Smart idea: In choosing basis nicely & the inverse.

$$\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

Any property: 'orthogonality'

'Binary – Involves only 1 & 0'

$$\boxed{Q^{-1} = Q^T}$$

'Not orthonormal'

↳ Divide by length.

$$\boxed{W^{-1} = W^T}$$

Is a fast way to multiply: The problem is solved

Other problem:

Is it any good?

'Good compression'

'Few basis vectors are enough to reproduce the image'

Fourier basis: JPEG 2000 (next standard) will include wavelets

---

Change of basis

W = 'New basis vectors'

Columns of W

$$[x]_{\text{old basis}} \longrightarrow [c]_{\text{new basis}}$$

$X = WC$

Transformation on matrices:

"$T \rightarrow$ linear transformation"

"8 by 8" with same basis.

T with respect to $v_1, \ldots, v_8$ $\rightarrow$ 1st basis
it has matrix A

with respect to $w_1, \ldots, w_8$ $\rightarrow$ 2nd basis
it has matrix B

what's the connection b/w A and B?

"for example: Rotation" (or) project (or) any other.

step: 1: Remind how we can create A

Step: 2: Remind " " B, Relate A & B (Same transformation)

step: 3: "A and B" are similar.

$$\boxed{B = M^{-1}AM}$$

$\hookrightarrow$ change of basis matrix.

Change to a different basis:

"Two things can happen"

1) Every vector has new coordinates
2) $X = WC$ (Every matrix changes, every transformation has a new matrix)

---

what is A? Using basis $v_1, \ldots, v_8$

1) If we know what transformation does to those eight basis vectors — we know everything about T from

$T(v_1), \ldots, T(v_8)$

$\therefore$ $T \rightarrow$ linear transformation.

Inputs: $v_1, v_2, \ldots, v_8$, outputs: $T(v_1), T(v_2), \ldots, T(v_8)$.

Output: like rotation, projection, etc....

<u>why linearity works?</u>

1) $x \to$ some combination of these basis.

$$x = c_1 v_1 + c_2 v_2 + \dots + c_8 v_8$$

$$T(x) = c_1 T(v_1) + c_2 T(v_2) + \dots + c_8 T(v_8)$$ $\to$ from linearity.

$T(v_1) \to$ some combination of 8 basis vectors.

$$T(v_1) = a_{11} v_1 + a_{21} v_2 + \dots + a_{81} v_8$$
$$T(v_2) = a_{12} v_1 + a_{22} v_2 + \dots + a_{82} v_8$$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{81} \\ \vdots & \vdots & & \vdots \\ a_{18} & a_{28} & \dots & a_{88} \end{bmatrix}$$

Inputs: Basis

compute: $T(v_1) \dots T(v_8)$ like for each basis.

Expand the result in the basis (that gives 64 numbers of the matrix)

<u>suppose: we have an Eigen vector basis</u>

$$T(v_i) = \lambda_i v_i$$

what's A?

Best basis: Eigen vector basis.
↓
Takes more computation than
* wavelet or
* Fourier basis.

First column

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots & \\ 0 & \lambda_2 & \dots & \\ \vdots & \vdots & \ddots & \\ \vdots & & & \lambda_8 \end{bmatrix}$$ $\to$ 'Diagonal matrix'

↳ Loved one in image processing!

1st input $v_1 \xrightarrow[\text{output}]{} \lambda_1 v_1$

<u>Recitation</u>

vector space of all polynomials in x of degree ≤ 2 has a basis $1, x, x^2$. let w_1, w_2, w_3 be a different basis, of polynomials whose values at $x=-1, 0, 1$ are given by:

x	w_1	w_2	w_3
-1	1	0	0
0	0	1	0
1	0	0	1

$\rightarrow$ when $x=-1$, $w_1 = 1, w_2 = 0, w_3 = 0$.

a) Express $y(x) = -x+5$ in the basis!

b) Find the change of basis matrices $(1, x, x^2) \rightleftarrows (w_1, w_2, w_3)$

c) Find the matrix of taking derivatives in both bases.

Solu:

a) $y(x) = \alpha\, w_1(x) + \beta\, w_2(x) + \gamma\, w_3(x)$.

Shortcut: using (b)

x	w_1	w_2	w_3	$y=-x+5$
-1	1	0	0	6
0	0	1	0	5
1	0	0	1	4.

(or)

$y(-1) = \alpha\, w_1(-1) + \beta\, w_2(-1) + \gamma\, w_3(-1)$

$y(-1) = \alpha$

$y(0) = \beta$

$y(1) = \gamma$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

$\alpha = 6, \beta = 5, \gamma = 4.$

$$y = 6w_1 + 5w_2 + 4w_3$$

b) Basis $(1, x, x^2) \rightleftarrows (w_1, w_2, w_3)$

$p = cW$

$\downarrow$ one basis $\quad \downarrow$ other basis

x	w_1	w_2	w_3	1	x	x^2
-1	1	0	0	1	-1	1
0	0	1	0	1	0	0
1	0	0	1	1	1	1

$$1 = w_1 + w_2 + w_3$$
$$x = -w_1 + w_3$$
$$x^2 = w_1 + w_3$$

$$w_1 = -\tfrac{1}{2}x + \tfrac{1}{2}x^2$$
$$w_2 = 1 - x^2$$
$$w_3 = \tfrac{1}{2}x + \tfrac{1}{2}x^2$$

$$A = \begin{array}{c c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ & \left[\begin{array}{ccc} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right] \begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \end{array}$$

$(1, x, x^2) \rightsquigarrow (w_1, w_2, w_3)$

$(w_1, w_2, w_3) \rightarrow$ Take inverse

$$A^{-1} = \begin{array}{c c} & \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ & \left[\begin{array}{ccc} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{array}\right] \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

$(w_1, w_2, w_3) \rightsquigarrow (1, x, x^2)$

c)
$$D_x = \begin{array}{c c} & \begin{array}{ccc} 1' & x' & (x^2)' \end{array} \\ & \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} 1 \\ x \\ x^2 \end{array} \end{array}$$

$1' = 0$

$x' = 1$

$(x^2)' = 2x$

$x' = 1 \rightarrow$ so 1 along 1

$(x^2)' = 2x \rightarrow$ so 2 along x.

$$D_w = A D_x A^{-1}$$

($\therefore$ D_x and D_w are similar)

$$= \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

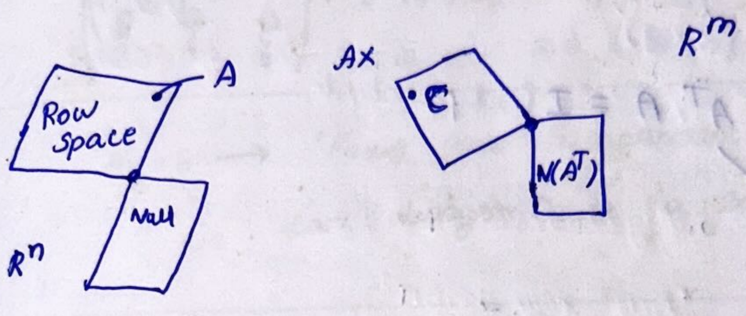
$$= \begin{bmatrix} -3/2 & 2 & -1/2 \\ -1/2 & 0 & 1/2 \\ 1/2 & -2 & 3/2 \end{bmatrix}$$

'we've done it explicitly'

Left and right inverses; pseudo inverses.

We'd like to able to 'invert A' to solve $Ax=b$. But A may have only a left inverse or right inverse (or no inverse). This one explains it.

Lecture-33



Two sided inverse:

'A matrix that gives the Identity'

$$AA^{-1} = I = A^{-1}A$$

What about m, n and r?

In invertible case $m = r = n$ (Full rank)

'Square matrix'

Full column rank: 'Left inverse'

$r = n$

'n columns are independent - what about null space'

↓

Just the zero vector. $= \{0\}$

∴ Rows = number of columns

$$\boxed{0 \text{ or } 1 \text{ solutions to } Ax=b}$$

$A^T A = \begin{bmatrix} \text{Invertible} \end{bmatrix}$

(4x3)(3x4)
$n \times m \; m \times n$

$(n \times n)$

<u>one sided inverse</u>

$$(A^T A)^{-1} A^T \rightarrow A^{-1} \text{ left}$$

$$\therefore \underbrace{(A^T A)^{-1} A^T}_{A^{-1}} A = I$$

$$A^{-1}(A^T)^{-1} A^T A = A^{-1} I A$$

$$= A^{-1} A \rightarrow \text{Left order}$$

$$= I$$

$$\cancel{A(A^{-1}(A^T)^{-1})A^T = I}$$

$$\boxed{\underset{n\times m}{A^{-1}} \; \underset{m \times n}{A} = I_{n\times n}}$$

Left inverse.

<u>For rectangular matrix:</u>

It can't be both side

<u>columns are invertible: rows are not:</u>

"Either A or A^T → has some null space"

$\boxed{\text{Right inverse}}$

Full row rank

$r = m < n$

'$N(A^T)$ has only $\{0\}$' → No combination of rows gives 0.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \end{pmatrix} \rightarrow C_1 + C_2 = C_3 \rightarrow \text{Not a Full rank column}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 4 & 6 \\ 3 & 5 & 6 \end{pmatrix} \rightarrow R_1 + R_2 = R_4 \rightarrow \text{Not a full row rank.}$$

$Ax = b$ → 'Rows are independent'

we can always have a solution → many solutions

∴ $n - m$ → Free variables

Right inverse:

$$A \underline{A^{-1}} = I$$

↳ Right inverse.

$AA^T = (3\times4)(4\times3)$

$= (3\times3)$

↳ square.

$(AA^T)^{-1} \; AA^T = I$

$(A^T)^{-1} A^{-1} A A^T = I$

$\boxed{I = I}$

Previous case

$A^TA \; (A^TA)^{-1} \neq I$

↓

Rows are not independent!

↓

Inverse is not possible

Left inverse

$$A^{-1}_{left} = (A^TA)^{-1} A^T$$

$AA^{-1}_{left} \rightarrow (4\times3)\big[(3\times4)(4\times3) \;\; (3\times4)\big]$

$\rightarrow (4\times3)\big[(3\times4)\big]$ ⟶ Full column case.

$\rightarrow (4\times4)$

$m\times m$ ⟶ 'Not independent'

Right Inverse :

$$A^{-1}_{left} A = (A^T A)^{-1} A^T \quad A$$

$$= \left((3\times4)(4\times3)\right)^{-1}(3\times4) \quad (4\times3)$$

$$= (3\times4)(4\times3)$$

$$= (3\times3)$$

Full column rank case.

$n\times n \rightarrow$ independent.

Right Inverse

'Full' row rank'

$$A A^{-1}_{right} = I_{(m\times m)}$$

$I \rightarrow$ Identity

$\hookrightarrow$ Independent

$$A^{-1}_{right} \quad A = (n\times n) \longrightarrow \text{Not identity}$$

$\hookrightarrow$ Not independent.

---

1. Full rank — Two sided (Just zero vector)
2. Full column $\rightarrow$ ($N(A)$ gone $\rightarrow$ Just zero vector)
3. Full row $\rightarrow$ ($N(A^T)$ gone $\rightarrow$ Just zero vector)
4. $r < m$ and $r < n$

$A(A^TA)^{-1}A^T \longrightarrow$ projection matrix

$\downarrow$

A^{-1}_{left}

(Trying to be an Identity matrix)

$$A^{-1}_{right} A = A^T (AA^T)^{-1} A$$

= Projection to the row space

P^T

Pseudo - Inverse

$r < n$, $r < m$

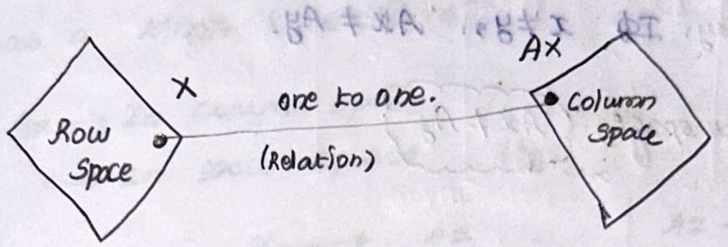
A and A^T both have null spaces.

what's the best inverse we could have:

$x \rightarrow$ From row space (multiply by A)

Ax = will be in the column space.

↳ combination of columns



r

'Both are r-dimensional'

Matrix A → killing the null space component.

If x & y are in row space then

$Ax \neq Ay$

---

screwing up: 'If a matrix takes a vector to zero there's no way an inverse can, like, bring it back'

'$r < n$, $r < m$'

↳ Have null spaces

Pseudo inverse

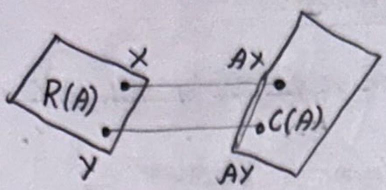
If $Ax = 0$ for some non-zero x, no way to find A^{-1}.

$\boxed{A^{-1}x = 0}$

$Ax \rightarrow$ Combination of columns

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 15 \\ 18 \end{pmatrix}$$

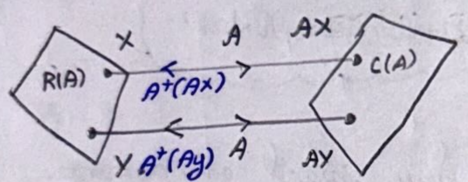
↳ In column space.



Nullspace components x ($Ax=0$) will be killed by A to send it to the column space.

Key: If $x \neq y$, $Ax \neq Ay$.

x, y are in row space, $Ax \neq Ay$



A → Invertible matrix (limited to row & column space)

↳ Inverse is pseudo inverse.

Pseudo inverse: $A^+(Ay)$

↳ Notation

$A \rightarrow$ kills $N(A)$ stuffs

$A^+ \rightarrow$ kills $N(A^+)$ stuffs.

B/w those two n-dimensional spaces of matrices are perfect.

Proof: suppose $x \neq y$ (Both in the row space)

$Ax = Ay$

$Ax - Ay = 0$

$A(x-y) = 0$

$x - y \neq 0$ ($\because x \neq y$)

so $Ax \neq Ay$

$\therefore$ $x \to$ In row space
$y \to$ In row space) $x-y$ will also will be in the row space.

$A(x-y) = 0$

only for the zero vectors

---

matrix A, nice mapping from row space to column space
(Invertible)

'That inverse - pseudo inverse'

$A \to$ has a single inverse A^{-1}

$Ax \to$ In column space
$x \to$ In row space

$x \xrightarrow{A} Ax$ | $Ax \xrightarrow{A^{-1}} x$

Row $\longrightarrow$ column | Column $\longrightarrow$ Row.

'If null spaces keep out of the way, then we have an inverse — pseudo inverse'
↓
Great applications.

Statistics : 'Least square'
'Repeat measurements $\to$ Incase they are not independent
↓ Singular
(we use pseudo inverse).

Find pseudo inverses A^+

① SVD (Start from this) $A = U D V^T$
orthogonal
Diagonal Σ

$$\Sigma = \begin{array}{c} m \text{ columns} \\ \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix} \end{array}$$

m rows
r (rank)

Rank = r singular.

↳ Pseudo inverse of this ?

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & 0 \\ & \ddots & & \\ & & 1/\sigma_r & \\ 0 & & & 0 \end{bmatrix}_{n \text{ by } m}$$

rank r.

In $r=m=n$ (full rank)

$$\Sigma^+ = \Sigma^{-1}$$

$$\underset{(m\times n)(n\times m)}{\Sigma\Sigma^+} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots\, 1 & \\ 0 & & & 0 \end{bmatrix}_{m\times m} \rightarrow \text{Projection to the column space}$$

$$\underset{(n\times m)(m\times n)}{\Sigma^+\Sigma} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots\, 1 & \\ 0 & & & & 0 \end{bmatrix} \rightarrow \text{Projection to the row space.}$$

Every matrix has a pseudo matrix.

Job of pseudoinverse:

It can't give the Identity, you get it's projection.

Two good spaces - row space & column space (wipes out the null space)

$$A = U\Sigma V^T$$

$$A^+ = (U\Sigma V^T)^{-1}$$

$$A^+ = (V^T)^{-1}\ \Sigma^+\ U^{-1}$$

$$A^+ = V\Sigma^+U^T$$

Orthogonal

$$Q^T = Q^{-1}$$

By means of SVD

we can find pseudoinverse faster.

(Puts all the problems in to diagonal

{Good pseudo inverse: minimal data. (minimal matrix)}

Recitation: pseudo inverse

$A = (1 \quad 2)$

i) what is A^+ ?

ii) A^+A & AA^+

iii) x is in $N(A)$ what is A^+Ax ?

iv) ~~If~~ x is in $C(A^+)$ what is A^+Ax ?

solu^n: $A = U\Sigma V^T$

$A^+ = V\Sigma^+ U^T$

'pseudo inverse exists for all matrices whether square or not'

$$\underbrace{\underset{1\times 2}{A} = \underset{(1\times 1)}{U}\ \underset{(1\times 2)}{\Sigma}}_{\text{Same dimensions.}}\ \underset{(2\times 2)}{V^T}$$

$U = [1]$
only orthogonal matrix.

$A^TA = (U\Sigma V^T)^T (U\Sigma V^T)$

$$\boxed{A^TA = V(\Sigma^T\Sigma)V^T}$$

$$\boxed{AA^T = U(\Sigma\Sigma^T)U^T}$$

$\begin{pmatrix}1\\2\end{pmatrix}(1 \quad 2) = V(\Sigma^T\Sigma)V^T$

$\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix} = V(\Sigma^T\Sigma)V^T$

↓

singular

($\lambda=0$ is one Eigen value)

$\lambda_1+\lambda_2 = 5$

$\boxed{\lambda_2 = 5}$

when $\lambda = 5$

$\begin{pmatrix}1-\lambda & 2\\2 & 4-\lambda\end{pmatrix} V = 0$

$\begin{pmatrix}-4 & 2\\2 & -1\end{pmatrix} V = 0$

$V = \begin{pmatrix}1\\2\end{pmatrix}$

↳ orthonormal

↓

$V = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$

when $\lambda = 0$

$V = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}$

$\therefore\ u = [1]$

$$A = [u]\begin{pmatrix}\sqrt{5} & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}^T$$

$u \quad \Sigma \quad V^T$

$$A = [1]\begin{pmatrix}\sqrt{5} & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ -2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}$$

$$A^+ = V(\Sigma^+)u^T$$

If matrix is invertible

$\Sigma^+ = \Sigma^{-1}$

$$A^+ = V\Sigma^{-1}u^T$$

Not necessary for our case (rectangular)

$\Sigma \to m \times n$

$\Sigma^+ \to n \times m$

$$A^+ = V\begin{pmatrix}1/\sqrt{5}\\ 0\end{pmatrix}[1]$$

$$A^+ = \begin{pmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}\begin{pmatrix}1/\sqrt{5}\\ 0\end{pmatrix}[1]$$

$$A^+ = \begin{pmatrix}1/5\\ 2/5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1\\ 2\end{pmatrix} \to \text{Approximation.}$$

i) AA^+

$$\begin{pmatrix}1 & 2\end{pmatrix}\frac{1}{5}\begin{pmatrix}1\\ 2\end{pmatrix} = \frac{1}{5}(5) = [1]$$

↳ Identity matrix

ii) A^+A

$$\begin{pmatrix}1\\ 2\end{pmatrix}\frac{1}{5}\begin{pmatrix}1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$$

iii) $N(A) = c\begin{pmatrix}-2\\ 1\end{pmatrix}$ (i.e $x = \begin{pmatrix}-2\\ 1\end{pmatrix}$

iii) $A^+Ax = ?$ $\quad x$ in $N(A)$

$$\frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2+2 \\ -4+4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A^+Ax = 0$$

$$\therefore A^+\underbrace{Ax}_{0} = 0$$

$$\boxed{A^+0 = 0}$$

iv) $A^+Ax \rightarrow x$ in $C(A^T)$

$$C(A^T) = C\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A^+Ax = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ 10 \end{pmatrix} = \frac{5}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} = x$$

$$A^+Ax = x$$

x in row space

$Ax \rightarrow$ projects to column space

$A^+(Ax) \rightarrow$ projects to row space.

If A is invertible

$$A^{-1}Ax = x$$

$$Ix = x$$

$$\boxed{x = x}$$

Exam-3 - review

'Elimination doesn't preserve Eigen values'

$A = A^T$ then factorize in to $Q\Lambda Q^T$

similar matrices → Represents the same Eigen values.

$\left(\text{'Powers of } A\text{'} = \text{'Powers of } B\text{'} \right) \rightarrow$ similar matrices

$$\boxed{B = M^{-1} A M}$$

1) $\frac{du}{dt} = Au = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ u

'Skew matrix' — 'singular' $\lambda_1 = 0$.

Solu: General solution

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

Eigen values:

$$\lambda_1 = 0$$

$$\begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix}$$

$\boxed{A = -A^T}$

↳ 'pure imaginary Eigen values'

$$-\lambda^3 - 2\lambda = 0$$

$$\lambda(-\lambda^2 - 2) = 0$$

$$\lambda = 0, \lambda^2 = -2$$

$$\lambda = \pm\sqrt{2}i$$

$$u(t) = c_1 e^{0t} x_1 + c_2 e^{-\sqrt{2}it} x_2 + c_3 e^{\sqrt{2}it} x_3$$

$$= c_1 x_1 + c_2 e^{(-\sqrt{2}i)t} x_2 + c_3 e^{(\sqrt{2}i)t} x_3$$

↓ magnitude 1 ↓ magnitude 1.

$$\boxed{\sqrt{\sin^2 x + \cos^2 x} = 1}$$

Doesn't blow up

'wanders in the unit circle'

<u>when does the solution returns to initial value:</u>

'periodic'

when $t = 0$

$$u(0) = c_1 x_1 + c_2 x_2 + c_3 x_3$$

Again repeat when $\left(e^{\sqrt{2}it} = e^{-\sqrt{2}it} = 1 \right)$

when $\sqrt{2}iT = 2\pi i$

$T = \frac{2\pi}{\sqrt{2}}$

$T = \sqrt{2}\pi$

$\sin 2\pi = 0$

$\cos 2\pi = 1$

$e^{2\pi i} = 1$

$e^{-2\pi i} = \cos(-2\pi) = 1.$

'Periodic with $T = \pi\sqrt{2}$

Note: Skew-symmetric matrices

'Eigen vectors of a symmetric or a skew-symmetric matrix are always orthogonal'

---

orthogonal Eigen vectors (x's)

when $AA^T = A^TA$

1) Symmetric
2) Skew (anti-symmetric)
3) orthogonal.

Symmetric

$A^2 = A^2$

Skew

$-A^2 = -A^2$

orthogonal

$QQ^T = Q^TQ$

$\boxed{I = I}$

$e^{At} = ?$

$u(t) = e^{At} u(0)$

(Fundamental matrix)

If A is diagonalizable,

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_m t} \end{bmatrix}$$

$\lambda_1 = 0$ $\quad \lambda_2 = c$ $\quad \lambda_3 = 2$

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

1) Is the matrix diagonalizable? (For all c's)

Solu:

Eigen vectors $\rightarrow$ Independent.

'orthogonal' too.

For all $c \to$ our matrix is diagonalizable

b). Symmetric? (For which c). $\left[c = ? \to \text{Not symmetric}\right]$

↳ For all real c's.

c) positive definite: (sub case of symmetric)

Eigen value test: Eigen value $= 0$ (+ve semi definite) could be

Semi +ve definite: If $c \geq 0$

d) markov matrix:

$\lambda_1 = 1$ other eigen values are smaller than 1.

Never a markov matrix?

e) $\frac{A}{2}$ is a projection matrix?

* Real-symmetric
* Eigen values are 0's and 1's

$P^2 = P$ so $\lambda^2 = \lambda \to$ Eigen values are 0 and 1.

$c = 2$ (or) $c = 0$

$\therefore \frac{A}{2} \to$ so $\frac{c=2}{2}$ will be 1.

If this is not orthogonal - symmetric, positive def, projection will be dead.

SVD

$A =$ (orthogonal) (diag) (orthogonal) $= U \Sigma V^T$

(Every A)

$AA^T = V \Sigma^T \Sigma V^T$

$A^T A = U \Sigma \Sigma^T U^T$

(symmetric)

Symmetric case

$U = V$

$V \to$ Eigen vector matrix for $A^T A$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

σ_i^2 = Eigen values of A^TA

$AA^T = u \Sigma \Sigma^T u^T$

$U \rightarrow$ Eigen vector matrix of AA^T

Problem: we have basis - we don't know which sign to use

$Av_i = \sigma_i u_i \rightarrow$ main point of SVD

$u_2 = Av_2$

$Av_2 = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} 3\sqrt{2}$ $\longrightarrow$ Ref SVD

↳ sign change

Example: $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$, $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

↓

Non singular. - Determinant not zero.

Singular values are not negative' $\sigma_i \neq (-)ve$

' Eigen values of symmetric matrices are allways real.

Rank 1.

Nullspace 1 (dimension)

$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$ singular

Rank = 1

dim N(A) = 1.

$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} x = 0$

$x = c\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\boxed{x = v_2} \Rightarrow V = (v_1 \; v_2)$

---

second column is the basis of nullspace of A.

<u>True False</u>

1) A is symmetric & orthogonal. (what can we say about Eigen values).

Symmetric $\rightarrow \lambda$ real

① Eigen values $\rightarrow$ $|\lambda| = 1$

'orthogonal matrices are like rotations $\rightarrow$ Not changing the length'

$Qx = \lambda x$

$||x|| = ||\lambda||\cdot||x||$

'orthogonal matrices don't change lengths'

① Eigen values $\rightarrow$ $\underbrace{1 \text{ and } (-1)}_{\text{Real, Magnitude}=1.}$

1) A $\rightarrow$ positive definite?

No $\rightarrow$ If $\lambda = -1$. $\rightarrow$ False

2) It has no repeated Eigen values $\rightarrow$

'In 3x3 or more $\rightarrow$ λ must be repeated'

↳ 'only 1 and -1 are allowed'

↓

3) Is it diagonalizable? (Repeated case)

yes! All Symmetric & orthogonal matrices are diagonalizable.

4) Non singular?

'orthogonal $\rightarrow$ always non-singular'

5) Show $\frac{1}{2}(A+I)$ is a projection matrix

Prop: $P^2 = P$

'symmetric'

$$\frac{1}{4}(A^2+2AI+I^2) = \frac{1}{4}(A^2+2A+I)$$

$$= \frac{1}{4}(A^2+2A+I) = \frac{1}{4}(2A+2I)$$

$$= \frac{1}{2}(A+I)$$

A $\rightarrow$ symmetric & orthogonal

$A^2 = I$

$A = A^T = A^{-1}$ $\rightarrow$ symmetric as well orthogonal

$A = A^T \rightarrow$ symmetric

$A^T = A^{-1} \rightarrow$ orthogonal

what are the Eigen values of $\frac{1}{2}[A+I]$

$\therefore$ W.K.t $\lambda = 1, -1$ for A.

$$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \lambda = 2 \text{ (or) } 0.$$

Divide by 2 $\rightarrow \lambda = 1$ [Eigen values of $P = 0, 1$]

Recitation

1) Find λ's and x's

i) $P = \frac{aa^T}{a^Ta}$, $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

ii) $Q = \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix}$ iii) $R = 2P - I$ $\downarrow$ Reflection matrix

Solu:

$$P = \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix}(3 \;\; 4)}{(3 \;\; 4)\begin{pmatrix} 3 \\ 4 \end{pmatrix}} = \frac{\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}}{9+16}$$

$$= \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

$Px = \lambda x$ $\qquad P = P^2$

$P^2 x = \lambda x$

$P(Px) = \lambda x$

$P(\lambda x) = \lambda x^2$

$\lambda^2 x = \lambda x$

$\lambda(\lambda - 1)x = 0$

$\lambda = 1, \lambda = 0 \rightarrow$ Eigen values of a P are 0 and 1.

$$Pa = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \longrightarrow \text{projection matrix}$$

$$= \frac{1}{25}\begin{pmatrix} 27+48 \\ 36+64 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} = a.$$

when $\lambda = 0$

$\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ $\perp^r$ vectors have $\lambda = 0$

when $\lambda = 1$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

verify

$$P b = \left\{ \frac{1}{25} \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \right\} \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $\rightarrow$ Normal vector

$Pb = 0.$

$\downarrow$

Normal to the plane.

ii) $Q = \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix}$

$\therefore 0.6^2 + (-0.8)^2 = 1 \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \rightarrow$ (Frame will be like this for some θ)

'Rotation matrix'

char eqn.

$$(0.6-\lambda)^2 + 0.8^2 = 0$$

$$\lambda = 0.6 \pm 0.8i$$

$(Q - \lambda I) u = 0$

For $\lambda = 0.6 + 0.8i$

$$\begin{pmatrix} -0.8i & -0.8 \\ 0.8 & -0.8i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$u = \begin{pmatrix} i \\ 1 \end{pmatrix} \text{ (or) } \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For $\lambda = 0.6 - 0.8i$

$$\bar{u} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

'complex conjugate'

iii) $R = 2P - I$

$$= \frac{2}{25} \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(or) shortcut

i) Eigen vectors of $P = 0, 1$

$2P$ has $\lambda = 2, 0$

$2P - I$ has $\lambda = 2 - 1$ | $\lambda = 0 - 1$

$\boxed{\lambda = 1}$ | $\lambda = -1$

"By shifting $2P$ by I → we doesn't move Eigen vector'
↓
only we move Eigen value.

So Eigen vectors of $2P-I$ same as P.

$e^{At} = S e^{\Lambda t} S^{-1}$

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \end{pmatrix}$$

$\lambda_1, \lambda_2, \lambda_3 = -1, 0, +1.$

e^{-t}, e^{0t}, e^{t}

Lecture - 35

Course Review

1) 3 by n matrix

$$Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{no solution}$$

$$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \text{Exactly 1 solution}$$

what can you tell me about m, n and r.

Solu:

$m = 3$ $\qquad (3\times\ )(\ \times 1) = (3\times 1)$

$Ax=b$ (no solution)

$\boxed{\text{rank} < m}$

∴ Some rows are combinations of others.

$Ax=b$ (Exactly 1 solution)

- Null space has only the zero vector in it'
- '$r=n$' → columns are independent.

$m=3,\ r=n=2$ (or) (1)

Example

Simply

$\boxed{m=3 > n=r}$

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \text{For solving } Ax=b$$

$b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ must in the column space.

(or) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

No solution $(0x_1 + 0x_2 \neq 1)$ Exactly one solution.

Cross out false facts:

1) $\det A^TA = \det AA^T$
2) A^TA is invertible
3) AA^T is +ve definite

Solu:

ii) $A^TA \to$ Invertible when $r = n$ (Independent columns of A)

↓

True (our case $r = n$)

$(4\times3)(3\times4) = (4\times4) \to$ Invertible when n is independent no. of columns.
$n \times n$

↓

Zero vector alone in the null space.

our example: $m = 3 > r = n$

$A^TA \longrightarrow (2\times3)(3\times2) = (2\times2) \to$ our example.

'Square'
'Identity'

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\det(A^TA) = 1$

'Invertible'

$AA^T = (3\times n)(n\times3)$

$= (3\times3)$
$m \times m$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank = 2   'Symmetric'

It has eigen value = 0

May be : (Positive semi definite)

$\det(AA^T) = 0$

$\det(A^TA) \neq \det(AA^T)$

↳ True incase of square matrices.

Square:

$\det(AB) = \det A \det B$.

prove $A^Ty = c$ atleast 1 solution for every c.

(only many solutions)

Solu:

$A^Ty = c$

↓

$(n \times m)$

Full row rank → always independent solvable.

$n = r$

$\dim N(A^T) = m - r > 0$ → more than 1 solutions.

$$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$$

a) Solve $Ax = v_1 - v_2 + v_3$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} x = v_1 - v_2 + v_3$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

b) Suppose $v_1 - v_2 + v_3 = 0$ → solution is not unique → True or false.

Solu:

c_3 → combination of c_1 and c_2 → dependent columns.

Rows → Full rank rows

$$\therefore \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} x = 0$$

↓

Null Space

$N(A)$ has some guys → A variety of solutions are available.

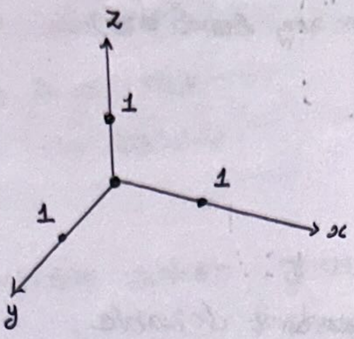
c) v_1, v_2, v_3 → orthonormal → what combination of v_1 & v_2 is closest to v_3

{orthonormal → Independent vectors.}

$-v_1 + -v_2 =$ closest to v_3

Key: v_1, v_2, v_3 be std x, y, z basis.

"v_1, v_2, v_3 are orthonormal"



"$x, y, z \rightarrow$ orthonormal to eachother"

"The closest one to z axis is 0"

$0v_1 + 0v_2 = 0$

---

3) Markov matrix

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix}$$

Column 1 + column 2 = 2 Column 3

<u>soln</u>:

1 Eigen value $\lambda = 0$ (singular)

$\lambda = 1$ (Markov matrix)

$$(0.2 + 0.2 + 0.4) - 1 = -0.2 = \lambda_3$$

$$u_k = A^k u(0) = A^k \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}$$

↓

$$u(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

<u>After k steps</u>

$$= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots + c_3 \lambda_3^k x_3$$

0 1 −0.2

<u>As $k \to \infty$</u>

$$c_2 (1)^\infty x_2 + \cdots c_3 (-0.2)^\infty x_3$$

↳ vanishes

$\boxed{u_\infty = C_2 x_2}$ → steady state problem.

Eigen vectors:

$\lambda = (1)$

$$\begin{bmatrix} -0.8 & 0.4 & 0.3 \\ 0.4 & -0.8 & 0.3 \\ 0.4 & 0.4 & -0.6 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \qquad u_\infty = C_2 \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

Beauty: total number of population never zero.

---

2 by 2 matrix

a) projects onto the line spanned by $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

Solu:

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{pmatrix} 4 \\ -3 \end{pmatrix}\begin{pmatrix} 4 & -3 \end{pmatrix}}{\begin{pmatrix} 4 & -3 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix}} = \frac{\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}}{25} = \frac{1}{25}\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}$$

b) matrix with $\lambda = 0, 3 \rightarrow x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

What about matrix A?

$$A = S\Lambda S^{-1}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

c) has real entries & can't be factored as B^TB for any B

$B^TB \rightarrow$ symmetric

Never could be factored $\boxed{A = B^TB}$

Key: A couldn't factored as B^TB

∴ If A is non-symmetric → Then $A = B^TB$

∴ $B^TB \rightarrow$ symmetric.

d) Not symmetric but orthogonal Eigen vectors

Solu:

"orthogonal Eigen vectors - Not symmetric" ↳ "could be skew symmetric"

↓

'orthogonal matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Examples.

"Eigen vectors → orthogonal"

skew symmetric

Least squares

5)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}}_{b}$$

Least square solution:

$$\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 11/3 \\ -1 \end{bmatrix}$$

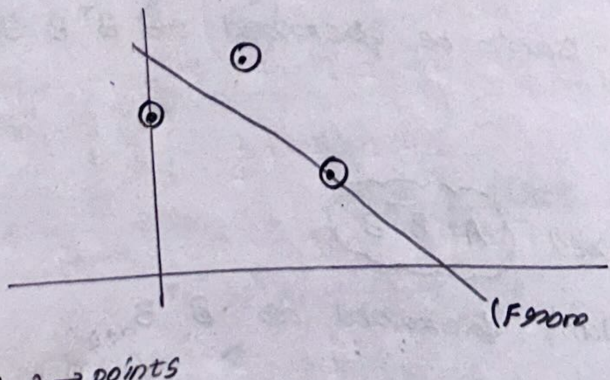
what's the projection P of this vector on to the column space.

① P (" " ")

↳ solved by least square solution

$$\frac{11}{3} c_1 - 1\, c_2 = \begin{bmatrix} 11/3 \\ 8/3 \\ 5/3 \end{bmatrix}$$

②



(From $\begin{pmatrix} 11/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

0, 1, 2 → points

1, 1, 1 → Difference b/w each point

3, 4, 2 → Heights.

c) Find a different b, least square solution changes to zero

$$\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$b \rightarrow$ orthogonal to the columns $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

eg: $b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\therefore$ projecting $\perp^r$ vectors $P(b) = 0$

Recitation

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Two eigen values $\lambda_1 = 1$, $\lambda_2 = 2$

First two points $d_1 = d_2 = 1$.

a) Find λ_3 and d_3

b) what's the smallest a_{33} that would make A positive semi-definite? what's the smallest c so that $A + cI$ is positive semi-definite?

solu:

$\lambda_3 = (1+1+0) - (1+2)$

$= 2 - 3$

$\boxed{\lambda_3 = -1}$

$d_1 d_2 d_3 = \det(A)$

$\downarrow$

pivots are not always the diagonal entries'

we have $\lambda_1, \lambda_2, \lambda_3 \neq 0$'

so,

$\det A = 1(-1) + 1(-1) = -2$

$d_1 d_2 d_3 = -2$

$\boxed{d_3 = -2}$

b) Smallest a_{33} that would make A +ve definite.

$\det A = -2 \rightarrow$ negative.