Column space of A: All multiples of $(1\ 1\ 1\ 1)$

$C(A) = R^1$

$\therefore$ $C(1) \longrightarrow$ Column space.

↳ constant

↳ 1 dimension.

Left Null Space

$N(A^T)$

$A^T y = 0$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (v_1\ v_2\ v_3\ v_4) = [0\ 0\ 0\ 0]$$

only when $v_1 = v_2 = v_3 = v_4 = 0$

$N(A^T) = \{0\} \rightarrow$ subspace (0-dimensional)

↓

No members in an empty space

dimension: Null space dim + Row space

: $3 + 1 = 4. = n \longrightarrow R^4$

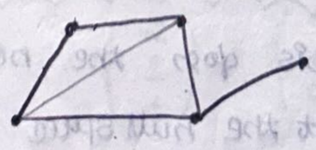
= Column space + $N(A^T)$

$= 1 + 0 = 1 = m \longrightarrow R^1$

$\boxed{\dim(R(A)) = \dim(C(A))}$

Small world graphs.

Graph: Bunch of nodes & edges connecting the nodes.



5 nodes - 6 edges

$G = \{\text{nodes, edges}\}$

↳ Graph is a collection of nodes & edges

example: * Each people in the country is a node

* Edges b/w two nodes - if they know each other.

'Distances Come down dramatically- By some shortcuts'

nodes: sites
Edges: links) world wide web.

Recitation- Matric spaces

Show that the set of 2×3 matrices whose null space contains $\begin{bmatrix}2\\1\\1\end{bmatrix}$ is a vector subspace, & find a basis for it. what about the set of those whose column space has $\begin{bmatrix}2\\1\end{bmatrix}$

Solu:

Null space:

Subspace: Sum in that space
scalar multiplication result in that space

$Ax = 0$

$A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $B\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Set of 2×3 matrices is a subspace?

$\therefore$ $\begin{bmatrix}2\\1\\1\end{bmatrix}$ which is in null space. Null space is a subspace vector space.

$\therefore$ $A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $B\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ → Assume.

$\therefore$ $A, B \rightarrow 2\times3$

Aim: Find a basis for the null space.
verify that the null space is a vector space.
Comment on the set of column space

Null space of 2×3 matrices:

$A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $B\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

Assume $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is the null solution to both A and B which are 2×3 matrices.

Sum:

$$(A+B)\begin{pmatrix}2\\1\\1\end{pmatrix} = A\begin{pmatrix}2\\1\\1\end{pmatrix} + B\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$(c(A))\begin{pmatrix}2\\1\\1\end{pmatrix} = c\left[A\begin{pmatrix}2\\1\\1\end{pmatrix}\right]$$

$$= c\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

closed under multiplication & addition.

so our null space - vector subspace

Basis:

$$A\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Any row will be of the format:

$$(a\ b\ c)\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

when $2a+b+c=0$

$c = -2a-b$

* must be $\begin{bmatrix}a & b & -2a-b\end{bmatrix} = \begin{bmatrix}a & 0 & -2a\end{bmatrix}$

$+$

$\begin{bmatrix}0 & b & -b\end{bmatrix}$

(Linear combination)

$(1 \ 0 \ -2), (0 \ 1 \ -1)$

Basis:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix},$$

Dimension: 4 $\quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$\therefore$ our basis is the linear combination of $(1 \ 0 \ -2), (0 \ 1 \ -1)$ vectors.

What about column space set having $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Is that a subspace:

IS this has zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

"Zero matrix doesn't have the column $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$"

So zero matrix doesn't belong to the column space.

"Not a vector subset" - our column space.

Lecture - 13

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} ? \end{bmatrix}$$

which matrices have the form AX for some X?

$$\therefore A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

Solu:

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

(Rank = 2)

Rank(A) = 2.

The columns of any matrix of the form Ax are linear combinations of the columns of A. That is, That is, they are vectors whose components all sum to zero:

$$Ax = B \text{ if and only if } B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}$$

As in a.

Graphs, Networks, Incidence matrices

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}, \quad x = ?$$

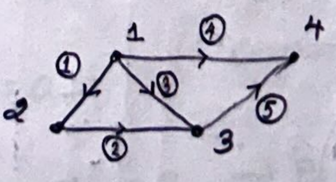
$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ -a & -b & -c \end{bmatrix}}_{\text{Linear Combination}} = \begin{bmatrix} a & b & c \\ d & e & f \\ -a-d & -b-e & -c-f \end{bmatrix}$$

Linear Combination of A (Sums up to zero)

Lecture-13

Applications of linear algebra : Graphs & Networks, Incidence matrices, Kirchoff laws.

Graphs : Nodes & edges.



4 nodes $m = 5$

5 edges. $n = 4$.

Current → +ve or −ve (Through this arrow)

Incidence matrix:

edge 1 leaves node 1 $\to -1$

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{l} \text{Edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

Every rows correspond to an edge.

Loops:

$\left.\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right\}$ Corresponding rows : Are those rows independent?
NO

Loops corresponds to linearly dependent rows.

∴ In case of large matrices,

* number of zeros will be high
* Only two ones per row.

Null space:

$$Ax = 0$$

Independent: only zero vector.

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$
$$-x_2 + x_3 = 0$$
$$-x_1 + x_3 = 0$$
$$-x_1 + x_4 = 0$$
$$-x_3 + x_4 = 0$$

$$\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

↓

Difference in potential b/w two nodes.

when
$x_2 = x_1 = x_3 = x_4 = 0$

↳ we can have $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$x = c\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rightarrow \text{constant potential.}$$

Basis:

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \rightarrow 1 \text{ dimensional.} \qquad \dim N(A) = 1$$

what does this mean: potential difference makes the current flows. constant potential difference avoids the flow of current.

we fix one of the potential: Ground that node.

$\text{Rank}(A) = 3$ → Any 3 are independent.
(nodes)

column space:

All combinations of columns.

Left null space:

$$A^T y = 0 \qquad N(A^T) = m - r = 5 - 3 = 2$$

$$\begin{bmatrix}-1 & 0 & -1 & -1 & 0\\ 1 & -1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & -1\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}_{n\times m}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\\y_5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

↳ potential difference zero.

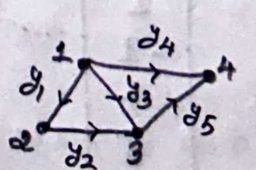
Currents y_1, y_2, y_3, y_4, y_5

Conductance × current

KCL:

$A^T y = 0$ (5 currents that satisfy KCL).

$-y_1 - y_3 - y_4 = 0$ → From graph



From graph :

① $-y_1 - y_3 - y_4 = 0$ → Node 1 → currents are leaving.

(Conservation law)

② $y_1 - y_2 = 0$

③ $y_2 + y_3 - y_5 = 0$

④ $y_4 + y_5 = 0$

we can find $N(A^T)$ by elemination — we can also go with out that:

* Rank will be 3.
* Fourth row will be zero.

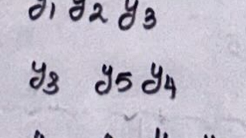
Basis for Null space:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

* Loop 1
* Loop 2.

$y_1 = 1, y_2 = 1$ → since current across the loop is 1A

$y_3 = ?$



we can take

Loop 1 $y_1\ y_2\ y_3$

Loop 2 $y_3\ y_5\ y_4$

Loop 3 $y_1\ y_2\ y_5\ y_4$

For Loop 1 2 3 (Say 1 A)

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ +1 \end{bmatrix}$$

$y_1 = y_2 = y_5$

$y_5 = -y_4$

$$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \rightarrow \text{Is a linear Combination of loop 1 \& 2.}$$

Rowspace: $\dim(R(A)) = r = 3$

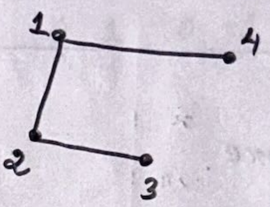
$R_1, R_2, R_3 \rightarrow$ Are not independent.

$R_1, R_2, R_4 \rightarrow$ Are independent.

$\therefore R_1, R_2, R_3 \longrightarrow$ Forms of a loop

Independence: once having no loop

Graph without loop:



'Tree' : Graph with no loops

$\dim(N(A^T)) = m - r$

$\downarrow$

Number of Independent loops $= \#\text{edges} - (\#\text{nodes} - 1)$

Rank $= n-1$ $\quad \therefore \dim(N(A^T)) = 1 = n - r$

$$\boxed{r = n-1}$$

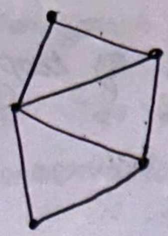
$$\#\text{nodes} - \#\text{edges} + \#\text{loops} = 1$$

$\#$nodes $\rightarrow$ 0d

$\#$edges $\rightarrow$ 1d (Line) connecting nodes

$\#$loops $\rightarrow$ 2d (Area: like)

'Euler's formula'



\# nodes = 5

\# Edges = 7

\# loops = 1

$5 - 7 + 3 = 1$

$e = Ax$ | $A^T y \rightarrow$ Kirchoff's current law

$y = Ce$

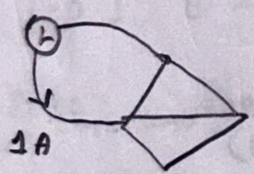
$e \rightarrow$ potential difference.

$y \rightarrow$ current.

$$\left.\begin{array}{l} e = Ax \\ y = Ce \\ A^T y = 0 \end{array}\right\} \text{Three basic equ.}$$

$A^T y = 0$
↳ No external source

$A^T y = f$
↳ External source.



$$A^T C A x = f$$

$A^T y = f$

$\therefore Ax = e, \; Ce = y,$

$A^T C e = f$

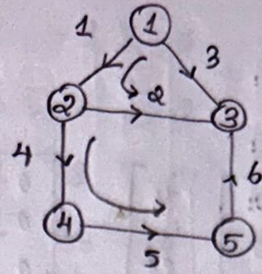
$A^T y = f$

$\boxed{f = f}$

Balanced equ: "Inequilibrium: Time isn't in this problem" — No newton laws.

"How currents are distributed?"

$A^T A \rightarrow$ Symmetric



6 nodes

6 edges

Find incidence matrix A ?

* N(A) , N(A^T)

* Trace ($A^T A$) = ?

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Solu:

$Ax = e$

Null Space N(A):

$Ax = 0$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x_1 + x_2 \\ -x_2 + x_3 \\ -x_1 + x_3 \\ -x_2 + x_4 \\ -x_4 + x_5 \\ x_3 - x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $N(A) = C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ → constant potential

Basis : $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$N(A^T) = ?$

$$\boxed{A^T y = 0}$$

$$\begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

current flowing through the node = current leaving the node.

$$-y_1 - y_3 = 0$$

$$y_1 - y_2 - y_4 = 0$$

$$y_2 + y_3 + y_6 = 0$$

$$y_4 - y_5 = 0$$

$$y_5 - y_6 = 0$$

From graph

$y_1 = y_2$

$$y = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

↳ Loop 1 (Above)

↳ Loop 2 (Below)

Entire one ↓ will be the superposition.

Trace:

m_1 m_2 → matrix

c_1 c_1 → column.

(1 1)

$A^T A$ → magnitude square of the columns. each entry is the

c_1 of m_1 & c_1 of m_2 gives c_1 of $A^T A$

Trace → sum of the diagonal entries. [Complex Eigen values]

↳ only defined for a square matrix.

$$\underset{(5\times 6)\,(6\times 5)}{A^T A} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 & 0 & 0 \\ 1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

$2+3+3+2+2 = 12$

$\text{Trace}(A^T A) = 12$

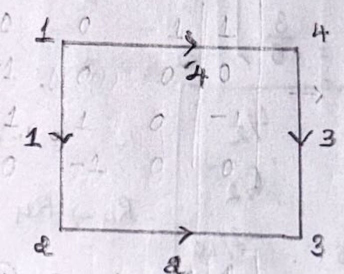
(or)

<u>Easy method:</u>

Sum of edges of each node. (degree)

$Tr(A^T A) = 2+3+3+2+2 = 12.$

<u>Preview</u>



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A^T C A x = f$

If $f = (1, 0, -1, 0) \longrightarrow$ find x & $y = -CAx$

<u>Solu:</u>

$$A^T C A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & -1 & 1 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -2 & 4 & -2 & -1 \\ -1 & 0 & -2 & 3 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1/2 \\ -1 & 0 & -2 & 3 & 0 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1$

$$= \begin{bmatrix} 1 & -1/2 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 5/2 & -2 & -1/2 & 1/2 \\ 0 & -1 & 2 & -1 & -1/2 \\ 0 & -1/2 & 0 & 5/2 & 1/2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ 0 & 5/2 & -2 & -1/2 & 1/2 \\ 0 & 0 & 6/5 & -6/5 & -3/10 \\ 0 & 0 & -\frac{2}{5} & \frac{12}{5} & \frac{3}{5} \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + \frac{5}{2} R_2 \\ R_4 \to R_4 + \frac{1}{5} R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ 0 & 5/2 & -2 & -1/2 & 1/2 \\ 0 & 0 & 6/5 & -6/5 & -3/10 \\ 0 & 0 & 0 & 2 & 1/2 \end{bmatrix} \quad R_4 \to R_4 + \frac{1}{3} R_3$$

Reduced row echelon form

$$= \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ 0 & 1 & -\frac{4}{5} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 1 & -1 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & 1/4 \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{1}{2} R_2 \\ R_3 \to \frac{1}{5} R_3 \\ R_4 \to \frac{1}{2} R_4 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ 0 & 1 & -4/5 & -1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 \end{bmatrix} \quad R_3 \to R_3 + R_4$$

$$= \begin{bmatrix} 1 & -1/2 & 0 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 \end{bmatrix} \quad R_2 \to R_2 + \frac{4}{5} R_3$$

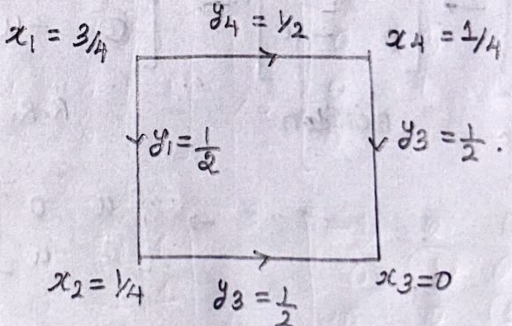
$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3/5 & 3/5 \\ 0 & 1 & 0 & -1/5 & 1/5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right] \quad R_1 \to R_1 + \frac{1}{2} R_2$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + \frac{3}{5} R_4 \\ R_2 \to R_2 + \frac{1}{5} R_4 . \end{array}$$

$$\qquad\qquad I \qquad\qquad\qquad x$$

$$x = \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix}$$

$$y = -CAx = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$



I) <u>Review -1</u>

u, v, w in R^7 are non-zero vectors. They span a subspace of R^7. What are the possible dimensions of that space.

* 1, 2 or 3 (couldn't be more than 3)
* vectors are non-zero → Not 0-dimensional.

II) 2) 5 by 3 matrix in reduced row echelon form has $r=3$ pivots.

1) <u>what's about null space</u>

$N(u)$ → only the zero vector.

(Rank = 3 → No linear combinations).

$Ax = 0 \rightarrow$ only for $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2) 10 by 3 matrix $\begin{bmatrix} u \\ 2u \end{bmatrix}$. what's the reduced row echelon form of B?

Solu:

Reduced row echelon form:

$$B = \begin{bmatrix} u \\ 0 \end{bmatrix} \quad \because 2u \rightarrow \text{combination of } u.$$

3) what's about rank of B:

$r \leq m \quad , \quad r \leq n$

so may be 3.

4) what's the Reduced row echelon form of $C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix}$

Solu:

$$C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix} \rightarrow \underset{\text{Echelon}}{\begin{bmatrix} u & u \\ 0 & -u \end{bmatrix}} \rightarrow \underset{\text{R.R. echelon.}}{\begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}}$$

$$\rightarrow \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

$u \rightarrow$ A matrix which may have some zero rows. we know $u \rightarrow$ we need to move that row/s to the bottom.

$$\text{Rank} = 6 \quad \because \begin{bmatrix} u \\ 0 \end{bmatrix} \rightarrow \text{Rank } 3 \;\; (10 \times 3) \text{ matrix.}$$

$$\boxed{\text{Rank}(C) = 6}$$

5) Dimension of Null space of (C^T)

dim $N(C^T)$

C's size (10×6)

$$\boxed{\therefore \; b = \begin{bmatrix} u \\ 2u \end{bmatrix} \rightarrow 10 \times 3}$$

$$\dim N(C^T) = m - r = 10 - 6 = 4.$$

3) Suppose $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

is a complete solution.

solu:-

Shape of matrix A:

3 × 3

$\therefore (3 \times 3)(3 \times 1) = (3 \times 1)$

what's about the dimension:

$\dim N(A) = 2$

$\therefore x = x_p + x_N$

$\hookrightarrow c \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

rank = ?

$\dim N(A) = n - r$

$r = 3 - \dim N(A)$

$\boxed{r = 1}$

what about A:

$c = 1, d = 1$

$Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$x = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

we don't know about c, d. So for every combination of c, d $\rightarrow Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$. So when $c = d = 0$

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \rightarrow$ manages,

when $c = 1, d = 0$

C_2 manages C_1

when $c = 0, d = 1$

$C_3 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow$ nullity.

$Ax = b$ can be solved if $\boxed{b = ?}$

If $b = c\begin{bmatrix}1\\2\\1\end{bmatrix}$

In this case we have a lot of null space

$\therefore$ Rank is 1 $\rightarrow$ so null space is large.

---

4) If the matrix is square, and the null space of A is {0}, what about the null space of A^T.

solu:

Also the {0}.

---

* Sum of two invertible matrices is not necessarily be a subspace.

so, invertible 5x5 matrix space $\rightarrow$ Not a subspace

If $b^2 = 0$, then $b = 0$?

NO!

$$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$n \times n$ matrix with independent columns - Does $Ax = b$ always solvable?

solu:

True: Rank = Full size

"square" - Invertible - Nothing goes wrong.

$$B = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1 & 2\\0 & 1 & 1 & -1\\0 & 0 & 0 & 0\end{bmatrix}$$

without doing multiplication.

solu:

'Basis for null space' $N(B) = ?$

$B_{3\times4}$ $N(B)$ is a subspace of R^4.

$$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix} \rightarrow \text{Invertible. (Independent)}$$

Is the null space of B is same that of D?

$\boxed{N(CD) = N(D) \text{ if } C \text{ is invertible.}}$

basis for $N(B) =$?

$\boxed{2 \text{ pivot in (columns) } D \quad [B = CD]}$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow$$ Free variable part is sitting here with all its signs reversed.

when $x_3 = 1, x_4 = 0$

$x_2 + 1 = 0$ | $x_1 - 1 = 0$

$x_2 = -1$ | $x_1 = 1$

$x_3 = 0, x_4 = 1$

$x_2 = 1$

$x_1 = -2.$

Solve $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ First column of our matrix agrees with R.H.S.

↳ Particular solution.

Solu:-

$$x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

↳ 1st Column of D

∴ we know what's the null solution (special solution)

'I need my answer $Bx = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ 1st column of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

'We may have any solution - as particular solution'

If $m = n$, is the row space is equal to the column space.

Solu:-

False.

eg: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Row space: multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

column space: multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In case of symmetric:

Rowspace = columnspace.

Matrices A & −A share the 4 subspaces ?

sol:

A and B have same 4 subspaces when B is a multiple of A? ⟶ False (Not necessarily)

$\boxed{A = cB} \longrightarrow \boxed{A = -(A)}$

$A \to 6 \times 6$ invertible matrix

Row space: R^6, columnspace : R^6

Null space : {0}, Left null space; {0}

A and B any invertible 6×6 matrices. Are they going to have the same 4 spaces? (subspace)

False.

Matrices A & −A → share the four subspaces

↓

True.

∴ If v is in a space, then −v will also be in the space.

---

1) If we exchange two rows which of the two subspaces can't change?

1) Row
2) Null
) Null space.

---

why $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ can't be in nullspace & be a row of A?

$$\begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↳ Any row 1 2 3

$\therefore$ $1+4+9 = 14 \rightarrow$ Not zero $\rightarrow$ So at a time same elements can be x and in the row.

Recitation

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 4 & k \end{pmatrix}$$, For which k does $Ax = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ have a unique solution?

b) which x - Infinitely many

c) when $k = 4$, Find LU decomposition

d) for all k, Find complete solution.

solu :-

unique solution: when A is invertible (Full rank)

ⓐ

$$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & k-3 & 1 \end{array}\right) \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

↳ b & c needs this (for solving time)

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-5 & 0 \end{array}\right) \quad R_3 \rightarrow R_3 - R_2$$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

when $k \neq 5 \rightarrow$ It's invertible.

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

ⓑ Infinitely many solutions when

Nullspace $\rightarrow$ Non trivial

when $k - 5 = 0$

$k = 5 \rightarrow$ 3rd row Augmented one is also zero

↓

So many solution.

when Augmented entry is non-zero $\rightarrow$ No solution.

c) when $K=4$ $[L\,U]$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$\therefore$ $u = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

we need to do

$\boxed{A = Lu}$

$\therefore$ $E_{32}\, E_{31}\, E_{21}\, A = u$

$\boxed{A = Lu}$

$L = E_{21}^{-1}\, E_{31}^{-1}\, E_{32}^{-1}.$

Flip: the entries of ~~all~~ near diagonal entries

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

d) Complete solution $x = x_p + x_N$.

Solu: $K \neq 5 \longrightarrow$ Invertible

$K = 5 \longrightarrow$ only many solutions.

$\underline{K \neq 5}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & K-5 & 0 \end{array}\right) \qquad Ax = b$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{K = 5}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rightarrow \text{particular}$$

↳ Free variable.

$\therefore$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1+c \\ 1-2c \\ c \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 2$

$x_2 + 2x_3 = 1$

$\boxed{x_3 = c}$

$x_2 = 1 - 2c.$

$x_1 = 1 + c$

Sum: 3×3 matrix reduces to I by row operations

$E_{21} : -4R_1 + R_2$ (From)

$E_{31} : -3R_1 + R_3$ "

$E_{23} : -R_3 + R_2$ "

solu:

$[A \mid I] \longrightarrow [I \quad A^{-1}]$

$A \downarrow I$

By row operations

$A \longrightarrow I$

$\therefore A^{-1} A = I$

$\therefore A^{-1} = I \xrightarrow[(E_{23}E_{31}E_{21})]{} A^{-1}$

$\therefore E_{23}\, E_{31}\, E_{21}\, A$

$A = I \xrightarrow[E_{21}^{-1}\, E_{31}^{-1}\, E_{23}^{-1}]{} A$

$A = LU$ $(U = ?)$ Reduce $A \rightarrow$ Row operations.

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} = A \quad \text{(Lower triangular matrix)}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$$A = LU \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$$

what's the lower triangular factor?

$A = LU$

$\boxed{L = ?}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} & ? & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$E_{31} E_{21} A = U$

$A = E_{21}^{-1} E_{31}^{-1} U$

$$L = E_{21}^{-1} E_{31}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}}_{L}$$

Unit - 2

Each component of a vector in $\mathbb{R}^n$ indicates a distance along one of the coordinate axis. This practice of dissecting a vector into directional components is an important one.

'Least squares' — collection of data.

Eigen values & vectors — calculations made easier

Vectors are easier to understand when they're described in terms of orthogonal bases. In addition, the four fundamental subspaces are orthogonal to eachother in pairs.

A - rectangular → $Ax=b$ (often unsolvable)

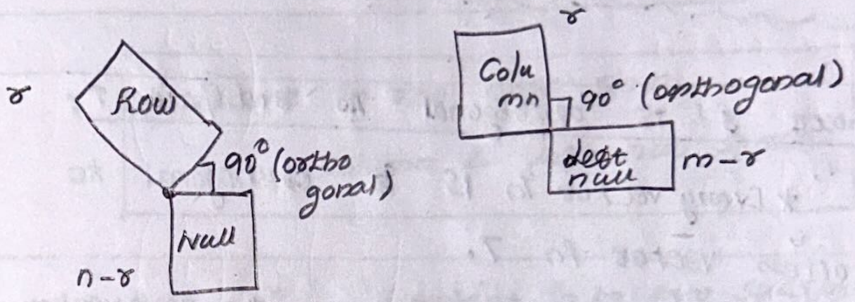
matrix A^TA → help us to find a vector $\hat{x}$ that comes as close as possible to solving $Ax=b$.

Lecture - 15

Orthogonal vectors & subspaces

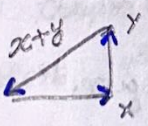
Null space ⊥ row space

$$N(A^TA) = N(A)$$



Orthogonal (perpendicular):

Angle b/w the vectors → 90°



"Pythagoras"

"Dot products"

$$\boxed{x^T y = 0}$$

↓ ↓

Column vectors

orthogonal: (Right triangle)

$$\|x^2\| + \|y^2\| = \|x^2+y^2\|$$

↳ vectors.

↓

$x^T x$

$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $x+y = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

$\|x\|^2 = 1+4+9 = 14.$ | $\|y\|^2 = 5.$ | $\|x^2+y^2\| = 19$

$x^T x + y^T y = (x+y)^T (x+y)$

↳ only true (right angle).

$x^T x + y^T y = x^T x + y^T y + x^T y + y^T x$

$\boxed{x^T y + y^T x = 0}$ → No difference.

$2x^T y = 0$

x → Any vector, y → zero vector:

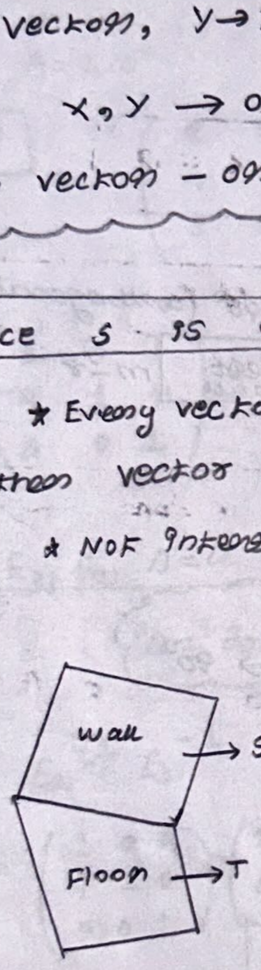
x, y → orthogonal

zero vector – orthogonal to every vector.

---

subspace S is orthogonal to subspace T:

* Every vector in S is orthogonal to every other vector in T.
* Not intersecting except at the zero vector.

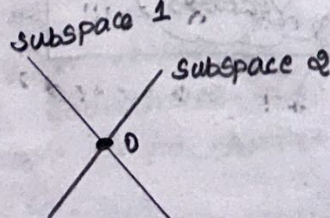


→ Are they orthogonal?

No – They have common intersection vectors.

* zero vector
* Line
* Entire-plane

) R^2 subspaces.



1, 2 → orthogonal (Line) → ∵ They only meet at zero.

Rowspace is orthogonal to nullspace:

Their dimensions add up to give the entire dimension

Null space: X solves $Ax = 0$	Row Space: Rows of A (Combinations)

$$A = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

x is orthogonal to $R_1, R_2, R_3 \dots R_n$.

$\therefore R_1, R_2, \dots R_n$ & their combinations are in the row space.

$c_1 (\text{row } 1)^T x = 0$

$c_2 (\text{row } 2)^T x = 0$

$$\boxed{(c_1 (\text{row } 1)^T + c_2 (\text{row } 2)^T + \dots) x = 0}$$

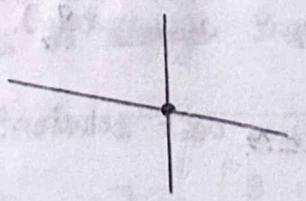
"same in the case for column & left null space"

R^n (considering in to two subspaces)

Null   Row

3 dimension: Couple of orthogonal vectors: That don't carve up the entire R^3

eg: Line



Are they be Row space & null space?

"Dimensions are not good"

$\therefore r + n - r = R^n$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \rightarrow \text{1 dimension (Row space)}$$

Null space $\dim(N(A)) = n - r = 3 - 1 = 2 \rightarrow$ They

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$\hookrightarrow$ free variables $\dim(N(A)) = 2$.

$\therefore$ They don't add up to give R^n

Null space & Rowspace are orthogonal:

* Their dimensions add up to give the entire dimension of the space.

* In specific: orthogonal complement

Null space & Rowspace are orthogonal complements in R^n

$\hookrightarrow$ Null space contains all vectors $\perp^r$ to the row space.

This chapter:

solve = $Ax = b$ which has no solution to its closest solution.

$\therefore m > n$

eg: satellite – some satellites – position – 1000 measurements (1000 eqn)

* pulse rate – (multiple rates)

(we can't able to solve it – accurately).

$\hookrightarrow$ what's the best solution?

$\downarrow$

one way – 'through away eqn that you get a nice square invertible system'

To get the best info – we need to take all measurements into account.

$A^T A$ – matrix:

$(n \times m)(m \times n) = (n \times n)$ matrix

'square matrix' $\rightarrow$ symmetric.

$(A^TA)^T = A^T A^{TT} = A^T A \rightarrow$ Is it invertible?

If not (Null space?)

Good eqn $\rightarrow$ $A^T A \hat{x} = A^T b$

$\hat{x} \rightarrow$ x hat $\rightarrow$ Best solution.

$\therefore x \rightarrow$ doesn't exist.

Example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \quad m=3, \; n=2 \quad \text{Rank} = 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we can't solve 3 eqn with 2 solutions — until the Right hand side is in the column space.

$\hookrightarrow$ Not the case.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$N(A^TA) = N(A)$ (Is it always invertible)

Suppose:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 9 & 27 \end{bmatrix}$$

Rank = 1 $\rightarrow$ Not invertible.

---

rank of A^TA = rank of A

$N(A^TA) = N(A)$

$A^TA \rightarrow$ Invertible exactly when A has independent columns.

S is spanned by (1 2 2 3) and (1 3 3 2)

i) Find a basis for $S^\perp$.

ii) Can every vectors in R^4 be written uniquely in terms of S and $S^\perp$?

Solu:

If x in $S^\perp$:

x is orthogonal to every vectors in S.

$$(1\ 2\ 2\ 3)\cdot X = 0$$

$$(1\ 3\ 3\ 2)\cdot X = 0$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduce → Find basis.

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \rightarrow \text{Rank } (2)$$

pivot

Let, $x_4 = b$, $x_3 = a$

$$x_2 + x_3 - x_4 = 0$$

$$x_2 = -x_3 + 4$$

$$\boxed{x_2 = -a+b}$$

$$x_1 = -2x_2 - 2x_3 - 3x_4$$

$$= 2a - 2b - 2a - 3b$$

$$= -5b$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5b \\ -a+b \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -a \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -5b \\ 1 \\ 0 \\ b \end{pmatrix}$$

$$= a\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Bases. $S^\perp$ → S perpendicular

ii) Yes!

$v = c_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$