Homogeneous: Seperation of variables

Inhomogeneous: variation of parameters

Integrating factors.

Superposition:

1) Multiplying a to the solution of

$$P_n(t) y^n + \cdots + P_0(t) y = q(t)$$ gives

the solution of

$$P_n(t) y^n + \cdots + P_0(t) y = a\, q(t).$$

2) Adding the solution of $P_n(t) y^n + \cdots + P_0(t) y = q_1(t)$ with the solution of $P_n(t) y^n + \cdots + P_0(t) y = q_2(t)$ gives the solution of $P_n(t) y^n + \cdots + P_0(t) y = q_1(t) + q_2(t)$

Existence & uniqueness theorem for a linear ODE:

Let $P(t)$ & $q(t)$ be continuous functions on an open interval $I$. Let $a \in I$, and let $b$ be a gn number. Then there exists a unique solution to the I order linear ODE.

$$\dot{y} + P(t) y = q(t)$$

Satisfying the Initial Condition

$$y(a) = b.$$

Caprice - lack of motivation

## Differential equations (DE)

Equation relating an unknown function and some of its derivative.

Guess a secret function $y(t)$ which satisfies

$$\frac{dy}{dt} = 3y$$

It can be satisfied by $y = 3e^{3t}, 7e^{3t}, -5e^{3t}, 0$, $e^{3t}$

$\therefore$

$$\frac{d}{dt}(3e^{3t}) = 3(e^{3t} \times 3)$$

$$9e^{3t} = 9e^{3t}$$

$\therefore$ A family of solutions are available.

General form = $y = Ce^{3t}$

---

Cell division

we'll model the no. of yeast cell in a batch of dough. (மாவுக்கு மாவு). As we work through this example, pay careful attention to the assumptions we make, and how the initial condition plays a role in the resulting differential equation.

System:

For our system, we assume a colony of yeast in a batch of dough,

$y$ – Number of cells

$t$ – Time measured in seconds.

$\therefore$ we need to get initial conditions $y_0$,

$\therefore$ There might be the numbers of cells in a yeast packet at $t=0$.

Differential model:

$\frac{dy}{dt}$ → Rate at which the no. of cells is growing. Then how $\frac{dy}{dt}$ depends on $y$?

In nature, cells given plenty of space & food tend to divide through mitosis regularly. If we assume that each cell is dividing independently of all other cells, then doubling the no. of cells should double the rate at which new cells are born. In fact, multiplying the no. of cells by any scalar factor should do the same to its derivative. So this directly implies that the growth rate of cells is proportional to the number of cells.

$$\boxed{y' \propto y}$$

By adding a proportionality constant $a$,

$$\frac{dy}{dt} = ay$$

exponential function:

$$e^0 = 1$$

$$e^{at+c} = e^c \cdot e^{at}$$

$$= e^{at} \rightarrow \text{Never zero}$$

If $a>0$ then $\lim_{t\to\infty} e^{at} = \infty$ and $\lim_{t\to-\infty} e^{at} = 0$

If $a<0$ then $\lim_{t\to\infty} e^{at} = 0$ and $\lim_{t\to-\infty} e^{at} = \infty$

b) For any +ve $a$, $e^{at}$ grows much faster than any polynomial.

---

<u>variables</u>:

$$f(x) = 3x^2 + 2x + 1$$

$x \rightarrow$ Independent variable

$y \rightarrow$ Dependent variable

<u>parameters:</u>

$$\int t^2\, dt = \frac{t^3}{3} + c$$

$c \rightarrow$ Anti derivative (Constant of integration)

---

Notations $\rightarrow$ $\frac{dy}{dx}$, $y'$ and $Dy$

$$\frac{d^2 y}{dx^2} = y'' = D^2 y$$

$$\frac{d^n y}{dx^n} = y^{(n)} = D^n y$$

$D^n$ – nth derivative

---

Differential equation is an equ expressing a relationship b/w a function & its derivatives.

$$D^2 x + 8Dx + 7x = 0 \quad \rightarrow (1).$$

when the function in the diff. equ has a single independent variable, we call it an ordinary diff equ.

ordeer of diff equ:

order of the largest derivative appearing in it.

Equ (1) is a second order diff equ.

Solving a diff equ:

Solving means finding a function that satisfies the equ.

eg: 1 Checking a solution by substitution.

verify $y(t) = e^{3t}$ is a solution to

$$\frac{dy}{dt} = 3y$$

$$\frac{d}{dt}(e^{3t}) = 3e^{3t}$$

$$3e^{3t} = 3e^{3t}$$

Satisfied.

$y = e^{3t}$ is a solution.

eg: 2 Rejecting a solution by substitution.

$y(t) = t^3$ is not a solution to the

diff equ

$$\frac{dy}{dt} = \frac{y}{t}$$

$$3t^2 = \frac{t^3}{t}$$

$$\boxed{3t^2 \neq t^2}$$

eg: 3 parametrizing the set of solutions of a diff equ

Diff equ have usually more than one solution.

we can describe them all at once using a parameter.

$$D^2x = 2t$$

Integrating,

$$Dx = \frac{2t^2}{2} + c_1$$

$$x = \frac{t^3}{3} + c_1 t + c_2$$

$c_1, c_2$ → Arbitrary Constants.

This gives a parametrized solutions.

$\therefore x = \frac{t^3}{3} + 2t + 1$ and $\frac{t^3}{3} + \pi t + 2.718$ are

both solutions.

---

Initial value problems:

Sometimes we have a diff eqn and initial conditions. Together they make up an initial value problem.

<u>eg</u>: Solve the initial value problem $D^2 x = 2t$ with $x(1) = 1$, $Dx(1) = 2$

$$x(t) = \frac{t^3}{3} + c_1 t + c_2$$

Differentiating,

$$x'(t) = \frac{3t^2}{3} + c_1$$

$$x'(1) = 1 + c_1$$

$$\boxed{2 = 1 + c_1}, \quad x(1) = \frac{1}{3} + c_1 + c_2$$

$$1 = \frac{1}{3} + c_1 + c_2$$

<u>Solving</u>:

$$2 = 1 + c_1$$

$$1 = \frac{1}{3} + c_1 + c_2$$

$$\boxed{c_1 = 1}$$

$$1 = \frac{2}{3} - c_2$$

$$c_2 = \frac{2}{3} - 1$$

$$\boxed{c_2 = -\frac{1}{3}}$$

$$x(t) = \frac{t^3}{3} + t - \frac{1}{3}$$

## Acronyms

1) Differential equation (DE)
2) Ordinary differential equ (ODE)
3) Initial value problem (IVP)
4) Initial conditions (IC)

---

$$\frac{dy}{dx} = 2y+1$$ (what will be the solution)

$$\boxed{y = 2xy + x}$$

$$y = ce^{2x} - \frac{1}{2}$$

$$2ce^{2x} = 2(ce^{2x}) - 1 + 1$$

$$\boxed{2ce^{2x} = 2ce^{2x}}$$

---

### Most important DE

1) A homogeneous linear ODE, is a diff equ such as

$$e^t \ddot{y} + 5\dot{y} + t^9 y = 0.$$

in which each summand is a function of t times one of $y, \dot{y}, \ddot{y}, \ldots$

Most general $n^{th}$ order homogeneous linear ODE:

$$P_n(t)\, y^n + P_{n-1}(t)\, y^{n-1} \ldots + P_1(t)\dot{y} + P_0(t)\, y = 0$$

$\therefore$ $P_n(t) \ldots P_0(t) \rightarrow$ coefficients.

2) An inhomogeneous linear ODE is the same except that it has also one term that is a function of t only.

$$e^t \ddot{y} + 5\dot{y} + t^9 y = 7\sin t$$

(Second-order inhomogeneous linear ODE)

$$P_n(t)y^n + P_{n-1}(t)\ y^{n-1} + \dots + P_1(t)\dot{y} + P_0(t)y = q(t)$$

3) A linear ODE is an ODE that can be rearranged into either of the two types above.

4) Either of the two forms above can be reduced further by dividing the entire DE by $P_n(t)$ so that the co.eff of the highest derivative $y^n$ becomes 1. A diff equ. written in either of the two forms above but with leading co.eff 1 is said to be in standard linear form.

$$y^n + P_{n-1}(t)\ y^{n-1} \dots + P_1(t)\dot{y} + P_0(t)y = 0$$

$$y^n + P_{n-1}(t)\ y^{n-1} \dots + P_1(t)\dot{y} + P_0(t)y = q(t).$$

---

* If $y=0$ is a solution, the ODE is homogeneous

* If $y=0$ is not a solution → Inhomogeneous

---

$\ddot{y} + 7t\,y\dot{y} = 0$ (Non linear terms — highlighted in pencil)

$\ddot{y} = e^t(y+t^2)$ → Linear

$$\boxed{\ddot{y} + (-e^t)y = e^t t^2}$$

$\dot{y} - y^2 = 0$

$y^2 - ty = \sin t$

$\dot{y} = \cos(y+t)$

$\dot{y} = ay \rightarrow$ Basic growth eqn

$\dot{y} = -ay \rightarrow$ Decay eqn.

Linear, I order, homogeneous diff eqn.

---

$\dot{y} = ay$ $\qquad\qquad$ $y' - ay = 0$

$\frac{dy}{dt} = ay.$

$\boxed{y(t) = Ce^{at}}$ $\qquad$ C-Constant

$\frac{d}{dt}(ce^{at}) = ace^{at}$

$ace^{at} = ace^{at}$

Exponential growth $(a>0)$

Decay $(a<0)$

---

Two kinds of modelling: Here we are discussing about math modelling.

Guidelines:

* Identify relevant quantities, both known & unknown, and give them symbols. Find the units for each
* Identify independent variable(s). The other quantities will be functions of them, or constants. Often time is the only independent variable.
* Write down eqn expressing how the functions change in response to small changes in the independent variable(s). Also write down any "laws of nature" relating the variables.

As a check, make sure that all summands in an eqn have the same units.

Often simplifying assumptions need to be made. The challenge is to simplify the eqn so that they can be solved but so that they still describe the real-world system well.

# A Savings account

I have a savings account earning Interest compounded daily, and I make frequent deposits or withdrawals into that account. Find an ODE with initial conditions to model the balance.

Step: 1: Simplification:

* Daily compounding is almost the same as continuous compounding. So let's assume that interest is paid continuously instead of at the end of each day.

* Similarly, let's assume that my deposits / withdrawals are frequent enough that they can be approximated by a continuous money flow at a certain rate, the net deposit rate (which is (−)ve when I am withdrawing).

Variables & Functions:

$P \rightarrow$ Initial amount that the account starts with (Dollars)

$t \rightarrow$ time from start

$x \rightarrow$ Balance

$I \rightarrow$ Interest rate (year$^{-1}$, 4%/year $= 0.04$/year)

$q \rightarrow$ The net deposit rate (Dollars/year)

$t \rightarrow$ Independent variable

$P \rightarrow$ Constant

$x, I, q \rightarrow$ Functions of $t$.

Equations:

We want to decide how the balance gets changed as time increases.

$x + \Delta x$ in time $t + \Delta t$

we can approximate the interest earned per dollar to be:

Interest earned per dollar $\approx I(t) \cdot \Delta t$

The balance in the account at time $t$ is $x(t)$

Total interest earned $= x(t) \cdot I(t)\, \Delta t$.

mean while money is put into the account at a rate $r(t)$, so

Net amount deposited } $\approx$ putting it all into the account

$$\Delta x \approx \text{Interest} + \text{deposit}$$

$$\Delta x \approx I(t)\, x(t)\, \Delta t + r(t)\, \Delta t \quad [\text{money deposit rate} \times \text{time}]$$

$$\frac{\Delta x}{\Delta t} \approx I(t)\, x(t) + r(t).$$

The smaller $\Delta t$ is, the better the approximation becomes, and in the limit as $\Delta t \to 0$.

$I(t) \to$ Not constant

$\hookrightarrow$ modeled itself as changing.

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$$\frac{dx}{dt} = I(t)\, x(t) + r(t)$$

$$\therefore \quad x(0) = P$$

But I is not only dependent on $t$, also on $x$.

---

<u>examples</u>

$$\frac{dy}{dx} = 2x.$$

$y(x) = x^2 + c \rightarrow$ General solution.

An expression like this, which parametrizes all the equ solutions is called general solution.

---

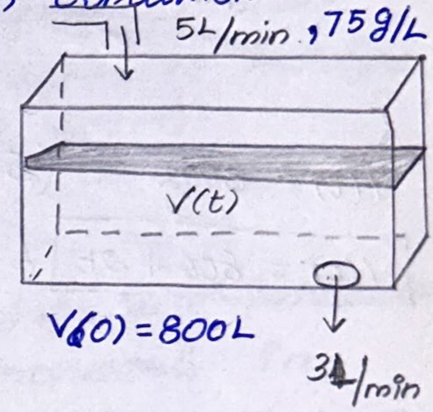
<u>Mixing Salt water for an ocean fish aquarium</u>

Here, fishes are used to analyze how cancer cells affect metabolism. So the conditions should be maintained. (concentration of salt).

Consider,

a fish living in a certain <u>condition</u>

34g / litre of salt conc.




<u>Solu</u>:

800 L of fresh water.

Flow in : 5L/min, Concentration 75g/L, Flow out 3L/min.

Assume : Instantaneously well-mixed.

Goal : model the change in salt conc over time.

Let,

$x(t) \rightarrow$ Total grams salt in tank.

$v(t) \rightarrow$ Total litres of Fluid in tank.

$$\text{Concentration} = \frac{x(t)}{v(t)}$$

$$\frac{dx}{dt}\text{ (salt amount)} = \text{rate (Salt in)} - \text{rate (Salt out)}.$$

[∴ pure water $v_0$]

$$\text{Rate in} = (5\ \text{L/min})(75\ \text{g/L})$$

$$\boxed{\text{Rate in} = 375\ \text{g/min}}$$

Rate out = (Existing conc) (3 L/min)

∴ The salt is mixed instantaneously. (Uniform)

$$\therefore \text{Concentration of salt} = \frac{x(t)}{V(t)}$$

$$\text{Rateout} = \left(\frac{x(t)}{V(t)}\right)\left(3 \text{ L/min}\right)$$

$$V(t) = 800L + (5-3)t$$

$$\boxed{V(t) = 800 + 2t} \; L$$

$$\therefore \text{Rate out} = 3\left(\frac{x(t)}{800+2t}\right) \text{ g/min}$$

$$\frac{dx}{dt} = \left(375 - 3\,\frac{x(t)}{800+2t}\right) \text{ g/min.}$$

---

From saving bank model:

$$\dot{x} = I(t)\,x(t) + v(t) \qquad \boxed{x(0) = P}$$

May be for financial planning I am interested in testing different saving strategies (different functions $v$) to see what balances $x$ they result in.

$$\dot{x} - I(t)\,x = v(t)$$

controlled by bank — Controlled by me.

In the 'System & signals' language of Engineering,

$v \rightarrow$ Input signal
Bank $\rightarrow$ system.
$x \rightarrow$ output signal.

Initial condt.

Input signal $\rightarrow$ System $\rightarrow$ output signal

The system may be mechanical system such as an automobile suspension or an electrical ckt, or an economic market. It is impacted by some external signal. we are interested in understanding the response of the system to the external stimulus.

Input signal $\rightarrow$ External signal stimulus. (It won't be so simple). It also determine the right hand side of the DE (In standard form).

<u>System response:</u>

(output signal) $\rightarrow$ measureable behaviour of the system that we are interested in. It is always the unknown function that we write a diff equ for.

All diff equ have many solutions. The solution of interest is often determined by the state of the system at the beginning. This initial state is gn by the initial conditions.

---

<u>Seperation of variables.</u> (Ref notes)

<u>Seperable:</u>

when algebra can seperate two variables.

$y' = x(y-1)$

<u>solu:</u>

$$\frac{dy}{dx} = x(y-1)$$

$$\frac{dy}{y-1} = dx$$

$$\int \frac{dy}{y-1} = \int x \, dx$$

$$\ln|y-1| = \frac{x^2}{2} + c_1$$

$$|y-1| = e^{c_1} \cdot e^{x^2/2}$$

$y-1 = \pm e^{x^2/2} \cdot e^{c_1}$

Let,

$y = 1 \pm e^{x^2/2} \cdot e^{c_1}$    $C = \pm e^{x^2/2}$

$y = 1 + Ce^{x^2/2}$

<u>Lost Solution:</u>

$\dot{y} = 2x(1-y)^2$

when $y=1$

$\boxed{0=0}$    $y=1$ is a solution.

<u>Also:</u>

$$\frac{dy}{(1-y)^2} = 2x\,dx$$

$$\frac{1}{1-y} = x^2 + c$$

$$y = 1 - \frac{1}{x^2+c}$$

$y(x)=1$ is not in the parametrized form. (So we don't use it).

$\therefore$ It is lost by seperation of variables.

$y(t) = Ce^{kt}$    $c = \pm e^{c_1}$

$\boxed{\dot{y} = ky}$

$\therefore y=0$ is a solution.

But $y(t) = ce^{kt}$ → It looks sneaky.

when $c=0$, $y=0$ ⇒ But $c = \pm e^{c_1}$ (exponential function never become 0) So we included the lost function

$$\frac{dy}{dx} = 2y+1$$

$$\int \frac{dy}{2y+1} = dx$$

$$\frac{1}{2}\ln|2y+1| + c_1 = x + c_2$$

$$\ln|2y+1| = 2x + c_3$$ (Amalgamate the constants)

$$|2y+1| = e^{c_3} e^{2x} \qquad c = \pm \frac{e^{c_3}}{2}$$

$$2y+1 = ce^{2x}$$

$$\boxed{y = ce^{2x} - \tfrac{1}{2}}$$

---

Is $y' + xy = x$ (seperable)

solu:

$$y' = x - xy$$

$$y' = x(1-y)$$

$$\frac{y'}{1-y)} = x$$

$$\frac{dy}{1-y} = x\,dx \rightarrow \text{Seperable.}$$

---

IVP- Initial value problem.

Solve:

$$\dot{y} = y^2, \quad y(0) = 1$$

solu:

$$\frac{dy}{y^2} = dx$$

$$-\frac{1}{y} = x + c$$

$$y = \frac{-1}{x+c}$$

$$1 = -\frac{1}{c}$$

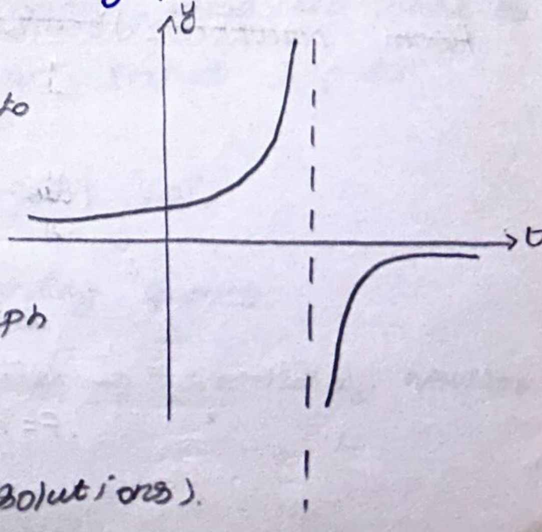
$$\boxed{c = -1}$$

$\therefore y(0) = 1$

$$y = \frac{-1}{(x-1)}$$

$$= \frac{1}{(1-x)}$$

The graph has vertical asymptote at $x=1$.



Starting $x=0$, the graph goes to infinity as $x \to 1$.

$\therefore$ Informally, we say $y$ blows up at $x=1$. The graph has two pieces.

$(-\infty, 1) \rightarrow 1$ piece

$(1, \infty) \rightarrow 2$nd piece. (2 solutions).

Linear insulation:

An insulator, which keeps the food warm.

Temperature inside = $x$

Temperature outside = $y$

$$\dot{x} = F(x, y)$$

where,

rate of change of temperature inside not directly depends on $x$ and $y$. But only on their difference.

$$\dot{x} = f(y - x)$$

when,

$x = y$, $f(0) = 0$ [No change in rate]

By linear app,

$$f(z) = Kz$$

$$f(y-x) = K(y-x)$$

$$\dot{x} = K(y-x)$$

$\dot{x} + Kx = Ky$ → Newton's law of cooling.

$K$ → coupling constant.

---

Newtonian mechanics

From Newton's second law:

$$F = ma$$

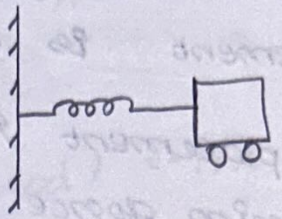
∴ (we are taking 1 dimension)

$$\boxed{a = \ddot{x}} \rightarrow \frac{d^2x}{dt^2}$$

$$F = m\ddot{x}$$

model a spring attached to a wall, with a mass on the other end sitting on a cart that moves without friction. we pull & release it. How does the mass on the cart behaves.

Solu:



$m \rightarrow$ mass

$K \rightarrow$ strength of the spring

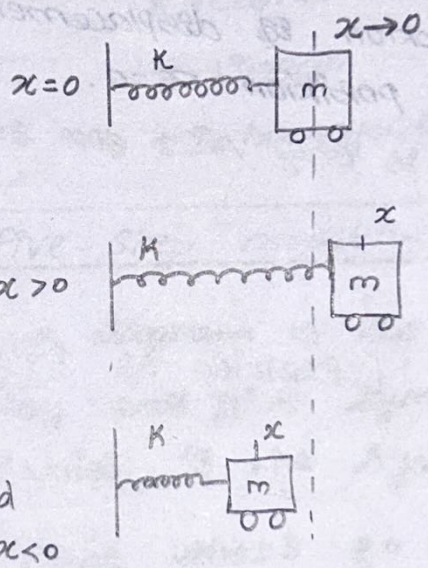
The variable of interest is the position of the mass. what means position of the mass? The most natural way is to include displacement ($x$).

spring is at rest (relax) $\rightarrow x = 0$

relaxed

stretched

compressed




System response $\longrightarrow$ Displacement of the mass.

Input signal?

Imagine: other forces also acting on the mass. like there is a sail on the mass, and the wind is blowing on the sail creating an input signal.

(Input signal = 0) $\longrightarrow$ No sail.

Stretching Force = Restoring force.

Let the force acting on the mass $\rightarrow$ Governed by Newton's second law:

$$F = m\ddot{x}$$

Assumption: Air resistance & Friction → Negligible.
but has initial conditions.

Another Force = Spring force.

For stretch → Spring force is (−)ve.

∴ Displacement is (+) ve

For compression: Displacement is (−)ve

Spring force is (+) ve.

∴ Restoring force is in opposite direction to the causing force.

$$F = -Kx$$

which is a function of displacement $x$ away from the neutral position $x = 0$.

(Linear model is only valid for small displacements)

<u>Initial conditions:</u>

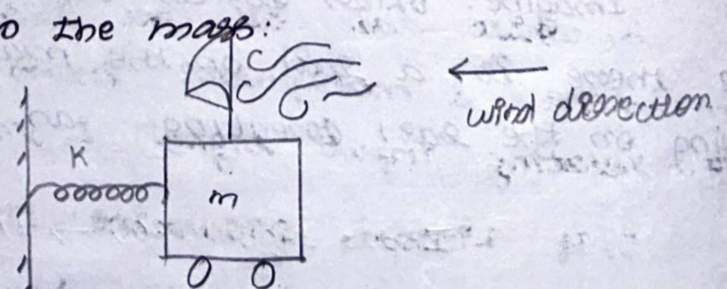
$x(0) = x_0$ [ Position at time $t = 0$)

$\dot{x}(0) = 0$

$$\therefore m\ddot{x} = -Kx \qquad (x(0) = x_0)$$

$$m\ddot{x} + Kx = 0 \qquad \dot{x}(0) = 0$$

Adding a sail to the mass:




mass experiences an additional external force.

(opposing)

$(m\ddot{x} - F_{wind}) = -Kx$ $\quad x(0) = x_0$

$\boxed{m\ddot{x} + Kx = F_{wind}}$ $\quad \dot{x}(0) = 0$

∴ Here the wind opposes the boat. (Reduces the effect of force).

$$(m\ddot{x} - F_{wind}) = -Kx$$

$$m\ddot{x} + Kx = F_{wind}.$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = 0$.

---

The above equ is (Assuming $m, K$ - Constants)

* second order
* linear
* In homogeneous

---

## Five-step modelling process

1) Draw a diagram of the system.

2) Identify and give symbols for the parameters & variables of the system.

3) Determine what's gn or can be measured (could be the input signal) and what is to be determined (system response). Identify initial conditions.

4) write down all equ relating the variables & parameters & manipulate them to obtain a diff equ relating input signal & response.

5) If linear, rewrite the equ in std. linear form (with initial conditions).

Natural growth: (modelling):

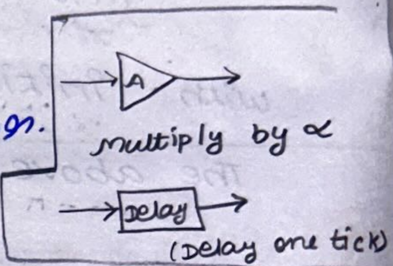
Oryx population has a natural growth rate of K years$^{-1}$. Assumed a constant harvesting rate of a oryxes/year.

1. write a differential equ (choose symbols & units)

Let,

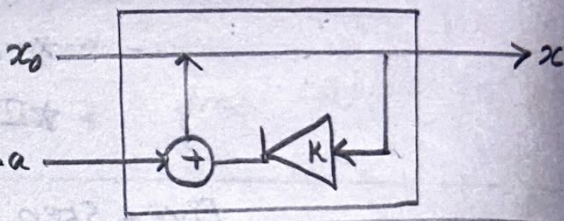
$x \rightarrow$ population of oryx.

$$\frac{dx}{dt} = (x \times K)$$

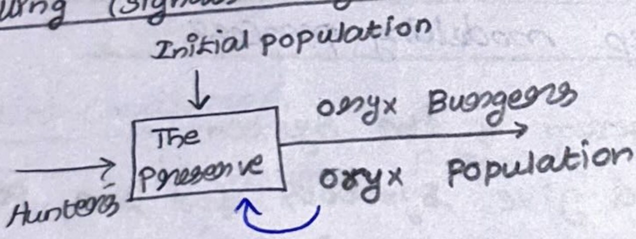


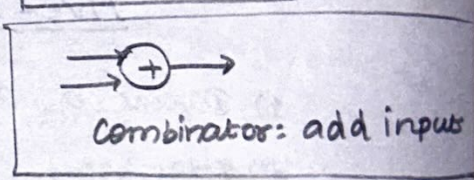

$\therefore$ There is a harvesting factor.

$$\frac{dx}{dt} = (xK) - a$$




Modelling (signals & systems)




3) Suppose $a = 0$; (No hunters) $\rightarrow$ what is the doubling time in terms of K.

(what's the relation b/w the population now & population after $K^{-1}$ years)?

Solu:

$$x(t + \Delta t) - x(t) \approx K\,x(t)\,\Delta t - a\Delta t$$

$$\frac{dx}{dt} = Kx - a. \qquad \text{when } a = 0.$$

$$\frac{dx}{dt} = Kx.$$

$$\cancel{x = \frac{Kx^2}{2} + C_1}$$

$Dx - Kx = 0$

$x(D-K) = 0$

$(D-K)x = 0$

solving:

$m - K = 0$

$\boxed{m = K}$

Here,

$x'(t) - x(t) \times K = 0$

$x'(t) - K\,x(t) = 0$

$\therefore m = K$

$\boxed{C.F = c_1 e^{Kt}}$

Seperable equation:

$$\frac{x'(t)}{x(t)} = K$$

(I order linear diff equ)

General solution

$\boxed{x = c_1 e^{Kt}}$

Doubling time = ?

$x(t) = c_1 e^{Kt}$

$2x(0) = 2c_1 e^{0}$

$2x(0) = 2c_1$

$\boxed{x(0) = c_1}$

---

Seperable equ:

a. Solve the IVP. $\frac{dy}{dx} = y^2$, $y(0) = 1$

b. Find General solution to the DE

solu:

$\frac{dy}{dx} = y^2$ , $\frac{dy}{y^2} = dx$ $\boxed{y \neq 0}$

$-\frac{1}{y} = x + c$

Constant of integration

(we only need C ∴ It's a I ODE).

$$\boxed{y = -\frac{1}{x+c}}$$

$$y(0) = -\frac{1}{x+c}$$

$$1 = -\frac{1}{x+c}$$

$$x+c = -1$$

$$c = -1-x$$

$\therefore x=0$

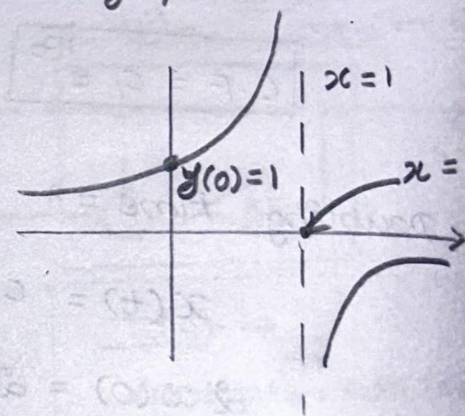
$y \rightarrow$ is in terms of $x$

$$\boxed{c=-1}$$

$\therefore$ Applying,

$$y(x) = \frac{-1}{-1+x}$$

$$\boxed{y = \frac{1}{1-x}} \rightarrow \text{vertical asymptote.}$$




Two portions.

$x \in (-\infty, 1) \Rightarrow y = \frac{1}{1-x}$

$x \in (1, +\infty) \Rightarrow y = \frac{1}{1-x}$.

b.) General solution:

$$\begin{cases} y = -\frac{1}{x+c} & ; \; y=0. \\ y \neq 0 \end{cases}$$

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onyx problem: $\frac{dx}{dt} = kx - a$

when, $a=0$ [No hunters]

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\,dt \qquad \boxed{x \neq 0}$$

$$\ln|x| = kt + c$$

$$|x| = e^{kt} \cdot e^{c} \Rightarrow x = \pm e^{kt} \cdot e^{c}$$

$\pm e^{c}$ = can be any non zero numbers.

$x(t) = ce^{kt}$, where $\boxed{c = \pm e^{c}}$

Identically,

$\boxed{x=0}$

$\dot{x} = kx$

$\boxed{0=0}$ → Satisfies but it is meaningless.

So It's a missing solution.

Initial value:

$x(0) = ce^{0}$

$\boxed{x(0) = c}$

3) Doubling time:

$x(T) = 2x_0$

$= 2c$

$= \cancel{\pm 2 e^{c}}$

⇒ Applying

$x(t) = ce^{kt}$

$x(T) = ce^{kT}$

$2c = ce^{kT}$

$2 = e^{kT}$

$\ln 2 = kT$

$T = \frac{\ln 2}{k}$

∴ $\boxed{c \neq 0}$

Provided ⇒ Both initial no. of oryxes & natural rate of growth k are both +ve.

∴ Doubling → ↑ (+ve).

3) Finding the general solution:

$\frac{dx}{dt} = kx - a$

If $k=0$

$\frac{dx}{dt} = -a$

Integrating,

$x(t) = -at + c$.

$\boxed{x(t) = -at + c}$

$\boxed{x(0) = c}$

Let $k \neq 0$:

$$\frac{dx}{kx-a} = dt$$

$$\frac{dx}{x-a/k} = k\,dt$$

$$\ln\left|\left(x-a/k\right)\right| = kt + c$$

$$\left|\left(x-\frac{a}{k}\right)\right| = e^{kt}\cdot e^{c}$$

$$x-\frac{a}{k} = \pm e^{kt}\cdot e^{c}$$

$$x-\frac{a}{k} = Ce^{kt}$$

$$\boxed{x = Ce^{kt} + \frac{a}{k}}$$

$$\boxed{x(0) = C + \frac{a}{k}}$$

here.

Missing solution:

$$\boxed{x \equiv \frac{a}{k}}$$

4) Check that the proposed solution satisfies the ODE

For $k = 0$, $\frac{dx}{dt} = \frac{d}{dt}x(t) = \frac{d}{dt}(c-at)$

$$= -a$$

For $k \neq 0$, $\frac{dx}{dt} = \frac{d}{dt}(x(t)) = \frac{d}{dt}\left(ce^{kt} + \frac{a}{k}\right)$

$$= cke^{kt}$$

$$= k(ce^{kt})$$

$$= k\left(x - \frac{a}{k}\right)$$

$$= kx - a \rightarrow \text{our model.}$$

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1. Find the general solution by seperation of variables.

1. $\frac{dy}{dx} = 2-y$, $y(0) = 0$

soln:

$$\frac{dy}{2-y} = dx \Rightarrow -\ln|2-y| = x + c \Rightarrow \ln|2-y| = -x-c$$

$$|2-y| = e^{-x} \cdot e^{-c}$$

$$2-y = \pm e^{-c} \cdot e^{-x}$$

$$2-y = c\,e^{-x}$$

$$-y = ce^{-x}$$

$$y = 2-ce^{-x}$$

$$y(0) = 2-ce^{-x} \qquad x=0$$

$$0 = 2-ce^{-x}$$

$$0 = 2-c$$

$$\boxed{c=2}$$

$$y = 2-2e^{-x}$$

$$y = 2(1-e^{-x}) \Rightarrow y = 2(1-e^{-x})$$

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Half life period

$$\therefore \quad N = N_0\, e^{-Kt} \qquad \therefore t = T_{1/2} \text{ (Half - life time)}$$

$$\frac{N_0}{2} = e^{-Kt} \times N_0 \qquad \therefore N = \frac{N_0}{2} \text{ (Half of initial)}$$

$$\frac{1}{2} = e^{-Kt}$$

$$2 = e^{Kt}$$

$$\ln 2 = \ln e^{Kt}$$

$$\ln 2 = Kt$$

$$\ln 2 = K\,T_{1/2}$$

$$K = \frac{\ln 2}{T_{1/2}}$$

Here

$$\boxed{K = \frac{0.693}{T_{1/2}}}$$

Half life time = 5 hours.

$$K = \frac{0.693}{5} \quad \left(\frac{1}{hr}\right)$$

$$\boxed{K = 0.1386} \ 1/hr$$

$\therefore \boxed{x(t) = x_0 e^{-Kt}}$ is decreasing.

$x(t) \leq x(1)$ for $t \geq 1$.

After 1 hr:

Atleast 50mg medicine per kg of body.

[patient is 60 Kg].

After an hour = 60×50

= 3000mg [needs to be in his body].

$$\boxed{x(1) = 3000mg}$$

$$x(1) = x_0 e^{-Kt}$$

$$\frac{3000}{e^{-Kt}} = x_0 \Rightarrow x_0 = 3000 \times e^{0.1386(1)}$$

$x_0 = 3445.99$ mg

$x_0 \approx 3446$

$x_0 \approx 3{,}446$ g.

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A → Rate of decay

$$\boxed{A = \frac{dN}{dt}}$$

$$A \propto N$$

N → Number of parent nuclei

$$\frac{dN}{dt} \propto N$$

$$\frac{dN}{dt} = -\lambda N$$

(− (due to decay))

λ → Decay constant

$$\int \frac{dN}{N} = -\lambda dt$$

$$\log_e N = -\lambda t + c$$

If N = $N_0$ (Initial amount)

$$\boxed{c = \log_e N_0}$$

$\boxed{t = 0}$

$$\ln N = -\lambda t + \ln N_0$$

$$\ln N - \ln N_0 = -\lambda t$$

$$\ln\left(\frac{N}{N_0}\right) = -\lambda t \Rightarrow \frac{N}{N_0} = e^{-\lambda t}$$

$$\boxed{N = N_0 e^{-\lambda t}}$$

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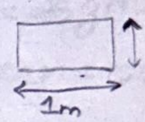
Early one morning it starts to snow. A snowplow sets off to clear the road. By 8AM it has gone 2 miles. It takes an additional 2 hours to go another 2 miles for another plow. Let $t=0$, when it begins to snow, Let $x$ denote the distance travelled by the plow at time $t$. Assuming the snowplow clears snow at a constant rate in cubic meters / hour.

i) Find DE modelling the value of $x$

ii) when did it start snowing.




Sol:

$K_1 \rightarrow$ Rate of snowfall (Height / hour), $K_2 \rightarrow$ Rate of snow clearance

Height of snow $= K_1 t$.

1min = 1 cubic m/hr
5min = 5 cubic m/hr

Snow clearance $= K_2 \Delta t = K_1 t \Delta x$

$\Delta t \rightarrow$ since we start our work

Only after some time of snow falling.


$$\frac{\Delta x}{\Delta t} \approx \frac{K_2}{K_1 t} \qquad \boxed{\frac{K_2}{K_1} = K}$$

$$\frac{\Delta x}{\Delta t} \approx \frac{K}{t}. \rightarrow \text{Modelled.}$$

using:

$$\int \frac{dx}{dt} = \int \frac{K}{t}$$

$$\int \frac{dx}{1} = \int \frac{K}{t}\, dt$$

$$x = K \ln t + C$$

Let the starting time be $t$. (Plowing)

At 7.A.m $= t = t$

At 8 Am $= t = t+1$

At 10 AM $= t = t+3$

Given 2 miles travelled b/w 7.AM & 8 AM

2 miles travelled b/w 7.AM & 10 AM.

$x = K\ln t + C$

i) B/w 7 Am & 8AM.

$$x(t+1) - x(t) = K\ln(t+1) - K\ln(t)$$

$$2 = K\left(\ln\left(\frac{t+1}{t}\right)\right)$$

ii) B/w 8.AM & 10 AM

$$x(t+3) - x(t+1) = K\ln(t+3) - K\ln(t+1)$$

$$2 = K\left(\ln\left(\frac{t+3}{t+1}\right)\right)$$

$$\therefore K\ln\left(\frac{t+1}{t}\right) = K\left(\ln\left(\frac{t+3}{t+1}\right)\right)$$

$$\frac{t+1}{t} = \frac{t+3}{t+1}$$

$$(t+1)^2 = t^2 + 3t$$

$$2t + 1 = 3t$$

$$\boxed{1 = t}$$

$\therefore \boxed{t=1} \rightarrow$ we started our work after 1 hours.

Snow falling started by 6.00 AM.

A tank holds 100 litres of water which contains 25 grams of salt initially. Pure water then flows into the tank. Salt water flows out of it. Both at 5 liters/minutes. The mixture is kept uniform at all times by stirring.

a) DE with IC for this situation.

b) How long will it take until only 1g of salt remains in the tank.

Solu:

$x(t)$ - Amount of salt in grams

$t$ - time in minute.

$$\frac{dx}{dt} = \text{Salt in} - \text{Salt out}$$

$$\frac{dx}{dt} = 0 - 5 \cdot \frac{x}{100} \Rightarrow \frac{dx}{dt} = -0.05x.$$

At

$x_0 = 25$ gms

water = 100 litre

$$\int \frac{dx}{x} = -0.05\,dt$$

$$\ln|x| = -0.05t + C$$

$$|x| = e^{-0.05t} \cdot e^C$$

$$x = Ce^{-0.05t}$$

when $x = 1$ in tank

$\boxed{t = ?}$

$$1 = Ce^{-0.05t}$$

$$e^{0.05t} = C$$

$\boxed{C = ?}$

$x_0 = 25g$

$t = 0$

$25 = Ce^0$

$\boxed{25 = C}$

$$e^{0.05t} = 25$$

$$0.05t = \ln 25$$

$$\boxed{t = 64.378 \text{ minutes.}}$$

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Solving I-order linear ODEs

i) Find the general solution to any homogeneous I order (linear) ODE using seperation of variables.

ii) Find a particular solution to any inhomogeneous I order linear ODE using variation of parameters

iii) Find the general solution to any inhomogeneous I order diff eqn by first finding the solution to the