Laplace transform

1) Find the Laplace transform of a function from the definition.

2) Identify the region of convergence of the Laplace transform of a function.

3) Use linearity & t-derivative rule - to find Laplace transforms.

Recap

Can we use the Transfer function to predict how the system will respond to other input signals?

- Suppose we have recorded the system response to a single (non-null) signal. Can we use that single response to determine the transfer function (and hence the system parameter).

- How can we extend our methods to cover the behaviour of more complicated systems involving feedback?

The only answer mechanism is Laplace transform.

Recalling LTI theory of mechanical & Electronic systems:

Considering a system responds to an exponential input signal - one of the form $y(t) = e^{st}$ for a fixed complex number s. We found that there is generally a system response that's a multiple of the input signal.

$$x(t) = H(s)\, e^{st}$$

- The system function (or) transfer function' - H(S) depends upon the exponential constant s (which is often a complex number) but is independent of t. The exception occurs when s is a pole of H(S) - the condition is termed as Resonance.

This provides one particular system response to e^{st}.

By superposition,

the other solutions are given by adding in some homogeneous solution (response to the zero input signal). For this theory to be useful, the system must be stable. (i.e. all these signals must die as t get large. We found that stability is equivalent to requiring all the poles of $H(s)$ to have negative real part. (i.e to lie in the left half plane. In this case, the null responses are 'transients', and all system responses to the same input signal become asymptotic as t grows large.

A second assumptions we make in this course is that the system is controlled by a diff equ: one of the form

$$P(D)x = Q(D)y$$

$$H(s) = \frac{Q(s)}{P(s)}$$

The transfer function thus contains, as coefficients, the system parameters characterizing the LTI system, as they enter in to the diff equ.

This is a beautiful theor, by taking $s = i\omega$ we obtain a complete understanding of how the system responds to sinusoidal signals; $H(i\omega)$ is the complex gain $G(\omega)$.

Laplace function

The Laplace transform will allows us to envision the effect of an LTI system on any function (subject to certain initial conditions) as multiplying by the transfer function.

Since our systems are time invariant, we might as well study of their behaviour at $t=0$, and this will be a standing convention in our work with the Laplace transform.

1st definition

3.1

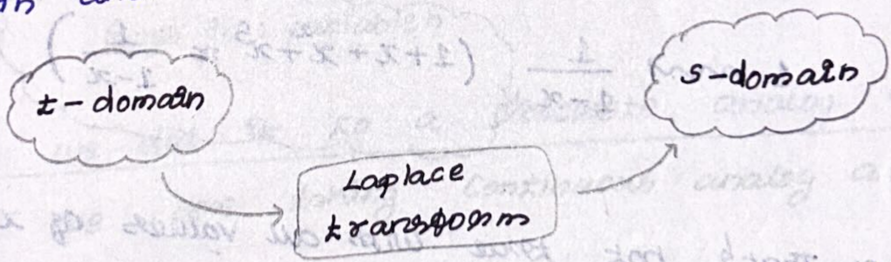
$$F(s) = \int_0^\infty f(t) e^{-st} \, dt$$

we will often use the notation

$$f(t) \rightsquigarrow F(s)$$

The Laplace transform converts functions of the real variable t to functions of a completely different variable, the complex variable s. It moves us from the time domain to what is often termed the frequency domain. (once we see the connection b/w Laplace transform and the system function, this terminology will seem more justified).

It's very important to recognize that the two functions $f(t)$ and $F(s)$ are functions of different variables. If we think of s as representing frequency, then the Laplace transform converts b/w the time domain and the frequency domain.



Transforms take in functions of one variable, and return functions of another variable.

Note: This definition of the Laplace transform probably seems abstract; the following video provides an analogy b/w Laplace transform & power series.

Power series

where from Laplace transform comes from?

Power series

$$\sum_0^\infty a_n x^n = A(x)$$

In computer notation:

$a_n = a(n)$

$$\sum_0^{\infty} a(n)\, x^n = A(x)$$

In Computer (Particular)

$\downarrow$

Discrete function.

n - Real number.

Taking the sequence of coefficients of the power series & associating that with the sum of the power series.

Gives

Taking the discrete function

$$a(n) \rightsquigarrow A(x)$$

Suppose: $a(n)$

$A(x)$

$$1 \rightsquigarrow 1+x+x^2+x^3+\ldots$$

$$\left(\sum_0^{\infty} 1.x^n = x^0 + x + x^2 + \ldots\right)$$

$$1 \rightsquigarrow \frac{1}{1-x} \quad \left(1+x+x^2+x^3 \approx \frac{1}{1-x}\right)$$

Problem: That's not true with all values of x but only true when x is such that the series converges. And that is only true when x lies b/w -1 and 1.

$$1 \rightsquigarrow \frac{1}{1-x}, \quad |x| < 1$$

If $x > 1$, the answer is it doesn't converge.

$$(1-x)^{-1} = 1+x+x^2+x^3+\ldots$$

valid when

$|-x| < 1$

$|x| < 1.$ (converges)

If $a(n) = \frac{1}{n!}$

$\frac{1}{n!} \rightsquigarrow e^{x}$

$\left[\sum_{0}^{\infty} \frac{1}{n!} x^{n} \ldots = e^{x}\right] \Rightarrow e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

↳ True for all x.

---

operation of summing a power series as taking a discrete function defined for positive integers, for non-negative integers, doing this process will give a continuous function.

$n = 0, 1, 2, \ldots\ldots$

Looking,

$a(n) \rightsquigarrow A(x)$

$1 \rightsquigarrow \frac{1}{1-x}$

$\frac{1}{n!} \rightsquigarrow e^{x}$

↳ Result in x.

Goes in : variable n

Above we did it to a discrete analog $a(n)$:

Here we are taking continuous analog $a(t)$

In continuous system, we can't sum like

$n = 0, 1, 2, \ldots \infty$
$\downarrow$
$t : 0 \leq t < \infty$
↳ +ve

$0 + 1 + 2 + \cdots$

(Since it has all values b/w $0 \leq t < \infty$)
Including each & every points.

$\therefore$

So Integration to the rescue:

$$\int_{0}^{\infty} a(t)\, x^{t}\, dt = A(x)$$

↳ making this little better.

Converting x to the base e.

$$x = e^{\ln x}$$

$$x^t = (e^{\ln x})^t$$

Problem: when $t \to \infty$

The $\int_0^\infty a(t)\, x^t \, dt$ → most unlikely

to converge. [∴ constant multiplication].

It happens only when x is less than 1.

(Otherwise – our integrate won't converge)

Better to have +ve, because if I allow it to be (–)ve

$(-1)^{1/2}$ → Imaginary one.

$$0 < x < 1.$$

∴ $\ln x = 0$ when $x = 1$

For $0 < x < 1$ → $\ln x$ will be negative.

($\ln 0.5 = -0.693$)

$(0 < x < 1)$: $\ln x < 0$

Let, $s = -\ln x$ $\quad (-s = \ln x)$

1) Nobody uses $\ln x$ as a variable

2) Convenient to work with $+s$ instead of $-s$.

$$\int_0^\infty f(t)\, e^{-st}\, dt = F(s)$$

Since $F(s)$ → we don't want to be like $A(e^{-s})$.

If

$-s = -0.693$

$\boxed{s = 0.693}$

$\boxed{-s = -0.693}$

Finally,

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \quad \left[\text{Continuous version}\right]$$

$\sum_{0}^{\infty} a_n x^n = A(x) \rightarrow$ discrete version.

$\int_{0}^{\infty} f(t)\, e^{-st}\, dt = F(s) \longrightarrow$ Laplace transform

<u>Other notation</u>

The Laplace transform is often denoted using the script letter $\mathcal{L}$.

for example, Laplace transform of a function $f(t)$ is the function

$$\mathcal{L}(f(t)).$$

But what is the variable of this transformed function? Because the variable of the transform is not always clear from context, we prefer the notation

$$\mathcal{L}(f(t); s)$$

which makes the variable of the transformed function explicit.

Sometimes we contract the notation even further and write $\mathcal{L}(f)$ for the Laplace transform. So here is all of the notation for the Laplace transform that you might see.

$$f(t) \rightsquigarrow F(s) = \mathcal{L}(f(t); s) = \underbrace{\mathcal{L}(f)}_{\text{simple.}} = \mathcal{L}(f(t))$$

<u>Laplace transform of 1.</u>

$$\int_{0}^{\infty} f(t) \cdot e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt \qquad (f(t) = 1)$$

Improper Integrals: (Complex functions)

$$\int_0^\infty e^{-st}\,dt = \lim_{R\to\infty} \int_0^R e^{-st}\,dt$$

$$= \lim_{R\to\infty} \left(\frac{e^{-st}}{-s} \right)_0^R$$

$$= \lim_{R\to\infty} \left(\frac{e^{-sR}}{-s} - \frac{1}{-s} \right)$$

$$= \frac{1}{s} \quad (s>0)$$

(True only when $s>0$)

$1 \rightsquigarrow \frac{1}{s}$.

Let's begin by computing the Laplace transform of the constant function with value 1:

$$\mathcal{L}(1;s) = \int_0^\infty e^{-st}\,dt.$$

This is an improper integral. If $s=0$, the integrant is the constant 1, and the integral diverses. If $s<0$, the situation is even worse: e^{-st} grows without bound as $t\to\infty$, and the integral again diverges. In the remaining case, $s>0$, we can calculate:

since $\lim_{t\to\infty} e^{-st} = 0$

$$\mathcal{L}(1;s) = \int_0^\infty e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s} - \frac{1}{s}\lim_{t\to\infty} e^{-st}$$

$$= \frac{1}{s}$$

But wait! The number s was supposed to be complex, not necessarily real: say $s=a+bi$; $a=\mathrm{Re}(s)$, $b=\mathrm{Im}(s)$. Then

$$e^{-st} = e^{-at}(\cos(-bt) + i\sin(-bt))$$

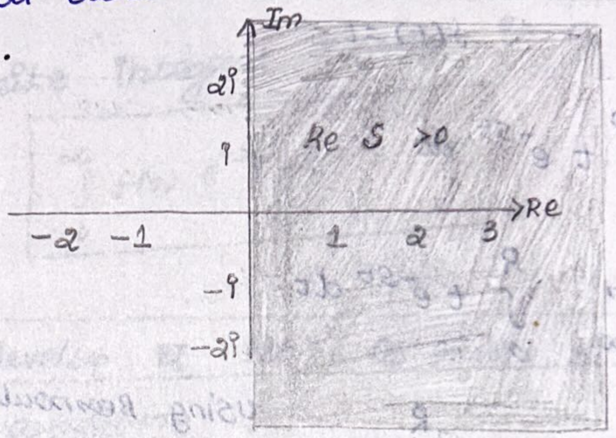
This again tends to zero as $t\to\infty$ if $a>0$.

It grows without bound. If $a=0$, e^{-st} runs around the $\circ^{le}$ (unit) as t increases. Both the latter two cases leads to divergent improper integrals.

So:

$$\mathcal{L}(1;s) = \begin{cases} 1/s & \text{if } Re(s) > 0 \\ \text{undefined} & \text{if } Re(s) \le 0 \end{cases}$$

This example exhibits a typical behaviour of the integral defining the Laplace transform: It diverges for all s to the left of some vertical line in the complex plane. The half plane where the integral does converge is the region of convergence.



Region of convergence $Re(s) > 0$

$$\mathcal{L}(e^{\gamma t};s) = \int_0^\infty e^{\gamma t} \cdot e^{-st}\, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{\gamma t} \cdot e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left(\frac{e^{(\gamma-s)t}}{(\gamma-s)} \right)_0^R$$

$$= \lim_{R\to\infty} \left(\frac{e^{(\gamma-s)R} - 1}{\gamma - s} \right)$$

The limit depends upon $\gamma - s$ is +ve or -ve.

$$\lim_{N\to\infty} \left| e^{(r-s)N} \right| = \begin{cases} 0 & \text{if } Re(s) > Re(r) \\ \infty & \text{if } Re(s) < Re(r) \end{cases}$$

when $Re(s) = Re(r)$, the $e^{(r-s)N}$ oscillates and doesn't converge. Therefore,

$$\mathcal{L}(e^{rt}; s) = \begin{cases} \dfrac{1}{s-r} & \text{if } Re(s) > Re(r) \\ \text{Diverges} & \text{if } Re(s) \le Re(r); \end{cases}$$

The reign of convergence is $Re(s) > Re(r)$

$f(t) = t.$

$$\mathcal{L}(t; s) = \int_0^\infty t e^{-st}\, dt$$

$$= \lim_{R\to\infty} \int_0^R t e^{-st}\, dt.$$

$$= \lim_{R\to\infty} \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{-s(-s)} \right]_0^R$$

using Bernoulli

$\int u\,dv = uv - \int v\,du$

$u = t \Rightarrow du = dt$

$dV = e^{-st}$

$V = \dfrac{e^{-st}}{-s}$

$$= \lim_{R\to\infty} \left[\frac{t}{-s} e^{-st} - \frac{e^{-st}}{s^2} \right]_0^R$$

$$= \begin{cases} 0 & \text{if } Re(s) > 0 \\ \infty & \text{if } Re(s) < 0. \end{cases}$$

when $Re(s) = 0$, the e^{-sR} oscillates & doesn't converge.

$$\therefore \mathcal{L}(t; s) = \begin{cases} \dfrac{1}{s^2} & \text{if } Re(s) > 0 \\ \text{Diverges} & \text{if } Re(s) \le 0. \end{cases}$$

Reign of convergence is $Re(s) > 0$.

Linearity

$$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$$

$$= \mathcal{L}(f) + \mathcal{L}(g).$$

$$\mathcal{L}(cf) = c\,\mathcal{L}(f)$$

The Laplace transform of the sum of two functions is the sum of the Laplace transforms of each of them seperately.

(True because of the form of the transform.

Fact: Definite integral itself is a linear operator.

$$\boxed{\int_0^\infty f(t)\, e^{-st}\, dt = F(s)}$$

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we will develop list of rules satisfied by the Laplace transform:

Linearity: For a and b constant:

$$\mathcal{L}(af(t) + bg(t); s) = a\mathcal{L}(f(t); s) + b\,\mathcal{L}(g(t); s)$$

or)

$$af(t) + b(g(t) \rightsquigarrow aF(s) + bG(s)$$

Proof (easy):

Multiplication by e^{-st} is linear & integration is linear.

The rules and the calculations combine to provide new evaluations of the Laplace transform.

Example: Let's combine linearity with

$$\mathcal{L}(e^{rt}; s) = \frac{1}{s-r}, \quad Re(s) > Re(r)$$

to evaluate $\mathcal{L}(\cos \omega t; s)$ using the inverse

Euler formula:

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\rightsquigarrow \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$$

$\left(e^{rt} = \frac{1}{s-r}\right)$

$$= \frac{1}{2} \frac{(s+i\omega)+(s-i\omega)}{(s^2+\omega^2)}$$

$$= \frac{s}{s^2+\omega^2}.$$

Each term had $Re(s) > 0$. as a reign of convergence, so this is the reign of convergence for this Laplace transform as well.

Each term had $Re(s) > 0$ as a reign of convergence, so this is the reign of convergence for this Laplace transform as well.

Compute $\mathcal{L}(\sin \omega t ; s)$

solu: using: Euler's Formula

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$= \frac{\cos \omega t + i\sin \omega t - \cos(-\omega t) - i\sin(-\omega t)}{2i}$$

$$= \frac{2i \sin \omega t}{2i} = \sin \omega t$$

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

$$= \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right)$$

$$= \frac{1}{2i}\left(\frac{2(i\omega)}{s^2+\omega^2}\right)$$

$$= \frac{s\omega}{s(s^2+\omega^2)} = \frac{\omega}{s^2+\omega^2}$$

The reign of convergence of this linear combination of transforms is the intersection of the two reigns of convergence. Since $\mathcal{L}(e^{i\omega t}; s)$ and $\mathcal{L}(e^{-i\omega t}; s)$ both have convergence $Re(s) > 0$ and the same is true for $\mathcal{L}(\sin\omega t; s)$.

$$\mathcal{L}((1+e^{-rt})^2) = \mathcal{L}(1+2e^{-rt} + e^{-2rt})$$

$$= \frac{1}{s} + \frac{2}{s+r} + \frac{1}{(s+2r)}$$

Observe that $\mathcal{L}$ doesn't preserve products

$$\mathcal{L}((1+e^{-rt})^2) \neq (\mathcal{L}(1+e^{-rt}))^2$$

Laplace transform of a damped sinusoid

$\mathcal{L}(e^{at}\cos(\omega t); s)$ a-real by the method used.

$\therefore$ Damped sinusoids are important signals!

$$e^{at}\cos\omega t = e^{at}\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)$$

$$= \frac{1}{2}\left(e^{(a+i\omega)t} + e^{(a-i\omega)t}\right)$$

$$\rightsquigarrow \frac{1}{2}\left(\frac{1}{s-(a+i\omega)} + \frac{1}{s-(a-i\omega)}\right) \quad Re(s) > a$$

$$= \frac{1}{2}\left(\frac{s-(a-i\omega)+s-(a+i\omega)}{(s-a)^2+\omega^2}\right)$$

$$= \frac{1}{2}\left(\frac{2s-2a}{(s-a)^2+\omega^2}\right) = \frac{s-a}{(s-a)^2+\omega^2}$$

$Re(s) > a$.

Exponential type and region of convergence

How can we guarantee that the Laplace transform of a function or its derivative even exists for some s, ∴ we need a condition that will guarantee that there is a half-plane of convergence.

Does the Laplace transform always exist?

Does the Laplace transform $\mathcal{L}(f(t);s)$ always exist on some reign?

Asking it another way: Does the integral $\int_0^\infty e^{-st} f(t)\,dt$ converge for some s?

Solu:

No, because this is an improper integral, and improper integrals don't always converge.

Condition: The condition that makes the Laplace transform definitely exist for a function is that f(t) shouldn't grow too rapidly. It can grow, because the e^{-st} is pulling it down, trying hard to pull it down to 0. (To make integral converge).

All we have to guarantee is that it doesn't grow rapidly that e^{-st} is powerless to pull it down.

(Growth condition)

How fast the function is allowed to grow?

f(t) of 'exponential type'

$|f(t)|$ (Absolute value) - shouldn't be greater than the rapidly growing exponent

$$|f(t)| \le Ce^{Kt}$$

C > 0 constant

K > 0 constant

all $t \ge 0$

$f(t) = \sin t$

$|\sin t| \le 1 \cdot e^{0t}$

$|t^n| \le M e^t$ $(K = 1,\ M > 0$ constant$)$

t^n is of exponential type.

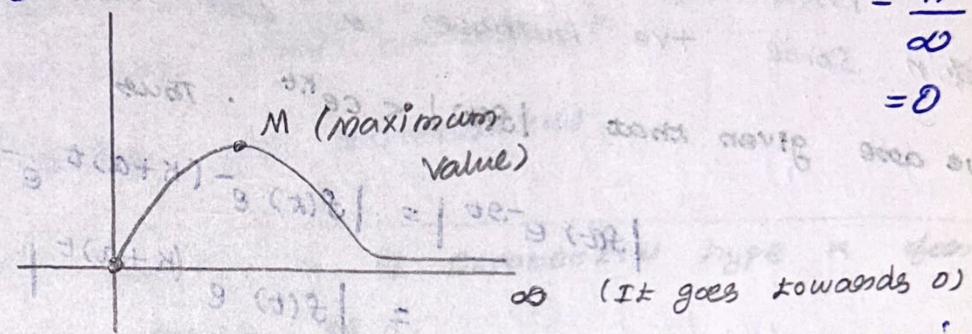
proof:

$\frac{t^n}{e^t} \le M$ (some number M)

As $t \to \infty$ (L'Hopital's rule)

$\frac{n t^{n-1}}{e^t}$ As it goes on

$\frac{n^{(a)} \cdot t^0}{e^t} = \frac{n^a}{e^t} = \frac{n^a}{\infty} = 0$



As it is continuous (Guarantees it has a maximum)

showing it's an exponential type.

Non-exponentials:

$\frac{1}{t}$ $\int_0^\infty e^{-st} \cdot \frac{1}{t}\, dt$

near $t = 0$

$e^{-st} \approx 1$

$\int_0^\infty \frac{dt}{t} \to$ At 0 doesn't converge.

So $\frac{1}{t}$ doesn't have a Laplace transform.

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e^{t^2} — grows too rapidly $\Rightarrow$ not of exponential type.

$e^{t^2} > e^{kt}$ if $t^2 > kt$, $t^2 > k$

As it grows rapidly, it doesn't have laplace transform.

Note:

A function $f(t)$ is of exponential type K, for some real number K, if for some constant $c > 0$:

$$|f(t)| \le c e^{Kt} \text{ for all } t \ge 0.$$

Theorem 8.1:

1) The Laplace transform $\mathcal{L}(f(t); s)$ exists if $f(t)$ is of exponential type.

2) If $f(t)$ is of exponential type K, then the Laplace transform $\mathcal{L}(f(t); s)$ converges for $Re(s) > K$.

proof: Suppose that $Re(s) > K$, that is $s = (K+a) + bi$ for some +ve number a and real number b. Thus we are given that $|f(t)| < ce^{Kt}$

$$|f(t) e^{-st}| = |f(t) e^{-(K+a)t} e^{-ibt}|$$

$$= |f(t) e^{-(K+a)t}| \, |e^{-ibt}|$$

$|e^{-ibt}| = 1.$

$$= |f(t)| e^{-(K+a)t}$$

$$\le ce^{Kt} \cdot e^{-(K+a)t}$$

$$= ce^{-at}$$

Since $\left|\int f(t)\, dt\right| \le \int |f(t)|\, dt$

$$\left|\int_0^M f(t) e^{-st}\, dt\right| \le \int_0^M |f(t) e^{-st}|\, dt$$

$$\le \int_0^M ce^{-at}\, dt.$$

The last integral has a limit $M \to \infty$, so the improper integral defining $\mathcal{L}(f(t); s)$ converges as long as $Re(s) > K$.

$1, t, t^2, \frac{1}{t+1}$, e^{at} for $a>0$.
e^{-at} for $a>0$

$e^{at} \cos \omega t$ for $a>0$

$e^{-at} \cos \omega t$ for $a>0$

$e^{t^n} \cos \omega t$ for $0<n<1$

} Have Laplace transform.

$\frac{1}{t-1}$ blows up in finite time $(t=1)$

So it's not exponential type (K since ce^{Kt} is finite at $t=1$). for all real numbers c and K.

IIIly $e^{t^n} \cos \omega t$ for $n>1$ is not of exponential type K for any real number K.

(All other options are exponential)

$e^{-2t} \cos(5t)$ is of exponential type K for the following values of K.

solu::

$$|e^{-2t} \cos 5t| \leq e^{-2t} < e^{-1.5t} < e^{0t} < e^{1.5t} < e^{2t} < e^{3t}$$

but there is no constant c such that

$$|e^{-2t} \cos(5t)| < ce^{-3t} \text{ for all } t.$$

If a function is of exponential type K and K' $>K$, then it's also of exponential type K'. This function $e^{-2t} \cos 5t$ is exponential of type K for any $K \geq -2$ and not of type K for any $K<-2$. $\therefore$ exponential type K for any

$3, 2, 1.5, 0, -1.5$ and -2 are correct answers.

Let take

$y = e^{2x}$, $y' = 2e^{2x}$ ($y' > y$)

3, 2, 1.5, 0, −1.5, −2.

let

$|e^{-2t} \cos(5t)|$

---

f(t) is of exponential type K, then the Laplace transform $\mathcal{L}(f(t);s)$ converges for $\text{Re}\, s > K$.

So, $K < \text{Re}(s)$ $\qquad -2 < \text{Re}(s)$

So the possible values are 3, 2, 1.5, 0, −1.5, −2

Laplace transform of a derivative.

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt \rightarrow$$ what turns this eqn in to f(t).

(Integrate by parts):

$\int u\,dv = uv - \int v\,du$

$u = e^{-st}$

$dv = f'(t)\,dt$

$du = -se^{-st}\,dt$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty f(t) \cdot (-s)e^{-st}\,dt$$

$$= -e^{-0} \cdot f(0) - \int_0^\infty f(t)(-s)e^{-st}\,dt$$

$$= -f(0) + s\int_0^\infty f(t) \cdot e^{-st}\,dt$$

$$\lim_{t \to \infty} \frac{f(t)}{e^{st}}$$ (what will make this limit 0? f(t) shouldn't go faster than e^{st})

f(t) is exponential

$|f(t)| < ce^{kt}$

$\lim_{K \to \infty} \frac{f(t)}{e^{St}}$ goes to zero when $S > K$.

$$= -f(0) + S \int_0^\infty f(t) e^{-St} \, dt$$

↳ Laplace transform

$$= -f(0) + SF(S).$$

$f'(t)$ $\rightsquigarrow$ $SF(S) - f(0)$

(Assuming exponential type)

The power of the Laplace transform in understanding differential equ arises from how it evaluates on derivatives. So let's think

$$\mathcal{L}(f'(t); S) = \int_0^\infty e^{-St} f'(t) \, dt$$

$u = e^{-St}$, $dv = f'(t)\, dt$

$$= \int_0^\infty e^{-St} f'(t) \, dt = e^{-St} f(t) \Big|_0^\infty - \int_0^\infty (-S) e^{-St} f(t) dt$$

The limit in the first term here will probably not converge for all S. But if $f(t)$ is of exponential type K, then it will converge as long as

$Re(S) > K$.

$$\therefore \quad = -f(0) + SF(S)$$

Remark 9:1:

Assume $f(t)$ was of exponential type K in this derivation. But after all, before we even started to think about $f'(t)$, we had to assume that $f(t)$ was of exponential type K in order to guarantee that the integral defining $F(S)$ converged for

$Re(S) > K$; so this is not a new assumption. What we do learn is that if $f(t)$ is of exponential type K then the region of convergence of $\mathcal{L}'(f'(t); s)$ has $Re(s) > K$.

9.2:

Just to verify that what need the $f(0)$ in the formula, let's work

$$f(t) = 1, \; f'(t) = 0. \text{ It follows}$$

$\mathcal{L}(0; s) = 0$ (so the t derivative rule gives)

$$0 = s \cdot \frac{1}{s} - 1.$$

$$\mathcal{L}(1; s) = \int_0^\infty 1\, dt \quad = 0$$

t-derivative rule:

By t-derivative rule.

$$\mathcal{L}(0; s) = sF(s) - f(0)$$

$$= s\left(\frac{1}{s}\right) - 1 \qquad \left(\frac{1}{s} \; (Re(s) > 0)\right.$$

Ret power series)

$$= 0$$

$$f(t) = e^{rt} \qquad f(0) = e^0$$

$$\mathcal{L}(f'(t); s) = sF(s) - f(0) \qquad = 1.$$

$$\mathcal{L}(e^{rt}; s) = \frac{1}{s-r}$$

$$\mathcal{L}(f'(t); s) = \frac{s}{s-r} - 1.$$

$$= \frac{s - s + r}{s-r} = \frac{r}{s-r}$$

The reign of convergence is the same as for $F(s)$ since multiplying by a nonzero constant doesn't change the reign of convergence:

$$Re(s) > Re(r)$$

$f(t) = e^{rt} \cos \omega t$ (r is a real number)

solu:

$$\mathcal{L}(e^{rt}\cos\omega t ; s) = \int_0^\infty e^{rt}\cos\omega t \cdot e^{-st}\, dt$$

$$= \int_0^\infty e^{(r-s)t} \cos\omega t \, dt$$

$$\therefore \mathcal{L}(e^{at}\cos\omega t ; s) = \frac{s-a}{(s-a)^2+\omega^2}$$

$$= \frac{s-r}{(s-r)^2+\omega^2}, \quad f(0) = 1$$

$$F'(s) = s(F(s)) - f(0)$$

$$= s\left(\frac{s-r}{(s-r)^2+\omega^2}\right) - 1$$

$$= \frac{r(s-r) - \omega^2}{(s-r)^2+\omega^2}$$

Reign of convergence is same as $F(s) : Re\, s > r$.

Laplace transform of higher derivatives

Integrate by parts twice:

or Hack method

$f''(t) = (f'(t))'$ → second derivative is the first derivative of the first derivative.

$$\mathcal{L}(f'(t), s) = sF(s) - f(0)$$

$$\mathcal{L}(f''(t); s) = s\,\mathcal{L}(f'(t), s) - f(0)$$

$$= s\left(\mathcal{L}(f(t), s)\cdot s - f(0)\right) - f'(0)$$

$$= s^2\mathcal{L}(f(t), s) - sf(0) - f'(0)$$

$$= s^2F(s) - s\,f(0) - f'(0)$$

---

we can go on to compute the effect of Laplace transform on the second derivative. To make calculation clearer, let's write $g(t) = f'(t)$ and $G(s)$ for its Laplace transform; So we know that

$$G(s) = sF(s) - f(0)$$

$$f''(t) = g'(t) \rightsquigarrow sG(s) - g(0)$$

$$= s(sF(s) - f(0)) - g(0)$$

$$= s^2F(s) - f(0)s - f'(0)$$

This process continues; we find that

$\mathcal{L}(f^{(n)}(t); s)$ depends upon $f(0), f'(0), \ldots f^{(n-1)}(0)$

$$f^{(n)}(t) \rightsquigarrow s^n F(s) - (f(0)s^{n-1} + f'(0)s^{n-2} + \cdots + f^{(n-1)}(0))$$

---

suppose that Laplace transform of a function $x(t)$ is $X(s)$. Moreover, suppose that

$$x(0) = 2$$
$$\dot{x}(0) = -1$$

use linearity,

$$y(t) = \ddot{x} + \dot{x} + 2x$$

$\mathcal{L}(y(t); s) = ?$

For $\ddot{x} \Rightarrow \mathcal{L}(\ddot{x}; s) = s^2X(s) - (x(0)s + x'(0))$

$$\mathcal{L}(\dot{x}; s) = sX(s) - x(0)$$

$$\mathcal{L}(x; s) = X(s)$$

$$\mathcal{L}(y(t); s) = s^2X(s) - (s(2) - 1) + sX(s) - 2 + 2X(s)$$

$$= \underline{s^2X(s)} - 2s + 1 + \underline{sX(s)} - 2 + \underline{2X(s)}$$

$$= s^2X(s) + sX(s) + 2X(s) - 2s - 3 + 2$$

$= s^2 X(s) + sX + 2X(s) - 2s - 1$

$= (s^2+s+2)X - (2s+1)$

Summary

$1 \rightsquigarrow \frac{1}{s}$ $\quad Re(s) > 0$

$e^{rt} \rightsquigarrow \frac{1}{s-r}$ $\quad Re(s) > Re(r)$

$\cos\omega t \rightsquigarrow \frac{s}{s^2+\omega^2}$ $\quad Re(s) > 0$

$\sin\omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}$, $\quad Re(s) > 0$

$t \rightsquigarrow \frac{1}{s^2}$, $\quad Re(s) > 0$

Rules:

linearity $\quad af(t) + bg(t) \rightsquigarrow aF(s) + bG(s)$

$f'(t) \rightsquigarrow sF(s) - f(0)$

$f''(t) \rightsquigarrow s^2F(s) - f(0)s - f'(0)$

$f^{(n)}(t) \rightsquigarrow s^nF(s) - (f(0)s^{n-1} + f'(0)s^{n-2} + \dots + f^{(n-1)}(0))$

Reign of convergence: If $f(t)$ is exponential of type K, then $\mathcal{L}(f)$ exists & converges for $Re\, s > K$.

Applying Laplace transform to solve DE

1) Solve IVP using Laplace transform.

2) Employ the method of partial fractions to find the inverse Laplace transform of $\frac{Q(s)}{P(s)}$ where

$\deg Q < \deg P$

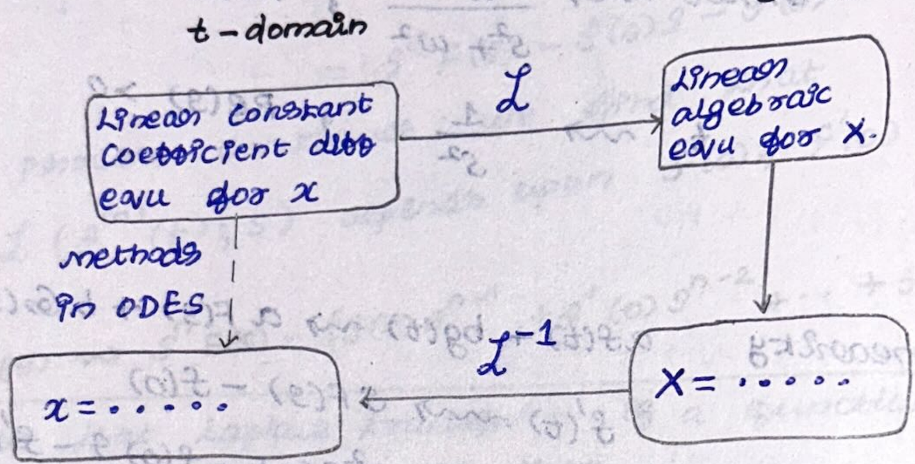
$P(s)$ has only simple roots.

Applications to differential equ:

The t-derivative rule shows that differentiation by s in the time domain becomes multiplication in the frequency domain (up to the annoying initial condition term.) This leads to the amazing fact

Fact: Laplace transform carries differential equations to algebraic equations.

So in using Laplace transform to solve a differential equ, the process is as follows:



The missing ingredient here is getting back from the s-world to the t-world. we will do this by using the rules & calculations of the Laplace transform to find $x(t)$ with the desired Laplace transform $X(s)$. To make this work, we need to know the following rule.

Inverse rule:

A continuous function $f(t)$ for $t \geq 0$ is determined by its Laplace form (transform) $F(s)$ (if it exists). we use the notation

$$\mathcal{L}^{-1}(F(s); t) = f(t) \quad \text{(or)} \quad \mathcal{L}^{-1}(F(s)) = f(t)$$

Exam: 2.1: Which function of t has $F(s) = \dfrac{1}{s-5}$ as its Laplace transform?

Solu:

$f(t) = e^{5t}$ has $F(s) = \frac{1}{s-5}$ as its Laplace transform. Every other continuous function that has this Laplace transform agrees with this one for $t \geq 0$. we write

$$\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = e^{5t} \quad \text{for } t \geq 0.$$

Inverse Laplace:

$1 \rightsquigarrow \frac{1}{s}$ $\quad$ $\text{Re } s > 0$

$e^{rt} \rightsquigarrow \frac{1}{s-r}$ $\quad$ $\text{Re } s > \text{Re } r$

$\cos \omega t \rightsquigarrow \frac{s}{s^2+\omega^2}$ $\quad$ $\text{Re } s > 0$

$\sin \omega t \rightsquigarrow \frac{\omega}{s^2+\omega^2}$ $\quad$ $\text{Re } s > 0$

$t \rightsquigarrow \frac{1}{s^2}$ $\quad$ $\text{Re } s > 0.$

Find a function whose Laplace transform is $\frac{3}{s+1}$

soln: $\frac{1}{s+1}$ is the Laplace transform of e^{-t},

so by linearity.

$$\mathcal{L}^{-1}\left(\frac{3}{s+1}\right) = 3e^{-t}.$$

$$\mathcal{L}(t, s) = \int_0^\infty t e^{-st} \, dt$$

$dv = e^{-st} \, dt$

$v = \frac{e^{-st}}{-s}$

$u = t$

$du = dt$

$$= \left. \frac{t(e^{-st})}{s} - \int \frac{e^{-st}}{-s} \, dt \right]_0^\infty$$

$$= \left. \frac{te^{-st}}{s} + \frac{1}{s} \frac{e^{-st}}{-s} \right]_0^\infty$$

$$= \left. \frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^\infty$$

$$= -\left(\frac{1}{s^2}\right) = \frac{1}{s^2}$$

$\mathcal{L}^{-1}\left(\left(\frac{s}{s^2+16}\right);t\right)$

Solu:

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4^2};t\right) = \cos 4t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+16};t\right)$$

By linearity:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+16}\right) = \frac{1}{4}\sin 4t$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}+\frac{1}{s^2+16};t\right) = t+\frac{1}{4}\sin\omega t$$

Example (solving LTI)

Solu: $\dot{x}+Kx=Ky$ (models Newtonian cooling or the water level in a bay by changing under the influence of ocean tides, where y is the external temperature, or the water level outside of the bay.

$$y=e^{-t}$$

In solving an ODE using Laplace transform, one must specify initial $(t=0)$ conditions. So here we suppose that $x(0)$ is known.

Solution:

Step: 1: Apply $\mathcal{L}$ to the diff equ

$$(sX(s)-x(0))+KX(s)=KY(s)$$

The result is a linear equ for $X(s)$, so it is easy to solve.

Step: 2: Solve the equation for $X(s)$.

First pull all terms not involving $X(s)$ as a factor on the right:

$$(s+k)\,X(s) = x(0) + kY(s)$$

Then solve:

$$X(s) = \frac{x(0) + kY(s)}{s+k}$$

step: 3: Find the inverse Laplace transform

$$\mathcal{L}^{-1}\,(X(s);\,t)$$

our input signal is $y(t) = e^{-t}$. we know the Laplace transform of $y(t)$ is given by

$$Y(s) = \frac{1}{s+1},$$

so

$$X(s) = \frac{x(0) + k/(s+1)}{s+k} = \frac{x(0)}{s+k} + \frac{k}{(s+1)(s+k)}$$

we want to find a function $x(t)$ whose Laplace transform is $X(s)$. By the linearity rule, we can treat each term seperately. The first one is easy; by linearity again, and using the known Laplace transform for an exponential function, we have

$$x(0)\,e^{-kt} \rightsquigarrow \frac{x(0)}{s+k}$$

The second term is trickier & actually has to be treated in two cases accordingly to whether $k \neq 1$ or $k = 1$. For now we'll do the more generic one & suppose that $k \neq 1$.

The relevant technique is now the method of partial fractions, which is important enough to get its own section. For the moment, we'll just recall the simplest case, which is what we need here: if $k \neq 1$, there are unique numbers a and b such that.

$$\frac{k}{(s+1)\,(s+k)} = \frac{a}{s+1} + \frac{b}{s+k}$$

The traditional way to find a and b is to cross

Multiply & set coefficients equal.

$$\frac{K}{(S+1)(S+K)} = \frac{a(S+K)+b(S+1)}{(S+1)\quad(S+K)}$$

$$K = a(S+K) + b(S+1)$$

$$K = aS + aK + bS + b$$

$$= aK + S(a+b) + b$$

$$K = a(S+K) + bS + b$$

$$\frac{K}{S+1} = (S+K)\frac{a}{S+1} + b$$

'Cover-up' method

$$\frac{K}{(S+1)\ (S+K)} = \frac{a}{S+1} + \frac{b}{S+K} \quad \text{(multiply by } S+1 \text{ set } S=-1\text{)}$$

$$\frac{K}{S+K} = a + (S+1)\frac{b}{S+K}$$

becomes

$$\frac{K}{S+K} = a \qquad (S=-1)$$

$$a = \frac{K}{K-1}$$

Also, multiply by $S+K$, set $S=-K$

$$\frac{K}{S+1} = (S+K)\frac{a}{S+1} + b$$

$$b = \frac{K}{S+1}$$

$$= \frac{K}{1-K}$$

$$\therefore \frac{K}{(S+1)(S+K)} = \frac{K}{K-1}\left(\frac{1}{S+1} - \frac{1}{S+K}\right)$$

The payoff is that it's easy to recognize this as the Laplace transform of a function. By linearity and using the known Laplace transform for an exponential function:

$$\frac{e^{-t}-e^{-kt}}{k-1} \rightsquigarrow \frac{1}{(s+1)(s+k)}$$

putting all this together, using linearity as usual,

$$x(t) = x(0)\,e^{-kt} + \frac{k}{k-1}\left(e^{-t}-e^{-kt}\right)$$

The presence of $k-1$ in the denominator confirms that the case $k = 1$ has to be treated seperately.

Now the structure of this expression: The exponential term $\frac{k}{k-1}e^{-t}$ is the "steady state". The exponential response given by the ERF. The two terms involving e^{-kt} together form the transient required to produce the given IC. One of these terms has the effect of cancelling the IV of the $\frac{k}{k-1}e^{-t}$ term, and other imposes the desired IV.

t domain $\qquad\qquad\qquad\qquad$ s-domain

$$\begin{tikzcd}
\boxed{\dot{x}+kx = ke^{-t}} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \boxed{(sX(s)-x(0))+kX(s) = k/s+1} \arrow[d] \\
\boxed{x(t) = x(0)e^{-kt} + \frac{k}{k-1}(e^{-t}-e^{-kt})} & \boxed{X(s) = \frac{x(0)}{s+k} + \frac{k}{(s+1)(s+k)}} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

The inverse Laplace transform involved using partial fractions to write the algebraic equations in a form that we could look up the inverse

Laplace function easily because we could recognize it as the Laplace transform of an exponential function. Of course we could have found this result using the ERF and finding the appropriate transient. The Laplace transform gives us another tool for solving diff eqn. More importantly, we will see that it provides a connection to the Transfer function of any LTI system.

$$\frac{K}{(s+1)(s+K)} = \frac{a}{s+1} + \frac{b}{s+K}$$

$$K = a(s+K) + b(s+1)$$

$$K = a(s+K) + b(s+1)$$ ⇒ So Applying roots is the way.

$$\cancel{a = \frac{K}{s+K}} \quad \Big| \quad \cancel{b = \frac{K}{s+1}}$$ (wrong)

$$\cancel{\frac{K}{(s+1)(s+K)} = \frac{K}{(s+K)(s+1)} + \frac{K}{(s+1)(s+K)}}$$

Partial fractions

Step 1: The degree of $Q(s)$ might be bigger than the degree of $P(s)$. In that case, use the division algorithm you learned about in high school algebra to write

$$\frac{Q(s)}{P(s)} = F(s) + \frac{R(s)}{P(s)}, \quad \deg R(s) < \deg P(s)$$

Step 2: The fundamental theorem of algebra assumes us that $P(s)$ factors into a product of linear & quadratic factors: terms of the form $(s+b)$ and (s^2+bs+K) where the quadratic terms don't have real roots. The partial fractions algorithm tells us that if $\deg Q(s) < \deg P(s)$ and there are not repeated factors in the denominator, then the rational function $Q(s)/P(s)$ can be written as a sum