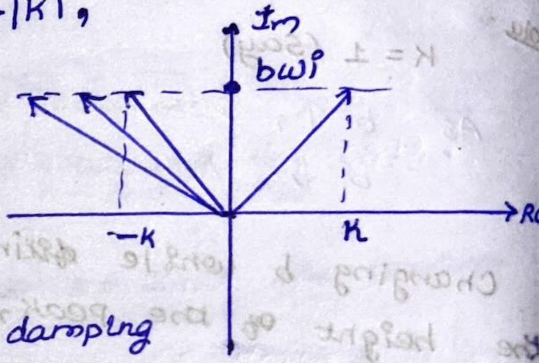
Minimum comfortable speed.

$K=2, b=1$ (Figure out minimum value of ω_{min} so that $g<1$.

ω min at when $g<0 = \omega = 2$.

$$G(\omega) = \frac{K + ib\omega}{(K-m\omega^2)+ib\omega}$$

The gain $|G|$ is less than 1 when the complex number in denominator has length strictly greater than the length of the complex number in the numerator.
This happens when $|K-m\omega^2| > |K|$,



$g<1$, when $\omega > \sqrt{\frac{2K}{m}}$.

$\therefore K=2, m=1, \omega_{min}=2$

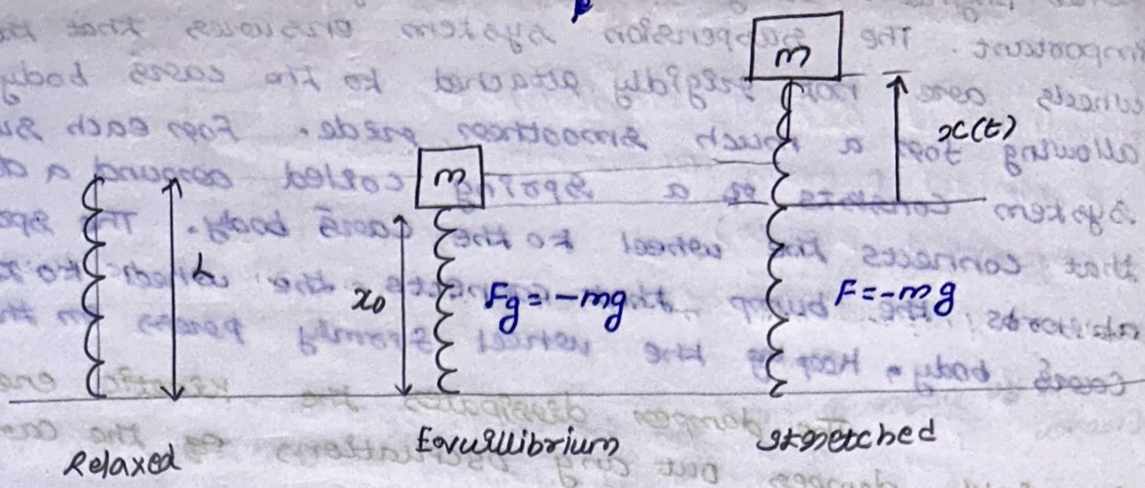
regardless of the values of the damping constant b.

When designing a suspension system, engineers typically fix the ω value that reflects the typical road conditions that the car will encounter. The engineers then choose the parameters K and b to ensure that the vehicle won't be in resonance with bumps in the road under normal conditions.

If $\omega = 2$, $K=4, b=1$ would be bad choice.
(since the car would be in resonance under these conditions)

ω_{min} → happens when both the input & response intersects eachother at the same x.
(Frequency of response is same as input) → No matter the phase lag.

Vertical mass system



Solu:

Relaxed $(x=0)$

$$\boxed{\begin{aligned} F_s &= K(0) \\ &= 0 \end{aligned}} \rightarrow \text{No effect of gravity.}$$

At equilibrium: ($F_s = F_g$ Balance each other).
$(a=0)$

$$0 = -K(x_0 - L) - mg$$

$$mg = -K(x_0 - L)$$

[since against gravity
↳ compressed]

At stretched:

$$m\ddot{x} = (-K(x_0 + x - L)) - mg$$

$$m\ddot{x} = -K(x_0 + x - L) + K(x_0 - L)$$

$$\boxed{m\ddot{x} = -Kx}$$ (or) simply

($x \rightarrow$ movement of the spring from equilibrium position)

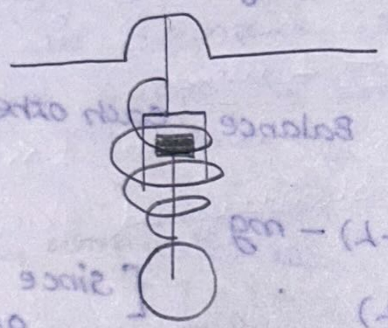
Same DE that describes a spring mass system with no gravity.

As long as we let $x(t)$ be the displacement from the equilibrium position, where the force of gravity & the spring force balance each other, the effect of gravity won't show up explicitly further computations.

when Engineers design a car wheel system — comfort is important. The suspension system ensures that the car's wheels are not rigidly attached to the car's body allowing for a much smoother ride. For each suspension system consists of a spring coiled around a damper that connects the wheel to the car's body. The spring absorbs the bump that connects the wheel to the car's body, keeping the wheel firmly press on the road.

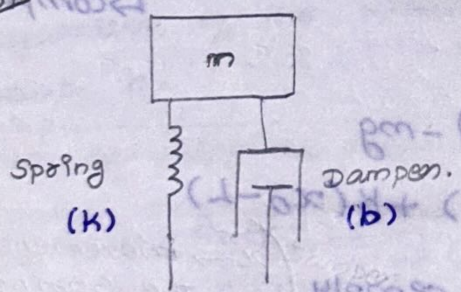
The damper dissipates the kinetic energy & quickly dampes out any oscillations of the car's body.

Modelling suspension system



→ only consider vertical motion of the car's body in wheels.

(single wheel)



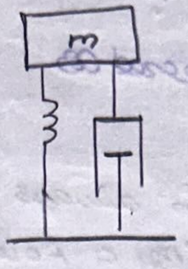
Assume: The forces produced by the spring & the damper can be always modeled using linear theory.
↓
valid as long as the deformation or as long as the deformation Compression & Elongations of the spring are small and as long as the Compression & Elongations the spring are small. (As long as the relative velocity b/w the car's body & wheel is small.

(As wheel is pressurized → when car goes over bumps it will deform) → wheel itself will act as a spring and dissipate Energy.

↳ In our model (negligible compared to spring & damper))

Assume: wheel is always in contact with road.

so, we can attach the spring & damper directly to the road,



Input & output

Input = height (directly beneath the surface of the wheel)

$y(t) \rightarrow$ Input

output (vertical position of the wheel)

The system will be in equilibrium!

when the spring is compressed slightly, upward spring force acting on the mass balances the force of gravity pulling the mass downwards.

↳ $x-y=0$ [In vertical equilibrium)

$x-y \rightarrow$ Represents the amount by which the spring has compressed or stretched away.

Equation of motion

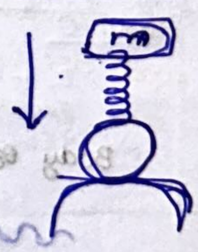
$m\ddot{x} = \Sigma F$ (vertical forces) → 1) Daming 2) Spring

$m\ddot{x} = -k(x-y) - b(x-y)$

↳ Downward motion

$\boxed{x<0}$

Same for damper



(In the direction of gravity)

[As effect of gravity is cancelled out by other parameters]

Standard form:

$$m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky$$

↳ we can see that the body of the car acts like a damped harmonic oscillator.

consider $y(t) = A\cos\omega t$

(sinusoidal pattern of bumps)

Gravel roads actually do develop a sinusoidal pattern of bumps (Referred as washboard roads)

Arbitrary periodic function $y(t)$ can always be decomposed into a sum of sines & cosines by using a Fourier series.

'when a damped harmonic oscillator is driven by a periodic driving force, the steady state response will also be periodic'

Understanding car's suspension requires the gain.

Gain 1 $\rightarrow$ means effective. (Absorbs bumps)

$g \gg 1$ means $\rightarrow$ It amplifies the effect

(phase can't be understood by a passenger)
$\hookrightarrow$ So neglected.

$$m\ddot{x} + b\dot{x} + Kx = b\dot{y} + Ky \qquad y = A\cos\omega t$$

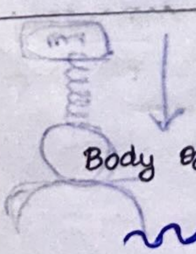
$$P(D)x = Q(D)y$$

$$Q(D) = bD + K$$

$$P(D) = mD^2 + bD + K$$

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)}$$

$$= \frac{K + ib\omega}{K - m\omega^2 + ib\omega}.$$



Drive fast or slow on a washboard road

Body of the car $\rightarrow$ ~

~ $\rightarrow$ Surface of the road.

The equ of motion is equivalent to the equ of motion we derived in the previous video. Recall y is the sinusoidal driving function resulting from the car moving on the bumps,

$x \rightarrow$ vertical position of the car.

Effect of driving slow:

$$\ddot{x} + b\dot{x} + kx = ky + b\dot{y} \quad (y = \cos\omega t)$$

$\omega \to 1, g(\omega) \to 1$ (Regardless of b & k)

$$G(\omega) = \frac{k + ib\omega}{k - m\omega^2 + ib\omega} \qquad \text{As } \lim_{\omega \to 0} G(\omega) = \frac{k}{k} = 1$$

The suspension system acts as if it were rigid & the car just moves up & down in the same way as the road beneath it. The limit of the complex gain implies not only that $\lim_{\omega \to 0} g(\omega) = 1$, but also that the input & response function are "in phase", and so are almost perfectly superimposed when $\omega \to 0$. This means that if you drive very slowly over a series of bumps, the suspension system acts as if it were rigid & the car just moves up and down in the same way as the road beneath it.

Driving very fast

As $\omega \to \infty$, $g \to 0$ This means that if you drive very quickly over a series of bumps, even though the wheels will be bouncing up & down very quickly, the body of the car will only have a small oscillation amplitude.

$P(r) = r^2 + 4$ ($r = \pm 2i$ → Fails to have solution of the form Ae^{at} if and only if a is root of the char polynomial. The problematic inputs in the list are e^{2it} & e^{-2it}.

The char polynomial $r^n - 1 \Rightarrow 1, e^{2\pi i/n}, e^{4\pi i/n}, \dots e^{(n-1)2\pi i/n}$

$\cos 2t$ ($\ddot{x} + 4x = r(t)$) → $\cos \omega t$

$P(r) = r^2 + 4 \Rightarrow A\cos(2t - \phi)$ would be solution to $\ddot{x} + 4x = 0$ instead of $\ddot{x} + 4x = r(t)$.

$\cos 2t$ → can't be written in $A\cos(2t - \phi)$

$\ddot{x} + b\dot{x} + 2x = \cos t$ (b starts at 1 and ↑)

Soln: Phase lag increases $[P(i) = i^2 + bi + 2 = 1 + bi]$ As $b \uparrow$, arg $P(i)$ also ↑.

Amplitude $= \dfrac{1}{|P(i)|} = \dfrac{1}{r^2 + br + 2}$ As $b \uparrow$, $A \downarrow$.

Changes sign infinitely → $\ddot{y} + b\dot{y} + Ky = 0$ $(t \to \infty)$ (what's true about m, K, b)

$b < 0$, $K > 0$, $b^2 - 4K < 0$ ∴ $r^2 + br + K$ has complex conjugate roots with real part. This is only possible when $b^2 - 4K < 0$ and $b < 0$

If $K < 0$ or zero $b^2 - 4K$ could possibly zero (∴ $K > 0$) must

---

$y''' + 5y'' + 8y' + 6y = f(x)$ → For which $f(x)$ the response resonant?

$e^{-x}\cos x$, $e^{-x}\sin x$, e^{-3x} give rise to resonant system response

$r^3 + 5r^2 + 8r + 6 = (r+3)(r+1-i)(r+1+i)$

$P(r) = 0$ for $r = -3, r = -1, r = -1 \pm i$

---

method of undetermined coefficients

Consider

$P(D)x = F(t)$ when $P(D) = D^2 + 2D + 2$.

Assume

$r^2 + 2r + 2 = 0$ → Not needed.

$$\frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \sqrt{-1}$$

$$= -1 \pm i$$

∴

For some classes of functions F, we can guess an expression for a particular solution. For example, if $F = \sin t$, since the derivatives of $\sin t$ are always either $\pm\cos t$ or $\pm\sin t$, It's reasonable to guess

$$x_p(t) = A\sin t + B\cos t$$

~~$P(D)x_p = 1$~~

$$P(D)x_p = (D^2+2D+2)(A\sin t + B\cos t)$$

$$= \sin t\,(A-2B) + \cos t\,(2A+B)$$

Setting,

$$P(D)x_p = F$$

$$\sin t\,(A-2B) + \cos t\,(2A+B) = \sin t$$

$$A-2B = 1 \quad (\because \text{coeff of } \sin t \text{ is } 1 \text{ in R.H.S})$$

$$2A+B = 0$$

$$\therefore x_p = \frac{\sin t}{5} - \frac{2\cos t}{5}.$$

Solving. $A = 1/5$, $B = -2/5$

When $F = \cos t$

$$\sin t\,(A-2B) + \cos t\,(2A+B) = \cos t$$

① $\rightarrow$ $A-2B=0$

② $\rightarrow$ $2A+B=1$

mul ① by 2 $\rightarrow$ $2A-4B=0$

$2A+B=1$

$-5B=-1$

$B = 1/5$

$A = \frac{2}{5}$

$$\frac{2\sin t}{5} + \frac{\cos t}{5},$$

$F = \sin 2t$ (Assume $A\sin 2t + B\cos 2t = x_p$)

$$P(D)x_p = -4A\sin 2t + 4A\cos 2t + 2A\sin 2t - 4B\cos 2t - 8B\sin 2t + 2B\cos t$$

$$= \sin 2t\,(-4A+2A-4B) + \cos 2t\,(4A-4B+2B)$$

$$= \sin 2t\,(-2A-4B) + \cos 2t\,(4A-2B)$$

$-2A-4B = 1$

$8A-4B=0.$

$-10A = 1$

$A = -1/10$

$-4B = 1 - \frac{2}{10}$

$B = \frac{8}{5\times 4}$

$B = -\frac{1}{5}$

$$\Rightarrow -\frac{\sin 2t}{10} - \frac{\cos 2t}{5}$$

$F(t) = t^2$, $x_p(t) = At^2 + Bt + C$

$P(D)\, x_p = (D^2 + 2D + 2)(At^2 + Bt + C) = F = t^2$

$= 2At^2 + (4A + 2B)t + (2A + 2B + 2C)$

$2A = 1$, $A = 1/2$ | $4A + 2B = 0$, $2A + 2B + 2C = 0$. $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$.

$x_p(t) = \frac{t^2}{2} - t - \frac{1}{2}$.

Linear time invariant operators

LTI:

Consider a homogeneous diff equ

$$Lx = 0$$

for an LTI operator L.

As an example, define a new operator T_h — shift operator

$$T_h\, x(t) = x(t+h)$$

An LTI operator can be then be defined as a linear operator L such that

$$LT_h = T_h L$$

$\therefore L T_h x = T_h L x$ for all functions x.

$T_h \rightarrow$ Itself is an LTI operator. [Not in the form $P(D)$ for some polynomial P with constant coeff

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$x(t) = \cos t$ is a solution to the equ $Lx = 0$, then which of the following functions are also solutions?

Solu: $-\sin t$, $\sin(t + \pi/15)$, $3\sin t - 5\cos(t+1)$

$$LD = DL \qquad (LT_h = T_h L)$$

Solu:

$$Dx(t) = \lim_{h \to 0} \frac{T_h x - x}{h}$$

To see $LD = DL$ for an LTI operator LTI,

$$DLx = \lim_{h \to 0} \frac{T_h Lx - Lx}{h}$$

$$= \lim_{h \to 0} \frac{L(T_h x - x)}{h}$$

$$= \lim_{h \to 0} L \frac{T_h x - x}{h}$$ $\therefore$ provided the extra condition

that is, if (L is compatible with limits)

$$\lim_{h \to 0} L \frac{T_h x - x}{h} = L \lim_{h \to 0} \frac{T_h x - x}{h}$$

then $\boxed{DL = LD}$

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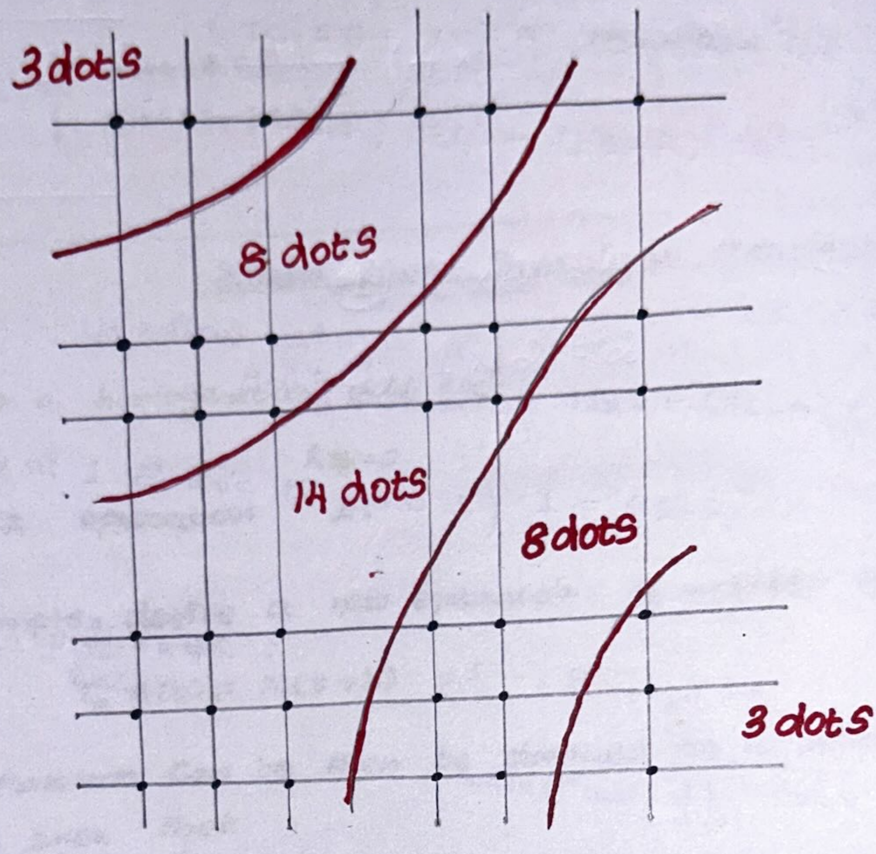
1) If $x(t)$ is a solution, then $Lx = 0$, since $LD = DL$
then $Lx' = LDx = DLx = D(0) = 0$. This means $x'(t)$ is also a solution.

2) If $x(t) = t^2 e^t$ is a solution, then $x(t+1) = (t+1)^2 e^{t+1}$ is also a solution. secondly $x(t) \to$ being a solution implies $x'(t) = 2te^t + t^2 e^t$ is a solution, Hence the linear combination $x'(t) - x(t) = te^t$ is also a solution. meaning & in turn implies

$(te^t)' = e^t + te^t \to$ Solution

$(te^t)' - te^t = e^t$ (is a solution).

$123 \times 321 = 39483$



Rule: If the number of dots is greater than 9 in a particular region, add the tenth value of that number with the previous one.

Here, the third position has 14 dots.

so → 8 is the previous region's dot count.

$8+1 = 9$

Answer: 39483.

By $||^{n}$ arguments, if $t^n e^t$ is a solution to $Lx = 0$, then $t^m e^t$ for any integer $0 \le m < n$ is also a number.

AM radio (RLC) $v = \sin\omega t$

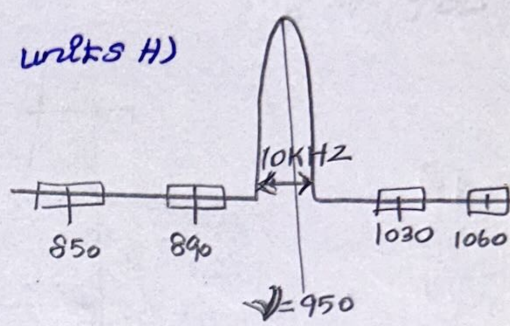
V_R – system response

you would like to know the relations b/w R, L, C in order to tune a station at $\nu = 950$ KHz. and with good enough reception

By good enough reception, you mean the gain V_R should drop to a fraction (say $\frac{1}{10}$), of the maximum gain at the two boundary frequencies ν_1 KHz, ν_2 KHz. The Band width of the AM radio station is about 10KHz.

solu:

Find c (μF) in terms of L (with units H) and $\nu = 950{,}000$ KHz. ($\omega \to$ rad/s). equivalent of $\left(\frac{1}{2\pi} \text{ Hz}\right)$



solu:

$C = ?$

$\omega_r = \frac{1}{\sqrt{LC}}$, $g = 1(\omega)$ at freq ν

$$G_R(\omega) = \frac{q(i\omega)}{p(i\omega)} = \frac{iR\omega}{\left(\frac{1}{c} - L\omega^2\right) + iR\omega}$$

$$\omega_r \text{ (rad/sec)} = \frac{1}{\sqrt{LC}} \Rightarrow c = \frac{1}{L(2\pi\nu)^2} = \frac{1}{L(2\pi 950)^2 (10)^6}$$

$$= \frac{1}{L(2\pi 950)^2} \; \mu F$$

Secondly, at the two boundary ν ν_1, ν_2 drops to (gain) $\frac{1}{10}$ of maximum.

since max gain is 1, Let $g = \frac{1}{10}$ (at boundaries)

$$g^2(\omega) = \frac{R^2\omega^2}{\left(\frac{1}{c} - L\omega^2\right)^2 + R^2\omega^2} = \frac{1}{100}$$

$$100R^2\omega^2 = \left(\frac{1}{c} - L\omega^2\right)^2 + R^2\omega^2$$

$$99R^2\omega^2 = \left(\frac{1}{c} - L\omega^2\right)^2$$

$$\sqrt{99}R\omega = \pm\left(\frac{1}{c} - L\omega^2\right)$$

$$0 = L\omega^2 \pm \sqrt{99}R\omega - \frac{1}{c}$$

$$\omega = \frac{-\sqrt{99}R \pm \sqrt{99R^2 + \frac{4L}{c}}}{2L} \quad \text{(or)} \quad \omega = \frac{\sqrt{99}R \pm \sqrt{99R^2 + \frac{4L}{c}}}{2L}$$

$$
\begin{array}{r}
12 \\
\times\ 12 \\
\hline
24 \\
120 \\
\hline
144
\end{array}
$$

$24 \rightarrow (4\times10^0 + 2\times10)$

① $2\times2=4$

② $1\times2=2$

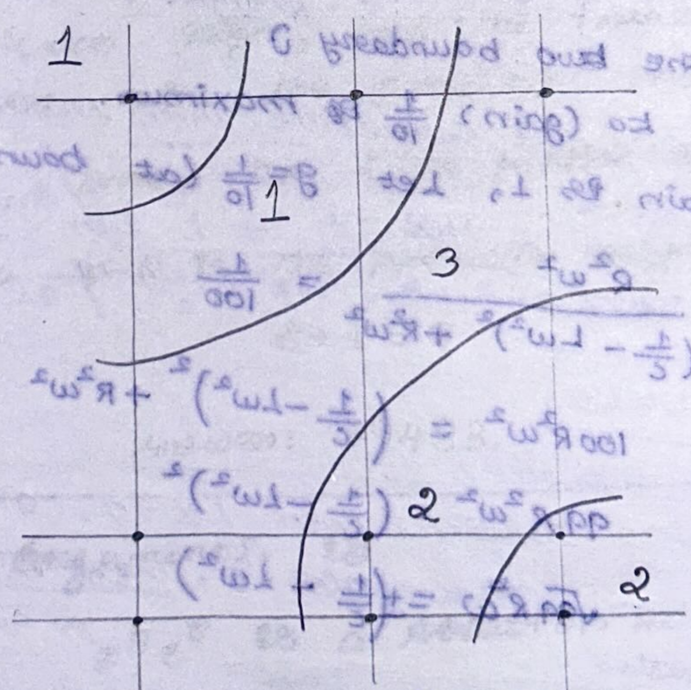
$0 \rightarrow$ placeholder

③ 2×1

④ 1×1

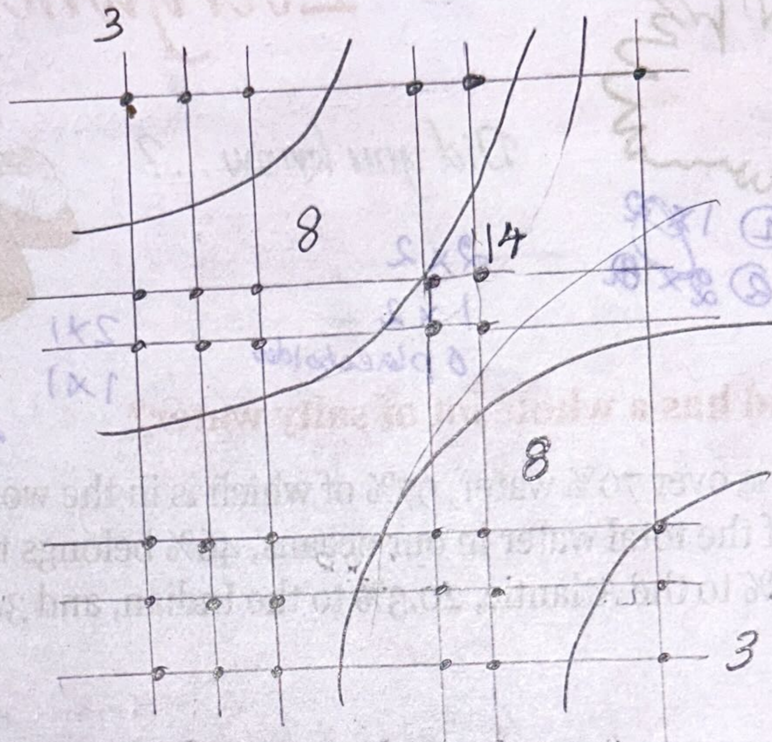
⑤ Add $\rightarrow$ 144

$102 \times 111 = 11988$ 11322



1 2 3 × 3 2 1

3 9 4 8 3



$$\omega_1 = \frac{-\sqrt{99}\,R + \sqrt{99R^2 + \frac{4L}{C}}}{2L}$$

$$\omega_2 = \frac{\sqrt{99}\,R + \sqrt{99R^2 + \frac{4L}{C}}}{2L}$$

Finally

$$\frac{|\omega_1 - \omega_2|}{2\pi} = 10000\ Hz$$

$$\frac{\sqrt{99}\,R}{L} = 10000\ Hz \Rightarrow R = \frac{20000\pi L}{\sqrt{99}}\ \Omega$$

$$= \frac{20\pi L}{\sqrt{99}}\ k\Omega$$