$$\int \frac{dy}{y(3-y)} = \int dt \qquad y \neq 0 \ \& \ y \neq 3$$

Evaluate the integral on the left using partial fraction omitting the constant of integration.

$$\int \frac{dy}{y(3-y)} = \int \left(\frac{1/3}{y} + \frac{1/3}{3-y} \right) dy$$

$$\frac{1}{y(3-y)} = \frac{a}{y} + \frac{b}{3-y}$$

$$1 = a(3-y) + by$$

$$3a - ay + by = 1 \qquad \Rightarrow \quad 3a = 1, \ a = 1/3 \quad \Bigg| \quad b - a = 0, \ b = a$$

$$3a + y(b-a) = 1$$

$$= \frac{1}{3}\left(\ln|y| - \ln|3-y|\right)$$

$$= \frac{1}{3} \ln\left|\frac{y}{3-y}\right|$$

$$\therefore \quad \frac{1}{3} \ln\left|\frac{y}{3-y}\right| = t + c$$

$$\ln\left|\frac{y}{3-y}\right| = 3t + C$$

$$\frac{y}{3-y} = c_1 e^{3t} \qquad (c_1 \neq 0)$$

$$y = 3c_1 e^{3t} - y c_1 e^{3t}$$

$$y(1 + c_1 e^{3t}) = 3c_1 e^{3t}$$

$$y = \frac{3c_1 e^{3t}}{1 + c_1 e^{3t}}$$

$$= \frac{3}{1 + ce^{-3t}} \qquad (c \neq 0)$$

To bring back the solution $y = 3$, we allow $c = 0$.

Determine the model.

In an environment without constraints, the population of a species of frogs is described by the DE

$$\dot{y} = 3y$$

where the constant 3 is the natural birth rate minus the death rate (in month^{-1})

The same species of frogs living in a pond grows according to the natural logistic equ. The population eventually reaches an equilibrium of 6000 frogs. Find $f(y)$ such that ~~diff~~ equ $\dot{y} = f(y)$ models the population of frogs in a pond, measured in kilofrogs.

solu:-

The logistic equ has growth rate $K(y) = a - by$. The constant a is the constant growth rate of frogs in an environment without constraints, given to be 3.

$$\therefore \quad \dot{y} = (3 - by)y$$

for some constant $b > 0$.

$$\therefore \quad \dot{y} = (3 - by)y = 0$$

$$3y - by^2 = 0$$

$$y(3 - by) = 0$$

$3 - 6b = 0$

$\boxed{b = \frac{1}{2}}$

$\boxed{b = \frac{a}{y_0}}$

From gn,

$\dot{y} = 0$ (when $y = 0$, $y = 6$ (6 kilofrogs)

$$\therefore \quad 6(3 - 6b) = 0$$

$$3 - 6b = 0$$

$\boxed{b = \frac{1}{2}} \rightarrow$ meaning $\boxed{\frac{a}{y_0}}$

y_0 - Non zero equilibrium population.

Renaming a to K_0 - Natural growth constant is the environment is unconstrained, the growth rate in the logistic equ can be written as

$$K(y) = K_0\left(1 - \frac{y}{y_0}\right).$$

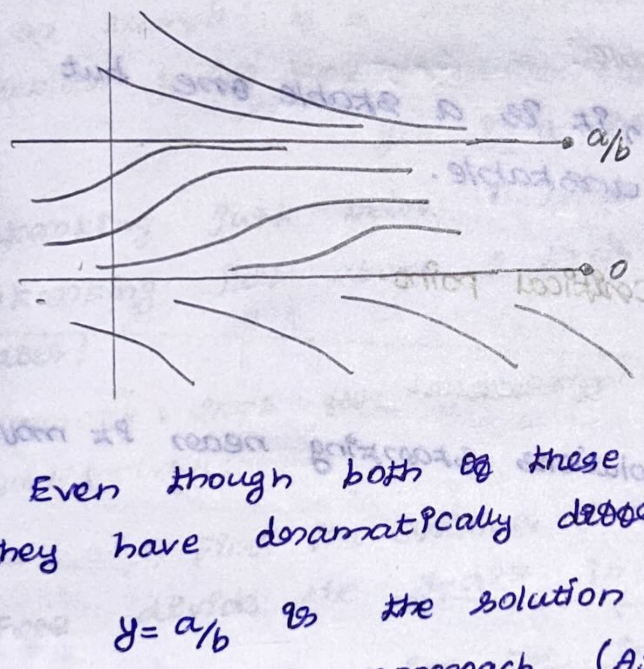
units of K_0 is month^{-1}. and the units of y_0 is kilo frog.

$$K(y) = a - by$$

$$K(y) = K_0 - \frac{K_0}{y_0} \cdot y$$

$$K(y) = K_0\left(1 - \frac{y}{y_0}\right)$$

Stability of critical points.



Even though both of these are constant solutions, they have dramatically different behaviours.

$y = a/b$ is the solution where all other solutions are trying to approach. (As $t \to \infty$)

$y = 0 \to$ repulsive (as it were, any solution that starts near 0)

meaning that

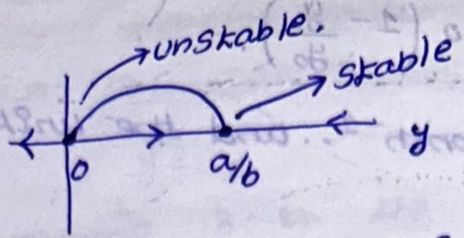
Any solutions start at zero will remain at zero (every time)

But any solution that starts above 0 will try to increase to a/b. This is known as stable solution. (Because everybody tries to get closer & closer to it).

$\therefore$ $a/b \to$ stable solution (critical point)

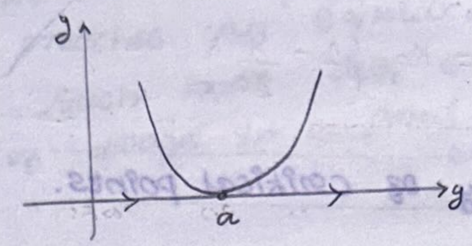
$0 \to$ unstable solution (critical point)

Particularly → stable & unstable critical points.

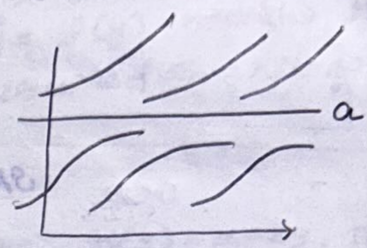


If arrow pointing towards (stable)
away (unstable)

<u>one example:</u>



⇒



Is it stable or unstable?

From one side, it is a stable one but from other, it is unstable.

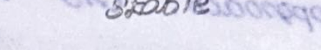
So, it is semi stable critical point.

A critical point $x = a$ is called

- stable if solutions starting near it move towards it

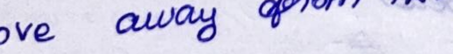
stable

- unstable - If solutions starting near it move away from it.

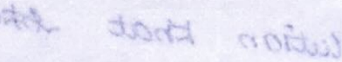


unstable.

- Semistable: If the behaviour depends on which side of the critical point the solution starts.

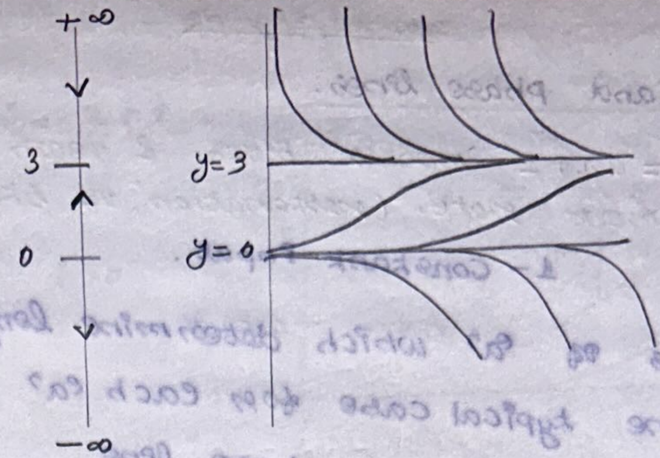


unstable



Semistable.

8.1



critical points 0, 3. The phase line shows that 0 is unstable and 3 is stable.

Remark 8.2:

A solution corresponding to an unstable critical point is an example of a separatrix because it seperates solutions having very different fates. In the example above, $y=0$ is a separatrix. A solution starting just below 0 tends to $-\infty$. while a solution starting just above 0 tends to 3: very different fates!

Summary: steps for understanding solutions to $\dot{y} = f(y)$ qualitatively:

1) Find the critical points $f(y) = 0$ (by solving). These divide the y-axis into open intervals.

2) Determine the intervals of y in which $f(y) > 0$ and $f(y) < 0$, by either graphing $f(y)$ or evaluating $f(y)$ at one point in each interval.

3) Draw the phase line, which consists of a line marked with $-\infty$ the critical points, $+\infty$ and arrows b/w these.

4) Solutions starting at a critical point are constant.

5) Other solutions tend to the limit that the arrow point to as t increases. As t decreases, solutions tend to the limit that the arrow originates from.

worked example

Autonomous eqn and phase lines:

$\dot{x} = ax + 1$ models birth & death rate and fixed replenishment rate. (restoration of stock)

1- Constant input.

a) Find intervals of 'a' which determine long term stability

b) Illustrate one typical case for each 'a'

i) phase line

ii) Sketch $x(t)$ vs t.

Solu:

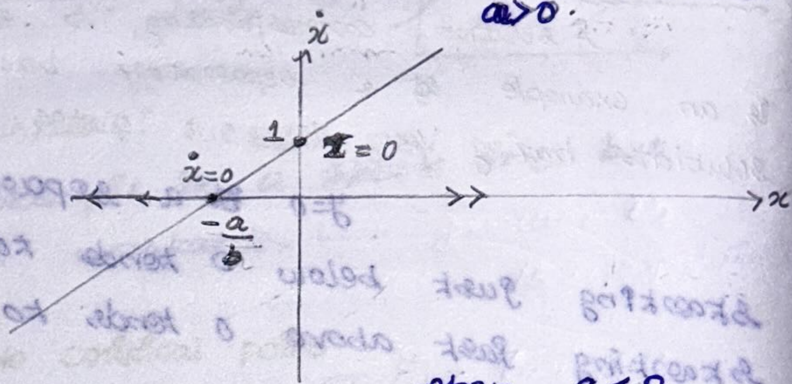
$\dot{x} = ax + 1$ (Autonomous eqn).

a>0.

$ax + 1 = 0$

$ax = -1$

$x = -1/a$



when a is (-ve)

when a<0

$x = 1/a$.

Test:

$\dot{x} = ax + 1$

(For a > 0)

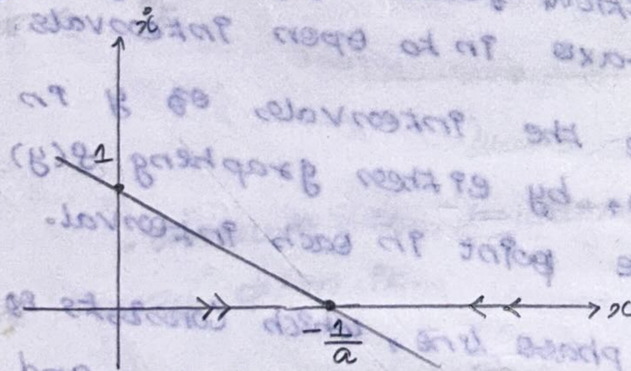
$\dot{x} = x + 1$

(↑)

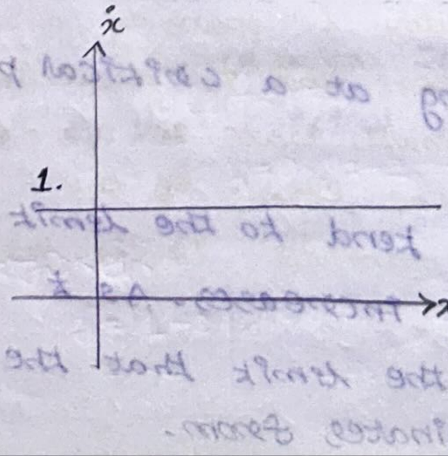
$\dot{x} = -x + 1$

(Slope is ↓)

a<0.

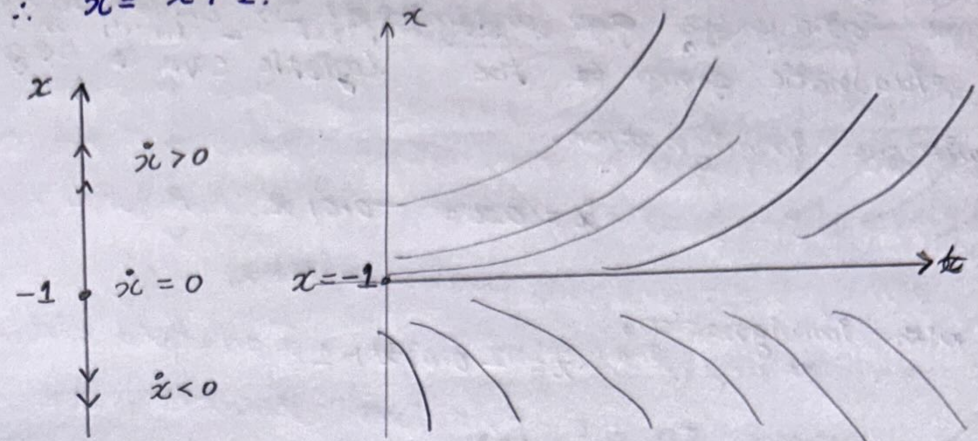


a=0:



b) Let $a=1$

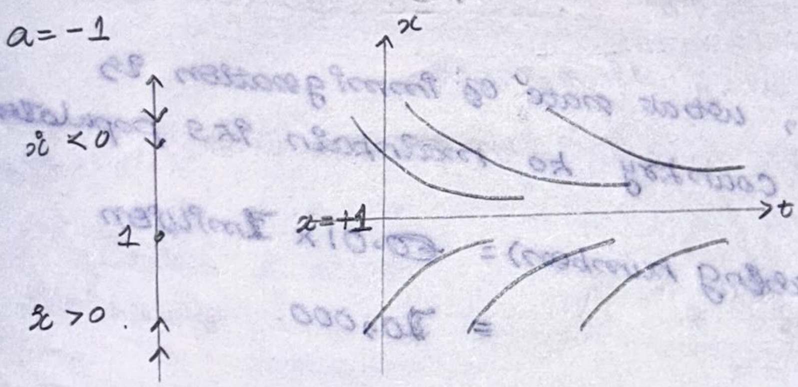
$\therefore \quad \dot{x} = x+1.$



$\therefore$ All the solutions on a region are autonomous (time independent) $\therefore$ $x=-1 \rightarrow$ unstable critical points.

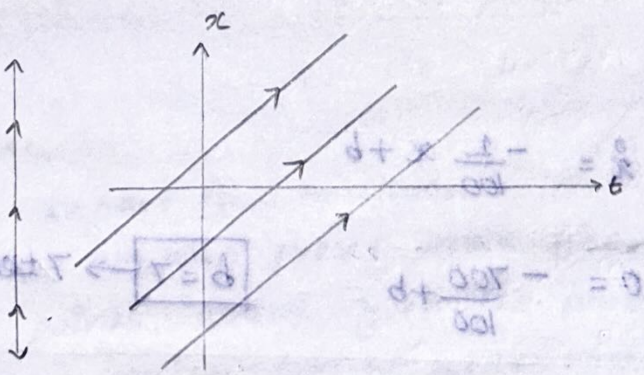
$\hookrightarrow$ Translation of one another.

$a=-1$



$\longrightarrow$ stable C.P.

$a=0$ (No critical points)



(Forever $\uparrow$)

$\therefore$ No C.P.

9.1: The population of a wealthy & sparsely populated country is currently 7 million and is declining from birth rate. The natural growth rate is $a=-0.01$. (In year^{-1}). What will be the effect of the population if the country admits 10,000 immigrants every year.

Solu:

$x(t) \rightarrow$ population (In ten thousands) at time t (in years).
Since the country is sparsely populated & wealthy, we can assume resources are unlimited. In other word, the quadratic form of the logistic equ is negligible.

$\therefore$ without immigration,

$$\dot{x} = ax = -0.01x$$

with immigration,

$$\dot{x} = -0.01x + 1$$

(1 - 10 thousand immigrants)

The population (current) $\rightarrow$ 7 million

$a = -0.01$. The current population will be decreasing while tending to 1 million.

<u>Maintaing population:</u>

In this, what rate of immigration is needed for the country to maintain its population.

(Decreasing number) $= 0.01 \times 7$ million

$= 70,000$.

$\therefore$ Allowing 70000 immigrants will give a stable equilibrium.

$$\dot{x} = -\frac{1}{100}x + b$$

$$0 = -\frac{700}{100} + b \qquad \boxed{b = 7} \rightarrow 7 \times \text{ten} \times \text{thousand}$$

$$\dot{y} = y^2 + 2y$$

$$y(y+2) = 0$$

$$y = 0, y = -2$$

when $-2 < y < 0$

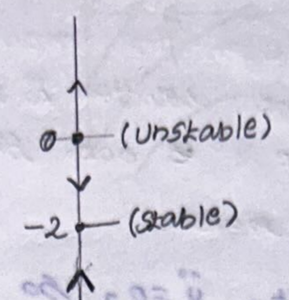
$\dot{y} = 1 - 2$ $= -1$	$\dot{y} = 0.01 - 0.2$ $= -ve$.

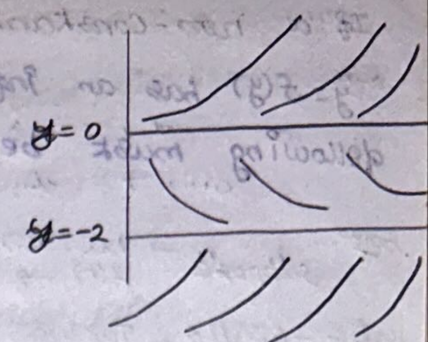
$y > 0$

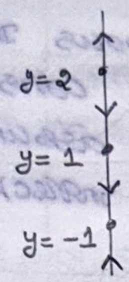
$\dot{y} = 1 + 2 = 3$

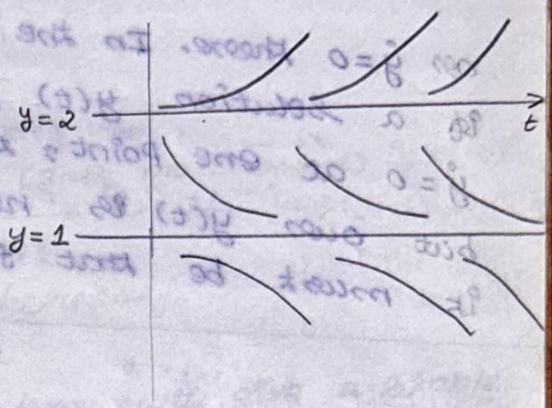
$\dot{y} > 0$ for $y < -2$ and $y > 0$

$\dot{y} < 0$ for $-2 < y < 0$

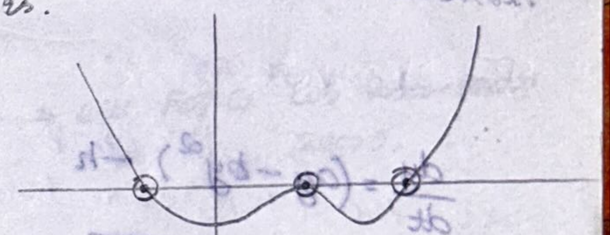




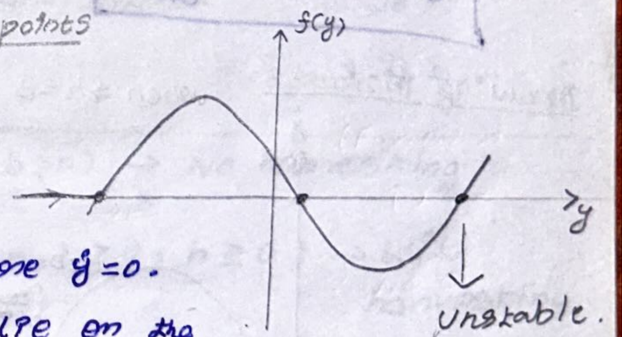




$f(y) \rightarrow$ Along horizontal axis.



critical points



For a solution,

to have local maxima,

it has to contain points where $\dot{y} = 0$.

But all points where $\dot{y} = 0$ lie on the 0-isocline, which for an autonomous equ consists of the constant solutions. Then by the existence & uniqueness theorem, other solution curves can't contain points where $\dot{y} = 0$. Therefore, no non-constant solutions can have a local maximum (or local minimum). Geometrically,

non-constant solution curves can't intersect the 0-isocline.

If a non-constant solution $y(t)$ to the autonomous DE $\dot{y} = f(y)$ has an inflection point at $t = a$, which of the following must be true at $t = a$?

Solu:

Suppose $y(t) \rightarrow$ is a nonconstant solution to the DE. So $\dot{y} = f(y) \neq 0$. Taking the derivative of both sides w.r.t. to t yields $\ddot{y} = f'(y)\dot{y}$ (by chain rule).

At any inflection point, $\ddot{y} = 0$, so either $f'(y) = 0$ or $\dot{y} = 0$ there. In the previous solution, we saw that if a solution $y(t)$ to an autonomous DE satisfies $\dot{y} = 0$ at one point, then $y(t)$ is a constant function, but our $y(t)$ is nonconstant by assumption. So it must be that $f'(y) = 0$ at the inflection point.

Harvesting model (salmon fish)

Harvest: At a constant time rate.

(400 pounds of salmon each day).

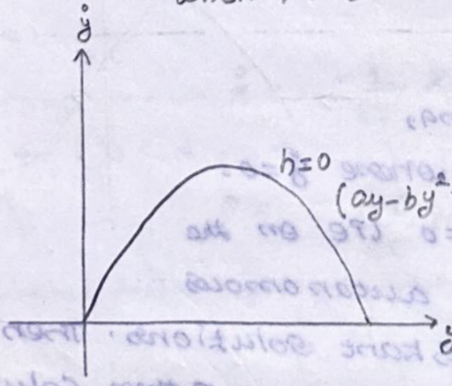
$\hookrightarrow$ Not a certain fraction.

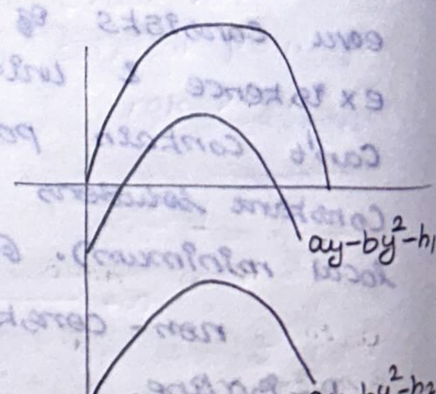
$\therefore$

$$\frac{dy}{dt} = (ay - by^2) - h$$

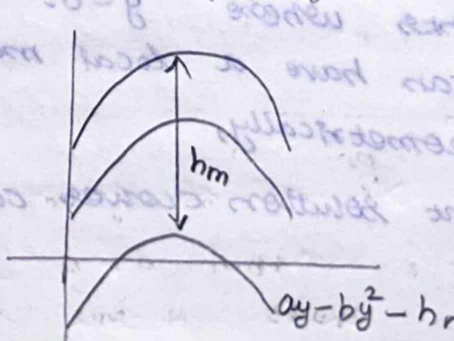
$$\boxed{ay - by^2 - h = \frac{dy}{dt}}$$

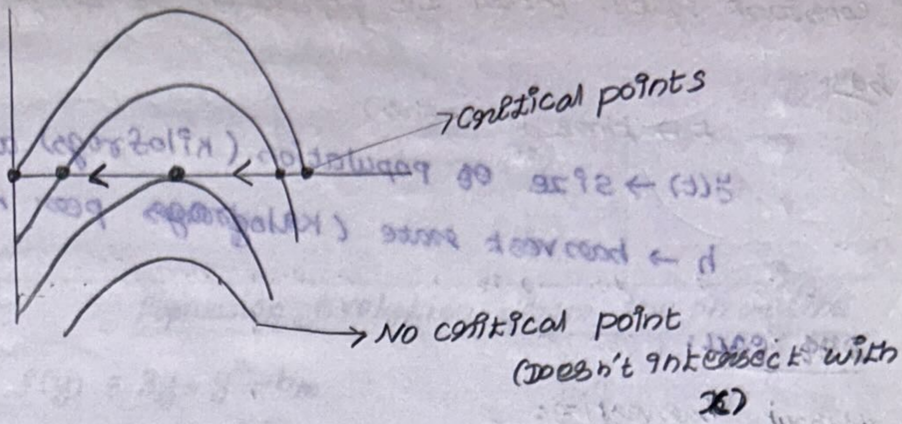
Drawing pictures: when $\Rightarrow h = 0$



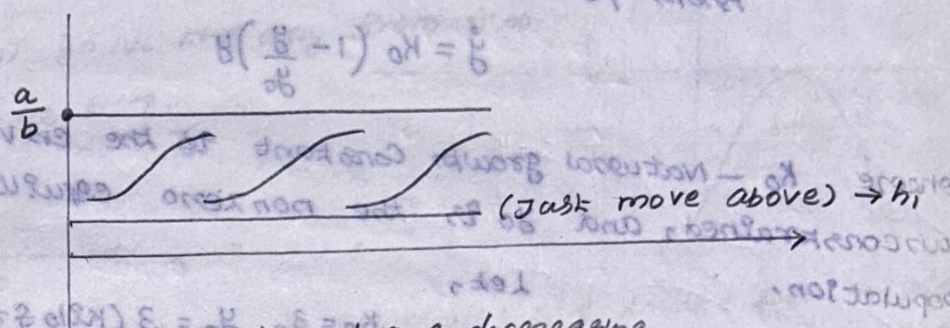


when h is lowered $(ay - by^2 - h_1)$

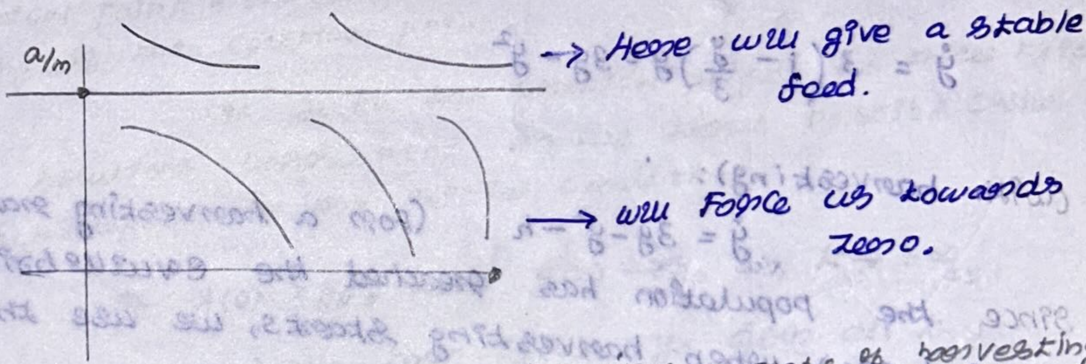




For h_1



solution always decreasing



For h_m – maximum rate of harvesting.

(maximum rate of harvesting – without letting there curve to go to zero).

$$\dot{y} = ay - by^2 \quad (a, b > 0) \rightarrow \text{No harvesting.}$$

$$\dot{y} = ay - by^2 - h \quad (a, b > 0,\ h \geq 0) \rightarrow \text{with harvesting}$$

This is an infinite family of autonomous eqn, one for each value of h, and each has its own phase line. we will explore how the phase line changes with h in the example & problems below.

12.1: Frogs grow in a pond according to the logistic eqn with natural growth constant 3 (in $month^{-1}$) and eventually the population reaches an equilibrium of 3000 frogs. Then the frogs are harvested at a

Constant rate. model the population of frogs.

Solu:

$t \to$ time (months)

$y(t) \to$ size of population (kilofrogs) at time t

$h \to$ harvest rate (kilofrogs per month).

Diff eqn:

without harvesting:

From previous notes,

$$\dot{y} = k_0 \left(1 - \frac{y}{y_0}\right) y$$

where k_0 — Natural growth constant if the environment is unconstrained, and y_0 is the nonzero equilibrium population.

Let,

$k_0 = 3$, $y_0 = 3$ (kilofrogs).

$$\dot{y} = 3\left(1 - \frac{y}{3}\right)y = 3y - y^2$$

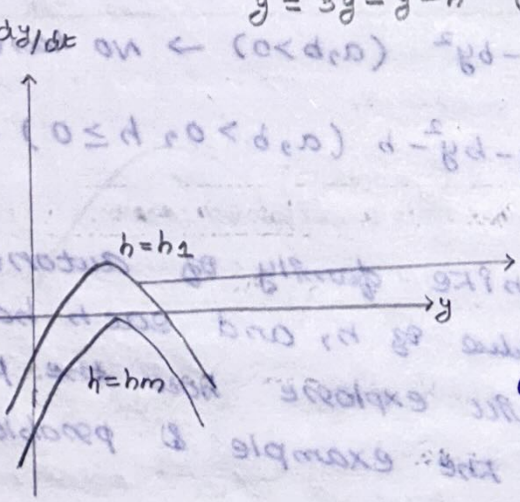
(with harvesting):

$$\dot{y} = 3y - y^2 - h$$ (For a harvesting rate h.

since the population has reached the equilibrium of 3 (kilofrogs) when harvesting starts, we use the IC $y(0) = 3$.

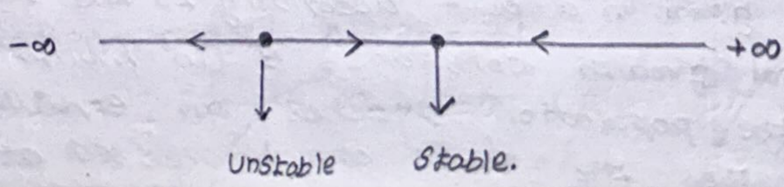
Draw the phase lines for two different harvesting rates:

$$\dot{y} = 3y - y^2 - h \quad (h > 0)$$

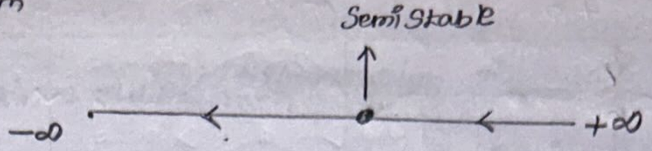


$\dot{y} = 3y - by^2 - h \quad (h = h_1)$

The exact location of each critical point is not required

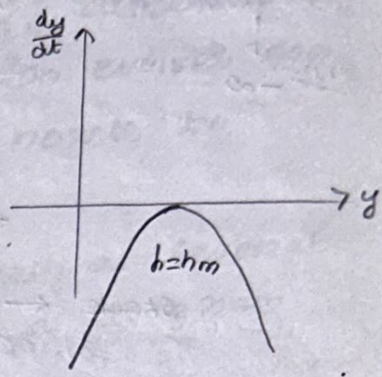


For $h = h_m$



Population evolution from the phaseline.

Let $h = h_m \Rightarrow f(y) = 3y - y^2 - h_m$.



According to phase line $h = h_m$,

can $y(t)$ go all the way from $+\infty$ to $-\infty$ as $t \uparrow$.

Solu.

There are three types of solutions: one to the left of the critical point, one at the critical point, and one to the right of the critical point.

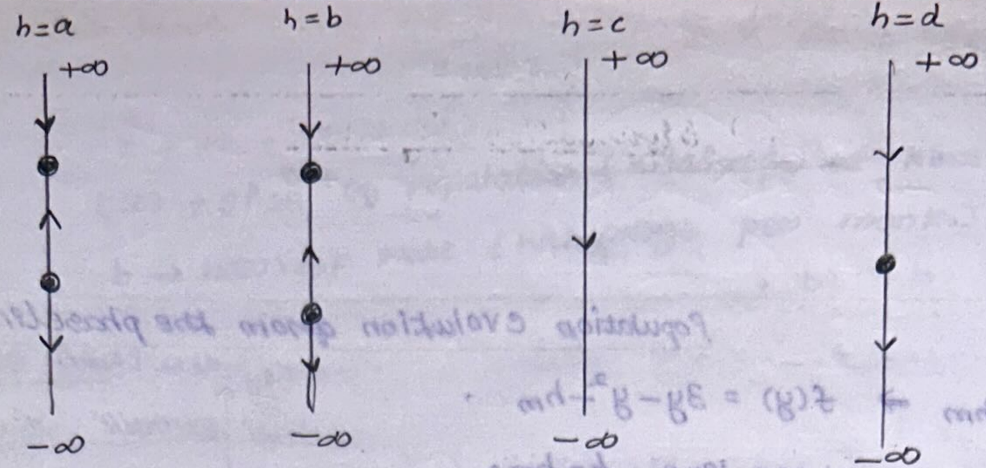
Let y_0 be the critical point. The three types of solutions correspond to the three possible evolutions which depend on the initial conditions.

- If $y(0) > y_0$, then $y(t) \to y_0$ as $t \to +\infty$
- If $y(0) = y_0$, then $y(t) = y_0$ for all t
- If $y(0) < y_0$, then $y(t) \to -\infty$ (we interpret this as a population crash: the from population reaches 0 in finite time, the part of the trajectory with $y < 0$ is not part of the population model.

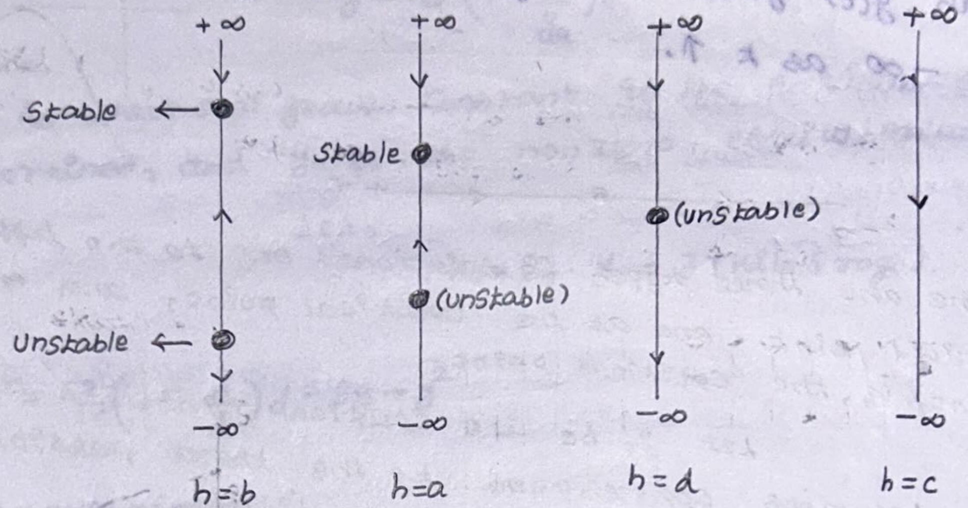
Phase lines for different harvesting rates

$\dot{y} = 3y - y^2 - h \quad (h > 0) \longrightarrow$ consider again

The 4 phase lines in below are for 4 different values of h: $h = a, b, c, d$ (order a, b, c, d in increasing order)



$b < a < d < c$



$h=b \rightarrow$ The value of h is very large. The solution is translated downwards.

<u>Number of critical points</u>

$\dot{y} = 3y - y^2 - h \quad (h \geq 0)$

<u>solu:</u> Let h_m be the value of h for which the DE has exactly 1 critical point & let y_0 be the value of that critical point. Find h_m and y_0.

The quantities h_m and y_0 are of physical importance; they correspond to the maximal harvesting rate & the equilibrium population at the maximum harvesting rate & the equilibrium population at the maximum harvesting rate respectively.

<u>solu:</u> $h_m = ?$ $\quad 3y - y^2 - h = 0 \Rightarrow y^2 - 3y + h = 0$

$$y_0(h) = \frac{3 \pm \sqrt{9-4h}}{2}$$

$\therefore$ DE has exactly 1 critical point $y_0 = \frac{3}{2}$ when $9-4h=0$ (The DE has 2 critical points when

$9-4h>0$ and no critical point when $9-4h<0$.

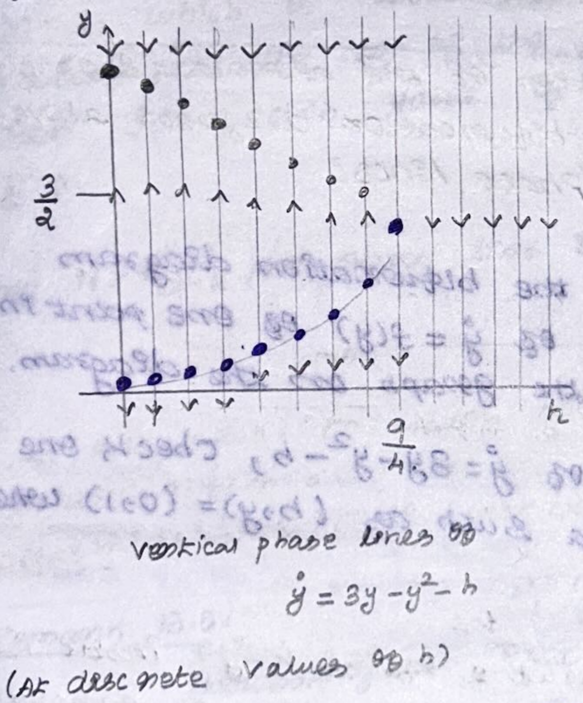
Let us continue with 13) Bifurcation diagrams

$$\dot{y} = 3y - y^2 - h \quad (h \geq 0)$$

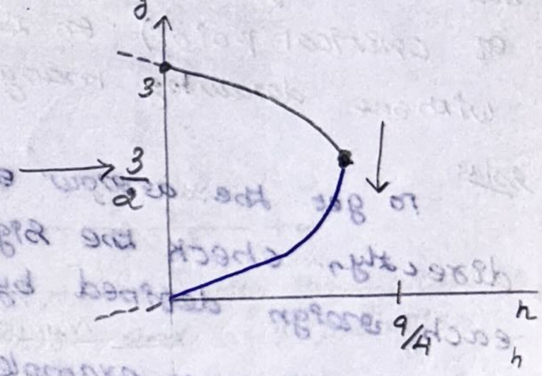
we have just seen that the phase lines corresponding to different values of h, can be very different. They tells us how the frog population evolves for different harvesting rate. Our goal now is to summarize all of these diagrams.

If we draw phase line vertically at several h-values in the (h, y)-plane, we get a diagram on the left below.

• stable
• unstable



Vertical phase lines of
$\dot{y} = 3y - y^2 - h$
(At discrete values of h)



Bifurcation diagram for
$\dot{y} = 3y - y^2 - h$

The diagram on the right is called a bifurcation (இருகூறாக்கு) diagram.

Definition 13.1: A bifurcation diagram of a family of autonomous equ depending on a parameter h is a plot of the values of the critical points as functions of h, along with one arrow in each region in the (h, y) plane defined by the curve, indicating whether solutions are ↑ or ↓ in that region.

As usual, the arrows tell us the stability

of the critical points. In the diagram, we have also explicitly indicated the stability of the critical points by colors: Stable → • } semistable → •
unstable → •

What's the shape of the curve on the bifurcation diagram, we have also explicitly for $\dot{y} = 3y - y^2 - h$?

Solu:

The curve on this bifurcation diagram is a parabola. Defined by $3y - y^2 - h = 0$, equivalently

$h = 3y - y^2$, a sideways parabola.

Remark: The shape of the curve on the bifurcation diagram depends on the diff eqn, and so there are many possibilities. We will see bifurcation diagrams with other curves in the mathlet & video.

How do we find the direction of the arrows (& stability of critical point) on the bifurcation diagram above without drawing many phase lines?

Solu:

To get the arrow on the bifurcation diagram directly, check the sign of $\dot{y} = f(y)$ of one point in each region defined by the graph on the diagram.

In our example of $\dot{y} = 3y - y^2 - h$, check one point inside the parabola such as $(h, y) = (0, 1)$ where

$\dot{y} = 3y - y^2 - h = 2 > 0.$

This gives an up arrow for the entire region inside the parabola. |||ly check one point outside the parabola such as $(h, y) = (1, 0) \Rightarrow \dot{y} = 3y - y^2 - h = -1 < 0$.

This gives a downward arrow for the entire region outside the parabola.

The upper branch of the parabola has stable critical points, and the lower branch has unstable critical points, when $9 - 4h < 0$ (when $9 - 4h = 0 \rightarrow$ the critical point is semistable)

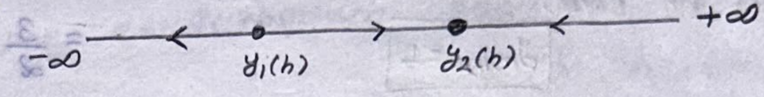
13.4: What's the maximum sustainable harvest rate?

(Sustainable — Harvesting doesn't cause the population to crash to 0, but that instead $\lim_{t\to\infty} y(t)$ is positive, so that the harvesting can continue indefinitely.

Solu: The maximum sustainable harvesting rate is $\frac{3}{2}$ meaning 2250 frogs/month. (So that the harvesting rate can continue indefinitely.)

Why

- $h < 9/4$, the phase line is



∴ The starting population of frogs is 3000. We have IC $y(0) = 3$, which is above the top (blue) branch of the parabola. Let the top of the parabola is called $y_{top}(h)$. Then for each value of $h < 9/4$, $y(t) \to y_{top}(h)$ as $t \uparrow$.

For $h > 9/4$, the phase line is.

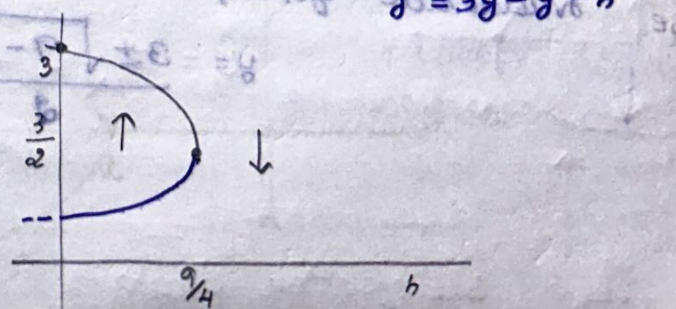
In this case, no matter what the starting population is, $y(t)$ will reach 0 in finite time and a population crash is inevitable (over harvesting).

Remark 13.5:

Harvesting at exactly the maximum rate is a little dangerous, however because if after a while y becomes very close to $\frac{3}{2}$ and a little kid comes along & takes a few more frogs out of the pond. So that the population is just below $3/2$, the whole frog population will crash

Avoiding population crash:

$$y' = 3y - y^2 - h$$



A farmer would like to harvest 2000 frogs per month. What's the minimum starting population y_{min} needed in order to avoid a population crash?

Solu:-

$$\dot{y} = 3y - y^2 - 2 \rightarrow y(3-y) = 2$$

$$-y^2 + 3y - 2 = 0$$

$$y^2 - 3y + 2 = 0$$

$$= \frac{3 \pm \sqrt{9-4(2)}}{2}$$

$$= \frac{3}{2} \pm \sqrt{\frac{1}{4}}$$

$$= \frac{3}{2} \pm \frac{1}{2}$$

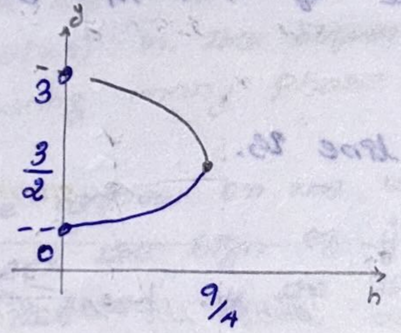
$y = \frac{4}{2} = 2$

$y = \frac{2}{2} = 1.$

$\boxed{y_{min} = 1}$

---

minimum population for different harvesting rates.



$\dot{y} = 3y - y^2 - h.$

Give a formula for the minimum starting population $y_{min}(h)$ of frogs required for a farmer to harvest at a rate of h (kilofrogs) per month without crashing the population. Assume $h < 9/4$.

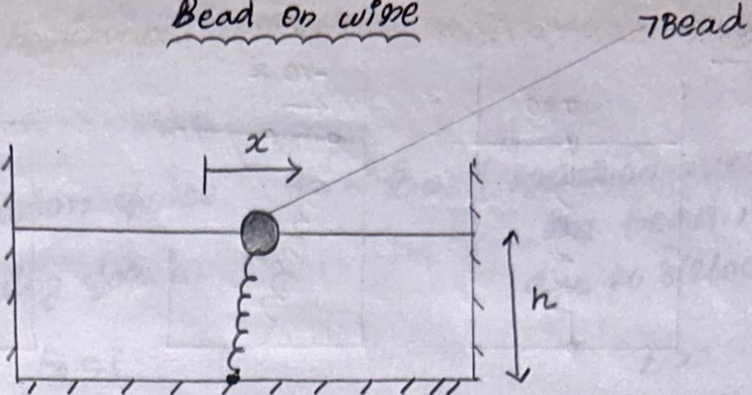
Solu:

For any harvesting rate, $h < 9/4$, the minimum starting population of frogs is the value of the lower branch of the parabola at h, which is the smaller root of $3y - y^2 - h = 0$. Solving for y in terms of h by the quadratic form,

$$y = \frac{3 \pm \sqrt{9-4h}}{2} \Rightarrow y_{min} = \frac{3 - \sqrt{9-4h}}{2}$$

$$\left(h < \frac{9}{4}\right)$$

Bead on wire



A bead is tied using a horizontal wire.

h = height of wire.

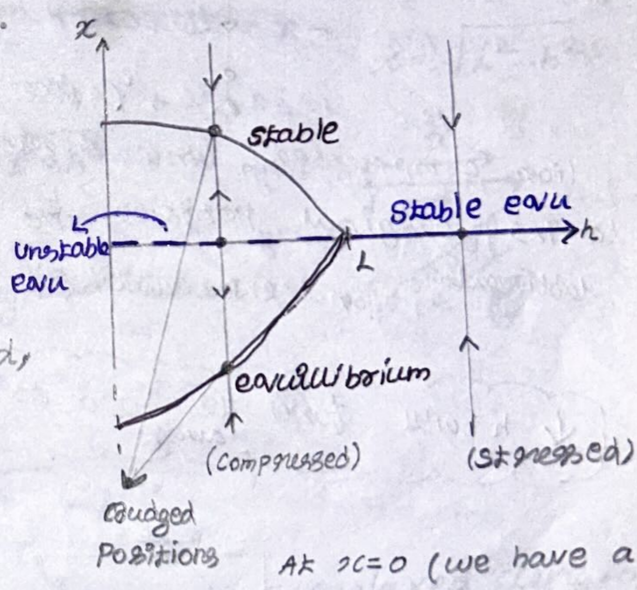
x = displacement of bead from centre.

What are the equilibrium positions of the bead?

Solu:

Bifurcation diagram:

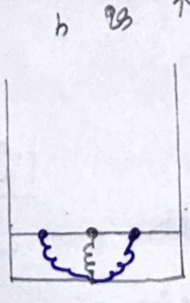
Equilibrium positions will be dependent upon the relationship b/w h & the relaxed length of the spring is just the length at which the spring is being neither stretched nor compressed.



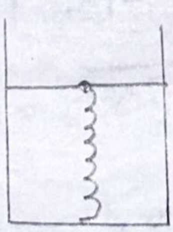
Let,

L - Relaxed length of spring

If the spring is stretched, h is ↑ed.



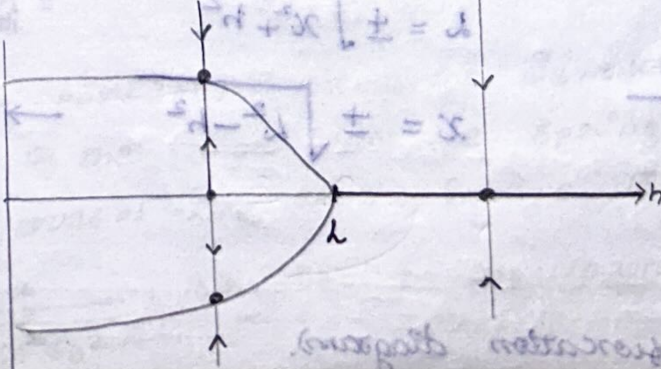
$h<L$

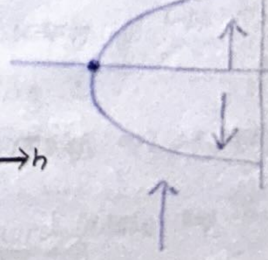


$h>L$

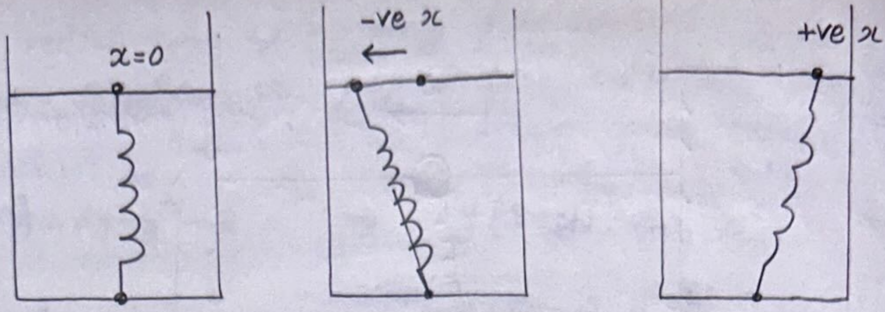
At $x=0$ (we have a solid equilibrium position)

Nudging (moving) → the bead



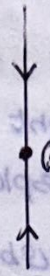


Stressed:

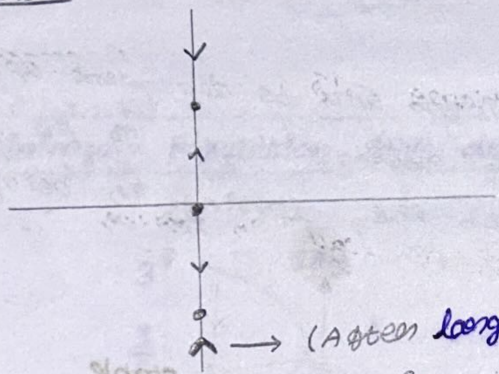


As $x=0$

Because, the $-x$ & $+x$ are trying to approach $x=0$ (stable equ).



For Compressed:

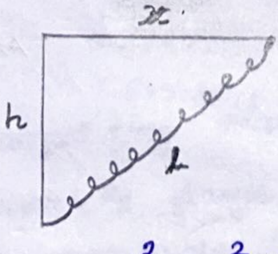


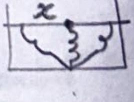
(After longer nudging, it will come to position where it came to equilibrium)

(For compressed, the spring will try to come back to its original position by moving the bead right or leftwards.

For expanded, it will try to compress.

↓ h will force the bead to move away

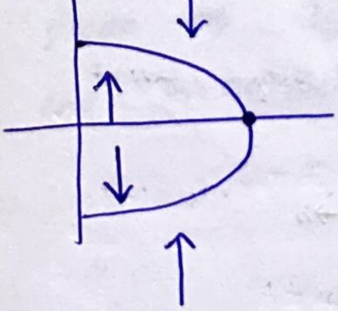




$$L^2 = x^2 + h^2$$

$$L = \pm\sqrt{x^2 + h^2}$$

$$x = \pm\sqrt{L^2 - h^2}$$ → Rev equ modeled.



(From bifurcation diagram).

modelling DE: (Horizontal motion).

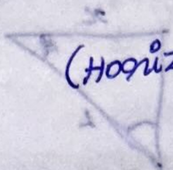
Two forces:

1) Friction force. $f = -\dot{x} b$ (Friction acts against the bead motion) due to velocity of bead.

$b > 0$.

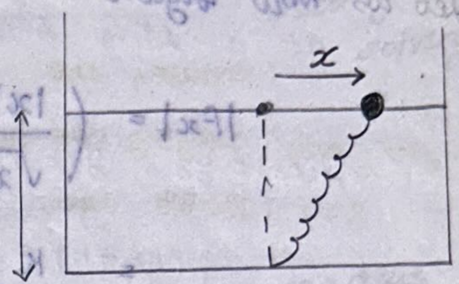
2) Spring force:

$F_s = ?$



(Horizontal) → Component alone

$F_s = ?$



K - Spring constant

L - Relaxed length of the spring

h - Height of the wire from the bottom of the box.

x - Horizontal displacement of the bead from the Center of the wire.

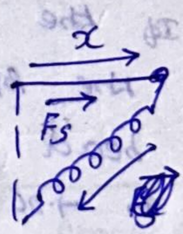
Find the spring force on bead in terms of K, L, h, x.

Solu: F_x → Spring force

$x = \sqrt{L^2 - h^2}$

~~$F_x = -Kx$~~

~~$F_x = -K\sqrt{L^2-h^2}$~~



The vector $\vec{F_s}$ has a magnitude

$$|\vec{F_s}| = K|d|$$

$$d = \sqrt{x^2+h^2} - L$$

d - The difference b/w stretched & relaxed length of the spring

$F = -Kx$
↓
stretched or Comp...

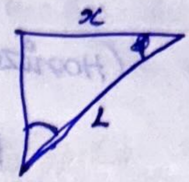
~~F_s~~ =

To find the horizontal Component, we multiply by the sine of the angle b/w the spring & the dotted vertical line in the image. This quantity is gn by $\frac{x}{\sqrt{x^2+h^2}}$. This implies the magnitude of the horizontal component F_x of the spring force is

$$|F_x| = \left(\frac{|x|}{\sqrt{x^2+h^2}}\right)|\vec{F_s}|$$

$$= k\frac{|x||d|}{\sqrt{x^2+h^2}}$$

Let us now figure it.



$$|F_x| = \left(\frac{|x||d|}{\sqrt{x^2+h^2}}\right)\cdot|\vec{F_s}|$$

$$= k\frac{|x||d|}{\sqrt{x^2+h^2}}$$

$$\sin\theta = \frac{x}{L} = \frac{x}{\sqrt{x^2+h^2}}$$

- If the bead is to the right of the centre ($x>0$) and the spring is stretched ($d>0$) then F_x pulls the bead towards the centre of the wire, which is in the $-x$ direction.
- If the bead is to the right of the centre ($x>0$) but the spring is compressed ($d<0$), F_x pushes the bead away from the centre of the wire (which is in $+x$ direction.

$$\therefore \quad F_x = -k\frac{x\left(\sqrt{x^2+h^2}-L\right)}{\sqrt{x^2+h^2}}$$

$$= -kx\left(1-\frac{L}{\sqrt{x^2+h^2}}\right)$$

Two forces:

$$f = -b\dot{x} \quad (b>0)$$

$$F_x = -kx\left(1-\frac{L}{\sqrt{x^2+h^2}}\right), \quad k>0$$

Newton's II law:

$$m\ddot{x} = -b\dot{x} - kx\left(1-\frac{L}{\sqrt{x^2+h^2}}\right)$$

(we are approximating $m \approx 0$)

$$b\dot{x} + kx\left(1-\frac{L}{\sqrt{x^2+h^2}}\right) = 0$$

$$\dot{x} + \frac{k}{b}x\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right) = 0$$

$\boxed{\frac{k}{b} > 0}$

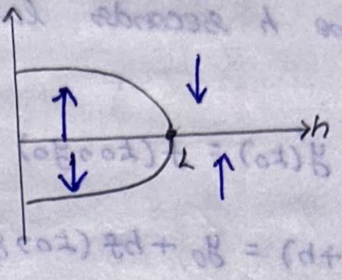
Bifurcation diagram: (Equilibrium $\dot{x} = 0$.

$x_0^* = 0$ [1 equilibrium position]

$x_{\pm}^* = \pm\sqrt{L^2 - h^2}$

$\left[1 - \frac{L}{\sqrt{x^2+h^2}} = 0\right]$ setting & solving.

Equ of semi circle



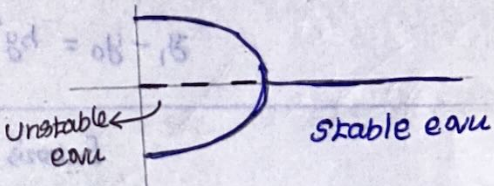
Inside semicircle $\sqrt{x^2+h^2}$ is less than radius.

$\therefore \left(\frac{L}{\sqrt{x^2+h^2}} - 1\right) \rightarrow$ gives +ve.

Out side semi cle

$\sqrt{x^2+h^2}$ is > than radius

$\left(\frac{1}{\sqrt{x^2+h^2}} - 1\right)$ will be −ve.



Numerical methods

1) Apply Euler's method to both linear & nonlinear diff equ with a given step size to approximate values of the solution function near some given initial values.

2) Describe how concavity or convexity contribute to the error using Euler's method.

3) Gain a sense of how higher order methods reduce the error.

Eulers method.

Consider an ODE $\dot{y} = f(t, y)$. It specifies a slope in the (t, y) plane, and solution curves follow the slope field. Suppose that we are given a starting point (t_0, y_0) and we are trying to approximate the solution curve through it.

2.1: where, approximatively, will be the point on the solution curve at a time h seconds later?

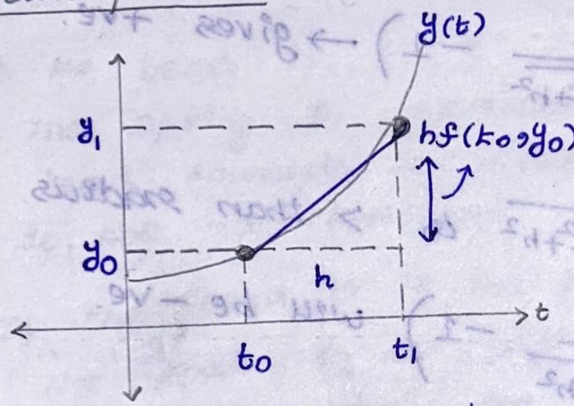
Solu:

$y(t_0) = y_0$, $y'(t_0) = f(t_0, y_0)$ using linear app,

$$y(t_0 + h) = y_0 + hf(t_0, y_0)$$

$$y' = \frac{f(t_0+h) - f(t_0)}{h}$$

Geometric picture:



$hf(t_0, y_0) = f(t_0 + h)$

$f(t_0 + h) = y_0 + hf(t_0, y_0)$

(APP $h \approx 0$)

$$y_1 - y_0 = hy'(t_0) = hf(t_0, y_0).$$

Eulers method.

In real life, most of the diff equ are solved by numerical methods. (The computer was calculating the solutions numerically & plotting it)

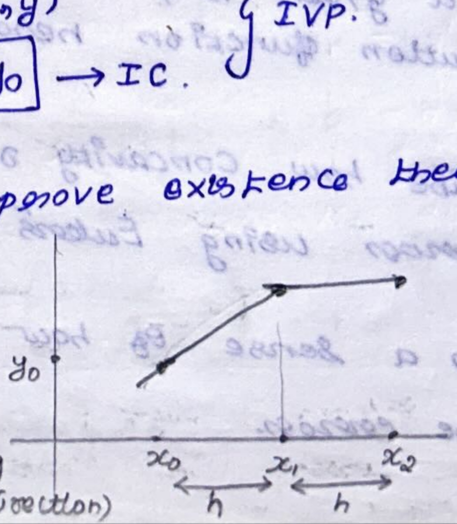
IVP problem

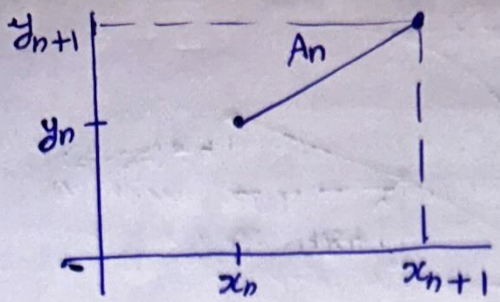
$$\left.\begin{array}{l} y' = f(x, y) \\ \boxed{y(x_0) = y_0} \rightarrow IC. \end{array}\right\} IVP.$$

Euler used this to prove existence theorem.

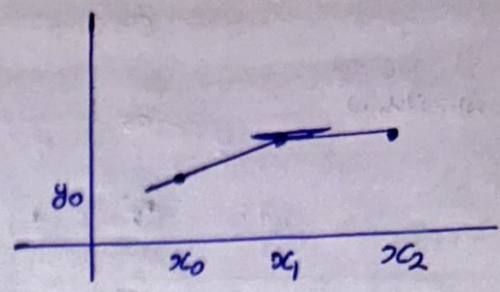
(It gives a broken line visual)

we have slope line at that point, (Continuing & drawing the curve at that direction)

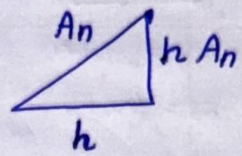




For n $\leftarrow$



h - stepsize.



$\therefore$ Slope $= \dfrac{\text{Rise}}{\text{Run}} = \dfrac{x}{h}$

$\boxed{A_n \cdot h = x}$

$y_{n+1} - y_n = h A_n$

$y_{n+1} = y_n + h A_n$

$\boxed{A_n = ?}$

Euler's equation:

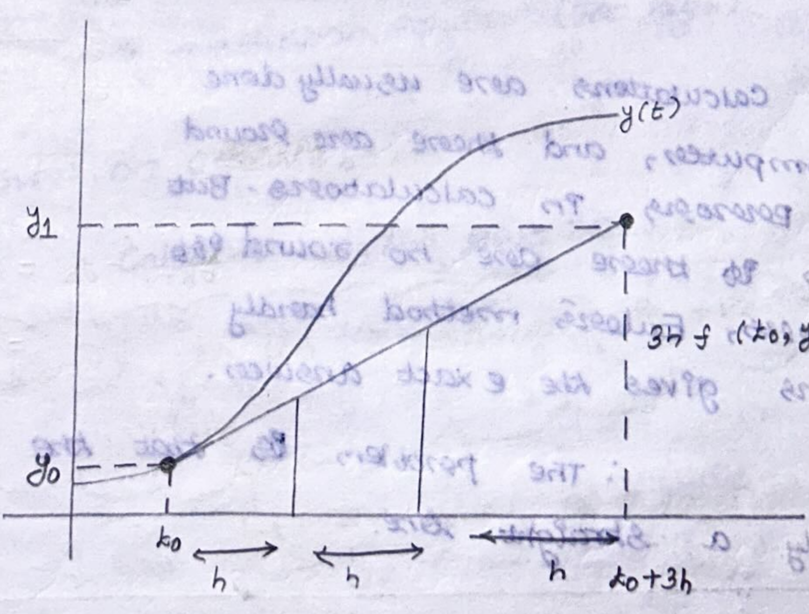
$x_{n+1} = x_n + h$ $\quad A_n = f(x_n, y_n)$

$y_{n+1} = y_n + h A_n$ $\quad$ Slope.

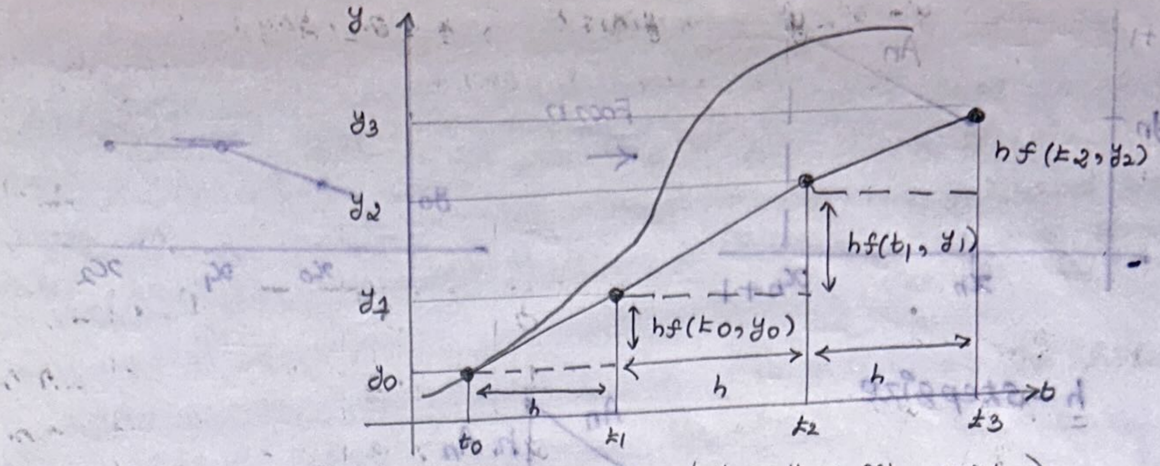
where, app will be the point on the solution curve at time $t_0 + 3h$?

Soln:
The crude answer would be to take 3 steps each using the initial slope $f(t_0, y_0)$.

Geometrically,



The more refined answer is called Euler's method: Takes 3 steps, but reasses the slope after each step. Using the slope field at each successive position:



$t_1 = t_0 + h$, $t_2 = t_1 + h$, $t_3 = t_2 + h$

$$\begin{pmatrix} y_1 = y_0 + f(t_0, y_0)h \\ y_2 = y_1 + f(t_1, y_1)h \\ y_3 = y_2 + f(t_2, y_2)h \end{pmatrix}$$

The sequence of line segments from (t_0, y_0) to (t_1, y_1) to (t_2, y_2) is a piecewise linear approximation to the solution curve. The more refined answer to the question is (t_3, y_3).

Euler's method:

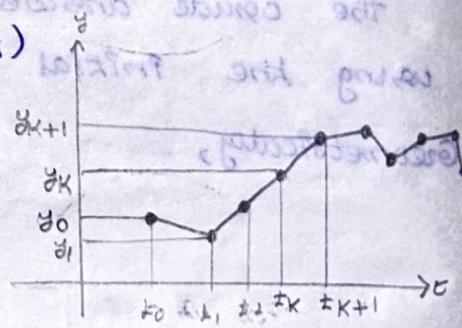
Given an initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$

and a choice of step size h (in seconds if time is the independent variable). The Euler method gives an app. to the solution curve b/w $t = t_0$ and $t = t_0 + (n+1)h$, by a sequence of line segments connecting the points $(t_0, y_0), (t_1, y_1), \ldots, (t_n, y_n), (t_{n+1}, y_{n+1})$, where for each $0 \le k \le n$,

$$t_{k+1} = t_k + h$$

$$y_{k+1} = y_k + hf(t_k, y_k)$$



These calculations are usually done by computer, and there are round off errors in calculators. But even if there are no round off errors, Euler's method hardly ever gives the exact answer.

$$y_{k+1} - y_k = hy'(t_k)$$

$$= hf(t_k, y_k)$$

∴ The problem is that the actual solution is rarely a straight line.

$y' = x^2 - y^2$, $y(0) = 1.$, $h = 0.1$ (say)

Non trivial (can't be solved in terms of elementary functions)

solu:

$y' = x^2 - y^2$

n	x_n	y_n	A_n	$A_n \cdot h$ Slope×h
0	0	1	-1	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82	-0.63	-0.063
3	0.3	0.757	-0.483	-0.048

$y_2 = y_1 + h\dot{y}$

$= 1 + (-0.1)$

$= 0.9$

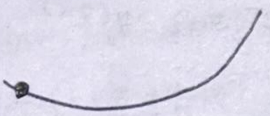
$y_3 = 0.9 - 0.08$

$= 0.82$

$$\begin{array}{r} 0.90 \\ -\ 0.08 \\ \hline 0.82 \end{array}$$

$y_4 = 0.82 + (-0.063)$

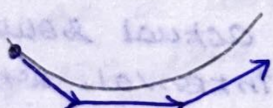
$= 0.757$ → Is this too high (or) too low?



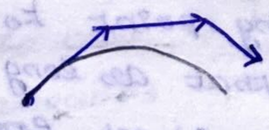
$y_1(x)$ → The solution is convex

→ concave.

eg:



— Euler's solution (Too low)



→ Euler's Solution (Too high).

How to find Convex or concave?

Calculus to the rescue.

A curve is convex if its II derivative is +ve

$y'' > 0$

↳ The slope of I derivative is ↑ (+ve).

$y'' < 0$ → concave.

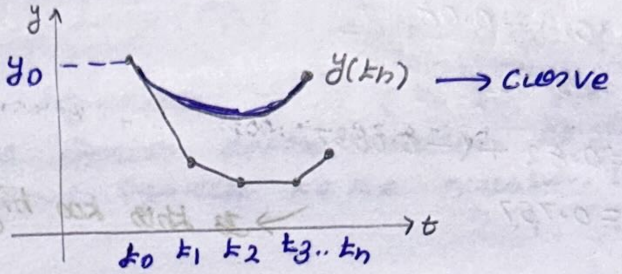
$$y' = x^2 - y^2 , \quad y'' = 2x - 2yy'$$

At $(0,1)$

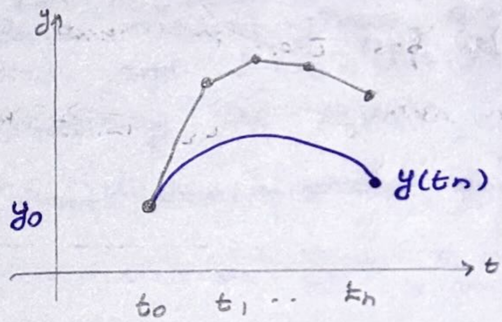
$$y(0) = 1, \quad y'(0) = -1$$

$$y''(0) = 0 - 2(1)(-1)$$
$$= 2 > 0 \text{ (convex)}$$ $\longrightarrow$ The answer will be too low.

For convex or concave up, if $\ddot{y} > 0$, in the interval $[t_0, t_n]$ and in a region along the Euler polygon $\ddot{y}(t)$, then $\bar{y}(t_n) < y(t_n)$.



For concave, $\ddot{y} < 0$, in the interval $[t_0, t_n]$, and in a region along the Euler polygon $\bar{y}(t)$, then $\bar{y}(t_n) > y(t_n)$



often we don't know whether the actual solution $y(t)$ is convex or concave in an entire interval. What we are able to compute without knowledge of the solution is the value of $\ddot{y}$ at the starting point t_0. This is still useful because this means that as long as we carry out Euler's method for a short enough time interval, we can still predict whether the Euler approximation overshoots or undershoots.

To find $\ddot{y}$ at the starting point t_0, we differenciate the DE $\dot{y} = f(t, y)$ to find $\ddot{y}$. The differentiation is often called implicit because the variable y depends on t. This gives us a formula for $\ddot{y}$ in terms of t, y and $\dot{y}$. To evaluate $\ddot{y}(t_0)$, plug in the initial condition t_0, $y(t_0)$ as well as $\dot{y}(t_0) = f(t_0, y_0)$.

Error & step size:

Euler's method is not the world's best method.

Concavity & convexity meaning → systematic error.

Better method of getting right answers:

Use smaller step size.

(Bigger the step size → More the error)

error depends upon step size. $e \sim c_1 h$

Euler is a ~~first~~ first order method → Because It's not the first order because it's $y' = f(x, y)$.

The I order means the fact that h occurs to the 1st power.

c_1 – constant (half the h size). (halve the step size ↳ halve the error)

To improve the approximation by the Euler method, we can use a smaller step size h, so that the slopes of the line segments are reassessed more frequently.

The cost of this, however, is that to ↑ t by a fixed amount, more steps will be needed.

Under reasonable hypotheses on f – the right hand side of the DE $\dot{y} = f(t, y)$ – one can prove that in the limit as $h \to 0$, this process converges and produces an exact solution curve. This is one way to prove that the solution to the IVP exist. Infact, this is the way Euler proved it.

Error of approximation:

Let $\bar{y}(t)$ – Approximate solution given by Euler's method with step size h.

Let the error of approximation e over the interval $[a, b]$ be

$$e = \max_{a \le t \le b} |y(t) - \bar{y}(t)|$$

That is, e is the maximum abolute difference b/w the actual solution $y(t)$ and $\bar{y}(t)$. The error e is bounded above by a linear function of h

$$e \le h \cdot C$$

where $C \rightarrow$ constant depending on f. Because h occurs as a first power above, not $h^{1/2}$ or h^2, Euler's method is called a first order method.

Euler's method mathlet:

Convex function:

Choose the ODE $y' = 0.5y + 1$.

Use $(-2, -1) \rightarrow$ Initial point $\rightarrow$ construct & compare Euler's solutions using step size $h = 0.5, 0.25, 0.125$. Then compare to the actual solutions.

$\hookrightarrow 0.125 = h$ is the better option (Lower error).

$\therefore$ All the solutions are undershooting.

Concave Function & error:

$y' = y^2 - 2y + 1$. Make a prediction before using the mathlet. From the initial point $(-1, -1)$. will the Euler approximation be overshooting or undershooting.

Sol:

$$y'(-1) = 1 - 2(-1) + 1$$
$$= 1 + 2 + 1 = 4.$$

$\rightarrow$ Over shoot

($\because$ The solution function is concave, and thus is curving downwards, below the tangent (concave). The approximation everywhere.

$$y'' = 2yy' - 2y'$$
$$= 2(-1)(4) - 2(4)$$
$$= -8 - 8 = -16 < 0$$

The Euler approximation is everywhere overshooting.