(the gain). So instead of graphing $H(s)$, we will graph $|H(s)|$. This is now a surface lying over the complex plane.

Let (undamped spring system)

$$m\ddot{x} + Kx = Ky$$

(Harmonic oscillator with natural frequency)

$$\omega_n = \sqrt{\frac{K}{m}}.$$

In std form:

$$P(D)x = Q(D)y$$

where,

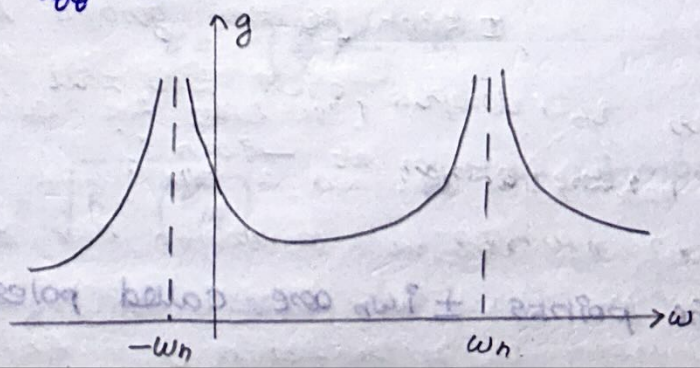
$$P(s) = ms^2 + K, \quad Q(s) = K$$

So, Transfer function is

$$H(s) = \frac{K}{ms^2+K} = \frac{K/m}{s^2 + K/m} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

$$g(\omega) = |G(i\omega)| = |H(i\omega)| = \frac{\omega_n^2}{\sqrt{(i\omega)^2+\omega_n^2}} = \frac{\omega_n^2}{\sqrt{-\omega^2+\omega_n^2}}$$

when $\omega = \pm\omega_n$, the denominator vanishes, $G(\omega)$ isn't defined. This is resonance! When ω is near to $\pm\omega_n$ but not quite equal to it, the value of the gain is large; the sinusoidal system response has amplitude much larger than the amplitude of the input signal. As $|\omega|$ gets large & large, the gain falls to zero.



This is the amplitude response curve of the system.

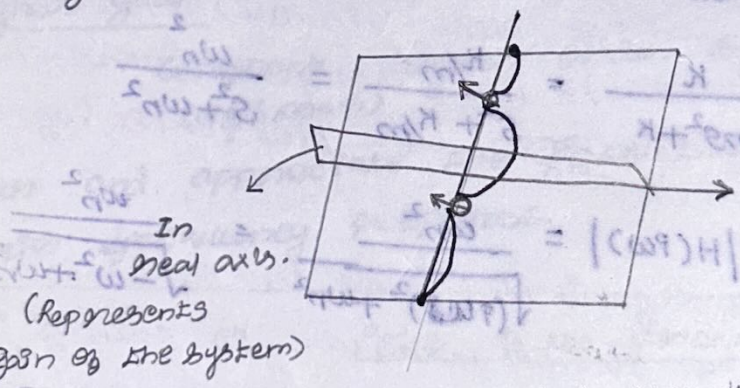
Note: The frequency enters here only through $i\omega$ square, we lose nothing, and gain quite a bit, by formally allowing ω to take on negative values as well as +ve ones.

Now, absolute value of the full transfer function,

$$|H(s)| = \frac{\omega_n^2}{|s^2+\omega_n^2|}$$

The polynomial $s^2+\omega_n^2$ has two roots, $s = \pm i\omega_n$. At these values of s, the transfer function is not defined, when s is near to $\pm i\omega_n$. The value of $|H(s)|$ is large; when $|s|$ gets big, $|H(s)|$ falls to small values.

The graph looks like this.



The black curve lies over the imaginary axis. The black curve on the surface is precisely the amplitude response curve.

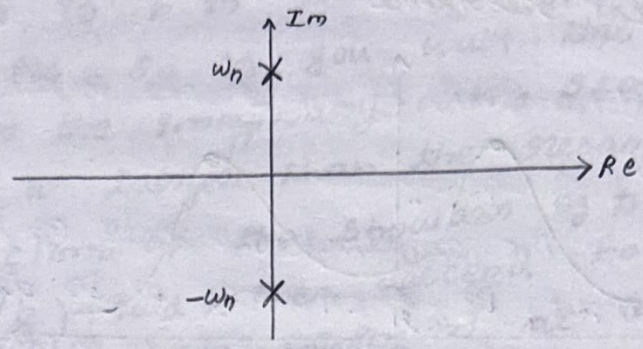
If you slice the graph of $H(s)$ with a plane, above the imaginary axis, you get the graph of the gain which is the amplitude response curve - the graph of $g(\omega) = |H(i\omega)|$.

I think of this graph like a tent. The canvas is draped over two tall poles, one at $i\omega_n$, the other at $-i\omega_n$.

Poles:

The points $\pm i\omega_n$ are called poles, perhaps because

they lie at the base of the tent poles. The most important features of the graph of $|H(s)|$ and so of the gain curve - are captured by the location of the poles. A diagram of the complex plane with the location of the poles marked by X's is called a pole diagram of $H(s)$.



In our cases the poles of $H(s)$ are exactly the roots of the denominator, $p(s)$ (assuming $p(s)$ and $q(s)$ have no common factors).

Introducing some damping, $m=1$: (To save on symbols).

$$\ddot{x} + b\dot{x} + Kx = ky$$

$$P(s) = s^2 + bs + K, \quad Q(s) = K$$

$$H(s) = \frac{Q(s)}{P(s)} = \frac{K}{s^2+bs+K}$$

Roots: $s^2+bs+K = (s+b/2)^2 + (K - b^2/4)$

s is a root if $(s+b/2)^2 = -(K-b^2/4)$ with b small (smaller than $2\sqrt{K}$), the system is underdamped, the right hand side is negative.

Roots are

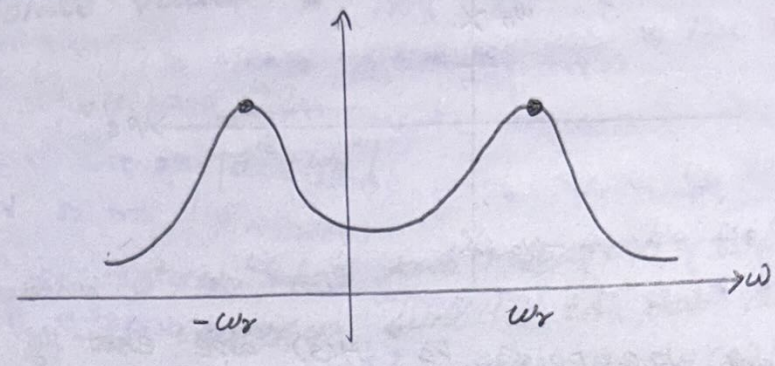
$$r = -\left(\frac{b}{2}\right) \pm i\omega_d$$

where,

$$\omega_d = \sqrt{K - \left(b^2/4\right)}$$ is the damped angular frequency. The roots have also acquired a

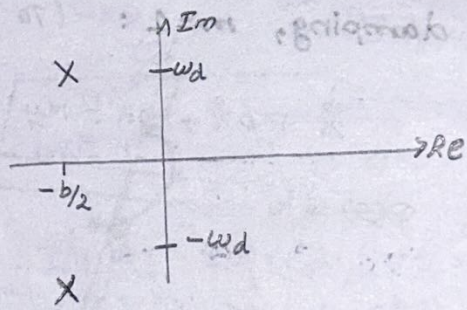
real part that is negative. The damping has pushed them off the imaginary axis!

Because of damping – there is no pure resonance. Engineers call the frequency that produces the largest gain (at least if b is small relative to K) 'practical resonance'. The amplitude response curve is

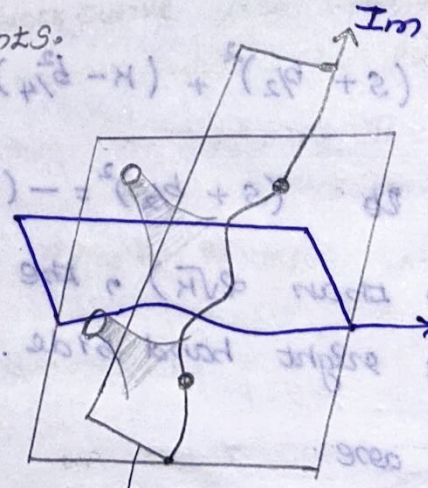


T.F:

$$H(s) = \frac{K}{s^2+bs+K} \qquad \left(s = -\frac{b}{2} \pm j\omega_d\right)$$



The graph $|H(s)|$ exhibits two only high peaks over these two points.



poles no longer alongen Imaginary axis.

→ Reason word.

Curve lies over the imaginary axis. (Precisely the amplitude response curve)

we have also outlined a box in the plane above the imaginary axis for emphasis. The blue curve – lies over the real axis. (It represents the gain of the system if it is fed a real exponential input).

slicing the graph of $H(s)$ with a plane above the imaginary axis, you get the graph of the gain, which is the amplitude response curve - the graph of $g(\omega) = |H(j\omega)|$.

These points are no longer on the imaginary axis, but if b is small relative to K, they are not far off. So if you walk this terrain on a path over the imaginary axis, starting from $j\omega$ for ω larger than the resonant frequency, you will climb up the shoulder of the mountain near $-\left(\frac{b}{2}\right) + j\omega_d$; then descend to a valley that reaches a relative minimum at $\omega = 0$ and then goes through a symmetrical elevation gain & loss for negative ω.

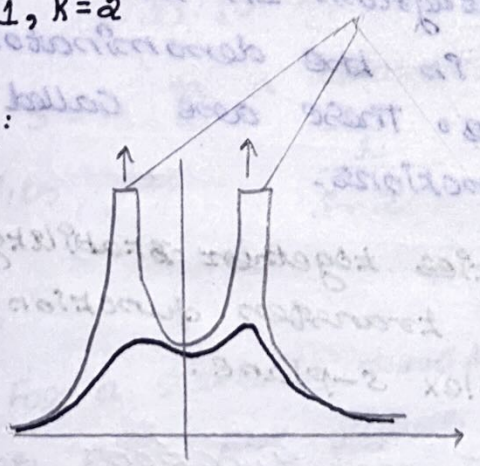
The hiking imagery is $\left|\frac{(j\omega)^2}{\ }\right|$ very useful. It explains how poles influence the gain curve.

Mathlet - Amplitude response & pole diagram

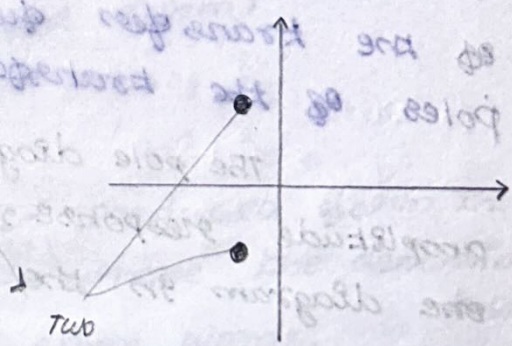
summary: pole (roots - where denominator becomes zero).
↳ $H(\omega)$ doesn't exist.

$b = 1$, $K = 2$

Side:



Top



Two inifty point

Poles, Amplitude response, Connection to ERF

LTI system,

$$P(D)x = Q(D)y$$

$y(t) \rightarrow$ Input, $x(t)$ - system response.

Stability: It is determined by the roots of the characteristic polynomial. $P(r)$

Amplitude response: The amplitude response of the system to a sinusoidal input of frequency ω is determined by the characteristic polynomial P and the polynomial Q via the equation

$$g(\omega) = \left| \frac{Q(i\omega)}{P(i\omega)} \right|$$

The transfer function is

$$H(s) = \frac{Q(s)}{P(s)}$$

we see that the roots of P give us important info about the system. The roots of $P(s)$ are the poles of $H(s)$. So all of this information is present in the pole diagram. In summary, the roots of P occur in the denominator of the transfer functions. These are called poles of the transfer functions.

The pole diagram lies together stability, amplitude response, and transfer function in one diagram in the complex s-plane.

The pole diagram of the transfer function:

1) shows whether the system is stable.
2) whether the unforced system is oscillatory
3) Rough picture of the amplitude response & practical resonance of the system.

∴ Pole diagram is a standard tool used by Engineers in understanding & designing systems. Since every LTI system has a transfer function, everything we learn here will apply to all such systems.

Defining Poles

Rational functions:

Def 4.1: A rational function is a ratio of polynomials $Q(s)/P(s)$ where P is not the zero function.

Def 4.2: The following are all rational functions

$$\frac{s^2+1}{s^2+3s+1}, \quad \frac{1}{ms^2+bs+K}, \quad \frac{s^5+s^3+s+1}{s^3}$$

If the numerator & denominator have no roots in common, then the rational function

$$\frac{Q(s)}{P(s)}$$ is in reduced form.

4.3: The three functions in the example above are all in reduced form.

4.4: $(s-2)/(s^2-4)$ is not in reduced form, ∴ $s=2$ is a common root. We can rewrite this as $\frac{1}{s+2}$ (Reduced form)

Poles:

For a rational function in reduced form the poles are the values of s where the denominator is equal to zero; or, in other words, the points where the rational function is not defined. We allow the poles to be complex numbers here.

Remark 4.5:

For the system $p(D)x = q(D)y$ with T.F. $H(s) = \frac{q(s)}{p(s)}$, the poles of the T.F are among the roots of $p(s)$. That is, they are among the characteristic roots used to solve the homogeneous eqn $p(D)x = 0$.

---

find the poles

1) $\frac{1}{s^2+8s+7}$, $s = -7, -1$

$7 \quad 1$
8

2) $\frac{s-2}{s^2-4} = \frac{1}{s+2}$ $(s=-2)$

$s^2+4=0$

$s^2=-4$

3) $\frac{1}{s^2+4} = \frac{1}{(s^2+4)}$ $s = \pm 2i$

$s = \sqrt{-4}$

$= 2i$

4) $s^2+1 \Rightarrow$ None

5) $\frac{1}{(s^2+8s+7)(s^2+4)}$ $s = -7, -1, 2i, -2i$

6) $\frac{3s}{(s^2+6s+10)(s-2)}$ $s = 2, -3+i, -3-i$

10
6

$\frac{-6 \pm \sqrt{36-40}}{2}$

$-3 \pm i$

---

poles of $W(s) = \frac{1}{s-2} + \frac{1}{s+3}$

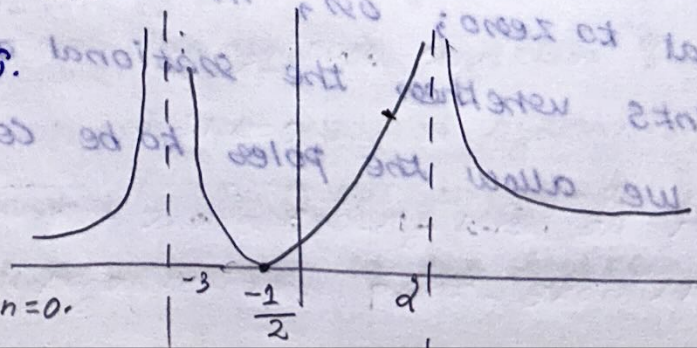
$= \frac{s+3+s-2}{(s-2)(s+3)} = \frac{2s+1}{(s-2)(s+3)}$

$(s-2)(s+3) = s^2+3s-2s-6$
$= s^2+s-6.$

$s = 2, -3$

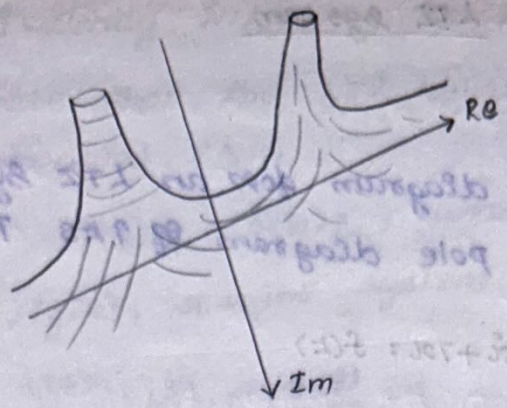
Sketch for real s

$W(s) = \frac{1}{s-2} + \frac{1}{s+3} \Rightarrow$ Sketch the graph of $|W(s)|$ for real s.



$\frac{2s+1}{(s-2)(s+3)}$

when $s = -\frac{1}{2} \Rightarrow$ function $= 0$.

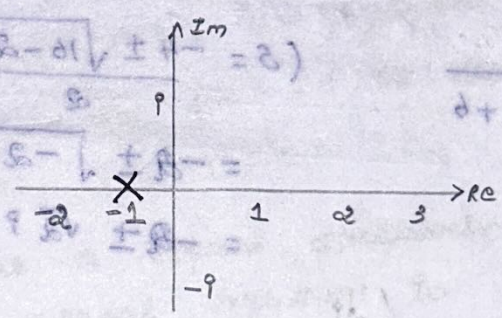


pole diagrams

Def: 5.1: The pole diagram of a function $F(s)$ is simply the complex s-plane with an x marking the location of each pole of $F(s)$.

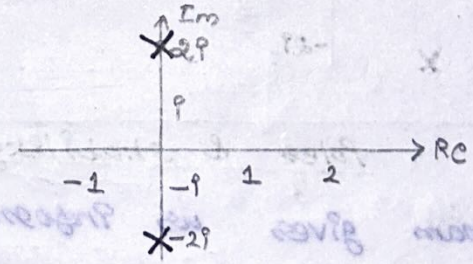
Exam: 5.2:

$F(s) = \frac{1}{s+2}$ has a pole $s = -2$. The pole diagram is a copy of the complex plane with an x at the location $s = -2$.



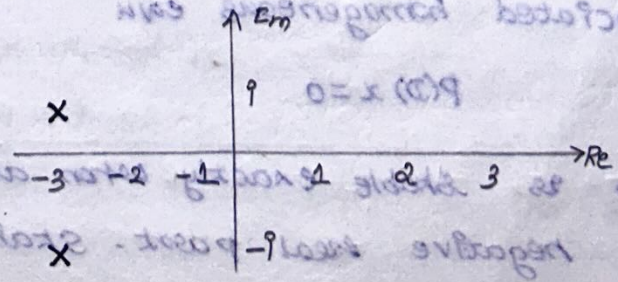
Exam: 5.3:

$F(s) = \frac{1}{s^2+4}$ has poles at $s = \pm 2j$. $(s = -2j, 2j)$



5.4

$F(s) = \frac{s}{s^2+6s+10}$ $\quad s = -3 \pm j$



LTI system

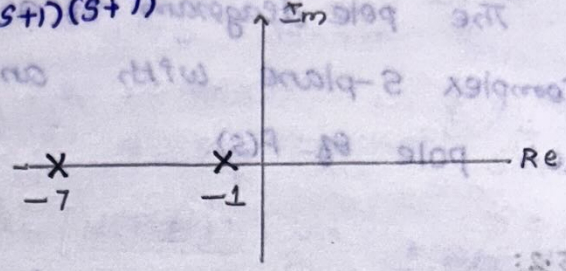
Pole diagram:

Def 6.1: The pole diagram of an LTI system is defined to be the pole diagram of its T.F.

Exam: 6.2 $\ddot{x}+8\dot{x}+7x = f(t)$

solu:- $f(t)$ – Input, $x(t)$ – output

$$W(s) = \frac{1}{s^2+8s+7} = \frac{1}{(s+1)(s+7)} \qquad s=-1,\ s=-7.$$



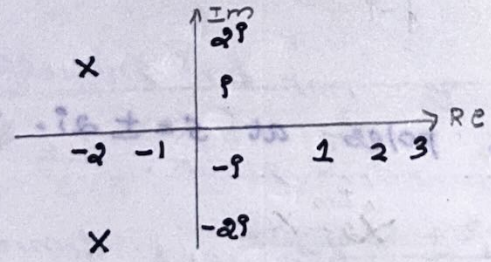
$\ddot{x}+4\dot{x}+6x = \dot{y}$

solu:-

$$W(s) = \frac{s}{s^2+4s+6} \qquad \left(s = \frac{-4 \pm \sqrt{16-24}}{2}\right.$$

$$= -2 \pm \sqrt{-2}$$

$$= -2 \pm \sqrt{2}\, i$$



Poles & stability

The pole diagram gives us information about the stability of a system.

$$P(D)\,x = Q(D)\,y$$

has an associated homogeneous eqn

$$P(D)\,x = 0$$

The system is stable exactly when all the roots of P have negative real part. Stability can be

Stated equivalently in terms of poles.

Suppose that the LTI system is modeled by the differential eqn,

$$P(D)x = Q(D)y \text{ , where}$$

y - Input signal, x - The system response.

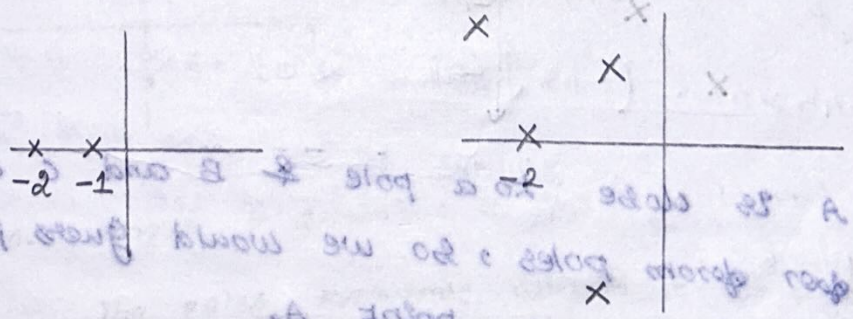
$$\therefore H(s) = \frac{Q(s)}{P(s)}$$

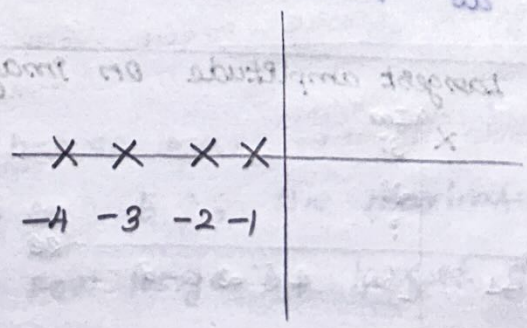
Then the poles of H(s) are among the zeros of P(s), and are exactly the roots of P(s) as long as Q(s) doesn't share any roots with P(s). We will make the general assumption that P(s) & Q(s) have no roots in common.

* The system is stable if all its poles have negative real part.

* Equivalently, the system is stable if all its poles strictly lie in the left part of the complex plane, $(Re(s)) < 0$.

This gives us a visual criterion for identifying whether the signal system is stable from the pole diagram.





The above all are stable.

Poles & amplitude response

How a pole diagram gives us a sense of the amplitude response, that is of the gain $g(\omega)$. We will see that the pole diagram gives us a useful graphical tool

1) We have seen that the graph of $|H(s)|$ is large near the poles of H.

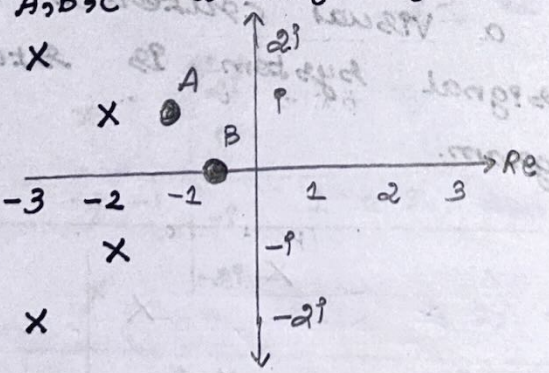
2) Thus, the pole diagram provides a crude graph of $|H(s)|$: roughly speaking, $|H(s)|$ will be large for values s near the poles.

3) The gain of a system with transfer function $H(s)$ is given by $g(\omega) = |H(j\omega)|$.

4) Thus, by restricting our attention to the imaginary axis $s = j\omega$, the pole diagram also gives us a crude sense of the gain $g(\omega)$.

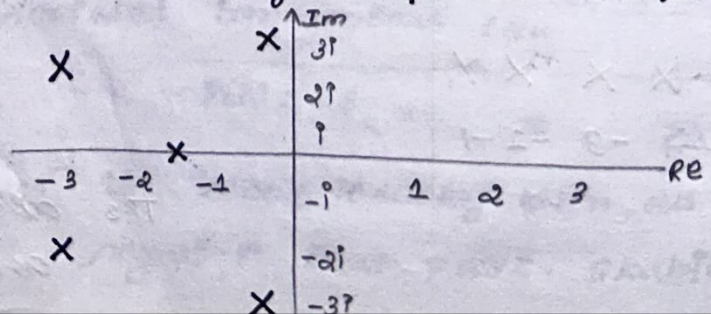
Where is amplitude largest

Among A, B, C would you guess $|H(s)|$ is largest.



point A is close to a pole & B and C are both far from poles, so we would guess point $|H(s)|$ is largest at point A.

Largest amplitude on imaginary axis



The pole diagram of a function $H(s)$ is shown in the picture. At what point s on the +ve imaginary axis would you guess that $|H(s)|$ is largest.

Soln:

s should be close to $3i$, which is near a pole. There is not enough information in the pole diagram to determine the exact location of the maximum, but it is most likely to be near the pole.

Amplitude response & Transfer Function.

$P(D)x = Q(D)y$, the transfer function is $H(s) = \frac{Q(s)}{P(s)}$. The gain is given by

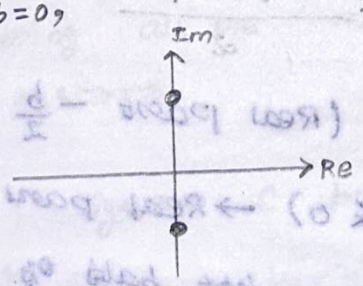
$$g(\omega) = |H(i\omega)|$$ (ratio of amplitudes: response over input)

Note: Rephrasing in graphical terms: we can graph the magnitude of the system function $|H(s)|$ as a surface over the s-plane. The amplitude response of the system $g(\omega) = |H(i\omega)|$ is given by the part of the system function graph that lies above the imaginary axis.

Amplitude response & pole diagram.

$$\ddot{x} + b\dot{x} + Kx = K\cos\omega t$$

when $b=0$,



(poles are in the imaginary axis)

ii) Fix $K=0.5$, move b from 0 to its maximum.

The poles traverse curves converging to the real axis, then move apart along the real axis.

3) $K=0.5$, ($b=0$ to max) → what about amplitude?

- As $b \uparrow$, the maximum gain $\downarrow$ (or stays the same)
- For large b, largest gain at $\omega=0$
- For any +ve input frequency, when you $\uparrow$

the damping constant you decrease the gain

A) Keep $K=0.5$ fixed, we'll compare observations of the gain curve with observations about the position of the poles. still with reference to the behaviour when the b changes: It appears that as you increase the b, two things happen at same time:

- The resonant peak moves to $\omega=0$ and
- The poles collide on the real axis.

The resonant peak moves to $\omega=0$ first

$\therefore$ $H(s)$ occurs when denominator vanishes;

$s^2+bs+\frac{1}{2} \Rightarrow s=\frac{b^2}{4}-\frac{1}{2} \geq 0$ (or) $b \geq \sqrt{2}$, For $b>\sqrt{2}$.

the roots are distinct and real. on the other hand, we determined above that the resonant peak collapses to $\omega=0$ when $b=1$. $\therefore$ The resonant peak collapses before the poles collide with real axis.

Stability

For $b>0$, $K>0$, There is some chance of b and K such that the poles of T.F lie in the right half plane.

Soln:

T.F has poles $\frac{-b \pm \sqrt{b^2-4K}}{2}$

If $b^2-4K>0 \Rightarrow$ Real roots (Real part $-\frac{b}{2}<0$)

$b^2-4K>0$. $(-b+\sqrt{b^2-4K} < 0) \rightarrow$ Real part.

$\therefore$ Always in the left half of the complex plane.

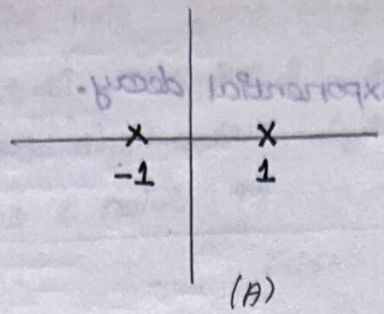
Worked example.

1) Consider the pole diagrams of LTI system $p(D)y=f$

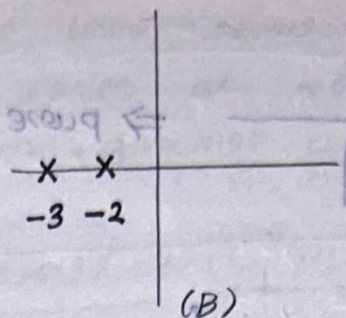
a) Best all stable systems

b) Fastest decay

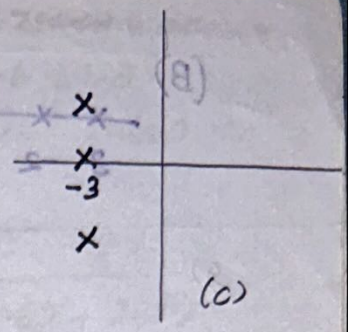
c) Fastest decay without oscillations.



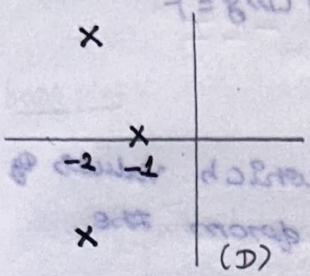
(A)



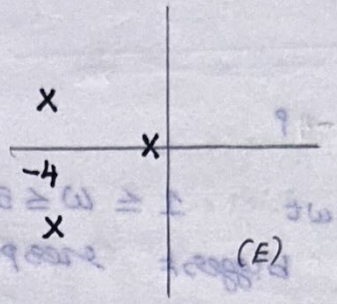
(B)



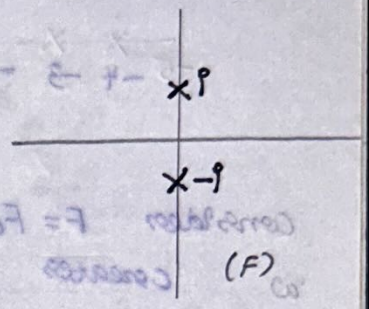
(C)



(D)



(E)



(F)

* List all the stable system.
* Choose the stable system whose the transient has the fastest decay
* Choose the stable system whose the transient decays as fast as possible without oscillation.

Solu:

Stable systems B, C, D, E (Realpart < 0)

more negative - Faster decay.

Comparing all the most right words one eg the stable systems,

ii) System C has e^{-3t} decaying

↳ Dominant.

(Fastest)

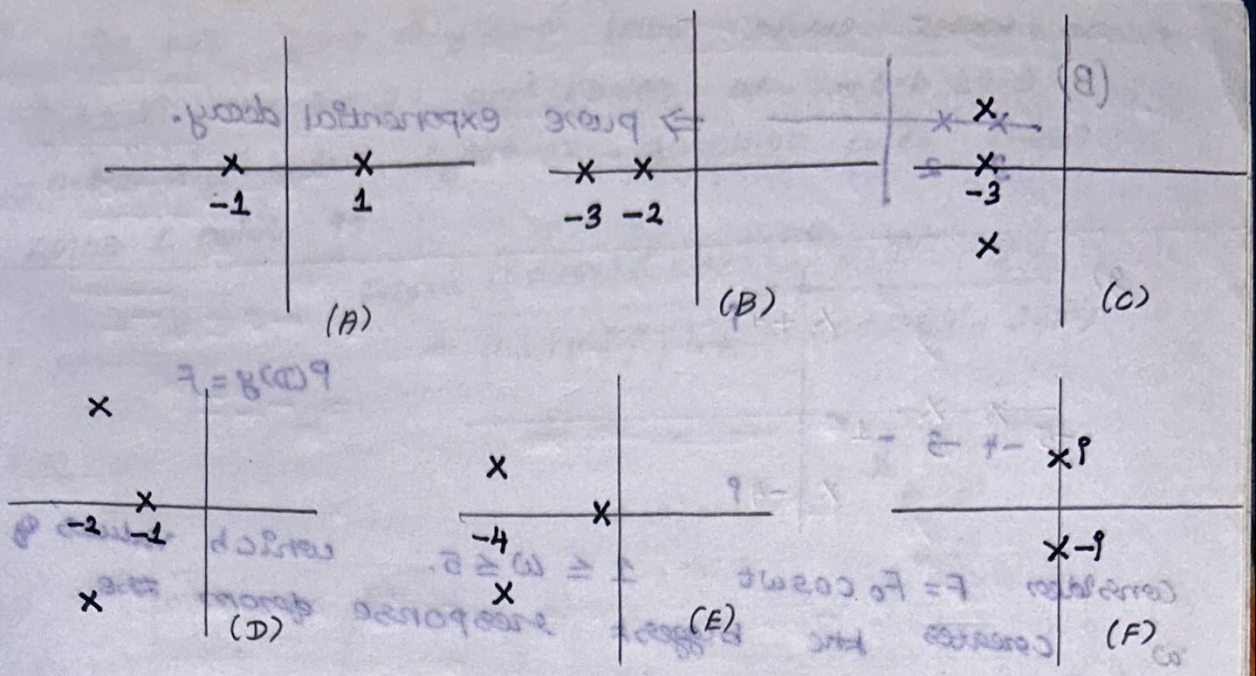
System B ⇒ e^{-2t}, system D ⇒ e^{-t}

System E ⇒ $e^{-0.5t}$

iii) <u>without oscillation:</u>

Pure Real part ⇒ Completely decaying function.

$\cos 3t + \sin 3t \longrightarrow$ oscillatory.



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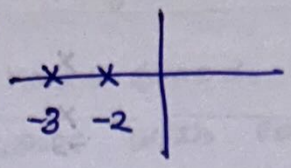
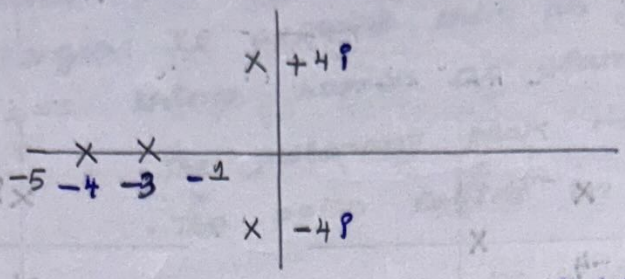
(B)



$\Rightarrow$ pure exponential decay.

---

g)



$P(D)y = F$

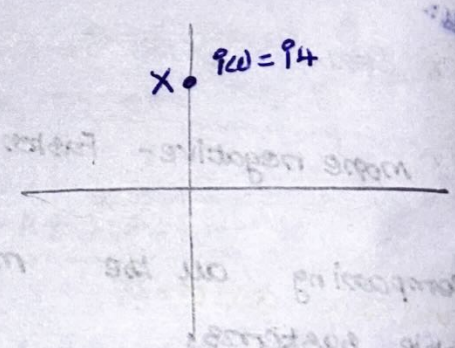
Consider $F = F_0 \cos\omega t$ $\quad 1 \le \omega \le 5$. which values of ω creates the biggest response from the system.

Solu:

Amplitude response $= \dfrac{1}{|P(i\omega)|}$ (from ERF formula)

Amplitude is largest when $|P(i\omega)|$ is very small.

$\boxed{\text{Let } i\omega = i4}$ $\rightarrow$ very close to the gn pole



Idea: Resonance.

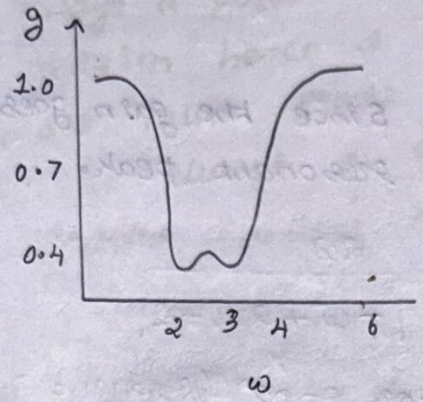
Zeros of the transfer functions:

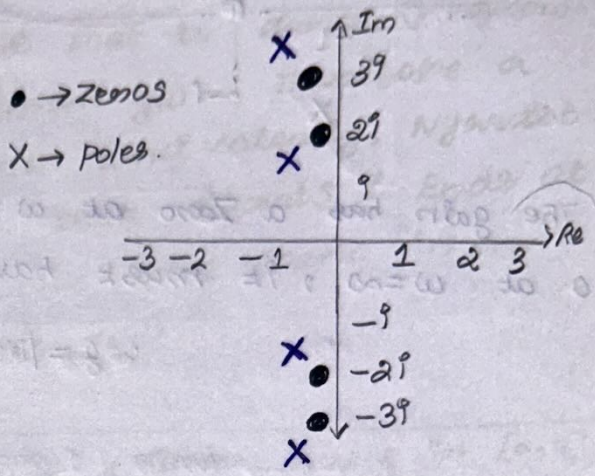
The zeros of a rational function are the values of s for which $H(s) = 0$. If $H(s) = Q(s)/P(s)$ is in reduced form, the zeros are exactly the roots of $Q(s)$. If an input frequency is near a zero then we would expect the gain to be small. Thus the zeros of the system also give us important information about the system. It's common practice to mark the zeros on the pole diagram with a circle.

11.1 The pole zero diagrams below shows zeros at $-0.3 \pm 2i$, $-0.3 \pm 3i$ and poles at $-0.6 \pm 1.8i$, $-0.6 \pm 3.2i$. One system function with these zeros & poles is

$$H(s) = \frac{((s+0.3)^2+4)((s+0.3)^2+9)}{((s+0.6)^2+1.8^2)((s+0.6)^2+3.2^2)}$$

Bode plot:





(when ω (not iω) is near 2 or 3, the gain is small)

Note that when iω is near the zeros the gain is small. The poles were chosen to make the gain as seen as ω is outside the range [2,3], Because the number of zeros equals the number of poles the gain goes asymptotically to a non-zero value (in this case 1) as ω gets large. The net result is a Band stop filter. Any input with frequency in the frequency band [2,3] is suppressed because the gain is near 0. Otherwise frequencies are passed through because the gain is near 1.

Remarks 11.2:

The location of zeros has nothing to do with stability. A rational T.F is determined by its pole-zero diagram up to a constant. (For this we are assuming that there is no repeated poles or zeros)

---

consider $m=1$, $b=2$, $k=2$

$$\ddot{x} + 2\dot{x} + 2x = 2y \quad (y = \cos\omega t)$$

Find pole-zero diagram:

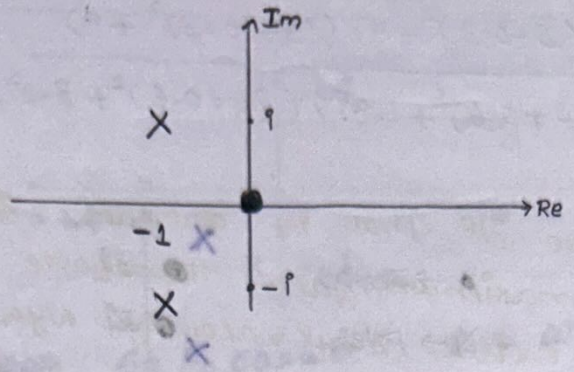
$$H(s) = \frac{2s}{s^2+2s+2}$$

$s = 0 \rightarrow$ zeros

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm \sqrt{-1}$$

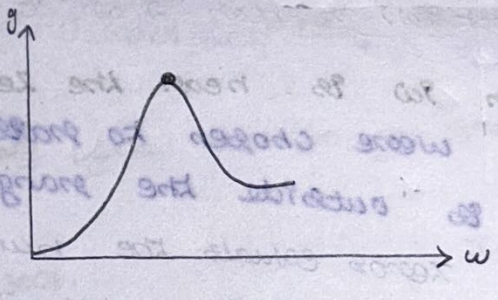
$$= -1 \pm j$$



The gain has a zero at $\omega = 0$, since the gain goes to 0 at $\omega = \infty$, it must have a resonant peak.

$$\therefore g = |H(j\omega)| = \frac{2\omega}{|-\omega^2 + 2j\omega + 2|}$$

when $\omega = \infty$, $g = \infty$ (Resonant peak)



The existence of the zero at the origin pushes the resonant peak away from 0. To see that it is greater than 1, we do some algebra

$$H(s) = \frac{2s}{(s+1)^2+1} \Rightarrow H(j\omega) = \frac{2j\omega}{(j\omega+1)^2+1}$$

$$= \frac{2j\omega}{(2-\omega^2)+2j\omega}$$

Note: The numerator is purely imaginary, the imaginary part of the denominator is exactly equal to the imaginary part of the numerator.

$\therefore$ The magnitude of gain is always less than or equal to 1. The gain is maximal precisely when the gain is 1. which happens only if the real part of the denominator is zero.

i.e. $\omega_r = \sqrt{2} > 1$. (Here)

$m\ddot{x} + b\dot{x} + Ky \rightarrow m, b, K$ (+ve) The system has a +ve resonant ω_r.

Soln:

$$\frac{bj\omega}{K - m\omega^2 + bj\omega}$$

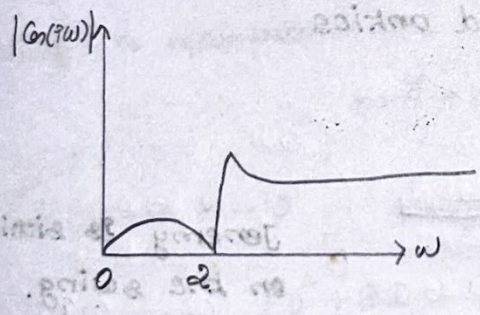
(Let, the Nyquist plot is a circle of radius $\frac{1}{2}$ centered at $z = \frac{1}{2}$)

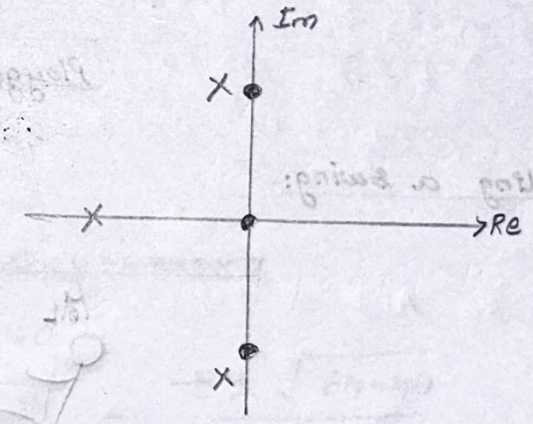
$$\left|z - \frac{1}{2}\right| = \frac{1}{2}$$ (Proved in the previous)

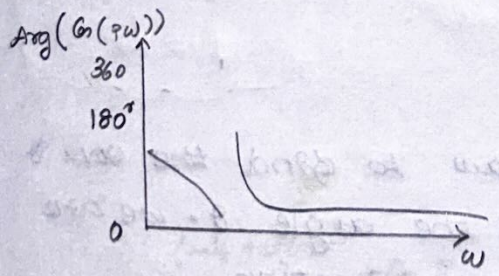
∴ Always a point on the circle that is farthest from the origin, hence a maximum gain. Therefore a resonant value must exist. Equivalently, Nyquist plot is always a circle that starts & Ends at the origin. (Achieve max value some where in b/w. Thus it always has a +ve ω_r).

Stable filter — 3 poles 3 zeros (Filters freq in [0,2]) and has gain near one for $\omega > 2$.

One possible answer: zeros at $0, \pm 2$ & pole pair near ± 2 & a real pole to help modulate the amplitudes in the linear bode plot.



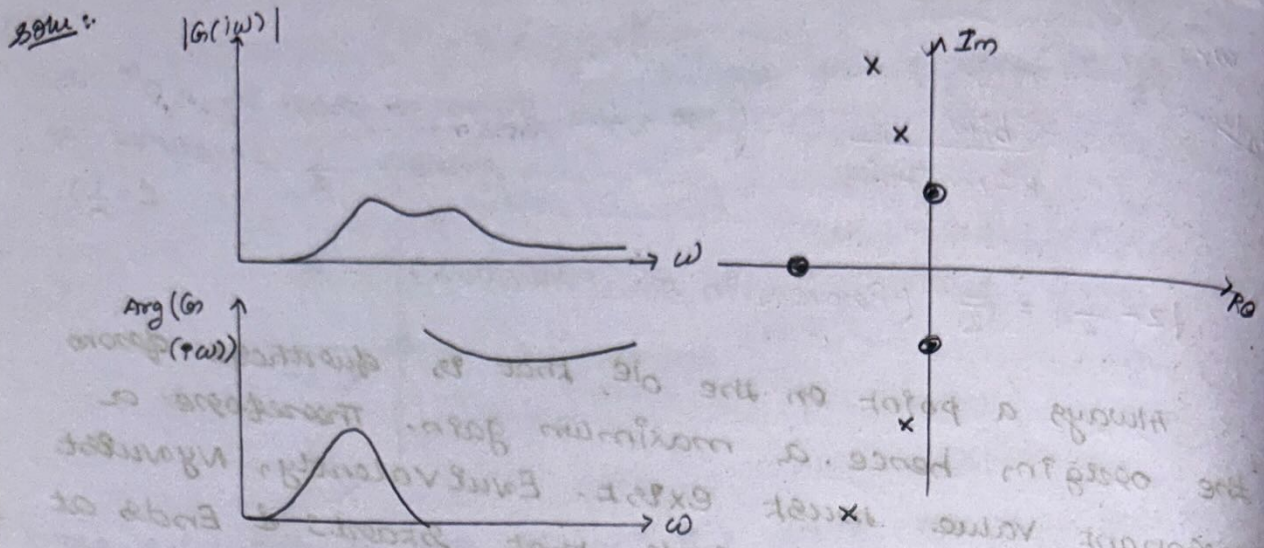




Stable pass-band filter that suppresses freq less than 2, having gain close to 1 for freq b/w 2 and 4, whose gains graph falls to zero after that: Using 2-pole

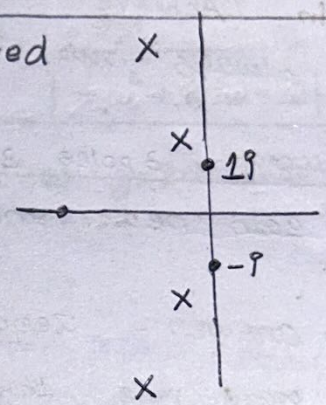
poles & some number of zeros.

Soln:



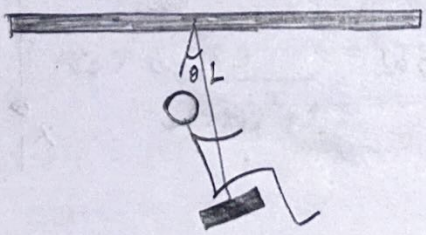
Here $\alpha - 4$ allowed

Near i & $-i$ → Gain (1 app)



Playground antics

Modelling a swing:



Jeremy is sitting on the swing.

1) Use geometry & Newton's II law to find the eqn of motion for Jeremy in terms of the angle θ. we are going to make a series of approximation.

- Approximate Jeremy as a point mass M situated on the seat of the swing.
- other parts of the swing (cables, seat) are massless.
- The only force present is gravity (no friction or wind resistance)

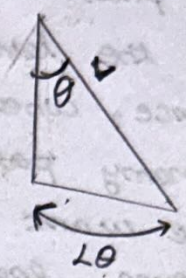
2) For small angles θ, you can express the eqn you found in the form.

$$m\ddot{\theta} + K\theta = 0 \longrightarrow ①$$

Find m and K in terms of the parameters L, g & M.

Soln:

$$F = Ma = -Mg\sin\theta \longrightarrow ④ \quad \text{(horizontal)}$$



(Radian formula)

Displacement $= L\theta$

$$v = L\dot{\theta}, \quad a = L\ddot{\theta}$$

$$\therefore \; Ma = M(L\ddot{\theta}) = -Mg\sin\theta$$

$$ML\ddot{\theta} + Mg\sin\theta = 0$$

Assuming, θ is very small. $\sin\theta \approx \theta$.

$$ML\ddot{\theta} + Mg\theta = 0 \longrightarrow ②$$

comparing ① and ②

$$m = ML, \quad K = Mg$$

Adding a damper:

$$m\ddot{\theta} + b\dot{\theta} + K\theta = 0$$

$m = 1, b = 8, K = 10$

$$(D^2 + 8D + 10)\theta = 0$$

$$= \frac{-8 \pm \sqrt{64-40}}{2}$$

$$= -4 \pm \sqrt{6}$$

$$\theta = c_1 e^{(-4+\sqrt{6})t} + c_2 e^{(-4-\sqrt{6})t}$$

IC

$$\theta(0) = c_1 + c_2 = 0$$

$$\dot{\theta}(0) = c_1(-4+\sqrt{6}) + c_2(-4-\sqrt{6}) = \sqrt{6}$$

$$c_1 - c_2 = 1 \qquad \therefore \; c_1 = 1/2, \; c_2 = -1/2.$$

Jeremy's friend Alice wants to set the swing into motion by periodically pushing & pulling it. Add a force to the system with input signal $f(t) = \cos\omega t$. This changes the eqn you found for small angles θ. From experience, Alice learned that Jeremy will gain most from her actions if she forces the swing at a phase lag of $\frac{\pi}{2}$ w.r.t $\theta(t)$. (i.e. when Jeremy is at the left / right most position, she applies no force, whereas she pushes or pulls most when Jeremy passes through the lowest point $\theta = 0$. Here we want to explore if this intuitive idea truly gives largest gain.

Solu: $m=1$, $b=0.2$, $k=2.0$, $\omega=1.5$ (By moving Bode amplitude Plot, measure amplitude of the sys response. what's the gain.

Gain $= 2.56$

Input amplitude $= 1$
output amp $= 2.56$

For $\omega_r = 1.4$,

$\phi(\omega_r) = 82$ degrees.

Keeping $m=1$, find the phase lag at resonance for 2 other b & k. Do you think Alice's intuition of $\phi = \frac{\pi}{2}$ when $\omega = \omega_r$ is correct?

Solu:

For $m=1$
$b = 0.73$
$k = 1.22$
$\omega_r = 0.95$
$\phi = 66°$.

For $m=1$,
$b = 0.4$,
$k = 1$
$\omega_r = 0.95$
$\phi = 74°$.

So Answer is
NO

Analysis. Complex gain $G(i\omega)$

$$\ddot{x} + b\dot{x} + kx = \cos(\omega t)$$

Solu:

$$\ddot{z} + b\dot{z} + kz = e^{i\omega t}$$

$$z_p = \frac{e^{i\omega t}}{(P(i\omega))} = \frac{e^{i\omega t}}{-\omega^2 + bi\omega + k}$$

$$x_p = Re\left(\frac{e^{i\omega t}}{-\omega^2+bi\omega+K}\right)$$

$$= Re\left(\frac{e^{i\omega t}\left((-\omega^2+K)-bi\omega\right)}{(K-\omega^2)^2-(bi\omega)^2}\right)$$

$$= Re\left(\frac{-\omega^2 e^{i\omega t}+Ke^{i\omega t}-bi\omega\, e^{i\omega t}}{(K-\omega^2)^2+b^2\omega^2}\right)$$

$$= \frac{-\omega^2\cos\omega t+K\cos\omega t+b\omega\sin\omega t}{(K-\omega^2)^2+b^2\omega^2}$$

Assume $b<\sqrt{2K}$

$b^2<4K$

$$r^2+br+K=0$$

$$= \frac{-b\pm\sqrt{b^2-4K}}{2}$$

$$= \frac{-b\pm i\sqrt{4K-b^2}}{2}$$

$$A(s) = \frac{1}{s^2+bs+K}$$

$$H(i\omega) = G(\omega)$$

$$= \frac{1}{K-\omega^2+ib\omega}$$

The gain squared is

$$g(\omega)^2 = |G(\omega)|^2$$

$$= \frac{1}{(K-\omega^2)^2+b^2\omega^2}$$

Gain is maximized when the gain squared is maximized,

$$-4\omega(K-\omega^2)+2b^2\omega=0$$

$$\omega(-4K+4\omega^2+2b^2)=0$$

$$\omega=\sqrt{\frac{K-b^2}{2}} \quad (\text{or})\ 0$$

$b<\sqrt{2K}$

$$G\left(\sqrt{\frac{K-b^2}{2}}\right) = \frac{1}{\frac{b^2}{2}+bi\sqrt{\frac{K-b^2}{2}}}$$

Phase lage $= \pi/2$ when the real part of the complex gain is 0.

$$G(\omega) = \frac{1}{K-\omega^2+j6\omega}$$

The real part is zero only when $\omega^2 = K$ (or) $\omega = \sqrt{K}$