$$u_h(x,0) = u(x,0) - u_{st}(x)$$

$$u_h(x,0) = x - (1-x)$$

$$\boxed{u_h(x,0) = 2x-1}$$

$$u_{h,k}(x,t) = \vartheta_k(x)\, w_k(t)$$

$$\frac{\partial w_k}{\partial t} = 2 w_k \cdot \lambda$$

$$s = -2n^2$$

$$c_1 e^{-2n^2 t}$$

$$\frac{\partial^2 \vartheta_k}{\partial x^2} = \vartheta_k \lambda$$

$$s^2 = -n^2$$

$$s = \pm in$$

$$c_2 \cos nx + c_3 \sin nx$$

" $u_h(x,t)$ " $\rightarrow$ homogeneous (Boundary conditions 0)

$$\vartheta_k(0) = 0 = c_2 \cos 0 + c_3(0)$$

$$\boxed{c_2 = 0}$$

$$\vartheta_k(1) = 0 = c_2 \sin n$$

$$\boxed{n = k\pi}$$

$$\lambda = -n^2$$

$$\lambda = -k^2\pi$$

$$\vartheta_k(x) = c_2 \cos(k\pi x) + c_3 \sin(k\pi x) \qquad \boxed{c_2 = 0}$$

$$\vartheta_k(x) = c_3 \sin(k\pi x).$$

$$w_k(t) = e^{-2k^2\pi^2 t} \quad \text{(constant multiple)}$$

$$u_h(x,t) = c_3 \sin(k\pi x) \cdot c_1 e^{-2k^2\pi^2 t}$$

$$\therefore u_h(x,0) = 2x-1 = \sin(k\pi x) \quad [\text{constant multiple } c_3]$$

$$u_h(x,0) = \sum c_k v_k(x) \qquad 0 < x < 1$$

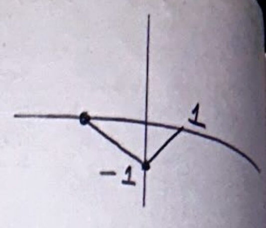
$$u_h(x,0) = \sum c_k \sin(k\pi x)$$

$$2x - 1 = \sum c_k \sin(k\pi x)$$

$$b_n = \frac{2}{L}\int_0^L \left(f(x) \sin \frac{k\pi x}{L}\right) dx = 2\int_0^1 (2x-1) \sin\left(\frac{k\pi x}{L}\right) dx.$$

$$= 2\left(\frac{2\sin(\pi k) - \pi k\,(\cos(\pi k)+1)}{\pi^2 k^2}\right)$$

$$= \frac{4\sin(\pi k) - 2\pi k\left((\cos(\pi k)) + 1\right)}{\pi^2 k^2}$$



"Even"

$$\therefore\ \therefore\ 2x = 1 = \frac{4\sin \pi k}{\pi^2 k^2} - \left(\frac{2\cos \pi k}{\pi k} + \frac{2}{\pi k}\right)$$

Heat equation in MATLAB

Simple numerical method to solve the heat equation:

$$\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2} \qquad 0 < x < L,\quad t > 0$$

with the boundary conditions $\theta(0, t) = f(t)$ and $\theta(L, t) = g(t)$ and I.C $\theta(x, 0) = h(x)$.

we will use forward in time, centered in space numerical scheme. Let θ_j^i denote the solution at time $i\Delta t$ and position $j\Delta x$.

Then a discrete forward time derivative is

$$\frac{\partial \theta}{\partial t} = \frac{\theta_j^{i+1} - \theta_j^i}{\Delta t} + \text{(higher order term in } \Delta t)$$

and a discrete centered space derivative is

and a discrete centred space derivative is

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\theta_{j+1}^i - 2\theta_j^i + \theta_{j-1}^i}{\Delta x^2} + \text{(higher order terms in } (\Delta x)^2)$$

Substituting the discrete time & space derivatives into the heat equ gives

$$\frac{\theta_j^{i+1} - \theta_j^i}{\Delta t} = \vartheta\, \frac{\theta_{j+1}^i - 2\theta_j^i + \theta_{j-1}^i}{\Delta x^2} + \text{higher order terms.}$$

$$\theta_j^{i+1} = \theta_j^i + \frac{\vartheta \Delta t}{(\Delta x)^2}\left(\theta_{j+1}^i - 2\theta_j^i + \theta_{j-1}^i\right) + \text{higher order terms}$$

In matrix notation,

$$\begin{pmatrix} \theta_1^{i+1} \\ \theta_2^{i+1} \\ \vdots \\ \theta_{N-1}^{i+1} \\ \theta_N^{i+1} \end{pmatrix} = \begin{pmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{pmatrix} \begin{pmatrix} \theta_1^{i} \\ \theta_2^{i} \\ \vdots \\ \theta_{N-1}^{i} \\ \theta_N^{i} \end{pmatrix}$$

$$\boxed{r = \frac{\vartheta \Delta t}{\Delta x^2}}$$

where at each time step i we impose the boundary conditions $\theta_1^i = f(i\Delta t)$ and $\theta_N^i = g(i\Delta t)$

Condition for numerical stability

$$\frac{\vartheta \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

one way to find the condition for numerical stability is to find the Eigen values of the matrix for this system and find conditions on r so that all the Eigen values must have a magnitude less than 1.

Wave equation

1) Describe the assumptions & simplifications that go in to the model for the wave equation.

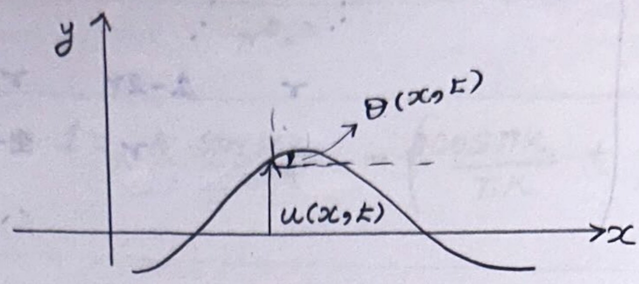
2) Apply Fourier's method to solve the wave equation with fixed end points.

Modelling:

* our Ears detect pressure waves.
* our Eyes detect Electromagnetic waves.

Transverse waves on a String:

'Transverse motion is $\perp^r$ to the length of the String'.



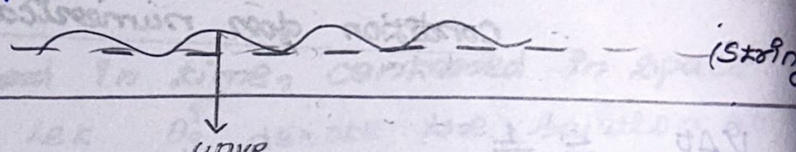
Assumptions:

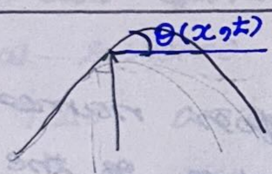
* only transverse motion (Y-axis)

$u(x,t) \rightarrow$ Transverse displacement

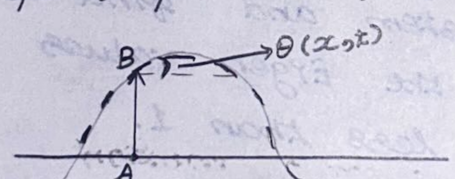
$\theta(x,t) \rightarrow$ Angle of the String (horizontally).

'what governs the speed at which the wave pulse moves along the String?





$\theta(x,t) \rightarrow$ Depends upon both time & position of the spring.



A → Initial position of the String

B → Final position of the String.

for large amplitudes → motion will be in x, y directions

for small amplitudes → motion is well constrained in only 1 direction.

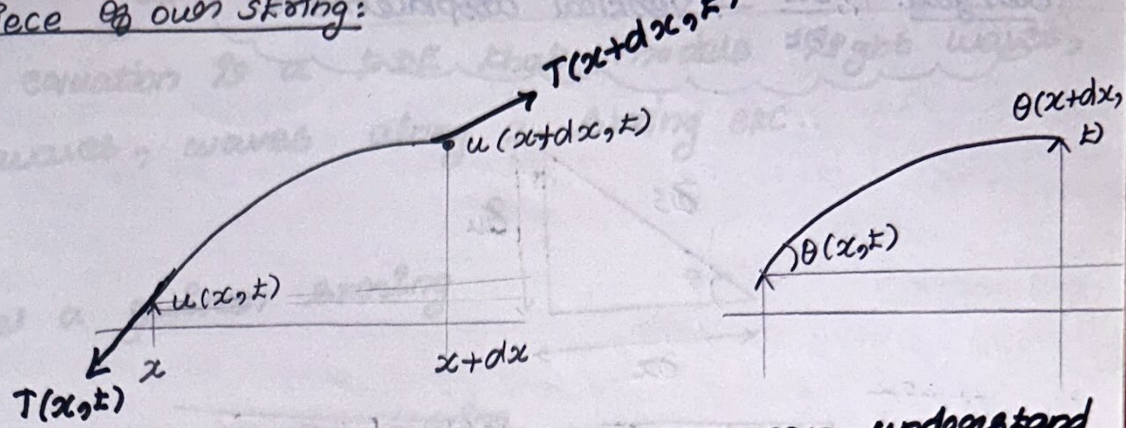
Mathematical modelling: 'Small amplitude' → means motion with θ closer to zero.

Motion of the String is governed by tension (which keeps the String together:

$T(x,t)$: tension

μ : mass per unit length.

Small piece of our string:



understand motion of this segment: we can understand the motion of our entire string:

The small string segment is connected to its neighbouring string pieces by the force of tension.

we have forces of tension acting on both ends of our string tangentially.

'Mass × horizontal acceleration acting on the string' = Horizontal force acting on the String.

∴ we have taken: Transverse motion

'No horizontal force — x direction'.

So, the horizontal forces balances each other

$$T(x+dx)\cos(\theta(x+dx)) = T(x)\cos(\theta x)$$

y-direction:

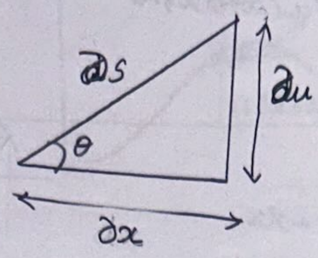
mass × vertical acceleration = Vertical Forces.

$$\mu ds \times \frac{\partial^2 u}{\partial t^2} = \sum F_y \rightarrow \text{(vertical Forces)}$$

Mass = mass per unit length × arc length (ds)
μ

$$\left(\mu\, ds * \frac{\partial^2 u}{\partial t^2}\right) = T(x+dx)\sin(\theta(x+dx)) - T(x)\sin(\theta x)$$

our goal: $\frac{\partial u}{\partial t} \rightarrow$ vertical displacement.



$$\partial s = \sqrt{\partial x^2 + \partial u^2} = (\partial x)\sqrt{1 + \frac{\partial u^2}{\partial x^2}}$$

For small angles

$$\frac{\partial u}{\partial x} \approx 0 \qquad \cos\theta = \frac{\partial x}{\partial s}$$

$$\boxed{\partial s \approx \partial x} \qquad \approx \frac{\partial x}{\partial x} \approx 1$$

$$\boxed{\sin\theta = \frac{\partial u}{\partial s} \approx \frac{\partial u}{\partial x}}$$

Equ: 1

$$T(x+dx)\cos(\theta(x+dx)) = T(x)\cos(\theta x)$$

For any position x and any position t, tension produced is a constant.

$$\boxed{T(x+dx) = T(x)} \qquad \therefore \boxed{\cos\theta = 1}$$

Equation: 2:

$$\left(\mu\, \partial x * \frac{\partial^2 u}{\partial t^2}\right) = T\left(\frac{\partial u}{\partial x}\right)_{x+dx} - T\left(\frac{\partial u}{\partial x}\right)_{x}$$

$$\mu \frac{\partial^2 u}{\partial t^2} = T \frac{\left(\frac{\partial u}{\partial x}(x+dx) - \frac{\partial u}{\partial x}(x)\right)}{\partial x} \rightarrow \text{second derivative } u \text{ w.r.t } x$$

$$\mu \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2}\right)$$

"Is this have any physical sense"

wave equation is a PDE that models light waves, sound waves, waves along a string etc..

2.1:

model a guitar string

solu:

$L \to$ Length of the spring

$\mu \to$ mass per unit length

$T \to$ magnitude of the Tension force.

$t \to$ time

$x \to$ position along the string (from 0 to L)

$u \to$ vertical displacement of a point on the string.

<u>why tension along a string is constant:</u>

"The tension in the rope is constant if its force doesn't have to be used to accelerate everything else, including itself". Therefore, it has negligible mass and is held taut b/w two points, the tension will be considered constant throughout.

"mass less string" $\to$ when we apply a Force F from both sides of the string, the tension remains constant"

"our string is at rest"

Tension in left segment is balanced by the right.

Here,

$L, \mu, T \to$ Constants

$x, t \to$ Independent variables;

$u = u(x,t)$ defined for $x \in [0, L]$ and ...

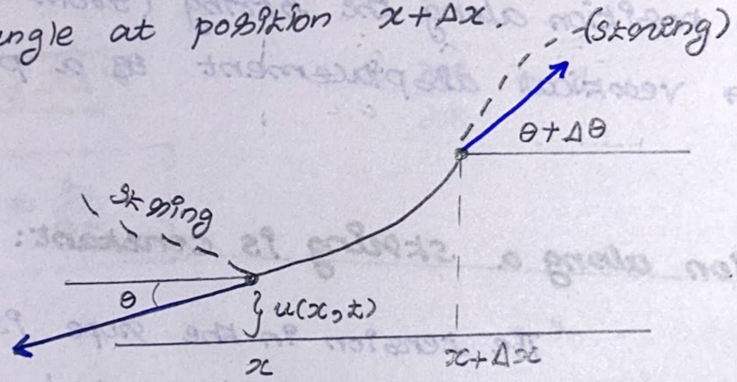
vertical displacement is measured relative to the equilibrium position in which the string makes a straight line.

At any given time t, the string is in the shape of the graph of $u(x,t)$ as a function of x.

Assumption:

The string is taut, so the vertical displacement of the string is small, and the slope of the string at any point is small.

Consider the piece of string b/w positions x and $x + \Delta x$. Let θ be the (small) angle formed by the string and the horizontal line at position x and let $\theta + \Delta\theta$ be the same angle at position $x + \Delta x$.



Derivation:

From Newton's II Law:

$$\boxed{ma = F}$$

Taking vertical component:

$$\underbrace{\mu\, dx}_{\text{mass}} \; \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{Acceleration}} = \underbrace{T\sin(\theta + d\theta) - T\sin(\theta)}_{\text{vertical Component of Force.}}$$

$$= T\, d \sin\theta$$

$\therefore\ d(\sin\theta) = \cos\theta \, d\theta$

$d(\tan\theta) = \dfrac{1}{\cos^2\theta}\, d\theta$

$$\therefore \cos\theta = 1 - \frac{\theta^2}{2} + \dots \approx 1$$

(As θ is very small)

$$d(\sin\theta) \approx \underbrace{d(\tan\theta)}_{\text{slope of the string}} = d\left(\frac{\partial u}{\partial x}\right)$$

Substituting:

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = T d\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = T\mu^{-1} \frac{d\left(\frac{\partial u}{\partial x}\right)}{dx}$$

$$\frac{\partial^2 u}{\partial t^2} = T\mu^{-1} \left(\frac{\partial^2 u}{\partial x^2}\right)$$

$\therefore$ we define a new constant $c : \sqrt{T\mu^{-1}}$, then this becomes

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Note: observe that T is a force, so it has dimension of mass times length over time squared $[mL/t^2]$ and μ is mass per unit length $[m/L]$ so the constant

$c = \sqrt{T/\mu}$ has dimension $\left[\frac{L}{T}\right]$

$$\sqrt{\left[\frac{mL}{t^2} \times \frac{1}{m/L}\right]} = \sqrt{\frac{L^2}{t^2}} = \frac{L}{T}$$

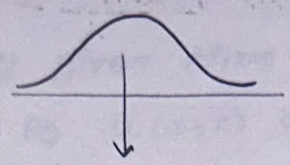
velocity.

The resulting eqn describing the evolution of the wave over time is gn by the following PDE:

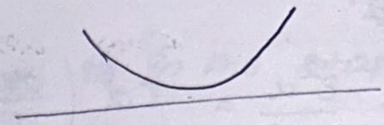
wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

This makes sense intuitives, since at all places where the graph of the string is concave up

$\left(\frac{\partial^2 u}{\partial x^2} > 0\right)$ the tension pulling on both sides should combine to produce an upward force, and hence an upward acceleration.



Concave down



concave up.

Comparing units of both sides of the wave equation shows that the units for c are m/s. The physical meaning of c as a velocity will be explained later.

The ends of a guitar string are fixed. So we have boundary conditions

$$u(0, t) = 0 \quad \text{for all } t \geq 0$$
$$u(L, t) = 0 \quad \text{for all } t \geq 0.$$

$$\mu \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}$$

Is it has physical sense?

$$\left(\frac{\text{mass} \times \cancel{\text{length}}}{\text{unit length}}\right) \times \left(\text{vertical acceleration}\right) = \frac{\text{vertical Force}}{\text{Length}}.$$

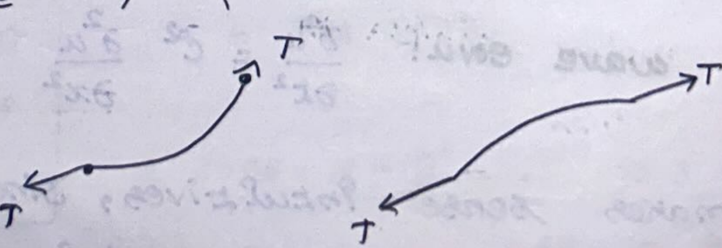
$$\frac{\text{mass} \times \text{v.acc}}{\text{Unit length}} = \frac{\text{vertical Force}}{\text{Length}}.$$

Newton's second law?

Geometrically:

1) II derivative $\rightarrow$ +ve  concave up

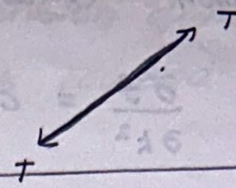
2) -ve  concave down



case : 3 :

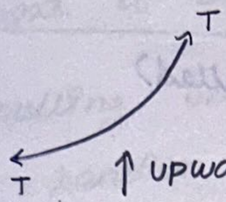
II derivative = 0

'No curvature'



How these three cases are related with $\frac{\text{mass} \times \text{V.acc}}{\text{Length}}$

1)



↑ upward motion (upward net force)

∴ $\mu \rightarrow$ always +ve

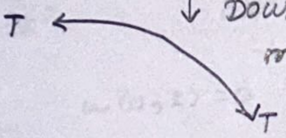
$T \rightarrow$ always +ve

when

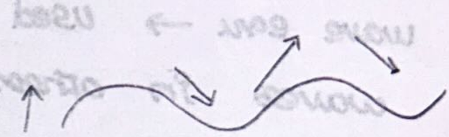
$\frac{\partial^2 u}{\partial t^2} \rightarrow +ve$

$\frac{\partial^2 u}{\partial x^2} \rightarrow +ve$

2)



↓ Downward motion (Downward net force)



(Force lines)

'motion will be towards the direction of force'

when $\frac{\partial^2 u}{\partial t^2} \rightarrow -ve$, $\frac{\partial^2 u}{\partial x^2} \rightarrow (-)ve$.

3)

'No Force'

'No motion'

No vertical acceleration = No Force (vertical)

$\mu(0) = T(0)$

$0 = 0$.

wave equation (general form)

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

$c \rightarrow$ wave speed (At which waves propogate

Examples:

string $\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 u}{\partial x^2}$

when Tension is large (more tightly pulled)

waves move very faster

$T \rightarrow$ Large, μ - small (weightless string)

If we have a very heavy string

(wave propagation will be much slower)

wave eqn $\rightarrow$ used to describe a variety of waves in other applications.

· Sound waves : $\frac{\partial^2 P}{\partial t^2} = c_P^2 \frac{\partial^2 P}{\partial x^2}$

$P \rightarrow$ pressure

$c_P \rightarrow$ speed of sound (depends on atmosphere).

EMF waves (Light)

1) Electric field $\frac{\partial^2 E}{\partial t^2} = c_L^2 \frac{\partial^2 E}{\partial x^2}$

2) Magnetic field $\frac{\partial^2 B}{\partial t^2} = c_L^2 \frac{\partial^2 B}{\partial x^2}$

$c_L \rightarrow$ speed of light.

1. pulling a string along the tension forces, the string will accelerate upwards toward a straight configuration. (upwards)

2) pulling a string along the tension forces, downwards will accelerate the string downwards toward a straight configuration.

3) pulling on a straight segment string along the tension forces, the string won't accelerate at all, and will remain straight.

Seperation of variables for the wave equation

Let $u = u(x, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where}$$

$$\left.\begin{array}{l} u(0, t) = 0 \\ u(L, t) = 0 \end{array}\right\} BC$$

$u(x, 0) = f(x) \longrightarrow$ Initial Shape

$\frac{\partial}{\partial t}(x, 0) = g(x) \longrightarrow$ Initial speed

Seperation of variables:

$$u(x, t) = X(x)\ \phi(t)$$

Assuming $\rightarrow$ our solution can be factored into a function just depending on x, and a function just depending upon t.

$$\frac{\partial^2}{\partial t^2}(X(x)\ \phi(t)) = c^2 \frac{\partial^2}{\partial x^2}(X(x)\ \phi(t))$$

$$X(x) \frac{\partial^2 \phi}{\partial t^2}(t) = c^2\ \phi(t) \frac{\partial^2 X(x)}{\partial x^2}$$

$$\frac{1}{c^2\ \phi(t)} \frac{\partial^2 \phi}{\partial t^2}(t) = \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}$$

L.H.S → Everything only depends on t

R.H.S → " " x

Imagine,

If we have fixed point on x

constant = time changing

$$\underbrace{\frac{1}{c^2 \phi(t)} \frac{\partial^2 \phi}{\partial t^2}(t)} = \underbrace{\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}}$$

when L.H.S is fixed, R.H.S is allowed to change.

when R.H.S is fixed, L.H.S is allowed to change.

So the equation,

$$\frac{1}{c^2 \phi(t)} \frac{\partial^2 \phi}{\partial t^2}(t) = \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}$$

never be satisfied.

So each side needs to be equal to a constant.

$$\boxed{\frac{1}{c^2 \phi(t)} \cdot \frac{\partial^2 \phi}{\partial t^2}(t) = \frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2} = \lambda}$$

↓

So we are dependent on

t → L.H.S

x → R.H.S

Our PDE is a ODE

$$\frac{1}{c^2 \phi(t)} \frac{d^2 \phi}{dt^2}(t) = \frac{1}{X(x)} \frac{d^2 X}{dx^2}(x) = \lambda$$

Solve:

$$\frac{d^2 X}{dx^2} = \lambda X(x)$$

$\boxed{r^2 = \lambda}$ $\boxed{r = \pm\sqrt{\lambda}}$

Case : 1 $\quad \lambda > 0$

$x(0) = 0$

$x(L) = 0$

$r^2 = \pm\sqrt{\lambda}$

$$\boxed{c_1 e^{\sqrt{\lambda} t} + c_2 e^{-\sqrt{\lambda} t} = X(x)}$$

when $\boxed{c_1 = c_2 = 0}$

case : 2 : $\lambda = 0$

So case : 1 $\rightarrow$ Trivial solution

$$\boxed{r^2 = 0, \quad r = 0, 0}$$

$X(x) = (c_1 + c_2 x)e^0$

$x(0) = 0$

$x(L) = 0.$

$= (c_1 + c_2 x)$

$c_1 = c_2 = 0 \rightarrow$ Trivial solution.

case : 3 $\quad \lambda < 0$

$r^2 = -\lambda$

$r = \pm i\sqrt{\lambda}$

$X(x) = c_1 \sin(\sqrt{-\lambda}\, x) + c_2 \cos(\sqrt{-\lambda}\, x)$

$x(0) = 0, \quad x(L) = 0$

$0 = c_2 \cos(\sqrt{-\lambda}(0))$

$\boxed{c_2 = 0}$

$c_1 \neq 0.$

$c_1 \sin(\sqrt{-\lambda x}) = 0$

$\sin(\sqrt{-\lambda}(L)) = 0$

$\boxed{\sin(\sqrt{-\lambda}\, L) = 0}$

$L\sqrt{-\lambda} = n\pi$

$-\lambda = \frac{n^2\pi^2}{L^2}$

$\lambda = -\frac{n^2\pi^2}{L^2}$

$\therefore$

$X(x) = c_1 \sin\left(\frac{n\pi x}{L}\right) + c_2 \cos\left(\frac{n\pi x}{L}\right)$

$= c_1 \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \qquad \boxed{c_2 = 0}$

$\sqrt{+\lambda} \rightarrow$ will give this very same result.

$$X(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

(most general solution)

Time equation:

$$\frac{1}{c^2\,\phi(t)} \frac{d^2\phi}{dt^2}(t) = \lambda$$

$$\frac{d^2\phi}{dt^2}(t) = \lambda \cdot c^2 \cdot \phi(t)$$

$$s^2 = \lambda c^2$$

$$s^2 = -\frac{n^2\pi^2 c^2}{L^2}$$

$$\boxed{s = \pm i\left(\frac{n\pi c}{L}\right)}$$

$$\boxed{-\lambda = \left(\frac{n\pi}{L}\right)^2}$$

$$\sqrt{-\lambda} = \frac{n\pi}{L}$$

$$-\lambda = \frac{n^2\pi^2}{L^2}$$

$$\boxed{\lambda = -\frac{n^2\pi^2}{L^2}}$$

Same case: like previous

$\lambda \rightarrow$ needs to be negative.

$\therefore$ w.k.t $\left(\frac{n^2\pi^2c^2}{L^2}\right) \longrightarrow$ +ve.

$$\phi(t) = c_1 \sin\left(\frac{n\pi c}{L} t\right) + c_2 \cos\left(\frac{n\pi c}{L} t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \left[\left(\sin\left(\frac{n\pi x}{L}\right)\right)\left(\alpha_n \sin\left(\frac{n\pi c}{L} t\right) + \beta_n \cos\left(\frac{n\pi c}{L} t\right)\right)\right]$$

combining co-efficients,

$$u(x,t) = \sum_{n\geq 1} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi x}{L}\right)\right] +$$

$$\sum_{n\geq 1} \left[b_n \sin\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi x}{L}\right)\right]$$

Problem : 1

$\{X(0)=0, X(L)=0\}$

$c=1, L=\pi$

$$u(x,t) = \sum_{n\geq 1}\left[a_n \cos(nt)\sin(nx)\right] + \sum_{n\geq 1}\left[b_n \sin(nt)\sin(nx)\right]$$

Initial conditions

$u(x,0) = f(x) \rightarrow$ Initial shape

$\frac{\partial}{\partial t}(x,0) = g(x) \rightarrow$ Initial speed.

$$f(x) = \sum_{n\geq 1}^{\infty}\left(\sin\left(\frac{n\pi x}{L}\right)(1) + 0\right)\cdot \beta_n$$

$$f(x) = \sum_{n=1}^{\infty}\left(\sin\left(\frac{n\pi x}{L}\right)\right)\cdot \beta_n$$

From knowledge in fourier series:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} L/2 & \text{if } n=m \\ 0 & \text{if } n\neq m.\end{cases}$$

$\sin\left(\frac{n\pi x}{L}\right) \rightarrow$ Fourier basis.

(All are orthogonal to each other)

$$f(x) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{\beta_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

Substitute $f(x)$, get β_n.

$\{\boxed{t=0}$ at 1st deri value

Second Initial condition:

$$g(x) = \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left(\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)a_n\right) + \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left(\sin\left(\frac{\pi n c t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right)b_n$$

$$g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \alpha_n \left(\frac{n\pi c}{L}\right)$$

$$\frac{\partial}{\partial t}(x,0) = g(x)$$

$$\alpha_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

How:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\alpha_n \sin\left(\frac{n\pi c}{L} t\right) + \beta_n \cos\left(\frac{n\pi c}{L} t\right)\right]$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\alpha_n \cos\left(\frac{n\pi c}{L} t\right) \cdot \left(\frac{n\pi c}{L}\right) - \beta_n \sin\left(\frac{n\pi c}{L} t\right) \left(\frac{n\pi c}{L}\right)\right]$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \alpha_n \left(\frac{n\pi c}{L}\right) (1)$$

$$g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \alpha_n \left(\frac{n\pi c}{L}\right).$$

$$\frac{L}{n\pi c} g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cdot \alpha_n$$

$$\alpha_n = \frac{2}{L} \times \frac{L}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

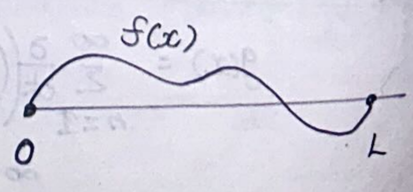
$$\boxed{\alpha_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx}, \quad n = 1, 2, 3, \ldots$$

$n = 1, 2, 3, \ldots$

Analyzing equation:

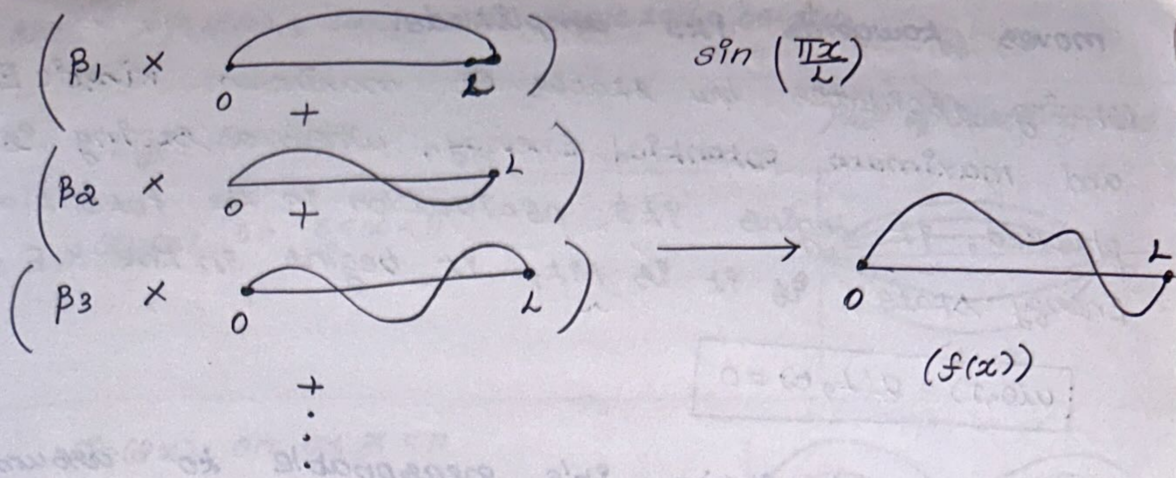
$$\underbrace{u(x,0)}_{f(x)} = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right)$$

what this mean physically?



The $\sin\left(\frac{n\pi x}{L}\right) \rightarrow$ decomposes

β_n to give $f(x)$



To specify a unique solution, we need two initial conditions: not only the initial position $u(x,0)$, but also the initial velocity

$$\frac{\partial u}{\partial t}(x,0), \text{ at each position of the string.}$$

(That two initial conditions needed is related to the fact that the PDE is second-order in the t variable.)

For a plucked string, it's reasonable to assume that the initial velocity is 0.

plucked — pull with a sudden force or with a sudden force?

The strings of a musical instrument can either be plucked or hit in order to begin a vibration, producing — if we're lucky — a note. A guitar player plucks strings but a piano player makes little hammers hit strings inside the instrument. The difference is that the free oscillation begins when the guitar player releases the string in a position away from its rest (equilibrium) position. Similarly, the guitar string begins to accelerate towards the relaxed position, while the piano string is at maximum velocity as soon as the hammer hits it and then decelerates as it moves towards its amplitude.

(String osciuates b/w states of maximum kinetic Energy and maximum potential Energy, when a string is plucked, it begins its osciuation in the potential Energy state; if it is hit, it begins in the K.E state.

$$\boxed{u(0,t) = u(L,t) = 0}$$

For a plucked string - it's reasonable to assume

$$\frac{\partial u}{\partial t}(x,0) = 0$$

what condition does this impose on the a_n and b_n? well, for the general solution above,

$$\frac{\partial u}{\partial t} = \sum_{n \geq 1} -n a_n \sin(nt)\sin(nx) + \sum_{n \geq 1} n b_n \cos(nt) \sin(nx)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n \geq 1} n b_n \sin(nx)$$

So the Initial condition says that $\boxed{b_n = 0}$ for every n; in other words,

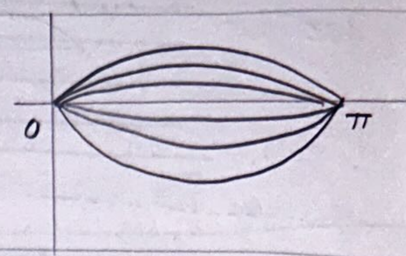
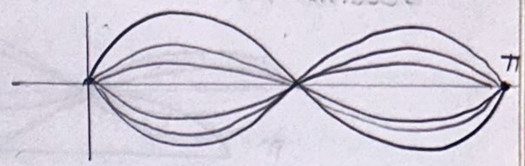
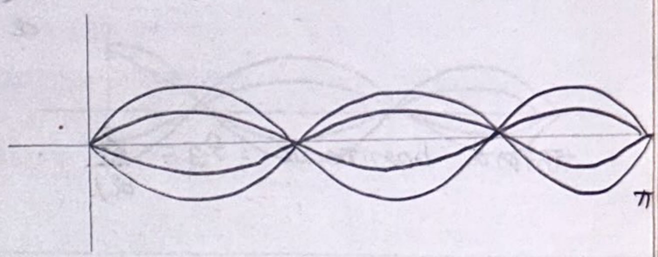
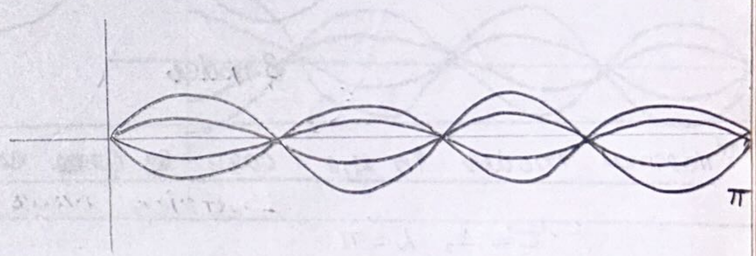
$$u(x,t) = \sum_{n \geq 1} a_n \cos(nt) \sin(nx)$$

If we also knew the initial position $u(x,0)$, we could solve for the a_n by extending to an odd, period 2π function of x and using the Fourier coefficient formula.

'use mathlet'

Normal modes: standing waves

In the previous example of transverse waves in a string with fixed ends, the eigen functions $c_n v_n(x)$ correspond to standing waves. These standing waves are the normal modes for this system, and are depicted below

n	Eigen Function	Plot of Standing wave
1	$\sin(x)$ on $0 < x < \pi$	
2	$\sin(2x)$ on $0 < x < \pi$	
3	$\sin(3x)$ on $0 < x < \pi$	
4	$\sin(4x)$ on $0 < x < \pi$	

'Demo of standing transverse waves in a different physical setup: the ends of the string are free. In this case the same partial differential equation models the situation but the boundary condition is determined by there being no tension force at the end of the string, and the boundary conditions reduce to

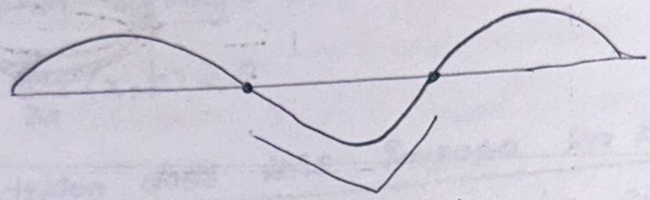
$$\frac{\partial u}{\partial x}(0, t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L, t) = 0.$$

observing how the normal modes with free ends differ from the case of fixed ends.

Fundamental frequency = $f_1 = \frac{v}{2L}$

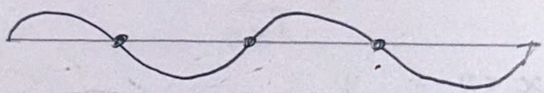
1 node.

Second Harmonic $f_2 = \frac{v}{L}$



2 nodes

Third harmonic : $f_3 = \frac{3v}{2L}$



3 nodes

Normal modes in the case of free ends:

$c = 1, \ L = \pi$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \pi$$

BC: $\frac{\partial u}{\partial x}(0,t) = 0$

$\frac{\partial u}{\partial x}(\pi,t) = 0$

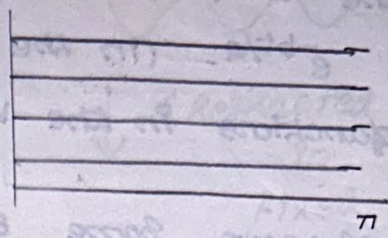
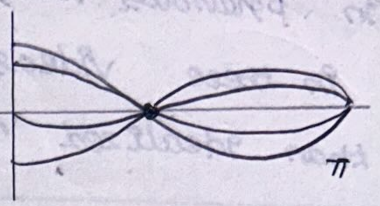
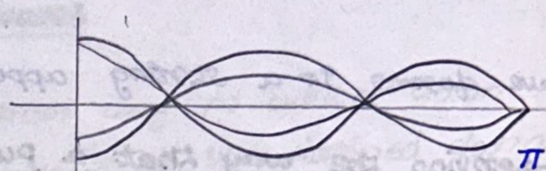
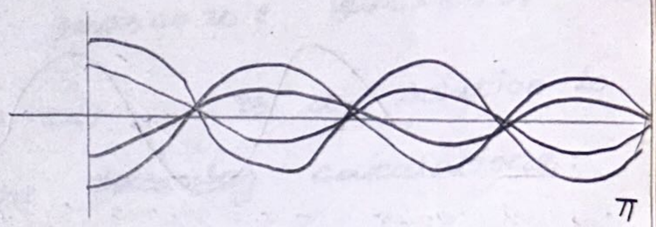
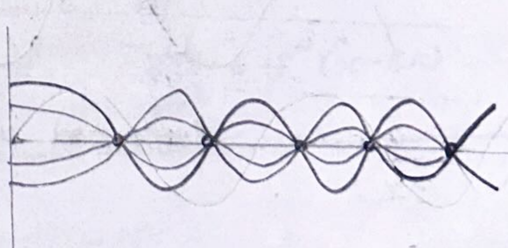
For our previous case: $\lambda = -\frac{n^2\pi^2}{L^2}$

Soln:

$$u(x,t) = v(x) \cdot w(t)$$

$\frac{\partial^2 w}{\partial t^2}(t) = \lambda w(t)$ | $\frac{\partial^2 v}{\partial x^2}(x) = \lambda v(x)$

Non-zero solutions exist only when $\lambda = -n^2$ for some non-negative integer n, and in this case $v = \cos(nx)$ [times a scalar]

n	Eigen Function	Plot of standing wave
0	1 on $0 < x < \pi$ $\cos 0 = 1$	
1	$\cos(x)$ on $0 < x < \pi$	
2	$\cos(2x)$ on $0 < x < \pi$	
3	$\cos(3x)$ on $0 < x < \pi$	
4	$\cos(4x)$ on $0 < x < \pi$	

Real-life waves

In real life, there is always damping. This introduces a new term in to the wave equ

$$\boxed{\text{Damped wave equation: } \frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

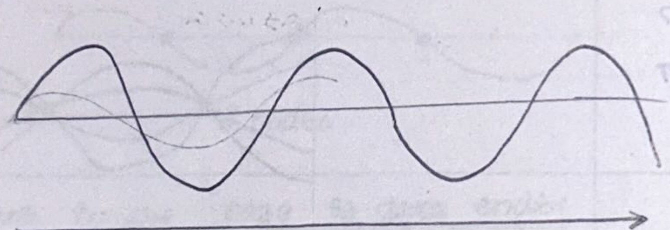
seperation of variables still works, but in each normal model the w(t) is a damped sinusoid involving a factor $e^{-bt/2}$ (in the underdamped case). But the Eigen functions in the variable x are the same!

we can observe some of the features that we've learnt in previous session. But not that you see damping in these videos as they are real-life rather than idealized model we've been exploring.

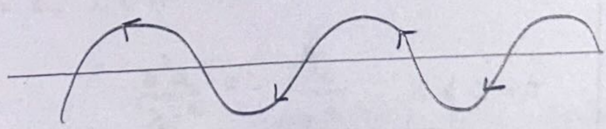
D' Alembeart's Solution

wave forms in a string appear to travel along the string.

'observing the way that a pulse appears to travel along the string':



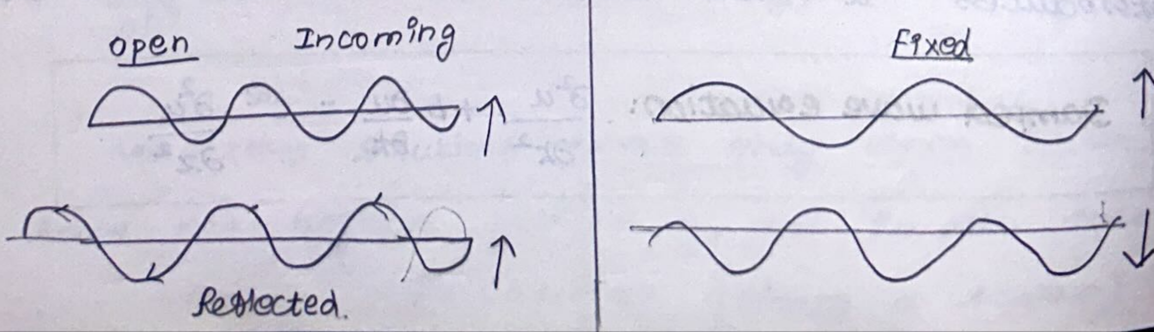
Incoming wave.

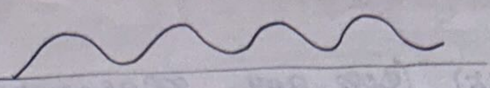


Reflected wave

'upon reflection at an open end the polarity of the wave is +ve'

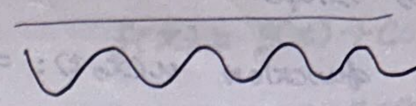
'upon reflection at a fixed end the polarity of the wave is negative'



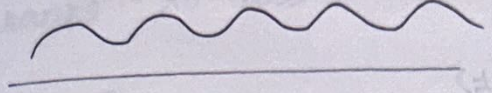


+ve ↑

(Incoming wave)



↓ (Reflected wave in fixed ends)



+ve ↑

(Reflected wave in open ends) (Free)

Traveling waves

D' Alembert figured out another way to write down solutions, in the case when $u(x,t)$ is defined for all real numbers x instead of a finite interval $0 \le x \le L$. Then, for any reasonable function f,

$u(x,t) := f(x-ct)$ is a solution to the PDE, as shown by the following calculations:

$$\boxed{\begin{array}{ll} \dfrac{\partial u}{\partial t} = (-c)\, f'(x-ct) & \dfrac{\partial u}{\partial x} = f'(x-ct) \\ \dfrac{\partial^2 u}{\partial t^2} = (-c)^2\, f''(x-ct) & \dfrac{\partial^2 u}{\partial x^2} = f''(x-ct) \end{array}}$$

So,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

what's the physical meaning of this solution at $t=0$,

we have $u(x,0) = f(x)$, so $f(x)$ is the initial position.

For any number t, the position of the wave at time t is the graph of $f(x-ct)$, which is the graph of f shifted ct units to the right. Thus the wave travels at constant speed c to the right, maintaining its shape.

The function $u(x,t) := g(x+ct)$ (for any reasonable function $g(x)$) is a solution too, a wave moving to the left. It turns out that the general solution is a superposition

$$u(x,t) = f(x-ct) + g(x+ct)$$

There is a tiny bit of redundancy: one can add a constant to f and subtract the same constant from g without changing u.

It's important to note that these solutions assume that the wave is defined for all $-\infty < x < \infty$. However, there are ways to extend these general solutions to the case of finite intervals with boundary conditions. This is why we are able to show demo videos of finite strings to see some of the phenomenon described here.

Remark: 8.1):

Note that the D'Alembert solution is what allows us to easily understand the co-efficient c in the PDE as being the wave speed.

Example: $c = 1$, that the initial position is $I(x)$, and that the initial velocity is 0. What does the wave look like?

Solu:

$$u(x,0) = I(x)$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{become}$$

$$f(x) + g(x) = u(x,0)$$

$$\boxed{f(x) + g(x) = I(x)}$$

$$-c\, f'(x-ct) + c\, g'(x+ct) = u'(x,t)$$

$$u'(x,0) = -cf'(x) + c\,g'(x)$$

$0 = -cf'(x) + cg'(x)$ $\qquad$ $f'(x) = g'(x)$

$f(x) = g(x) + \text{Constant}$

$$\boxed{-f'(x) + g'(x) = 0}$$

The second equation says that $g(x) = f(x) + \text{const}$ some constant con, and we can adjust f and g by constants to assume that $c = 0$. Then $f(x) = \frac{Ix}{2}$.

The wave

$$u(x,t) = \frac{I(x-t)}{2} + \frac{I(x+t)}{2}$$

consists of two equal waveforms, one travelling to the right and one travelling to the left. We can observe the wave an initial pulse of zero velocity splits into two half amplitude pulse travelling in opposite directions.

$\therefore$ $f(x) = g(x) + \text{cons}$

Assume constant $= 0$

$$\boxed{f(x) = g(x)}$$

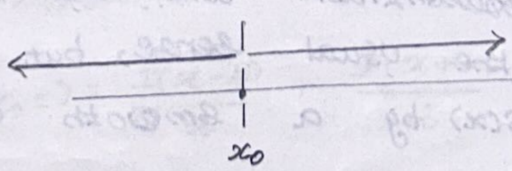
$f(x) + g(x) = I(x, 0)$

$2f(x) = \frac{I(x)}{1}$

$$\boxed{f(x) = \frac{I(x)}{2}}$$

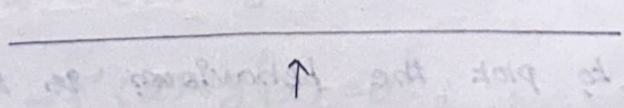
$u(x,t) = f(x-ct) + g(x+ct)$

$u(x,t) = \frac{I(x-t)}{2} + \frac{g(x+t)}{2}$

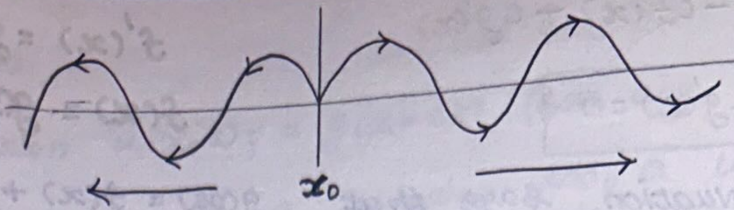


(Initial wave travelling towards left & right)

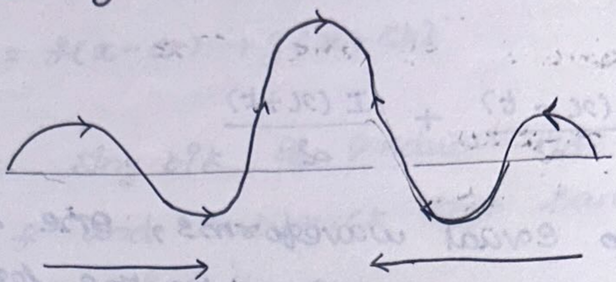
Superposition



Give a pulse of Force in the middle of the 'wave machine'



"A single pulse creates 2 waves of $\frac{1}{2}$ amplitude that add constructively when they pass each other."



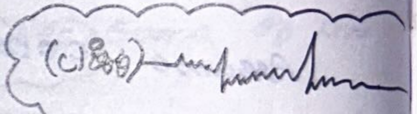
Reflection waves.

"Added constructively, when they pass each other"

"Use mathlet" → To see left & right waves

Wavefronts

Step function



$$s(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

and consider the solution

$u(x,t) = s(x-t)$, to the wave equation.

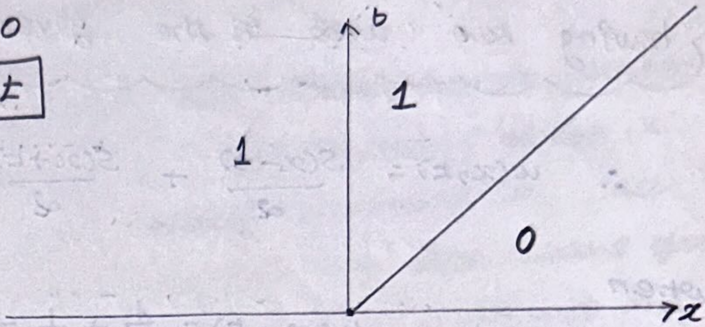
This is a "cliff shapped wave" travelling to the right. (You would be right to complain that this function is not differentiable and therefore can't satisfy the PDE in the usual sense, but you can imagine replacing $s(x)$ by a smooth approximation a function with very steep slope. The smooth approximation also makes more physically: a physical wave wouldn't actually have a jump discontinuity

Another way to plot the behaviour is to use a space-time diagram. In a plane with axes x (space) and t (time). (Usually one draws only the part with $t \geq 0$). Divide the (x, t)-plane into regions according to the value of u. The boundary b/w the region is called the wavefront.

In the example above $u(x,t) = 1$ for points to the left of the line $x - t = 0$, and $u(x,t) = 0$ for the points to the right of the line $x - t = 0$. So the wave front is the line

$x - t = 0$

$\boxed{x = t}$



wavefront:

Infinite string satisfies the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , t > 0$$

suppose that the initial position $u(x, 0) = S(x)$,

Initial velocity is $\frac{\partial u}{\partial t}(x, 0) = 0$

into how many regions is the $t \geq 0$ part of the space-time diagram divided.

Solu:

From previous problem

$$u(x,t) = \frac{I(x-t)}{2} + \frac{I(x+t)}{2} \quad \left[\text{superposition}\right]$$

$$= \frac{S(x-t)}{2} + \frac{S(x+t)}{2}$$

$u(x, 0) = S(x) = f(x) + g(x)$

$\frac{\partial u}{\partial t}(x, 0) = -f'(x) + g'(x) = 0$

$f'(x) = g'(x)$

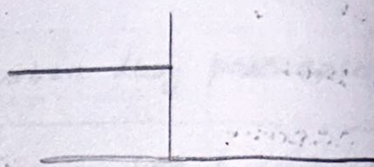
∴ From our knowledge

Infinite String

"Even in Guitar, we pluck in the middle - or somewhere else of the string → Not in ends" -

"A single pulse creates an amplitude of $\frac{1}{2}$ having two waves of the given force"

$$\therefore \quad u(x,t) = \frac{S(x-t)}{2} + \frac{S(x+t)}{2} \quad \text{for } t \geq 0.$$



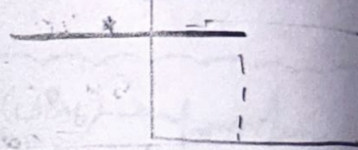
S(x)

when

$x < -t \qquad u(x,t) = \frac{1}{2} + \frac{1}{2} = 1.$

$-t < x < t \qquad u(x,t) = \frac{1}{2} + 0 = \frac{1}{2}$

$x > t \qquad u(x,t) = 0 + 0 = 0.$



t

S(x-t)

case: 1

$$S(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Case: 1

$x < -t$ ((-)ve (smaller than $-t$))

$\boxed{-x > t}$

$$\frac{S(x-t)}{2} + \frac{S(x+t)}{2}$$

$$\frac{S(-ve)}{2} + \frac{S(-ve)}{2}$$

$$\frac{1}{2} + \frac{1}{2}.$$

1) $x - t \rightarrow -ve$

$x + t \rightarrow -ve$

case: 2

x values b/w $-t$ & t.

$-t < x < t$

$$\frac{S(-ve)}{2} + \frac{S(+ve)}{2}$$

$$\frac{1}{2} + 0$$

$x - t \rightarrow -ve$

$x + t \rightarrow +ve$

Case: 3

$$\frac{S(+ve)}{2} + \frac{S(+ve)}{2}$$

$x - t \rightarrow +ve$

$x + t \rightarrow -ve$

$x > t$