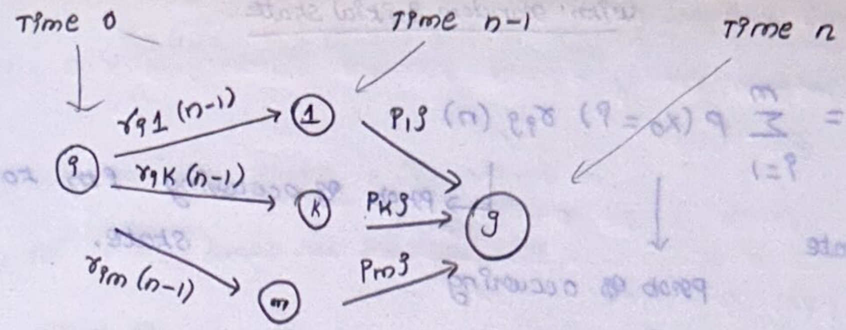


$$\boxed{\begin{array}{ll} P_{ij}(0) = 1 & \text{if } i = j \\ P_{ij}(0) = 0 & \text{if } i \neq j \end{array}}$$

$P_{ij}(1) = P_{ij}$

$(0) \rightarrow$ State.

'Different paths → for coming to state 1 in Time $n-1$'

* we are going to condition on the state at time $(n-1)$
* Finally j is the state.

Coming to j → from 1 → from k → from m

Markov assumption: only depends on present

∴ Taking $(n-1)$ state alone

Key recursion:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1)\, P_{kj}$$

$r_{ij}(n) = P(X_n = j \mid X_0 = i)$

go to state j
↓
Prob of $(n-1)$ state × prob of $(n-1)$ state going to j

$\xrightarrow{r_{i1}(n-1)}$ 1 $\xrightarrow{P_{1j}}$ → prob of occuring j th state from ① state

$r_{i1}(n-1)$ → Prob of occuring ① in $n-1$

'Total prob' — j can occurs from ①, ⓚ or ⓜ.

with random initial state

$$P(X_n = j) = \sum_{i=1}^{m} P(X_0 = i)\, r_{ij}(n)$$

$P(X_n = j)$ ↓ Final state

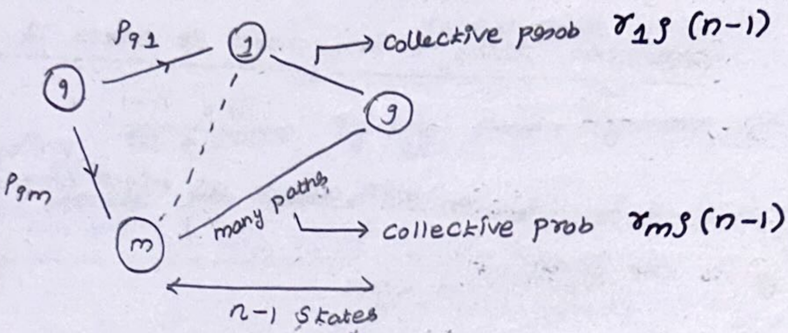
$P(X_0 = i)$ ↓ prob of occurring ith state

$r_{ij}(n)$ ↳ prob of occurring ith to jth state.

"we don't know about the initial state" → Take as random.

↓ (may be any) [some of all possibilities]

"Total prob theory" _ conditioned

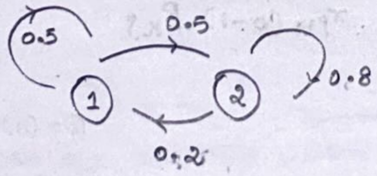


"collective prob?"

$$r_{ij}(n) = \sum_{k} p_{ik}\, r_{kj}(n-1)$$

Previous: conditioned one step before final state

Now: Conditioned right after the first step.



	$n=0$	$n=1$	$n=2$	$n=100$	$n=101$
$r_{11}(n)$	1	0.5	0.35	$\approx 2/7$	$\approx 2/7$
$r_{12}(n)$	0	0.5	0.65	$\approx 5/7$	$\approx 5/7$
$r_{21}(n)$	0	0.2	0.26	$\approx 2/7$	$\approx 2/7$
$r_{22}(n)$	1	0.8	0.74	$\approx 5/7$	$\approx 5/7$

Same
↓
Describes
(After long time what the state doesn't care about initial state)
↓
Due to randomness

$$r_{11}(n) = r_{21}(n-1)\cdot(0.2) + r_{11}(n-1)\cdot 0.5$$

$r_{21}(n-1)$ → from 2 to 1

$r_{11}(n-1)$ ↓ from 1 to 1 in $(n-1)$ state

$r_{12}(n) = r_{12}(n-1) \cdot 0.5 + r_{22}(n-1) \cdot 0.8$

① $r_{11}(0) \to$ At zero state $\to$ Certain at 1

② $r_{12}(0) \to$ zero state (no moving) $\to 0$

③ $r_{21}(0) \to 0$

④ $r_{22}(0) \to 1$

$\underline{n=1}$

$r_{11} \to 0.5$

$r_{12} \to 0.5$

$r_{21} \to 0.2$

$r_{22} \to 0.8$

$\underline{n=2}$ $\to P_{ik}$ $\to r_{kj}(n-1)$

$r_{11}(2) \to r_{11} \cdot r_{11} + r_{12} \cdot r_{21}$

$= (0.5 \times 0.5) + (0.5 \times 0.2)$

$= 0.35.$

$P_{11} r_{12} + P_{12} r_{22}$

$r_{12}(n) \to (r_{12}\ r_{22}) + (r_{11} \cdot r_{12}).$

$\to (0.5 \times 0.8) + (0.5 \times 0.5)$

$= 0.65$

$r_{21} \to (r_{22} \cdot r_{21}) + (r_{21} \cdot r_{11})$

$= (0.8 \times 0.2) + (0.2 \times 0.5)$

$= 0.16 + 0.10$

$= 0.26$

$r_{22} \to (r_{22}\ r_{22}) + (r_{21}\ r_{12})$

$\to (0.8)^2 + (0.2 \times 0.5)$

$= 0.74.$

$r_{11}(101) = P_{11} \cdot r_{11}(100) + P_{12}\, r_{21}(100)$

$= (0.5) \cdot \left(\frac{2}{17}\right) + \frac{5}{7}\left(\frac{1}{5}\right)$ $\quad (r_{11} + r_{12} = 1)$

$\downarrow$ Remaining prob

$= 2/7$ (Saturates)

'Markov chains enters in to a steady state'

X_n: doesn't become steady, what becomes steady is the probability that describes X_n.

(still 1 to 0, 0 to 1) → keep happening.

why? $r_{12}(n) > r_{11}(n)$

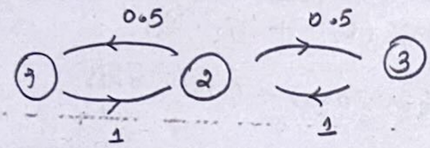
↳ 'state 2' is sticky → harder to get out.

Nice things

* Limit exist → (settles)
* Initial state doesn't matter $\left(r_{11} = \frac{2}{7},\ r_{21} = \frac{2}{7}\right)$

Is this the case always?

* Does $r_{ij}(n)$ converge to something?



'Is the initial state got forgotten in a long run?'

* Nice chains - Both true
* peculiar or unique structure - This may not true.

n: odd : $r_{22}(n)^{\infty} \to 0$

n: even: $r_{22}(n)^{\infty} \to 1$

without convergence: up & down → periodic behaviour

Start state 2 → 1, → 3   next state back to 2

'Go out - then come' → periodic

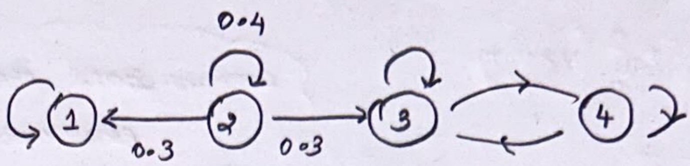
'After even no. of steps: certain to be back at 2'

odd: No way to be back at initial state → 1, → 3 (Doesn't converge)

mechanism: periodic structure: convergence fails

If we have convergence whether the initial state matters?

Situation



$r_{11}(n) = 1$ (Forever) $\longrightarrow$ Initial state matters (Long term

$r_{31}(n) = 0$ (No way to move to 1.

$r_{21}(n) = 1/2$ (As 0.5 $n \to \infty$)

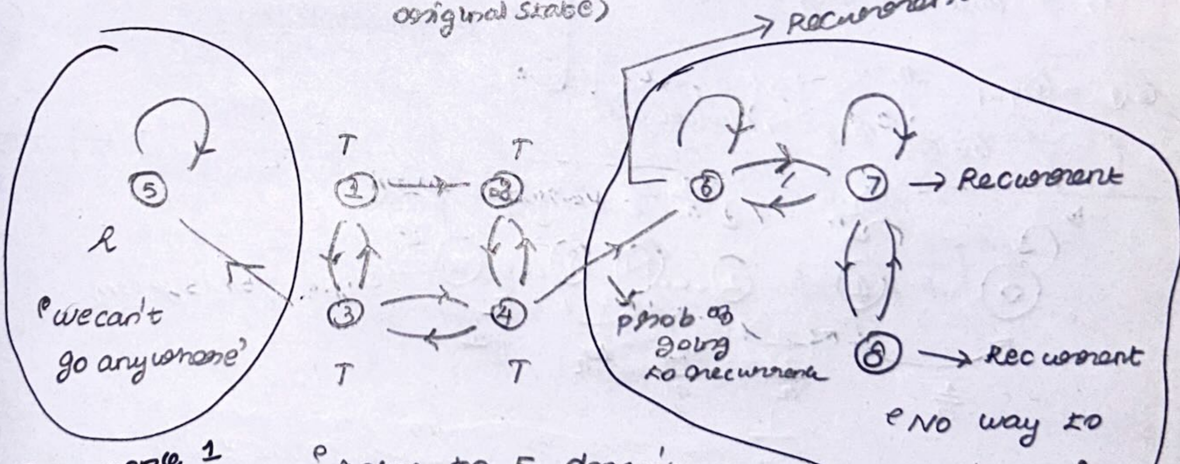
By symmetry

0.3 (1)   0.3 (3)

'Affected by where you start from'

Recurrent & transient states

- There is a way to return it (returns to original state)
- Not recurrent — transient.



'going to 5 doesn't get back'

Recurrence 2

'going to 6 doesn't get back' → Here - get to recurrence & stuck even.

Both Recur 1 & 2 has no connection

'Initial position matters' → Else we dont know where we are.

Setting up a Markov chain

n fish → Green
n fish → Blue

1) catch 1 fish/day (equally likely)
2) If green, paint blue
3) Return fish.

Current state affected by previous state

G_i = { i green fish left }

After each day

$G_i \to G_{i-1}$ (if caught green) → count of green ↓ by 1

$G_i \to G_i$ (Blue caught)

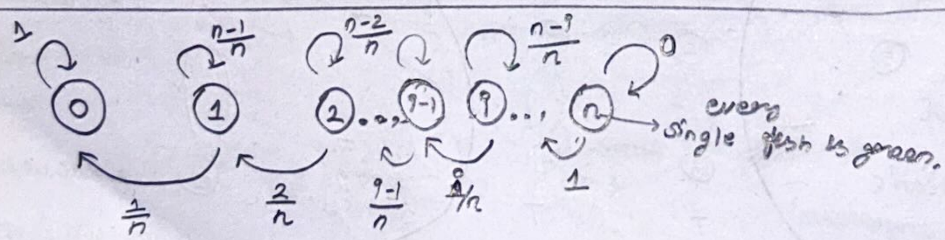
① ① ② ③ . . . . . . ⓝ → Every single fish is a green fish.

P_{ij} = P(j green fish tomorrow | i green fish today)

$$\begin{cases} \frac{n-i}{n} & : \text{Blue fish} \to j=i \\ \frac{i}{n} & : \text{Green fish} \to j=i-1 \\ 0 & : \text{otherwise} \end{cases}$$

If G_i today G_j tomorrow
↓
Blue fish caught & returned

$G_j = G_{i-1}$ then
Green caught → turned blue.



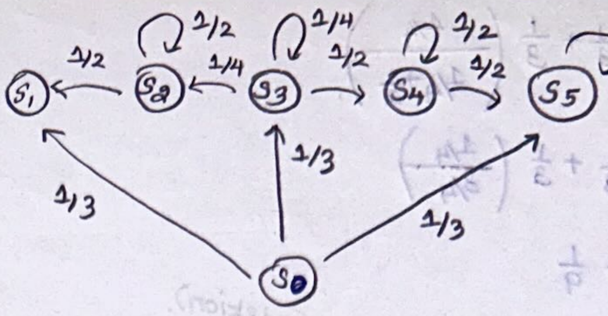
∴ all green → No rotation (Blue fish)

0 green fish: Blue fish then return forever.

n → once it left (no way to come back)

* ∴ ① to ⓝ all are transients

* ⓪ → Along recurrent. (Always sta
↓
Absorbing (Every state is absorbed by ⓪)



State S_0 before Trial →

a) Process enters S_2 for the first time as the result of kth trial,

b) Never enters S_4

c) Enter S_2 then leaves S_2 on the next trial

d) Process enters S_1 for the first time on the third trial

e) S_3 immediately after nth trial

Sol:

a) S_1, S_5 → Recurrence

Enters left → Can't able to enter to the right.

a) A_k : Process enters S_2 for the 1st time at k^{th} trial.

$k=1$, $P(A_1)=0$

$k=2$, $P(A_2) = \frac{1}{3}\times\frac{1}{4}$ (Straight)

(or)

$$k=2,3,\ldots \Rightarrow P(A_k) = \underbrace{P_{03}}_{\text{Enter } 3}\cdot\Big(\underbrace{P_{33}^{k-2}}_{\text{Revolve } 3}\Big)\underbrace{\left(\frac{1}{4}\right)}_{\substack{\text{Enter } 2\\ \text{(at } k^{th} \text{ trial)}}} = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-2}\left(\frac{1}{4}\right)$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^{k-1}$$

b) Process never enters S_4. [Never] ↱ Info gn: Conditioned.

$$P(B) = P_{01} + P_{05} + P_{03}\cdot P\left(\begin{array}{c}\text{transition}\\ \text{to } S_2\end{array}\middle|\, \begin{array}{c}\text{in } S_3 \text{ \& State}\\ \text{Change}\end{array}\right)$$

$$= \frac{1}{3}+\frac{1}{3}+\frac{1}{3}\left((S_3\to S_2 \mid S_3\to S_2 \text{ or } S_3\to S_4)\right)$$

↓ Avoiding revolving around S_3

$$= \frac{1}{3}+\frac{1}{4}+\frac{1}{3}P\left(\frac{S_3\to S_2}{(S_3\to S_2)+(S_3\to S_4)}\right)$$

$$= \frac{1}{3}+\frac{1}{4}+\frac{1}{3}\left(\frac{1}{3}\right)$$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\left(\frac{1/4}{1/4+\frac{1}{2}}\right)$

$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\left(\frac{1/4}{3/4}\right)$

$= \frac{2}{3} + \frac{1}{9}$

$= 7/9$

c) process enters S_2 and leaves S_2 on next trial

(Enters S_2 (condition).) (conditioned.)

$P(C) = P\{\text{enters } S_2\} \, P(\text{leaves } S_2 \text{ on next trial} \mid \text{in } S_2\}$

$= \left(P_{03}\left(P_{32} \mid S_3 \to S_4 \text{ or } S_3 \to S_2\right)\right) \cdot S_{45}$ → $P(S_4 \to S_5 \mid \text{state transition } (S_4 \to S_5))$

$= \frac{1}{3}\left(\frac{1/4}{1/4+1/2}\right) \cdot \left(\frac{1}{2}\right)$

$= \frac{1}{3}\left(\frac{1/4}{3/4}\right)\left(\frac{1}{2}\right)$ → don't have count

$= \frac{1}{6}\left(\frac{1}{3}\right)$

$= 1/18 = 1/18$

→ $P(\text{enters } S_2 \cap (\text{leaves } S_2 \text{ on next trial} \mid \text{in } S_2))$

$= \left[\sum_{K=2}^{\infty} P(A_K)\right] \cdot \frac{1}{2}$

$= \sum_{K=2}^{\infty} \frac{1}{3}\left(\frac{1}{4}\right)^{K-1} \cdot \frac{1}{2}$

$= \frac{1}{6}\left(\frac{\frac{1}{4}}{1-\frac{1}{4}}\right)$

$= \frac{1}{18}$

d) D: process enters S_1 for 1st time on 3rd trial.

$= \cancel{P_{03} \cdot (P(S_3 \to S_2 \mid S_3 \to S_2 \text{ or } S_3 \to S_4)) \cdot P_{21}}$

$= \cancel{P_{03} \cdot}$ 3 trials alone: must pass $P_{03} \cdot P_{32} \cdot P_{21}$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$

$= \frac{1}{24}$

Previous: we don't have limited counts.

Here only 3.

"conditioned"

e) E: process in S_3 immediately after N^{th} trial

Sdu. $S_0 \to S_3 \to (S_3)$ for $n-1$ trials

$= \frac{1}{3} \times \left(\frac{1}{4}\right)^{n-1} = \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}$