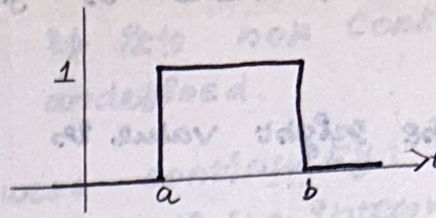
The a, b window has the graph



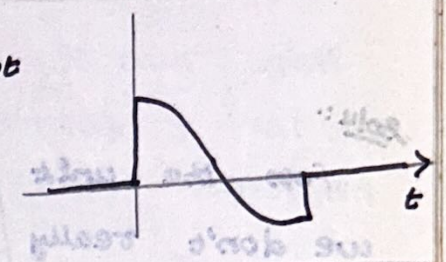
It's given by the formula

$u(t-a) - u(t-b)$

In thinking about functions built from step functions, read them from left to right: turns (window) on at time a & off at time b.

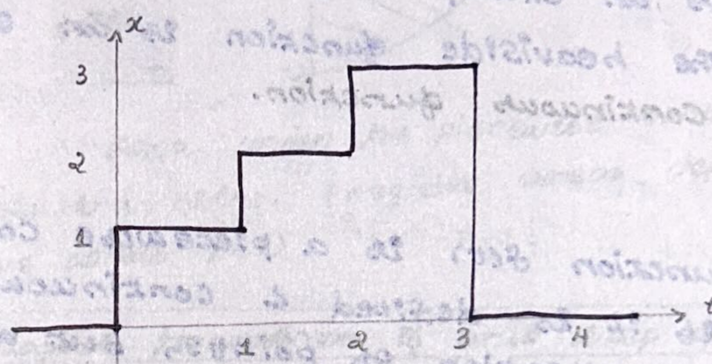
5. We can clip a function past

$(u(t) - u(t-\pi))\cos t$

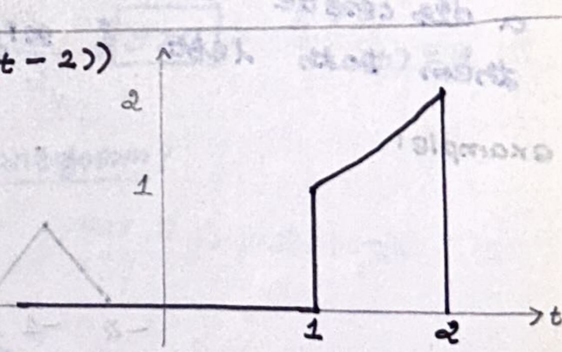


Graph $u(t) + u(t-1) + u(t-2) - 3u(t-3)$

solu:



$t\,(u(t-1) - u(t-2))$



Piecewise continuity

It's good to be precise about what kind of functions we are willing to consider in our study of systems & signals. While we want to accept certain discontinuities, we don't want functions like $f(t) = 1/t$ that blow up to infinity in finite time.

In order to control what happens near discontinuities we recall the definition of one sided values of a function:

The left value is $f(a^-) = \lim_{t \to a^-} f(t) = \lim_{t \uparrow a} f(t)$

The right value is $f(a^+) = \lim_{t \to a^+} f(t) = \lim_{t \downarrow a} f(t)$

Here $t \uparrow a$ denotes the limit from below

$t \downarrow a$ denotes the limit from above.

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

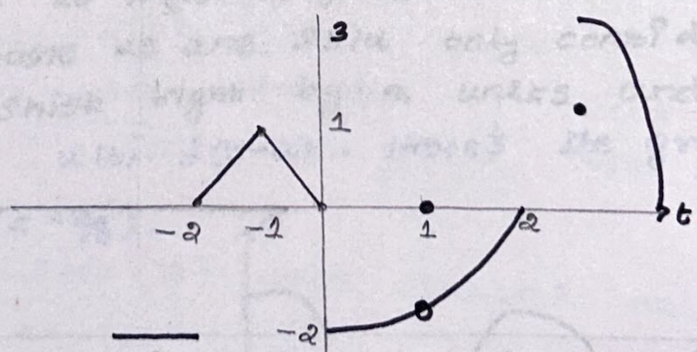
solu:

For the unit step function, $u(0^-) = 0$ while $u(0^+) = 1$ we don't really care about the value of $u(t)$ at $t = 0$ and have left it undefined. The heaviside function is an example of a piecewise continuous function. The heaviside function is an example of a piecewise continuous function.

Definition:

A function $f(t)$ is a piecewise continuous function if it is defined & continuous except at a discrete collection of points, but at each of them both left & right limits exist.

example:

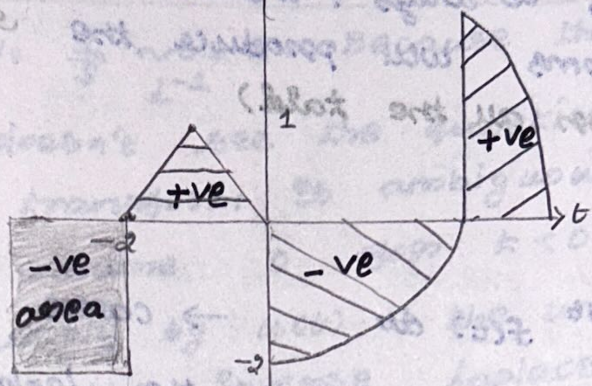


Notice that a piecewise continuous function is allowed to be undefined at some points, of course if $f(t)$ is not defined at $t = a$, then we require that the left and right limits exist at a. If f is continuous at a then the left & right limits are the same & both equal $f(a)$. So in a sense we never care about the value of $f(t)$ at any single point. If $f(t)$ is continuous at $t = a$

then we can reconstruct the value f(a) from knowing neighbouring values & if it's not continuous then we can just leave f(a) undefined.

Here's a nice feature of piecewise continuity: If f(t) is piecewise continuous in the finite interval [a,b] then

$\int_a^b f(t)\,dt$ exists (and has a finite value)



Area under the piecewise continuous curve is well defined and finite. (Negative areas depicted as well as +ve areas too)

Laplace transform of unit step function

$$\mathcal{L}(u(t)) = \int_0^\infty e^{-st}\,u(t)\,dt = \frac{1}{s}; \boxed{s>0} \quad t>0 \;\; u(t) = 1.$$

What's the inverse laplace transform?

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = ?$$

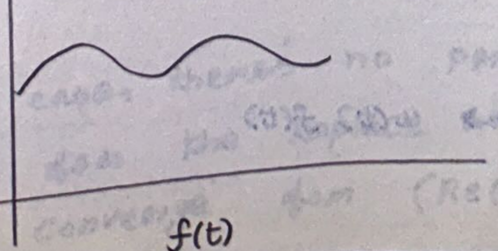
Investigate:

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

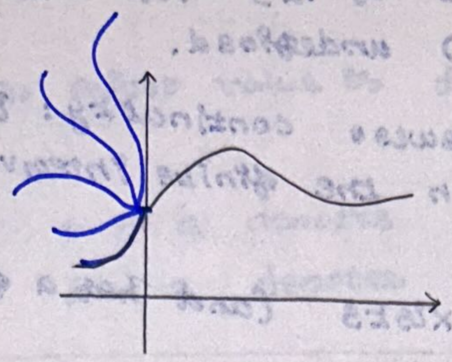
$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t) \quad \text{[But what's the case here]}$$

?

Let:



we can complete our function f(t) in other ways too.



Here out of ∞ ways, we've drawn 5 possible ways. All the 5 forms will produce the same Laplace transform (for all the tails)

why?

$$\int_0^\infty e^{-st} f(t)\, dt \rightarrow \text{cares only +ve}$$

values. (In general, we use the Laplace transform only for problems for future time. That's why the Engineers & physicists use it habitually.

(Don't have to know about past)

If you have to know past too!

It's the Fourier transform.

In Laplace (we are starting at t=0 & going forward).

so we are declaring our function (forcefully), by brute force:

our Function is in 0 for t<0.

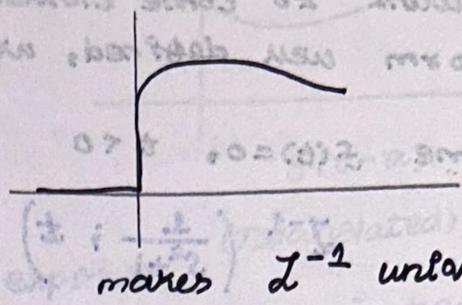
(That makes it unique).

<u>How do we make f(t) for negative values of t.</u>

∴ By multiplying it by the unit step function.

$$F(s) \xrightarrow[\mathcal{L}^{-1}]{} u(t)\, f(t)$$

multiply it by 1 for +ve values
0 for -ve values.



(picks the least t, at (-)ve values)
↳ $t=0$.

makes $\mathcal{L}^{-1}$ unique.

up until now, $\frac{1}{s} \xrightarrow[\mathcal{L}^{-1}]{} 1$. Because the Laplace transform doesn't see the function for $t<0$. The inverse transform is ambiguous.

we assume 0 for $t<0$.

That is multiply by $u(t)$ → the unit step function. Now the inverse Laplace transform of

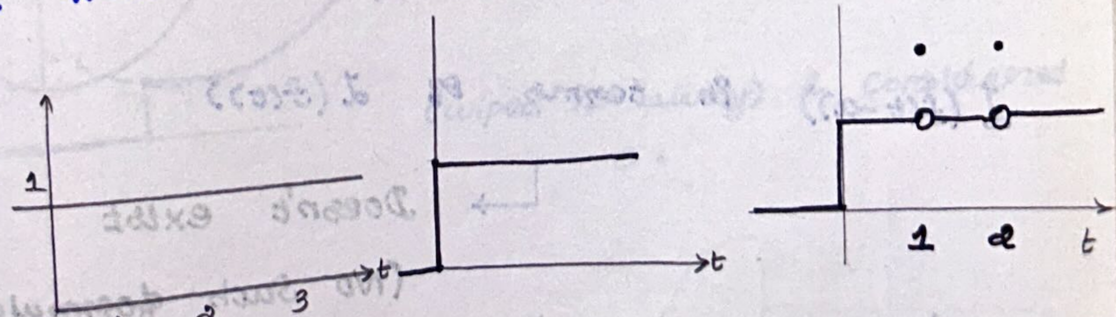
$$\frac{1}{s} \xrightarrow[\mathcal{L}^{-1}]{} u(t)$$

unfortunately,

this still leaves some ambiguity. Both the Laplace transform is defined as an integral and because integral range is over $t>0$. There are several limitations on how completely reconstruct $f(t)$ from its Laplace transform $F(s)$

- you can't say anything about $f(t)$ $(t<0)$
- " " about $f(a)$ for any specific value $t=a$.

eg:



In each case, there's no problem defining the Laplace transform, and all three integrals for the Laplace (Re(s) > 0) and converge integral converge to the same values. The Laplace transform of all three functions is $\frac{1}{s}$ (for $Re(s) > 0$)

If we want to come closer to making the inverse transform well defined, we can do several things.

1) Assume $f(t) = 0, \quad t < 0$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}; t\right) = \sin t$$

we can say

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}; t\right) = u(t)\sin t$$

2) while we can't recover $f(a)$ itself for any specific value of $a \geq 0$. It turns out that we can determine both left & right limits at a.

$$f(a^-) = \lim_{t \uparrow a} f(t) \quad \text{and} \quad f(a^+) = \lim_{t \downarrow a} f(t)$$

(Not obvious, but true, at least assuming $f(t)$ is piecewise continuous.) So if these two limits coincide, we can specify that our function should be continuous at $t = a$ & declare the value $f(a)$ to be that common value. If the two values differ, the function $f(t)$ exhibits a jump discontinuity at $t = a$, and it's best to leave $f(a)$ undefined.

<u>t-shift rule</u>

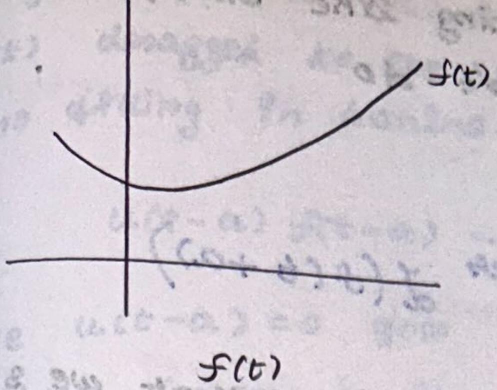
$\mathcal{L}(f(t-a))$ in terms of $\mathcal{L}(f(t))$

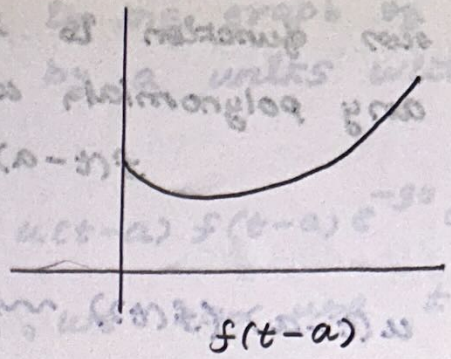
→ Doesn't exist

(No such formula)

<u>why can't be such a formula:</u>

Assume $f(t)$ has a Laplace transform.





why it's impossible to express the laplace transform of the new one in terms of the old one.

Reason:

$\mathcal{L}(f(t); s)$ → only cares about +ve values.

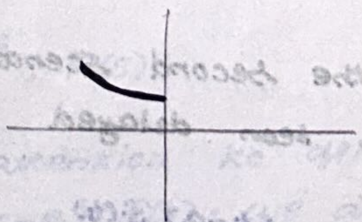
we can't retrieve the past data (−)ve values from it. since it's lost.

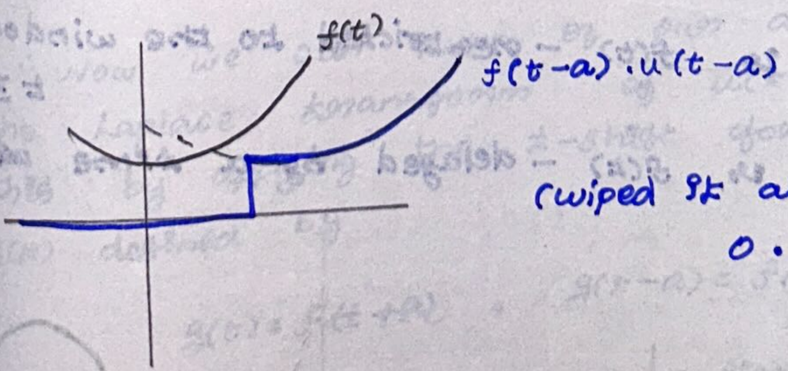
∴ we need the past info if we translate it which is not possible. (The given function will be corrupted - some parts are missing)

we can't express one in terms of other

The right way:

$u(t-a)\,f(t-a)$ → wiped out the (negative region) past from the translated function.





(wiped it away) & considered 0.

$u(t-a)\,f(t-a) \xrightarrow[\mathcal{L}]{} e^{-as}\,F(s)$

If our function is something like $\sin t$, $\sin^2 t$, and any polynomial, we can't do

$$f(t-a),$$

so,

$$u(t-a)(f(t)) \rightsquigarrow e^{-as} \mathcal{L}(f(t+a))$$

we have made a general agreement that we start observing our system at $t=0$. But perhaps the action actually begins a little later! So perhaps the music starts to play not at $t=0$, but rather at $t=a$.

There are two ways this might happen.

1) perhaps the music was playing all along, but we turned on the amplifier at $t=1$. If $\sin(wt)$ represents the sound (pressure) wat at our ear, what we hear is

$$u(t-1)\sin wt.$$

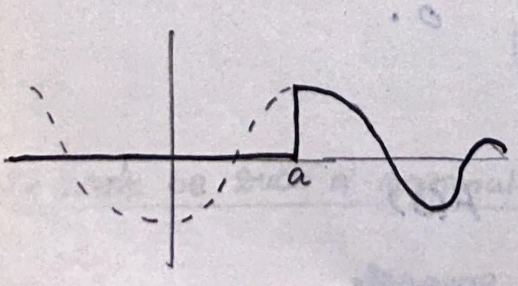
2) on the other hand, perhaps the band really only began to play at time $t=1$. Then what we hear is modeled by

$$u(t-1)\sin(w(t-1))$$

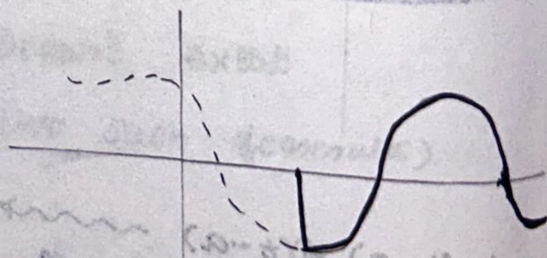
In the second scenario, we say that the signal $\sin(wt)$ has been delayed by 1 second.

$u(t-a) f(t)$ is $f(t)$ - restricted to the window $t \geq a$

$u(t-a) f(t-a)$ is $f(t)$ - delayed by a time units.



Restricted to $t > a$



Delayed by a units.

The graph of $u(t-a)f(t-a)$ is the graph of the $f(t)$ dragged to the right by a units with zero filling in behind.

$$u(t-a)f(t-a) \rightsquigarrow \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt$$

since $u(t-a)=0$ for $t<a$, we can use $t=a$ for the lower limit & drop the factor $u(t-a)$

$$u(t-a)f(t-a) \rightsquigarrow \int_a^\infty f(t-a)\,e^{-st}\,dt$$

∴ The integral calls for the change of variables.

$$\tau = t-a, \quad t=\tau+a, \quad dt=d\tau$$

so the lower limit of τ is now 0, and

$$u(t-a)f(t-a) \rightsquigarrow \int_0^\infty f(\tau)\,e^{-s(\tau+a)}\,d\tau$$

$$u(t-a)f(t-a) \rightsquigarrow e^{-as}\int_0^\infty f(\tau)\,e^{-s\tau}\,d\tau$$

The integral is over τ, and s is constant, so we can pull out the factor e^{-as}.

∴ <u>t-shift rule:</u>

$$u(t-a)f(t-a) \rightsquigarrow e^{-as}\cdot F(s)$$

∴ Now we can turn of our attention to finding the Laplace transform of $u(t-a)f(t)$. We do this by applying the t-shift formula to the function $g(t)$ defined by

$$g(t)=f(t+a), \quad g(t-a)=f(t)$$

↳ left shift by a

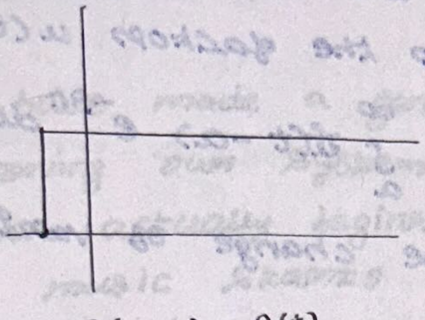
↳ right shift by a

Equilibrium position.

$$u(t-a)\, f(t) = u(t-a)\, g(t-a) \rightsquigarrow e^{-as}\, \mathcal{L}(g(t); s)$$

$$= e^{-as}\, \mathcal{L}(f(t+a); s)$$

$\therefore\ f(t+a) = g(t)$



$f(t+a) = g(t)$

Let

~~[illegible]~~

$$u(t-a)\, f(t) \rightsquigarrow e^{-as}\, \mathcal{L}(f(t+a))$$

$\therefore$

$$u(t-a)\, g(t-a) \rightsquigarrow e^{-as}\, \mathcal{L}(g(t); s).$$

This is the second form of the t-shift rule.

The first form expressed

$$\mathcal{L}(u(t-a)\, f(t-a)) = F(s) \cdot e^{-as}$$

unfortunately, there is no such expression for $\mathcal{L}(u(t-a) f(t); s)$. So we must express it $e^{-as}\, \mathcal{L}(f(t+a); s)$

For 1st expression:

I need $F(s)$. But after Laplace transform, I completely lost past values. So it's not applicable.

---

1) $f(t) = 1$.

when $f(t) = 1$

$f(t+1) = 1$.

Solu:

$$u(t-a) \rightsquigarrow e^{-as}\, \mathcal{L}(1; s)$$

$$\rightsquigarrow e^{-as} \cdot \frac{1}{s}$$

$$\rightsquigarrow \frac{e^{-as}}{s}.$$

---

2) $f(t) = \cos(\omega t)$ and $a > 0$. Then

Solu:

$$u(t-a)\cos(\omega(t-a)) \rightsquigarrow e^{-as}\mathcal{L}(\cos\omega t\,;s)$$

$$= \frac{s e^{-as}}{s^2+\omega^2}.$$

5.3) $\mathcal{L}(u(t-a)\cos\omega t\,;s)$

Let $\phi = -\omega a$

Solu:

$$\cos(\omega(t+a)) = \cos(\omega t+\omega a)$$

$$= \cos(\omega t - \phi)$$

$$= \cos(\phi)\cos(\omega t) + \sin(\phi)\sin(\omega t)$$

$$u(t-a)\cos(\omega t) \rightsquigarrow e^{-as}\mathcal{L}(\cos(\omega(t+a))\,;s)$$

$$= e^{-as}\mathcal{L}\big((\cos(\phi)\cos\omega t) + \sin\phi\,\sin\omega t)\,;s\big)$$

$$= e^{-as}\,\frac{\cos(\phi)\,s + \sin\phi)\,\omega}{s^2+\omega^2}.$$

Summary:

we have t-shift rule expressed in two ways.

$$u(t-a)f(t-a) \rightsquigarrow e^{-as}F(s)$$

$$u(t-a)f(t) \rightsquigarrow e^{-as}\mathcal{L}(f(t+a))$$

Note: 1: This is the first function in the frequency domain that's not a rational function that we have encountered

Note: 2: e^{-as} has neither zeros nor poles, for any values of a. So multiplying by it doesn't change the pole diagram at all. As far as the pole diagram of the Laplace transform is concerned, the long term behaviour of $f(t)$ is just the same as that of its shift

$$u(t-a)f(t-a).$$

Repeat

$$g(t) = f(t+a), \quad g(t-a) = f(t)$$

$u(t-a) \cdot f(t) = u(t-a)\, g(t-a) \rightsquigarrow e^{-as} \mathcal{L}(g(t); s)$

$= e^{-as} \mathcal{L}(f(t+a); s)$

Proof:

$\int_0^{\infty} e^{-st}\, u(t-a)\, f(t-a)\, dt$

$t_1 = t - a$

$\boxed{t = t_1 + a}$

when,

$t_1 = t - a$

when

$t = 0,\ t_1 = -a$

$t = \infty,\ t_1 = \infty$

solu:

$= \int_{-a}^{\infty} e^{-s(t_1+a)} \cdot u(t_1)\, f(t_1)\, dt_1$

$= \int_{-a}^{\infty} e^{-st_1} \cdot e^{-sa} \cdot u(t_1)\, f(t_1)\, dt_1$

$= \left[\int_{-a}^{\infty} e^{-st_1}\, u(t_1) \cdot f(t_1)\, dt\right] \cdot e^{-sa}$

$u(t) = 1$ for +ve values

$u(t) = 0$ for $t_1 < 0$.

$= e^{-as} \int_0^{\infty} e^{-st_1}\, f(t_1)\, dt$

$= e^{-as}\, \mathcal{L}(f(t_1); s)$

$= e^{-as}\, F(s) \quad \rightarrow$ Doesn't exist.

ii) $u(t-a)\, f(t-a) \rightsquigarrow e^{-as}\, \mathcal{L}(f(t)) \quad \leftarrow$ Replace t by $t+a$

$u(t-a)\, f(t)$

$\parallel$

$u(t-a)\, f(t-a+a) \rightsquigarrow e^{-as}\, \mathcal{L}(f(t+a)) \rightarrow$ Changing the input signal accessing output.

$u(t-a)\, f(t) \rightsquigarrow e^{-as}\, \mathcal{L}(f(t+a))$

Find the Laplace transform of

$u(t-a)\,(t-a)^2$

solu:

Replace $t_1 = t-a$

$u(t)\, t^n \rightsquigarrow \frac{n!}{s^{n+1}}$

$u(t-a)\,(t-a)^2 \rightsquigarrow ?$

$u(t-a)\, f(t-a) \rightsquigarrow e^{-as}\, F(s)$

$u(t-a)\, f(t-a) \rightsquigarrow e^{-as}\left(\frac{2}{s^3}\right)$

$u(t-a)t^2 \rightsquigarrow e^{-as} \mathcal{L}(f(t+a);s)$

$e^{-as} \mathcal{L}((t+a)^2;s)$

$e^{-as} \mathcal{L}(t^2+2at+a^2;s)$

$e^{-as}\left(\frac{2}{s^3}+\frac{2a}{s^2}+\frac{a^2}{s}\right)$

what works? what not?

$u(t-a)f(t-a) \rightsquigarrow e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau$

There is not any expression as answer

$\mathcal{L}(u(t-a)f(t);s)$

↳ It involves past values not exist.

exist:

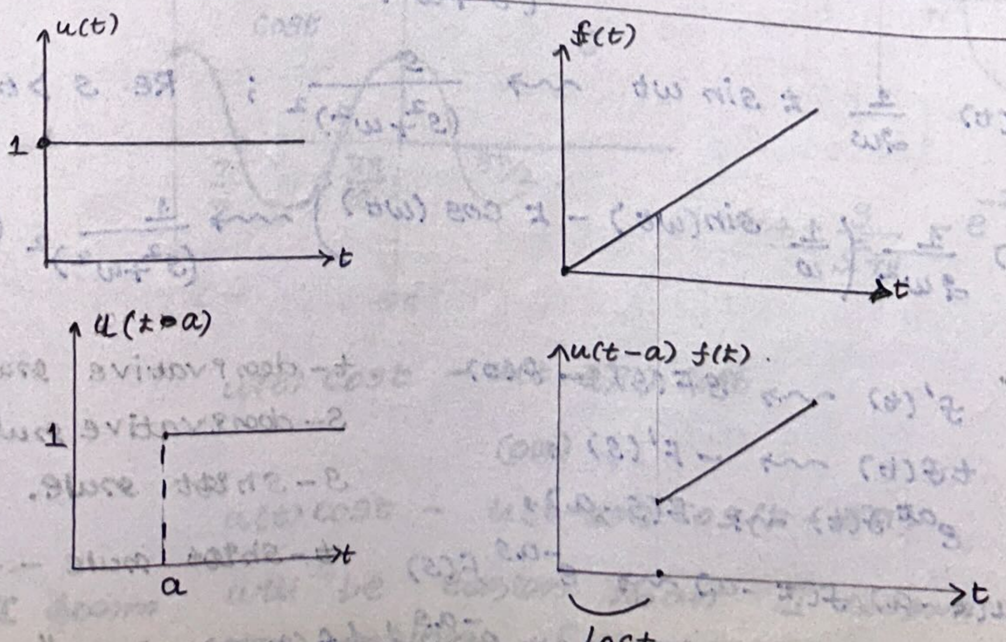
$u(t-a)f(t-a) \rightsquigarrow e^{-as}F(s)$

$u(t-a)f(t) \rightsquigarrow e^{-as}\mathcal{L}(f(t+a);s)$

No such answer like

some function $\rightsquigarrow \mathcal{L}(u(t-a)f(t);s)$

↳ Involves past values.



Examples

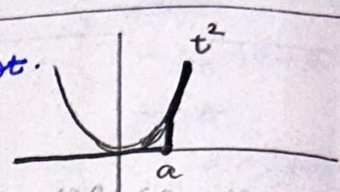
$\therefore \mathcal{L}(u(t)) = \frac{1}{s}$

1) $u_{ab}(t) = u(t-a) - u(t-b)$

$$\overset{\leadsto}{\mathcal{L}} \; e^{-as} \cdot \frac{1}{s} - e^{-bs}\left(\frac{1}{s}\right) \rightarrow \text{t-shift formula.}$$

$$= \frac{e^{-as} - e^{-bs}}{s}$$

$u(t-1)\,t^2 \rightarrow$ II form of t-shift.



$u(t-a)\cdot f(t)$

solu:

$$e^{-as}\left(\frac{2}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}\right)$$

---

calc:

$u(t) \leadsto 1/s \quad \operatorname{Re} s > 0$

$u(t)\, e^{\gamma t} \leadsto \frac{1}{s-\gamma} \quad \operatorname{Re} s > \operatorname{Re} \gamma$

$u(t) \cos \omega t \leadsto \frac{s}{s^2+\omega^2} \quad \operatorname{Re} s > 0$

$u(t) \sin \omega t \leadsto \frac{\omega}{s^2+\omega^2}, \quad \operatorname{Re} s > 0$

$u(t)\, t \leadsto \frac{1}{s^2} \quad \operatorname{Re} s > 0$

$u(t)\, t^n \leadsto \frac{n!}{s^{n+1}} \quad \operatorname{Re} s > 0$

$u(t)\; t \sin(\omega t) \leadsto \frac{2\omega s}{(s^2+\omega^2)^2} \quad \operatorname{Re} s > 0$

$u(t)\; t\cos(\omega t) \leadsto \frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \quad \operatorname{Re} s > 0$

$u(t)\; \frac{1}{2\omega}\, t \sin \omega t \leadsto \frac{s}{(s^2+\omega^2)^2}; \quad \operatorname{Re} s > 0$

$u(t)\; \frac{1}{2\omega^2}\left(\frac{1}{\omega}\sin(\omega t) - t\cos(\omega t)\right) \leadsto \frac{1}{(s^2+\omega^2)^2} \quad (\operatorname{Re} s > 0)$

Rules:

$f'(t) \leadsto sF(s) - f(0)$ t-derivative rule

$t f(t) \leadsto -F'(s)$,   s-derivative rule

$e^{at} f(t) \leadsto F(s-a)$,   s-shift rule.

$u(t-a)\, f(t-a) \leadsto e^{-as} F(s)$ t-shift rule - I form

$u(t-a)\, f(t) \leadsto e^{-as} \mathcal{L}(f(t+a); s)$ " II form

$\mathcal{L}(u(t-3)t \,;\, s)$

Soln: $\rightsquigarrow e^{-3s}\,\mathcal{L}(f(t+3)\,;\,s) = e^{-3s}\,\mathcal{L}((t+3)\,;\,s)$

$$= e^{-3s}\left(\frac{1}{s^2}+\frac{3}{s}\right)$$

$\mathcal{L}(f)$ for $f(t) = \begin{cases} \cos t & 0 < t < 2\pi \\ 0 & t > 2\pi \end{cases}$

Soln:

$$= \int_0^{2\pi} \cos t \cdot e^{-st}\,dt + \int_{2\pi}^{\infty} 0 \cdot e^{-st}\,dt$$

last $\rightsquigarrow \left(\frac{s}{s^2+1}\right)$ $\qquad \therefore \cos\omega t = \left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)$

$$= \int_0^{2\pi} \cos t \cdot e^{-st}\,dt$$

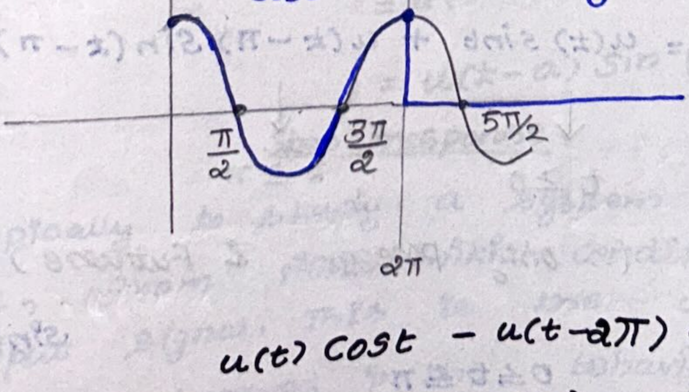
$$= \frac{1}{2}\left(\frac{e^{(i\omega t - st)}}{i\omega - s} + \frac{e^{(-i\omega t - st)}}{-i\omega - s}\right)_0^{2\pi}$$

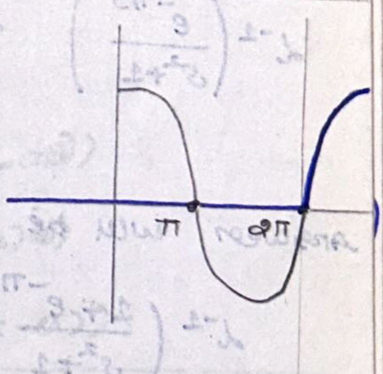
$$= \frac{1}{2}\left(\frac{e^{(i\omega - s)2\pi}}{i\omega - s} + \frac{e^{(-i\omega - s)2\pi}}{-i\omega - s} - \frac{1}{i\omega - s} - \frac{1}{-i\omega - s}\right)$$

$$= \frac{s}{s^2+1} - e^{-2\pi s}\,\frac{s}{s^2+1}$$

$f(t) = (u(t) - u(t-2\pi))\cos(t)$

$= u(t)\cos t - u(t-2\pi)\cos(t-2\pi)$





Ans: $\frac{s}{s^2+1} - e^{-2\pi s}\left(\frac{s}{s^2+1}\right)$

$u(t)\cos t - u(t-2\pi)\cos t$

(or)

$u(t)\cos t - u(t-2\pi)\cos(t-2\pi)$

$\therefore$ I form will be easier than II form for cosines (Also $\cos t = \cos(t-2\pi)$)

FYI theorem:

Every polynomial can be written in terms of powers of $t-1$ (or a in order to be more general)

t^2 in terms of $(t-1)^2$ will be

$$t^2 = ((t-1)+1)^2$$

Binomial theorem:

$$(x+y)^2 = x^2 + 2xy + y^2$$

Let $x = (t-1)$
$y = 1$

$$t^2 = ((t-1)+1)^2 = (t-1)^2 + 2(t-1) + 1$$

So Laplace made easy!

FYI - For your Information

Troubles:

$$\mathcal{L}^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right)$$

$\frac{1}{s^2+1} \rightsquigarrow \sin(t).u(t)$

Solu:

$$\mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) \rightsquigarrow u(t-\pi)\sin(t-\pi)$$

$u(t-a)f(t) \rightsquigarrow e^{-as}\,\mathcal{L}(f(t+a);s)$

$u(t-a)f(t-a) \rightsquigarrow e^{-as}F(s)$

Answer will be

$$\mathcal{L}^{-1}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = \underbrace{u(t)\sin t}_{t \geq 0} + \underbrace{u(t-\pi).\sin(t-\pi)}_{t \geq \pi}$$

(∴ only present & future)

$\sin(t-\pi) = -\sin t$

$$f(t) = \begin{cases} \sin t & 0 \leq t \leq \pi \\ \sin t + \sin(t-\pi) = \sin t - \sin(t) = 0 & t \geq \pi. \end{cases}$$

(Inverse answer)

$$f(t) = \begin{cases} \sin t & 0 \le t \le \pi \\ 0 & t \ge \pi. \end{cases}$$ Jump discontinuities occurs at $t = \pi$.

whenever $t < \pi$, the Laplace transform of $u(t-\pi)\sin(t-\pi)$ doesn't exist.

Integration → will make functions more continuous

Differenciation → more discontinuous.

In $t < \pi$, only the first part is operational.

when $t = \pi$ ① → $= \sin \pi$ $0 \le t \le \pi$

$= 0$

② → 0 $t \ge \pi$

(Both cases match at $t = \pi$)

$$\mathcal{L}^{-1}\left(\frac{e^{-as}}{\frac{s^2}{\omega} + \omega}\right)$$

Solu.

$$\mathcal{L}^{-1}\left(\frac{e^{-as}}{\frac{1}{\omega}(s^2+\omega^2)}\right) = \mathcal{L}^{-1}\left(\omega\left(\frac{e^{-as}}{s^2+\omega^2}\right)\right)$$

$$= \mathcal{L}^{-1}\left(e^{-as}\left(\frac{\omega}{s^2+\omega^2}\right)\right)$$

$f(t) = \sin\omega t$

$f(t-a) = \sin\omega(t-a)$

$$= u(t-a).(\sin\omega t - a\omega)$$

$$= u(t-a)\,(\sin(t-a)\omega)$$

$$= u(t-a)\sin(\omega(t-a))$$

Step response

It's typically to study a system by studing its response, from rest initial conditions, to a unit step input signal. This is the unit step response. In this, we observed in the behaviour of the mascot: The shift shaft drive moved abruptly to a different position, where it stayed, and the system osciuated in response.

In terms of block diagrams,

$$u(t) \longrightarrow \boxed{\text{system}} \longrightarrow \text{system response.}$$

we can write it in terms of frequency domain.

$u(t) \rightsquigarrow \frac{1}{s}$. If $H(s)$ is the system function, then the Laplace transform of the output response is $\frac{H(s)}{s}$.

$$\xrightarrow{1/s} \boxed{H(s)} \xrightarrow{H(s)/s}$$

Q.1: Suppose you get a job at $t=0$. Your salary is \$S /month, and you have a total of \$E expenses & taxes each month.

Net income = \$S − E = A/month.

Fortunately, S > E. So you open a bank account & your employer deposits money directly into the account at a rate of A.

The diff eqn modelling your bank account with interest rate r is

$$\dot{x} = A \cdot u(t) - rx$$

$$\dot{x} + rx = A \cdot u(t)$$

$u(t)$ – step unit function. (Your employer is kind enough to continually put money into your account at this rate).

At $t=0$ (The bank account is empty).

Solu:

$$sX(s) + rX(s) = \frac{A}{s}$$

$$X(s)\,[s-r] = \frac{A}{s}$$

$$X = \frac{A}{s(s-r)}$$

$$X = \frac{A}{s(s-r)} = \frac{a}{s} + \frac{b}{s-r}$$

$$\frac{A}{s-r} = \frac{a+bs}{s-r} \qquad s=0.$$

$$a = \frac{A}{s-r}$$

$$a = -\frac{A}{r} \rightarrow \text{coverup } a$$

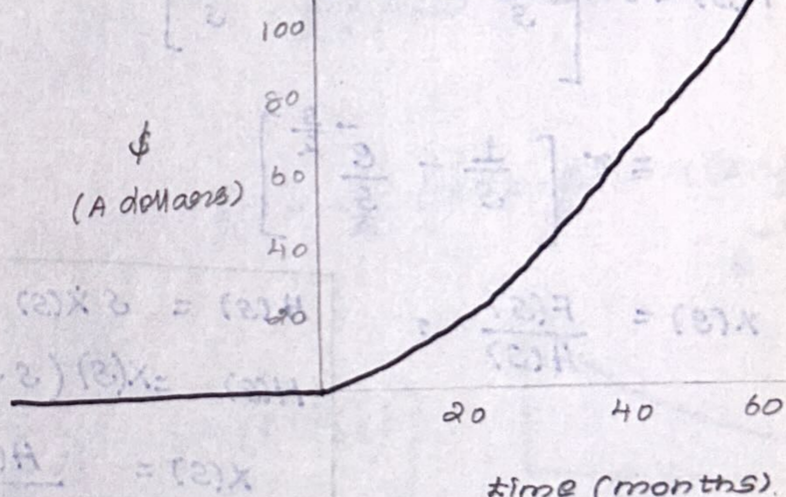
$$\frac{A}{s} = \frac{a(s-r)+b}{s} \qquad s-r=0$$
$$s=r$$

$$b = \frac{A}{s}$$

$$\boxed{b=\frac{A}{r}} \rightarrow \text{coverup } b.$$

Taking the inverse Laplace transform, and using the s-shift rule on inverse table look up, we get.

$$x(t) = Au(t)\left(-\frac{1}{r} + \frac{1}{r}e^{rt}\right)$$



Step rate $r=0.01$

Step response

The fish population in a lake is not reproducing fast enough and the population is decaying exponentially with decay rate k. A program is started at $t=0$, to stock the lake with fish at a constant rate of r units of fish/year. Unfortunately after $t=1/2$ year the funding is cut & the program ends.

In the series of questions that follow, we model this situation & solve the resulting DE for the fish population as a function of time.

$x(t)$ - Fish population

$x(0^-) = A$ (Initial population)

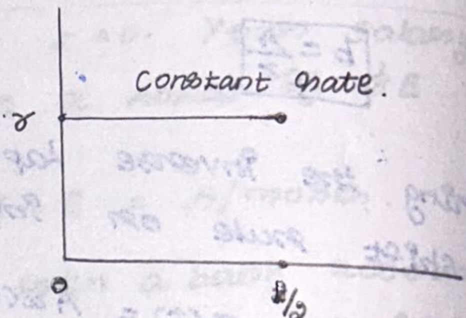
Since the fish population is decaying exponentially with rate k, the population is modeled by

$$\dot{x} + kx = f(t), \quad x(0^-) = A$$

what's the input signal?

$$f(t) = \begin{cases} r & 0 < t < 1/2 \\ 0 & t < 0, \; 1/2 < t. \end{cases}$$

$$f(t) = r\left[u(t) - u(t - 1/2)\right]$$



By Laplace transform:

$$F(s) = r\left[\frac{1}{s} - e^{-\frac{1}{2}s} \cdot \frac{1}{s}\right]$$

$$= r\left[\frac{1}{s} - \frac{e^{-\frac{s}{2}}}{s}\right]$$

$$X(s) = \frac{F(s)}{H(s)}, \quad H(s) = s\,X(s) + K\,X(s)$$

$$H(s) = X(s)(s+K)$$

$$X(s) = \frac{H(s)}{(s+K)}$$

(or) $H(s) = X(s)(s+K)$

$$(X(s))^2 = \frac{\frac{r}{s}\left(1 - e^{-s/2}\right)}{(s+K)}$$

$$-x(0) + s \cdot X(s) + K\,X(s) = F(s)$$

$$-A + s.X(s) + KX(s) = \frac{r}{s}\left(1 - e^{-s/2}\right)$$

$$X(s) = \frac{\left(\frac{r}{s}\left(1 - e^{-s/2}\right) + A\right)}{(s+K)}$$

$$x(s) = \frac{A}{s+K} + \frac{r}{s(s+K)}\left(1-e^{-s/2}\right)$$

$$= \frac{A}{s+K} + \frac{r}{s(s+K)} - \frac{r e^{-s/2}}{s(s+K)}$$

$$= Ae^{-Kt}\cdot u(t) + \mathcal{L}^{-1}\left(\frac{r}{s(s+K)}\right) - \mathcal{L}^{-1}\left(\frac{r e^{-s/2}}{s(s+K)}\right)$$

$$\mathcal{L}^{-1}\left(\frac{r}{s(s+K)}\right) \rightsquigarrow ?$$

$$\frac{r}{s(s+K)} = \frac{a}{s} + \frac{b}{s+K}$$

$$\frac{r}{s+K} = a$$

$$\text{so } \frac{r}{K} = a$$

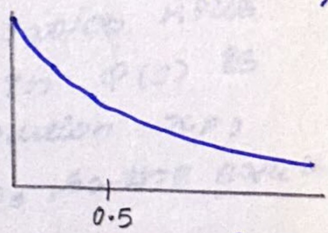
$$\frac{r}{s} = b$$

$$b = -\frac{r}{K}$$

$$s+K=0$$
$$s=-K$$

$$\mathcal{L}^{-1}\left(\frac{r}{Ks} - \frac{r}{K(s+K)}\right) = \frac{r}{K}\left(u(t) - u(t)e^{-Kt}\right)$$

$$\mathcal{L}^{-1}\left(\frac{r e^{-s/2}}{s(s+K)}\right) \rightsquigarrow u(t-1/2)\cdot\frac{r}{K}\left(u\left(t-\frac{1}{2}\right) - u\left(t-\frac{1}{2}\right)e^{-K(t-\frac{1}{2})}\right)$$



$$= u(t-1/2)\left(\frac{r}{K}\left(1-e^{-K(t-1/2)}\right)\right)$$

$$\therefore u(t) = (u(t))^2.$$

$$x(t) = Ae^{-Kt} + \frac{r}{K}\left(1-e^{-Kt}\right) - \frac{r}{K}u\left(t-\frac{1}{2}\right)\left(1-e^{-K(t-\frac{1}{2})}\right)$$

$$= \begin{cases} Ae^{-Kt} + \frac{r}{K}\left(1-e^{-Kt}\right) & \text{for } 0 < t < 1/2 \\ Ae^{-Kt} + \frac{r}{K}\left(e^{-Kt} + e^{-K(t-\frac{1}{2})}\right) & \frac{1}{2} < t. \end{cases}$$

when $t > 1/2$

$$u(t-1/2) = 1.$$

=Ans.

Spring system: (Res * Initial condn)

$m=2,\ b=4,\ K=20.$

$m\ddot{x}+b\dot{x}+Kx=f(t)$

$(ms^2+bs+K)\,X(s)=KF(s)$

$(2s^2+4s+20)\,X(s)=KF(s)$

$$X(s)=\frac{20F(s)}{2s^2+4s+20}$$

$$=\frac{\mathcal{L}(u(t))\times 20}{2s^2+4s+20}$$

$$=\frac{20}{s(2s^2+4s+20)} = \frac{a}{s}+\frac{b(s+1)+c}{2s^2+4s+20}$$

$(s=0)$

$$a=\frac{1}{2s^2+4s+20}$$

$$a=\frac{1}{20}$$

$\cancel{bs+c=\frac{1}{s}}$

$2s^2+4s+20=$

$s^2+2s+10=0$

10

2

$$=\frac{a}{s}+\frac{b(s+1)+c}{(s+1)^2+9}$$

$(s+1)^2=-9$

$s+1=3i$

$s=-1+3i$

$$b(s+1)+c=\frac{1}{s}$$

$$b(3i)+c=\frac{1}{-1+3i}$$

$$20(3Bi+c)=-1-3i$$

$$B=-1/20$$

$$c=-1/20$$

$$\therefore\ 20\,\frac{B(s+1)+c}{(s+1)^2+9}=-\frac{s+1}{(s+1)^2+9}+\frac{-1}{(s+1)^2+9}$$

Taking inverse Laplace

$u(t) \rightsquigarrow \frac{1}{s}$

$u(t)\, e^{-t} \cos 3t \rightsquigarrow \frac{s+1}{(s+1)^2+9}$

$u(t)\, e^{-t} \frac{\sin 3t}{3} \rightsquigarrow \frac{1}{(s+1)^2+9}$

$$x(t) = u(t) \left(1 - e^{-t} \cos 3t - \frac{e^{-t}}{3} \sin 3t\right)$$

For $t > 0$

$$x(t) = 1 - e^{-t} \cos(3t) - \frac{e^{-t}}{3} \sin(3t).$$

The steady state is easy:

1) with out laplace transform, we used to find transients by

$P(D)x = Q(D)y$ (The term 1 is the steady state solution, and the damped sinusoid is the transient required to produce rest initial condt. The steady state response to a unit step input signal is constant) → modeled by

$$P(D)x = Q(D)y.$$

we are interested in $y(t) = 1$. → response ($u(t)$) Differenciation kills this function, and the constant term in $Q(s)$ is $Q(0)$. If we look for a constant solution x_p, the derivatives in $P(D)$ will kill it too, so the equ is

$$P(0)x_p = Q(0)$$

$$x_p = \frac{Q(0)}{P(0)}.$$

with laplace transform, with $y = u(t)$, we get

$$P(s)X = \frac{Q(s)}{s}$$

$$X = \frac{Q(s)}{s\,P(s)}$$

Apply partial fractions:

$$X = \frac{a}{s} + \frac{R(s)}{P(s)} \quad \text{using Coverup}$$

$$a = \frac{q(0)}{p(0)}.$$

Impulse & Impulse response

1) Incorporate the delta function into a description of signals.

2) Graph generalized functions.

3) Describe & compute the unit impulse response of an LTI system using laplace transform.

4) Distinguish b/w pre & post- initial conditions, & identify post- initial condit of the unit impulse response.

Derivative of unit step

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$u(t) \rightarrow$ non differenciable at $t = 0$; it's not even continuous there. But we need to talk about its derivative

Solu:

Case: 1:

Bank account modeled by

$$\dot{x} - Ix = r(t)$$

I - Interest rate; $r(t)$ — Net rate of deposit. one day you deposit a large amount of money. This total deposit is best modeled by a step function, Thus the deposit rate $r(t)$ is modeled by a (unit) step function's derivative.

2) Suppose we give a soccer ball a good swift kick. Its velocity increases abruptly from $v = 0$ to a +ve value v_0. The velocity can be modeled by $v(t) = v_0 u(t)$. what we can say about the force applied by my foot?

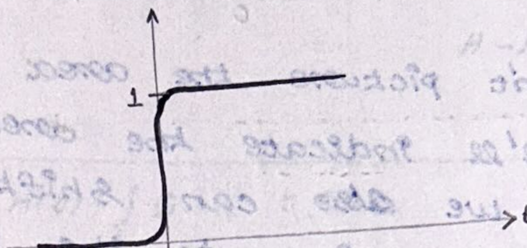
Force is the time rate of change of momentum. momentum is mass times velocity.

$mv(t) = mv_0 u(t)$.

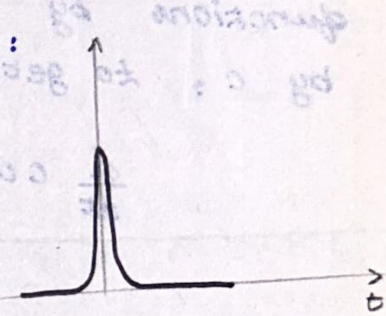
So force is the derivative of $mv_0\, u(t)$.

While the ideal step function has no derivative, our treatment of the step function lets us deal with its derivative.

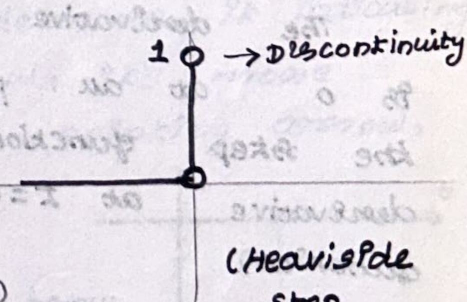
We saw that in reality the unit step function is an idealization of a smooth function that increases in value from 0 to 1 in a very short period of time. Viewed in that light, the derivative of $u(t)$ is easy to picture. Expanding the t-axis so we can able to see how $u(t)$ increases in value from 0 to 1, we get a graph like this:



The derivative curve look like this:



$\therefore$ $u(t)$ increases in value by 1, the area under the graph $\dot{u}(t)$ is 1, It's very narrow tall spike. The same kind of idealization that we used to replace the smooth function of $u(t)$ with the heaviside step function leads to Dirac-delta function.



Heaviside functions can only take values of 0 and 1.

(Discontinuous function)

(Heaviside step function)

<u>Dirac delta function:</u>

δ function → Introduced by paul Dirac

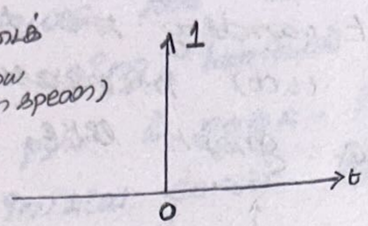
Used to model the density of an idealized point mass or charge as a function equal to zero everywhere except for zero & whose integral over the entire real line is equal to 1.

$$\delta(t) = \dot{u}(t)$$

The delta function is secretly just a smooth function which takes on non-zero values only in a neighbourhood of 0 that's small for us (too small) to resolve & has total integral 1.

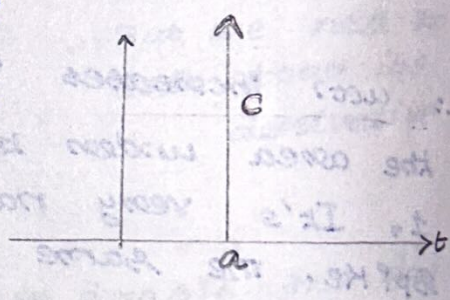
When we graph, we will denote the delta function by a 'harpoon'

(திமிங்கல வேட்டைக் கருவி - arrow with spear)



Since we can't picture the area under the harpoon, we'll indicate the area by a number next to it. We also can shift the delta functions by a units to the right & scale it by c, to get

$$\frac{d}{dt} c\,u(t-a) = c\,\dot{u}(t-a)$$
$$= c\,\delta(t-a).$$

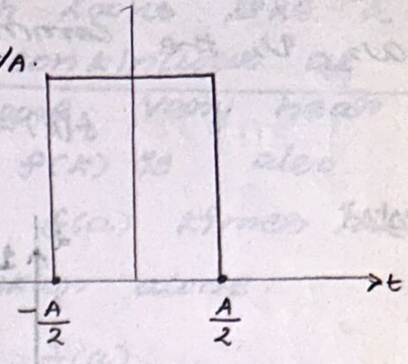


The derivative of unit step (or) step function is 0 at all places except at $t=0$. At $t=0$, the step function is discontinuous. So its derivative at $t=0$ is infinite. (The impulse function)

Impulse:

Special function often used by certain engineers to model certain events. An impulse function is not realizable, in that by definition the output of an impulse function is infinity at certain values. An impulse function is also known as 'delta function', although there are different types of delta functions that each have slightly different properties. Specifically this unit-impulse function is known as the Dirac term 'impulse' is unambiguous because there is only one definition of the term impulse'.

$$D(t) = \frac{1}{A}\left[u(t+A/2) - u(t-A/2)\right]$$



$$\delta(t) = \lim_{A\to 0} \frac{1}{A}\left[u\left(t+\frac{A}{2}\right) - u\left(t-\frac{A}{2}\right)\right]$$

1) $\delta(t) = 0$ for $t \neq 0$

2) $\delta(t) = +\infty$ for $t = 0$

3) $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

H - Heaviside step function.

$$H(t) = \begin{cases} 0 & ; \ t < 0 \\ 1 & ; \ t \geq 0 \end{cases}$$

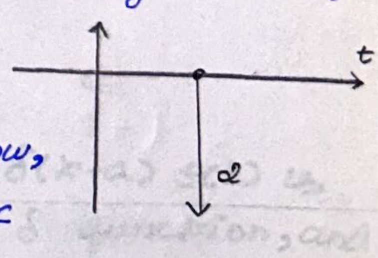
$$\int_{-\infty}^{\infty} \delta(t)\,dt = H(t)\Big]_{-\infty}^{\infty} = 1.$$

Conventions

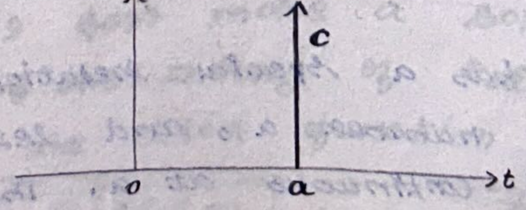
$-2\,\delta(t-1)$

solu:

while you might decide to draw a upwards arrow at $t=1$ with a -2 decorate it indicating a value of -2 for the integral, it's more common to draw a downwards pointing arrow, decorated with a positive 2.



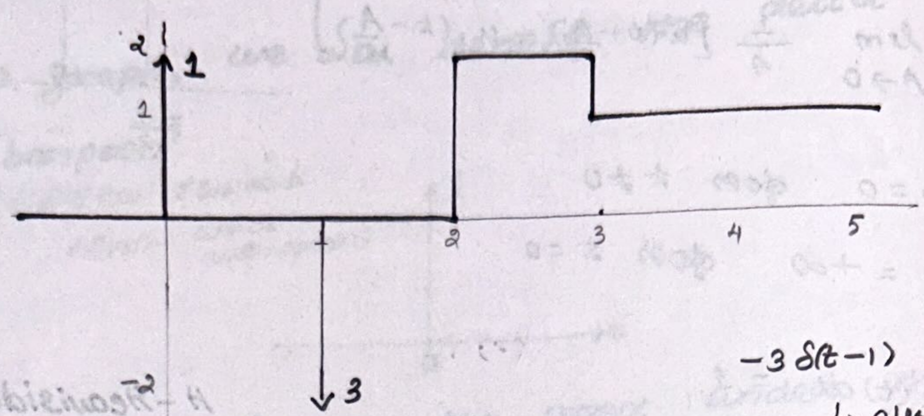
However, note that in the image below, we don't require that the constant c be positive.



Instead, we allow the possibility that c be negative, but keep the convention of drawing the delta function pointing upwards. If we know that c is (-) ve, we instead draw the δ function putting downwards as is the common convention.

$$f(t) = \delta(t) - 3\delta(t-1) + 2u(t-2) - u(t-3)$$

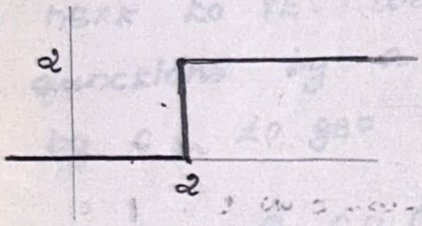
solu:



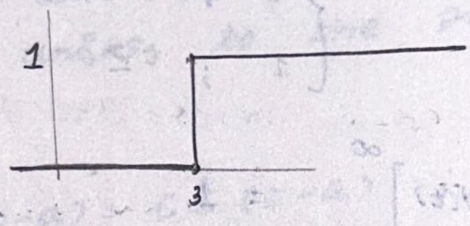
$-3\,\delta(t-1)$

↳ shifted by 1 right & ↑ by 3. downwards

$2u(t-2)$



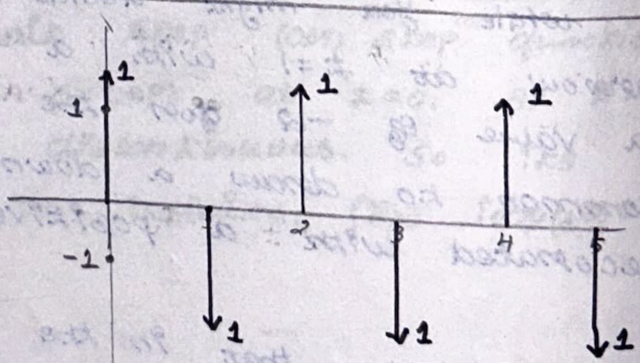
$u(t-3)$



Derivative of the square wave

$$u(t) = u(t-1) + u(t-2) - u(t-3) + \cdots$$

Ans: $\delta(t) - \delta(t-1) + \delta(t-2) - \delta(t-3) + \cdots$



Integral of the delta function

The δ function has a special relationship with integration. Fix a number a, and let f(t) be any function that's continuous at a. Then for any numbers b and c such that $b < a < c$

$$\int_b^c \delta(t-a)\, f(t)\, dt = f(a)$$

To see this just remember that $\delta(t-a)$ is zero except very near to a, and near to a it becomes very large with area 1. Its graph looks like a very tall rectangle. Since $f(t)$ is continuous at a, the values of $f(t)$ near a are very near to $f(a)$. So the graph of $\delta(t-a)\,f(t)$ is also very narrow rectangle that is $f(a)$ times taller than the one for the delta function alone.

Hence $\delta(t-a) \cdot f(t)$ is $f(a)$.

eg:

$$\int_{-\infty}^{\infty} \delta(t-a)\,f(t)\,dt = f(a)$$

$$\int_{a-}^{a+} \delta(t-a)\,f(t)\,dt = f(a).$$

These formulas reflect the actual practice of measuring the value of a function. When you measure voltage, or current, or position, or temperature, you can't make an instantaneous measurement. Your instrument accumulates the effect of the quantity you are measuring over some small time interval - micro seconds if your time scale is in seconds & averages over that small interval. This averaging process, heavily weighting the values near $t=a$, is precisely the process of forming the integral.

$$\int_{b}^{c} \delta(t-a)\,f(t)\,dt$$

This expression for the integral of $\delta(t-a)\,f(t)$ is a characteristic property of the δ function, and is the basis for more a sophisticated development of the theory of "distributions", which covers the delta function & much more.

$\delta(t) = 0 \;, 1 < t < \infty$

$$\int_{1}^{\infty} \delta(t)\,f(t)\,dt = 0$$

(since $\delta(t)$ has value only at $t=0$)

$t \neq 0$ means $\delta(t) = 0$

$$\int_{-\infty}^{\infty} \delta(t) \cos(\omega t) = \cos 0 = \underline{1}$$

$$\int_{-\infty}^{\infty} \delta(t) \tan(t+\pi/4) \left(\frac{e^{-3t}}{1+t^2}\right) dt = \tan\left(\frac{\pi}{4}\right)\left(\frac{1}{1}\right)$$
$$= \tan(\pi/4) = 1$$

$$\int_{-\infty}^{\infty} \delta(t-\pi)\, e^{t^2-5} \cos(5t-1)\, dt = e^{\pi^2-5} \cos(5\pi-1)$$

$$\int_{-1}^{1} (\delta(t) + \delta(t+1/2) + \delta(t-1/2))(e^{t} \cdot t)\, dt$$

$$= 0 + \int_{-1}^{1} \delta(t+1/2)\, e^{t} \cdot t\, dt + \int_{-1}^{1} \delta(t-1/2)\, e^{t} \cdot t\, dt$$

$$= e^{-1/2} \cdot \left(-\frac{1}{2}\right) + e^{1/2} \cdot 1/2$$

---

2)
$$\int_{0^-}^{5} \delta(t) \arctan(t+1)\, dt = \arctan(1)$$
$$= \pi/4$$

$$0^- \longrightarrow 0 \longrightarrow 0^+$$

$$\int_{3-}^{3+} t^3 (\delta(t) + \delta(t-1)\cos\pi t + \delta(t-2)\cos(2\pi t) + \dots$$
$$+ \delta(t-k)\cos(k\pi t) + \dots)dt$$

$= \delta(t-3)(t^3 \cos 3\pi t)$ is the only surviving term.

$$= (+27)\cos(9\pi)$$

$$= 27(-1) = -27.$$

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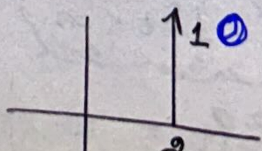
a) $\int_{0^-}^{\infty} \delta(t)\, e^{t^2}\, dt$

$\boxed{e^0 = 1.}$

b) $\int_{0^-}^{1} \delta(t-2)\, e^{t^2} \sin t \cos 2t$



$\left[\because \delta(t-2) = 0 \text{ at } t \neq 2\right]$