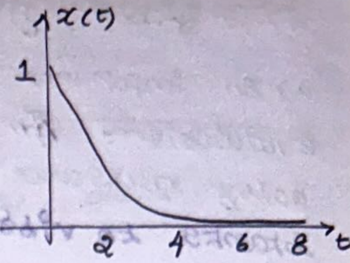
$c_2 = -1/2$, $c_1 = 3/2$



∴ $e^{-t} \to 0$ more slowly than $e^{-3t} \to 0$.

$\ddot{x} + 4\dot{x} + 4x = 0$.

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$ ⇒ critically damped.

$x(0) = 1$, $\dot{x}(0) = 0$

∴ $x(t) = e^{-2t}(c_1 + t c_2)$

$1 = c_1 + 0$

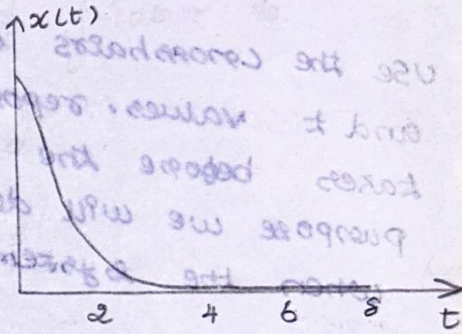
$\boxed{c_1 = 1}$

$\dot{x}(t) = e^{-2t}(c_2) + -2e^{-2t}(c_1 + t c_2)$

$0 = c_2 - 2c_1$

$\boxed{c_2 = 2}$

∴ $x(t) = e^{-2t}(1 + 2t)$



Comparing damping qualitatively:

Comparing qualitatively $\ddot{x} + b\dot{x} + x = 0$.

$x(0) = 1$, $\dot{x}(0) = 0$. $b = 1, 2, 3$.

Soln:

when $b = 1$ $\quad x_1(t) = e^{-t/2}\left(\cos \frac{t\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \sin \frac{t\sqrt{3}}{2}\right)$

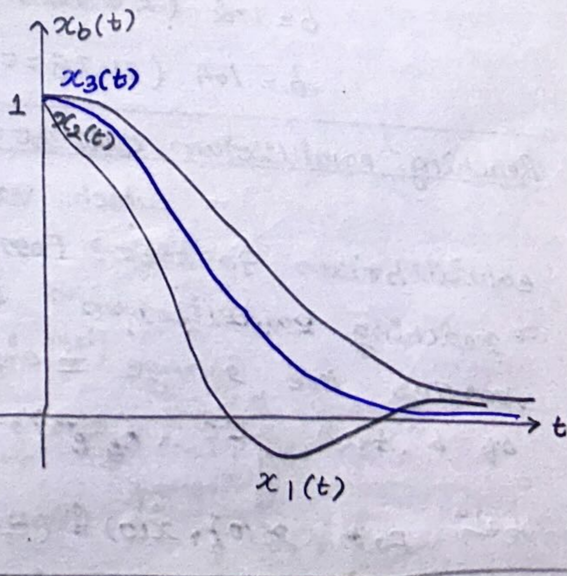
when $b = 2$ $\quad x_2(t) = e^{-t}(1 + t)$

when $b = 3$ $\quad x_3(t) = \left(\frac{1}{2} + \frac{3}{2\sqrt{5}}\right) e^{\frac{-3+\sqrt{5}}{2}t} + \left(\frac{1}{2} - \frac{3}{2\sqrt{5}}\right) e^{\frac{-3-\sqrt{5}}{2}t}$.

$x_1 \to$ underdamped

$x_2 \to$ critically damped

$x_3 \to$ overdamped



Application

a) An important use of damping is to bring a system into equilibrium. In many mechanical systems, vibrations are a noisy nuisance or even dangerous. If your airplane wing starts to vibrate, then you want it to settle down promptly. The spring-mass-dashpot

$$m\ddot{x}+b\dot{x}+Kx=0.$$ (Basic way of modelling vibrations. we will study how the damping constant b affects the rate at which vibrations settle down in the mathlet damped vibration.

Use the basic way. (sliders to set $m=1/2$, $b=1$, $K=1/2$ (Drag $x(0)=1$, $\dot{x}(0)=-1$.

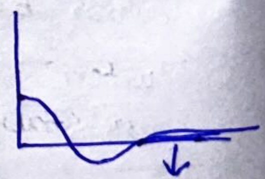
Use the crosshairs by mousing over the graph to see x and t values. report to two decimals the time it takes before the value settles to equilibrium. For this purpose we will define (reaching equilibrium' to tend when the system reaches the range ± 0.005.

(Look at the system at scale $x=1$, to get the overall picture and then zoom in to scale 0.01 to get more accurate view $(x=0)$. near.

Try $b=0.6, 0.8, 1.0, 1.2, 1.4$ ($b=1 \rightarrow$ case of critical damping)

solu:

$b=0.6$ (the curve settles at $t=8.33$)



$\left.\begin{array}{l} b=0.8 \ (t=6.13) \\ b=1.0 \ (t=5.29) \\ b=1.2 \ (t=9.11) \end{array}\right\}$ settles

$b=1.4$ $(11.78=t)$

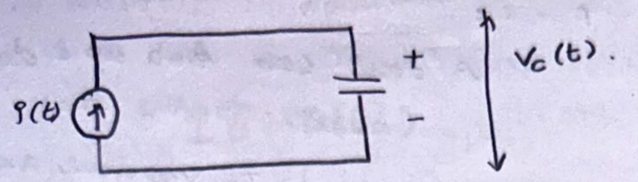
Reaching equilibrium fastest:

which value of b brings the system to equilibrium fastest? For this purpose we will define "reaching equilibrium" to mean when does the system reaches the range ± 0.005. Report this (best' value of b for IC $(x(0), \dot{x}(0)) = (1,-1))$, $(0,-1)$ and $(1,0)$

solu: For $x(0), \dot{x}(0) = (1,-1)$ $b=$ (best value) $=0.93$

$(0,-1)$ $b=0.82$, $(1,0) = b=0.86$

The best value of b changes by 12% in varying these Initial conditions.



$x(t) = i(t)$, output $= V_c(t)$.

Soln:

$$i = C\frac{dv}{dt} \Rightarrow V = \frac{1}{C}\int_0^T i\,dt + V_0.$$

i) ~~causal~~

ii) ~~Memory (having)~~

iii) ~~Non linear.~~

iv)

$$V = \frac{1}{C}\int_0^T i\,dt + V_0.$$

$V_0 \rightarrow$ Initial voltage across $t=0$

Assume ::

Before $t<0 \rightarrow$ No pulse

$t = 0 \longrightarrow V_0 = 0$

$$\cancel{V(T) = \frac{1}{C}\int_0^T i\,dt.}$$

$$\cancel{= \frac{i}{C}\int_0^T dt}$$

simply

$$V(T) = \frac{1}{C}\int_0^T i\,dt + C_1$$

C_1 – Constant

$C \rightarrow$ Capacitance

$$V = \frac{1}{C}\int_0^T i\,dt + C_1$$

1) stable,

2) Memory

3) Non linear

4) Causal

5) Time invariant

$$\int d(t-K)$$

$C \rightarrow$ Capacitance

$C_1 \rightarrow$ Constant (based on Initial value of this I-order homogeneous eqn

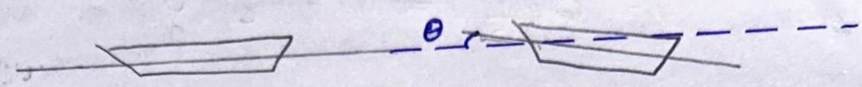
2) A series of problems related to oscillations of a boat in the water can be found interspersed b/w a series a series of videos that introduce the problem, context & model.

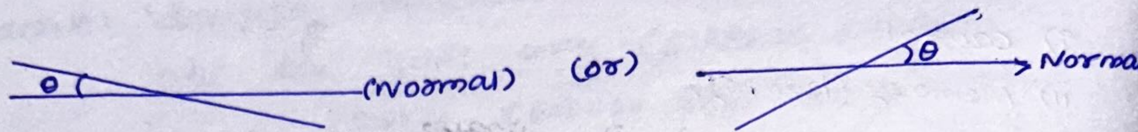
A boat can bob up & down (horizontally too)
↳ In various axes.

Solu:

Taking single element alone:

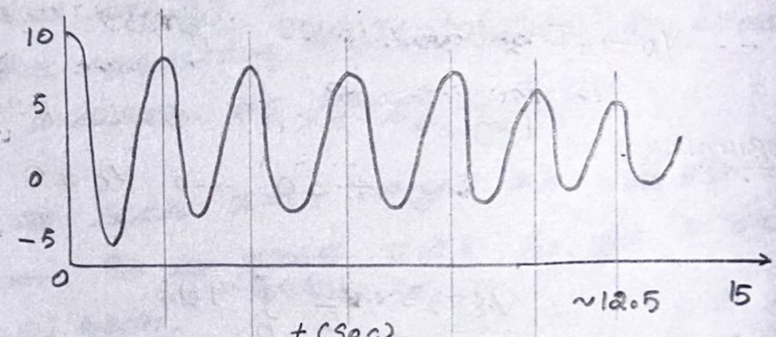
θ, t





Flat = Equilibrium position.

Boat oscillation (Using computer software):

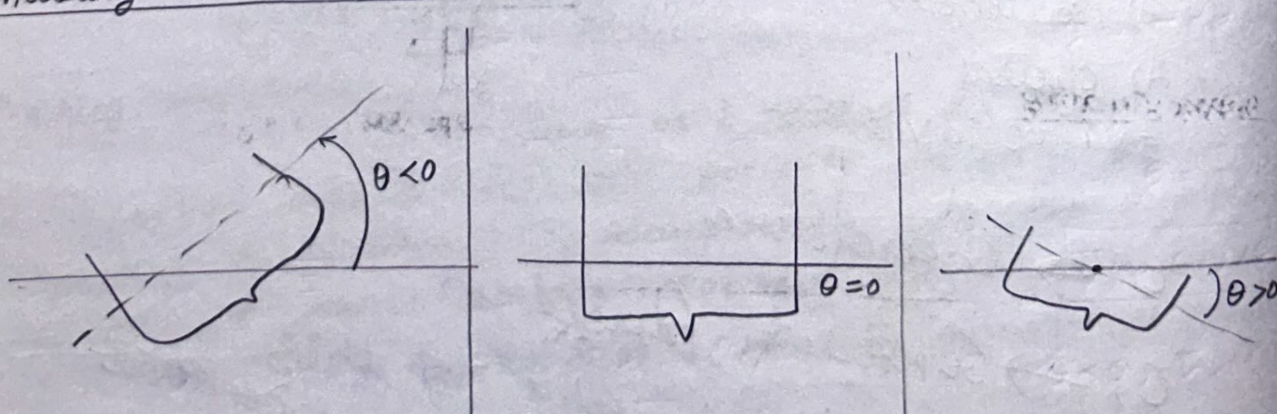


$T = \frac{12.5}{6} \approx 6.25$ Sec

If we want to build a ship with different shape
(It must stay in equilibrium position)

The Oscillations must be lower (for comfort of passengers)

Modeling a Rocking boat:



Newton's II law: $m\ddot{x} = F$

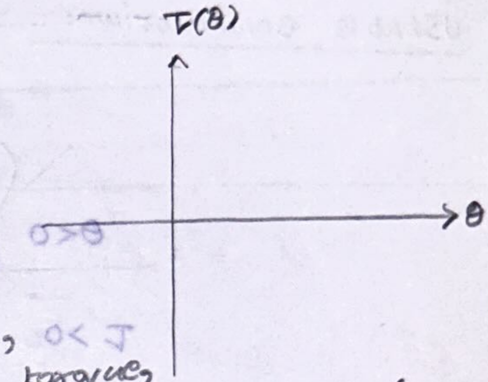
$\underset{\text{Rotational}}{\hookrightarrow}$ $I\ddot{\theta} = \tau$ (Larger the I, harder it will rotate)

$\therefore \quad I\ddot{\theta} = \tau(\theta)$

How τ depends on θ:

In Rocking boat at equilibrium, no rotation

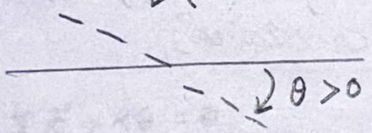
$\tau(0) = 0.$

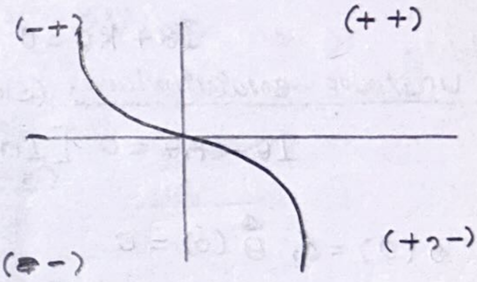


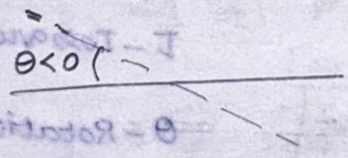
Since water around the boat, will act as a spring.

For +ve rotational angles, the boat experienced a negative torque, acting to rotate the boat back in to its equilibrium position.

$\tau(\theta>0) < 0$

(+,-) 



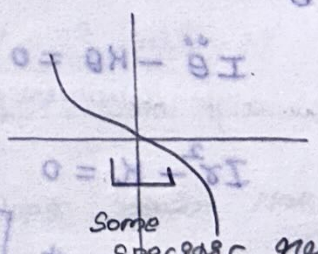


$\tau(\theta<0) > 0$

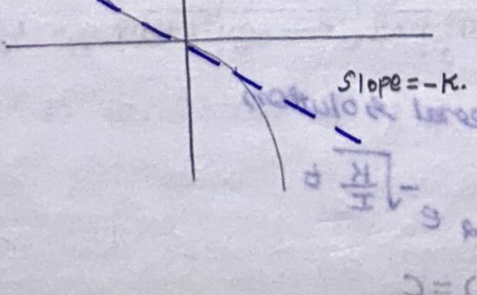
(we are more interested in how the boat behaves in its equilibrium position) (nearby)

(-,+)

So we need to care about



Approximating



$\tau(\theta) \approx -K\theta$

when $\theta > 0, \tau < 0$

$\theta < 0, \tau > 0.$

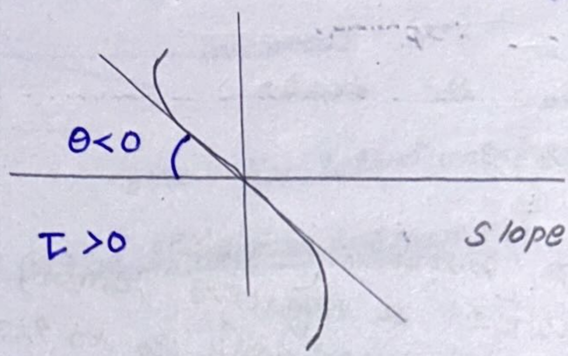
$\tau(\theta) \approx -K\theta$, $\theta << 1$. (smaller reign alone)

$I\ddot{\theta} + K\theta = 0$ → simple harmonic oscillator.

$P = \dfrac{2\pi}{\sqrt{K/I}}$ (period) $\boxed{P = \dfrac{2\pi}{\omega_n}}$

(considering without damping)

<u>Ustable equilibrium:</u>



∴ Slope = (−)ve.

$\theta > 0$
$\tau < 0$
$\boxed{\tau \sim -K\theta}$
$K = K, \theta =$

$I\theta + K\theta = 0$ [In equilibrium]

<u>Unstable equilibrium:</u> (slope > 0)

$I\theta - K\theta = 0$ [In unstable condition]

$\theta(0) = 0$, $\dot{\theta}(0) = C$

I - moment of Inertia
τ - Torque exerted on the boat
θ - Rotation angle.

$\boxed{\tau(\theta) \sim K\theta}$ $(K > 0)$

∴ Then the DE describing the rotation angle is

$$I\ddot{\theta} - K\theta = 0 \quad (K>0)$$

$$I r^2 - K = 0$$

$$r = \pm\sqrt{\frac{K}{I}}$$

where $\dfrac{K}{I} > 0$

when
$\theta > 0$, $\tau < 0$
$\tau \sim -K\theta$
$(K = +ve,$
$\theta = +ve)$
$\boxed{\tau \sim (-)ve}$
$\theta < 0$, $\tau > 0$
$(K = -ve,$
$\theta = -ve)$
$\tau \sim (-)ve.$

∴ This gives a general solution

$$\theta(t) = c_1 e^{\sqrt{K/I}\, t} + c_2 e^{-\sqrt{\frac{K}{I}}\, t}$$

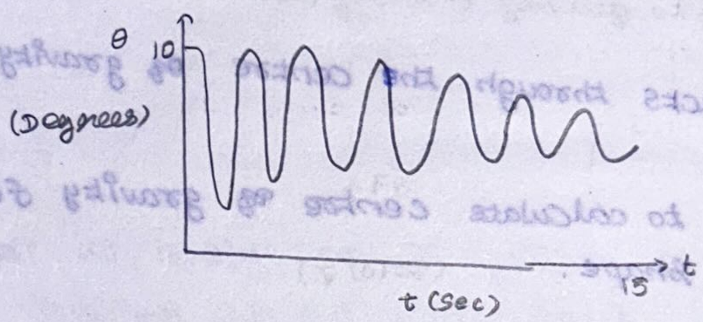
$\theta(0) = 0$, $\dot{\theta}(0) = C$

$$\boxed{c_1 = -c_2 = \frac{c}{2}\sqrt{\frac{I}{K}}}$$

$$\therefore \quad \theta(t) = \frac{c}{2}\sqrt{\frac{I}{K}}\left(e^{\sqrt{\frac{K}{I}}t} - e^{-\sqrt{\frac{K}{I}}t}\right)$$

The term with the growing exponential will dominate for $t > 0$. So the boat will rotate away from the flat position. This is why we call the flat position in this case an unstable equilibrium position

→ one side tipping happens.



$\frac{K}{I} = ?$

Solu:

$I\ddot{\theta} + K\theta = 0 \qquad (K > 0,\ \theta << 1).$

$\omega = \sqrt{\frac{K}{I}}, \quad P = \frac{2\pi}{\omega} \Rightarrow \quad P = 2\pi\sqrt{\frac{I}{K}}$

(From graph $P = 2s$)

$$\sqrt{\frac{I}{K}} = \frac{2}{2\pi}$$

$$\sqrt{\frac{I}{K}} = \frac{1}{\pi}$$

$$\frac{K}{I} = \pi^2$$

In unstable conditions, the slope near the equilibrium position.

unstable equilibrium → Equilibrium position won't be stable.

Gravity & Buoyancy

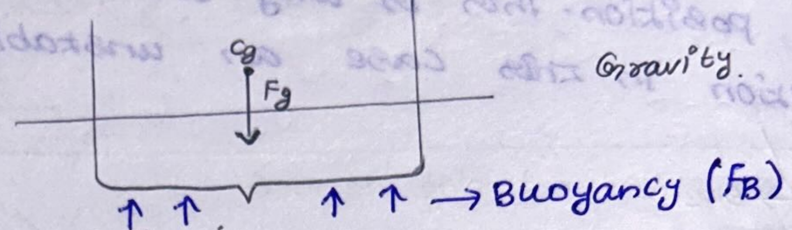
Stable equilibrium:

Effect of Boat shape

$$I(\theta) \approx -K\theta \quad , \theta << 1$$

$$I\ddot{\theta} - K\theta = 0$$

How k depends on physical parameters:



cg – centre of Gravity

Fg – force due to gravity (Pointing downwards)

Gravity (Newton): $\vec{F}_G$ acts through the centre of gravity, C_G.

From calculus: we need to calculate centre of gravity for a body of an arbitrary shape.

case:1: our boat (In equilibrium position) → doesn't move up or down.

∴ Net force on the boat = 0 [So the other force must be acting upwards]
↳ Force of Buoyancy

Buoyancy:

This is the result of all the pressure forces that act on the submerged surface of the boat.

Just like we are able to treat the force of Gravity as a single force acting through a single point, there is a theorem called 'Archimedes' principle which states that 'we can treat all of these pressure forces as a single force of buoyancy that points up'. (F_B). And it acts as a single force namely point of Buoyancy. (Centre of Buoyancy is purely geometrical. Doesn't depend on how the mass is distributed)

$\therefore$ Magnitude of F_B = Magnitude of F_G

$$|\vec{F_B}| = |\vec{F_G}|$$

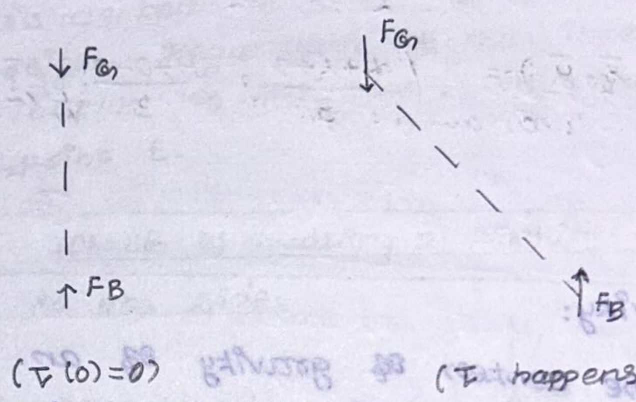
We can now understand why the boat is in rotational equilibrium)

$\therefore$ The F_B and F_G are acting in a single line.

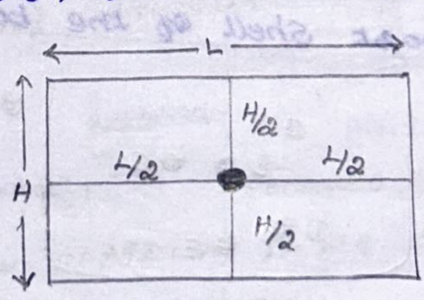
$\therefore$ They are acting along the same line.

$\therefore$ So the can't produce a torque.

$\therefore$ $\tau(0) = 0$



Centroid: From calculus that the centroid of the plane is the average position of all points of the shape. For example, the centroid of a rectangle is its centre.



For a shape described as a reign b/w the graph of a function $y = f(x)$ and the x axis, the x coordinate $\bar{x}$ of the centroid is given by the average value of x over the reign

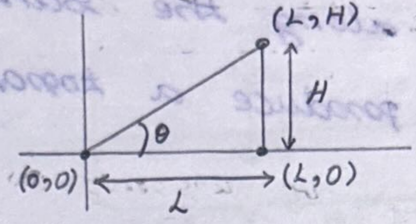
$$\bar{x} = \frac{\int x f(x)\, dx}{\int f(x)\, dx}.$$

where the integrals are over the entire reign. lll^ly the y-coordinate of the centroid is the average value of y over the entire reign. You can easily verify that these formulas give the correct location of the centroid of the rectangle.

These formulas also give a simple formula for the centroid of a triangle whose 3 vertices A, B and C are at (x_A, y_A), (x_B, y_B) and (x_C, y_C):

$$(\bar{x}_\Delta, \bar{y}_\Delta) = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}\right)$$

For a Δ^{le}:



$\tan\theta = \frac{H}{L}$

$H = L\tan\theta$

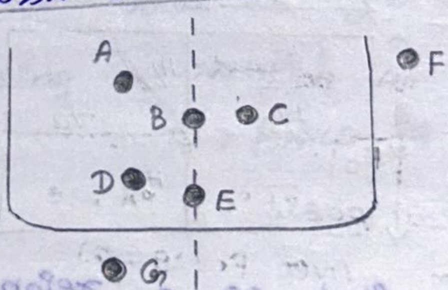
$$\text{Centroid } (\bar{x}_\Delta, \bar{y}_\Delta) = \left(\frac{L+L+0}{3}, \frac{H+0+0}{3}\right) = \left(\frac{2L}{3}, \frac{H}{3}\right)$$

$$= \left(\frac{2L}{3}, \frac{L\tan\theta}{3}\right)$$

Centre of gravity:

The center of gravity of an object is its centre of mass. It is determined by the objects shape & mass distribution.

Assume our boat has the same cross-section along its length. The outermost shell of the boat is



without information on the interior construction of the boat or the mass distribution of the exterior shell of the boat, which of the points (A – G) can be centre of gravity?

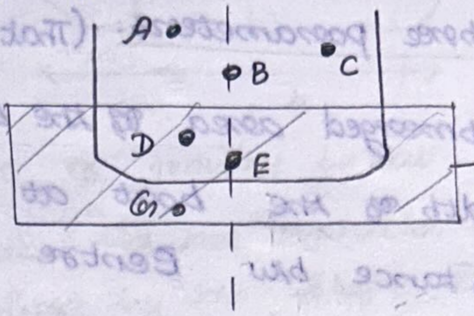
Solu:

Ans: A, B, C, D, E (since we don't know the mass distribution of the interior of the boat, the centre of gravity can be anywhere in the interior.

Center of buoyancy:

The center of buoyancy of an object submerged in water is the centre of mass of the water body being displaced.

Our boat from the previous question is now submerged in water as depicted in the figure which of the following points can be the boat's center of buoyancy?

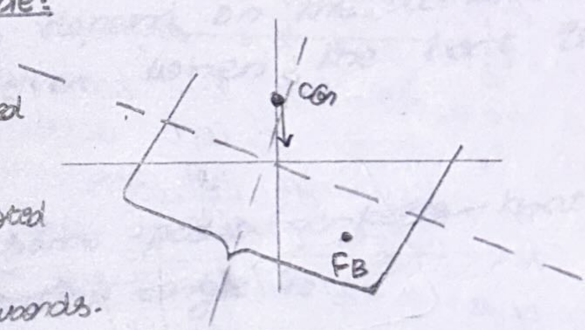


Ans: E → The center of Buoyancy is the centroid of the submerged region, regardless of the interior construction or mass distribution of the boat, and therefore can only be at the point E.

<u>Details of modeling a rocking boat</u>

<u>Boat is rotated to one side:</u>

∴ position of C_G is unchanged (Like in flat).



As the boat submerged is shifted rightwards, the centre of buoyancy will be moved rightwards.

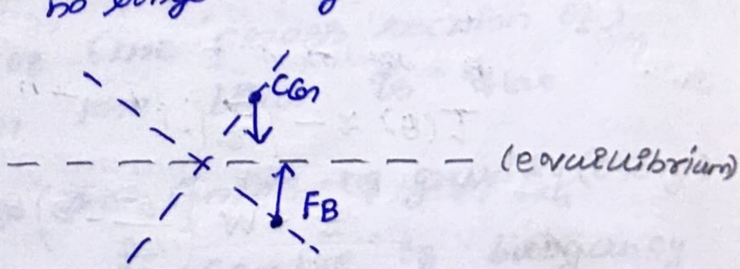
(shape - unchanged).

Eventhough the boat is rotated:

$$|\vec{F_B}| = |\vec{F_C}|$$

(Because the boat will always adjust its vertical position in the water until it is in vertical equilibrium. (Net Force = 0)

However, F_B and F_C are no longer aligned. Therefore they generate a lever.



From physics:

Torque produced by two equal & opposite

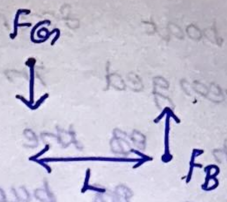
(C_G & F_B will try to realign the boat at equilibrium)

$$\tau = |\vec{F_G}| \cdot L$$

(Torque = Any one of Force × distance b/w the two forces)

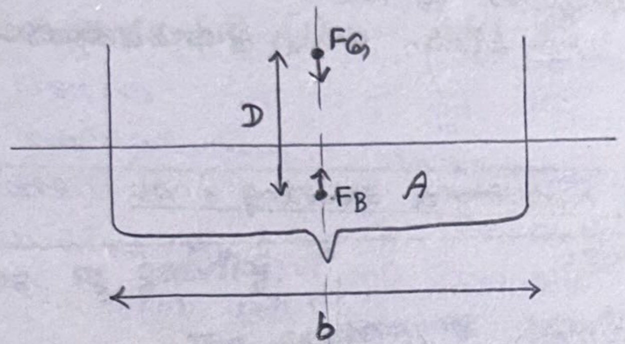
Here

$T = -|\vec{F_G}| \cdot L$



<u>Lever depends on three parameters:</u> (That affects boat shape)

* Submerged area of the boat (A)
* width of the boat at the waterline (b)
* Distance b/w Centre of Gravity & centre of buoyancy.

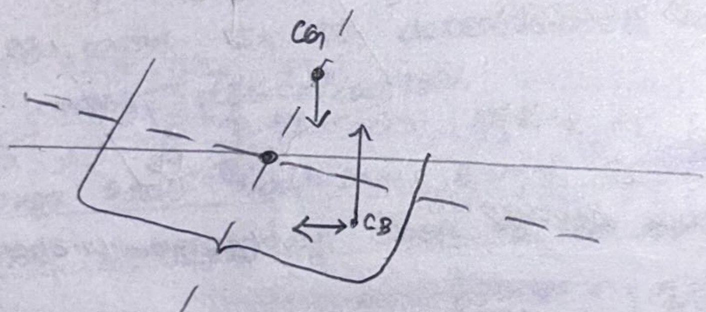


W: weight of the boat

b: width of the boat at waterline

D: distance b/w C_G & C_B. (At $\theta = 0$)

A: Area of water submerged



Distance b/w C_G & C_B is determined by A, D, b, θ

$$L \approx \left(\frac{b^3}{12a} - D\right)\theta$$ → only valid for $\theta << 1$.

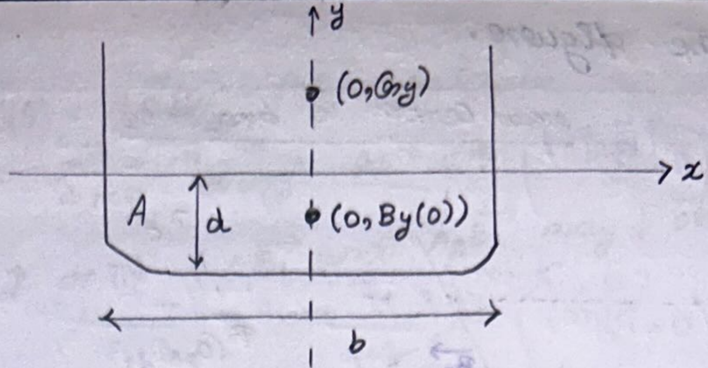
$$T(\theta) \approx -|\vec{F_G}| \cdot L$$

$$\approx -W\left(\frac{b^2}{12a} - D\right)\theta$$

$\boxed{\because \vec{F_G} = W}$

<u>The setup:</u>

Consider a boat with a symmetrical cross-section along its length). Take the origin (0,0) of the coordinate system to be at the bottom center of the boat, and the x-axis to be along the bottom of



Let the Centre of gravity be at the coordinates $(0, G_y)$.

$G_y \rightarrow$ Independent of the rotation angle θ.

(Determined by the shape & mass distribution of the boat).

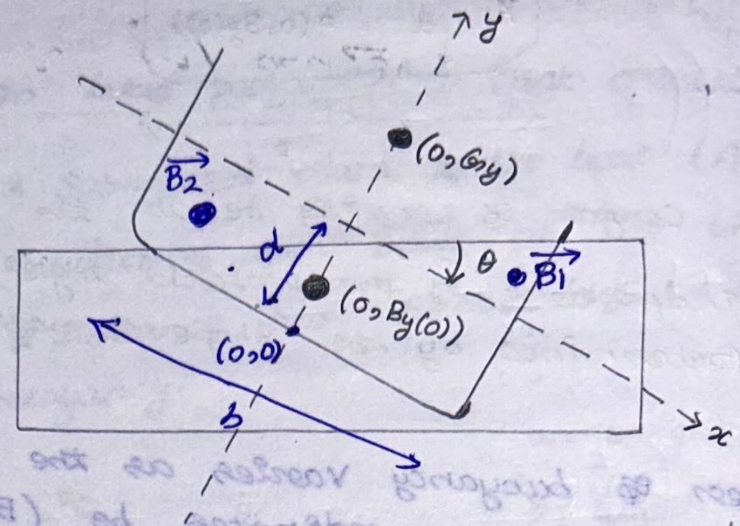
The center of buoyancy varies as the rotation angle θ varies. Let its coordinates be $(B_x(\theta), B_y(\theta))$. The centre of when the boat is rotated by θ. The centre of buoyancy is a purely geometric computation once you know where the waterline is. But where the waterline is does depend on the distribution of mass of the boat, even when the boat is at equilibrium.

Below are the list of some parameters that are independent of the rotation angle θ,

* The weight of the boat, W (In Newtons)
* The beam of the boat at the waterline, b (in meters)
* The height of the water line at the center of the boat, d (in meters)
* The area of (the cross section of) displayed fluid when the boat is flat.
* Location of the Centre of gravity $(0, G_y)$
* The location of centre of buoyancy when the boat is sitting flat. $(B_x(0), B_y(0))$ each coordinate in meters

Assume: The boat has rectangular sides in the range of rotation angles that are considered, as shown in the figure.

New center of buoyancy.



Our boat as described in the setup is rotated clockwise by an angle θ, at which the boat has rectangular sides. Find the new center of Buoyancy $(B_x(\theta), B_y(\theta))$ as an algebraic expression of the boat's parameters A, b, d, $B_y(\theta)$ and the angle θ. This expression will be helpful to determine the torque later.

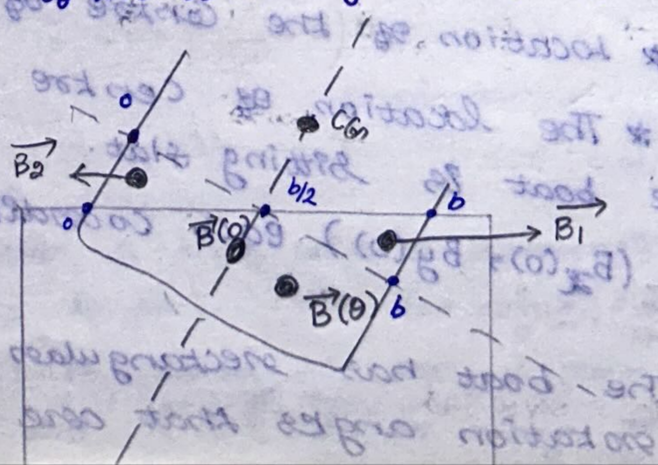
Remember we are using the coordinate system in which the origin O is at the bottom centre of the system boat, and x-axis is along the bottom of the boat.

(Hint: Think of how the centroid of the submerged region is affected by the decrease in area on the left and increase in area on the right. You can start with writing an eqn relating the old & new centres of Buoyancy & the centroids of the two Δ^{les}. The eqn will involve the area A and the area of the Δ^{les}.

solu: Answer → In terms of A, b, d, $B_y(0)$, and θ

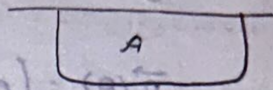
$A_1, A_2 \rightarrow$ Area of Δ^{les} having points $\vec{B_1}$ & $\vec{B_2}$.

$A \rightarrow$ Area of the submerged region (Boat – Flat position)



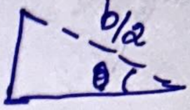
By definition,

$$\vec{AB}(\theta) = \vec{AB}(0) + A_1\vec{B_1} - A_2\vec{B_2}$$



for every rotation, ($A_1\vec{B_1}$ and $A_2\vec{B_2}$ may ↓ or ↑)

$\bar{AB} \rightarrow$ may ↑ or ↓



$$A_1 = A_2 = \frac{1}{2}\left(\frac{b}{2}\right)\left(\frac{b}{2}\tan\theta\right)$$

$$= \frac{1}{2}\left(\frac{b^2}{4}\right)\tan\theta$$

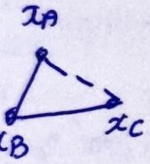
$$= \frac{b^2}{8}\tan\theta.$$

$\tan\theta = \frac{\text{opp.sid}}{b/2}$

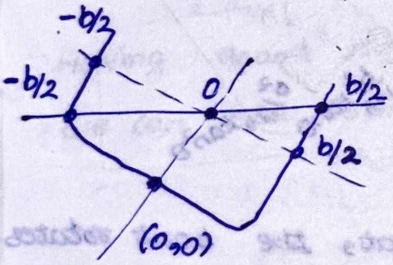
$\frac{b}{2}\tan\theta = \text{opp.si}$

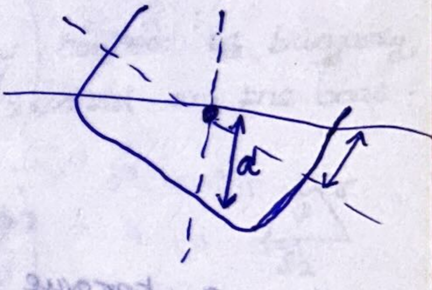
The centroid of the Δ^{le} having $\vec{B_1}$

$$\vec{B_1} = \left(\frac{1}{3}(x_A + x_B + x_C), \frac{1}{3}(y_A + y_B + y_C)\right)$$

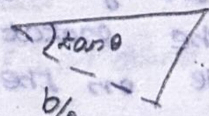


$$\vec{B_1} = \left(\frac{1}{3}\left(0 + \frac{b}{2} + \frac{b}{2}\right), \frac{1}{3}\left(d + d + x_3\right)\right)$$



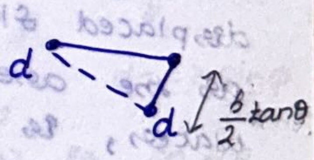


x values



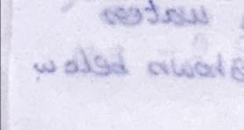
oppside $= \frac{b}{2}\tan\theta$

y values.



$$\vec{B_1} = \left(\frac{b}{3}, \frac{1}{3}\left(d + d + \left(d + \frac{b}{2}\tan\theta\right)\right)\right)$$

$$= \left(\frac{b}{3}, \left(d + \frac{b}{6}\tan\theta\right)\right)$$



$$\vec{B_2} = \left(-\frac{b}{3}, \left(d - \frac{b}{6}\tan\theta\right)\right)$$

Applying in eqn (1): $\vec{AB}(\theta) = \vec{AB}(0) + A_1\vec{B_1} - A_2\vec{B_2}$

$$\vec{B}(\theta) = \vec{B}(0) + \left(A_1\vec{B_1} - A_2\vec{B_2}\right)\frac{1}{A},$$

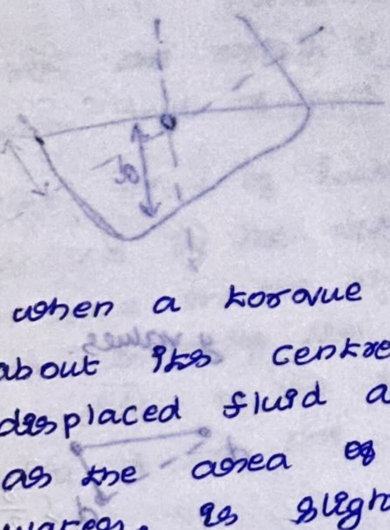
$$\vec{B}(\theta) = \left(0, B_y(0)\right) + \left(\frac{b^2}{8}\tan\theta\right)\left(\vec{B_1} - \vec{B_2}\right)\frac{1}{A}$$

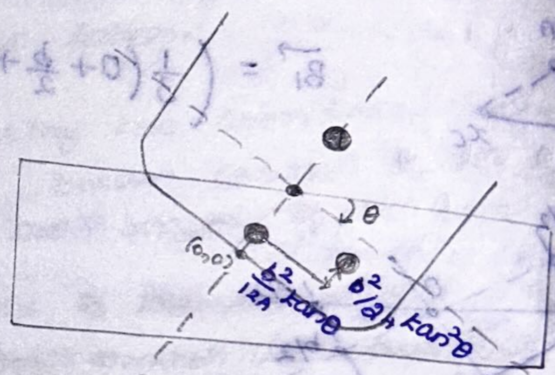
$$= \left(0, B_y(0)\right) + \left(\left(\frac{b^3}{24A}\tan\theta - \left(\frac{-b^3}{24A}\tan\theta\right)\right),\right.$$

$$\left.\left(\frac{b^2}{8}\tan\theta d - \frac{b^2}{8}\tan\theta d + \frac{2b^3\tan^2\theta}{6\times 8A}\right)\right)$$

$$= \left(0, B_y(0)\right) + \left(\frac{b^3}{12A}\tan\theta, \frac{b^3\tan^2\theta}{24A}\right)$$

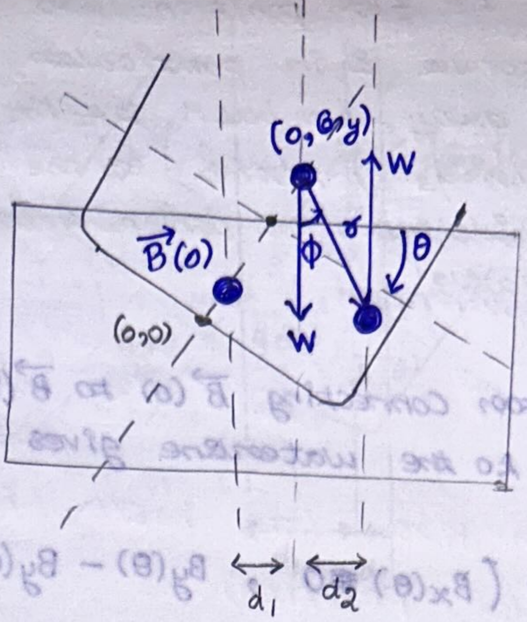
$$= \left(\frac{b^3}{12A}\tan\theta, B_y(0) + \frac{b^3}{24A}\tan^2\theta\right)$$





when a torque is exerted on the boat, the boat rotates about its centre of gravity. Note that the area of displaced fluid after the rotation now less than A, as the area of the Δ^{le} on the left that leaves the water, is slightly greater than the area of the Δ^{le} on the right, that enters the water.

This means that the upward force of Buoyancy is now less than the downward force of gravity. Therefore, the boat will move down in the water until the displaced area is again A, as shown below.



$\vec{B}(\theta) \rightarrow$ centre of Buoyancy (submerged region).

An unaligned pair of equal & opposite forces creating a torque.

$\phi > 0$ if the center of buoyancy is to the right (In water's frame) of the CG.

$\phi <$ " " " is to the left " of the center of gravity.

Having found the coordinates of the new center of buoyancy, we can now compute the torque τ exerted on the boat:

$$\tau = -Wd_2 \quad (d_2 = \gamma \sin\phi)$$

$\frac{d_2}{\gamma} = \sin\phi$

$d_2 = \gamma \sin\phi$

$W \rightarrow$ weight of the boat

$d_2 \rightarrow$ length of the lever arm.

Negative sign is present because if $d_2 > 0$, meaning the center of buoyancy $(B_x(\theta), B_y(\theta))$ is to the right of the CG, the torque will rotate the boat in the $-\theta$ direction (counterclockwise in our convention).

d_2 depends on the locations of both CG and C_B

$\therefore$ It depends on various boat parameters & it's a function of θ. The sign of d_2 is very important. It corresponds to whether CG is to the right or left of CG (In water's frame). When the boat is rotated away from the flat position. This determines the direction of the torque & in particular whether the boat rotate further away or back to the flat position. The expression for d_2 in terms of the boat parameters will give us guidelines to design a boat that won't capsize immediately.

Projecting the vector connecting $\vec{B}(0)$ to $\vec{B}(\theta)$ on to a unit vector parallel to the waterline gives

$$d_1 + d_2 = \left(B_x(\theta) - 0,\ B_y(\theta) - B_y(0)\right) \cdot (\cos\theta, \sin\theta)$$

(Adding horizontal & vertical components)

<u>From previous ones:</u>

$$\vec{B}(\theta) = \left(\frac{b^3}{12A}\tan\theta,\ B_y(0) + \frac{b^3}{24A}\tan^2\theta - B_y(0)\right)$$

$$\therefore\ d_1 + d_2 = \left(\frac{b^3}{12A}\tan\theta \cdot \cos\theta,\ \cancel{B_y(0)\sin\theta} + \frac{b^3}{24A}\tan^2\theta\sin\theta - \cancel{B_y(0)\sin\theta}\right)$$

$$= \left(\frac{b^3}{12A}\sin\theta,\ \frac{b^3}{24A}\tan^2\theta\sin\theta\right)$$

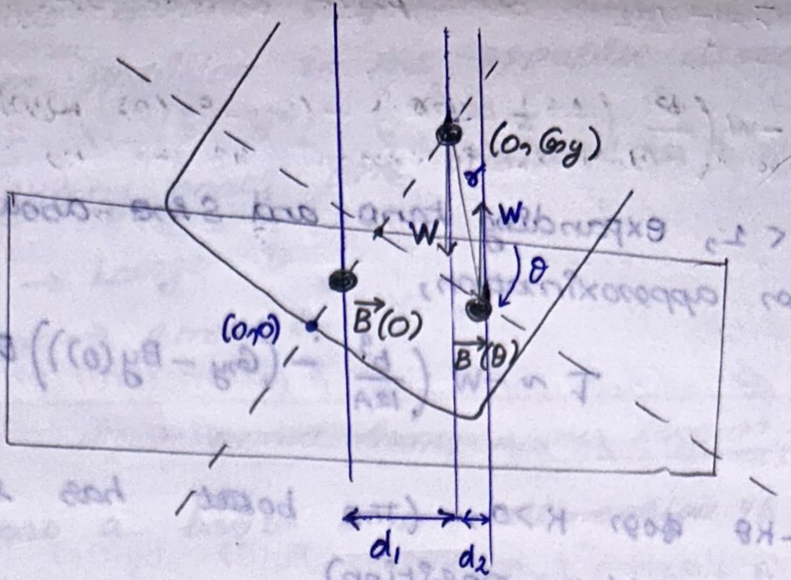
At $B_y(0) \Rightarrow \theta = 0^\circ,\ \sin 0 = 0$

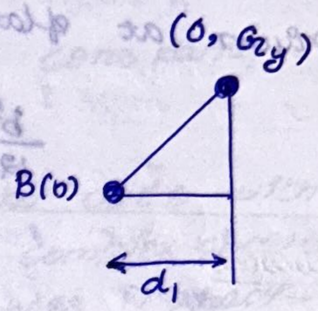
$$d_1 + d_2 = \frac{b^3}{12A}\sin\theta + \frac{b^3}{24A}\tan^2\theta\sin\theta$$

(Adding x & y component)

$$= \frac{b^3}{12A}\sin\theta\left(1 + \frac{1}{2}\tan^2\theta\right)$$

But, $\therefore$ $d_1 =$



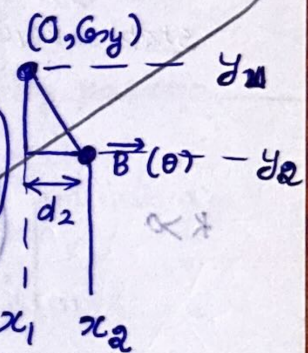


$$\frac{d_1}{(G_y - B_y(0))} = \sin\theta$$

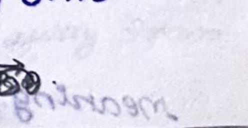
$$d_1 = (G_y - B_y(0))\sin\theta.$$

$$d_2 = r\sin\theta.$$

$$= \left(\left(-0 + \frac{b^3}{12A}\tan\theta\right), \left(G_y + \frac{b^3}{24A}\tan^2\theta + B_y(0) + r\right)\right)$$



$$= \left(\frac{b^3}{12A}\tan\theta + \frac{b^3}{24A}\tan^2\theta - G_y + B_y(0) + r\right)$$



$$= \left(\frac{b^3}{12A}\tan\theta\left(1 + \frac{1}{2}\tan\theta\right) - (G_y - B_y(0)) + r\right)\sin\theta$$

$$d_2 = \frac{b^3}{12A}\sin\theta\left(1 + \frac{1}{2}\tan^2\theta\right) - (G_y - B_y(0))\sin\theta$$

$$= \left(\frac{b^3}{12A}\left(1 + \frac{1}{2}\tan^2\theta\right) - (G_y - B_y(0))\right)\sin\theta$$

$$\tau = -W d_2$$

$$\tau = -W\left(\frac{b^3}{12A}\left(1+\frac{1}{2}\tan^2\theta\right) - (G_y - B_y(0))\right)\sin\theta$$

For $\theta << 1$, expanding $\tan\theta$ and $\sin\theta$ about $\theta = 0$ gives the linear approximation,

$$\tau \sim -W\left(\frac{b^3}{12A} - (G_y - B_y(0))\right)\theta \quad (\theta << 1)$$

$\therefore$ $\tau \sim -K\theta$ for $K > 0$, (The boat has stable equilibrium at flat position)

$$\frac{b^3}{12A} - (G_y - B_y(0)) > 0.$$

Conclusions:

$$\tau(\theta) \approx -W\left(\frac{b^3}{12A} - D\right)\theta$$

$$K = W\left(\frac{b^3}{12A} - D\right)$$

Stable equilibrium:

$W > 0$ (Always)

$K > 0$ $\left(\frac{b^3}{12A} - D\right) > 0$

$\therefore$ [The C_B is at right. Torque will try to rotate it in $-\theta$]

$\therefore$ Short & wide boats have stable equilibrium

$\therefore$ Wide boat b will be large

For short boat D will be small.

meaning $\Rightarrow$ The lever is large $\Rightarrow$ Torque that the boat generates is large.

$\therefore$ Stable.

Unstable equilibrium:

$\left(\frac{b^3}{12A} - D\right) < 0.$ (means C_B is located left to the C_G).

Two forces still generate a torque, which will try the boat further & further away from its equilibrium position in the opposite direction

Tall, skinny boats are unstable

Tall boat → Large D

Skinny boat → small b.

Higher order homogeneous linear ODE

1) Know what a basis & the dimension of a vector space is

2) Solve the homogeneous ODEs with constant co.eff of order n.

3) Find a real basis of solutions to ODEs with constant co.eff with atleast two complex char. roots

4) Apply the existence & uniqueness theorem for linear ODEs when solving initial value problems

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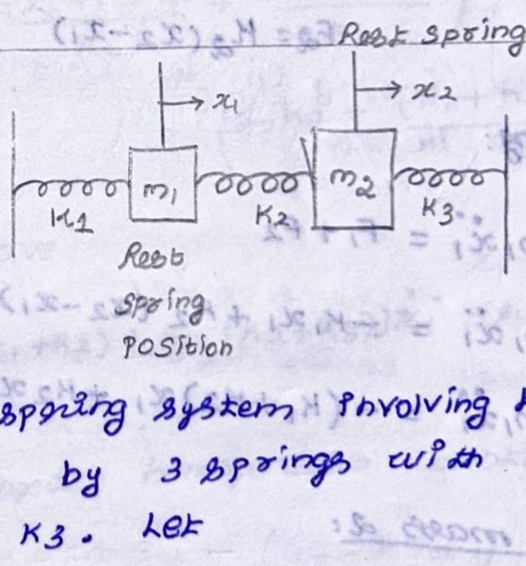
The Coupled oscillator:



Consider this ideal spring system involving two masses, m_1 and m_2 connected by 3 springs with spring constants K_1, K_2 & K_3. Let

$x_1 \rightarrow$ Displacement of the mass m_1 (away from the equilibrium position)

$x_2 \rightarrow$ " " of m_2 " "

The case $x_1 = x_2 = 0$ corresponding to the situation where neither mass is displaced from its equilibrium position. The displacements x_1, x_2 satisfy two simultaneous II order eqn involving both variables.

we can eliminate x_2 to get a single fourth order homogeneous ODE with constant Co.eff for x_1

Soln:

$$m_1 m_2 x^4 + (m_2(K_1+K_2) + m_1(K_2+K_3))\ddot{x}_1 + (K_1K_2 + K_1K_3 + K_2K_3)x_1 = 0$$

Evaluation

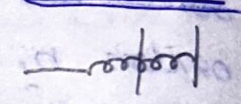
Extension or Compression of spring $1 = x_1$

$2 = x_2 - x_1$

$3 = -x_2$

Here: No damping.

why $-x_2$



$x=0$

(During expansion it expands in (-) direction

$\therefore x_2 = -(-x_2)$

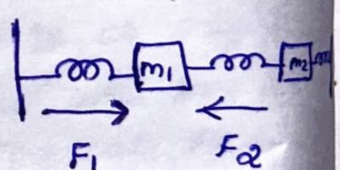
$= x_2$ units)

Force on mass 1:

Two spring forces acting on mass 1: The Force F_1 due to spring 1 & the force F_2 due to spring 2.

Gn by $F_1 = -K_1 x_1$;

$F_2 = K_2(x_2 - x_1)$



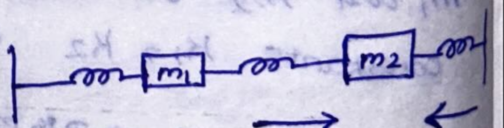
Combining:

$$m_1\ddot{x}_1 = F_1 + F_2$$

$$m_1\ddot{x}_1 = -K_1x_1 + K_2(x_2 - x_1)$$

$$m_1\ddot{x}_1 = -(K_1+K_2)x_1 + K_2x_2$$

Force on mass 2:



$$F_2 = -K_2(x_2 - x_1)$$

$$= K_2(x_1 - x_2)$$

$F_3 = -K_3 x_3$ (x_3 moves backwards)

$$m_2\ddot{x}_2 = -K_2x_2 + K_2x_1 - K_3x_2$$

$$= -x_2(K_3+K_2) + K_2x_1$$

The two eqn above together form a coupled system of diff eqn:

$$m_1 \ddot{x}_1 = -(K_2 + K_1) x_1 + K_2 x_2$$

$$m_2 \ddot{x}_2 = K_2 x_1 - (K_2 + K_3) x_2.$$

These two eqn together is a system of II order linear diff eqn, each with two dependent variables x_1 and x_2. The approach we will take for now is to eliminate one of the dependent variables to obtain a single eqn involving only the other dependent variable.

Notice that we can write x_2 in terms of x_1 & $\ddot{x}_1$ using the first DE:

$$x_2 = \frac{m_1}{K_2} \ddot{x}_1 + \frac{K_1 + K_2}{K_2} x_1$$

plugging it in the second DE, regrouping items,

$$\frac{m_1 m_2}{K_2} x_1^{4} + \left(\frac{m_2 (K_1 + K_2)}{K_2} + \frac{m_1 (K_2 + K_3)}{K_2} \right) \ddot{x}_1 + \left(-K_2 + \frac{(K_1 + K_2)(K_2 + b_3)}{K_2} \right) x_1 = 0$$

multiplying by K_2, we have

$$m_1 m_2 x_1^{(4)} + (m_2 (K_1 + K_2) + m_1 (K_2 + K_3)) \ddot{x}_1 + (K_1 K_2 + K_1 K_3 + K_2 K_3) x_1 = 0$$

This is the fourth order homogeneous constant co-eff ODE for x_1.

How do we solve a higher order ODE

use char polynomials in the same way as we did to solve II order ODEs with constant co-eff. The char polynomial of the DE above is of 4th order. It has 4 (possibly complex) roots counted with multiciplicity.

For II order linear homogeneous ODEs, we need 2 linearly independent solutions corresponding to the 2 char roots. For an n^{th} order linear ODEs, we need n linearly independent solutions.

what does linear independence mean for a set of n functions?

It means much more sophisticated notion than linear independence of 2 functions.

Span:

Span is the linear algebra term for the phrase "All linear combinations"

Def: 3.1:

Suppose that $f_1, \dots, f_n$ are functions. The span of $f_1, \dots, f_n$ is the set of all linear combinations of $f_1, \dots, f_n$.

$$\text{Span}(f_1, \dots, f_n): \{\text{All functions } c_1 f_1 + \dots + c_n f_n\}$$

where $c_1, \dots c_n$ Complex numbers

(Sometimes $c_1, \dots, c_n$ be complex and sometimes we allow to be real.)

Span (t^2) is the set of all functions of the form ct^2, where c is a number, It is an infinite set of functions

$$\text{span}(t^2) = \{\dots, -3t^2, 4t^2, 0, -t^2, \pi t^2, \dots\}$$

Is this true that

$$\text{Span}(2t^2) = \text{span}(t^2)$$

Yes. They are equal. Every function that's a number times $2t^2$ is also a (different) numbers times t^2, and vice versa.

Span (e^{2t}, e^{-3t}) is a set of all linear combinations of e^{2t} & e^{-3t}. It is also the set of all solutions to $\ddot{y} + \dot{y} - 6y = 0$. It is an infinite set of functions:

$$\text{Span}(e^{2t}, e^{-3t}) = \{\dots, 5e^{2t} + (-7)e^{-3t}, 0e^{2t} + e^{-3t},$$

$\pi e^{2t} + 9e^{-3t}, \; 0e^{2t} + 0e^{-3t}, \dots \}$

Is $(e^{2t}, e^{-3t}, e^{2t} + e^{-3t})$ = Span of (e^{2t}, e^{-3t})

Yes. They are equal.

$\therefore$ Any linear combination in the left side such as

$$5e^{2t} + 6e^{-3t} + 2(e^{2t} + e^{-3t}) = 7e^{2t} + 8e^{-3t}$$

is also the linear combination in the r.h. side & viceversa

$\therefore$ Technically it's correct to say,

$c_1 e^{2t} + c_2 e^{-3t} + c_3 (e^{2t} + e^{-3t})$ is the general solution to the homogeneous equ $\ddot{y} + \dot{y} - 6y = 0$

(But this is silly, since the third function $e^{2t} + e^{-3t}$ is redundant, because it's a linear combination of the other two.

Vector spaces

The span of a set of functions is always a vector space.

What's a vector space:?:

Exam: 4.1: The 2dimensional (x,y) plane is a vector space. The vectors are the ordered pairs (x,y) that add component-wise.

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. There is a zero element $(0,0)$, we can also scale vectors by real number:

$$a(x,y) = (ax, ay)$$

4.2:

Suppose that S is a set of elements called vectors equipped with two laws: a law for adding vectors in S, called vector addition.

law for multiplying vectors in S by real numbers called scalar multiplication. Then S is a real vector space if all of the following are true.

1: There is a zero vector in S: a vector 0 in S such that $v+0=v$ for every vector v in S.

2. (closed under scalar multiplication) multiplying any one vector in S by a real number gives another vector in S: if v is in S and a is a real number, then av is in S too.

3. (closed under vector addition). Adding any two vectors in S give another vector in S. If v is in S and a is a real number, then av is in real too.

If v and w are in S, then v+w is also in S.

4.3: The set of all real valued functions form a real vector space. The constant 0-function is the zero vector. vector addition of functions point wise

$$(f+g)(x) = f(x)+g(x).$$

Scalar multiplication is multiplying a function by a real number at each points.

eg: $-2f(x) = (-2)(f(x))$

Non example:

The set of all non-negative functions is not a real vector space because x^2 is in this set,

Theorem: 4.4: The set of all homogeneous solutions to a linear ODE is a vector space.

Subspace:

A subspace is itself a vector space but is also a subset of a large vector space. So say V-vector space & W-subspace of V. Then V and W both satisfy the prop. of being vector spaces ($W \subset V$)

Linear Independence:

Def 5.1: Vectors $v_1, v_2, \dots, v_n$ are linearly dependent if at least one of them is a linear contribution of others. Otherwise, call them linearly independent.

Def 5.2 In a vector space consisting of functions, each function is a vector. The functions

$$e^{2t}, e^{-3t}$$ are linearly independent.

Note: For a set of 2 vectors, the definition of linear independence & linear independences reduces to what we have learned before: v_1, v_2 are linearly dependent if one is a scalar multiple of the other. Otherwise, they are linearly independent.

Exam: 3: The functions $f_1 = e^{2t}$, $f_2 = e^{-3t}$, $f_3 = e^{2t} + e^{-3t}$ are linearly dependent. The third function is a linear combination of the

$$f_3 = f_1 + f_2$$

Infact, each function is a linear combination of the other two:

$$f_1 = e^{2t} = (e^{2t} + e^{-3t}) - e^{-3t} = f_3 - f_2$$

|||ly $f_2 = f_3 - f_1$

Equivalent definition:

vectors $v_1, v_2, \dots, v_n$ are linearly dependent if and only if there exist numbers $c_1, \dots, c_n$, not all zero such that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$\therefore$ In example,

$$f_3 = f_1 + f_2, \quad f_1 = f_3 - f_2, \quad f_2 = f_3 - f_1$$

$\therefore$ A more symmetric way to describe this is

$$f_1 + f_2 - f_3 = 0$$

Exam: 5.4:

The three functions $e^{2t}, e^{-3t}, 5e^{2t} + 7e^{-3t}$ are linearly dependent, because

$$(-5)e^{2t} + (-7)e^{-3t} + 1(5e^{2t} + 7e^{-3t}) = 0$$

The meaning of linear dependence is that atleast one of the functions on the list is redundant (and can be expressed as a linear combination of the others.)

Basis

In discussing II - order diff ODES, we called a set of 2 linearly independent homogeneous solutions a basis for all homogeneous solutions.

Definition 6.1:

A basis of a vector space S is a list of vectors $v_1, v_2, \dots, v_n$ such that

1) span $= (v_1, v_2, \dots, v_n) = S$ and

2) The vectors $v_1, v_2, \dots, v_n$ are linearly independent.

Example: 6.2: The vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ form a basis for R^3 (a real space vector) This is because any real vector (x, y, z) can be written as (real) linear combinations of these 3 basis vectors, and these 3 basis vectors are linearly independent:

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = 0$ if and only if c_1, c_2 and c_3 all equal to zero.

The vector spaces that are relevant to diff equ are vector spaces of functions. A basis of a vector space of functions is given as a list of functions $f_1, f_2, \dots f_n$. Think of the functions in the basis as 'basic building blocks'. Condition: 1 says that every function in S can be built from $f_1, f_2, \ldots$

condition 2 says that there is no redundancy in the list (no building block could have been built from the others).

Exam: 6.3: The functions e^{2t}, e^{-3t} form a basis for the space of solutions to the homogeneous eqn $\ddot{y} + \dot{y} - 6y = 0$.

Solu:

Key point: The vector space of solutions to a homogeneous ODE consists of infinitely many solutions. To describe it compactly, we give a basis of the vector space. In this case, the basis has only 2 functions.

The plural of basis is bases, pronounced BAY-seas.

Fact: A vector spaces has many different bases.

Exam: 6.4: The vectors $(1,0), (0,1)$ is a basis for R^2 (a real vector space), but so is any other pair of linearly independent vectors, such as $(1,0), (1,1)$

$\therefore$ The space of solutions to $\ddot{y} + y = 0$ is spanned by both by the basis e^{it}, e^{-it}, and the basis $\cos t$, $\sin t$ (Here, we allow complex linear combinations of the basis functions).

Basis for $\dot{y} = 3y$

The function e^{3t} is by itself a basis ($2e^{3t} \rightarrow$ Also)

$$\frac{dy}{y} = 3$$

$$\ln y = 3t$$

$$y = ce^{3t}$$

$\therefore$ The basis is supposed to consist of linearly independent functions such that all the solutions can be built from them. (The functions in the basis are supposed to be linearly independent.)

Dimension

It turns out that although a vector space can have different bases, each basis has the same number of vectors in it.

Def: 7.1: The dimension of a vector space is the number of vectors in any basis.

7.2: The vector space R^3 is a 3-dimensional real vector space, since there are 3 vectors in the basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

7.3: The space of solutions to the homogeneous equ $\ddot{y}+\dot{y}-6y=0$ is a 2 dimensional, since the basis e^{2t}, e^{-3t} has 2 elements (Every other basis for the same vector space also has 2 elements).

7.4: The space of solutions to $\dot{y}=3y$ is 1-dimensional.

In the examples involving diff equ above, the dimension equals to the order of the homogeneous linear ODE. It turns out that this holds in general.

Dimensional theorem: The dimension of the space of solutions to an n^{th} order homogeneous ODE with constant co.efficients is n.

In other words, the no. of parameters needed in the general solution to an n^{th} order homogeneous ODE with constant co.eff is n. The theorem is true for n^{th} order homogeneous linear ODEs with variable co.eff provided that they are continuous & the leading co.eff is never 0.

(This theorem is the consequences of the existence and uniqueness theorem as we will see shortly)