Euler's numbers.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

solu:

$$\ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right)$$

$$\Delta x = \frac{1}{n}$$

$$\therefore \boxed{\Delta x \to 0}$$

$$\therefore \ln\left(1+\frac{1}{n}\right)^n = \frac{1}{\Delta x}\ln(1+\Delta x)$$

$$\ln\left(1+\frac{1}{n}\right)^n = \frac{1}{\Delta x}\left[\ln(1+\Delta x) - \ln(1)\right]$$

∴ substracting 0 from it

∴ $\ln(1) = 0$.

$$\therefore \frac{d}{dx}(\ln(x))\Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1.$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\left[\lim_{n\to\infty}\ln\left[\left(1+\frac{1}{n}\right)^n\right]\right]}$$

$$\therefore e^{\ln x} = x$$

$$e \approx \left(1+\frac{1}{100}\right)^{100}$$ (numerical app)

$$\lim_{\Delta x\to 0}\ln\left(1+\frac{1}{n}\right)^n =$$

$$\lim_{\Delta x\to 0}\frac{\ln(1+\Delta x)-\ln(1)}{\Delta x}$$

$$= \frac{d}{dx}\ln x\Big|_{x=1}$$

$$= \frac{1}{x}\Big|_{x=1} = 1.$$

$$\therefore \lim_{n\to 0}\left(1+\frac{1}{n}\right)^n = e^1$$

$$= e$$

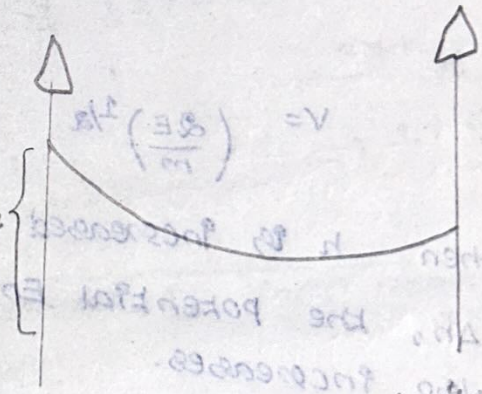
Approximations

eg: zipline (cable)

↳ small change in how dimension affect process

↳ understood by linear approximation.

zipline design



V - maximum velocity achieved during ride.

Conservation of Energy:

E = Potential Energy + Kinetic Energy

$$E = mgh + \frac{1}{2}mv^2$$

$$\boxed{E - \text{Constant}}$$

Now, we are ignoring the effects of heat, friction etc...

velocity will be maximum when

$$E = \frac{1}{2} m v^2$$

$$v = \sqrt{2E/m}$$

$$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

linear approximation

$(1+u)^r$ near $u = 0$

$$f(u) = f(0) + f'(0)(u-0)$$

$$f(u) = (1)^r + r(1+0)^{r-1}(u)$$

$$= 1^r + r(1)^{r-1} u$$

$$= 1 + ru \qquad \because \boxed{\begin{matrix} 1^r = 1 \\ 1^{r-1} = 1 \end{matrix}}$$

$$v = \left(\frac{2E}{m}\right)^{1/2}$$

when h is increased by Δh, the potential Energy also increases.

$$\therefore E \to E + mg\Delta h$$

$$v + \Delta v = \left(\frac{2(E + mg\Delta h)}{m}\right)^{1/2}$$

$$= \left(\frac{2E}{m} + 2g\Delta h\right)^{1/2}$$

$$= \left(\frac{2E}{m}\right)^{1/2} \left(1 + \frac{2g\Delta h}{\frac{2E}{m}}\right)^{1/2}$$

$$v + \Delta v = \underbrace{\left(\frac{2E}{m}\right)^{1/2}}_{\text{units of velocity}} \underbrace{\left(1 + \frac{mg}{E}\Delta h\right)^{1/2}}_{\text{unitless}}$$

$$(1+u)^r = 1 + ru$$

(near 0)

$$r = \frac{1}{2}, \quad u = \frac{mg}{E}\Delta h$$

$$v + \Delta v \approx \left(\frac{2E}{m}\right)^{1/2} \left(1 + \frac{mg}{2E}\Delta h\right)$$

$$v = \left(\frac{2E}{m}\right)^{1/2}$$

$$v + \Delta v \approx v\left(1 + \left(\frac{m}{2E}\right) g\Delta h\right)$$

$$v + \Delta v \approx v + v \cdot \left(\frac{m}{2E}\right) g \Delta h$$

caring about Δv alone:

$$\Delta v = v \cdot \left(\frac{m}{2E}\right) g\Delta h$$

$$\therefore v = \left(\frac{2E}{m}\right)^{1/2}$$

$$v^2 = \left(\frac{2E}{m}\right)$$

$$\frac{1}{v^2} = \left(\frac{m}{2E}\right)$$

$$\therefore \Delta v = v \cdot \left(\frac{1}{v^2}\right) g\Delta h$$

$$\Delta v = \frac{1}{v} g\Delta h$$

$$\therefore \Delta v = \frac{g}{v}\Delta h$$

By knowing v and Δh, we can analyse the change in Δv.

with this added assumption suppose we change the height the height displaced during the slide by 2%.

$$\frac{\Delta h}{h} = 0.02$$

what % did v change by?

$\frac{\Delta V}{V} = ?$

∴ The rider starts completely at rest. This means, the K.E is zero, so the total Energy is gn by $E = mgh$

solu:

$$E = mgh$$

$$\therefore V = \sqrt{\frac{2E}{m}} \quad \because E = mgh$$

$$V = \sqrt{\frac{2mgh}{m}}$$

$$\boxed{V = \sqrt{2gh}}$$

$$V + \Delta V = \sqrt{2g(h+\Delta h)}$$

$$V + \Delta V = \sqrt{2gh + 2g\Delta h}$$

$$V + \Delta V = \sqrt{2gh}\left(1 + \frac{2g\Delta h}{2gh}\right)^{1/2}$$

$$V + \Delta V = (2gh)^{1/2}\left(1 + \frac{\Delta h}{h}\right)^{1/2}$$

$$\therefore (1+u)^r = 1 + ru$$

$$V + \Delta V = (V)\left(1 + \frac{1}{2}\frac{\Delta h}{h}\right)$$

$$\Delta V = V \cdot \frac{1}{2}\frac{\Delta h}{h}$$

$$\frac{\Delta V}{V} = \frac{1}{2}\frac{\Delta h}{h}$$

$$\frac{\Delta V}{V} = 0.01$$

Linear approximation (sinx) near 0.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Near $x_1 = 0$:

$$\sin x \approx \sin(0) + \cos(0)(x)$$

$$\approx x$$

$$\cos x \approx \cos 0 - \sin(0)(x)$$

$$\approx 1$$

$$e^x \approx e^0 + e^0 x = 1 + x$$

$$\ln(1+x) \approx \ln 1 + \frac{1}{1}x$$

$$\approx x.$$

Near $x = 0$

$$(1+x)^r \approx 1 + ru$$
$$\sin x \approx x$$
$$\cos x \approx 1$$
$$e^x \approx 1 + x$$
$$\ln(1+x) \approx x$$

Examples: (near $x = 0$)

$$\sin x + \ln(1+x) \approx 2x$$

$$3e^x \approx 3(1+x)$$

zipline example:

$$(1+ax)^{1/2} = (1+u)^{1/2}$$
$$= 1 + \frac{1}{2}u$$
$$= 1 + \frac{ax}{2}$$

$ax = u$
(near 0)

Example: 1: $\ln(u)$ near $u = 1$

solu:

near $u = 1$, $u = 1 + x$ ↓ near 0.

$$\ln(u) = \ln(1+x) \approx x = u - 1$$

$$\ln(u) = \ln(1) + \ln'(1)(u-1)$$

$$= \frac{1}{1}(u-1)$$

$$= u - 1$$

Confirmed.

ex: $(4+\sin x)^{3/2}$ near $x=0$.

$= 4^{3/2}\left(1+\frac{\sin x}{4}\right)^{3/2}$

$u = \frac{\sin x}{4}$

$= 4^{3/2}\left(1+\frac{3}{2}\cdot\frac{\sin x}{4}\right) \rightarrow$ app.

$= 4^{3/2}\left(1+\frac{3}{8}\sin x\right)$

$= \sqrt{64}\left(1+\frac{3}{8}x\right)$

$\sin x \approx x$

$= 8\left(1+\frac{3}{8}x\right)$

$= (8+3x)$

(near 0)

$(1+x^2)^{-1}$ near $x=0$.

$\therefore = (1-x^2)$ $\quad x^2 \approx 0$

(near 0)

$= 1.$

---

proof: $(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \cdots$

$(1+x^2)^{-1} = 1 - x^2 + \frac{(-1)(-2)}{2}x^4 + \cdots$

$= 1 - x^2 + x^4 + \cdots$

(Neglecting higher terms)

$(1+x^2)^{-1} \approx 1-x^2.$

Approximation,

$f(u) \approx f(0) + f'(0)\cdot u$

$f(g(x)) \approx f(g(0)) + f'(g(0))\cdot g(x)$

---

e^{-x} (app) $\rightarrow$ near 0.

$f(u) \approx f(0) + f'(0)\cdot u$

$\approx 1 - u$

(or)

$f(x) \approx f(0) + f'(0)\cdot u = 1-x.$

---

i) e^{x^3} (near) $x=0$

solu:

$f(u) = f(0) + f'(0)\cdot u$

$= 1 + u$

$= 1 + x^3$

ii) $\ln\sqrt{1+x^2}$ near 0.

solu:

$\frac{1}{2}\ln(1+x^2)$

$\sqrt{1+x^2} = (1+x^2)^{1/2}$

Approximating,

$\left(1+\frac{1}{2}x^2\right)$

$\ln\left(1+\frac{1}{2}x^2\right) \approx \frac{1}{2}x^2$

$(\because \ln(1+x) \approx x)$

Time dilation

satellite

$T \rightarrow$ Time of pt.

$T' \rightarrow$ me

(watch) (on surface of Earth)

$T' = \frac{T}{\sqrt{1-v^2/c^2}}$

(Difference b/w two times).

$T' = T\left(1-v^2/c^2\right)^{-1/2}$

$= T(1-u)^{-1/2}$

$\approx T\left(1+\frac{1}{2}u\right)$

$T' \approx T\left(1+\frac{1}{2} v^2/c^2\right)$

$u = v^2/c^2$

∴

Since due to dilation of time, is there any problem on transmission?

∴ $V = 4$ km/sec

$C = 3 \times 10^5$ km/s

∴ $\boxed{u = \frac{v^2}{c^2} \approx 10^{-10}}$ → Error will be in mm.

Since the clock is different when the satellite broadcast its radio frequency, that frequency would be shifted, would be offset.

They decided that the fidelity was so important so they would sent the satellites with exactly this offset to compensate the way the signal is so good from the view of good reception on your little GPS device, they change the frequency at which the transmitter in the satellites according to exactly the above rule.

↓

Neglected all kinds of higher order quadratics error

∴ $u \approx 10^{-20}$ (In nano metres) ← Completely negligible in any kind of measurement

$\frac{e^{-3x}}{\sqrt{1+x}}$ at $x = 0$. Linear approximation

Solu:

$e^{-3x}(1+x)^{-1/2}$

$e^{-3x}\left(1-\frac{1}{2}x\right)$

$(1-3x)\left(1-\frac{1}{2}x\right)$ $\quad$ $e^x \approx 1+x$

$= 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2$

$= 1 - \frac{7}{2}x + \frac{3}{2}x^2$

$= 1 - \frac{7}{2}x$ $\quad$ $x^2 \approx 0$ when $x \to 0$

Why killing up quadratic term?

Two reasons:

1) Negligible ($\because x \approx 0$)

If x is of order $\frac{1}{100}$

$x^2 \approx \frac{1}{10000}$ (very small)

2) The quadratic term doesn't mean any sense. It doesn't have any meaning.

3.2m $\boxed{16.04\,m^2}$

4.7m

Only two digits are significant. This is due to the inherent error in our measurement.

Linear approximations of products

$h(x) = f(x)\,g(x)$ near $x=0$

It is suffice to find a linear approximation for $f(x)$ as well as $g(x)$ near $x=0$ and $h(x)$ is the product of these two approximations where we cancel all of the terms that are quadratic

$$h(x) = g(x)\cdot f(x)$$

$$f(x) \approx f(0) + f'(0)x$$

$$g(x) \approx g(0) + g'(0)x$$

$$h(x) \approx (f(0) + f'(0)x)(g(0) + g'(x)x)$$

$$\approx f(0)g(0) + f'(0)g(0)x + g'(0)x \cdot f(0) + f'(0)g'(0)x^2$$

$$\approx f(0)g(0) + x(f'(0)g(0) + g'(0)f(0)) + f'(0)g'(0)x^2$$

$$\approx f(0)g(0) + x(f'(0)g(0) + g'(0)f(0))$$

$\frac{\ln(1-x)}{(1+x)^2}$ near $x=0$

solu:

$$h(x) = \ln(1-x)\cdot(1+x)^{-2}$$

$$(1+x)^{-2} \approx 1-2x$$

$$\ln(1-x) \approx (-x)$$

$$(1+x)^{-2}\ln(1-x) \approx (1-2x)(-x)$$

$$\approx -x + 2x^2$$

$$\approx -x.$$

$\frac{\sin x}{\cos^2(x)}$ near $x=0$

solu:

$\sin x \approx x$ (near x)

$\cos x \approx 1 + 0x.$

$$\frac{\sin x}{\cos^2 x} = x(1)^{-1}$$

$$\approx x$$

$\ln(x)\sin(x-1)$ near $x=1$

solu:

$x=1$

$u = x-1$

$\boxed{\text{as } u \to 0,\ x=1}$

$$\ln x = \ln(u+1) \approx u$$

$$\sin(x-1) = \sin u \approx u$$

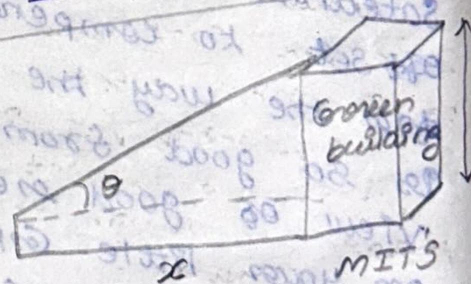
$$\ln(x)(\sin(x-1)) = u^2 = (x-1)^2$$

$$= 1 + x^2 - 2x$$

$$\approx 1 - 2x$$

(or)

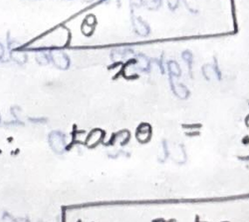
$\boxed{u^2 \approx 0}$



MITS Largest building

$$y = h + x\tan\theta$$

(How?)



$$\therefore \tan\theta = \frac{?(y)}{x}$$

$\boxed{y = x\tan\theta}$

Height of eye ball $h = 4.9$ feet (± 0.1 feet error)

Distance to green building $x = 175$ feet (± 3 feet)

Angle $\theta = 57^\circ$ ± 3 degrees

error:

* The error in the measurement of Jen's height comes from rounding error and inherent error in using a tape measure to enough accuracy.

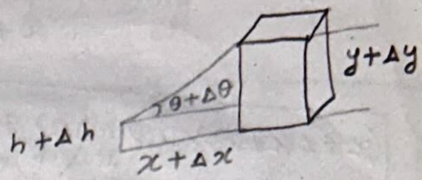
* The error in the distance to green building is coming from a few places: error in the actual length of the rope which may differ by few inches, error in our walking in a perfectly straight line as we measure and error coming from the fact that the rope when we pull on it, stretching the rope. so in fact we expect our measurement is likely an overestimate.

* The error in the angle is coming from wind & the string not hanging straight down, error in the construction of the device, and error in holding the device to read the measurement.

$\therefore$ y (height of Green building)

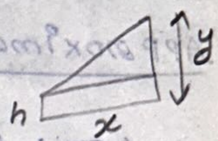
$$y = h + x\tan\theta$$
$$= 4.9 + 175\tan 57^\circ$$
$$= 274 \text{ (3 sig fig).}$$

Error in estimate:



$$\tan(\theta+\Delta\theta) = \frac{(y+\Delta y)-(h+\Delta h)}{x+\Delta x}$$
$$= \frac{(y-h)+\Delta y-\Delta h}{x+\Delta x}$$
$$= \frac{x\tan\theta+\Delta y-\Delta h}{x+\Delta x}$$

$$\therefore \boxed{\tan\theta = \frac{y-h}{x}}$$



$$\therefore \cancel{y+\Delta y = h+}\Delta h + \cancel{x\tan(\theta+\Delta\theta)}$$

$$y+\Delta y = (h+\Delta h) + x\left(x\tan\theta + \frac{\Delta y-\Delta h}{x+\Delta x}\right)$$

$$\therefore \boxed{y = h + x\tan\theta}$$

$$\Delta y = (h+\Delta h) +$$

$$y+\Delta y = (h+\Delta h) + (x+\Delta x)\tan(\theta+\Delta\theta)$$

$$y+\Delta y = (h+\Delta h) + (x+\Delta x)\frac{x\tan\theta+\Delta y-\Delta h}{x+\Delta x}$$

$$y+\Delta y = h+\Delta h + x\tan\theta + \Delta y - \Delta h$$

$$\tan(\theta+\Delta\theta) = \frac{\Delta y - \Delta h + x\tan\theta}{x+\Delta x}$$

$$\boxed{(x+\Delta x)\tan(\theta+\Delta\theta) + \Delta h - x\tan\theta = \Delta y}$$

To get a sense of the order of this error term, let's find a linear approximation of $\tan(\theta+\Delta\theta)$ near $\theta = 57^0$.

Approximation:

$\tan(57+\Delta\theta) \approx$

$\cancel{f(x) = f(0) + f'(0)\cdot x}$

$\boxed{x = 57+\Delta\theta}$ $\Delta\theta \to 0$ near $\boxed{\theta = 57^0}$

$\cancel{f(\theta+\Delta\theta)} =$

$f(\theta+\Delta\theta) = \cancel{f(\theta)} + \cancel{f'(\theta)\cdot}$

$u = 57+\Delta\theta$

$\boxed{u=57}$ as $\Delta\theta \to 0$

$$f(u) = f(57) + f'(57)(u-57)$$

$$f(57+\Delta\theta) = \tan 57 + \sec^2(57)\,\Delta\theta$$

$$\Delta y = \Delta h + (x+\Delta x)\tan(\theta+\Delta\theta) - x\tan\theta$$

$$\Delta y \approx \Delta h + (x+\Delta x)(\tan\theta + \sec^2(\theta)\cdot\Delta\theta) - x\tan\theta$$

$$\approx \Delta h + x\tan\theta + \Delta x\tan\theta + x\sec^2\theta\,\Delta\theta + \Delta x\sec^2\theta\,\Delta\theta - x\tan\theta$$

$$\approx \Delta h + \Delta x\tan\theta + x\sec^2\theta\,\Delta\theta + \Delta x\sec^2\theta\cdot\Delta\theta$$

$$\approx \Delta h + \Delta x\tan\theta + x\sec^2\theta\cdot\Delta\theta$$

(cancelled quadratic term)

Error

The term Δh is very small. To compare the other two terms, we could do so explicitly comparing numbers.

The green building is the tallest building on MIT campus, my angle θ was rather large (close to $\pi/3$), meaning both $\tan(\theta)$ and $\sec\theta$ are both very large.

$\therefore$

$$\tan\theta < \sec\theta < \sec^2\theta \quad (\pi/4 < \theta < \pi/2)$$

|||ly $x\Delta\theta$ is large compared to Δx.

$x \Delta\theta \sec^2(\theta) \gg \Delta x \tan(\theta)$

So the largest contribution to error is $x \sec^2(\theta) \cdot \Delta\theta$

$$\frac{\Delta h}{x\tan\theta}, \quad \frac{\Delta x \tan\theta}{x \tan\theta}, \quad \frac{x \sec^2\theta \cdot \Delta\theta}{x \tan\theta}$$

simplify to

$$\frac{1}{\tan\theta} \frac{\Delta h}{x}, \quad \frac{\Delta x}{x}, \quad \frac{\sec^2(\theta)}{\tan\theta} \cdot \Delta\theta$$

$$\therefore \frac{\Delta h}{x} \approx \frac{0.01}{175}$$

$$\frac{\Delta x}{x} \approx \frac{3}{175}$$

$$\Delta\theta \approx 3 \frac{\pi}{180}$$

(Radian)

$\therefore$ $\frac{\Delta h}{x}$ is multiplied by $\frac{1}{\tan\theta} < 1$, and the term $\Delta\theta$ is multiplied by $\frac{\sec^2\theta}{\tan\theta} > 1$. So the error coming from the angle measurement is the largest error contributing factor.

↳ By considering each of the 3 dimensionless quantities corresponding to the source of error.

$y \approx h + x \tan\theta$

$\boxed{y \approx x \tan\theta}$

and its relative magnitude

Compared to the height y of the building.

$\Delta y \approx \Delta h + \Delta x \tan\theta + x \Delta\theta \sec^2\theta$

$$\leq 0.01 + 3 \tan\left(\frac{19\pi}{60}\right) + 175\left(\frac{3\pi}{180}\right)\sec^2\left(\frac{19\pi}{60}\right)$$

≈ 35 feet.

$\therefore 295 - 274 = 21$ (Actual error) → within expected error bound.

$\therefore 57^\circ \to$ Radian

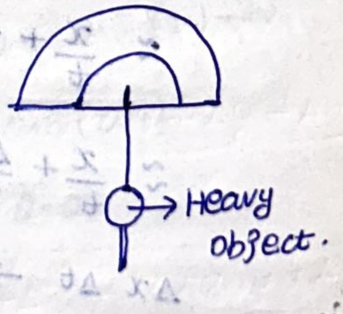
$\to 57 \times \frac{\pi}{180}$

$\to \frac{19}{60}\pi$

Altimeter (Astrolabe)

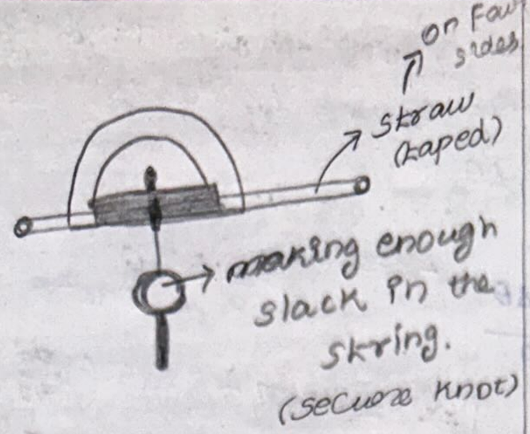
(degree taking device).

1) protractor
2) straw
3) string
4) Heavy object & tape.



(exactly at the measuring line)

(For stiffness of the thread put tape)



The thread shows the angle.

Suppose that you measure the velocity of an object by measuring that it takes 1 second to travel 1.2 metres. The measurement error is 0.001 meters in distance, and the error in time is 0.01 second. What is the absolute value of the error in the linear approximation for the velocity?

Solu:

$$v + \Delta v = \frac{x + \Delta x}{t + \Delta t}$$

$$= (x + \Delta x)(t + \Delta t)^{-1}$$

$$= (x + \Delta x)\frac{1}{t}\left(1 + \frac{\Delta t}{t}\right)^{-1}$$

we pulled the t out of the second term.

$$v + \Delta v \approx (x + \Delta x)\frac{1}{t}\left(1 - \frac{\Delta t}{t}\right)$$

$$\approx \frac{x}{t} + \frac{\Delta x}{t} - \frac{x\Delta t}{t^2} - \frac{\Delta x \Delta t}{t^2}$$

$$\approx \frac{x}{t} + \frac{\Delta x}{t} - \frac{x\Delta t}{t^2}$$

$\Delta x \, \Delta t \rightarrow$ ignored as it is quadratic.

Taking Δv alone

$$\Delta v \approx \frac{\Delta x}{t} - \frac{x\Delta t}{t^2}$$

$\therefore \Delta x \le \pm 0.001$ meters

$\Delta t < \pm 0.01$ seconds.

The total error Δv is bounded by the largest possible error.

$$|\Delta v| = \left|\frac{\pm 0.001}{1} - \frac{(1.2)(\pm 0.01)}{1}\right|$$

$$\le |\pm 0.001| - |\pm 0.012|$$

$= 0.013$ meters per second.

Suppose you measure the velocity of a mosquito to be 1.3 m/s with an error of 0.2 m/s. We approximate the mosquito app equal to a grain of rice, which is on the order of magnitude of 10^{-6} kg. What is the error in the kinetic energy. Assume there is no rotational velocity.

Soln:

$$K.E = \frac{1}{2}mv^2 \quad \left(\frac{kg\,m^2}{s^2}\right)$$

$$E + \Delta E = \frac{1}{2}m(v + \Delta v)^2$$

$$= \frac{1}{2} \times 10^{-6}(1.3 + 0.2)^2$$

(or) (with app)

$$= \frac{1}{2}m(v^2 + 2v\Delta v)$$

$$= \frac{1}{2}mv^2 + 2v\Delta v$$

$$= K + mv\Delta v.$$

$$\Delta E = mv\Delta v$$

$$\Delta E = (10^{-6})(1.3)(0.2)$$
$$= 2.6 \times 10^{-7} \ \frac{Kg\, m^2}{s^2}$$

observe that near $x=0$, the argument $\frac{\pi}{2}(1+x)^{3/2}$ is near $\frac{\pi}{2}$. So we need a linear app. of $\cos(u)$ near $\pi/2$.

$$\cos u \approx \cos\left(\frac{\pi}{2}\right) + \left(-\sin\frac{\pi}{2}\right)\cdot(u - \pi/2)$$
$$\approx \left(\frac{\pi}{2} - u\right)$$

$$\therefore \frac{\pi}{2}(1+x)^{3/2} \approx \frac{\pi}{2} + \frac{3\pi}{4}x,$$

substituting, $\frac{\pi}{2} + \frac{3\pi}{4}x$ for u,

$$\cos\left(\frac{\pi}{2}(1+x)^{3/2}\right) \approx \frac{\pi}{2} + \frac{3\pi}{4}x$$

(90° rotated)

$$\boxed{\frac{\pi}{2}\left(1+\frac{3}{2}x\right) \Rightarrow \frac{\pi}{2} + \frac{3\pi}{4}x}$$

$$\cos\left(\frac{\pi}{2} + \frac{3\pi}{4}x\right) \approx \frac{\pi}{2} - u$$
$$\approx \frac{\pi}{2} - \frac{\pi}{2} - \frac{3\pi}{4}x$$
$$\approx -\frac{3\pi}{4}x$$

$$\frac{\pi}{2}(1+x)^{3/2} \rightarrow \text{near } 0$$

$$\frac{\pi}{2}\left(1+\frac{3}{2}x\right) \approx \frac{\pi}{2} + \frac{3\pi}{4}x$$
$$\approx \frac{\pi}{2}$$ near $x=0$

$$(w(x)+1)\, e^{w(x)} = x$$

Linear app

$$w(x) \approx w(1) + w'(1)(x-1)$$
$$\therefore w(1) = 0$$

$$\frac{d}{dx}(w(x)+1)e^{w(x)} = \frac{d}{dx}(x)$$

$$w'(x) e^{w(x)} + w'(x) \cdot e^{w(x)} w(x)$$
$$+ w'(x) e^{w(x)} = 1$$

$$w'(x) e^{w(x)} + (w(x)+1) e^{w(x)} w'(x) = 1$$

$$w'(x)\left[e^{w(x)}(w(x)+1+1)\right] = 1$$

$$w'(x) = \frac{1}{e^{w(x)}[w(x)+2]}$$

$$w'(1) = \frac{1}{e^{w(1)}[w(1)+2]}$$
$$= \frac{1}{2}$$
$$= 0.5$$

$$w(x) \approx w(1) + w'(1)(x-1)$$

$$w(1.1) \approx 0 + 0.5(1.1-1)$$
$$\approx 0.5(0.1)$$
$$\approx 0.05.$$

Quadratic approximation

Linear approximation:

∴ Linear app of a function at a point is a line. The quadratic app of a function at a point is parabola. (Less error compared to linear approximation).

compare the function $f(x)$ and this tangent line through the point $x=a$.

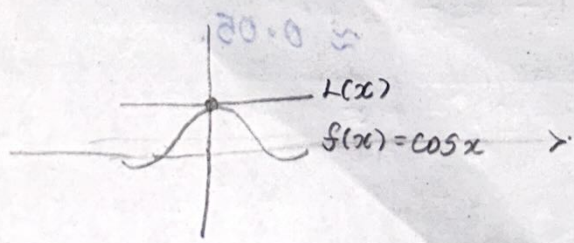
$$L(x) = f(a) + f'(a)\,(x-a)$$

$$\therefore \quad f(a) = L(a)$$
$$f'(a) = L'(a)$$

∴ The tangent line has the same value as the function at the point a, and it has the same derivative at a. unless the second derivative of f is zero at a.

It is unlikely that the second order derivatives agree. At nearby points, we can assume that the value of the function and the tangent line are close, but they are certainly not exactly equal.

Intuition



Linear approximation = best fit line

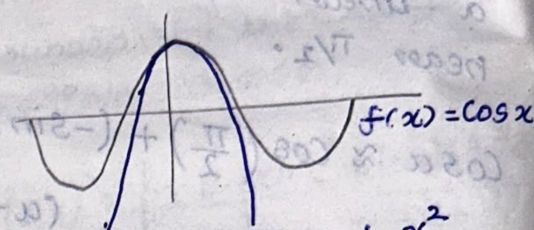
$L(0) = f(0)$.

$L'(0) = f'(0)$

we don't able to get much information about the curvature.

so,

quadratic approximation = best fit parabola



$$P(0) = f(0)$$
$$P'(0) = f'(0)$$
$$P''(0) = f''(0)$$

The best fit quadratic to function $f(x)$ at the point $x=0$ is the quadratic function $q(x)$ whose values agrees with the value of f at $x=0$, and whose first and second derivatives agree with the first and second derivatives of f at $x=0$.

$$f(0) = q(0)$$
$$f'(0) = q'(0)$$
$$f''(0) = q''(0)$$

Quadratic approximation near 0

Suppose that you have a function $f(x)$. The best fit quadratic function near $x=0$ is gn by

$$v(x) = ax^2 + bx + c$$

solu:

$$\therefore f(0) = v(0) = c$$

$$f'(0) = v'(0) = b$$

$$f''(0) = v''(0) = 2a.$$

$$\therefore f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

quadratic approximation:

near $x = 0$:

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2} x^2$$

near $x = a$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

For $\cos x$

$$\cos x \approx \cos 0 - \sin(0).x - \frac{\cos(0)}{2}.x^2$$

$$\cos x \approx 1 - \frac{x^2}{2}$$

$$\sin x \approx \sin 0 + \cos 0\,(x) - \frac{\sin 0}{2} x^2$$

$$\approx x.$$

$$\ln(1+x) \approx \ln 1 + \frac{1}{1} x + \frac{\ln''(1)}{2} x^2$$

$$\approx x - \frac{x^2}{2}.$$

x^2 $\quad f'(x) = ?$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

$2x$

$$\lim_{\Delta x \to 0} \frac{2(x+\Delta x) - 2x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2.$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n.$$

$$f(x) = \frac{f(0)}{0!} x^0 + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots +$$

$\cos 62^\circ = ?$ $\quad \boxed{2^\circ = \frac{\pi}{90}}$

$\therefore 60^\circ = \pi/3$ radian.

$62^\circ = \frac{\pi}{3} + \frac{\pi}{90}$ radians

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{90}\right) \approx \cos\left(\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{3}\right)\frac{\pi}{90} - \frac{\cos(\pi/3)}{2}\left(\frac{\pi}{90}\right)^2$$

$$\approx \frac{1}{2} - \frac{\sqrt{3}}{2}\frac{\pi}{90} - \frac{1}{4}\frac{\pi^2}{8100}$$

$$\approx 0.469.$$

$$\varphi(f) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2.$$

$\therefore$

$$\varphi(fg) = \varphi(\varphi(f)\cdot\varphi(g)).$$

Example:

$f(x) = e^x$, $g(x) = \sin x$.

$$\varphi(f) = e^0 + e^0\cdot x + \frac{e^0\cdot x^2}{2}$$

$$= 1 + x + \frac{x^2}{2}.$$

$$\varphi(g) = x$$

$$(\varphi(e^x \sin x)) = \varphi\left(\left(1 + x + \frac{x^2}{2}\right)x\right)$$

$$= \varphi\left(x + x^2 + \frac{x^3}{2}\right)$$

$$= x + x^2$$

(Neglecting higher order terms).

Near 0, $\varphi\left(x + x^2 + \frac{x^3}{2}\right)$ $\downarrow$ will be all the terms up to the quadratic term.

Cheat sheet:

$$(fg)' = f'g + g'f$$

$$(fg)'' = f''g + f'g' + g''f + g'f'$$

$$= f''g + 2(f'g') + g''f$$

$$\varphi(f) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2$$

$$\varphi(g) = g(0) + g'(0)\,x + \frac{g''(0)}{2}x^2$$

R.H.S

$$\varphi(f)\,\varphi(g) = f(0)g(0) + (f'(0)g(0) + g'(0)f(0))\,x + \left(\frac{fg''(0)}{2} + \frac{gf''(0)}{2} + \frac{f'g'(0)}{1}\right)x^2$$

$$+\left(\frac{f'g''(0)}{2} + \frac{g'f''(0)}{2}\right)x^3 + (\ldots)\,x^4.$$

$$\varphi(\varphi(f)\,\varphi(g)) = fg(0) + (f'g(0) + fg'(0))\,x$$

$$+\left(\frac{fg''(0)}{2} + \frac{gf''(0)}{2} + f'g'(0)\right)x^2$$

L.H.S:

$$\varphi(fg) = fg(0) + (fg)'(0)\,x + \frac{(fg)''(0)}{2}x^2$$

$$= fg(0) + (f'g(0) + fg'(0))\,x + \frac{(f'g(0) + fg'(0))'}{2}x^2$$

$$= fg(0) + (f'g(0) + fg'(0))\,x + \left(f'g'(0) + f''g(0) + f'g'(0) + fg''(0)\right)\frac{x^2}{2}$$

$$= fg(0) + [f'g(0) + fg'(0)]\,x + \left(\frac{fg''(0)}{2} + \frac{gf''(0)}{2} + f'g'(0)\right)x^2$$

L.H.S = R.H.S.

example

$f(x) = e^{x+x^2}$, near $x = 0$?

(quadratic appl)

Solu:

$Q(f) = f(0) + f'(0)\,x + \frac{f''(0)}{2} x^2$

$f(0) = e^0 = 1.$

$f'(0) = (1+2x) \cdot e^{x+x^2}$

$f''(0) = (2) e^{x+x^2} + (1+2x)^2 e^{x+x^2}$

$\cancel{Q(x) = 1 + (1+2x) e^{x+x^2} + \left(e^{x+x^2} + \frac{(1+2x)^2}{2} e^{x+x^2}\right)}$

At $x = 0$

$\cancel{Q(0) = 1 + e^0 + \left(1 + \frac{1}{2}\right)}$

$\cancel{= 3 + \frac{1}{2}}$

$\cancel{= \frac{7}{2}.}$

Near $x = 0$.

$f(0) = 1$, $f''(0) = 2 + 1$
$= 3$

$f'(0) = 1$

$Q(f) = 1 + x + \frac{3}{2} x^2$

Other way:

$\boxed{f(x) = e^x \cdot e^{x^2}}$

$Q(e^x \cdot e^{x^2}) = Q(Q(e^x) \cdot Q(e^{x^2}))$

$Q(e^x) = e^0 + e^0 \times x + \frac{e^0}{2} x^2$

$= 1 + x + \frac{x^2}{2}.$

$\cancel{Q(e^{x^2}) = 1 + 2x(e^{x^2})x + \frac{(2e^{x^2} + 2x(2x)e^{x^2})x^2}{2}}$

$\cancel{Q(e^{x^2}) = 1 + 2x^2(e^{x^2})}$

$Q(e^{x^2}) = 1 + 0 + (2+0)\frac{x^2}{2}$

$= 1 + x^2$

$Q(f) = Q\left(\left(1 + x + \frac{x^2}{2}\right)\left(1 + x^2\right)\right)$

$= Q\left(1 + x^2 + x + \frac{x^2}{2} + \dots + \dots\right)$

$= 1 + x + \frac{3x^2}{2}.$

Error estimates.

Near $x = 0$:

$(1+x)^5 \approx 1 + 5x$ (Linear)

$\approx 1 + 5x + 10x^2$ (quadratic)

$\therefore$ There must be some error.

$\therefore (1+x)^5 \approx 1 + 5x + 10x^2 + \text{error}.$

$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$

$\boxed{\text{error} = 10x^2 + 10x^3 + 5x^4 + x^5}$

Ex.:

x	x^2	x^3	x^4	x^5
1.1^5	$.1$	$.01$	$.001$	0.0001 10^{-5}

$\therefore$ since x is near 1.

$\therefore$ $\boxed{2 \text{ is greater than } 1}$

(Linear app)

$\therefore$ error $= 10x^2 + 10x^3 + 5x^4 + x^5$

$\leq 11x^2 + 5x^4 + x^5$

$\leq 12x^2$

(All comes under $12x^2$)

Big O notation:

error $= O(x^2)$ near $x = 0$

(order of magnitude)

Definition

$f(x) = O(x^2)$ near $x = 0$

is $|f(x)| \leq kx^2$ for some constant k.

Near $x = 0$:

$f(x)$ error of linear approximation is $O(x^2)$.

$\therefore$ $f(x) = f(0) + f'(0)x + O(x^2)$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

A function $f(x)$ is on the order x^n near $x = 0$, which is denoted using big "O" notation as $f(x) = O(x^n)$ near $x = 0$, if

$|f(x)| \leq Kx^n$.

Car loan

Suppose that you take out a \$ 15,000 loan to purchase a car. The loan has an interest rate of 3% per year, compounded monthly.

The formula for the amount of interest accrued over a time period T measured in number of months is

$$A = 15000\left(1 + \frac{0.03}{12}\right)^T - 15000$$

Use quadratic app to approximate the amount of interest accrued over 4 years.

solu:

(In months)

4 years = 48 months

$$A = 15000\left(1 + \frac{0.03}{12}\right)^{48} - 15000$$

~~By approximation:~~

~~$Q(A) = A(0) + A'(0)x + \frac{A''(0)}{2}x^2$~~

~~$A(0) = 15000 - 15000 = 0$~~

~~$A'(0) = ?$~~

~~$\boxed{\frac{d}{dx} a^x = \ln a \cdot a^x}$~~

~~$A'(T) = 15000 \times \ln\left(1 + \frac{0.03}{12}\right) \cdot \left(1 + \frac{0.03}{12}\right)^T -$~~

$$A = 15000\left(1+\frac{1}{400}\right)^{48} - 15000$$

Quadratic approximation:

$$q(f) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

$\therefore$ Approximating,

$$15000\left(1+\frac{1}{400}\right)^{48}$$

$$\left(1+\frac{1}{400}\right)^{48} \approx \left(1+\frac{48}{400}\right)$$

$\hookrightarrow$ Linear

Quadratic

$$\left((1+x)^n\right) = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2$$

$$\left(1+\frac{1}{400}\right)^{48} = 1 + \frac{48}{1!}\times\frac{1}{400} + \frac{48\,(48-1)}{2!}\times\frac{1}{(400)^2}$$

$$= 1 + \frac{48}{400} + \frac{48\times47}{2(400)^2}$$

Quadratic app:

$$= 15000\left(1+\frac{48}{400}+\frac{141}{20000}\right) - 15000 + \text{error}$$

$$\approx 15000(1+0.12+0.00705) - 15000$$

$$\approx 1905.75.$$

Actual

$$15000\left(1+\frac{1}{400}\right)^{48} - 15000$$

$1909.920316.$

Linear term:

$$\text{order} = K|x|$$

$$= 15000(0.12)$$

$$\approx 1800$$

$$\boxed{\text{order} \approx 10^3}$$

Quadratic term:

$$\text{order} = K|x^2|$$

$$= 15000(0.00705)$$

$$= 105.75$$

$$\boxed{\text{order} \approx 10^2}$$

Error in this approximation is $O(x^3)$.

$$\left(\frac{1}{400}\right)^3 \approx 1.5625 \times 10^{-8}$$

$$\boxed{\text{order of magnitude} \approx 10^{-8}}$$

Absolute value of error:

Exact − APP =

$$= 1909.92 - 1905.75$$

$$= \$\,4.17.$$

$$\boxed{\text{order} \approx 1}$$

$\therefore$

$$\frac{4.17}{15000}$$

$$\therefore |f(x)| = Kx^n.$$

$$= Kx^3$$

$\therefore$ error magnitude $O(x^3) = 1.$

$$\therefore K = 10^8$$

$K x^3 = 10^8 \times 10^{-8}$

$= 1.$

what does $O(x^3)$ mean?

f is $O(x^3)$ near $x = 0$

is x^3 is dominant.

eg: $e^{\sin x} = e^{x + O(x^3)}$

$\therefore$ $x + O(x^3) \rightarrow$ quadratic approximation of sine x.

$= e^u$

quadratic app of e^u:

$= 1 + u + \frac{u^2}{2} + O(u^3)$

$= 1 + (x + O(x^3)) + \frac{(x + O(x^3))^2}{2} + \underbrace{O((x + O(x^3))^3)}_{\text{negligible}}$

$\approx 1 + (x + O(x^3)) + \frac{(x + O(x^3))^2}{2}$

$\therefore$ $e^{\sin x} = 1 + (x + O(x^3)) + \frac{x^2}{2} + \frac{O(x^3)^2}{2} + \frac{2x\, O(x^3)}{2}$

$= 1 + (x + O(x^3)) + \frac{x^2}{2} + \frac{O(x^3)^2}{2} + 2x\, O(x^3)$

$= 1 + x + \frac{x^2}{2} + O(x^3)$ (All higher order terms)

2) $\frac{e^{-3x}}{\sqrt{1+x}}$

$e^{-3x} = 1 - 3x + \frac{9x^2}{2} + O((-3x)^3)$

$= 1 - 3x + \frac{9x^2}{2} + O(x^3)$

$\therefore$ the constant term at $O((-3x)^3)$ not going to alter the dominant behaviour.

$(1+x)^{-1/2} \simeq 1 - \frac{x}{2} + \frac{3}{8}x^2 + O(x^3)$

$\frac{e^{-3x}}{\sqrt{1+x}} = (1 - 3x + \frac{9x^2}{2} + O(x^3)).(1 - \frac{x}{2} + \frac{3}{8}x^2 + O(x^3))$

Any time I get a term that is x^3 or higher, I'm just throwing that in to this error term.

$= 1 - \frac{x}{2} + \frac{3}{8}x^2 + O(x^3) - 3x + \frac{3}{2}x^2 + O(x^3) + \frac{9}{2}x^2 + O(x^3)$

$= 1 - \frac{7}{x}x + \frac{51}{8}x^2 + O(x^3)$

$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$e^x = 1 + x + \frac{x^2}{2} + O(x^3)$

$\sin x = x + O(x^3)$

$\cos x = 1 - \frac{x^2}{2} + O(x^3)$

$\ln(1+x) = x - \frac{x^2}{2} + O(x^3)$

$(1+x)^r = 1 + rx + \frac{r(r-1)}{2}x^2 + O(x^3)$

Newton's method

The linear & quadratic app are only effective when the function near our point

$f(x) = ax^2 + bx + c$,

root:

* points x where $f(x) = 0$
* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
* points where the graph has an x-intercept.

Approximating roots of functions.

$f(x) = 2 - x^2$

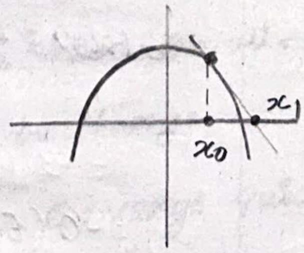
Roots : $x = \pm\sqrt{2}$

$= 1.41421356237\ldots$

Guess : $x_0 = 1$

we can app it, by its tangent line (linear app) near $x = 1$, Then we can app by x intercept of the tangent line.

$x_0 = 1$



Find tangent line

$L(x)$ to $f(x) = 2 - x^2$ at $x_0 = 1$

solu:

$f'(x) = -2x$

$f'(x_0) = -2$

$\boxed{y = mx + c}$

$(y - y_1) = m(x - x_1)$

$f(x_0) = 2 - 1$
$= 1$

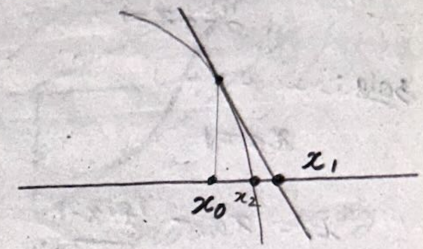
$(y - 1) = -2(x - 1)$

$y = -2x + 2 + 1$

$\boxed{y = -2x + 3}$

roots:

$(2x = 3)$ $x = 3/2$



$x_0 = 1$, $x_1 = 3/2$

Newton-Raphson method:

* make a guess x_0
* Find x-intercept x_1 of tangent to f at x_0.

$y = f(x_0) + f'(x_0)(x - x_0)$

$0 = f(x_0) + f'(x_0)(x - x_0)$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

step:3 : Repeat : $0 \to 1$, $1 \to 2$.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

General formula :

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x) = 2 - x^2$

$f'(x) = -2x$

$x_0 = 1$

$x_1 = x_0 - \frac{2 - x_0^2}{-2x_0} = 1 - \frac{1}{-2.1}$

$= 1.5$

$x_2 = 1.416$

$x_3 = 1.41425, \ x_4 = 1.4142135$

$x_4 = 1.4142135$

$\therefore \boxed{\text{Root} = 1.4142135}$

$y = 4x - 1$

Solu:

$x_0 = 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{3}{4}$

$= 0.25$

$x_2 = 0.25 - 0$

$= 0.25$

$x_3 = 0.25 - 0$

$= 0.25$

Square root of 5

$f(x) = x^2 - 5, \ f'(x) = 2x$

Solu:

$x_0 = 2$

$x_1 = 2 - \frac{f(2)}{f'(2)}$

$= 2 + \frac{1}{2(2)}$

$= 2.25$

$x_2 = 2.25 - \frac{f(2.25)}{f'(2.25)}$

$= 2.236111$

$x_3 = 2.236 - \frac{f(2.236)}{f'(2.236)}$

$= 2.2360679$

Convergence rate

why Newton's method work:

f - function

$x^* \rightarrow$ root

$\therefore f(x^*) = 0$

APP	Error
x_0	$\varepsilon_0 = x^* - x_0$
x_1	$\varepsilon_1 = x^* - x_1$

$\therefore \varepsilon_1 = O(\varepsilon_0)^2$

using linear approximation to check its performance:

$f(x) = f(x_0) + f'(x_0)(x - x_0) + O((x - x_0)^2)$

$\boxed{x = x^*}$

$\therefore 0 = f(x_0) + f'(x_0)(x^* - x_0) + O((x^* - x_0)^2)$

$0 = f(x_0) + f'(x_0)\varepsilon_0 + O(\varepsilon_0^2)$

$f(x_0) = -f'(x_0)\varepsilon_0 + O(\varepsilon_0^2)$

$\therefore -O(\varepsilon_0^2) = +O(\varepsilon_0^2)$

what matters is only the magnitude.

$\therefore$ 3000, 2000 has the same magnitude 10^3.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \varepsilon_1 = x^* - x_1$$

$$\boxed{x_1 = x^* - \varepsilon_1}$$

$$\varepsilon_1 = x^* - x_0 + \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \boxed{\varepsilon_0 = x^* - x_0}$$

$$\varepsilon_1 = \varepsilon_0 + \frac{f(x_0)}{f'(x_0)}$$

$$\varepsilon_1 = \varepsilon_0 + \frac{1}{f'(x_0)}\left[-f'(x_0)\varepsilon_0 + O(\varepsilon_0^2)\right]$$

$$\varepsilon_1 = \varepsilon_0 - \varepsilon_0 + \frac{O(\varepsilon_0^2)}{f'(x_0)}$$

$$\boxed{\varepsilon_1 = \frac{O(\varepsilon_0^2)}{f'(x_0)}}$$

$$\therefore \boxed{\varepsilon_{n+1} = \frac{O(\varepsilon_n^2)}{f'(\varepsilon_n)}}$$

$$\therefore O(\varepsilon_n^2) = \varepsilon_{n+1} = O(\varepsilon_{n-1}^4)$$

If $f'(\varepsilon_n)$ is too small, $O(\varepsilon_n^2)$ will be large.

So it will converge quickly. (Approximations will work so quickly).

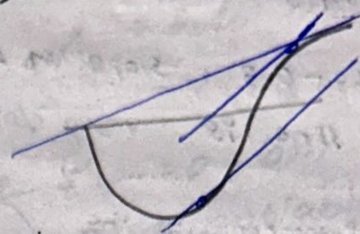
How can this method fail?

★ Bad guess x_0

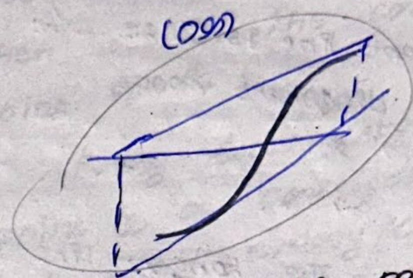
$\varepsilon_0 \to$ big

$\varepsilon_1 = O(\varepsilon_0^2) \to$ Bigger.

(It may converge at some other roots)



(or)



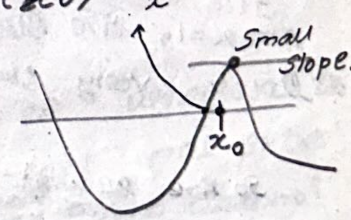
The roots may circle with this values alone.

★ $\varepsilon_1 = \frac{O(\varepsilon_0^2)}{f'(x_0)}$

Good guess x_0

$\varepsilon_0 \to$ small.

$\varepsilon_1 = \frac{O(\varepsilon_0^2)}{f'(x_0)}$



The tangent slope at x_0 may be very very small.

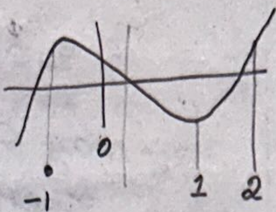
So $O(x^2) \to$ Bigger

(This method may fail to converge)

or it may converge at incorrect roots.

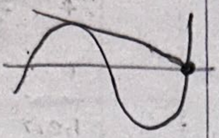
$$\boxed{\begin{aligned} x_0 &= 1 \\ x_0 &= 2 \end{aligned}}$$

$x_0 = -0.5$.



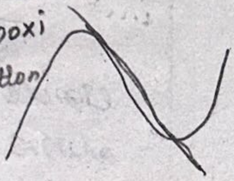
At $x = -0.5$, slope of the tangent line is very close to zero

$\therefore$ x intercept of the tangent line is close to a different root, near $x = 2$



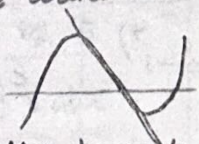
$x = -0.5$

At $x = 0$, The tangent line is very close approximation to the function near the root of interest,

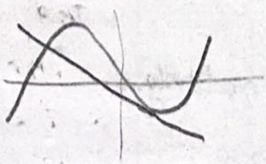


and the second approximation is already very close to the root we want to find.

* At $x = 0.5$, the tangent line is very close to the function and the second app is very close to the root we want to find.

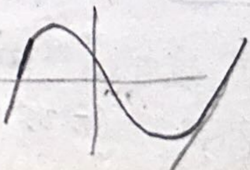


At $x = 1$, the slope of the tangent line is very close to zero,



So the second app will have a tangent line close whose root lies along negative y axis, and will not make it back to the root of interest

At $x = 2$, very close to another root



So, At

$x_0 = -0.5$

$x_0 = 1, 2$

we won't able to find the root of our choice.

Numerical approximation for π can be found by functions

$\sin(x)$

$\cos(x) + 1$

$\tan x$ which

have a root at $x = \pi$.

$f(x) = \cos x + 1$

(This is the wrong fun to use to find this root.) Newton's methods very, very slowly converging.

$\therefore \cos(x) + 1$ has double root at $x = \pi$.

<u>solu</u>:

$x_0 = 1$.

$$\boxed{\begin{aligned} f(\pi) &= 0 \\ f'(\pi) &= 0 \end{aligned}}$$

$x_0 = 3.14$

$x_1 = 3.14 - \dfrac{f(3.14)}{f'(3.14)}$

$= 3.140796327$

$x_2 = 3.14119449$

$x_3 = 3.141393572$

$x_4 = 3.14149311277$

Graphing

$f'(x) > 0$ on (a,b)

f is increasing on (a,b)

$f'(x) < 0$ on (a,b)

f is decreasing on (a,b)

$f(x) = 2x - 4\sin x$

solu:

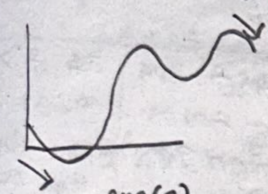
$f'(x) = 2 - 4\cos x$

Decreasing (2π)

$4\cos x = 2$

$\cos x = 1/2$

$x = \cos^{-1}(1/2)$



Decreasing (0)

$x = \frac{\pi}{3}, \frac{4\pi}{3}$.

$f'(0) = 2 - 4\cos(0)$

$= -2$ (Decreasing)

$f'(2\pi) = 2 - 4\cos(2\pi)$

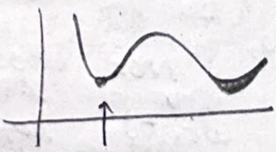
$= -2$ (decreasing

Critical points:

$f'(x) = 0$ (or)

$f'(x)$ doesn't exist.

Local extrema + critical points



Local minimum.

f has a local minimum at $x = a$ if $f(a) \le f(x)$ for all x "near a" (within some small distance of a).

Local maximum:

if $f(a) \ge f(x)$

" "

Local extrema.

$\rightarrow f$ is not differentiable.

If $f'(a) = 0$ (or) f is not differentiable at a. Then a is the critical point of f.

Every local extremum point is a critical point but not every critical point.

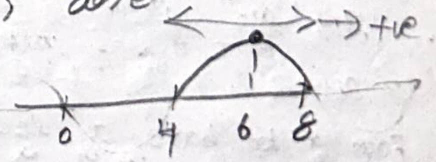
Let's imagine that we have a continuous function f. we've calculated its derivative f'; the only three critical points of f are at $x=0$, $x=4$, $x=8$.

Assume f' is continuous at all other points

solu:

If $f'(6) > 0$

$\therefore$ The places where f' is zero (or might not exist and have a jump discontinuity) are $x = 0, 4, 8$



At $f'(6) > 0$ increas

so

f' is +ve at $x=6$,
So it must remain +ve at
$4 < x < 8$.

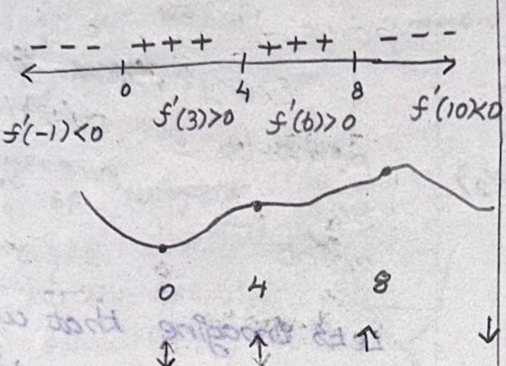
f' is +ve at $x=5$, $x=7$.

So, we don't know anything about other points.

Interval Sampling

∴ function f will change at critical points.

$x = 0, 4, 8$



f has a local minimum at $x=3$

Define a function $g(x)$ and an interval I such that

$g(x) = f(x)$ for x in I

$g(x) \neq f(x)$ for x not in I

Identify all of the intervals I defined below for which g must have a local maximum at $x=3$

Solu:

$I = (2, 3)$

$I = [2.9, 3.1)$

$I = [0, 3.5]$

For g to have a local maximum at $x=3$, $g(3)$ must be greater than or equal to all values of $g(x)$. Thus, any interval I which contains values just to the right & just to the left of $x=3$ will guarantee that the local maximum property is kept intact.

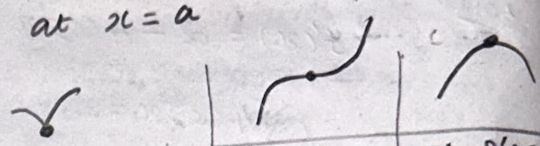
Local min

$(f'(0) = 0)$

$f'(0) = DET$

First derivative test:

Suppose f has a critical point at $x=a$ and it is continuous at $x=a$



If $f'<0$ just to the left of a & $f'>0$ just to the right of a. ⇓ Local min.	If f' has the same sign just to the left & right of a ⇓ Neither	If $f'>0$ just to the left of a & $f'<0$ just to the right of a ⇓ Local max

Second derivative test

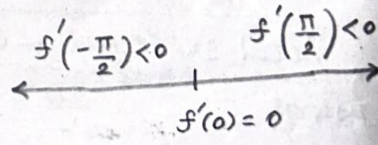
$f(x) = x^2 + 3\cos x$

$f'(x) = 2x - \sin x$

at $x=0$

$f'(x) = 0$

$f'(\frac{\pi}{2}) > 0$, $f'(-\frac{\pi}{2}) < 0$



Not enough to understand what's happening.

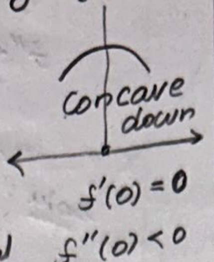
Solving:

$2x - 3\sin x = 0$

↳ can't able to solve algebraically.

$f''(x) = 2 - 3\cos x$

$f''(0) < 0 \Rightarrow f'$ going downwards.



Suppose $f'(a) = 0$

If $f''(a) < 0$, then a is a local max.

If $f''(a) > 0$, then a is a local min.

If $f''(a) = 0$, then... its not a point of inflection.

$$f(x) = \frac{x^5}{20} + \frac{x^4}{4}$$

$$f'(x) = \frac{5x^4}{20} + \frac{4x^3}{4}$$

$$= \frac{x^4}{4} + x^3$$

$$f''(x) = \frac{12x^3}{4} + 3x^2$$

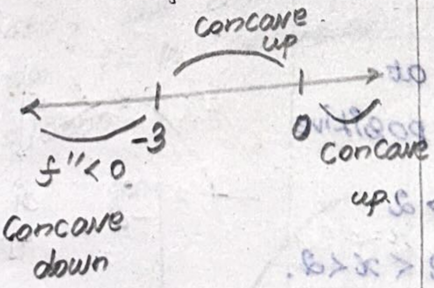
$$= x^3 + 3x^2$$

$$= x^2(x+3)$$

$f''(-4) < 0$

$f''(-1) > 0$

$f''(1) > 0$

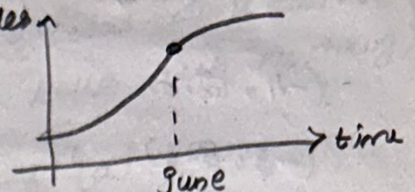


Inflection point: where the second derivative switches its sign.

Here, − + +

Not an inflection point.

'Sales hit an inflection point in June'



meaning: Sales were accelerating concave up and then in June they started to decelerate, concave down.

proper use of inflection point.

eg: The fall of the Berlin wall was a political inflection point.

↳ Not very mathematical.

An inflection point is a point where the concavity of the function changes.

∴ second derivative $f''(x)$ changes sign

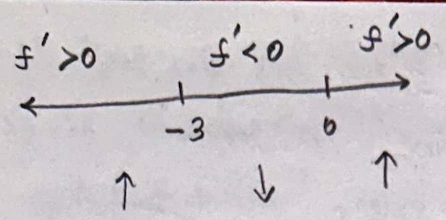
$f''(x) > 0$ just to the left of x and $f''(x) < 0$ just to the right of x (vice versa).

$$f(x) = \frac{x^5}{20} + \frac{x^4}{4}.$$

$$f'(x) = \frac{x^4}{4} + x^3$$

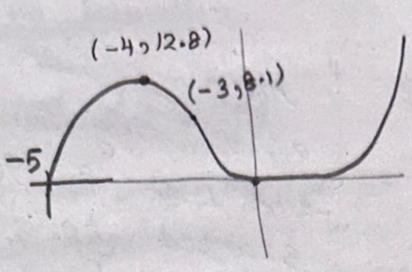
$$0 = \frac{1}{4}x^3(x+4)$$

$x = 0$, $x = -4$.

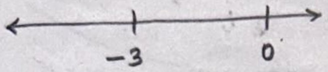


$(-4, 12.8)$

$(-3, 8.1)$



∴



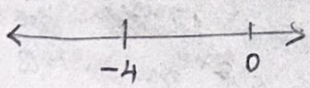
\- + +

Concave up

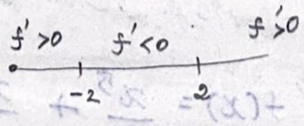
Inflection point (up to 0)

∴ By First derivative, The function reaches local maxima at $x = -4$.

∴



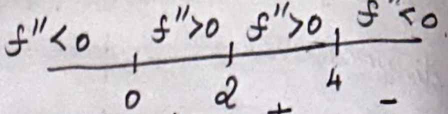
Suppose $f'(x)$ is zero at $x = -2$ and $x = 2$, is positive for $x < -2$ and $x > 2$ and negative for $-2 < x < 2$.



Suppose $f''(x) = 0$ at $x = 0$, $x = 2$, is +ve on the $0 < x < 2$ and $2 < x < 4$ and is -ve for $x < 0$ and $x > 4$.

solu:-

Local minimum & maximum



\- + + -

concave up concave down

solu:

maxima : (Local)

$x = -2, x = 2$

↳ $f'(x) = 0$

$f' > 0$ at $x = -2$

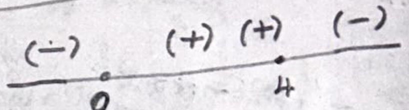
so

Local maximum is $x = -2$

Local minimum

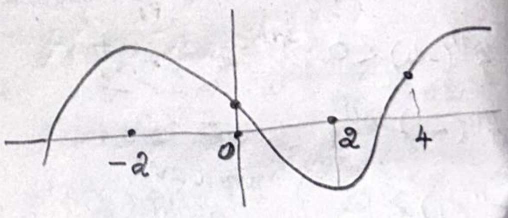
$x = 2$, $f' < 0$.

Inflection points:



Concave up concave down

$x = 0, x = 4$.



(one of the possible Answers)

Suppose that a function f is defined and continuous for all real numbers x, and

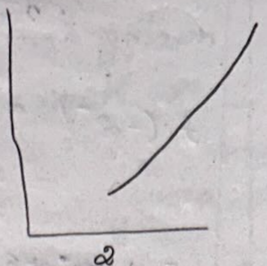
$f(2) = 1$

$f'(x) > 0$ for all $x \geq 2$

$f''(x) > 0$ for all $x \geq 2$.

The graph is $\uparrow$, (concave up)

$f''(x) > 0$



Is there a point $x = a$ where $f(a) = 10000$

yes.

Suppose that g is defined & continuous for all real numbers x, and

$g(2) = 1$

$g'(x) > 0$ for all $x \geq 2$.

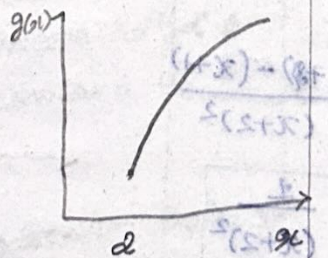
$g''(x) < 0$ for all $x \geq 2$

Is there a point $x = a$ where $g(a) = 10000$?

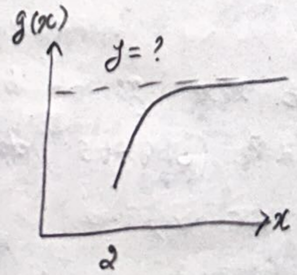
Solu:

$\therefore$ Graph is increasing,

The graph is $\uparrow$. But it might have a horizontal asymptote or it might be unbounded. If it has a horizontal asymptote, we don't know whether it is above or below $y = 10000$.



(or)



Suppose we have a critical point $x = a$ where $f'(a) = 0$, $f''(a) = 0$ (so the second derivative test tells us nothing).

$f'''(a) > 0$.

Solu:

What can we conclude about $f''(x)$ when x is just to the right of a?

It is +ve.

What can we conclude about $f''(x)$ when x is just to the left of a?

It is -ve.

$\therefore$ we have $f'''(a) > 0$

$\therefore$ Indicating, the second derivative f'' must be $\uparrow$ near $x = a$.

$f'(a) = 0$, f'' just to the right of $x = a$, what can we conclude about $f'(x)$ when x is just to the right of a?

$\therefore$ second derivative is $\uparrow$.

$f''(x) > 0$ when x is just to the right of a.

$\therefore$ $\boxed{f'(a) = 0}$

$\therefore$ $f' > 0$ just to the right of a.

$f''(x) < 0$ [x is just left of a.

meaning, f' is ↓ on some small interval just to the left of a.

$f'(a) = 0$.

∴ $f' > 0$ just to the left of a.

Critical point ($x = a$):

we discovered that $f' > 0$ just to the right of a, and $f' > 0$ just to the left of a.

∴ By First Derivative Test, f has neither a local min or max at $x = a$.

Big picture.

'Long term behaviour'

1) Chemist: Steady state
2) Computer scientists: 'Algorithms' (Large data sets)
3) economist: Future behaviour of stocks.

Sketching

$f(x) = \frac{x+1}{x+2}$, $f'(x) = \frac{(x+2)-(x+1)}{(x+2)^2}$

$$\boxed{0 \neq \frac{1}{(x+2)^2}}$$

'plot the graph'

To find x values where f(x) is undefined.

'plot points'

$x = -2$ (where the function is not defined).

$f(-2^+) = \frac{}{-2^+ + 2}$ small

$= -\infty$

Right

-2

$f(-2^-) = \frac{-2+1}{-2^- + 2} = +\infty$

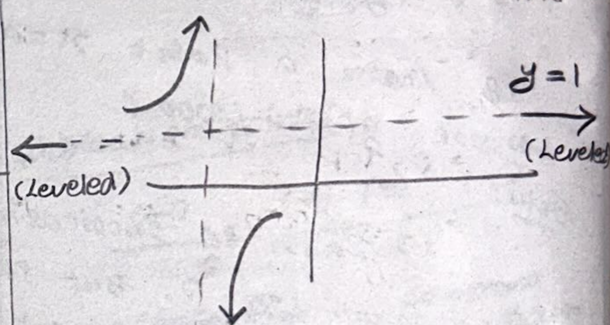
End points:

$x \to \pm\infty$

$f(x) = \frac{x+1}{x+2} = \frac{1 + 1/x}{1 + 2/x}$

$= \frac{1 + 1/\infty}{1 + 2/\infty}$

$f(\pm\infty) = 1.$



$x = -2.$

$f(-2^+) = -\infty$

$f(-2^-) = \infty$

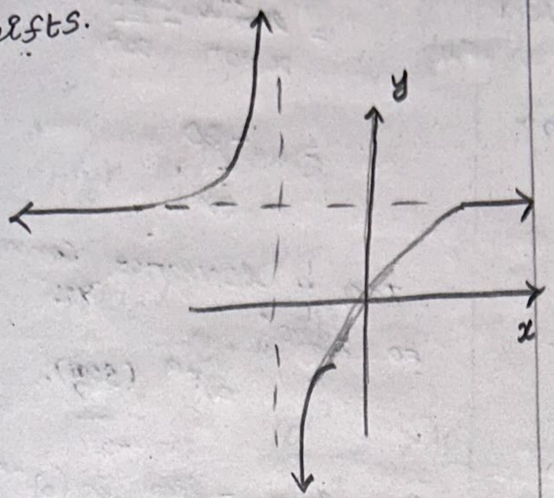
Is the graph back track? (Shifts from ↑ to ↓)

∴ $f'(x) = \frac{1}{(x+2)^2} \Rightarrow$

graph is increasing.

So the graph must be

$f'(x) \neq 0$, the graph doesn't back track or shifts.



Plot:

* Discontinuities (especially ∞ ones)
* End points (or $x \to \pm\infty$)
* Easy points ($x = 0$, or $y = 0$) (optional)

2) Plot critical points & values. Solve $f'(x) = 0$ or undefined.

3) Decide whether $f' < 0$ or $f' > 0$ on each interval b/w endpoints, critical points, and discontinuities.

4) Identify where $f'' < 0$ and $f'' > 0$ (concave up or down).

(Identify inflection points)

5) Combine the graph

Consider $\frac{x}{(\ln x)^2}$.

Solu:

$$y' = \frac{(\ln x)^2 - x \times 2 \times \frac{1}{x}}{(\ln x)^4}$$

$$y = \frac{2\ln x - 2}{4\ln x}$$

$$y' = \frac{\ln x - 1}{2\ln x}$$

$$f(0^+) = \frac{+0^+}{(\ln(0^+))^2}$$

$$= +\infty$$

$$f(0^-) = \frac{0^-}{2\ln(0^-)}$$

$$= +\infty$$

End points: $\boxed{x = +\infty}$

$$f(x) = \frac{x}{(\ln x)^2}$$

$f(+\infty) =$ (Right side)

$= +\infty$

Left end point:

$x = 0 \to$ alone.

∴ For function $\ln x$ is defined for $x > 0$ alone.

∴ $\boxed{x = 0}$

$$f(x) = \frac{x}{(\ln x)^2}$$

$0 < x < 1$, $1 < x < \infty$

$f(1^+) = \infty$

$f(1^-) = \infty$

$$f(0^+) = \frac{0^+}{-\infty}$$

$$\lim_{x\to 0} \ln(x) = -\infty$$

Proving that $\ln(x)$ is monotone increasing:

$x_1, x_2 \in R^+$, with

$x_2 > x_1$.

$$\ln x_2 = \ln\left(\frac{x_2}{x_1} \cdot x_1\right)$$

$$\ln x_2 = \ln\left(\frac{x_2}{x_1}\right) + \ln x_1$$

$$\boxed{x_2 > x_1}$$

$$\therefore \frac{x_2}{x_1} > 1$$

$$\ln\left(\frac{x_2}{x_1}\right) > 0.$$

$$\ln(x_2) > \ln(x_1)$$

$\therefore$ since it's monotone $\uparrow$. $\ln x$ and has a limit for $x \to \infty$ and since the function is not bounded this limit must be $+\infty$. So:

$$\lim_{x\to\infty} \ln x = +\infty$$

$$\therefore \ln\left(\frac{1}{x}\right) = -\ln x$$

$\therefore$ logarithm is defined only for $x>0$

$$\lim_{x\to 0} \ln x = \lim_{x\to 0^+} \ln x$$

$$y = \frac{1}{x}$$

$$\lim_{x\to 0^+} \ln x = \lim_{y\to\infty} \ln\left(\frac{1}{y}\right) = \lim_{y\to\infty} -\ln(y)$$

$$= -\infty$$

(or)

$$L = \lim_{x\to 0^+} \ln x$$

$$= \lim_{x\to 0^+} \frac{1}{x}$$

$$= \lim_{x\to 0^+} -\frac{1}{x^2}.$$

$$= -\infty$$

(or)

Take a sequence converging to zero.

2^{-n} (say).

$$\lim_{x\to 0} \ln x = \lim_{n\to\infty} \ln(2^{-n})$$

$$= -\lim_{n\to\infty} n \ln 2$$

$$= -\infty$$

$$y = \ln x$$

$$\boxed{e^y = x}$$

$x \to 0$, $e^y \to 0$ (as)

$$\boxed{y \to -\infty}$$

$$\therefore \lim_{y\to-\infty} \ln(e^y)$$

$$\therefore = \lim_{y\to-\infty} y$$

$$= -\infty.$$

$$f(x) = \frac{x}{(\ln x)^2}$$

$0 < x < 1$, $1 < x < \infty$

Solu:

$f(1^+) = \infty$

$f(1^-) = \infty$