$$\boxed{\therefore -\sin(\omega x) = \sin(-\omega x)}$$

conclusion:

There exist non-zero solutions if & only if $\lambda = -n^2$ for some positive integer n; integer n. in that case, all solutions are of the form $b \sin nx$.

Analogy with Eigen-value & Eigen vector problems

To describe a function $v(x)$, one need to give only many numbers, namely its value at all the different input x values. Thus $v(x)$ is like a vector of infinite length.

The linear differential operator $\frac{d^2}{dx^2}$ maps each function to a function, just as a 2×2 matrix defines a linear transformation mapping each vectors in R^2. to another vector in R^2. Thus $\frac{d^2}{dx^2}$ is like an $\infty \times \infty$ matrix.

The ODE $\frac{d^2}{dx^2} v = \lambda v$ (with boundary conditions) is like a system of equations: the ODE amounts to an infinite system of equations at each x consists of one equality of numbers, and boundary conditions are in the interval $(0, \pi)$, and boundary conditions are at the endpoints. Thus the ODE with boundary boundary conditions is like a system of equations

$$\boxed{Av = \lambda v}$$

Non-zero solutions $v(x)$ to $\frac{d^2}{dx^2} v = \lambda v$ exists only for special values of λ, namely

$$\lambda = -1, -4, -9, \ldots$$

The nonzero solutions $v(x)$ to $\frac{d^2}{dx^2}(v) = \lambda v$ satisfying BC, are called Eigen functions. Since they act like Eigen vectors.

Summary:

Eigen vector problem	Eigenfunction problem
vector v	Function $v(x)$
$n \times n$ matrix A	The linear operator $\frac{d^2}{dx^2}$
Eigen value - Eigen vector problem $Av = \lambda v$	BVP $\frac{d^2}{dx^2} v = \lambda v \quad (0 < x < \pi, \; v(0) = 0, v(\pi) = 0)$
No more than n eigen values λ	Eigen values $\lambda = -1, -4, -9, \ldots$
No more than n Eigen vector v	Eigen functions $v(x) = \sin(\sqrt{-\lambda}\, x)$, $\lambda = -1, -4, -9, \ldots$

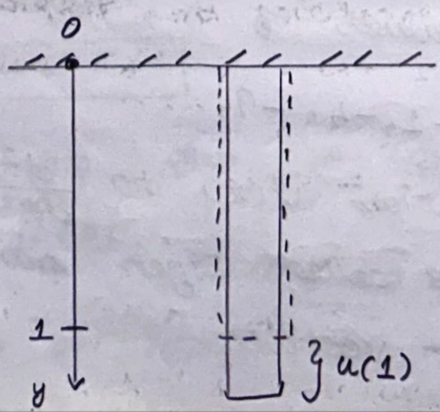
Revise 2x2 systems

Linear Elasticity

Boundary condt are most common when the independent variable is space rather than time. In time, we can typically assign initial condt. In space, it makes sense to specify conditions on the end points of a spatial object.

Linear Elasticity for vertical beams:

Suppose we have a bar hanging vertically with the top end attached to a solid surface. The y coordinate is the spatial variable running along the length of the bar, so that each point on the non-elongated bar exists on the interval $y \in [0, 1]$

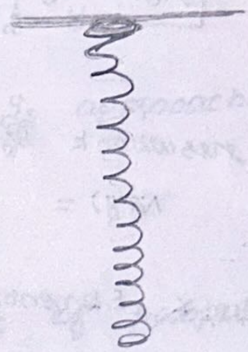


{Spatial- Relating to space}

under its own weight, or an external force $f(y)$ pulling downwards, the bar will tend to lengthen. Thus each point on the bar will move to a new location, displaced to a new location, displaced by some amount u. So that the new point y is $y+u$

The displacement u is a function of position, $u = u(y)$.

example: Take a slinky (precompressed helical spring), and hang it on one end from the ceiling (or the under side of a shelf?)



Then the rings seperate, with the seperation b/w the rings decreasing as you go down from the top. The u in our eqn is then the distance from where a particular ring is at, to the position it would have if the slinky rings were placed right next to each other as if it were floating in zero gravity.

→ zero gravity (As in our hands)

Then the diff eqn describing the displacement in terms of the external stress per coil (of slinky) $f(y)$ is

$$\frac{d^2u}{dy^2} = \frac{1}{E} f(y)$$

where the constant E is determined by the material Elasticity.

When the external force is gravity,

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{E}$$

$\rho \rightarrow$ Density of the beam, g - constant of gravitational acceleration.

$\boxed{\text{Stress} = \frac{\text{Force}}{\text{Area}}}$ $\boxed{\text{Density} = \frac{\text{mass}}{\text{Volume}}}$

Elasticity $\rightarrow$ measure of how difficult it is to stretch an object.

unit: $ML^{-1}T^{-2}$, Density: ML^{-3}, Gravity: LT^{-2}

Physics of Linear elasticity:

$f(y) \rightarrow$ External force per unit volume,

$\boxed{f(y) = \rho g}$ $\boxed{\therefore g\rho = \frac{\text{mass} \times g}{\text{Volume}}}$

$N(y) \rightarrow$ Internal resultant of these external forces,

which is defined as $N(y) = \int_y^1 f(w)\,dw$

'Take - 2.01 X Elements of structures'

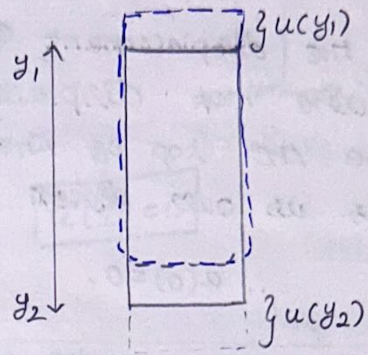
There is a force per unit area on the bar, which arises from the deformation of the material & its elasticity. There is an internal resultant of this force, $N(y)$, which is felt by the cross sectional area A through any position y on the vertical axis of the bar, is defined as stress linear elasticity tells us that the stress is proportional to the strain.

Stress $\propto$ Strain

Let E be this constant of proportionality. The strain is defined as the ratio of the relative change in length in any length ("infinitesimal") cross-section of the bar relative to its rest state.

In terms of the slinky, the strain is the ratio of the increment in seperation b/w the coils under the stress, and their seperation at rest.

To find the strain, choose two points in the non-elongated bar: y_1 and y_2. In the elongated bar, the displacement of y_1 is $u(y_1)$, and the displacement of y_2 is $u(y_2)$. So the total change in length b/w these points is $u(y_2) - u(y_1)$. The initial length of the bar b/w these points is $y_2 - y_1$.



The ratio is $$\frac{u(y_2) - u(y_1)}{y_2 - y_1} = \frac{\Delta u}{\Delta y}$$

Taking the limit as y_2 approaches y_1, we get the strain $$\left.\frac{du}{dy}\right|_{y=y_1}$$

Let E be the constant of proportionality relating the stress $N(y)/A$ to the strain $\frac{du}{dy}$

$$\boxed{EA \cdot \frac{du}{dy} = N(y)}$$

$\boxed{\text{Stress} \propto \text{Strain}}$ → Resultant (Internal) forces)

$\therefore \frac{N(y)}{A}$ (Resultant force per area)

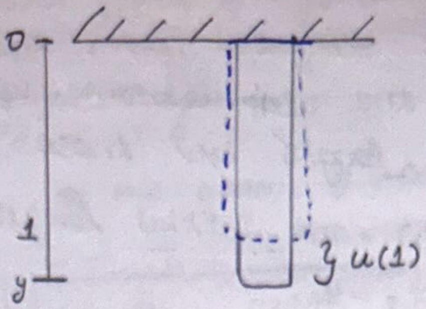
$$\frac{du}{dy} \propto N(y)$$

$$\frac{du}{dy} = \frac{1}{E} \cdot \frac{N(y)}{A}$$

$$\boxed{EA \frac{du}{dy} = N(y)}$$

<u>Boundary conditions</u>

Case: 1: Fix one end, other end hanging free.



$u(0)=0, \quad \frac{du}{dy}(1)=0.$

u- represents the displacement of a point, and so points on the bar that are not displaced in the stretching won't move. since the top of the bar is firmly set in place, this gives us our first boundary condition.

$$u(0)=0.$$

At any cross section along the beam, Force per unit area (stress) is equal (bar to be just enough to support the weight of the bar below this point.

Thus:

$$EA \cdot \frac{du}{dy} = g\,W(y)$$

$W(y) \rightarrow$ mass of the bar below y. suppose the density ρ of the beam & the cross sectional area A are both constant along the beam.

$$\text{density} = \frac{\text{mass}}{\text{Volume}}$$

$$\text{mass} = \text{Density} \times \text{Volume}.$$

$$\boxed{\int \text{Area} = \text{volume}}$$

$\therefore$

$$\therefore \quad W(y) = \int_y^1 \rho A \cdot dx = \rho A \,(x)_y^1 = \rho A(1-y)$$

$\boxed{\therefore\ 0 \text{ to } 1 \rightarrow \text{object doesn't displace}}$

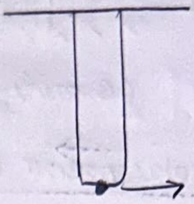
$\therefore$ where $1 \rightarrow$ length of the beam.

$$\therefore \quad AE\,\frac{du}{dy} = (\rho A - \rho A y)g$$

Taking derivative w.r.t y

$$\boxed{AE \frac{d^2u}{dy^2} = -PAg}$$

At the end of the rod:



No more mass to hang.

$\therefore AE \cdot \frac{du}{dy} = 0$

$$\boxed{\frac{du}{dy} = 0} \qquad \boxed{u'(1) = 0}$$

Q:1: what's the formula for the displacement $u(y)$ if the force is gravity acting along a uniform beam?

Solu:

$$\boxed{W(y) = \text{gravity then}}$$

[spring motion]

$$\frac{d^2u}{dy^2} = -\frac{1}{E} \cdot \frac{gP}{1}$$

$$\frac{du}{dy} = -\frac{1}{E} \cdot gP \cdot y + c_1$$

$$u = -\frac{1}{2}\frac{gP}{E} y^2 + c_1 y + c_2$$

sub Boundary Conditions,

$u(0) = 0$,

$\frac{du}{dy}(1) = 0$

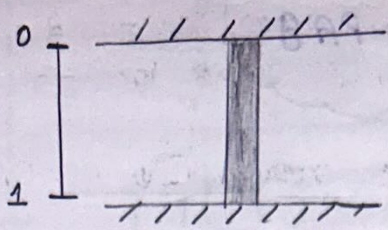
① ⇒ $\boxed{0 = c_2}$

② ⇒ $0 = -\frac{1}{E} gP + c_1$

$$\boxed{c_1 = \frac{gP}{E}}$$

$$\therefore \quad u(y) = -\frac{1}{2}\frac{gP}{E} y^2 + \frac{gP}{E} y.$$

Case: 2 : Fixed at both ends



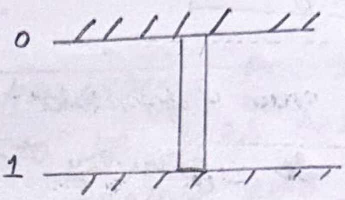
(Gravity alone acting, it's at steady state)

<u>$u \rightarrow$ represents displacement (change in length)</u>

$u(0) = 0, \quad u(1) = 0 \rightarrow$ No displacements.

<u>Boundary value problem:</u>

'A vertical beam fixed at both ends - Force of gravity alone acting - steady state'



$$\therefore \boxed{u(y) = -\frac{1}{2}\frac{gP}{E}y^2 + c_1 y + c_2}$$

$$u(0) = c_2 \Rightarrow \boxed{c_2 = 0}$$

$$u(1) = -\frac{1}{2}\frac{gP}{E} + c_1$$

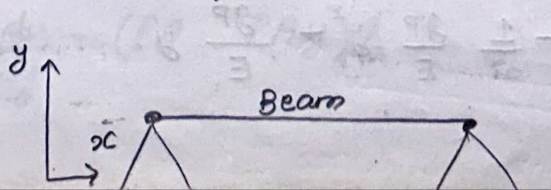
$$\boxed{c_1 = \frac{1}{2}\frac{gP}{E}}$$

$$\boxed{u(y) = -\frac{1}{2}\frac{gP}{E}y^2 + \frac{1}{2}\frac{gP}{E}y}$$

<u>Beam bending equation</u>

'Fourth order diff eqn that describes the static bending of a slender, horizontal beam due to a distributed load. The main simplifying assumption we are making is that we are only modelling the steady state, static bending. That's our model doesn't change in time, so there won't be no time derivatives. The other simplifying assumption is that the beam doesn't bend very much.

'Draw & mark variables'

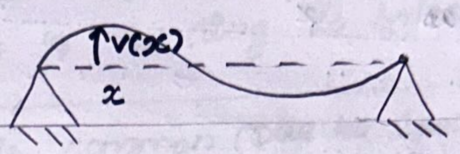


(undeformed beam)

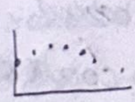
'Two ends placed (pinned) struck into the ground'.

vertical direction → y axis
horizontal direction → x axis.

<u>vertical displacement be v(x)</u> [at every point x along the beam].



Deformed beam

Geometric quantities of interest	variable name
vertical deflection of the beam at each point.	$v(x)$
slope of the beam at each point	$\frac{dv}{dx}(x)$
curvature of the ~~beam~~ at each point (2nd derivative)	$\approx \frac{d^2v}{dx^2}(x)$
(Assuming v(x) is small.	 $A = \int\int dx.$

<u>Goal:</u>

* Loading on the beam.
* Boundary conditions (constraints)
* material of the beam
* Geometry of the beam.

(Deformation of a beam — Bending
(slender beams).)

Beam in structural mechanics:
carries load which are orthogonal to its longitudinal axis.

Slender: Length of the beam is very much larger than any of the other dimensions.

Static $\rightarrow$ No acceleration

$\boxed{\text{Linear} \rightarrow \text{when bending is small.}}$

<u>2d - representation</u>

<u>Deflection $= v(x)$</u>

$$\text{Slope} \quad v(x) = \frac{dv}{dx}(x)$$

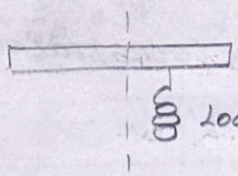
$$\text{curvature} = \frac{1}{\text{radius of curvature}} \quad (x) \longrightarrow \text{at any section } x.$$

$\boxed{\text{when radius is large (Flat)} \rightarrow \text{curvature zero.}}$

$$\text{curvature} = \frac{dv^2}{dx^2}(x)$$

<u>we can analyze</u>

1) Know external loading
2) Boundary condition
3) Knowing constraints (pinned), material, geometry.

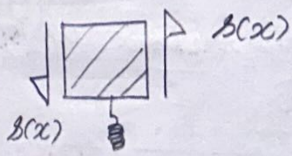


when we cut (That portion will fall)

so there must be an internal force which prevents the portion from falling.

(Resultants)

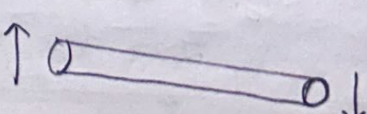
<u>Shear Force resultants $\boxed{S(x)}$</u>



Small portion

<u>Torque:</u>

'Force couple'

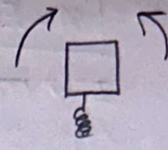


'Equal & opposite force'.

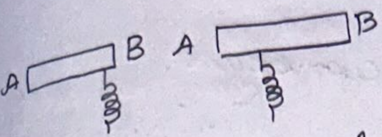
Two resultants:

Shear force $S(x)$

moment of bending $M(x)$



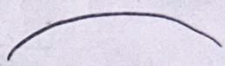
'Bending'



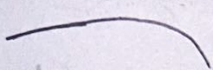
As we place load far & far away from B, we need more moment of bending to balance is high.

∴ Shear force is common (Due to load). → constant

But moment of bending is not.



Curvature is large.



Curvature is small.

$$\therefore \; M(x) \Leftrightarrow \frac{1}{\rho}(x) = \frac{d v^2}{dx}(x)$$

(moment of bending is directly related with curvature of the bending.)

How we are going to relate:

'Elastic theory of deformation'

'property (Stiffness) of the material determines how much it can be bend' easily.

How much load is required → To get a particular curvature.

↓

Depends on material.

$M(x) = E$ (Also depends on cross section).

$\boxed{\text{E- Young's Modulus}}$ → Large (Difficult to deform)

$\boxed{M = EI} \times \frac{1}{\rho(x)}$ $I = \int y^2 dA$

$I \rightarrow$ moment of inertia

'Stiffness the geometry', how far the area is away from the midline of the beam?

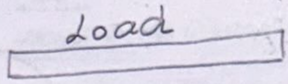
$\boxed{M = EI} \times \frac{1}{\rho(x)}$ ⟶ curvature.

$I = \int y^2 \, dA$

⟶ Area of cross section

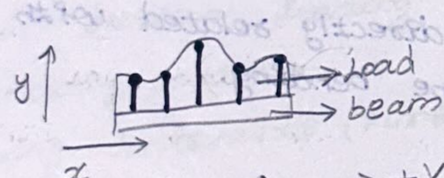
⟶ Area (Far) away from mid-line.

Deformation for load:



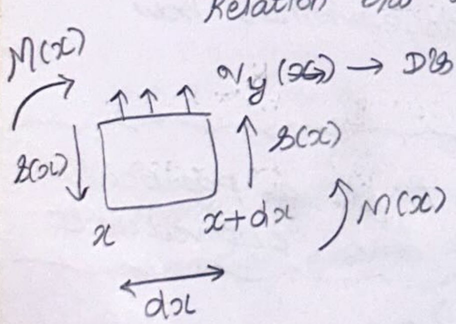
(Load is distributed throughout) /unit length

'At every point - a mass will be there, force will be acting' (may not be equal, but a force is there'.

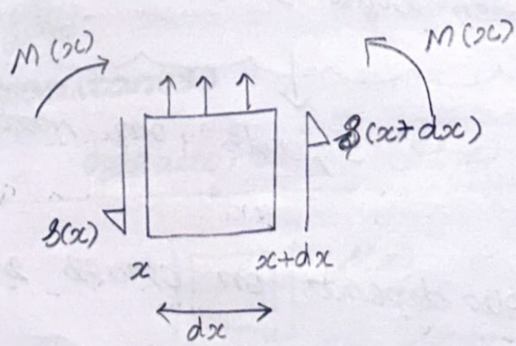


$q_y(x)$ ⟶ +ve & upward

Relation b/w distributive load & resultant forces

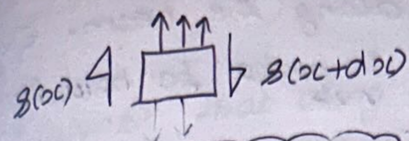


$q_y(x)$ → Distributive load.



Shear force resultant:

'Shear force resultant needs to change along the longitudinal direction according to how much extra load you put in each little dx slice.



$$\frac{ds}{dx} = -q_y(x)$$

moment of bending:

$$\boxed{\frac{dM}{dx} = -s(x)}$$

'Constant shear force'.

$$\therefore \quad q_y(x) = \frac{ds}{dx} = \frac{d^2 M}{dx^2} = \frac{d^2}{dx^2}\left(EI \times \frac{d^2 v}{dx^2}\right)$$

special case: EI $\rightarrow$ uniform (constant)

$$EI\left(\frac{d^4 v}{dx^4}\right) = q_y(x)$$

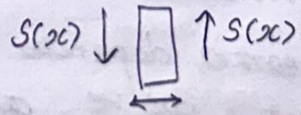
In order to determine how the load will affect beam bending, we need to understand all of the resultant forces inside of the beam that occurs due to external loading.

Two main resultants (Important to us)

1) Shear force $S(x)$
2) Bending moment $M(x)$ [equivalent to torque]

'Internal force preventing the reign from falling - Shear force'.

Internal to the beam equal & opposite to the resultant force.



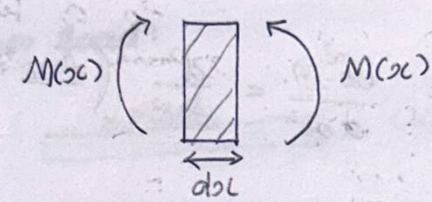
infinitesimal slice of beam

Shear force S with +ve orientation of

In our example:

Shear force is constant along the beam. Exactly cancelling the effect of the weight in order to hold it up.

If we hold the beam further & further away from the point where the mass is hung, we not only have to apply a vertical shear force, but the further away from the point of contact, the force also creates a torque, and the beam must supply a torque. (or bending moment as it is called in the mechanics of beam) to prevent the beam from rotating.



Bending moment M with +ve orientation on an infinitesimal slice of the beam.

The more the beam is trying to curve, the more bending moment must be applied to counter the rotation. In our toy example, we see that the further we are from the shear force, the more moment we need to counteract the effect.

$$\frac{d}{dx} M(x) = -S(x)$$ [Change in moment of beam bending = Shear force]

Additionally,

The bending moment is exactly what causes the beam to bend. Thus it is proportional to the curvature, & the relationship b/w the bending moment & curvature is gn by

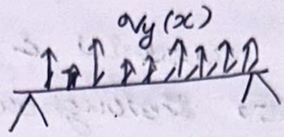
$$\boxed{M(x) = EI \frac{d^2 v}{dx^2}}$$

E → Depends on material of the beam

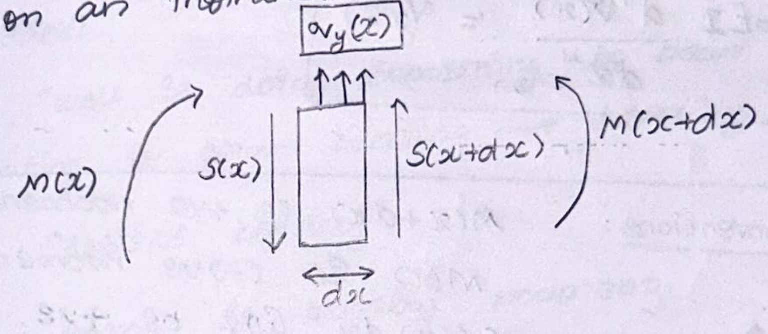
I → moment of Inertia depends upon geometry of the beam, Now that we've all the forces.

How are resultant forces affected by a distributed load?

Suppose that along our beam, we have some distributed load $v_y(x)$ [units of (force)/(Length)], where the subscript y is used to denote the fact that the +ve direction points in the +ve y direction.



To see, how this load affects the resultant forces inside the beam, we need to do a force & torque balance on an infinitesimal segment of our beam.



On an infinitesimal segment of beam of width dx, we can assume that the load is the constant value $v_y(x)$. Thus the total force is $v_y(x)\,dx$. Force balance tells us that

$$S(x+dx) - S(x) + v_y(x)\,dx = 0$$

$$v_y(x)\,dx \rightarrow \frac{\text{Force due to mass}}{\text{Length}} \qquad \frac{kg\,ms^{-2}}{s}$$

$$\boxed{\therefore \text{Force} = kg\,ms^{-2} \quad \therefore v_y(x)\,dx}$$

$$\therefore \frac{S(x+dx) - S(x)}{dx} = -v_y(x)$$

As $dx \to 0$

$$\frac{d}{dx}(S(x)) = -v_y(x)$$

$$\therefore \quad \frac{d}{dx} M(x) = -S(x)$$

$$q_y(x) = -\frac{d}{dx} S(x) = \frac{d^2}{dx^2} M(x)$$

$$= \frac{d^2}{dx^2}\left(EI \frac{d^2 v}{dx^2}\right)$$

In this case that the material & the geometry of the beam are constant throughout, this reduces to the fourth order differential equation relating the deflection of the beam to the external loading

$$\boxed{EI \frac{d^4 v(x)}{dx^4} = q_y(x)}$$

<u>Conventions:</u> $M(x+dx)$ is +ve moment

$M(x)$ is (−)ve moment

$S(x)\,dx$ will be +ve.

 $\longrightarrow$ +ve convention

(Beam bending - smiling)

<u>Horizontal beams & boundary conditions</u>

$$EI \frac{d^4 v}{dx^4}(x) = q_y(x)$$

$$M(x) = EI \frac{d^2 v}{dx^2}(x)$$

$$S(x) = \frac{dM(x)}{dx} = -EI \frac{d^3 v(x)}{dx^3}$$

$$\theta(x) = \frac{dv}{dx}(x)$$

<u>Boundary conditions:</u>

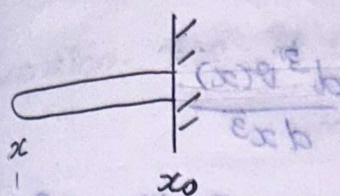
'If we want to solve a specific problem in beam bending - we also need to know how to impose the correct boundary conditions.'

(essential boundary condition)

'supporting the beam':

'Fixed boundary conditions'

case:1 Fixed at wall.



We know that

$\boxed{v(x_0) = 0}$

slope at x_0

$\frac{dv(x_0)}{dx} = 0$

Don't know:

'wall is doing something to beam'

preventing it from bending → bending moment.

'Resists bending'

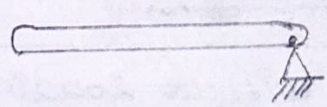
1) shear force & bending moment.

$\frac{d^3 v(x_0)}{dx^3}$ (shear force), $\frac{d^2 v(x_0)}{dx^2}$ (moment of bending)

Case:2

Pin support:

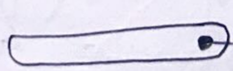
'Frictionless pin'



This frictionless pin prevent moving left to right.

(moving)

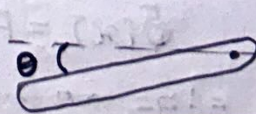
$\boxed{\text{But it can't prevent beam from rotating.}}$



→ stuck by a pin.

(The beam can rotate).

hinge (Frictionless pin)



Known:

$v(x_0) = 0$, $M(x_0) = 0$.

$\boxed{\text{Even without loading it may rotate.}}$

No displacement at x_0 $\quad v(x_0) = 0.$

$$M(x_0) = 0 \quad \left(\frac{d^2 v}{dx^2}(x_0) = 0 \right)$$

Unknown

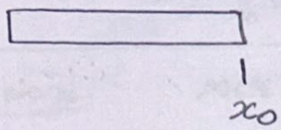
Slope at x_0 (Deformed)

$$\frac{dv}{dx}(x_0) = ?$$

$\boxed{\text{Shear force} = ?} \Rightarrow S(x) = \dfrac{d^3 v(x)}{dx^3}$

$\therefore$ No displacement at x_0. So the pin resists vertical motion due to mass.

Case : 3 $\qquad$ Free (without shear force)
↳ No load.



Unknown :

$$\frac{dv}{dx}(x_0) = ? \qquad v(x_0) = 0$$

No Force :

At free end $\quad M(x_0) = 0, \; S(x_0) = 0$

$$\frac{d^3 v}{dx^3}(x_0) = 0 \;, \; \frac{d^2 v}{dx^2}(x_0) = 0.$$

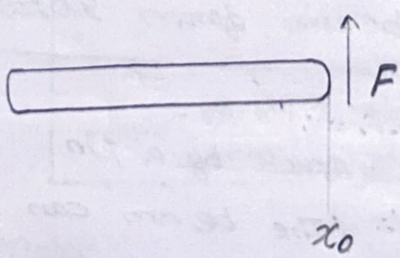
---

case 1, 2, 3 $\longrightarrow$ without loading.

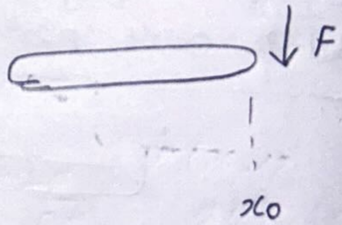
(Torque) Force couples $\rightarrow$ Same Bending moments

Loading (concentrated loads)
↳ Forms a Force couple)

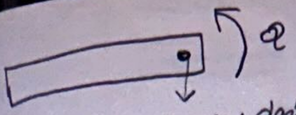
Case : 4 :



$S(x) = F.$



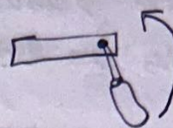
$S(x) = -F$



Screw driver. → Applying a concentrated moment.

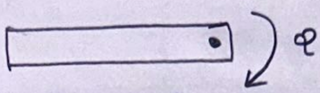
(How)

$\boxed{M(x_0) = Q}$



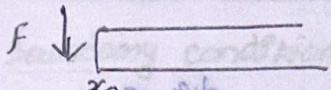
Then rotate screw driver. (Q)

'Convention: +ve Q (moment) we made the beam to smile'.

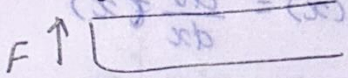


$M(x_0) = -Q$

If Free end is on the left



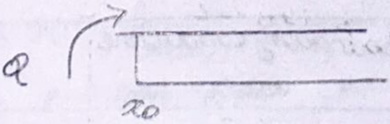
$\longrightarrow S(x) = F$



$\longrightarrow S(x) = -F$

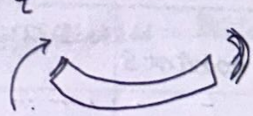
'Reverse sign'.

Downward force → +ve



$M(x_0) = Q$

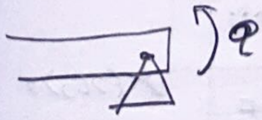
(smile



$-Q$

At pinned

(we Apply a moment by screw driver



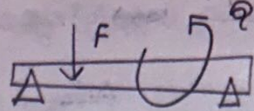
$v(x_0) = 0$

$M(x_0) = Q$

$$\boxed{\frac{dv^2}{dx^2}(x_0) = \frac{Q}{EI}}$$

∴ curvature & Moment are related by Material & Geometrical properties

Advanced



Shear Force to the resultant left & to the right of the applied load is to be different.

concentrated moment at a particular section of the beam, what's going to jump from left to right of the point of application of the moment is the bending moment resultant.

Note:

Equation governing the static deflection of a slender horizontal beam under a load $q_y(x)$ is gn by)

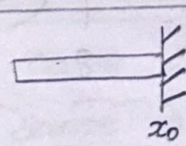
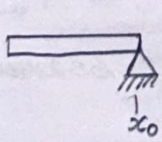
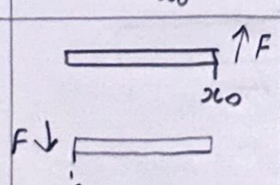
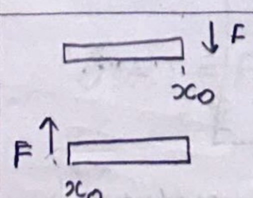
$$EI \frac{d^4 v(x)}{dx^4} = q_y(x).$$

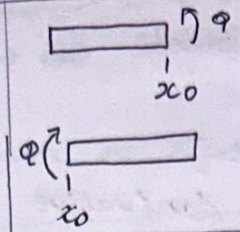
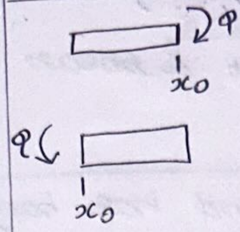
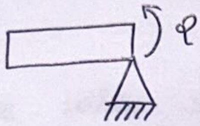
where

Angle of deflection $\theta(x) = \frac{dv}{dx}(x)$

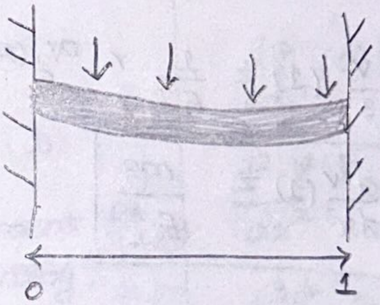
Bending moment $M(x) = EI \frac{d^2 v}{dx^2}(x)$

Shear forces $S(x) = -EI \frac{d^3 v}{dx^3}(x).$

constraints	Drawing	Boundary conditions	unknown
Fixed (in wall)		$v(x_0) = 0$, $\frac{dv}{dx}(x_0) = 0$	$\frac{d^2v}{dx^2}(x_0)$ & $\frac{d^3v}{dx^3}(x_0)$
pinned (on hinge)		$v(x_0) = 0$, $\frac{d^2v}{dx^2}(x_0) = 0$	$\frac{dv}{dx}(x_0)$ and $\frac{d^3v}{dx^3}(x_0)$
Free with applied shear		$\frac{d^2v}{dx^2}(x_0) = 0$ $\frac{d^3v}{dx^3}(v_0) = -\frac{F}{EI}$	$v(x_0)$, $dv/dx\,(x_0)$
Force		$\frac{d^2v}{dx^2}(x_0) = 0$ and $\frac{d^3v}{dx^3}(x_0) = \frac{F}{EI}$	$v(x_0)$ $\frac{dv}{dx}(x_0)$

Free		$\frac{d^2v}{dx^2}(x_0)=0$ $\frac{d^3v}{dx^3}(x_0)=0$	$v(x_0)$ & $\frac{dv}{dx}(v_0)$
Free with applied torque		$\frac{d^2v}{dx^2}(x_0)=\frac{\varphi}{EI}$ $\frac{d^3v}{dx^3}(x_0)=0$	$v(x_0)$ $\frac{dv}{dx}(x_0)$
		$\frac{d^2v}{dx^2}(x_0)=-\frac{\varphi}{EI}$ $\frac{d^3v}{dx^3}(x_0)=0$	$v(x_0)$ $\frac{dv}{dx}(x_0)$
Pinned with applied torque		$v(x_0)=0$ $\frac{d^2v}{dx^2}(x_0)=\frac{\varphi}{EI}$	$\frac{dv}{dx}(x_0)$, $\frac{d^3v}{dx^3}(x_0)$

Boundary conditions for a horizontal beam.



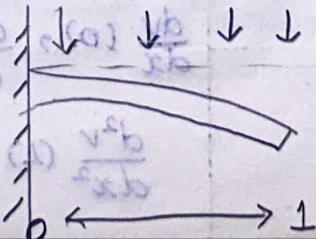
$v(0)=0$, $v(1)=0$. Both right & left sides are clamped to the wall. Because the bar is perpendicular to the wall.

$$\frac{dv}{dx}(0)=0, \quad \frac{dv}{dx}(1)=0.$$

$$\boxed{\begin{aligned} v(1) &= \frac{dv}{dx}(1)=0 \\ v(0) &= \frac{dv}{dx}(0)=0 \end{aligned}}$$

One fixed, one free end:

$-EIq$



$0 =$ (No load).

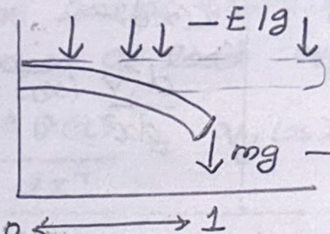
$v(0) = \frac{dv}{dx}(0) = 0$. And the left endpoint is fixed to the wall. Bar hangs free on the right side.

Right end:

$$\frac{d^2v}{dx^2}(1) = 0 \text{ (Shear force)}$$

$$\frac{d^3v}{dx^3}(1) = 0.$$ ∴ Second & 3rd derivative terms are proportional to the bending moment & shear force.

3. One end fixed, one end has hanging mass



$$v(0) = \frac{dv}{dx} = 0$$

$$\frac{d^2v}{dx^2}(1) = 0, \quad \frac{d^3v}{dx^3}(1) = \frac{1}{EI} \times \alpha_y(x)$$

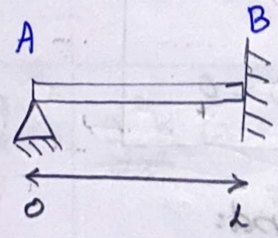
$$\boxed{\frac{d^3v}{dx^3}(1) = \frac{mg}{EI}}$$

mg → magnitude of the point force

E → material constant of elasticity relating stress & strain.

I → moment of inertia.

Boundary condition - 1



Pinned

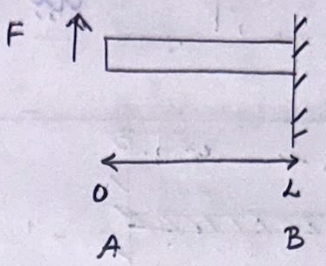
Known		Function
$v(0)=0$	$v(L)=0$	$\frac{dv}{dx}(0)$, $\frac{d^3v}{dx^3}(0)$,
$\frac{d^2v}{dx^2}(0)=0$	$\frac{dv}{dx}(L)=0$	$\frac{d^2v}{dx^2}(L)$, $\frac{d^3v}{dx^3}(L)$.

End A → pinned (No displacement, Rotation)

End B → wall (No displacement, No slope)

pinned → shear force. but no bending moment.

Boundary condition 2

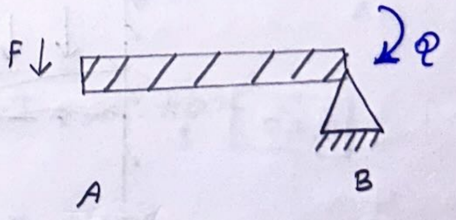


End A → Free with shear force.

End B → wall.

Known	Unknown
$v(L) = 0$	$v(0) = ?$
$\frac{dv}{dx}(L) = 0$	$\frac{dv}{dx}(0) = ?$
Free end: No moment or bending,	$\frac{d^2 v}{dx^2}(L) = ?$
$\frac{d^2v}{dx^2}(0) = 0$	$\frac{d^3 v}{dx^3}(L) = ?$
$\frac{d^3 v}{dx^3}(0) = \frac{F}{EI}$ (shear force)	

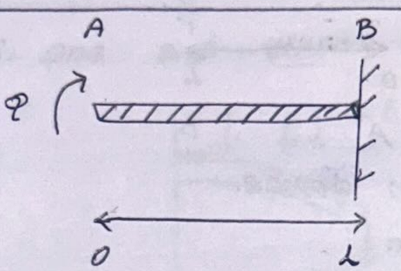
Boundary condition 3



End A → Free with shear force

End B → pinned.

Known	Unknown
$v(L) = 0$	$\frac{dv}{dx}(L) = ?$
$\frac{d^2v}{dx^2}(L) = Q/EI$	$\frac{dv}{dx}(0) = ?$
$\frac{d^3v}{dx^3}(0) = -\frac{F}{EI}$	$v(0) = ?$
$\frac{d^2v}{dx^2}(0) = 0.$	$\frac{d^3v}{dx^2}(L) = ?$



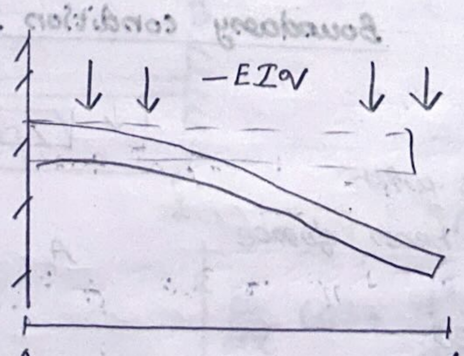
Free left end with a moment of bending Q,

End B → attached to the wall

Known	Unknown
$v(L)=0$	$\frac{d^2v}{dx^2}(L) = ?$
$\frac{dv}{dx}(L)=0$	$\frac{d^3v}{dx^2}(L) = ?$
$\frac{d^2v}{dx^2}(0)=Q/EI$	$v(0) = ?$
$\frac{d^3v}{dx^3}(0)=0$	$\frac{dv}{dx}(0) = ?$

Solving the beam equation.

10.1 Q



Find the vertical displacement for the beam in the image above. In other words, Solve the following boundary value problem:

$$EI \frac{d^4 v}{dx^4}(x) = -EI\alpha, \ x \in [0,1]$$

why $EI\alpha \longrightarrow$ To cancel out EI in L.H.S.

$$v(0) = 0 \qquad \frac{d^2 v}{dx^2}(1) = 0$$

$$\frac{dv}{dx}(0) = 0 \qquad \frac{d^3 v}{dx^3}(1) = 0.$$

$$\frac{d^4 v}{dx^4}(x) = -\alpha$$

$$\frac{d^3 v}{dx^3}(x) = -\alpha x + a$$

$$\frac{d^2 v}{dx^2}(x) = -\frac{\alpha x^2}{2} + ax + b$$

$$\frac{dv}{dx}(x) = -\frac{\alpha x^3}{6} + \frac{ax^2}{2} + bx + c$$

$$v(x) = -\frac{1}{24}\alpha x^4 + \frac{ax^3}{2\times 3} + \frac{bx^2}{2} + cx + d$$

Taking as constants,

$$v(x) = -\frac{1}{24}\alpha x^4 + ax^3 + bx^2 + cx + d$$

Applying Boundary values:

$d = c = 0$, so

$$v(x) = -\frac{1}{24}\alpha x^4 + ax^3 + bx^2$$

Taking second derivative gives

$$\frac{d^2 v}{dx^2}(x) = -\frac{1}{2}\alpha x^2 + 6ax + 2b.$$

and again,

$$\frac{d^3v}{dx^3}(x) = -qx + 6a$$

$$\frac{d^3v}{dx^3}(1) = 0 \quad \therefore \boxed{a = \frac{q}{6}}$$

we plug $x=1$, in 2nd derivative

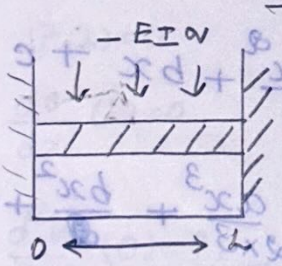
$$0 = -\frac{1}{2}q + 6\frac{q}{6} + 2b$$

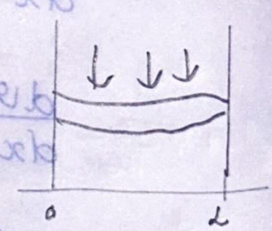
$$\boxed{b = -\frac{q}{4}}$$

Final solution:

$$\boxed{v(x) = -\frac{q}{24}x^4 + \frac{q}{6}x^3 - \frac{q}{4}x^2}$$

Practice problems.





Solu: $v(x) = -\frac{1}{24}qx^4 + ax^3 + bx^2 + cx + d.$

$$\frac{dv}{dx}(0) = v(0) = 0$$

$$\frac{dv}{dx}(1) = \frac{dv}{dx}(0) = v(1) = 0.$$

$\therefore \boxed{d=0}$, $\rightarrow$ I boundary condition.

$$\frac{dv}{dx}(x) = -\frac{4}{24}qx^3 + 3ax^2 + 2bx + c$$

$$\frac{dv}{dx}(0) = 0 = c \qquad \boxed{c=0}$$

$$v(x) = -\frac{1}{24}\alpha x^4 + a x^3 + b x^2$$

$$0 = v(1) = -\frac{1}{24}\alpha + a + b \longrightarrow ①$$

$$\frac{dv}{dx}(1) = -\frac{\cancel{4}}{6}\alpha + 3a + 2b$$

$$0 = -\frac{\alpha}{6} + 3a + 2b. \longrightarrow ②$$

Solve:

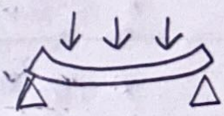
$$-\alpha + 24a + 24b = 0$$
$$-\alpha + 18a + 12b = 0$$
$$(-) \quad (-) \quad (-)$$
$$6a + 12b = 0$$

$$\boxed{a + 2b = 0.}$$

3
18
4
2

6 12
72
48
24

$$-2\alpha + 48a + 48b = 0$$
$$-4\alpha + 72a + 48b = 0$$
$$(-) \quad (-) \quad (-)$$
$$2\alpha - 24a = 0$$



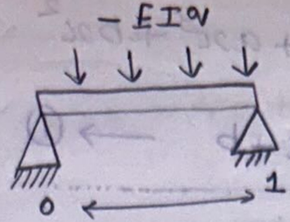
$$24a = 2\alpha$$

$$\boxed{a = \frac{\alpha}{12}}$$

$$b = \frac{1}{2}\left(\frac{\alpha}{6} - 3a\right)$$
$$= \frac{1}{2}\left(\frac{\alpha}{6} - 3\left(\frac{\alpha}{12}\right)\right)$$
$$= \frac{\alpha}{12} - \frac{\alpha}{8}$$
$$= \frac{2\alpha - 3\alpha}{24} = -\frac{\alpha}{24}.$$

4 | 8, 12
2 | 2, 3
3 |

$$v(x) = -\frac{1}{24}\alpha x^4 - \frac{\alpha}{24}x^2 + \frac{\alpha}{12}x^3$$



$\alpha y = -EI\alpha$

$v(0) = 0$ | $v(1) = 0$

$\frac{d^2v}{dx^2}(0) = 0$ | $\frac{d^2v}{dx^2}(1) = 0$

$$v(x) = -\frac{1}{24}\alpha x^4 + ax^3 + bx^2 + cx + d$$

$$\frac{d^2v}{dx^2}(x) = -\frac{1}{2}\alpha x^2 + 6ax + 2b.$$

i) $d = 0$, ii) $2b = 0 \Rightarrow b = 0$

iii) $0 = -\frac{1}{2}\alpha + 6a$

$$a = \left(\frac{1}{2}\alpha\right)\frac{1}{6}$$

$$a = \frac{1}{12}\alpha$$

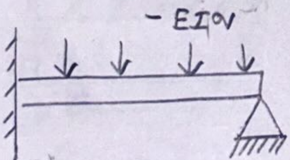
iv) $0 = -\frac{1}{24}\alpha x^4 + \frac{1}{12}\alpha x^3 + cx$

$$c = \frac{1}{24}\alpha - \frac{1}{12}\alpha$$

$$= -\frac{1}{24}\alpha$$

$$v(x) = -\frac{1}{24}\alpha x^4 + \frac{1}{12}\alpha x^3 - \frac{1}{24}\alpha x.$$

$\alpha y(x) = -EI\alpha$



$v(1) = 0$ | $v(0) = 0$

$\frac{d^2v}{dx^2}(1) = 0$ | $\frac{dv}{dx}(0) = 0$

$\boxed{d=0}$,

$$\frac{d^2v}{dx^2}(1) = -\frac{\alpha}{2} + 6a + 2b$$

$$v(1) = -\frac{\alpha}{24} + a + b$$

Solving,

$$a = \frac{5\alpha}{48}$$

$$b = -3\alpha/48$$

$$\frac{dv}{dx}(x) = -\frac{4}{24}\alpha x^3 + 3\left(\frac{1}{12}\right)\alpha x^2 + c$$

$$\frac{dv}{dx}(0) = c$$

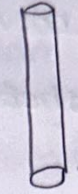
$$\boxed{c=0}$$

$$v(x) = -\frac{\alpha x^4}{24} + \frac{5\alpha x^3}{48} - \frac{3\alpha x^2}{48}.$$

A beam bending equ for both transverse & axial loading:

An equation for beam buckling due to compression

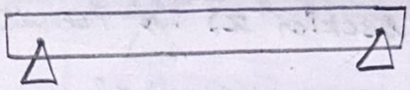
Compressive load:



Buckling: Deviate from the straight line (collapse).

'Trusses in bridges – Foundation'

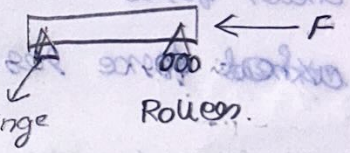
So we need to know loading limit.

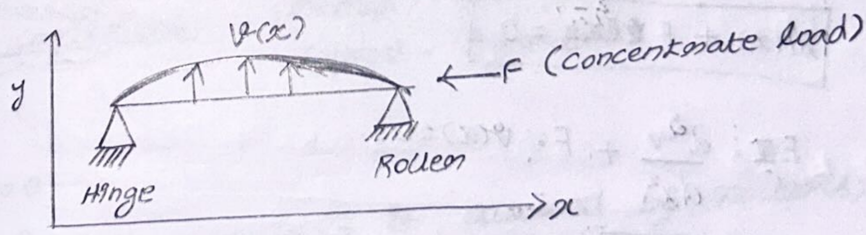


(pins are not allowed to move).

when they move → buckling happens).

Imagine rollers – permits horizontal movement.





'compress – buckle'

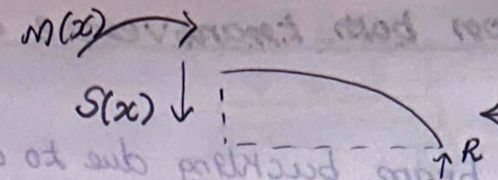
Equilibrium constraints:

'whole beam in equilibrium'

Isolating the buckling portion:

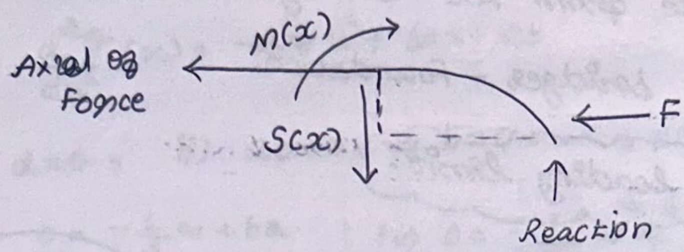
'The beam (buckled) doesn't move' → so must be in equilibrium

(No acceleration).



(Buckled piece of beam)

1) 'pin prevents → up & down movement'
2) 'Concentrated force' → shear force.
3) Bending moment (⌒)



'Axial force (at section x) → Resultant Force 'counters reaction'

Even in the buckled state → It's in equilibrium.

* Shear force balanced by reaction.
* ~~axial~~ force resultant balances applied load

Rotational equilibrium: (we are interested in)

$$\boxed{M(x) + F\,v(x) = 0}$$

moment = force × displacement

$$EI \cdot \frac{d^2v}{dx^2} + F \cdot v(x) = 0.$$

$$\boxed{EI\,\frac{d^4 v}{dx^4} = q_y(x)}$$ → If we want to relate this & our previous equation.

Assume: EI is uniform along the length of the beam.

$$EI\,\frac{d^4 v}{dx^4} + F\frac{d^2 v}{dx^2} = 0 \quad \left[\text{Differenciating twice}\right]$$