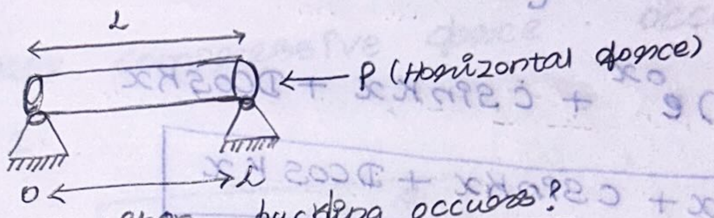
$$EI \frac{d^4 v(x)}{dx^4} + F \frac{d^2 v}{dx^2}(x) = q_y(x)$$

If $q_y(x) = 0$, we will have our previous expression (No distributive load) $\longrightarrow$ only have compressive axial load.

for transverse loading $(F=0) \Rightarrow F \frac{d^2 v}{dx^2}(x)$ will be zero.

so, we will have a general equation.

'Special load — under which buckling occurs'.



critical P when buckling occurs?

$$\boxed{k^2 = \frac{P}{EI}}$$

$$\frac{d^4 v}{dx^4}(x) + k^2 \frac{d^2 v}{dx^2} = 0$$

P → Applied horizontal force.

∴ No distributive or vertical loading. (R.H.S = 0)

Boundary conditions: (pinned – Left End, Rollers – Right End)

1) $v(0) = 0$.

2) (Horizontal movement is allowed by rollers) No vertical movement.

$v(L) = 0$.

$v(x) \rightarrow$ vertical displacement

'pin supports is frictionless'

$$\frac{d^2 v}{dx^2}(0) = 0 \text{ (No moment of bending)}$$

$$\frac{d^2 v}{dx^2}(L) = 0$$

Solutions:

'our equation has constant co.eff → we can expect exponential solutions'.

$$\boxed{v(x) \propto e^{\lambda x}}$$

$\lambda \to$ obeys characteristic equ for this differential equation.

Fourth order polynomial $\Rightarrow$ 4 roots.

Two roots $\begin{cases} \lambda = 0 & \text{repeated} \\ \lambda = \pm iK & \text{(complex roots)} \end{cases}$

$$v(x) = (A+Bx)e^{0x} + C\sin Kx + D\cos Kx$$

$$\boxed{v(x) = A + Bx + C\sin Kx + D\cos Kx}$$

$$\frac{d^2 v}{dx^2}(x) = -K^2 C\sin(Kx) - K^2 D\cos(Kx)$$

$\boxed{\cos 0 = 1}$

1) $v(0) = 0 \Rightarrow \boxed{A + D = 0}$

2) $v''(0) = 0$

$\boxed{D = 0}$, $\boxed{A = 0}$

$A = 0$
$D = 0$

$v(L) = 0 \Rightarrow A + BL + C\sin KL + D\cos KL$

$$\boxed{BL + C\sin KL = 0}$$

$v''(L) = 0 \Rightarrow -K^2 C\sin(KL) - K^2 D\cos(KL) = 0$

$-K^2 C\sin(KL) = 0$	$BL = 0$
Either $C = 0$	$\therefore B = 0$
$\sin(KL) = 0$	$\therefore L \neq 0$

If $C = 0$, $v(x) = 0$

physically meaning no vertical displacement anywhere in our beam;

$C=0$ (when happens, if we didn't try to compress our beam too hard).

so $\sin KL = 0$ $\therefore KL = 0, \pi, \ldots$

$KL = n\pi$

$\therefore \boxed{P = EIK^2}$ $\therefore K^2 = \frac{P}{EI}$ $\boxed{L = \frac{n\pi}{K}}$

$$P = EI\left(\frac{n\pi}{L}\right)^2$$

The smallest compressive force occurs when $n=1$.

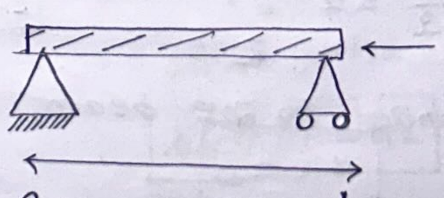
$$\therefore \boxed{P_{critical} = EI\left(\frac{\pi}{L}\right)^2}$$

$$\boxed{P_{critical} = EI\left(\frac{\pi}{l}\right)^2}$$

'This is the critical value for compressive forcing, buckling becomes possible'. It's not possible to obtain this amplitude from our theory.

'so for buckling to occur, we are expecting buckle a sinusoidal shape'

'For different physical set ups
↓
Different Boundary conditions'



$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0,$$

$$\boxed{\lambda^4 + k^2 \lambda^2 = 0}$$

$\lambda = 0, 0, \quad \lambda = \pm ik.$

$$v(x) = A + Bx + C\sin kx + D\cos kx.$$

A, B, C, D constants

$A = B = D = 0$

$$v(x) = C\sin(k(x))$$

$v(x) = 0$ when $C = 0$, $\sin(kx) = 0$

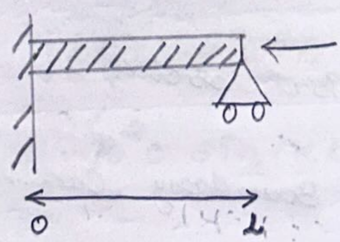
when $kx = n\pi$

$$\boxed{k = \frac{n\pi}{x}}$$

where the amplitude C can be any number. Hence the amount of vertical displacement can't be determined from this model. Our model is a linear theory, and is only valid when the bending of the beam is small.

This critical value of P is known as Euler's critical load. This value is determined by the boundary conditions.

Problem



Soln:

$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0, \qquad k^2 = \frac{P}{EI}$$

$P \rightarrow$ Axial load applied to the beam, ($k \rightarrow$ for simplification)

$$\vartheta(x) = A + Bx + C\sin Kx + D\cos(Kx)$$

Soln:

$$\vartheta(0)=0 \qquad \vartheta(L)=0$$

$$\frac{d\vartheta(0)}{dx}=0 \qquad \frac{d^2\vartheta}{dx^2}(L)=0$$

① ⇒ $\boxed{A+D=0}$

② ⇒ $\vartheta'(x) = B + CK\cos Kx - DK\sin(Kx)$

$\vartheta'(0) = B + CK$

$\boxed{B+CK=0}$

③ ⇒ $A + BL + C\sin(KL) + D\cos(KL) = 0$

④ ⇒ $\vartheta''(L) = -K^2C\sin(KL) - K^2D\cos(KL)$

$C\sin(KL) + D\cos(KL) = 0$

$KL = \frac{\pi}{4}$

$K = \frac{\pi}{4L}$

$\boxed{D=-A, \quad B=-KC}$

$0 = -D - KCL + C\sin(KL) + D\cos(KL) \longrightarrow ③$

$0 = -K^2C\sin(KL) - K^2D\cos(KL) \longrightarrow ④$

$C\sin(KL) + D\cos(KL) = 0 \longrightarrow ⑤$

sub ⑤ in ③

$\boxed{D = -KCL}$

$\therefore C\sin(KL) - KCL\cos(KL) = 0$

$C(\sin KL - KL\cos(KL)) = 0$

$C = 0$ (or) $\sin KL - KL\cos(KL) = 0$

$\sin KL = KL\cos(KL)$

$\boxed{\tan(KL) = KL}$

The smallest such values, i.e critical value happens when

(By numerical calculators

$\hookrightarrow \tan x = x$ when $x = 4.4934..$ (minimum)

$kL \approx 4.493$ (or)

$$\boxed{k \approx \frac{4.493}{L}}$$

Recitation

Matlab:

'Solving linear systems'

Solving linear equations $\rightarrow$ Is so important.

'Important in designing a stable bridge'

Converting linear equ $\longrightarrow$ Matrix equation $Ax = b.$ $\downarrow$ Solve using numerical solvers

1)
$$1.5x_1 + x_2 = 3$$
$$x_3 = 4x_2$$
$$4 - x_1 + x_2 = x_3$$

$$\boxed{\begin{aligned} 1.5x_1 + x_2 + 0 &= 3 \\ 0x_1 - 4x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= -4 \end{aligned}} \rightarrow \text{organized.}$$

$$\underset{A}{\begin{bmatrix} 1.5 & 1 & 0 \\ 0 & -4 & 1 \\ -1 & 1 & -1 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}}$$

$$\boxed{Ax = b}$$

If a, x & b are scalars, we can divide them,

$$\frac{a}{a}x = \frac{b}{a}$$

$$\boxed{x = \frac{b}{a}}$$

$$\frac{Ax}{A} = \frac{A}{b}$$

$$\boxed{x = \frac{A}{b}}$$

Matrix multiplication → computed using algorithms.

Commands:

$\gg x = A\cancel{/}b$ (A\b)

x =

1.4286
0.8571
3.4286

Large data:

coeff <300x300 double>
force <300x1 d>
tension <300x1 d>

\>> load bridgedata
\>> tension = coeff \ force

tension =

1.2516
0.9787
⋮

Generate random 10x10 matrix A

$Ax = b$, column vector 10x1 column vector b,

Solu:

<u>code</u>

```
>> A = rand (10,10);
>> b = rand (10,1);
>> x = rand (10,1);
>> x = A\b
```

eye → Identity matrix

zeros → matrix of all 0's

randn → Normally distributed random numbers

diag → Diagonal matrix

ones → Matrix of all 1's

rand → uniformly distributed random numbers

randi → Uniformly distributed random integers

linspace → Evenly spaced vector.

```
>> I = eye (10);
```

zeros (10) → Square matrix.

'Check documentation'

↳ For our required matrix.

<u>Two uses</u>

1) D = diag(V);

creates an n×n diagonal matrix with the elements of V along the diagonal.

'If V is a vector of length n.

2) x = diag (A)

↳ column vector x out of the elements from the main diagonal of A. (A → n×n vector)

```
>> A= rand (15,15);
>> V= diag(A)
>> D = diag V
```

V = diag (A) → has diagonal elements of A (column vector)

D = diag (V) → has (n×n – matrix) with diagonal elements as element of vector V.
10×10

Super & Sub diagonal

Let V be a vector of length n, then

$D = diag(v, 1)$

creates an $(n+1) \times (n+1)$ matrix with the elements of v above the main diagonal.

$v = ones(4,1)*5;$ → vector with 4 entries all equal to 5

$A = diag(V, 1);$ → 5x5 matrix

$$A = \begin{pmatrix} 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$V = ones(3, 1) * -1$

$A = diag(V, -2)$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

---

* column vectors v with entries 1, 2, 3, 4 in that order.
* create a matrix A with the entries of v on the 3rd super diagonal.
* create a matrix B with the entries of v on the (-1) st sub diagonal.

```
>> v= [1, 2, 3, 4];
>> A= diag(v, 3)
>> B = diag(v, -1)
```

a =

```
0 0 1 0 0 0
0 0 0 2 0 0
0 0 0 0 3 0
0 0 0 0 0 4
0 0 0 0 0 0
0 0 0 0 0 0
```

b =

```
0 0 0 0 0 0
1 0 0 0 0 0
0 2 0 0 0 0
0 0 3 0 0 0
0 0 0 4 0 0
```

Tridiagonal matrix

Matlab has a way of creating tridiagonal matrices (matrices with entries only on the diagonal, and super & sub diagonals). These are saved as 'sparse matrices' which are a data type that only saves the location of non-zero entries.

Create a 10×10 matrix A with entries 1 along the main sub diagonal, 2 along the diagonal, and 4 along the super diagonal as follows. Attempting to find the entry in the (1, 1) position will give the following error.

```
>> n= 10;
>> A = gallery('tridiag', n, 1, 2, 4);
>> B = full(gallery('tridiag', n, 1, 2, 4));
```

5×5 tridiagonal matrix A with entries 1, 2, 4 as a sparse matrix

B → with as a full matrix

```
>> n=5;
>> A = gallery('tridiag', n, 1, 2, 4)
>> B = full (gallery ('tridiag', n, 1, 2, 4)).
```

A =

Sparse matrix.

(1, 1)	2
(2, 1)	1
(1, 2)	4
(2, 2)	2
(3, 2)	1
(2, 3)	4
(3, 3)	2
(4, 3)	1
(3, 4)	4
(4, 4)	2
(5, 4)	1
(4, 5)	4
(5, 5)	2

B =

$$\begin{matrix} 2 & 4 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{matrix}$$

2 4 0 . . . .

1 2 4 . . . . . (up to 10)

0 1 2 4

1 2 4

1 2 4

1 2 4

. . .

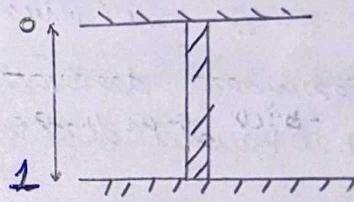
. . . (up to 10)

3 diagonals

Solving boundary value problems numerically.

ODE 45 → can solve only a I order ODE

(we won't able to use ODE 45 → It requires initial conditions. Here we will see how one can discretize the problem and reduce it to a problem in linear algebra instead!)



vertical elastic beam, fixed at both ends, with force acting along its vertical axis.

∴ This system satisfies our differential equation

$$\boxed{\frac{d^2u}{dy^2} = \frac{f(y)}{E}} \qquad \therefore u(0) = 0,\ u(1) = 0.$$

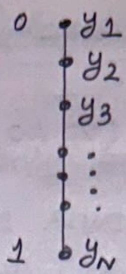
u - vertical displacement, $f(y)$ → stress acting per unit length on the beam.

E - Youngs modus (constant).

step:1:

I step is to discretize the beam. The height y is a continuous variable from 0 to 1. Create a series of evenly spaced points $y_1, \dots y_N$ with $y_N = 1$. Then $\boxed{\Delta y = y_{i+1} - y_i}$

$u_i = u(y_i)$



Discrete version of $\frac{du}{dy}$ at the point y_i? This is a subtle question because there isn't just one answer.
The approach is to use a secant line approximation.
The question is which points to use in the appr.
Two equally good answers that seem the most natural.

Answer 1

$$\frac{u_{i+1} - u_i}{\Delta y}$$

Answer 2

$$\frac{u_i - u_{i-1}}{\Delta y}$$

Answer 1 → Forward derivative.
Answer 2 → Backward derivative

Second derivative at y_i:

(Now more answers) → more options.

$$\frac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta y)^2}$$ Forwards

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2}$$ Centered

$$\frac{u_i - 2u_{i-1} + u_{i-2}}{(\Delta y)^2}$$ Backwards

We will use centered second derivative here.
Let's see how to write our diff eqn as a system of linear eqn,

For each i, we get

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2} = \frac{f(y_i)}{E}.$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{f(y_i)\,\Delta y^2}{E}$$

we can write this as

$$\begin{pmatrix} ? & ? & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & ? & ? \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \frac{(\Delta y)^2}{E} \begin{pmatrix} f(y_1) \\ f(y_2) \\ \vdots \\ f(y_N) \end{pmatrix}$$

what ~~to~~ do do with end points?

1) Enforce the boundary conditions $u(0) = u_1 = 0$

$u(1) = u_N = 0.$

∴ The first eqn in the matrix must be an eqn setting this boundary condt, and the last eqn must set the other boundary condt.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \vdots & \vdots & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = \frac{(\Delta y)^2}{E} \begin{pmatrix} 0 \\ f(y_2) \\ \vdots \\ f(y_{N-1}) \\ 0 \end{pmatrix}$$

← coefficient matrix

'can be solved using Matlab'

∴ I derivative ⟶ $\dfrac{u_{i+1} - u_i}{\Delta y}$

II derivative $= \dfrac{(u_{i+2} - u_{i+1}) - (u_{i+1} - u_i)}{(\Delta y)^2}$

$= \dfrac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta y)^2}$

(or) II derivative: $\dfrac{u_i - u_{i-1}}{\Delta y}$

$$\text{II derivative} = \frac{(u_{i+1} - u_i) - (u_i - u_{i-1})}{(\Delta y)^2}$$

$$= \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta y)^2}$$

Centered.

(or) I derivative: $\dfrac{u_{i-1} - u_{i-2}}{\Delta y}$

$$\text{II derivative} = \frac{(u_i - u_{i-1}) - (u_{i-1} - u_{i-2})}{(\Delta y)^2}$$

$$= \frac{u_i - 2u_{i-1} + u_{i-2}}{(\Delta y)^2}$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{f(y_i)}{E} \cdot \Delta y^2$$

$$\boxed{\begin{aligned} u(y_1) &= 0 \\ u(y_N) &= 0 \end{aligned}}$$

$(n \times n)\ (n \times 1) = (n \times 1) \rightarrow$ matrix

---

MATLAB

Differential equation with boundary conditions:

coefficient matrix

Solve $\dfrac{d^2u}{dy^2} = -\dfrac{0.01}{E}$, $0 < y < 1$.

$u(0) = 0$, and $u(1) = 0$.

Steps:

1) Create a 10x10 matrix A corresponding to the discrete second derivative & boundary conditions. (as seen above).

2) Create a 10 element vector b that is 0 at the ends and -0.01 everywhere else.

3) $Au = \frac{(\Delta y)^2}{E} \times b$ for $E = 3.2$, Find u.

4) Create a vector y of containing 10 evenly spaced points from 0 to 1.

5) Plot u on the vertical axis, and y on the horizontal axis.

Our matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ & & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

» Equ will be

1) $u_2 - 2u_1 + u_0$

2) $u_3 - 2u_2 + u_1$

3) $u_4 - 2u_3 + u_2$

4) $u_5 - 2u_4 + u_3$

5) $u_6 - 2u_5 + u_4$

6) $u_7 + \ldots$

$$\begin{array}{cccccc} u_0 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{l} u_1 = u(y_1) = 0 \\ u_N = u(N) = 0 \end{array}}$$

code:

```
>> A = full(gallery('tridiag', 10, 1, -2, 1));
>> a(1,1) = 1;   a(1,2) = 0;
>> a(10,10) = 1;  a(10,9) = 0;
>> b = ones(10,1) * (-0.01);
>> b(1,1) = 0;   b(10,1) = 0;
```

A

$$\begin{array}{llllll} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -2 & 1 & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \\ & \vdots & & & & -2 \end{array}$$

b =

```
     0
-0.0100
-0.0100
-0.0100
-0.0100
-0.0100
-0.0100
-0.0100
-0.0100
     0
```

$$Au = \frac{(\Delta y)^2}{E} \cdot b$$

$$u = \frac{(\Delta y)^2}{E} \cdot \frac{b}{A}$$

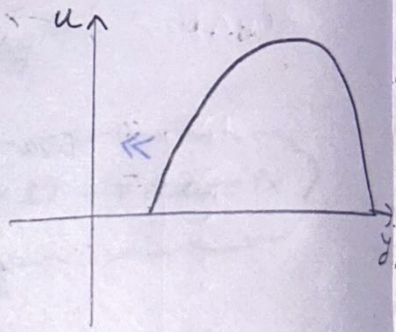
$$= \frac{(\Delta y)^2}{E} \cdot A \backslash b$$

```
>> E = 3.2;
>> Delta = 0.1;
```

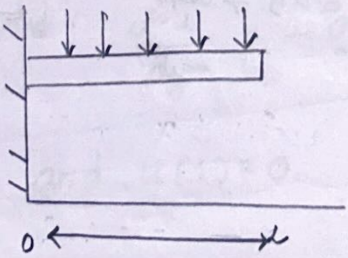
$\boxed{\text{Here} \quad \Delta y = \frac{1}{10} = 0.1}$

$\therefore$ $\boxed{\begin{array}{l} Ax = B \\ x = A\backslash B \end{array}}$

```
>> u = (a)\((b*(delta^2))/E);
>> y = [0.1: 0.1: 1];
>> y = transpose (y);
>> plot (y, u)
```



Horizontal beam & higher derivatives



We will use linear algebra to discretized solve the problem of a cantilever - a metal beam stuck into a wall at one end, and free at other end.

solving,

$$\boxed{\frac{d^4 v}{dx^4} = \frac{v(x)}{EI}}$$

For solving numerically, we need to discretize the fourth derivative. Additionally, we will have to find the formulas for the boundary conditions.

$\boxed{L=1}$

$$v(0)=0 \qquad \frac{d^2v}{dx^2}(L)=0$$

$$\frac{dv(0)}{dx}=0 \qquad \frac{d^3v}{dx^3}(L)=0.$$

Discrete Formulas

$v_i = v(x_i)$, we have $N=10$, that's we have discretized x in 10 evenly spaced points $x_1, \dots, x_{10}$.

Let $v_i = v(x_i)$

Solu:

Reducing the formula to linear relationships, involving the v_i alone and let $v_i = v(x_i)$ are homogeneous.

↳ possible → conditions

$$\frac{d^4v}{dx^4} = \frac{w(x)}{EI}$$

$$\frac{d^3v}{dx^3} = \frac{w(x)}{EI}x + a$$

$$\frac{d^2v}{dx^2} = \frac{w(x).x^2}{2EI} + ax + b$$

$$\frac{dv}{dx} = \frac{w(x)}{6EI}x^3 + ax^2 + bx + c$$

$$v = \frac{w(x)}{24} \cdot \frac{x^3}{EI} + ax^3 + bx^2 + cx + d$$

$\boxed{d=0}$, $\boxed{\frac{w(x)}{24} \cdot \frac{1}{EI} + a + b + c = 0}$

$$0 = \frac{w(x)}{EI} + a \qquad 0 = \frac{w(x)}{2EI} - \frac{wx}{EI} + b$$

$$\boxed{a = -\frac{wx}{EI}} \qquad \boxed{b = \frac{-w(x)}{2EI}}$$

$$c = -a - b - \frac{\omega(x)}{24EI}$$

$$c = \frac{\omega(x)}{EI} + \frac{\omega(x)}{2EI} - \frac{\omega(x)}{24EI}$$

$$c = \frac{3\omega(x)}{2EI} - \frac{\omega(x)}{24EI}$$

$$c = \frac{36\omega(x) - \omega(x)}{24EI} \Rightarrow c = \frac{35\,\omega(x)}{24EI}.$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{f(y_i)}{E}(\Delta y)^2$$

Apply boundary condition:

1) I derivative $\rightarrow \dfrac{u_{i+1} - u_i}{\Delta y}$

II derivative $\rightarrow \dfrac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta y)^2}$

(At $x=0$) $\longrightarrow$ Forward

$x=0$

$\boxed{v_{i1} = v(x_i)}$

$\boxed{v_1 = 0}$

$\boxed{V_1 - V_2 = 0}$

$x = 1$

$\boxed{-v_7 + 3v_8 - 3v_9 + v_{10} = 0}$

$\boxed{v_8 - 2v_9 + v_{10} = 0}$

In our example

$v(0) = 0,$

$\dfrac{dv(0)}{dx} = 0$

$\dfrac{d^2 v}{dx^2}(1) = 0$

$\dfrac{d^3 v}{dx^3}(1) = 0$

$i=0$

Proof

our equ

$$\frac{d^4 v}{dx^4} = \frac{q(x)}{EI}$$

$$\boxed{\begin{aligned} \therefore\ v_2 &= 0 \\ v_1 &= 0 \end{aligned}}$$

1) $v(0) = 0$

meaning $\boxed{x_1 = 0}$ $\quad \therefore \boxed{v_1 = 0}$

x_1
x_2
$\vdots$
x_{10}

2) $\frac{dv(0)}{dx} = 0$

Forward alone

$$= \frac{v_{i+1} - v_i}{\Delta x} = 0$$

$$\boxed{0 = v_2 - v_1}$$

3) $x = 1$

meaning $v(1) = v(x_{10}) = v_{10}$.

$\frac{d^2 v}{dx^2}(1) = 0$

(backward)

$I \to$ derivative $\quad = \frac{u_i - u_{i-1}}{\Delta y}$

$$0 = \frac{(u_i - u_{i-1}) - (u_{i-1} - u_{i-2})}{(\Delta y)^2}$$

$$0 = u_i - 2u_{i-1} + u_{i-2}$$

$$\boxed{0 = v_{10} - 2v_9 + v_8}$$

$\frac{d^3 v}{dx^3}(1) = 0$

$$= \frac{(u_i - u_{i-1}) - 2(u_{i-1} - u_{i-2}) + (u_{i-2} - u_{i-3})}{(\Delta y)^3}$$

$$0 = u_i - 3u_{i-1} + 3u_{i-2} - u_{i-3}$$

$$\boxed{0 = v_{10} - 3v_9 + 3v_8 - v_7}$$

Fourth derivative $v(x_i)$

$$\frac{v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}}{(\Delta x)^4}$$

Putting all together.

$$\frac{d^4 v}{dx^4} = \frac{q(x)}{EI}$$

Solu:

$$\begin{pmatrix} * & * & 0 & 0 & & 0 & \cdots & 0 \\ * & * & 0 & 0 & & 0 & \cdots & 0 \\ \bullet & \bullet & \bullet & \bullet & 0 & \cdots & & 0 \\ 0 & \bullet & \bullet & \bullet & \bullet & \cdots & & 0 \\ & \ddots & & \ddots & & \ddots & & \\ 0 & \cdots & \bullet & \bullet & \bullet & & \bullet & \bullet \\ 0 & \cdots & 0 & 0 & * & & * & * \\ 0 & \cdots & 0 & 0 & * & & * & * \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ \vdots \\ v_{N-1} \\ v_N \end{pmatrix} = \frac{\Delta x^4}{EI} \begin{pmatrix} 0 \\ 0 \\ q(x_3) \\ \vdots \\ q(x_{N-2}) \\ 0 \\ 0 \end{pmatrix}$$

$0 \Rightarrow \underline{v_2 - 4v_1} + 6v_0 - 4v_{-1} + v_{-2}$

$1 \Rightarrow v_3 - 4v_2 + 6v_1 - 4v_0 + v_{-1}$

$2 \Rightarrow v_4 - 4v_3 + 6v_2 - 4v_1 - v_0$

$3 \Rightarrow v_5 - 4v_4 + 6v_3 - 4v_2 - v_1$

$*$ → Boundary conditions

$\bullet$ → Rows of $\circ$ correspond to the 4th derivative.

Fourth order system with boundary conditions.

$A,\ b,\ v$

$Av = b$

$\boxed{v = A \backslash b} \rightarrow \boxed{A^{-1} b}$ meaning.

Heat equation

1) Apply superposition & seperation of variables to find a general solution to the I order heat equation with homogeneous boundary conditions.

2) Set $t=0$ in the general solution to obtain the Fourier series solution describing the Initial condition. Apply the I.C to determine the Fourier co.eff of that series.

3) Find a particular solution to the 1-d heat equ with in homogeneous boundary conditions.

4) Recognize the diffusion equ as analogous to the heat equ, and solve using Fourier methods with different boundary conditions.

Introduction to the heat equation.

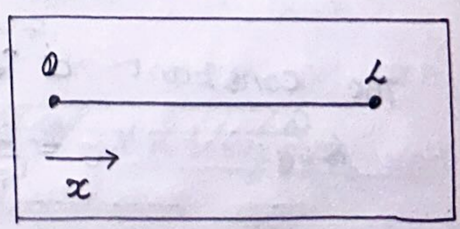
In this section, we meet our first partial differential equation (PDE)

$$\boxed{\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2}}$$

This is the equation satisfied by the temperature $\theta(x,t)$ at position x and time t, of a bar depicted as a segment,

$$0 \le x \le L, \quad (t \ge 0)$$

The constant ϑ is the heat diffusion co.efficient, which depends upon the material of the bar. we will focus on one physical experiment. Suppose that the initial temperature is 1, and then the ends of the bar are put in ice.



we write this as

$$\theta(x,0) = 1, \quad 0 \le x \le L$$

$$\theta(0,t) = 0, \quad \theta(L,t) = 0.$$

$t > 0$

The values of $\theta = 1$ at $t = 0$ are called initial conditions. The values at the ends are called end points on boundary conditions. we think of the initial and end point values of θ as the input, and the temperature

$$\theta(x, t) \text{ for } t > 0, \quad 0 < x < L \text{ as}$$

the response.

<u>Remark 2.1</u>

For simplicity, we assume that only the ends are exposed to the lower temperature. The rest of the bar is insulated, not subject to any external change in temperature. Fourier's techniques also yield answers even when there is heat input over time at points along the bar.

As time passes, the temperature decreases as cooling from the ends spreads toward the middle. At the point (mid-point)

$\frac{L}{2}$, one finds Newton's law of cooling

$$\theta\left(\frac{L}{2}, t\right) \approx ce^{-t/T}, \quad t > T.$$

The so-called characteristic time T is inversely proportional to the conductivity of the material. If we choose units so that $T = 1$ for copper, then according to wikipedia

$$T \sim 7 \text{ (cast iron)}$$

$$T \sim 7000 \text{ (dry snow)}$$

The constant c on the other had is universal:

$$c = \frac{4}{\pi} \approx 1.3.$$

It depends only on the fact that the shape is a bar (modeled as line segment).

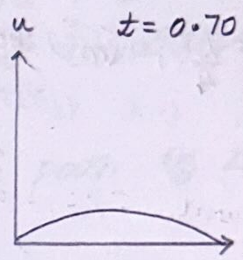
Fourier figured out not only how to explain C using differential equations, but the whole temperature profile:

$$\theta(x,t) \approx e^{-t/\tau}\, h(x); \quad h(x) = \frac{4}{\pi} \sin\left(\frac{\pi}{L} x\right),$$

$$\boxed{t > \tau}$$

The shape of h shows that the temperature drop is less in the middle than at the ends. It's natural that h should be some kind of hump, symmetric around $L/2$

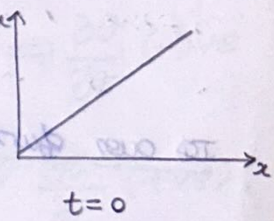
$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$



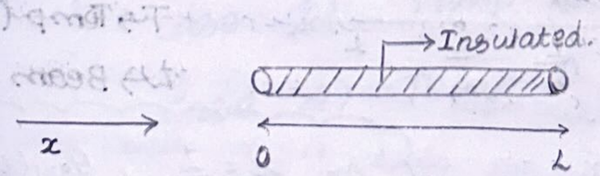
• Use Mathlet

Heat equations

Ramp (IC)



<u>Deriving the heat equ</u>

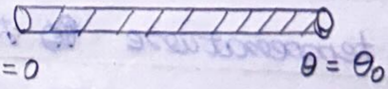


1-D bar. (line)

Length is more long.

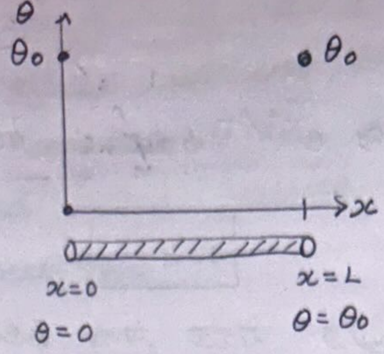
$\theta(x,t) \rightarrow$ temperature.

• Heating or cooling at ends alone



External temperature doesn't affect insulated region.

Focus: Equilibrium: $(t \to \infty)$



At $x=0, \theta=0$
$x=L, \theta=\theta_0$

Temperature profile:

Average = Difference b/w left & right end.

At middle $x=\frac{L}{2}$, Bar to be at the average temp $\frac{\theta_0 - 0}{2}$

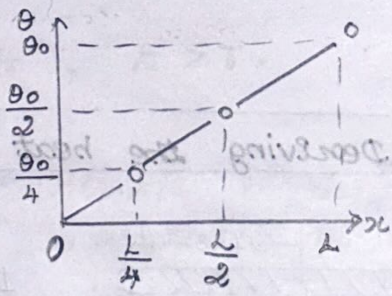
At $\frac{L}{4}$, (mid way b/w $x=0$ and $x=\frac{L}{2}$)

At $\frac{L}{4} \Rightarrow \boxed{\theta = \frac{\theta_0}{4}}$

we can continue this as,

$$\frac{L}{8} \Rightarrow \frac{\theta_0}{8}, \quad \frac{L}{16} \Rightarrow \frac{\theta_0}{16}$$

If our function is continuous:



(or)

$$\theta_{equ}(x) = \frac{T}{L} x$$

$T \to$ Temp (θ_0)

$L \to$ Beam length

$$\theta_{eq}(x) = \frac{\theta_0 x}{L}$$

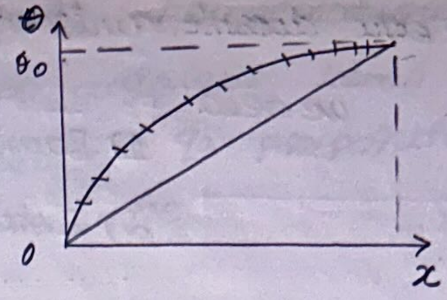
At $x = \frac{L}{4}$, $\theta_{eq}(x) = \frac{\theta_0 L}{4L} = \frac{\theta_0}{4}$

$\therefore$ valid only when our system had attained it's equilibrium value.

How temperature of the bar evolves over time?

At $x=0$, $\theta=0$

$x=L$, $\theta=\theta_0$



If our initial,

temperature profile

as concave time.

As time increases,

$\rightarrow$ concave down curve $\} \rightarrow$ will tend to move as a straight line (equilibrium)

$\rightarrow$ concave up $\rightarrow$ will $\uparrow$ to achieve the equilibrium

to be in the path of the concave down (decreases) line.

From single variable calculus:

1) c. down $\dfrac{\partial^2 \theta}{\partial x^2} < 0$ (when concave down) $\Rightarrow \dfrac{\partial \theta}{\partial t}$ (−)ve

2) c. up. $\dfrac{\partial^2 \theta}{\partial x^2} > 0$ (when concave up) $\Rightarrow \dfrac{\partial \theta}{\partial t}$ (+ve).

case 1) Temp $\downarrow$ as $t \uparrow$

case 2) Temp $\uparrow$ as $t \downarrow$.

Simplest possible relationship:

Linear relationship b/w time derivative & space derivative.

$$\boxed{\frac{\partial \theta}{\partial t} = \vartheta \frac{\partial^2 \theta}{\partial x^2}} \rightarrow \text{Heat equation.}$$

$\vartheta \rightarrow$ bar's property

Heat eqn doesn't uniquely specifies how heat evolves

we need

1) Boundary } conditions.
2) Initial }

Here we used BVP

$\theta(0,t)=0$ | $\theta(x,0)=f(x)$
$\theta(L,t)=0$ | ↳ Initial condition.

Fourier introduced heat equation, solved it, and confirmed in many cases that it predicts correctly the behaviour of temperature in experiments like the one with the metal bar.

Actually, Fourier crushed the problem, figuring out the whole formula for $\theta(x,t)$ and not just when the initial value is $\theta(x,0)=1$, but also when the initial temperature varies with x. His formula even predicts accurately what happens when $0<t<T$.

Another derivation

we are going to look at one way to derive a partial differential equation describing the evolution of temperature in an insulated, uniform bar of length L. we first looked at this problem in the course Linear Algebra and NxN systems of differential eqn, when we were studying nxn systems, Initially, we considered a rod of length L, with two thermometers placed evenly along its length, at the points $x_1=\frac{L}{3}$ and $x_2=\frac{2L}{3}$. The left end of the rod was held at the temperature θ_L and the right end was held at θ_R. we used Newton's law of cooling, which said that the rate of change of temperature at x_1, $\frac{d\theta_1}{dt}$ is affected by the adjacent temp θ_L and θ_2. The left contribution is proportional to the difference in temperature

$$\theta_L - \theta_1.$$

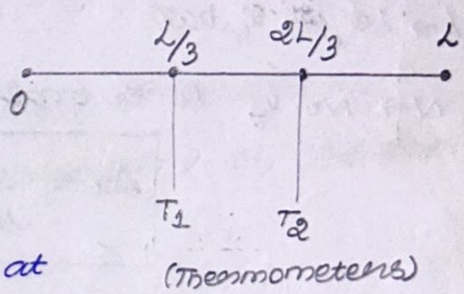
while the right contribution is proportional to $\theta_2 - \theta_1$. This meant that we could write down a system of two eqn for θ_1 and θ_2.

$$\frac{d\theta_1}{dt} = K(\theta_L - \theta_1) + K(\theta_2 - \theta_1)$$

$$= K(\theta_L - 2\theta_1 + \theta_2)$$

$$\frac{d\theta_2}{dt} = K(\theta_1 - \theta_2) + K(\theta_R - \theta_2)$$

$$= K(\theta_1 - 2\theta_2 + \theta_R)$$



Note: we are placing many thermometers on the bar & track the interaction of temperatures at many points, then we should adjust the constant K for the distance b/w points.

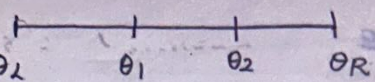
Let's suppose that we place N thermometers at equally spaced points $x_n = n\Delta x$, $n = 1, 2, \ldots N$. along the bar, with

$$\boxed{\Delta x = \frac{L}{N+1}}$$

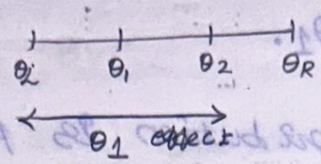
If $\theta_n(t)$ is the temperature at x_n, then the correct scaling for the influence of θ_{n+1} and θ_{n-1} on θ_n is the following version of Newton's law.

$\theta_L \rightarrow$ Left end temp

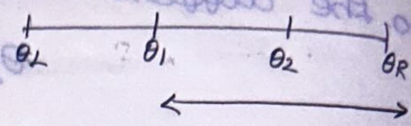
$\theta_R \rightarrow$ Right end temp



Effect of θ_1



Effect of θ_2

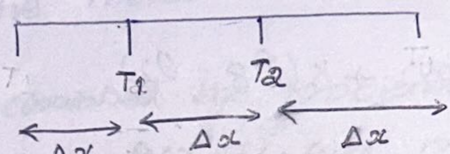


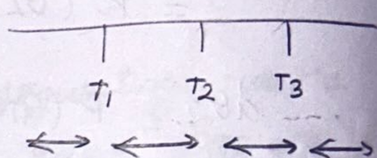
$$\frac{d\theta_1}{dt} = K(\theta_L - \theta_1) + K(\theta_2 - \theta_1) = K(\theta_L - 2\theta_1 + \theta_2)$$

$$\frac{d\theta_2}{dt} = K(\theta_1 - \theta_2) + K(\theta_R - \theta_2) = K(\theta_1 - 2\theta_2 + \theta_R)$$

Placing thermometers:

$$\Delta x = \frac{L}{N+1}$$





$L \rightarrow$ Length of bar

$N \rightarrow$ No. of thermometers.

$$\boxed{\Delta x = \frac{L}{\text{No. of segments}}}$$

By Newton's law of cooling:

$$\frac{d\theta_n}{dt} = \frac{\vartheta(\theta_{n-1} - \theta_n)}{(\Delta x)^2} - \frac{\vartheta(\theta_n - \theta_{n+1})}{(\Delta x)^2}$$

$$\boxed{K = \frac{\vartheta}{(\Delta x)^2}}$$

$\vartheta \rightarrow$ property of the material.

(Inverse square law is what Fourier discovered by testing bars of metals of various lengths. He then applied the insight that calculus brings, namely that the same rule should work at infinitesimal scales.) We can now simplify the right hand side to obtain

$$\frac{d\theta_n}{dt} = \nu \frac{(\theta_{n+1} - 2\theta_n + \theta_{n-1})}{(\Delta x)^2}$$

$\Delta x \to 0$, $N \to \infty$.

we are interested in deriving a partial der eqn for the temperature $\theta(x,t)$ at any position x along the bar.

$\therefore$ we therefore need to choose n so that $x_n \to x$ as $N \to \infty$.

$$\boxed{\text{As } N \to \infty,\ \Delta x \text{ goes to } 0}$$

Possible choice:

$$n = \left\lfloor \frac{x}{\Delta x} \right\rfloor = \left\lfloor \frac{x(N+1)}{L} \right\rfloor \quad \text{as } N \to \infty,$$

the right hand side of the eqn for $\frac{d\theta_n}{dt}$ becomes $\frac{\partial^2 \theta}{\partial x^2}(x)$. In this limit,

we have the following der eqn describing the evolution of temperature in the bar:

Continuous

$$\boxed{\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}}$$

Boundary cond:

"Ends"

"Impose fixed temp for all time at both ends"

$$\boxed{\theta(0,t) = \theta_L, \quad \theta(L,t) = \theta_R}$$

The initial condition can be any smooth function

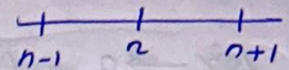
$\theta_0(x)$

so that $\theta(x, t=0) = \theta_0(x)$

Boundary along with IC can give a unique solution

$\theta(x, t)$ for all future time $t > 0$.

$$n = \left\lfloor \frac{x}{\Delta x} \right\rfloor$$

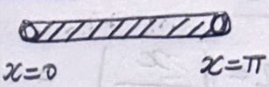


$$\Delta x = \frac{L}{N+1}$$

$$= \left\lfloor \frac{x(N+1)}{L} \right\rfloor \quad \text{As } N \to \infty, \ \Delta x \to 0.$$

Seperation of variables: Solving a PDE with Inhomogeneous boundary conditions.

$$\boxed{\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}}$$



$x=0, x=\pi \rightarrow$ Just for simplification.

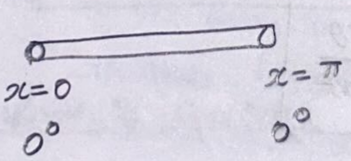
i) $\theta(x, 0) = 1$ (Initial)

> Initially our bar is held at a temperature 1°
> ↳ At every point of our bar $\theta(x, 0) = 1$

Boundary condt:

$L = \pi, \nu = 1,$

$\theta_0 = 1$



$$\boxed{\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}}$$

(Immersed at ice)

$$\boxed{\begin{aligned} \theta(0, t) &= 0 \\ \theta(\pi, t) &= 0 \end{aligned}}$$

method - seperation of variables

$\theta(x, t) = a(x)\, b(t)$ } For using this method we need to have a solution in this form.

Step: 1:

Sub $\theta(x, t) = a(x)\, b(t)$ in our partial expression

$$\therefore \frac{\partial}{\partial t}\left(a(x)\, b(t)\right) = \frac{\partial^2}{\partial x^2}\left(a(x)\, b(t)\right)$$

$$\boxed{a\frac{\partial b}{\partial t} = b\frac{\partial^2 a}{\partial x^2}}$$

Divide by ab

$$\boxed{\underbrace{\frac{1}{b}\frac{\partial b}{\partial t}}_{\text{Functions of time}} = \underbrace{\frac{1}{a}\frac{\partial^2 a}{\partial x^2}}_{\text{Functions of space.}}}$$

How a function of time = Function of space

only when they are constants

$$\boxed{\frac{1}{b}\frac{\partial b}{\partial t} = \lambda, \quad \frac{1}{a}\frac{\partial^2 a}{\partial x^2} = \lambda}$$

Solving:

$$\frac{\partial^2 a}{\partial x^2} = a\lambda$$

$$\theta(0,t) = 0, \quad \theta(\pi, t) = 0$$

$$\theta(0,t) = a(0)\cdot b(t) = 0$$

$$\therefore \boxed{a(0) = 0}$$

$$\boxed{a(\pi) = 0}$$

only has non zero solutions if

$$\boxed{\lambda = -n^2, \quad n = 1, 2, 3, \ldots}$$

$$a(x) = A\sin(nx) \qquad \therefore \text{Complex root}$$

$$\frac{1}{b}\,\frac{\partial b}{\partial t} = \lambda$$

$$\boxed{\lambda = -n^2}$$

$$\frac{\partial b}{\partial t} = b\lambda$$

$$\boxed{b(t) = Be^{-n^2 t}}$$ → General solution.

Initial condition:

$$\theta(x,t) = A\sin(nx)\cdot Be^{-n^2 t}$$

$$\theta(x,t) = C\,e^{-n^2 t}\cdot \sin(nx)$$

$\theta(x,0) = 1$ → But here not possible

Family of values are available, $(n = 1, 2, 3, \dots)$

∴ $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial x^2}$ → Linear → we can use superposition. (Linear combination).

$$\theta_n(x,t) = c_n\,e^{-n^2 t}\sin(nx)$$

$$\theta(x,t) = c_1 e^{-t}\sin x + c_2 e^{-4t}\sin 2x + c_3 e^{-9t}\sin 3x + \dots$$

$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} c_n\,e^{-n^2 t}\sin(nx)}$$ → more general solution.

Adding all the possible solutions,

our answer → need to satisfy both IC & BC

Note: we are making a slight change in notation.
'Substituting into the PDE'

(Then solving) $\boxed{\begin{matrix} \frac{1}{b}\frac{db}{dt} = \lambda \\ \frac{1}{a}\frac{d^2a}{dx^2} = \lambda \end{matrix}}$

Simply:

$$v(x)\,\mathring{w}(t) = w(t)\,v''(x)$$

$$\frac{\mathring{w}(t)}{w(t)} = \frac{v''(x)}{v(x)}$$ (Atleast where $w(t)$ and $v(x)$ are non zero)

$$v''(x) = \lambda v(x), \quad \mathring{w}(t) = \lambda w(t)$$

Sub $\theta(x,t) = v(x)\,w(t)$ in to the BC,

$\theta(0,t) = 0$, $v(0)\,w(t) = 0$, for all t,

$w(t)$ is not the zero function, so this translates in to $v(0) = 0$. III^{ly} the II boundary condition $\theta(\pi,t) = 0$ translates into $V(\pi) = 0$.

$\therefore$ we have already solved,

$$v''(x) = \lambda v(x) \qquad v(0) = 0,\ v(\pi) = 0.$$

Nonzero solutions $v(x)$ exist only if

$$\boxed{\lambda = -n^2}$$

what's the matching possibility for w?

$$\mathring{w} = -n^2 w$$

$w \rightarrow$ is a scalar times $e^{-n^2 t}$

$$\theta_n(x,t) = e^{-n^2 t}\sin(nx) \qquad n = 1, 2, 3 \ldots$$

Each solution is called a normal mode.

Boundary conditions are homogeneous.

$$\theta(x,t) = b_1 e^{-t} \sin x + b_2 e^{-4t} \sin 2x + \cdots$$

$$\therefore \left(\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2} \right)$$

general solution.