complex gain $\frac{1}{(49-n^2)+(0.1n)i}$ and gain

$$g_n = \frac{1}{|(49-n^2)+(0.1n)i|}$$

$$x = Im\left(\frac{1}{(49-n^2)+(0.1n)i} e^{int}\right)$$

This is a sinusoid of amplitude g_n, so

$$x = g_n \cos(nt - \phi_n) \quad \text{for some } \phi_n$$

The input signal,

$$\frac{\pi}{4} sq(t) = \sum_{n \geq 1, odd} \frac{\sin nt}{n},$$

elicits the system response,

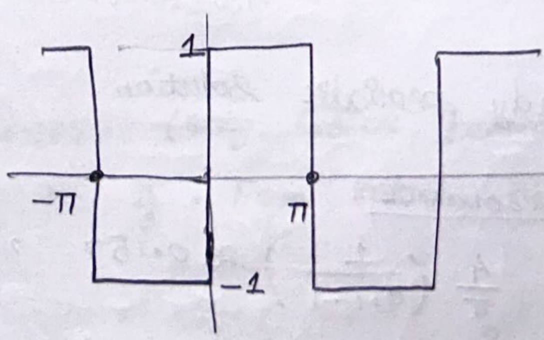
$$x(t) = \sum_{n \geq 1, odd} g_n \frac{\cos(nt - \phi_n)}{n}$$

$$\approx 0.021 \cos(t - \phi_1) + 0.008 \cos(3t - \phi_3)$$
$$+ 0.008 (\cos(5t - \phi_5))$$
$$+ 0.204 \cos(7t - \phi_7) + 0.003 \cos(9t - \phi_9)$$
+ even smaller terms.

The system response is almost indistinguishable from a pure sinusoid of angular frequency 7.

Practice problems

$f(t)$ be the odd square wave of period 2π with $f(t)$ $= 1$ for $0 < t < \pi$



$$\ddot{x} + 9.1x = f(t)$$

Solu:

$$f(t) = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

$$= \frac{4}{\pi} \sum_{n\,odd} \frac{\sin nt}{n}$$

$$\ddot{x} + 9.1x = f(t)$$

$$\ddot{x} + 9.1x = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \dots\right)$$

Solving

$$\ddot{x}_n + 9.1x_n = \frac{\sin nt}{n}$$

$$\ddot{z}_n + 9.1z_n = \frac{1}{n} e^{int}$$

$$z_p = \frac{1}{n} \frac{e^{int}}{(-n^2+9.1)}$$

$$x_{n,p}(t) = \frac{1}{n} \frac{\sin nt}{(9.1) - n^2}$$

From superposition:

$$\ddot{x} + 9.1x = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \dots\right)$$

$$x_{sp}(t) = \frac{4}{\pi}\left(x_{1,p}(t) + x_{3,p}(t) + x_{5,p}(t) + \dots\right)$$

$$= \frac{4}{\pi}\sum_{n\,odd} x_{n,p}(t)$$

$$= \frac{4}{\pi}\left(\frac{\sin t}{9.1-1} + \frac{\sin 3t}{3(9.1-9)} + \frac{\sin 5t}{5(9.1-25)} + \dots\right)$$

$$= \frac{4}{\pi}\sum_{n\,odd} \frac{\sin nt}{n(9.1-n^2)}$$

Steady periodic solution

Near resonance:

$$\frac{4}{\pi}\left(\frac{1}{9.1-1}\right) \approx 0.157, \quad \frac{4}{\pi}\left(\frac{1}{3(9.1-9)}\right) \approx 4.244$$

$$\frac{4}{\pi}\left(\frac{1}{5(9.1-25)}\right) \approx -0.016.$$

for $n=1,3,5$ resp. $n=3 \rightarrow$ Resonance occurs.

(Steady state periodic response $x_{sp}(t)$ has the biggest amplitude).

$n=1, \rightarrow$ second largest.

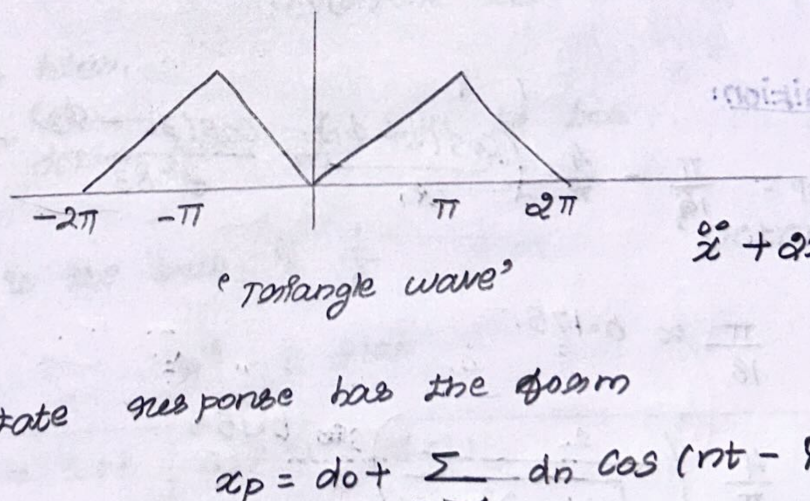
$\therefore \sqrt{9.1} \approx 3$; and so the system has a resonant-type response to the embedded third harmonic $\frac{\sin 3t}{3}$ in the input signal.

$\boxed{\text{Input fundamental frequency} = 1}$

So the presence of 3rd harmonic is not apparent to the eye, and yet the driven oscillator picked it out in its response, which has a dominant frequency three times the fundamental freq. of the input.

we can push a pendulum swing into resonance even if you give it a push only on every third swing, instead of pushing it every swing.

Largest damping.



$\ddot{x}+2\dot{x}+9x=f(t)$

'Triangle wave'

steady state response has the form

$$x_p = d_0 + \sum_{n\geq 1} d_n \cos(nt-\phi_n)$$

Soln.

$f(t)$ is an even function. (only cosine terms). The average value of $\frac{a_0}{2}$ is $\frac{\pi}{2}$. For $n\geq 1$ (we can use $n=0$ too),

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt = \frac{2}{\pi}\int_0^{\pi} t\cos nt\, dt$$

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos \pi + \frac{\cos 3t}{3^2} + \frac{\cos 5t}{5^2} + \cdots\right)$$

Following the same steps:

$\ddot{x}_n + 2\dot{x}_n + 9x_n = \cos nt$ for each n.

If $n=0$ we get, $\boxed{x_{n,p} = \frac{1}{9}}$

$$\ddot{z}_n + 2\dot{z}_n + 9z_n = e^{int}$$

$$z_{n,p} = \frac{e^{int}}{9-n^2+2ni},$$

Gain is $\frac{1}{|9-n^2+2in|} = \frac{1}{\sqrt{(9-n^2)^2+4n^2}}$.

Let

$R_n = \sqrt{(9-n^2)^2+4n^2}$ to simplify notation.

Thus, $z_{n,p} = \frac{e^{int}}{9-n^2+2in}$, which implies that $x_{n,p} = \frac{\cos(nt-\phi_n)}{R_n}$.

Superposition:

$$x_{sp} = \frac{\pi}{18} - \frac{4}{\pi}\left(\frac{\cos(t-\phi_1)}{R_1} + \frac{\cos(3t-\phi_3)}{3^2 R_3} + \cdots\right)$$

$$\frac{\pi}{18} \approx 0.175.$$

$$\frac{4}{\pi}\left(\frac{1}{\sqrt{(9-1)^2+4}}\right) \approx 0.154$$

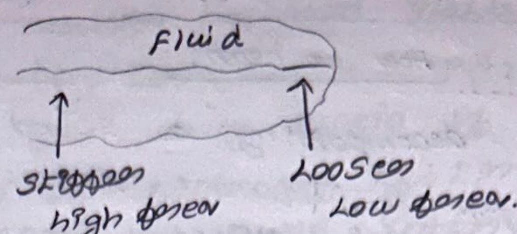
$$\frac{4}{\pi}\left(\frac{1}{9\sqrt{(9-9)^2+4(9)}}\right) \approx 0.024$$

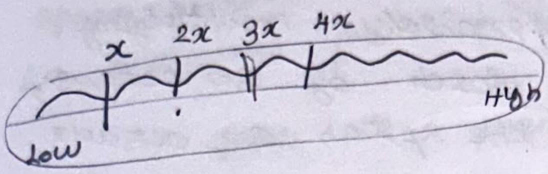
$$\frac{4}{\pi}\left(\frac{1}{25\sqrt{(9-25)^2+4(25)}}\right) \approx 0.003$$

$\therefore$ $n=0$, $n=1$ terms have largest amplitude

Listening to Fourier Series

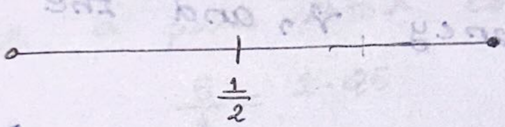
Cochlea





A frequency $2x$ will vibrate at the same place as x did (may be more strong) → "our brain will feel that they are almost the same" overlap"

Pythagoras



1	$1/2$	$1/3$	$1/4$	$1/5$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	(Keys)
C	C	G	C	E	G	B♭	C	D	F#	Notes.

"overtone series"

$\frac{1}{2}$ – octive higher.

$\therefore \frac{1}{2}$ is octive higher, $\frac{1}{4}$ will be too.

"octaves"

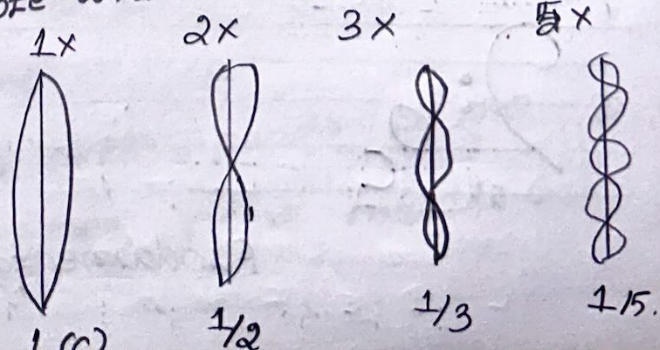
$\frac{1}{6}$ is the half of $\frac{1}{3}$

B♭ – B flat

C : G = 1 : 3 | C : F# = 1 : 11

C : E = 1 : 5 | C : C# = 1 : 17

* Note & the note with twice the freq are alike.
* chordes



1 A440 → 440 times vibrating , A880 (1/2)

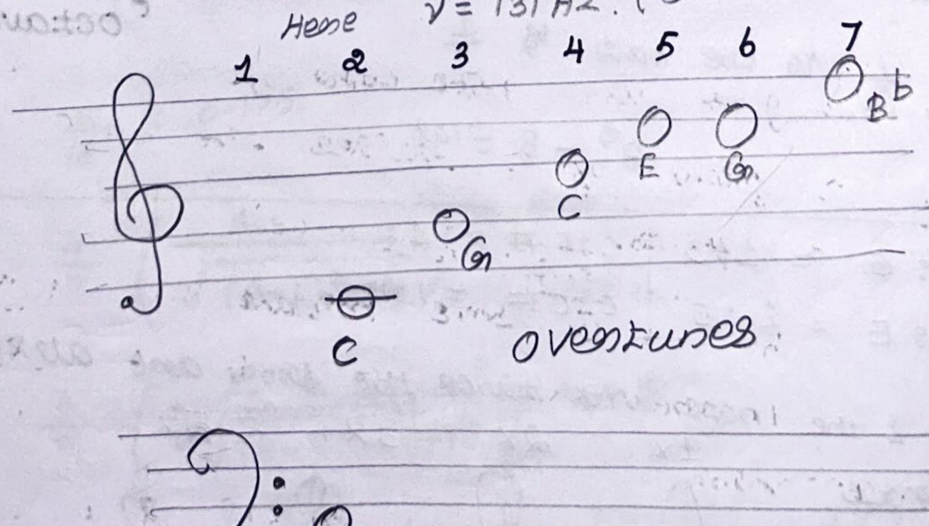
How we hear?

Ear is capable of decomposing a sound wave in to its Fourier components of different freq. Each freq corresponds to a certain pitch. Increasing the freq produces higher pitch. more precisely, multiplying the freq by 2 raises the pitch by an octave, and multiplying by 3 raises the pitch an octave plus a perfect fifth.

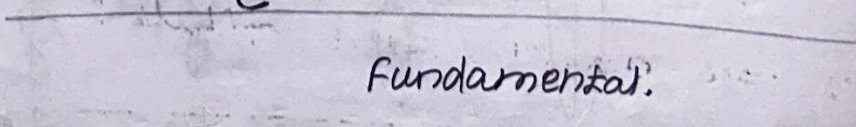
When an instrument plays a note, it's producing a sound wave (periodic) in which typically many of the Fourier co.eff are non zero. In a general F.S, the combination of the 1st two non-constant terms (a$_1$ cost + b$_1$ sint), if the period is 2π) is a sinusoid of frequency ν, and the next combination,

(a$_2$ cos2t + b$_2$ sin2t) has freq 2ν, and so on: the frequencies are the +ve integer multiples of the lowest frequency ν. The note corresponding to the frequency ν is called the fundamental, and the notes corresponding to 2ν, 3ν are called the over tunes.

Here ν = 131 Hz. (C below middle C)



overtunes



Fundamental.

Foot notes:

most modern keyboard instruments divide the octave into twelve (12) - half - steps, each of which represents a frequency ratio of $2^{1/12}$. This means that intervals on such an instrument are only approximated to the pure intervals corresponding to rational number ratios. For instance,

- A fifth on a piano has 7 step- halves, for example C to G, hence frequency ratio of $2^{7/12} \approx$ 1.4983, where a pure fifth corresponds to $\frac{3}{2} = 1.5$.

- A major third on a piano has 4 half steps, C to E (example) $\Rightarrow$ frequ ratio = $2^{4/12} \approx 1.2599$ where as a pure third corresponds to a ratio $\frac{5}{4} = 1.25$

- Fourth major has 5 half steps G to C, hence a frequ ratio $2^{5/12} \approx 1.3348$, pure one $\frac{4}{3} = 1.333...$

Book : Temperament by Stuart Isacoff.

1.2.1

Can you guess what note corresponds to 9? multiply a frequ by 3, raises the pitch by an octave + a fifth. $\boxed{O\ G}$ → see music notes.

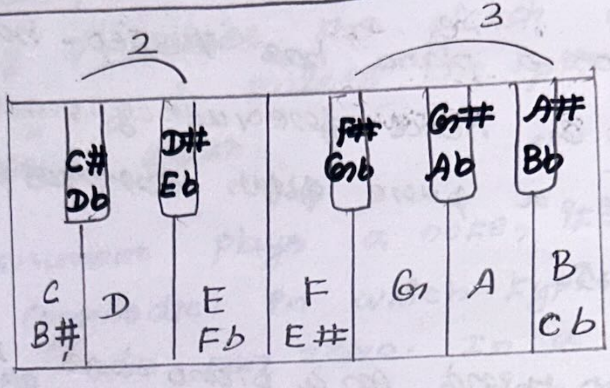
3? → G above middle C, we go up an octave to the next G, and then up a fifth from there → meaning

9? → corresponds to the D a little over two octaves above middle C.

Full piano has 88 keys → 7 octaves plus a minor third. Every pitch will sound different. The notes repeat in a series ranging from A–G.

The distance b/w a note & the next time that note repeats on the keyboard is called an octave.

$\boxed{\text{C–C is an octave}}$



'This pattern repeats seven times'

Black keys broken into a set of 2 and 3.

b → Flat notes

→ Sharp notes

'music moves in steps going up or down in pitch'

The space b/w notes that are next to each other on the keyboard is a half step.

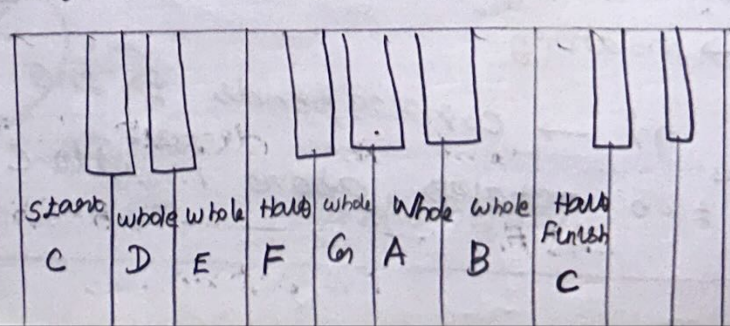
B/w E and F and B and C → No black key

'Half step'

The space b/w those two sets of notes is also a half step. The first notes are a half step & the second notes are a whole step

$\boxed{\text{C to D\#}}$

(specially C to D)



If you play the white keys from C to C will give you a C Major scale.

---

octave – Interval whose frequency (note has a) of sound-wave vibration twice than of its lower note.

---

52 – white keys, 36 black → 88 keys.

12 steps in an octave

Tempo:

"Fast or slow" → measured in beat. (Foot step)

rythm, clock (osciUate).

~ 40 – 250 BPM (normally used).

Beats to count : $\frac{4}{4}$ (Beats in a bar)

Length of note to count :

Duration

Frequency (pitch)

Node

pitch — Frequency.

A	B	C	D	E	F	G
440 HZ	493	1046	587	659	698	783 HZ.

880 = 440×2 (played with twice the speed)

Octave :

we call them the distance (gap) b/w

440 HZ ⟷ 880 HZ

as octave.

we differenciate them by

A4	A5
440HZ	880HZ.

<u>Notes</u>: (88 keys)

Ao	Bo	Co	Do	Eo	Fo	Go	2, 3, 4, 5, 6, 7 . . . . . .
1	2	3	4	5	6	7	

(Repeated – white keys).

<u>Black keys:</u>

named based on their position)

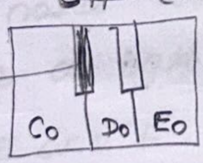
We use fancy term to name them $\boxed{\text{Sharp } \#}$ & $\boxed{\text{Flat } b}$

Sharp means – we go up to the next closest key

Flat means – we go down to the next closest key.

<u>Gap b/w</u> each key is a half step or semitone.

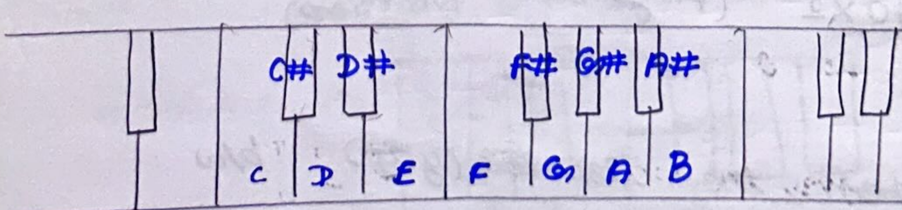
~~C#~~ (or) Dbo → D flat



C#
(or)
Dbo → Black key (shaded) directly to the right of C &
one semitone above C.

Also it lies one semitone below Do.

So also known as $\boxed{Db}$
D flat.



No. of notes in an octave : 12

C C# D# E F F# G G# A A# B

12 semitones in an octave.

Scales:

collection of nodes (which may sound fine)

F2 → Not in same scale as the other notes.
(Not in Emaj scale)

scales
↙ major (happy)
↘ minor (sad)

eg: . Cmaj.

Cmaj → only uses white keys. CDEF.GABC (eg:)

---

minor → starting at A & ending at A.

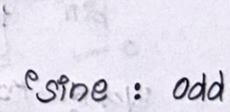
Chords:

multiple nodes played in harmony. at the same time

Triad:. (3 notes)

$f(t) = |\sin(t)|$

$f(t) = |\sin t|$



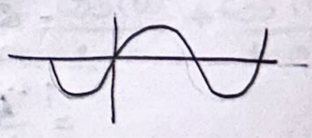
smallest period = pi

$\therefore$ when $F.S = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(w_n t) + b_n \sin(w_n t)$

$\boxed{w_n = ?}$

$\boxed{w_n = 2n}$

'sine : odd function'



Use symmetry:

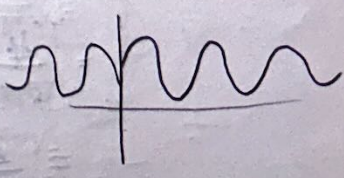
'only sine & $\frac{a_0}{2}$'

$|f(t) = |\sin(t)|$

$|\sin(t)|$

bn terms goes to 0.

$\boxed{b_n, \ n \geq 1}$ goes to zero.



(even function)

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

Assuming the period as 2π

$$\frac{a_n}{2} = \frac{2}{2\pi}\int_{-\pi}^{\pi} \sin t \cdot \sin(nt)\, dt$$

$$= \frac{4}{\pi}\int_{0}^{\pi} \sin t \sin nt\, dt$$

$$= \frac{4}{\pi} - \int_{0}^{\pi} \frac{\cos(t+nt) - \cos(t-nt)}{2}\, dt$$

$\sin A \sin B + \cos A \cos B = \cos(A-B)$

$-\sin A \sin B + \cos A \cos B = \cos(A+B)$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

$\frac{1}{2}(\cos(A+B) - \cos(A-B)) = -\sin A \sin B$

$$\sin t \sin nt = \frac{1}{2}(\cos(t-nt) - \cos(t+nt))$$

$$= \frac{4}{2\pi}\int_{0}^{\pi}\left[\cos(t-nt) - \cos(t+nt)\right]_0^{\pi} dt$$

$$= \frac{4}{2\pi}\left[\frac{\sin(t-nt)}{1-n} - \frac{\sin(t+nt)}{1+n}\right]_0^{\pi}$$

$$= \frac{4}{2\pi}\left[\frac{\sin(\pi - n\pi)}{1-n} - \frac{\sin(\pi + n\pi)}{1+n}\right]$$

$$= \frac{4}{2\pi}\left[\frac{\sin\pi\cos n\pi - \sin n\pi \cos\pi}{1-n} - \left(\frac{\sin\pi\cos n\pi + \sin n\pi\cos\pi}{1+n}\right)\right]$$

$\boxed{\cos\pi = -1}$

$$= \frac{4}{2\pi}\left[\frac{+\sin n\pi}{1-n} - \left(\frac{-\sin n\pi}{1+n}\right)\right]$$

$$= \frac{4}{2\pi}\left[\frac{\sin n\pi + n\sin n\pi + \sin n\pi - n\sin n\pi}{1-n^2}\right]$$

$$= \frac{4}{2\pi}\left[\frac{2\sin n\pi}{1-n^2}\right] = \frac{2}{\pi}\left[\frac{\sin(n\pi)}{1-n^2}\right]$$

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin t\,dt$$

$$= \frac{1}{\pi}\left(\frac{-\cos t}{1}\right)_0^{\pi}$$

$$= \frac{1}{\pi}((1) + \cos 0)$$

$$= 2/\pi$$

$$-(-1-1)/pi$$

$$= \frac{2}{pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

sol:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin t \cos(nt)\,dt$$

$$\boxed{\begin{aligned} &\sin A \cos B + \cos A \sin B = \sin(A+B) \\ &\sin A \cos B - \sin B \cos A = \sin(A-B) \end{aligned}}$$

$$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$$

$$\sin t \cos(nt) = \frac{1}{2}(\sin(t+nt) + \sin(t-nt))$$

$$\therefore = \frac{2}{\pi}\left(\frac{1}{2}\right)\int_0^{\pi} (\sin(t+nt) + \sin(t-nt))\,dt$$

$$= \frac{1}{\pi}\left[-\frac{\cos(t+nt)}{1+n} - \frac{\cos(t-nt)}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(0)}{1+n} + \frac{\cos(0)}{1-n}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(\cos\pi\cos n\pi - \sin\pi \sin n\pi)}{1+n} - \frac{(\cos\pi\cos n\pi + \sin\pi\sin n\pi)}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1-n+1+n}{1-n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\cos n\pi + 2}{1-n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi + 1}{1-n^2}\right]$$

when $n = 1, 3, 5, \ldots$ $\cos n\pi = -1$

when $n = 2, 4, \ldots$ $\cos n\pi = 1$

$$= \frac{2}{\pi}\left[\frac{2}{1-n^2}\right]$$

Pure resonance

$$\cos(5(t+2\pi)) = \cos(5t+2\pi) = \cos 5t$$

Solu:- function $f(t)$ is periodic of period p if

$$\boxed{f(t+p) = f(t) \text{ for all } t}$$

$\sin^2(t) \longrightarrow$ even

$$\sin^2(-t) = (-\sin t)^2 = \sin^2 t$$

It's not odd. $\therefore$ It fails when $t = \pi/2$

$\therefore \sin^2 t = -\sin^2 t$ (Always) $\rightarrow$ Not possible

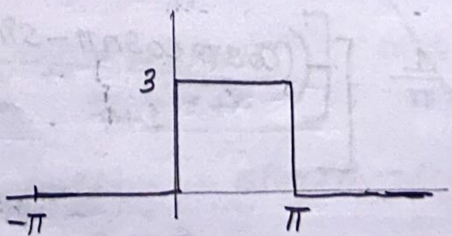
---

$f(t) \rightarrow$ function of period 2π

$$f(t) = \begin{cases} 0 & -\pi < t \le 0 \\ 3 & 0 < t \le \pi \end{cases}$$

what's the value of the constant co.eff in the F.S

Solu:-



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi} 3\,dt\right)$$

$$= \frac{1}{\pi}(3t)_0^{\pi} = \frac{3\pi}{\pi} = 3$$

$$\boxed{\frac{a_0}{2} = \frac{3}{2}.}$$

$f(t)$ be periodic (2π), $f(t) = |t|$ for $-\pi \le t < \pi$

solu:: co. eff of $\sin 7t = 0$

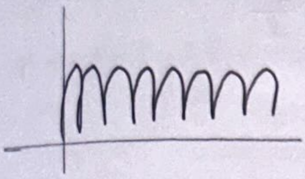
e It's an even function?.

$|\sin(4\pi t)|$

period $= \dfrac{2\pi}{\omega_0}$

$= \dfrac{2\pi}{4\pi} = \dfrac{1}{2}$

solu::



$|\sin(t+\pi)| = |-\sin t| = |\sin t|$ The period can't be smaller than π.

$|\sin 4\pi(t+P)| = |\sin 4\pi t|$

when $P = 1/4$ (or) 0, multiples of $\frac{1}{4}$.

$\boxed{\sin(t+\pi) = \sin t}$

$f(t)$ periodic function of period 6 such that $f(t) = |t|$ for $-3 \le t < 3$. Find a_0 such than $\frac{a_0}{2}$ is the constant term of the F.S of f

solu::

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$= \frac{1}{2(3)}\int_{-3}^{3} |t|\,dt$$

$$= \frac{1}{3}\int_{0}^{3} |t|\,dt = \frac{1}{3}\left(\frac{t^2}{2}\right)_0^3$$

$$= \frac{1}{3}\left(\frac{9}{2}\right) = \frac{3}{2}.$$

$\boxed{a_0 = 3}$

f be a period 2π

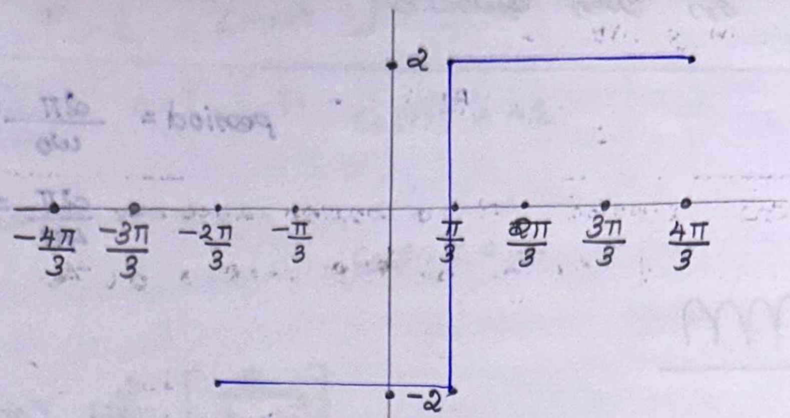
$$f(t) = \begin{cases} -2 & -2\pi/3 < t < \pi/3 \\ 2 & \pi/3 < t < 4\pi/3 \end{cases}$$

Find the co. eff of $\sin 5t$ in the F.S of $f(t)$

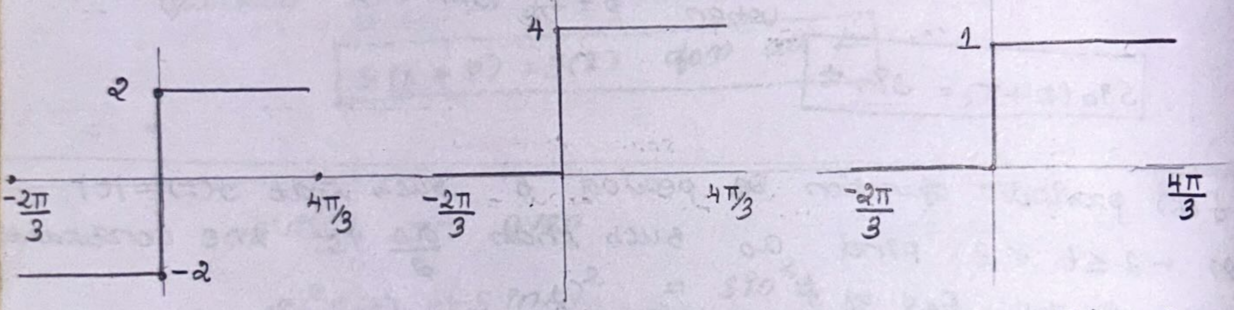
(The std. square wave $Sq(t)$, defined as the period 2π odd function such that $Sq(t) = 1$ for $0 < t < \pi$, has

F.S $\frac{4}{\pi} \sum_{n \geq 1 \text{ odd}} \frac{\sin nt}{n}$. Express $f(t)$ in terms of sq) ($\sin 5t$ in FS

Solu:

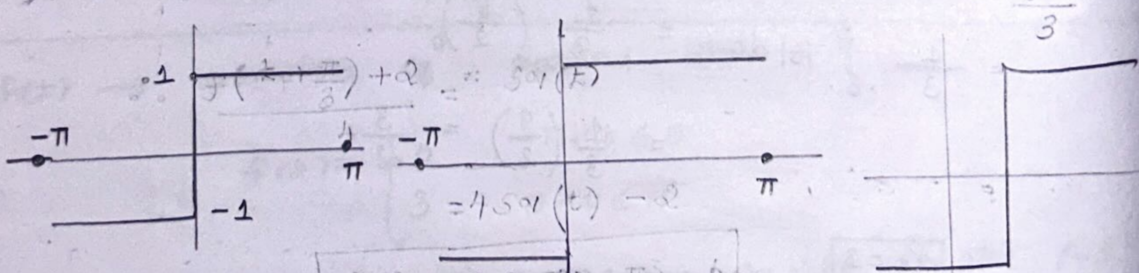


1) (Shifting the signal leftwards. ($\pi/3$)
2) Shifting the signal upwards by 2.
3) Reducing the signal (scaling) by $\left(\frac{4}{4}\right)$



$f(t+\pi/3)$ $\qquad$ $f(t+\frac{\pi}{3})+2$ $\qquad$ $\frac{f(t+\frac{\pi}{3})+2}{4}$

Std square wave



$2\,\mathrm{sq}(t)$ $\qquad$ $2\,\mathrm{sq}(t-\frac{\pi}{3})$

$$= \frac{8}{\pi} \sum_{\substack{n\geq 1\\ \text{odd}}} \frac{\sin n(t-\pi/3)}{n}$$

$$= \frac{8}{\pi} \sum_{\substack{n\geq 1\\ \text{odd}}} \frac{\sin nt - \frac{n\pi}{3}}{n}$$

For $a = 5$ ($\sin 5t$)

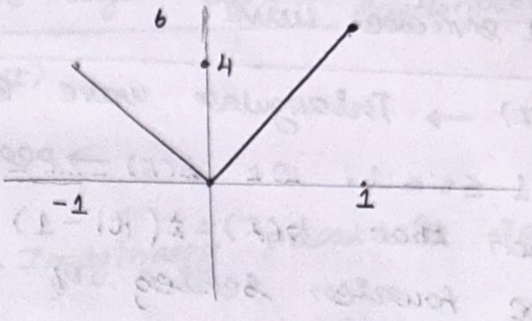
$$\frac{8}{\pi}\frac{\sin(5t - 5\pi/3)}{5} = \frac{8}{5\pi}\left(\sin 5t \cos\frac{5\pi}{3} - \cos 5t \sin\frac{5\pi}{3}\right)$$

So the co-eff of $\sin 5t$ is

$$\frac{8}{5\pi}\cos\left(\frac{5\pi}{3}\right) = \frac{4}{5\pi}$$

Let f be the period 2 function such that $f(t) = t+5$ for $t \in [-1, 1)$, and let g be its F.S, $g(1/2)$

solu:



$\therefore$ If f is continuous at $\frac{1}{2}$, we have $g(1/2) = f(1/2)$

$\therefore g$ is the F.S of f.

$\therefore g(1/2) = f(1/2) = \frac{1}{2} + 5 = \frac{11}{2}$.

Let f be the period 2 function such that $f(t) = t+5$ for $t \in [-1, 1)$. Let g be it's F.S. what's $f(1)\,g(1)$.

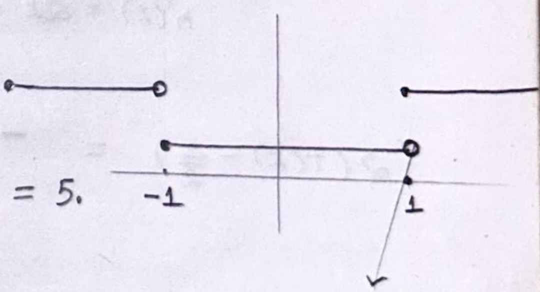
solu:

At $t = 1$, discontinuity occurs.

$$g(1) = \frac{f(1^-) + f(1^+)}{2}$$ (Convergence theorem)

$$= \frac{f(1^-) + f((-1)^+)}{2}$$

$$= \frac{(1+5) + (-1+5)}{2} = \frac{10}{2} = 5.$$



$f(1)\,g(1) = (4)(5) = 20.$

$\because$ f is of period 2, we have $f(1) = f(-1) = 5 + (-1)$
$= 4.$

By an antiderivative of a piecewise differentiable function f. we mean a continuous function F such that $F'(t)$ exists & equals $f(t)$ at any point t where f is continuous, which of the following functions $f(t)$ have a periodic antiderivative ?

The square wave $Sq(t)$. defined as the period 2π function such that $Sq(t) = 1$ for $0 < t < \pi$ and $Sq(t) = -1$, for $-\pi < t < 0$.

Solu: A piecewise differentiable periodic function f has a periodic antiderivative if and only if the constant term in its Fourier expansion is 0. That constant term is also the average value of f over one complete period. By symmetry average value is zero for a square wave.

$T(t) \rightarrow$ Triangular wave of period 2 gn by $T(t) = |t|$ for $-1 \le t \le 1$. Let $h(t) \rightarrow$ periodic function of period 2 such that $h(t) = t(|t| - 1)$ for $-1 \le t \le 1$. Given that the fourier series of $T(t)$ is.

$$T(t) = \frac{1}{2} - \sum_{\substack{n \ge 1 \\ odd}} \frac{4}{n^2\pi^2} \cos n\pi t.$$

Solu: co.eff of $\sin 3\pi t$, of F.S of $h(t)$

For $t \in (0, 1)$, we have

$$h(t) = t(t-1)$$

$$h'(t) = 2t - 1 = 2(T(t) - 1/2)$$

$$2(T(t) - \tfrac{1}{2}) = - \sum_{\substack{n \ge 1 \\ odd}} \frac{8}{n^2\pi^2} \cos(n\pi t)$$

Integrating,

$$h(t) = C - \sum_{\substack{n \ge 1 \\ odd}} \frac{8}{n^2\pi^2} \left(\frac{\sin n\pi t}{n\pi}\right)$$

$$= C - \sum_{n \ge 1} \frac{8}{n^3\pi^3} \sin n\pi t$$

For $\sin 3\pi t$, The co.eff is $\dfrac{-8}{3 \times 9\pi^3} = -\dfrac{8}{27\pi^3}$

for $\ddot{x}+\omega^2 x = Sq(t)$ has a periodic solution?

The answer is periodic for all +ve numbers that are not odd integers.

$Sq(t) \rightarrow$ creates pure resonance at $1, 3, 5, \ldots$

$P(r) = r^2+\omega^2$

$P(in) = -n^2+\omega^2$

$\omega^2 = n^2$ for some odd +ve integer n.

$\omega \rightarrow$ odd +ve integer else, a periodic solution occurs

---

$$\ddot{x}+2\dot{x}+\omega^2 x = Sq(t)$$

solu:

$P(r) = r^2+2r+\omega^2$

$P(in) = -n^2+2ni+\omega^2 \neq 0$ ($\because$ Imaginary part is $2n$)

pure resonance never occurs.

So periodic for all +ve numbers ω.

$Sq(t) \rightarrow$ odd periodic function of period 2π,

$Sq(t) = 1$, $0<t<\pi$, Let $x(t)$ be the steady state solution (i.e periodic solution to)

$$\ddot{x}+\dot{x}+8x = Sq(t)$$

solu:

Co.eff of $\cos 3t$

$P(r) = r^2+r+8$

$P(in) = -n^2+in+8$

$Sq(t) = \frac{4}{\pi}(\sin t + \ldots t)$

$\ddot{x}+\dot{x}+8x = \frac{4}{3\pi}\sin 3t \longrightarrow$ our sum.

$$x_p = \frac{4}{3\pi}\,\frac{e^{3it}}{(-9)+3i+8}$$

$$x_p = \frac{4}{3\pi}\,\frac{e^{3it}}{3i-1}$$

Taking complex conjugate

$$= -\frac{1}{10}\left(\frac{4}{3\pi}\right)\sin 3t - \frac{3}{10}\left(\frac{4}{3\pi}\right)\cos 3t$$

Co-eff of $\cos 3t$ is

$$-\frac{3}{10}\left(\frac{4}{3\pi}\right) = -\frac{2}{5\pi}$$

When a cellist plays an open string note on the A string (220 Hz). Which of the following are present among the overtones? (A cello note is rich enough that it has all physically possible overtones among those in the list below).

A $\to$ 440 Hz

$\therefore$ The frequencies present are those that are a +ve integer times the fundamental frequency of 220 Hz: which includes

440 Hz, 660 Hz, 880 Hz, 1100 Hz but not the others

Resonance

Pure resonance occurs when an undamped system is forced as ω_n.

$$3\ddot{x} + 27 = F(t), \quad F(t) = 1 \text{ on } 0<t<\pi, \; F = \text{odd}$$

$$\ddot{x} + 4x = F(t), \quad F(t) > 0 \text{ on } 0<t<\frac{\pi}{2}, \; F(t) = 0 \text{ on } \frac{\pi}{2} < t < \pi \text{ and } F \text{ is odd}$$

$$2\ddot{x} + 50x = F(t); \text{ where } F(t) = \pi t - t^2 \text{ on } 0<t<\pi, \text{ and } F \text{ is odd}$$

Soln:

a) $2\ddot{x} + 10x = F(t)$

No pure resonance (Input freq $n = 1, 2, 3, 4, 5, \ldots$

$\boxed{\text{But we need } \sqrt{5}}$

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$p(r) = 2r^2 + 10$ where $r = \pm i\sqrt{5}$

$p(in) = -2n^2 + 10$ (Imaginary)

b) $3\ddot{x} + 27 = F(t), \; F(t) = 1.$

$$sw(t) = \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \cdots\right)$$

$$3\ddot{x} + 27x = F(t) \qquad p(r) = 3r^2 + 27$$

$$\boxed{r = \pm 3i}$$

$\therefore$ Therefore there is a pure response since the co.eff b_3 in the F.S is non zero.

c) $\ddot{x} + 4x = F(t)$

$$p(r) = r^2 + 4 \qquad r = \pm 2i$$

Is $\boxed{b_2 \neq 0}$

$$b_2 = \frac{2}{\pi}\int_0^{\pi} F(t)\sin 2t\,dt = \frac{2}{\pi}\int_0^{\pi/2} F(t)\sin(2t)\,dt > 0$$

d) $2\ddot{x} + 50x = F(t)$

$$p(r) = 2r^2 + 50 \qquad r = \pm 5i$$

b_5 is nonzero?

$$b_5 = \frac{2}{\pi}\int_0^{\pi} F(t)\sin(5t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi t - t^2)\sin(5t) \neq 0$$

$\therefore$ pure resonance is available

---

e) $2\ddot{x} + (0.1)\dot{x} + 18x = F(t)$

$$p(r) = 2r^2 + 0.1r + 18$$

$$p(i\omega) = 18 - 2\omega^2 + 0.1i\omega \neq 0 \longrightarrow \text{Damped.}$$

(Img)

(No pure resonance)

Fourier Series

Consider the F.S of the function with period 2 in by t^2 if $0 < t < 2$. The F.S for the function

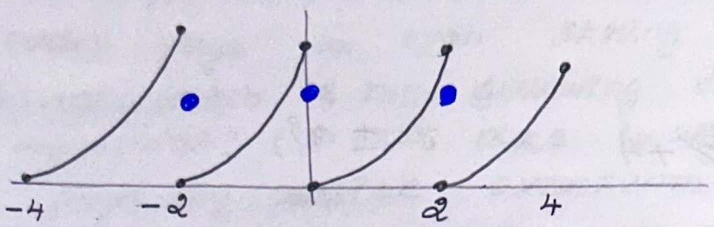
$$f(t) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\cos(n\pi t)}{n^2}\right. - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}$$

graph $f(t)$ on $-4 < t < 4$.

Soln:

The period 2 repeats the curve $f(t) = t^2$. The Fourier convergence theorem tells us that the value at $t = -2$, $t = 0$, $t = 2$.

(t^2 repeats every 2π interval)



• represents, it converges to the middle at discontinuities.

---

h with period 2π, $h(t) = 1 + t^2$ $(0 < t < 2\pi)$

Soln:

$$h(t) = 1 + \pi^2 f\left(\frac{t}{\pi}\right)$$ $\boxed{f(t) = t^2}$

$$h(t) = 1 + \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

Solving ODEs with periodic input

$$\ddot{x} + 0.1\,\omega_0 \dot{x} + \omega_0^2 x = \omega_0^2 f(\omega_0 t)$$

with input $f(\omega_0 t)$ where $f(t)$ is the 2π-periodic sawtooth wave $f(t) = t$ for $-\pi < t < \pi$ of period 2π. which has Fourier series

$$f(t) = 2\left(\sin t - \frac{\sin(2t)}{2} + \frac{\sin(3t)}{3} + \cdots\right)$$

The F.S representation of the steady state takes the form

$$x(t) = A_1 \cos(\omega_1 t - \phi_1) + A_2 \cos(\omega_2 t - \phi_2) + \cdots$$

Calculate the largest two amplitudes & then app. them by signs. check the 3rd largest amplitude is less than 1% of 1st largest

(Solution as a sum of two sine waves)

$x(t) \approx ?$

Solu:

$$f(w_0t) = 2\left(\sin w_0t - \frac{\sin(2w_0t)}{2} + \frac{\sin(3w_0t)}{3} \dots\right)$$

$$w_n = w_0 n$$

$$\therefore \quad (D^2 + 0.1 w_0 D + w_0^2)x = w_0^2 \sin(w_0 n t)$$

$$x_{p,n} = Im\left(\frac{w_0^2 e^{iw_0nt}}{(iw_0n)^2 + 0.1 w_0(iw_0n) + w_0^2}\right)$$

$$= Im\left(\frac{w_0^2 e^{iw_0nt}}{w_0^2(1 - n^2 + 0.1ni)}\right) = \frac{1}{\sqrt{(1-n^2)^2 + 0.01n^2}} \cos(w_0nt - \phi_n)$$

$\therefore$ steady state

$$x_p = 2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{1}{\sqrt{(1-n^2)^2 + 0.01n^2}} \cos(w_0nt - \phi_n)\right)$$

$\therefore$ Amplitude is proportional to $\frac{1}{n^3}$,

$\therefore$ So $n=1, n=2$ will be the largest terms.

$$A_1 = \frac{2}{\sqrt{0.01}} = 20$$

$$A_2 = \frac{1}{\sqrt{9 + 0.04}} = 0.3$$

$$A_3 = \frac{2}{3} \frac{1}{\sqrt{64 + 0.09}} = 0.08.$$

$$\therefore 20 \cos(w_0t - \phi_1) + 0.3 \cos(w_0 2t - \phi_2)$$

when: $\phi_1 = \pi$

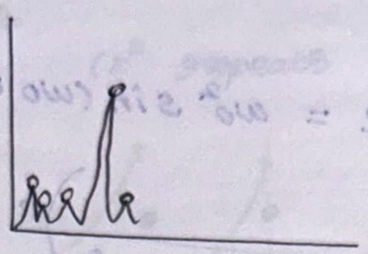
$$-20 \cos(w_0t) + 0.3 \cos(w_0 2t - \phi_2)$$

is also accepted.

(with no reference of phase shifts are OK)

Tiberian throat singing:

Technique of using the resonance of your mouth to sing two notes at the same time.



5 frequencies → 1710, 210, 330, 1590, 1380 } Hz

'Base note' → A, F#, G#

Closest notes corresponding to the five largest freq. ($t = 1049$)

G#	=	210
E	=	330
F	=	1382
G	=	1590
G#	=	1710

F = 1382 → closely correspond.

Decode the message

Fun exercise. : See the program.

Boundary value problems

1) Understand why boundary value problems don't satisfy the uniqueness & existence theorem.

2) Solve simple boundary value problems with specified boundary conditions.

Initial value problem

Find a solution to the following homogeneous, II order

IV problem:

$$\frac{d^2}{dt^2}(x) + 4x = 0, \qquad x(0)=0, \quad \frac{dx}{dt}(0)=1.$$

Soln: "simple harmonic oscillator". which starts at position $x=0$ (m). with a velocity 1(m/s) to the right.

$$\ddot{x} + 4x = 0$$

$$\ddot{x} = -4x \qquad\qquad \frac{\ddot{x}}{x} = -4$$

$$\dot{x} = -\frac{4x^2}{2} + c$$

$$x = -\frac{4x^3}{2\times 3} + cx + D$$

$x(0)=0$, $\dot{x}(0)=1$

$$1 = -\frac{4x^2(0)}{2} + c \qquad 0 = cx + D$$

$$\boxed{C=1} \qquad \boxed{D=0}$$

$$\therefore \quad x = -\frac{2x^3}{3} + x$$

$$\ddot{x} + 4x = 0 \qquad\qquad r^2 + 4 = 0$$

$$r^2 = -4$$

$$r = \pm 2i$$

$$x(t) = (A\cos 2t + B\sin 2t)$$

$$\therefore \quad x(0) = A\cos(0) + B\sin(0)$$

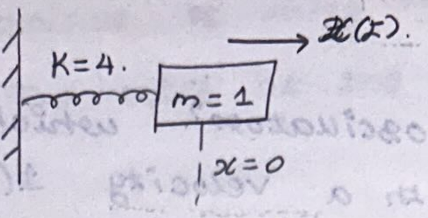
$$\boxed{A=0}$$

$$\dot{x}(0) = -A\sin(0) + B\cos(0)\cdot 2$$

$$\boxed{1/2 = B}$$

$$x(t) = \frac{1}{2}\sin(2t)$$

Existence / uniqueness theorem:



'pure sine'

Amplitude $= \frac{1}{2}$, $\boxed{\omega_n = 2}$

If we have n^{th} order ODE with n initial conditions then we will have unique solution exists.

If $x(t_1) = 1$ (At $x = 1$ or later time)

Find solu $0 \le t \le t_1$

'Time interval' $\rightarrow$ '$0 \le t \le t_1$'

Because we are specifying the conditions at the ends of our time interval, ~~this~~ is known as boundary value

General solution will be

$$x(t) = A\cos 2t + B\sin 2t.$$

Applying,

$$x(0) = 0, \quad x(t_1) = 1.$$

$A\cos 0 + B\sin 0 = 0$ | $A\cos(2t_1) + B\sin(2t_1) = 1$

$\boxed{A = 0}$ | $B = \dfrac{1}{\sin(2t_1)}$

$$\boxed{x(t) = \frac{\sin 2t}{\sin 2(t_1)}}$$

As $\sin(2(t_1)) = 0$

the $x(t) \rightarrow$ is invalid

$t_1 = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

Doesn't have a solution.

constrast to the initial value problems, where we got exact & unique solutions.

In case of boundary values, we have different possibilities

1) we have a unique Solution.

$\boxed{\text{eg: } x(t_1) = \text{ is provided such that } \sin(2t_1) \neq 0}$

2) we have no solution

when t_1 takes value such that $\boxed{\sin(2t_1) = 0}$

3) Multiple solutions (violating uniqueness part of the uniqueness theorem).

Generally, the boundary value problems arise when the independent variable is position, rather than time. Because it's more natural to want to specify conditions at different points in space, rather than different points in time.

---

1) $\ddot{x} + 4x = 0$

$x(0) = 0, \quad x(t_1) = 0$

solu:

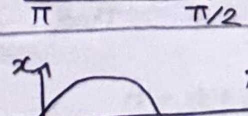
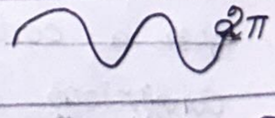
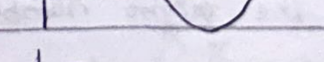
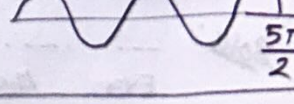
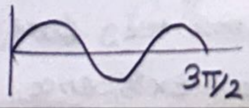
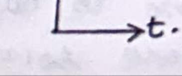
$x(t) = A\sin 2t + B\cos 2t$ | $\boxed{0 = B}$

$\boxed{0 = B}$

$\therefore \quad x(t) = A\sin 2t.$

satisfies the boundary conditions & differential eqn for all values of A. Therefore there are infinitely many solutions.

Graph of A $\sin(2t)$ for $A = 1$.

n	t_1		n	t_1	
1	$\pi/2$		4	2π	
2	π		5	$\frac{5\pi}{2}$	
3	$3\pi/2$				

Homogeneous & Inhomogeneous boundary conditions

4.1

$$\frac{d^2}{dx^2} v(x) = \lambda v(x), \quad v(0) = 3, \frac{dv}{dx}(0) = 5$$

Solu. $\lambda \to$ parameter taking any real value

There is only 1 solution by the uniqueness & existence theorem.

Uniqueness & existence theorem:

$P_{n-1}(t), \dots, P_0(t), q(t)$ be continuous functions on an open interval I. Let a be in the interval I, and let $b_0, \dots, b_{n-1}$ be given numbers. Then there exists a unique solution to the n^{th} order linear ODE.

$$y^{(n)} + P_{n-1}(t) y^{(n-1)} + \dots + P_1(t)\dot{y} + P_0(t) y = q(t)$$

satisfying the n initial conditions

$$y(a) = b_0, \ \dot{y}(a) = b_1, \dots, y^{(n-1)}(a) = b_{n-1}.$$

Existence $\longrightarrow$ Atleast one solution

uniqueness $\to$ only one solution.

This is an IVP, since the conditions are the value & I derivative at the same point. In contrast BVP, has conditions at different points.

Consider the following family of examples, one for each λ.

4.2: $v(x)$ on $[0, \pi]$ satisfying $\frac{d^2}{dx^2} v(x) = \lambda v(x)$ for a constant λ and satisfying the boundary conditions $v(0) = 0$ and $v(\pi) = 0$.

Solu: Even though there are the right number of conditions, there is no longer any guarantee that they specify a unique solution - there is no existence & uniqueness theorem for BVP. for most values of λ, it will turn out that this problem has a unique solution (namely, 0), but for special values of λ, we will see that it has ∞ly many solutions.

* $\frac{d^2}{dx^2} v(x) = \lambda v(x)$, $v(0) = 0$, $v(\pi) = 0$ is known as homogeneous BVP, and can have one or ∞ly many solutions.

* An inhomogeneous BVP (linear) could have one or zero or ∞ly many solutions.

* The situation for non-linear boundary value problems are more complicated.

our goal: Use BVP for PDES, but ODES can have BVP as well. In general, ODES are much easier to solve than PDES,

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Failure of existence & uniqueness theorem

Find all non-zero functions $v(x)$ on $[0,\pi]$, satisfying $\frac{d^2}{dx^2} v(x) = \lambda v(x)$ for a constant λ and satisfying the boundary conditions

$v(0) = 0$, $v(\pi) = 0$)

Solu:

$v''(x) = \lambda v(x)$ is a homogeneous linear ODE with char polynomial $r^2 - \lambda$.

case: I: $\lambda > 0$

$r^2 = \lambda$

$r = \pm\sqrt{\lambda}$

General solution $= a e^{\sqrt{\lambda}x} + b e^{-\sqrt{\lambda}x}$

From BVP

$0 = a + b \rightarrow ①$

$a e^{\sqrt{\lambda}\pi} + b e^{-\sqrt{\lambda}\pi} = 0$.

$$\det\begin{pmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\pi} & e^{-\sqrt{\lambda}\pi} \end{pmatrix} = \begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\pi} & e^{-\sqrt{\lambda}\pi} \end{vmatrix} = e^{-\sqrt{\lambda}\pi} - e^{\sqrt{\lambda}\pi}$$

$$\neq 0$$

The only solution to this linear system is $(a, b) = (0, 0)$, Thus there are no non-zero solutions v.

Case: 2:

$\lambda = 0$, $r^2 = 0 \Rightarrow r = 0, 0$.

General solution is $(a + bx)\, e^0 = a + bx$, and the boundary conditions say

$$\boxed{a = 0}$$

$$a + b\pi = 0$$

$$b\pi = 0$$

$$\boxed{b = 0}$$

Again only solution to this linear system is $(a, b) = (0, 0)$. Thus there are no non-zero solutions v.

Case: 3

$\lambda < 0$

$\lambda = -\omega^2$ for some $\omega > 0$. Then roots will be in the form $\pm i\omega$, and the general solution is

$$a \cos \omega x + b \sin \omega x.$$

$a = 0 \rightarrow$ from I boundary condt ($v(0) = 0$)

$$\therefore\ v = b \sin \omega x.$$

$v(\pi) = 0$

$$v(\pi) = b \sin \omega \pi$$

$$\boxed{b \sin \omega \pi = 0}$$

we are looking for non-zero solutions of v, so we can assume $\boxed{b \neq 0}$

$$\sin \omega \pi = 0, \text{ so}$$

ω is an integer n. $(n > 0)$.