b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & a_{33} \end{bmatrix} \rightarrow$ For +ve definite matrix

$a_{33} = ?$

$\det A_1 = 1$

$\det A_2 = 1$

$\det A_3 = 1(a_{33} - 1) + 1(-1)$

$\det A_3 = a_{33} - 2$

when $a_{33} > 2$

$A + CI \rightarrow$ Smallest c that makes +ve semi definite

solu:

A has Eigen values $1, 2, -1$.

$A + CI = 1 + c,\ 2 + c,\ -1 + c.$ (Eigen values)

For +ve semi definite:

Eigen values ≥ 0

$\therefore c \geq 1.$

---

c) Starting with $u_0 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

and with $u_{k+1} = \frac{1}{2} A u_k$ what's the behaviour of u_k as $k \rightarrow \infty$

solu:

$$\frac{1}{2} A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Eigen values of $A: 1, 2, -1$

$\frac{1}{2} A : \frac{1}{2}, 1, -\frac{1}{2}$

multiplication by a scalar doesn't change Eigen vectors

Eigen vectors. $\underset{\lambda = 1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, ($

$$u_k \xrightarrow{k \to \infty} u_\infty = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underbrace{u_{k+1} = \frac{1}{2} A u_k}$$

Sum of Entries = c. (doesn't change with iterations)
↓
same

$u_0 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, $u_\infty = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(sum=3) sum=3

2020 vision

① $A = CR \to$ Row space matrix
↓
column space matrix

* Not easy to compute (Not good for big computing).

$$BC = \begin{bmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} - c_1 - \\ - c_2 - \\ - c_3 - \end{bmatrix} = b_1c_1 + b_2c_2 + b_3c_3$$

'column times row'

② $PA = LU$ → upper, → lower } Δ^r matrices.
↓
permutation matrix

$Q^TQ = I_n$

$QQ^T = I$

$$Q = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

orthonormal columns $v_1, v_2, \ldots$

Q - doesn't change length.

$Q^TQ = QQ^T = I$ (For square matrices alone).

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\lambda_1 = \cos\theta + i\sin\theta$ $\quad |\lambda_1|^2 = 1$

$\lambda_2 = \cos\theta - i\sin\theta$ $\quad |\lambda_2|^2 = 1.$

'Length is preserved'

$\|Qx\|^2 = x^T Q^T Q x = x^T x = \|x\|^2 \rightarrow$ Length of x doesn't change.

Gram-Schmidt

$$A = QR$$

$Q^T A = R \rightarrow$ triangular.

$$q_i^T a_K = r_{iK}$$

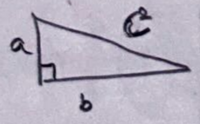
$a_1, a_n \rightarrow$ Independent

$q_1, q_n \rightarrow$ orthonormal.

'Fundamentals of linear & computation linear algebra'.

Least squares - Application of $A = QR$.

For right angle Δ^{le}, ($\perp^r$ ones)



$$a^2 + b^2 = c^2$$

$m > n$ m eqn $Ax = b$, n unknowns, minimize

$$\|b - Ax\|^2 = \|e\|^2$$

If Normal eqn for the best $\hat{x}$ (new solution):

$$A^T e = 0 \quad \text{(or)} \quad A^T A\hat{x} = A^T b$$

If $A = QR$ then

$$R^T Q^T Q R\hat{x} = R^T Q^T b$$ leads to

$$R\hat{x} = Q^T b$$

$S = S^T \rightarrow$ Real Eigen values & orthogonal Eigenvectors

$S = S^T$ has orthogonal eigen vectors $x^T y = 0$.

Proof:

$Sx = \lambda x \quad (Sy = \alpha y)$ $\quad \boxed{\lambda \neq \alpha \quad S^T = S}$

How to show $x^T y = 0$?

$Sx = \lambda x$	$Sy = \alpha y$
$x^T S^T = \lambda x^T$	$x^T S y = x^T (\alpha y)$
$x^T S = \lambda x^T$	$(\lambda x^T) y = \alpha x^T y$
	$\lambda x^T y = \alpha x^T y$

$\therefore \lambda \neq \alpha \rightarrow$ Both are equal only when $x^T y = 0$.

$S = Q \Lambda Q^T = Q \Lambda Q^{-1}$ $\quad Q^T = Q^{-1}$

$A = U \Sigma V^T$ is a sum of $\sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$ of rank 1 matrices

$\sigma_1, \sigma_2 \dots \sigma_r$ in Σ (diagonal matrix)

singular vectors in U and V.

$\underbrace{S = Q \Lambda Q^T \rightarrow \text{Sum of } \lambda_1 q_1 q_1^T + \dots + \lambda_r q_n q_n^T \text{ of rank 1 matrices}}$ with $S = A^T A$

"Every symmetric matric is the combination of $Q \Lambda Q^T$" $\longrightarrow$ spectral theorem.

$AX = X\Lambda \rightarrow$ "Non symmetric case"

$\Lambda \rightarrow$ Eigen value diagonal matrix.

$$\boxed{A = X \Lambda X^{-1}}$$

'Eigen values & vectors are the way to break a square matrix & find this diagonal matrix'

$$Ax = \lambda x$$

$$A^2 x = A(\lambda x) = \lambda(Ax) = \lambda^2 x$$

$$\boxed{A^n x = \lambda^n x}$$

$$A^2 = (X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^2 X^{-1}$$

$$\boxed{A^n = X\Lambda^n X^{-1}}$$

$A^n \to 0$, when $\Lambda^n \to 0$:

All $|\lambda_i| < 1$

$A^TA \to$ square, symmetric, non-negative definite

1) $(A^TA) = (n \times m)(m \times n) = n \times n \to$ square

2) $(BA)^T = A^TB^T$

$(A^TA)^T = A^TA \Longrightarrow$ symmetric

3) $S = S^T$ is nonnegative if

Eigen value test 1: All $\lambda \geq 0 \to Sx = \lambda x$

Energy test 2: $x^TSx \geq 0$ for every x

- Data: eg: No. of patients $\neq$ no. of medicines

↓

Rectangular matrix.

'Eigen values & vectors $\to$ Are for square matrices'

↓

Rectangular $\longrightarrow$ 'SVD'

we need some other

$A = U \Sigma V^T$ with $U^T U = I$

$V^T V = I$

$AV = U\Sigma$ means

$$A \begin{bmatrix} v_1 & v_2 & .. & v_n \end{bmatrix} = \begin{bmatrix} u_1 & \dots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}$$

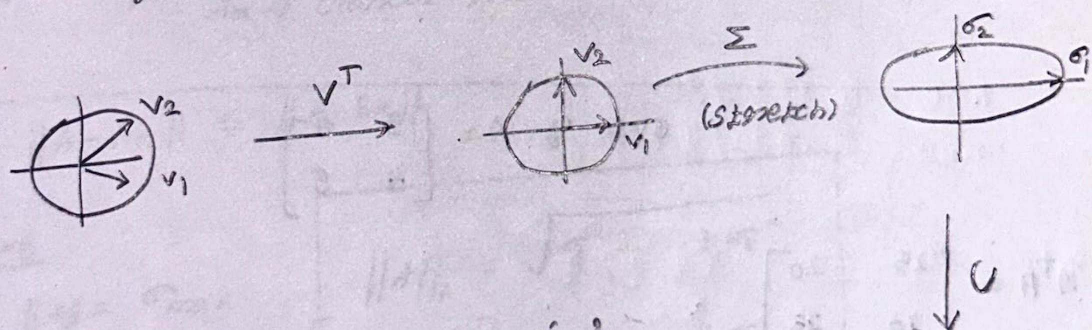
and $Av_i = \sigma_i u_i$

Singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$

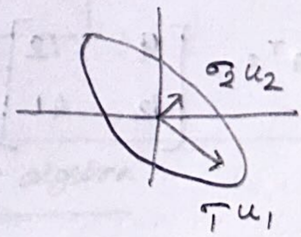
r = rank of A.

'Idea' : x in row space $\xrightarrow{x}$ Ax (column space)

V^T (orthogonal columns) (normal) → Turns the vectors $\perp^r$



'SVD – 'Rotation, stretch, Rotation'



'orthogonal matrices with orthonormal columns don't change length just rotate'

↓ Σ → stretches.

How to choose orthonormal v_i in the row space of A

v_i → Eigen vectors of A^TA

$V^TV = I$

$A^TAv_i = \lambda_i v_i = \sigma_i^2 v_i$ (v_i orthonormal)

u_i (column space) $= \dfrac{A v_i}{\sigma_i}$

$u_i \rightarrow$ orthonormal

$U^T U = I$

$$\left(\frac{A v_j}{\sigma_j}\right)^T \left(\frac{A v_i}{\sigma_i}\right) = \frac{v_j^T A^T A v_i}{\sigma_j \sigma_i} = \frac{v_j^T \sigma_i^2 v_i}{\sigma_j \sigma_i} = \begin{matrix} 1 \ (i=j) \\ 0 \ (i \neq j) \end{matrix}$$

'Proof they are orthogonal - v_i & v_j'

SVD - Best factorization of matrices.

$A = U \Sigma V^T$

$(m \times n) = (m \times m) \ (n \times n)$

u_{r+1} to u_m: Null space of A^T

v_{r+1} to v_n: Null space of A

$$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ & \sigma_r & \cdots \\ 0 & & 0 \end{bmatrix}$$

SVD of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

$A^T A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

$U = \dfrac{\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}}{\sqrt{10}} \rightarrow$ Eigen vectors of $A^T A$

$A A^T = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$

$V^T = \dfrac{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}{\sqrt{2}}$

$\Sigma = \begin{bmatrix} 3\sqrt{5} & \\ & \sqrt{5} \end{bmatrix}$

$$\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = \frac{3}{2}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

'Data science wants to know what's

matrix

$$\sigma_1 = \frac{3}{2} \geq \sigma_2 = \frac{1}{2}$$

our matrix is broken into two — 1 for each Eigen vectors).

$\sigma_1 \rightarrow$ Important (Big guy).

SVD $\rightarrow$ picks out important part of the matrix.

Low rank approximation — to a big matrix

Start from the SVD

$$A = U \Sigma V^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

Keep the k largest σ_1 to σ_k

$$A_k = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T$$

$A_k \rightarrow$ closest rank k matrix to A

$$\|A - A_k\| \leq \|A - B_k\|$$

Norms

$$\|A\| = \sigma_{max}$$

$$\boxed{\begin{aligned} \|A\|_F &= \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} \\ \|A\|_N &= \sigma_1 + \cdots + \sigma_r \end{aligned}}$$

Randomized Numerical linear algebra

For very large matrices, $\rightarrow$ Randomization has brought a revolution.

example:

Multiply AB with column-row sampling $(AS)(S^TB)$

$$AS = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} s_{11} & 0 \\ 0 & 0 \\ 0 & s_{32} \end{bmatrix} = \begin{bmatrix} s_{11} a_1 & s_{32} a_3 \end{bmatrix}$$

$$S^T B = \begin{bmatrix} S_{11} & b_1^T \\ S_{32} & b_3^T \end{bmatrix}$$

SS^T is not close to I. we can have

$$E\left[SS^T\right] = I$$

$$E\left[(AS)(S^T B)\right] = AB$$

<u>Norm-squared sampling:</u>

choose column-row with probabilities

$$\approx \|a_i\| \, \|b_i^T\|$$

This choice minimizes the sampling variance.

<u>18.065</u> → <u>Linear algebra and learning from data</u>

'second course.

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math.mit.edu/linearalgebra

math.mit.edu/learningfromdata.

<u>Schur's theorem:</u> $A = Q T Q^T$

T → Triangular

A → square real matrix with real Eigen values

<u>matrix picture:</u> → Ref 'matrix world'

'All symmetric orthogonal matrices

+ve det

projection

} Diagonal → square → matrices

'simple picture'.