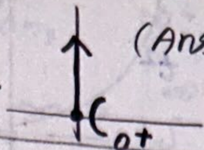
c) $\int_{0^+}^{\infty} \delta(t)\, e^{t^2}\, dt = 0$ (Answer)



$0^+ \to$ open integral.

d) $\int_{0^-}^{\infty} \delta(t-2)\, e^{t^2 \sin t \cos 2t}$

$e^{4 \sin 2 \cos 4}$

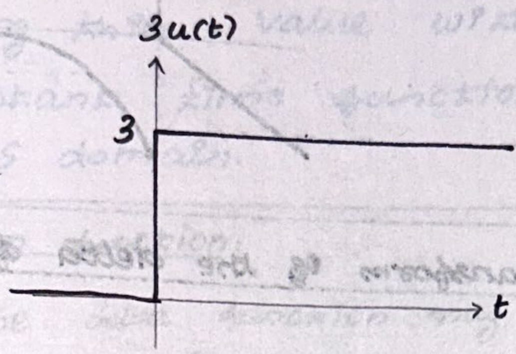
Find the generalized derivatives of

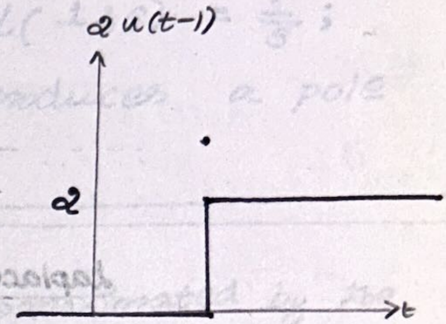
a) $f(t) = 3u(t) - 2u(t-1)$

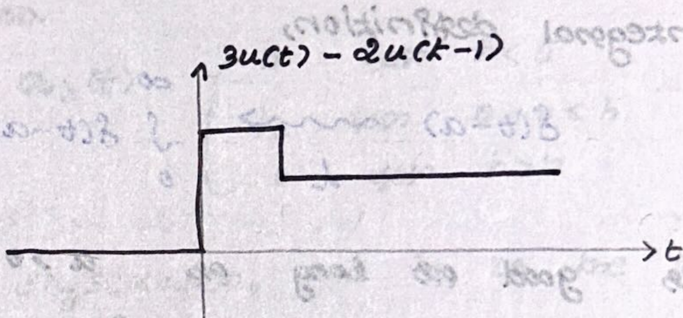
b) $f(t) = \begin{cases} t^2 & t < 0 \\ e^{-t} & t > 0 \end{cases}$

Solu:

a)







$f'(t) = 0 + 0 + 3\delta(t) - 2\delta(t-1)$

(continuity)

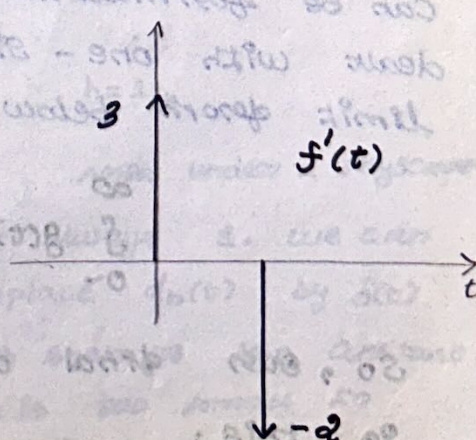
∴ Continuous parts:

1) $-\infty$ to 0
   0 to ∞

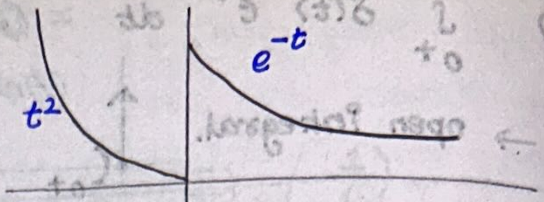
2) $-\infty$ to 1
   1 to ∞

Discontinuity at

1) $t = 0$ &
2) $t = 1$



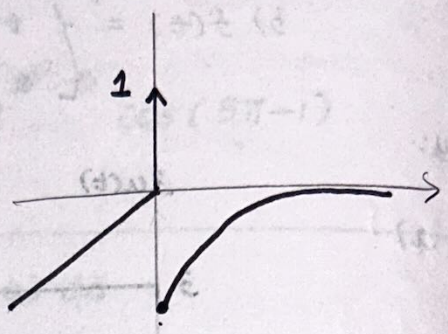
b) $f'(t) = \begin{cases} 2t & t < 0 \\ -e^{-t} & t > 0 \end{cases}$ +



$\delta(t) \rightarrow$ Discontinuous at $t = 0$.

$\therefore \boxed{e^0 = 1}$

$u'(t) = \hookrightarrow \delta(t)$



Laplace transform of the delta function

From integral definition,

$$\delta(t-a) \xrightarrow{\mathcal{L}} \int_0^\infty \delta(t-a)\, e^{-st}\, dt = e^{-as}$$

This is good as long as $a > 0$. To find the Laplace transform of $\delta(t)$ itself, though, we can't make sense of the lower limit - it asks us to evaluate $\delta(t)$ at $t = 0$.

To accomodate this we will make a slight adjustment in our integral definition of the Laplace transform. we want to include the whole of the delta function, so we want to start the integral at a point somewhat to the left of $t = 0$. This can be formalized in just the same way as we dealt with one-sided values of functions: take a limit from below: say

$$\int_{0-}^\infty g(t)\,dt = \lim_{a \uparrow 0} \int_a^\infty g(t)\,dt$$

So, our final definition of the Laplace transform is this:

$$\mathcal{L}(f(t);s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt.$$

with this definition,

$$\delta(t) \rightsquigarrow \int_{0^-}^{\infty} \delta(t)\, e^{-st}\, dt = \lim_{a \uparrow 0} e^{-sa}$$

$$= 1.$$

The Laplace transform of the Dirac delta function in the frequency domain with value 1. comparing this value with $\mathcal{L}(1;s) = \frac{1}{s}$; the constant time function produces a pole in the s domain.

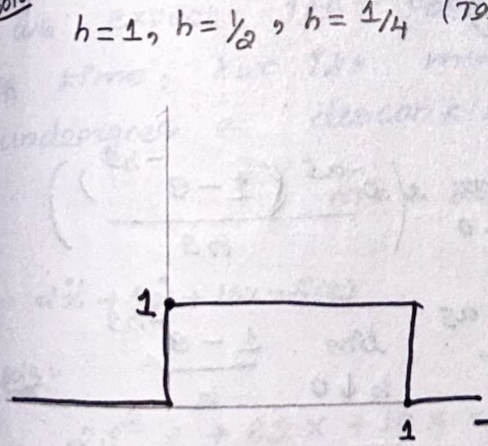
Skyscraper function:

The delta function may be approximated by the skyscraper function.

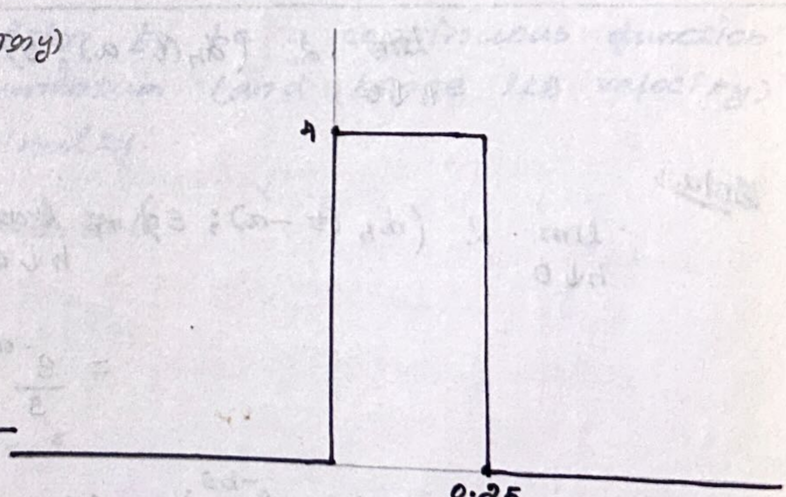
$$d_h(t) = \begin{cases} 0 & \text{for } t<0 \\ 1/h & \text{for } 0<t<h \\ 0 & \text{for } t>h \end{cases}$$

h - width of the skyscraper, and $1/h$ is the height.

Solu: $h=1$, $h=1/2$, $h=1/4$ (Try)

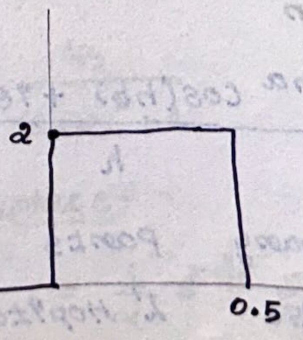


$h=1$



$h=1/4$

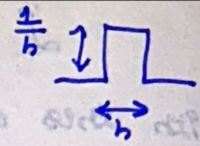
It's actually not a very good model for $\delta(t)$, since it's not smooth, but it will serve for most purposes.

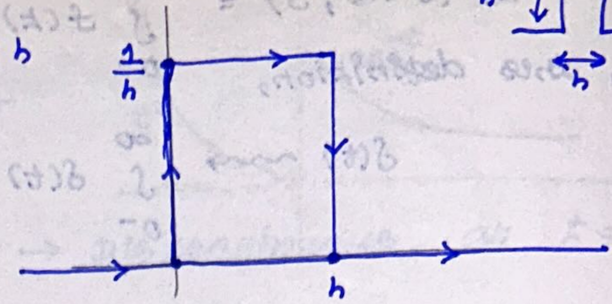


$h=1/2$

Area under a skyscraper is always 1. we can replace $d_h(t)$ by $\delta(t)$ as soon as the constant h is too small to resolve.

$$d_h(t) = \begin{cases} 0 & t < 0 \\ 1/h & 0 < t < h \\ 0 & t > h \end{cases}$$





$$= \frac{1}{h} u(t) - \frac{1}{h} u(t-h)$$

$$= \frac{u(t) - u(t-h)}{h}$$

Find the Laplace $\mathcal{L}(d_h(t); s)$ and $\mathcal{L}(d_h(t-a); s)$

solu:

i) $d_h(t) = \dfrac{u(t) - u(t-h)}{h} \overset{\mathcal{L}}{\leadsto} \dfrac{1}{h}\left(\dfrac{1}{s} - e^{-hs} \cdot \dfrac{1}{s}\right)$

↳ t-shift II form

$$= \frac{1 - e^{-hs}}{hs}$$

$$d_h(t-a) \leadsto e^{-as}\left(\frac{1 - e^{-hs}}{hs}\right)$$

↳ Again using t-shift rule I form.

$$\lim_{h \downarrow 0} \mathcal{L}(d_h(t-a); s)$$

solu:

$$\lim_{h \downarrow 0} \mathcal{L}(d_h(t-a); s) = \lim_{h \downarrow 0} \left(e^{-as}\frac{(1 - e^{-hs})}{hs}\right)$$

$$= \frac{e^{-as}}{s} \boxed{\lim_{h \downarrow 0} \frac{1 - e^{-hs}}{h}}$$

Computing $\displaystyle\lim_{h \downarrow 0}\left(\frac{1 - e^{-hs}}{h}\right)$. (Remember, s is a complex number. $s = a + bi$. then

$$\frac{1 - e^{-ha}\cos(hb) + ie^{-ha}\sin(hb)}{h}$$

Both real & imaginary parts tend to $\frac{0}{0}$ as h tends to 0. Use L' Hopitals method, (The linear part of the Taylor simply the definition of derivative,

$$\lim_{h\downarrow 0}\left(\frac{1-e^{-hs}}{h}\right) = \lim_{h\downarrow 0}\left(+\frac{s\,e^{-hs}}{1}\right)$$

$$= s\,e^{0}$$

$$= s = a+bi.$$

$\therefore$

$$\lim_{h\downarrow 0} \mathcal{L}\left(d_h(t-a); s\right) = \frac{e^{-as}}{s}\ (a+bi)$$

$$= e^{-as}$$

This agrees with the integral definition of $\mathcal{L}(\delta(t-a); s)$.

Impulse response

Response to a unit impulse, that is $\delta(t)$ as input signal. The example to keep in mind is a spring system, in which the mass is struck hard.

$$m\ddot{x} + b\dot{x} + kx = P\,\delta(t)$$

we expect its position to be a continuous function of time, but its momentum (and hence its velocity) undergoes a discontinuity.

$$m=2,\ b=8,\ k=10$$

$$2\ddot{x} + 8\dot{x} + 10x = \delta(t)$$

solu:-

$$2s^2X + 8sX + 10X = 1.$$

$$X = \frac{1}{2s^2+8s+10}$$

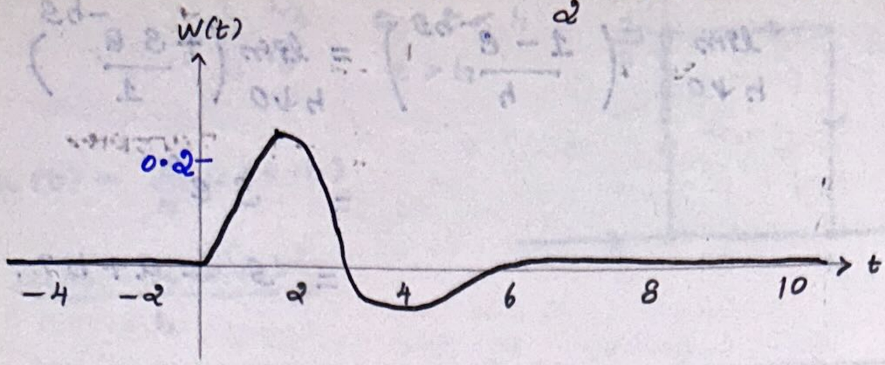
$$= \frac{1/2}{(s+2)^2+1}$$

Taking inverse laplace:

$$= \frac{1}{2}e^{-2t}\sin t.$$

... impulse response should vanish for $t<0$, so it's

$$w(t) = \frac{1}{2} u(t) e^{-2t} \sin t.$$



(Typical unit impulse response of an underdamped second order system.)

Stopping a pendulum with delta input

$y'' + y = \delta(t - \pi/2) \cdot A$ (Translated)

(spring kicked with impulse A. at $t = \pi/2$)

$y(0) = 1, \; y'(0) = 0.$

solu: Kicked in → Delivering a large impulse over an extremely

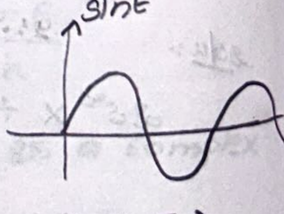
$$y'' + y = A\,\delta(t - \pi/2)$$

$$s^2 Y(s) - 1(s) - 0 + Y = A \cdot e^{-\frac{\pi}{2}s} \quad (1)$$

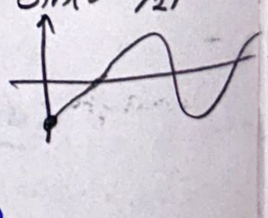
$$s^2 Y(s) - s + Y = A e^{-\frac{\pi}{2}s}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{A e^{-\frac{\pi}{2}s}}{s^2+1}.$$

Taking inverse Laplace:



$$y(t) = \cos t + A \, \mathcal{L}^{-1}\left(\frac{e^{-\pi/2}}{s^2+1}\right)$$



$$y = \cos t + A \sin(t - \pi/2) \cdot u(t - \pi/2)$$

$$y = \cos t + A \sin(t - \pi/2) \cdot u(t - \pi/2).$$

$$y = \begin{cases} \cos t & 0 \le t \le \pi/2 \\ (1-A)\cos t & t \ge \pi/2. \end{cases}$$

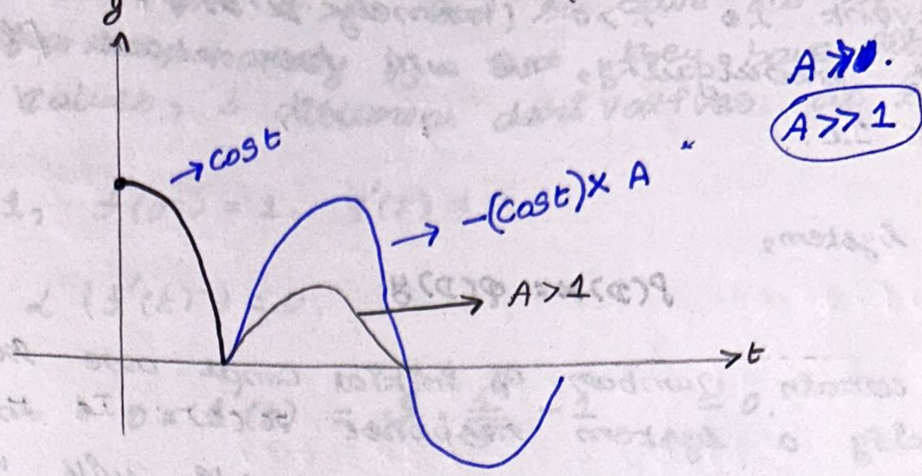
$\sin(t - \pi/2) = -\cos t$

At $t = \pi/2$

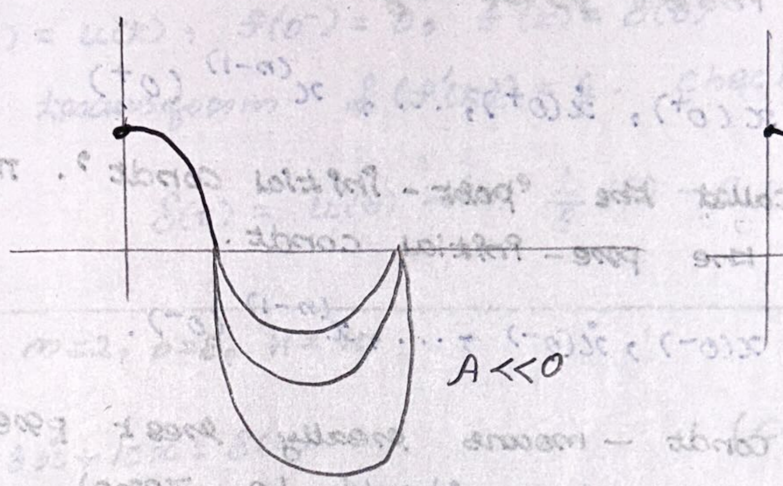
$\cos \pi/2 = 0$

① $\Rightarrow$ $\cos \pi/2 = 0$
② $\Rightarrow$ $(1-A) \cos \pi/2 = 0$ } working nice.

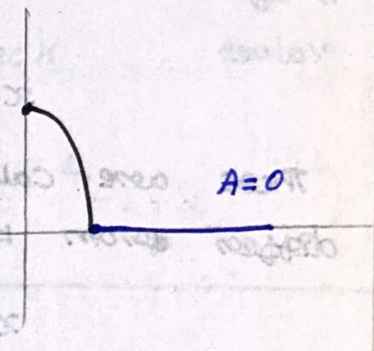
(Don't jump)



when $A < 0$



when $A = 0$



eg: pendulum (undamped) → Hitting it with a baseball bat at $x = 0$ (equilibrium position).

Pre- and post initial conditions

spring system in which the mass is struck hard,

$$m\ddot{x} + b\dot{x} + Kx = p\,\delta(t)$$

we expect this to be a continuous function of time, but its momentum (and hence its velocity) undergoes a discontinuity.

When the delta function is thought of as an input function it's often called a "unit impulse". The term comes from this physical example:

In physics, a impulse is a time integral of force, or a change in momentum.

one of the initial conditions for a second order case like this is the velocity $\dot{x}(0)$. But $\dot{x}$ has a discontinuity at $t=0$! which value shall we take, the one relevant to $t<0$ (namely $\dot{x}(0)=0$) or the one relevant to $t>0$ (namely $\dot{x}(0)=P/m$)? To clarify this ambiguity, we will talk about pre & post IC.

LTI System,

$$p(D)x = q(D)y$$

a certain number of initial condt are required to specify a system response $x(t)$. If the operator $p(D)$ is of order exactly n, we will need the n of them. more precisely we will need the values

$$x(0^+),\ \dot{x}(0^+),\ \ldots,\ x^{(n-1)}(0^+)$$

These are called the 'post-initial condt'. They may differ from the pre-initial condt.

$$x(0^-),\ \dot{x}(0^-),\ \ldots,\ x^{(n-1)}(0^-).$$

'Rest initial condt - means really rest pre-initial condt; all these values should be zero).

The pre initial condt of cos t are gn by $\cos(0^-)=1$, $\cos'(0^-)=0, \ldots$ since $\cos(t)$ is continuous, the post initial condt are the same. On the other hand, the pre-initial condt of $u(t)\cos(t)$ are all zero, while the post-initial condt are $\cos(0^+)=1$, $\cos'(0^+)=0$, $\cos''(0^+)=-1\ldots$

This plays into the t-derivative rule for the Laplace transform. Since we have refined the definition by using 0^- as lower limit, the proper form of the t-derivative rule is this:

<u>t-derivative rule:</u>

$$f'(t) \rightsquigarrow sF(s) - f(0^-)$$

$\parallel^{ly}$, all the derivatives occuring in the rule for the

Laplace transform for higher derivatives of $f(t)$ are to be evaluated at 0^-.

9.2: Refined rule: $f(t) = 1$, $f(t) = u(t)$. These two functions coincide for $t > 0$. and so have the same Laplace transforms, namely $1/s$. But they have different pre-initial values, & different derivatives at $t = 0$.

Solu: $f(t) = 1$, $f(0^-) = 1$, $f'(t) = 0$

$$\mathcal{L}(f'(t)) = 0.$$

$$0 = 1'(t) = s \cdot \frac{1}{s} - 1 \quad = 0.$$

with

$f(t) = u(t)$, $f(0^-) = 0$, $f'(t) = \delta(t)$ has the Laplace transform $\mathcal{L}(f'(t)) = 1$. check

$$\delta(t) = u'(t) \rightsquigarrow \frac{1}{s}(s) - 0 = 1.$$

9.3 $m = 2$, $b = 8$, $k = 10$

$$2\ddot{x} + 8\dot{x} + 10x = \delta(t)$$

(char polynomial Solution)

Solu:

we got the answer as $\boxed{e^{-2t}}$ $(a\cos t + b\sin t)$

$\therefore$ No force acting for $t > 0$, the solution is a transient for the system. The completion of the square we did shows that the roots of the char polynomial ~~are~~ $-2 \pm i$. The general transient is

$$e^{-2t}(a\cos t + b\sin t).$$

we need to figure out the post-initial condt. The unit impulse increases the momentum by one unit, and hence increases the velocity by $\frac{1}{2}$. so immediately after the kick, we have

$$x(0^+) = 0, \quad \dot{x}(0^+) = 1/2.$$

This implies $a = 0, b = 1/2$ → In Char polynomial solution.

$$\boxed{x(t) = \frac{1}{2} e^{-2t} \sin t}$$

↳ Same as Laplace transform

∴ Laplace cares about future.

Unit step & Impulse responses

I) Find unit impulse response of

a) $\dot{x} + 2x = f(t)$

b) $2\ddot{x} + 7\dot{x} + 3x = f(t)$

II) Find unit step response of I) a

a) $\dot{x} + 2x = f(t)$

Solu: $x(0^-) = 0$

$\dot{x} + 2x = \delta(t)$

$(s+2)X = 1$

$X = \frac{1}{s+2}$

Taking inverse laplace:

$$\boxed{x(t) = e^{-2t}}$$

quantity of radioactive chemical in a tank,

$\delta(t)$ → that we are given a high disturbance on a very short time on this system.

eg: Dumping a huge amount of chemical in that tank, and then just letting the system evolve after dumping all that chemical in a tank.

$x(0^-) = 0 \longrightarrow x(0^-) = 1.$

(Jump)

$\cancel{(s+2)x = 0}$

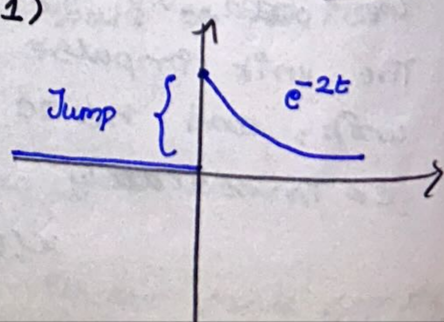
$s+2=0$

$\boxed{s=-2}$

$\dot{x} + 2x = 0 \quad (x(0^-) = 1)$

$x(t) = e^{-2t} \cdot u(t) \quad (t>0)$

$t \in R$



b) $2\ddot{x} + 7\dot{x} + 3x = f(t)$

UIR:

$2\ddot{x} + 7\dot{x} + 3x = \delta(t)$ with rest I.C.

The impulse will effect on the second largest derivative (or) introduces discontinuity on that part by inverse of the highest derivatives coefficient.

$\delta(t)$ introduces discontinuity on $\dot{x}$:

Just from $\dot{x}(0^-) = 0$ to $\dot{x}(0^-) = 1/2$.

$$2\ddot{x} + 7\dot{x} + 3x = 0 \qquad \begin{cases} x(0^-) = 0 \\ \dot{x}(0^-) = 1/2 \end{cases}$$

$$2r^2 + 7r + 3 = 0$$

$$r = \frac{-7 \pm \sqrt{49 - 4(2)(3)}}{4} = \frac{-7 \pm 5}{4}$$

$x(t) = c_1 e^{-3t} + c_2 e^{-1/2 t}$

$0 = c_1 e^{-3t} + c_2 e^{-1/2 t}$

$0 = c_1 + c_2$

$\dot{x}(t) = -3c_1 - \frac{1}{2} c_2$

$-1/2 = 3c_1 + \frac{1}{2} c_2$

$-1/2 = 3c_1 - \frac{1}{2} c_1$

$c_2 = \frac{1}{5} \left(c_1 = -\frac{1}{5}\right)$

$$\therefore\ x(t) = \frac{1}{5}\left(e^{-1/2 t} - e^{-3t}\right) \cdot u(t).$$

$t > 0$

$t \in R.$

Unit step response:

$\dot{x} + 2x = u(t)$

$\dot{x} + 2x = 1 \qquad t > 0$

$\dot{x} + 2x = 0$

$s + 2 = 0$

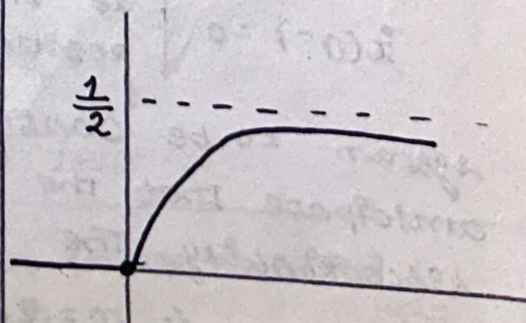
$\boxed{s = -2}$

$x(t) = c_1 e^{-2t} + \frac{1}{2}$
(P.I)

$\boxed{c_1 = -\frac{1}{2}}$

$x(t) = \frac{1}{2}(1 - e^{-2t})$

(From rest IC)



It's a smooth change. (transition) → I've started inputting the amount of the chemical in the tank, and in a non abrupt way.

$$x(0^-) = 0 \text{ then changes to } x(0^-) = \frac{1}{2}.$$

---

savings account with \$ 500 (at $t=0$). That day, you receive and immediately deposit \$ 1000. The savings account accures interest at a continuous rate of I. The money in your savings account can be modeled by

$$\dot{x} - Ix = 1000\, u(t)$$

soln:

u - unit step function.

In other words,

$$\dot{x} - Ix = 1000\,\delta(t)$$

Assuming that the response x is as smooth as possible

$$\boxed{\begin{aligned}(s-I)\,x &= 1000 \\ x &= \frac{1000}{s-I}\end{aligned}}$$

$x(0^-) = 500$

$x(0^+) = 1500$

pre initial condt $x(0^-) = 500$

post-initial condt, is the amount of money in the account right after you deposit the money, but before it starts accuring interest. In other words, $x(0^+) = 1500$.

$$2\ddot{x} + \dot{x} + Kx = 2\,\delta(t)$$

At rest before $t=0$

$x(0^-) = 0$

$\dot{x}(0^-) = 0$

Because we expect the solution x to be as smooth as possible, we expect the position of the mass in the spring system to be continuous $x(0^+) = 0$. However, we anticipate that the momentum experiences a jump discontinuity. The momentum $m\dot{x}(0^+) = 2$.

$\therefore m = 2,\ \dot{x}(0^+) = 1.$

$\therefore$ $2\delta t$ causes discontinuity by $\frac{1}{2}(2) = 1$.

$x(0^+) = 0$ to $\dot{x}(0^+) = 1$.

$$\dot{V} + KV = u(t) \qquad V(0^-) = 0$$

$$SV(S) + KV(S) = \frac{1}{S}$$

$$V(S) = \frac{1}{S(S+K)}$$

$$= \frac{a}{S} + \frac{b}{S+K}$$

$$= \frac{1}{KS} - \frac{1}{K(S+K)}$$

$$= \frac{1}{K}\left(1 - e^{-Kt}\right)$$

$$\frac{1}{S(S+K)} = \frac{a}{S} + \frac{b}{S+K}$$

$$a = \frac{1}{S+K}$$

$$a = \frac{1}{K},$$

$$b = \frac{1}{S}$$

$$= -\frac{1}{K}$$

$$V(\infty) = \lim_{S\to\infty} S\,V(S)$$

$$= \lim_{S\to\infty} S\left(\frac{1}{S(S+K)}\right)$$

$$= \lim_{S\to\infty} \frac{1}{S+K}$$

$$= 0.$$

$$\boxed{\text{As } \lim_{x\to\infty} \cancel{x}(t) = \frac{1}{K}(1) = \frac{1}{K}.}$$

Initial value

$$\lim_{S\to 0} \int_0^\infty f'(t)\,dt = \lim_{S\to 0} \left[SF(S) - f(0^-)\right]$$

NO S $\leftarrow$ $f(\infty) - f(0^-) = \lim_{S\to 0} SF(S) - f(0^-)$ $\rightarrow$ NO S

$$\boxed{f(\infty) = \lim_{S\to 0}\left[SF(S)\right]}$$

means

$$\lim_{t\to\infty} f(t)$$

$\rightarrow$ Refer Index

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