Calculus

Limits:

As B tends move closer to A.

At A, the line will be the tangent of the curve.
The slope will be its derivative.

Integrals - used to measure area of curvy regions.
The integral is the limit of the total area of the rectangles as the width tends to zero.

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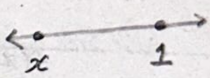
calculus is all about functions.

$x \to f(x) \to y$

Input output



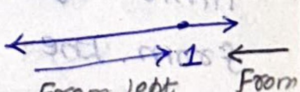
As $x \to 1$ (from the left)

we're only concerned with values of x that are near one but not equal to one.

when x moves, $f(x)$ moves too.

Let,

$$f(x) = \frac{\sqrt{3-5x+x^2+x^3}}{x-1}$$



Soln:: At $x=1$, $f(x)$ is not defined.

x	$f(x)$
0	$\sqrt{3}$ -1.73
0.5	≈ -1.87
0.9	-1.97
0.99	-1.997

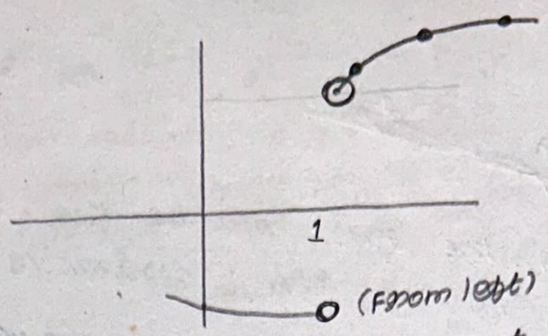
x	$f(x)$
1.1	2.024.
1.01	2.0024
1.5	2.12
2	2.24

$\therefore$ $f(x)$ moves towards -2. (But never achieves)

As $x \to 1^-$ (From left) $f(x) \to -2$ (From left)

As $x \to 1^+$ $f(x) \to 2$

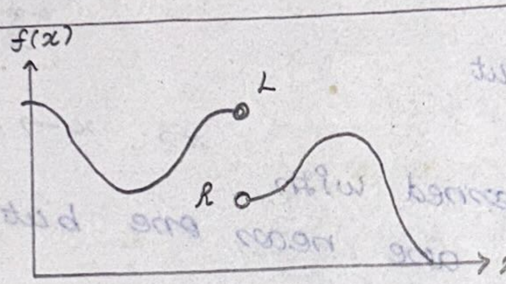
Graph:



(At $x=1$, $f(x)$ - not defined - put open circle)

Approaching a certain value → limit of $f(x)$ as $x \to 1^+$ is 2.

$$\lim_{x \to 1^+} f(x) \Rightarrow 2, \quad \Bigg| \quad \lim_{x \to 1^-} f(x) \Rightarrow -2.$$



Suppose $f(x) \to R$ for values of x that really get close to (but are not equal to) a from the right. Then we say R is the right hand limit of the function $f(x)$ as x approaches a from the right.

$$f(x) \to R \quad \text{as} \quad x \to a^+$$

(or)

$$\lim_{x \to a^+} f(x) = R.$$

If $f(x)$ gets really close to L for values of x that get close to (but are not equal to) a from the left, we say that L is the left-hand limit of the function $f(x)$ as x approaches a from the left.

we write,

$f(x) \to L$ as $x \to a^-$

(or)

$$\lim_{x \to a^-} f(x) = L$$

Lets $g(x) = \frac{x}{\tan(2x)}$ as $x \to 0^{\pm}$

x	$g(x)$
1	-0.458
0.5	0.321
0.1	0.493
0.05	0.498
0.01	0.4999

$$\lim_{x \to 0^+} g(x) = 0.5$$

$$\lim_{x \to 0^-} g(x) = 0.5$$

x	(Deg) $g(x)$	Radian
-1	28.636	-0.4576
-0.5	28.644	0.32104
-0.1	28.647	0.49331
-0.01	28.6478	0.4999

$h(x) = \frac{|x| + \sin x}{x^2}$ as $x \to 0^{\pm}$

Solu:

x	$h(x)$
-1.0	0.159
-0.5	0.082
-0.1	0.017
-0.01	0.002
-0.001	0.0002

x	$h(x)$
1	1.8414
0.5	3.9177
0.1	19.9833
0.01	199.9983
0.001	1999.9998
0.0001	19999.9998

$$\lim_{x \to 0^-} h(x) = 0$$

$$\lim_{x \to 0^+} h(x) = \infty$$

(+ve)

↳ Limit doesn't exist.

$g(x) = \sin\left(\frac{13}{x}\right)$ as $x \to 0$ from the right.

x	$g(x)$
1	0.4201
0.5	0.7625
0.1	−0.9301
0.01	−0.5805
0.001	0.0894
0.0001	0.78083
0.00001	0.4482
0.000001	0.9979
0.0000001	−0.5976

As $x \to 0^+$, $g(x)$ approaches neither a finite number L, nor $+\infty$ nor $-\infty$.

DNE
(Limit doesn't exist)

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possible limit behaviours:

* The rh and left limits may both exist & be equal.
* " " but fail to be equal.
* A right - and / or left - limit could fail to exist due to blowing up to $\pm\infty$ (example: consider the function $\frac{1}{x}$ near $x = 0$). In this case, we either say the limit blows up to ∞. we also say that limit doesn't exist because ∞ - is not a real number.
* A right - and / or left limit could fail to exist because it osciuates b/w many values and never settles down. In this case, we say the limit doesn't exist.

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overall limit.

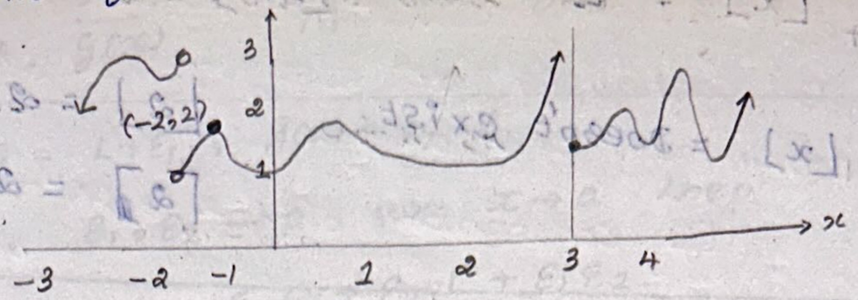
$\lim_{x \to a} f(x) = L$ where $\lim_{x \to a^-} f(x) = L$

and

$\lim_{x \to a^+} f(x) = L$

Definition:

If a function $f(x)$ approaches some value L as x approaches a from both the right & the limit, then the limit of $f(x)$ exists and equals L.



$\lim_{x \to (-2)^-} f(x) = 3$ (limit doesn't exist)

$\lim_{x \to (-2)^+} f(x) = 1$ $\quad f(-2) = 2$ (By graph) point $(-2,2)$

$\lim_{x \to 1^-} f(x) = 2$, $\lim_{x \to 1^+} f(x) = 2$, $\lim_{x \to 1} f(x) = 2$

$f(1) =$ Doesn't exist.

$\lim_{x \to 3^-} f(x) =$ Doesn't exist, $\lim_{x \to 3^+} f(x) = 1$

$\lim_{x \to 3} f(x) =$ DNE, $f(3) = 1$.

If we know $f(a)$ exists, this doesn't means that $\lim_{x \to a} f(x)$ exist.

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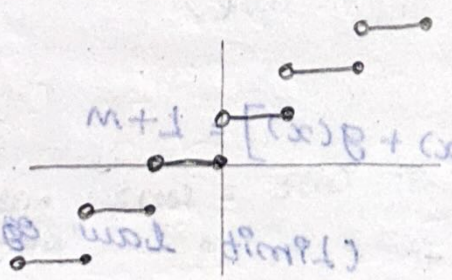
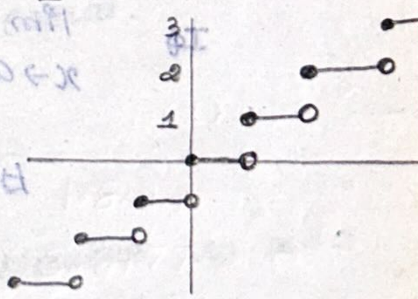
floor function $\to$ floor $(x) = \lfloor x \rfloor$

ceil function $\to$ ceil $(x) = \lceil x \rceil$



(ceil function)

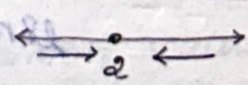


(Floor function)

floor $(2.4) = 2 = \lfloor 2.4 \rfloor$

ceil $(2.4) = \lceil 2.4 \rceil = 3$ while $\lceil 2 \rceil = 2$

$\lim_{x \to 2^-} \lfloor x \rfloor = 1$



eg: $\lfloor 1.5 \rfloor = 1$

$$\lim_{x\to 2^+} \lfloor x \rfloor = 2 \qquad \lfloor 2.5 \rfloor = 2$$

$$\lim_{x\to 2} \lfloor x \rfloor = \text{Doesn't exist.} \qquad \lfloor 2 \rfloor = 2. \to \text{Ground}$$

$$\lceil 2 \rceil = 2 \to \text{ceil.}$$

Limit laws

$$\lim_{x\to a} f(x) = 5 \quad , \quad \lim_{x\to a} g(x) = 3$$

combinations : $\lim_{x\to a} [f(x)+g(x)] = 5+3$
$= 8$

$x\to a$, $f(x) = 5+\varepsilon_1 \to$ small error

$x\to a$, $g(x) = 3+\varepsilon_2$

as $x\to a$, $\varepsilon_1, \varepsilon_2 \to 0$

$f(x)+g(x) = 8+\varepsilon_1+\varepsilon_2$ [as $x\to a$, $\varepsilon_1, \varepsilon_2 \to 0$]

$\boxed{f(x)+g(x) = 8}$ $\because \varepsilon_1+\varepsilon_2 \to$ very small.

Simply,

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$

then

$$\lim_{x\to a} [f(x)+g(x)] = L+M$$

(Limit law of addition)

III^{ly}

$$\lim_{x\to a} [f(x)-g(x)] = L-M$$

$$\lim_{x\to a} [f(x)\cdot g(x)] = L\cdot M$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad (\text{if } g(x) \neq 0)$$

$f(x) = L + \varepsilon_1$, $g(x) = M + \varepsilon_2$

where, $\varepsilon_1, \varepsilon_2 \to 0$ as $x \to a$ then

$f(x)\, g(x) = LM + \varepsilon_1 M + \varepsilon_2 L + \varepsilon_1 \varepsilon_2$

$\therefore$ As L & M are constants and $\varepsilon_1, \varepsilon_2$ tend to zero, all three error terms $\varepsilon_1 M$, $\varepsilon_2 L$, $\varepsilon_1 \varepsilon_2$ will go to zero as $x \to a$,

then,

$$\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$$

Continuity at a point.

f is continuous at $x = a$ if

$\lim_{x \to a} f(x) = f(a)$ [No break down]

$f(a) \to$ exist.

Continuous at $x = a$:

$\lim_{x \to a} f(x)$ exists

$f(a)$ exists

equal.

when

$\lim_{x \to a^+} f(x) = f(a) \to$ right - continuous at $x = a$

$\lim_{x \to a^-} f(x) = f(a) \to$ Left - continuous at $x = a$.

(Function may jump up or down, (or) DNE at $x = a$)

Rational function is the quotient of two polynomial functions.

Consider $g(x) = \dfrac{1}{x - 17}$. At which values of x is this function continuous?

overall Continuity.

If f is continuous at every point, we say f is continuous everywhere.

eg: constants, $g(x) = x$, $|x|$ — Absolute functions.

$\sin x$, $\cos x$.

$\tan x = \frac{\sin x}{\cos x}$ $\cos x \neq 0$ But $\boxed{\cos \frac{\pi}{2} = 0}$

$|x|$ — Non negative value of x.

A function $f(x)$ is continuous if for every point c in the domain of $f(x)$, the function f is continuous at the point $x = c$.

$\lim_{x \to 3} \lfloor x \rfloor = \lfloor 3 \rfloor \rightarrow$ False

∴ Floor functions are right continuous

$\lim_{x \to 3^+} \lfloor x \rfloor = 3 = \lfloor 3 \rfloor = 3$ $\boxed{\text{but } \lim_{x \to 3^-} \lfloor x \rfloor = 2}$

Limit laws + continuity.

$\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} g(x) = g(a)$

$h(x) = f(x) \cdot g(x)$

$x \to a$ $h(a) = f(a) \cdot g(a)$

$\therefore \lim_{x \to a} h(x) = h(a)$

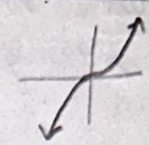
h is continuous at a.

$\therefore f, g$ continuous everywhere $\Rightarrow$ $\left.\begin{array}{l} f \cdot g \\ f+g \\ f-g \end{array}\right\}$ continuous everywhere

Basically any polynomial function will be continuous.

$\frac{f}{g}$ is continuous where defined.

$\tan x$ is continuous $[-\pi/2, \pi/2]$



If $f, g \to$ continuous
$f \circ g \to$ continuous } everywhere.

Let,
$f(x) = \sin x$, $g(x) = x^2+1$.
$f(g(x)) = \sin(x^2+1) \to$ continuous everywhere.
$\therefore$ $x^2+1 \to$ continuous everywhere.

The following functions are continuous at all real numbers:

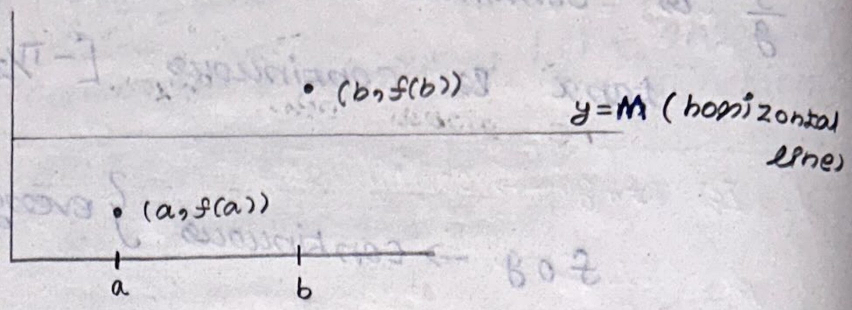
* All polynomials
* $\sqrt[3]{x}$
* $|x|$
* $\cos x$ and $\sin x$
* exponential functions a^x with base $a>0$

Right continuous:

* $\sqrt{x}$, $x \geq 0$
* $\tan x$ where all x, $\tan x$ is defined.
* logarithmic functions $\log_a x$ with base $a>0$, $x>0$

Intermediate value theorem

Two point on the graph of a function.



(Does the curve need to intersect)

Intermediate theorem: states that f is continuous and we have some value M that's b/w the values of f(a) and f(b), in other words, M is an intermediate value then there is atleast one point c b/w a and b such that f(c) = M

f → Don't need to be everywhere but atleast b/w a and b.

If f is continuous on the open interval (a, b), Right continuous at a, Left continuous at b.

If f is a function which is continuous on the interval [a, b], and M lies b/w the values of f(a) and f(b), then there is atleast one point c b/w a and b such that f(c) = M.

(A function f is continuous on a closed interval [a, b], if it is right continuous at a, left continuous at b, and continuous at all points b/w a and b).

Continuity at a point is a local information. [we only require the knowledge of the function's behaviour near that point alone). If we know this at every point, we can tell the global behaviour.

Using IVT

$f(x) = x^4 - x - 1 = 0$? $(M = 0)$

↳ continuous everywhere.

IVT: If f is continuous on $[a, b]$, and M is b/w $f(a)$ & $f(b)$, then there is some c b/w a and b.

$$f(c) = M.$$

Soln:

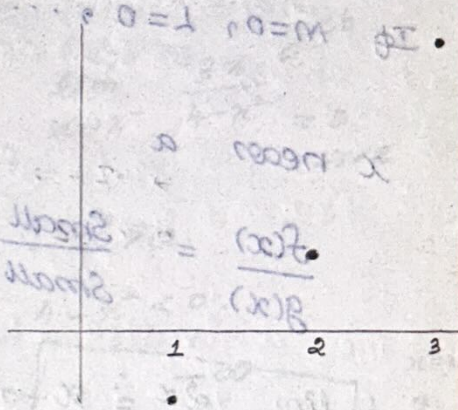
$M = 0$

$f(1) = -1$

$f(3) = 77$

we can say, the root is b/w 1 and 2. [Then 1 and 1.5]

↳ or by Newton's method.



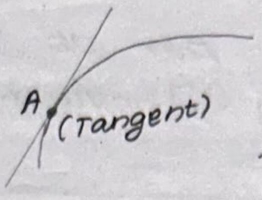
Descartes' rule of signs:

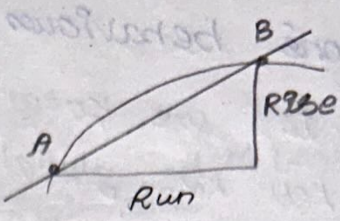
It asserts that the no. of +ve roots is atmost the no. of sign changes in the sequence of polynomial's coefficients (omitting the zero co.eff). The difference b/w these two numbers is always even.

∴ The no. of +ve roots of the polynomial is either equal to the no. of sign changes b/w consecutive non zero co.eff, or is less than it by an even number. A root of multiplicity k is counted as k roots.

Limit of quotients:

$\frac{f}{g}$ (Both are zero)





$\lim_{x\to a} f(x) = L$

$\lim_{x\to a} g(x) = M$

$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, if $M \neq 0$

If $M = 0$, $L \neq 0$

$x \to a$: $\frac{f(x) \to \text{not small}}{g(x) \to \text{small}}$ = huge!

If $M = 0$, $L = 0$, ... ?

x near a

$\frac{f(x)}{g(x)} = \frac{\text{Small}}{\text{Small}} = ?$ [we need to know how small - num & denominator]

$\boxed{\lim_{x\to 0} \frac{2x}{x} = x} \to \therefore x \neq 0$ $\boxed{x \to 0 \text{ (closer)}}$

$\lim_{x\to -1} f(x) = 0$, $\lim_{x\to -1} g(x) = 17$, $\lim_{x\to -1} h(x) = 0$

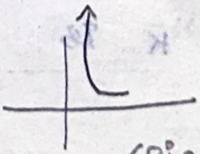
Limit & division

Limit doesn't exist means:

* $+$ve ∞
* $(-)$ve ∞
* may be function goes crazy.

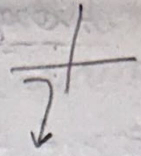
$\lim_{x\to 0^+} \frac{1}{x}$ $\frac{\nearrow 1}{\to 0}$

DNE



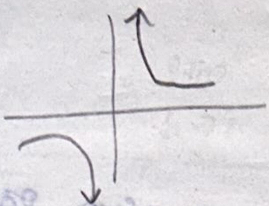
(Right)

$\lim_{x \to 0^-} \frac{1}{x} = DNE$ ($\to 1$, $\to -\infty$)

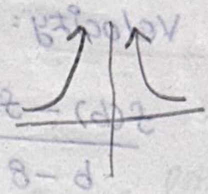


The $\lim_{x \to 0} \frac{1}{x}$ doesn't exist, and it is neither $+\infty$, nor $-\infty$).

$\lim_{x \to 0} \frac{1}{x}$



$\lim_{x \to 0} \frac{1}{x^2}$



+ve infinity alone

$\therefore$ Let $\lim_{x \to 2} \frac{3x}{4-x^2}$

$\lim_{x \to 2^-} \frac{3x}{4-x^2}$ ($3x \to \approx 6$, $4-x^2 = \approx 0$ +ve)

$\therefore 4 > x^2$

$\therefore = +\infty$

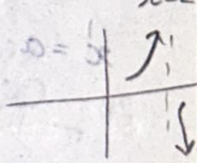
$\lim_{x \to 2^-} \frac{3x}{4-x^2}$ ($3x \approx 6$, $4-x^2 \approx 0$ (−)ve)

$\therefore x^2 > 4.$

$4 - x^2 < 0.$

$\therefore -\infty$

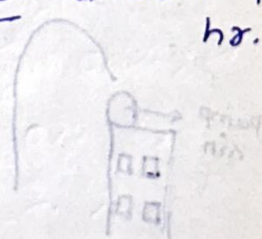
$x=2$



Average velocity

$8.00\,Am - 8.01\,Am. = \frac{f(8.01) - f(8.0)}{8.01 - 8.00}$

B/w 8Am and 10.Am $= \frac{220-50}{2} = 85$ miles/hr.



$\therefore f(8) = 50$ miles

$f(10) = 220$ miles.

Average rate of change.

$$\frac{51-50 \text{ mile}}{1\text{min}} = 1 \frac{\text{mi}}{\text{min}}$$

$$= 60 \text{ miles/hours.}$$

$$\text{Avg change} = \frac{\Delta f}{\Delta x} = \frac{f(b)-f(a)}{b-a}.$$

The derivative at a point.

Instantaneous Velocity:

$$\lim_{b \to 8} \frac{f(b)-f(8)}{b-8}$$

Instantaneous rate of change of $f(x)$ at $x=a$ $=$ The derivative of $f(x)$ at $x=a$

$$f'(a) = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}.$$

$$\left[\lim_{b \to 8} \frac{f(b)-f(8)}{b-8} \right]$$

↳ In car problem.

The derivative of a function $f(x)$ at a point $x=a$ is defined to be

$$f'(a) = \lim_{b \to a} \frac{f(b)-f(a)}{b-a}$$

(or)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Throwing a pumpkin

Height at time t seconds $= f(t) = 100 + 20t - 5t^2$ metres

100: Initial height $(t=0)$

$20t$: velocity

$-5t^2$: Effect of gravity.

Average velocity b/w $t = 0$ and $t = 1$

$$= \frac{f(1) - f(0)}{1 - 0}$$

$$= \frac{(100 + 20 - 5) - 100}{1}$$

$$= 115 - 100$$

$$= 15 \ m/s$$

Instantaneous velocity at $t = 1s$

$$f'(1) = \lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}$$

$$= \lim_{b \to 1} \frac{(100 + 20b - 5b^2) - 115}{b - 1}$$

$$= \lim_{b \to 1} \frac{20b - 5b^2 - 15}{b - 1}$$

$$= \lim_{b \to 1} \frac{-5(b-1)(b-3)}{b-1}$$

$$= \lim_{b \to 1} -5(b-3)$$

$$= -5(-2)$$

$$= 10 \ m/s.$$

$\therefore$ At $t = 1s$, gravity comes into the picture. (slower than average velocity)

Instantaneous velocity at $t = 3$

$\therefore f(3) = 100 + 60 - 45 = 115.$

$$f'(3) = \lim_{b \to 3} \cancel{-5(b-3)}$$

$$= 0 \ m/s \quad \text{(At the maximum) (peak)}$$

$$f'(3) = \lim_{b \to 3} \frac{f(b) - f(3)}{b - 3}$$

$$= \lim_{b \to 3} \frac{(100 + 20b - 5b^2) - 115}{b - 3}$$

$$= \lim_{b \to 3} \frac{20b - 5b^2 - 115}{b - 3}$$

$$= \lim_{b \to 3} \frac{-5(b-1)(b-3)}{b-3}$$

$$= -5(3-1)$$

$$= -5 \times 2 = -10 \ m/s \rightarrow \text{moving downwards.}$$

The sign of the derivative

$$f'(2) = \lim_{b \to 2} \frac{f(b) - f(2)}{b - 2}$$

$$= \lim_{b \to 2} \frac{100 + 20t - 5t^2 - (100 + 40 - 20)}{(b-2)}$$

$$= \lim_{b \to 2} \frac{100 + 20t - 5t^2 - 120}{(b-2)}$$

$$= \lim_{b \to 2} \frac{-20 + 20t - 5t^2}{b-2}$$

4
-2 -2
-4

$$= \lim_{b \to 2} -5 \frac{(t^2 - 4t + 4)}{b-2}$$

$$= \lim_{b \to 2} -5(b-2)$$

$$= -5(0)$$

$= +0\ m/s$. (At max → height

If $f'(a) > 0$, f is increasing at a

If $f'(a) < 0$, f is decreasing at a.

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3 interpretations:

1. physical interpretation. (Instantaneous rate of change)
2. Geometrical interpretation (slope of the tangent line)
3. sensitivity measurement

Tangent line.

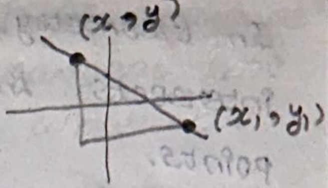
$y = f(x) \rightarrow$ line   point $(x_0, y_0) = (a, f(a))$

slope $\rightarrow m$

Tangent line:

$$y - f(a) = m(x - a)$$

$$m = \frac{y - y_1}{x - x_1}$$



$m(x - x_1) = y - y_1 \rightarrow$ Line formula (equ)

$\therefore \boxed{m = \frac{y - y_1}{x - x_1}}$ (Rate of change b/w two points)

$\therefore m = f'(a)$

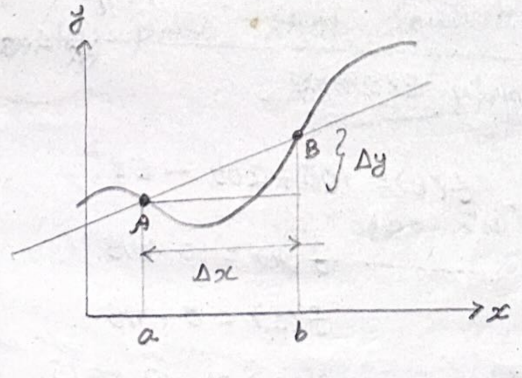
In high school that a tangent line is "a line that touches the curve in only one point." This is true if your curve is a circle, but for many other curves and functions, this is a terrible definition.

Intuition for tangent lines.

Tangent line is the one where both the slope of the S.L and the curve are the same.

The tangent line is only a good approximation for our function in this small zoomed in neighborhood. far away from this point, the tangent line and the function don't agree at all.

Secant line.



$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}.$$

Tangent :

The tangent line is "going in the same direction as the curve".

Secant line:

A line that intersects the curve at a minimum of two distinct points. In circle - exactly at two points.

In Geometry, a secant of a curve is a line that intersects the curve at a minimum of two distinct points.

Secant comes from the latin word (secare) meaning to cut.

$$\Delta y = f(b) - f(a)$$

$$\boxed{\Delta y = \Delta f}$$

$$\Delta x = b - a$$

$\therefore\ m = \text{slope} = \dfrac{f(b) - f(a)}{b - a}$ = Average rate of change of $f(x)$ w.r. to x.

when we move the point b closer and closer to a (at a, it will be the tangent).

limit as $b \to a$

Interpretations

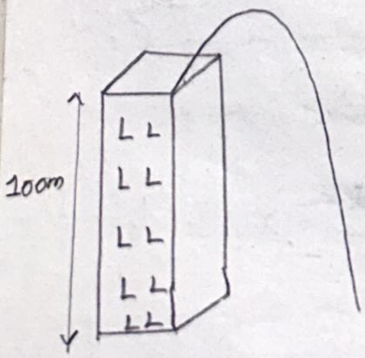
Geometric: secant line $\xrightarrow[b \to a]{}$ tangent line

slope slope

Symbolic: $\dfrac{f(b) - f(a)}{b - a} \xrightarrow[b \to a]{}$ $f'(a)$ (Derivative of the function at a)

Physical: Average rate of change of our function, $\xrightarrow{b \to a}$ (Instantaneous rate of change of our function at a)

Pumpkin example



$$f(t) = 100 + 20t - 5t^2$$

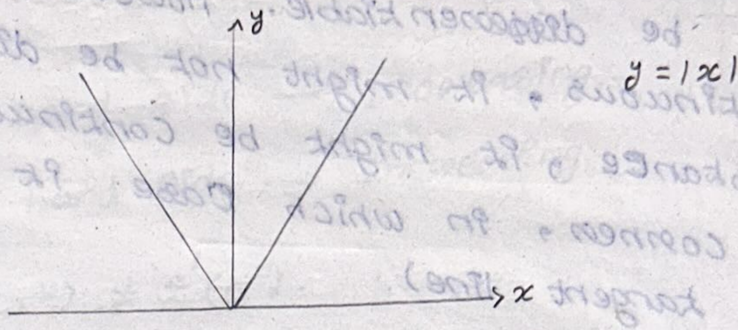
$$f'(1) = 10 \text{ m/s}$$

$$f'(2) = 0 \text{ m/s}$$

$$f'(3) = -10 \text{ m/s}$$

$\therefore$ By estimating the slope of the curve, we can estimate the derivative of the curve.

The slope of the tangent line, which is known as the derivative, only exists if the tangent line exists! Let's explore some cases, when the tangent line doesn't exist.



This is an example of a graph that doesn't have a tangent line at this point. (No slope) f' doesn't exist.

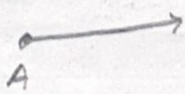
$$\lim_{x \to 0^+} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

$$\lim_{x \to 0^-} \frac{|x| - 0}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{-x} = -1.$$

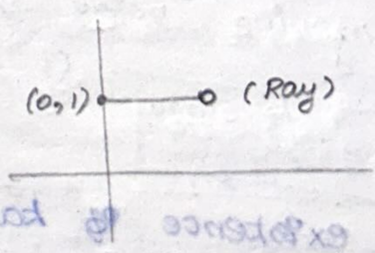
$\therefore$ Limit doesn't exist. Hence, the derivative also doesn't exist here.

$\therefore$ Since $f'(0)$ doesn't exist, we say that f is not differentiable at $x = 0$.

Ray - part of a line, that has a fixed starting point that doesn't have an end point.



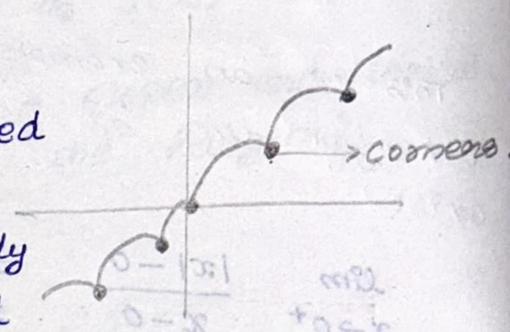
eg: step function (0, 1).



Hence technically, no tangent line there. f is not differentiable at this point (0, 1).

If f is not continuous at $x=a$, then f is not differentiable at a.

(If a function has a discontinuity at a point, then it can't have a tangent line and so it can't be differentiable. However, even if it is continuous, it might not be differenciable. for instance, it might be continuous but have a corner, in which case it doesn't have a tangent line)



Corner:

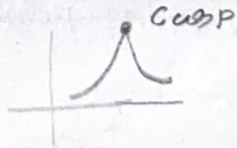
A derivative at a specified point is only defined for a function where there is only one slope at that specified point. A corner is one type of shape to a graph that has a different slope on either side.

∴ Because of a disagreement b/w right hand & left hand limits.

Cusp:

A cusp, where the derivative is not defined.

(A sharp point on a curve)



cusp - singularity.

Also known as spinode, is a point on a curve where a moving point must reverse direction.

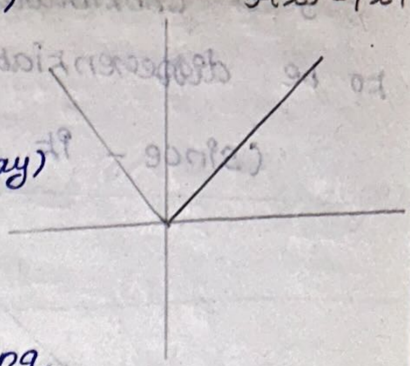
↳ singularity (Function goes to ∞ at some point)

The existence of tangent lines & derivatives.

No tangent line ⇒ No derivative.

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = f'(0^+) = 1 \text{ (say)}$$
(+ve)

$f(x) = |x|$



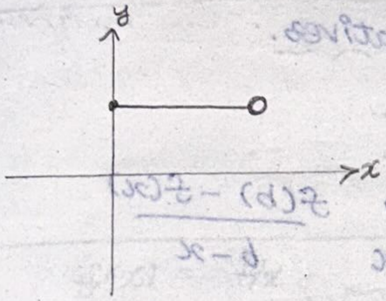
$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = f'(0^-) = -1 \text{ (say)}$$
(−ve).

$\therefore$ Left hand side $\longrightarrow$ Decreasing.

Right hand side $\longrightarrow$ Increasing

$\therefore$ $f'(0^+) \neq f'(0^-)$.

$\therefore$ $f'(0) \rightarrow$ doesn't exist.



Right-sided derivative

$f'(0^+) = 0$

Left hand derivative

$f'(0^-) = \infty$

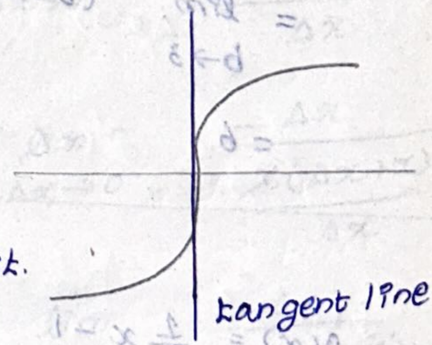
If tangent line exists, does the derivative exist?

yes, Except for vertical tangent lines.

eg: $f(x) = \sqrt[3]{x}$.

At $x = 0$, $m = \infty$

$f'(0) =$ doesn't exist.



$f(x) = 2x^2 + 3x$ at $x = 1$.

when $x = 1$, $y = 5$ (1, 5)

Slope (m) $= 4x + 3$ $(x = 1)$

$= 7$

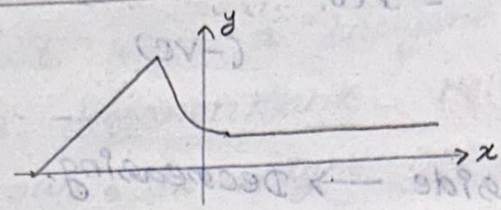
$y - 5 = 7(x-1)$

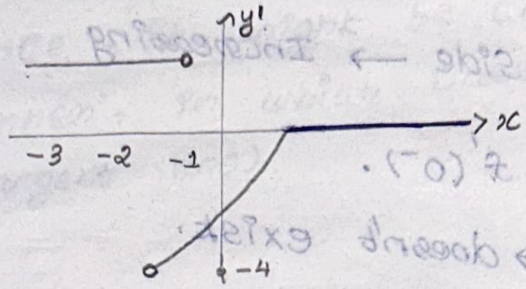
$y = 7x - 7 + 5$

$y = 7x - 2$

∴ f is continuous at $x = a$, then it need not to be differentiable there.

(since - it may have corners).





Calculating derivatives.

$f(x) = x^2$.

$$f'(3) = \lim_{b \to 3} \frac{f(b) - f(3)}{b - 3}$$

$$= \lim_{b \to 3} \frac{b^2 - 9}{b - 3}$$

$$= \lim_{b \to 3} (b+3)$$

$$= 6$$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$

$$= \lim_{b \to x} \frac{b^2 - x^2}{b - x}$$

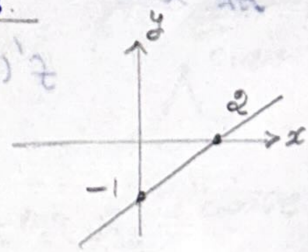
$$= \lim_{b \to x} (b + x)$$

$$= 2x.$$

∴ It's a Polynomial — Continuous.

Linear functions

$g(x) = \frac{1}{2}x - 1$



$g'(x) = \frac{1}{2}$.

$\therefore \quad g(x) = mx + b$

$$\boxed{g'(x) = m}$$

$$g'(x) = \lim_{c \to x} \frac{g(c) - g(x)}{c - x} = \lim_{c \to x} \frac{mc + b - mx + b}{c - x}$$

$$= \lim_{c \to x} \frac{mc - mx}{c - x} = m.$$

Special Case:

$$\text{If } g(x) = 0x + b$$
$$= b$$

then $g'(x) = 0.$ (A constant function, doesn't change)

Another notation.

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}.$$

$b - x = \Delta x$

$\therefore b = x + \Delta x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

why $\Delta x \to 0$

b moves closes to a

$\boxed{\Delta x \to 0}$

$f(x) = 1/x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x - x - \Delta x}{x(\Delta x + x)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{-\Delta x}{x(\Delta x + x)}}{\Delta x}$$

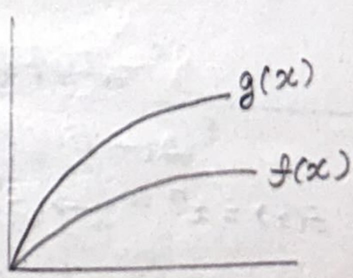
$$= \lim_{\Delta x \to 0} \frac{-1}{x(\Delta x + x)}$$

$$= -\frac{1}{x^2}$$

Derivatives of constant multiples.

$g(x) = 2f(x)$

$g'(a) = 2f'(a).$



If $g(x) = Kf(x)$ for all x, then

$$g'(x) = Kf'(x)$$

eg: $g(x) = -5x^2$

$g'(x) = -10x$.

Derivative of a sum

$h(x) = f(x) + g(x) \rightarrow h'(x) = f'(x) + g'(x)$

$h(x) = f(x) - g(x) \rightarrow h'(x) = f'(x) - g'(x)$

$$h(x) = \frac{1}{x} + (3x-7)$$

$$h'(x) = -\frac{1}{x^2} + 3$$

proof

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(f(x+\Delta x) + g(x+\Delta x)) - (f(x) + g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right)$$

$$= f'(x) + g'(x)$$

strategy

$f'(1) = ?$ [we need to differenciate first]

then need to apply $\boxed{x=1}$

$f(x) = -3x^2 + \frac{1}{x} - 2$

$f(1) = -4$

$f'(x) = -6x - \frac{1}{x^2}$

$f'(1) = -6 - 1$

$= -7$.

power rule.

$f(x) = x^n \rightarrow f'(x) = ?$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b-x} = \lim_{b \to x} \left(\frac{b^n - x^n}{b-x}\right)$$

$$b^n - x^n = (b-x)(b^{n-1} + b^{n-2}x + b^{n-3}x^2 + \dots + b x^{n-2} + x^{n-1})$$

$$f'(x) = \lim_{b \to x} (b^{n-1} + b^{n-2}x + b^{n-3}x^2 + \dots + b x^{n-2} + x^{n-1})$$

$$= x^{n-1} + x x^{n-2} + x^2 x^{n-3} + \dots + x \cdot x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + x^{n-1} + x^{n-1} + \dots$$

$$= n x^{n-1}$$

Factoring $b^n - x^n$

$$(b-x)(b+x) = b^2 - x^2$$

$$(b-x)(b^2 + bx + x^2) = b^3 - x^3$$

$$(b-x)(b^{n-1} + b^{n-2}x + b^{n-3}x^2 + \dots + b x^{n-2} + x^{n-1}) = b^n - x^n$$

Expanding

$$= (b^n + b^{n-1}x + b^{n-2}x^2 + \dots + b^2 x^{n-2} + b x^{n-1})$$
$$- (b^{n-1}x + b^{n-2}x^2 + \dots + b x^{n-1} + x^n)$$

$$= b^n - x^n$$

General power rule.

$f(x) = x^n$, $f'(x) = n x^{n-1}$

True when n is any fixed number.

warning: Doesn't apply to $f(x) = 2^x$

$g(t) = (\cos t)^3$

$h(x) = x^x$

Tangent line of a polynomial.

Compute the tangent line to the curve $y = x^3 - x$ at $(2, 6)$.

Solu:

$y' = 3x^2 - 1$

$y'(2) = 3(4) - 1$

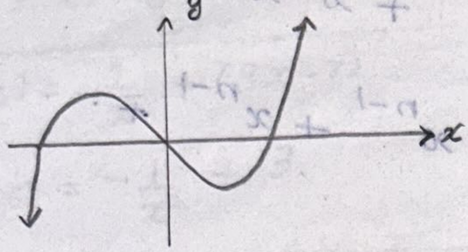
$\boxed{m = 11}$

$y - y_1 = m(x - x_1)$

$y - 6 = 11(x - 2)$

$y = 11x - 22 + 6$

$\boxed{y = 11x - 16}$



∴ Slope of both line & curve is the same incase of tangency.

Notations:

$f'(x) \rightarrow$ prime notation (Introduced by Newton).

$\frac{df}{dx} \rightarrow$ Liebniz notation (Introduced by (Spoiler alert) Leibniz.

Technically, Newton developed dot notation ($\dot{f}(t)$), which is only used with derivatives w.r.to time, and lagrange developed prime notation ($f'(x)$).

Introducing to Liebniz notation

$m = \frac{\Delta y}{\Delta x}$

$\Delta y = f(b) - f(a)$

$\Delta x = b - a$

$\frac{\Delta y}{\Delta x} \xrightarrow[\Delta x \to 0]{} \frac{dy}{dx}$ [Leibniz notation]

d – Infitesimal version (extremely small)

↓

An indefinitely small quantity; a value approaching zero.

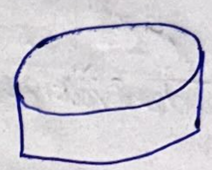
Δ – Small difference.

At $x=3$ can be written as

$$\left.\frac{df}{dx}\right|_{x=3}$$ ← d- limit of a difference

why do we like to use leibniz notation?

The biggest reason is that it reminds us what the input variable is. The derivative is measuring the instantaneous rate of change of the output variable of a function w.r.t the input variable.



circumference

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\boxed{\begin{aligned} A &= f(r) \\ A' &= f'(r) \end{aligned}}$

$C = 2\pi r$

$r = \frac{C}{2\pi}$

$A = \frac{\pi C^2}{4\pi^2}$

$= \frac{C^2}{4\pi}$

$\frac{dA}{dC} = \frac{2C}{4\pi}$

$= \frac{C}{2\pi}$

$\boxed{\begin{aligned} A &= g(C) \\ A' &= g'(C) \end{aligned}}$

we will be in track with physical quantities while using Leibniz notation.

<u>Properties of Leibniz notation:</u>

Units: If P has units of pressure, and t has units of time, then $\frac{dP}{dt}$ has units of pressure per time.

Evaluating at points: If we want to take the derivative at $x = 3$, then we use the notation

$$\left.\frac{df}{dx}\right|_{x=3}$$ (The bar is read as "evaluated at")

* Derivatives act on functions

$\frac{d(x^2)}{dx} \rightarrow$ For x^2 derivative

$\frac{d}{dy}(y^3 + 2y^2) \rightarrow$ For $(y^3 + 2y^2)$ derivative.

Introducing higher derivatives.

$(f')'(x) = f''(x)$ (or) $\frac{d}{dx}\left(\frac{df}{dx}\right) = \left(\frac{d}{dx}\right)^2 (f)$

$= \frac{d^2 f}{dx^2}$

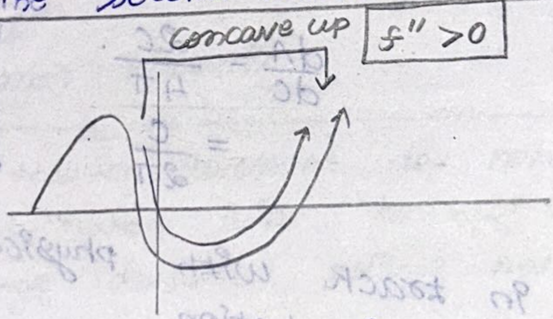
Third derivative $f'''(x) = \frac{d^3 f}{dx^3}$

In case of too many primes

$f^{(3)}(x) = \frac{d^3 f}{dx^3}$

Concave down $f'' < 0$

The second derivative & Concavity.



$f(x) = -2x^4 + 3x^3 + 1$

$f(1) = -2 + 3 + 1$

$= 2.$

$f'(x) = -8x^3 + 9x^2$

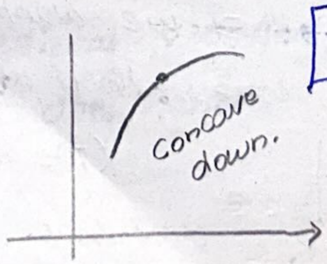
$f'(1) = -8 + 9$

$= 1$

$f''(x) = -24x^2 + 18x$

$f''(1) = -24 + 18$

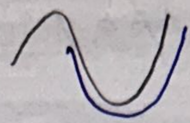
$\boxed{f''(1) = -6}$



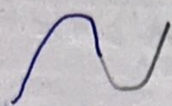
So $f''(x) < 0$ near 1

Concave down.

If $f'' > 0$, the function is concave up



$f'' < 0 \rightarrow f$ is concave down



Rate of job growth slows.

f'

↳ goes down.

$f' > 0$ [slower than previous quarter)

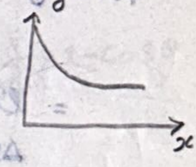
$f'' < 0$

$\boxed{\text{Concave down}}$

"population loss in capital city eases"

$p'(t) \rightarrow$ negative, $p''(t) \rightarrow +ve$

concave up



Acceleration.

$f(t) \rightarrow$ position, $f'(t)$ – velocity, $f''(t) \rightarrow$ acceleration

$f'' > 0 \rightarrow$ concave up

$f'' < 0 \rightarrow$ concave down

points where the graph of a function changes from concave up to down (or) vice versa, are called inflection point.

Trignometric functions.

Sound, light, and electricity all involve oscillations.

whole world is full of oscillating behaviours.

Spring →

Derivative of $\sin x$

$$\frac{\sin(x+\Delta x)-\sin x}{\Delta x} = \frac{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin x}{\Delta x}$$

$\sin(a+b) = \sin a\cos b + \cos a\sin b.$

$$= \frac{\sin x\,(1-\cos\Delta x) + \cos x\sin\Delta x}{\Delta x}$$

$$= \frac{\sin x}{\Delta x}(1-\cos\Delta x) + \cos x\,\frac{\sin\Delta x}{\Delta x}$$

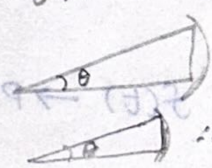
Taking limit for derivative.

$$= \lim_{\Delta x\to 0}\left(\frac{\sin x}{\Delta x}(1-\cos\Delta x) + \cos x\,\frac{\sin\Delta x}{\Delta x}\right)$$

At $x=0$

$$= \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}$$

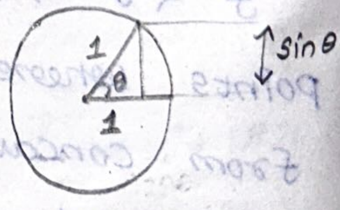
$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$



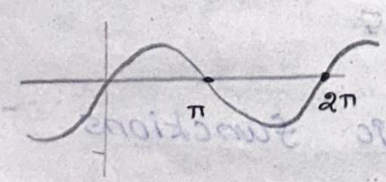
Geometric proof

Let $\Delta x\to\theta$, $\theta\to 0$

$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$ (As θ getting smaller & smaller)

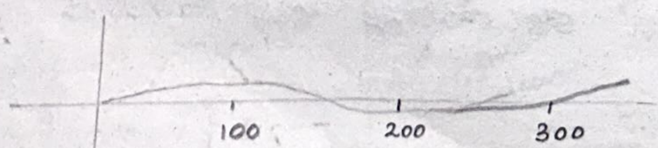


In radian



(Radian)

S



(In degrees - Hard & not so perfect)

(while taking slope)