$p_R(r) = ?$

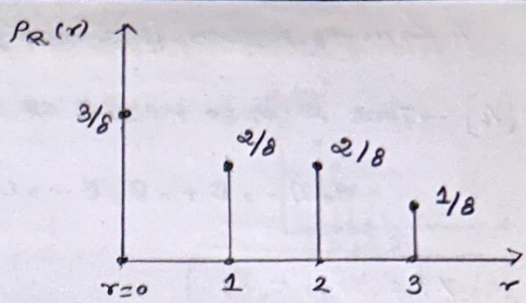
$$\begin{cases} 3/8 & , r=0 \\ 2/8 & , r=1 \\ 2/8 & , r=2 \\ 1/8 & , r=3 \end{cases}$$

$p_R(0) = 3/8$

$p_R(1) = 2/8$

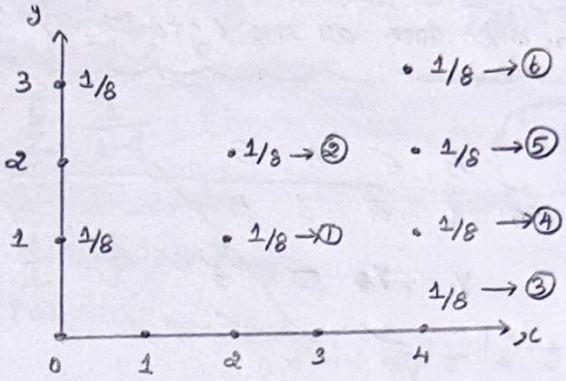
$p_R(2) = 2/8$

$p_R(3) = 1/8$



$A = \{x^2 \geq y\}$ Find $E[XY]$, $E[XY|A]$

Solu:



$A = \{x^2 \geq y\} \rightarrow (1,2), (2,2)$
$(4,0), (4,1), (4,2),$
$(4,3)$

$E[XY] = E[W] \qquad [W = XY]$

$$E[W] = 0\left(\frac{3}{8}\right) + 2\left(\frac{1}{8}\right) + 4\left(\frac{2}{8}\right) + 8\left(\frac{1}{8}\right) + 12\left(\frac{1}{8}\right)$$

$$= \frac{2}{8} + \frac{8}{8} + \frac{8}{8} + \frac{12}{8}$$

$$= \frac{30}{8} = \frac{15}{4}$$

$(0,1) \rightarrow 0$
$(0,3) \rightarrow 0$

$(2,1) \rightarrow 2 \longrightarrow A$
$(2,2) \rightarrow 4 \longrightarrow A$
$(4,0) \rightarrow 0 \longrightarrow A$
$(4,1) \rightarrow 4 \longrightarrow A$
$(4,2) \rightarrow 8 \longrightarrow A$
$(4,3) \rightarrow 12 \longrightarrow A$

$$E[W|A] = \sum_{w} w \, P_{W|A}(w)$$

$$= 0\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 4\left(\frac{2}{6}\right) + 8\left(\frac{1}{6}\right) + 12\left(\frac{1}{6}\right)$$

$$= \frac{2}{6} + \frac{8}{6} + \frac{8}{6} + \frac{12}{6}$$

$$= \frac{30}{6} = \frac{15}{3} = 5$$

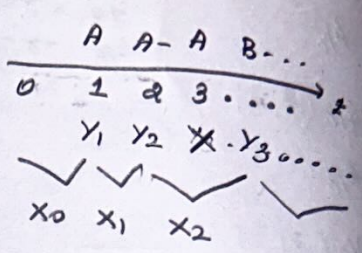
"Coupon collection problem"

$K \rightarrow$ coupons (grades) $\rightarrow K = 6$

$Y_i \rightarrow$ No. of papers [i th new grade]

$[Y_6] \rightarrow$ Time takes to receive all 6 grades

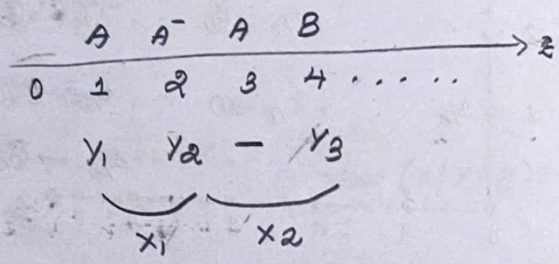
$\{A, A-, B+, B, B-, C+\}$



$$\boxed{X_i = Y_{i+1} - Y_i}$$

$$\boxed{Y_6 = \sum_{i=0}^{5} X_i}$$

Problem: when does all the 6 grades appear



$\boxed{X_i = Y_{i+1} - Y_i} \rightarrow$ No. of papers need to see $i+1$ th paper from i th paper.

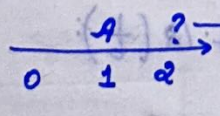
$$Y_6 = X_1 + X_2 + X_3 + X_4 + X_0 + X_5$$

$$\boxed{Y_6 = \sum_{i=0}^{5} X_i}$$

X_0
1st paper $= Y_1 - Y_0$
↓ one paper ↓ No paper

How many trials needed? → Geometric distribution

'equal probability'



→ Already 1 grade appeared. prob of another grade $= \frac{6-1}{6} = \frac{5}{6}$

$$\boxed{X_i = \text{Geometric}\left(\frac{6-i}{6}\right)}$$

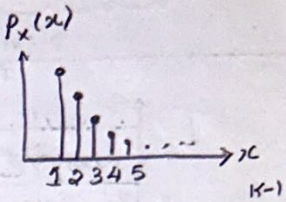
→ As times goes, new grades appear — prob of new grade ↓ (Geometric distribution)

$E[X_i] = ?$

Mean of Geometric distribution

$$E[X] = \sum_x x\, P_X(x)$$

$$E[X_i] = \frac{1}{p} = \frac{6}{6-i}$$



$P(\text{head in } K \text{ tosses}) = (1-p)^{K-1}\, p$

Head toss $= 1/2$

$P_X(1) = 1/2$

$P_X(2) = 1/4$

$P_X(3) = 1/8$

$E[X] = 1/p$

$$E[Y_6] = E\left[\sum_{0}^{5} X_i\right] = \sum_{i=0}^{5} \left[E[X_i]\right]$$

↓ Linearity

$$= \sum_{i=0}^{5} \left[\frac{6}{6-i}\right]$$

$$= 6 \sum_{i=0}^{5} \frac{1}{6-i}$$

$$\left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right)$$

$$E[Y_6] = 6 \sum_{i=1}^{6} \frac{1}{i} = 14.7$$

∴ when K grades:

$$\boxed{E[Y_K] = K \sum_{i=1}^{K} \frac{1}{i}} \simeq K \ln(K)$$

Lecture: 7 – multiple discrete RV

$$P_X(x) = \sum_y P_{X,Y}(x,y) \longrightarrow \text{marginal pmf}$$

In a conditioned universe (scaled prob)

$$\sum_x P_{X|Y}(x|y) = 1$$

$P(A|B) = P(A) \cdot P(B|A)$

$$\boxed{P[X=x \mid Y=y] = \frac{P_{XY}(x,y)}{P_Y(y)}}$$

[Possibly more than 3 random variable]

$P_{X,Y,Z}(x,y,z)$

$$\boxed{P_X(x) = \sum_{y,z} P_{X,Y,Z}(x,y,z)}$$

∴ x together occurs with y & z.

Multiplication rule for three variables:

$$\boxed{P_{X,Y,Z}(x,y,z) = P_X(x) \cdot P_{Y|X}(y|x) \cdot P_{Z|X,Y}(z \mid x,y)}$$

∴ If X, Y, Z are independent

$$P_{X,Y,Z}(x,y,z) = P_X(x)\, P_Y(y) \cdot P_Z(z) \quad [\text{For all } x,y,z]$$

$\therefore P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|B \cap A)$

4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		
	1	2	3	4

Independent: Knowing something doesn't change likelyhood of others.

$P_{X|Y}(x|y) = P_X(x) \rightarrow$ iff $P_Y(y) > 0$

$\therefore$ we can't condition with zero probabilities.

$P_{X|Y,Z}(x|y,z) = P_X(x)$

"Is X, Y independent" – Directly?

look at $x=2, y=1 \rightarrow 1/20$ (prob)

Are they independent?

value of Y – gives useful info of X.

If $Y=1$:

only one probability to $x \rightarrow x=2$

"telling $y=1$, tells $x=2$"

"Not independent"

"telling $y=2$, x may 2 or 3 or 4"

conditional independance:

$X \leq 2$ & $Y \geq 3$

4	1/20	2/20
3	2/20	4/20
	1	2

$\rightarrow$ A event

$\rightarrow X$

$P_{X,Y|A}(x,y|A) = ?$

(Rescaling) ★ 1 : 2 : 2 : 4 ratio

$P_{X,Y|A}(x,y|A) =$ Table

1/9	2/9
2/9	4/9

$\rightarrow A$

In this universe, X & Y are independent?

If $y = 3$:

$x = 1 \rightarrow 1/9$

$x = 2 \rightarrow 2/9 \longrightarrow$ Twice likely

$y = 4$

$x = 1 \rightarrow 2/9$

$x = 2 \rightarrow 4/9 \rightarrow$ Twice likely

Same in both cases

'conditional pmfs' $\boxed{P_{X|Y,A}(x|y) = P_{X|A}(x)}$

$\therefore$ X conditioned on A

same as marginal pmf of X in the new world (conditioned)

$$\boxed{P_{X|Y,A}(x|y) = P_{X|A}(x)}$$

Irrespective of y

why:

$P_{X|Y,A}(x|3) \rightarrow 2/9$ or $4/9$

$P_{X|Y,A}(x|4) \rightarrow 1/9$ or $2/9$

$P_{X|A}(1) = 3/9$

$P_{X|A}(2) = 6/9 \rightarrow$ Twice.

$\therefore x = 2$ has twice the prob of $x = 1$

'y doesn't give any useful info'

$\therefore$ Verify

Independent

$$\boxed{P_{X,Y}(x,y) = P_X(x)\, P_Y(y)}$$

		1	2
3/9 ←	4	1/9	2/9
6/9 ←	3	2/9	4/9
		↓ 3/9	↓ 6/9

$3/9 \times 3/9 = 1/9$

$3/9 \times 6/9 = 2/9$

$6/9 \times 3/9 = 2/9$

$6/9 \times 6/9 = 4/9$

Independent

'Inside this universe X, Y are independent'

↓

'conditionally independent'

* $E[X] = \sum_x x P_X(x)$

* $E[g(x,y)] = \sum_x \sum_y g(x,y) P_{X,Y}(x,y)$

* work with x, y don't with pmf of $g(x,y)$

* $E[g(x,y)] \neq g(E[X], E[Y])$

* $E[\alpha X + \beta] = \alpha E[X] + \beta$ [func of avg $\neq$ avg of func]

* $E[X+Y+Z] = E[X] + E[Y] + E[Z]$

$\therefore$ Expectations still behaving linearly.

* If X, Y are independent

* $E[XY] = E[X]E[Y]$

* $E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$

$E[XY] = \sum_x \sum_y xy \; P_{X,Y}(x,y)$

$\rightarrow$ Independent

$= \sum_x \sum_y xy \; P_X(x) \; P_Y(y)$

$= \sum_x x P_X(x) \sum_y P_Y(y)$

$= E[X] \; E[Y]$

$= \sum_x \sum_y g(x) h(y) \; P_X(x) \; P_Y(y)$

$= E[g(X)] \cdot E[h(Y)]$

'X & Y are independent'

X doesn't tell anything about both y and $h(y)$

$\therefore$ $g(x)$ & $h(y)$ are also independent.

---

x doesn't tell me about y

x doesn't tell me about y^2 too.

Variance

* $Var(aX) = a^2 Var(X)$

* $Var(X+a) = Var(X)$ $\rightarrow$ how wide a distribution is.

* $Z = X + Y$ (Independent)

$Var(X+Y) = Var(X) + Var(Y)$

'Adding constants – shifts the distribution' – width still same'.

$$var(x+y) = E[(x+y)^2] - (E[(x+y)])^2$$

$$= \sum_x \sum_y (x+y)^2 P_{X,Y}(x,y) - \left(\sum_x \sum_y (x+y) P_{X,Y}(x,y)\right)^2$$

$$= \left[\sum_x x^2 P_X(x) + \sum_y y^2 P_Y(y) + 2\sum_x x P_X(x) \cdot \sum_y y P_Y(y)\right]$$

$$- \left(\sum_x x P_X(x) + \sum_y y P_Y(y)\right)^2$$

$$\boxed{\begin{aligned} var(X) &= E[X^2] - (E[X])^2 \\ var(Y) &= E[Y^2] - (E[Y])^2 \end{aligned}}$$

$$= \left[E[X^2] - (E[X])^2\right] + \left[E[Y^2] - (E[Y])^2\right] +$$

$$2E[X]E[Y] - 2E[X]E[Y]$$

$$= var(X) + var(Y)$$

$\therefore$ Independent

---

* If $X = Y$, $var(X+Y) = var(2X) = 4\,var(X) \neq var(X) + var(X)$
* If $X = -Y$, $var(0) = 0$ [mean value] — Extremely dependent.
* $X, Y \rightarrow$ Independent

$Z = X - 3Y$ are also independent

$$var(Z) = var(X - 3Y) = var(X) + 9\,var(Y)$$

x & $-3y$ are independent (as x & y are independent)

Revisit

Binomial distribution: X = # of success in n independent trials

Probability of success P

$$E[X] = \sum_{K=0}^{n} K \binom{n}{K} P^K (1-P)^{n-K}$$

$$X_i = \begin{cases} 1, & \text{success} \\ 0, & \text{else.} \end{cases}$$

Indicators (occured or not)

'pattern of common solution'

HHT, THH, HTH

$\rightarrow$ Lose is easier $\rightarrow$ calc: 0's & 1's

$E[X_i]$ = win or lose

$= 1(p) + 0\cdot(1-p)$

$\boxed{E[X_i] = p}$ $\boxed{\sum X_i = X}$

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = p^n$$

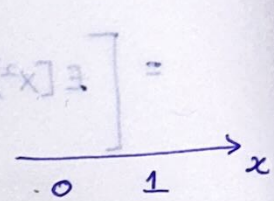
$\boxed{\therefore\ n \text{ tosses},\ p \to \text{head}\ (p \times p \times p \cdots p = p^n)}$

$$E[X] = \sum_{x} x\, P_X(x)$$

↳ particular event success p

$= \cancel{(\text{[illegible]})}$

$= (0\cdot(1-p) + 1\cdot p) + \cdots + ((1-p)\cdot 0 + p(1))$



(Loss or win)

$= n\cdot p$

$\boxed{E[X] = np}$

$\boxed{\begin{aligned}\sum X_i &= \text{Success} + \text{Failure} + \text{Success} + \cdots \\ &= \#\text{No. of success} \\ &= X\end{aligned}}$

Variance

$\boxed{\text{var}(X) = E[X^2] - (E[X])^2}$

$$E[X] = \sum_{x} x\, P_X(x) \qquad \left[E[X] = \sum E[X_i]\right]$$

$= (0\cdot(1-p)) + (1\cdot(p)) + \cdots\cdots + n$ terms (tosses)

[Each get you success or failure]

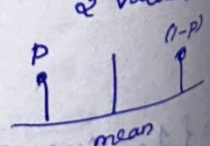
$E[X] = np$

$$\text{var}(X) = \sum_{x} (x - E[X])^2 \cdot P_X(x)$$

↓

how far x is from mean

2 values



$= p(1-p)^2 + (1-p)(0-p)^2$

$= p - 2p^2 + p^3 + p^2 - p^3$

$= P(1-P)$

(or)

$$E[X_i^2] - (E[X_i])^2 = \left[\sum_x x^2 P_X(x)\right] - (P)^2$$

Squaring 1 is 1

$$= P - P^2$$

$$= P(1-P)$$

$$Var(X) = \sum Var(X_i)$$

$\therefore$ for independent variables : $Var(X+Y) = Var(X) + Var(Y)$

$$\boxed{Var(X) = nP(1-P)}$$

Binomial distribution

Taking a single case: → win, → Loss

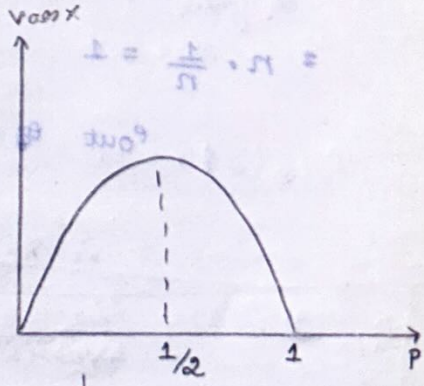
$$E[X_i] = P(1) + 0\cdot(1-P)$$
$$= P$$

$\therefore$ $E[X]$ = # no. of success = success at each trial sum

$$\boxed{E[X] = \sum E[X_i] = nP}$$

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$
$$= P - P^2$$
$$= P(1-P)$$

For $Var(X_i) = \sum X_i = nP(1-P)$



- most uncertain → when coins fair
- Extremely biased → certain (Low variance)

The hat problem

n people throw their hats in a box & then pick one at random.

X : number of people who get their own hat

$E[X]$: ?

$$x_i = \begin{cases} 1, & i \text{ selects own hat} \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{X = X_1 + X_2 + \dots + X_n}$$

person 1 – right hat (1)
wrong hat (0)

∴ Total = no. of people with right hat.

$P(X_i = 1) = 1/n$ [n hats – among n one is him]

$E[X_i] = 1 \cdot \frac{1}{n} + 0 \cdot \left(1 - \frac{1}{n}\right)$

$$\boxed{E[X_i] = 1/n}$$

$\therefore E[X] = ?$

"Are they independent? – Each hat choosing"

10 people:
i got their own hat, 10th also got his/her own hat

"Not independent"

$E[X] = \sum E[X_i] \longrightarrow$ we don't need independence

$= n \cdot \frac{1}{n} = 1$

"out of n only 1 is succeeding in average"

∴ "X_i's are not independent"

$Var(X+Y) \neq Var X + Var Y$

$Var(X) = E[X^2] - (E[X])^2$

$$\boxed{E[X^2] = ?}$$

$(1+2+3+4 \dots)(1+2+3+\dots)$

$(1^2 + 2^2 + 3^2 + \dots) + (2 + 3 + \dots)(1 + 3 + \dots)$

$X^2 = (\sum X_i)^2$

$X^2 = \sum X_i^2 + \sum_{i,j;\, i \neq j} X_i X_j$

square terms

cross terms

$E[x_q^2] = E[x_q]$

$\therefore$ x_q takes 0 or 1

$\therefore$ $E[x_q^2] = 1^2(1/n) + 0(1-1/n)$

$$\boxed{E[x_q^2] = 1/n}$$

$E[x_q x_s] = ?$ $\rightarrow$ Take a simple case

$x_q \rightarrow$ takes 0 or 1

$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$1 \cdot 0 = 1$
$1 \cdot 1 = 1$

$P(x_1 x_2 = 1) = P(x_1 = 1) \cdot P(x_2 = 1 \mid x_1 = 1)$
$\hookrightarrow$ 1 hat taken coz seek hat

$= \frac{1}{n} \cdot \left(\frac{1}{n-1}\right)$

$= E[x_q x_s]$ $\rightarrow$ as q & s [~~doesn't~~ will have the same]

$\therefore E[x_q x_s] = 0 \cdot \left(\frac{1}{n}\right) \cdot 0 \cdot \left(\frac{1}{n-1}\right) + 1 \cdot \left(\frac{1}{n}\right) \cdot 1\left(\frac{1}{n-1}\right) + 0 \cdot \left(\frac{1}{n}\right) \cdot 1\left(\frac{1}{n-1}\right)$
$+ 1\left(\frac{1}{n}\right) \cdot 0\left(\frac{1}{n-1}\right)$

$$\boxed{E[x_q x_s] = \frac{1}{n} \cdot \frac{1}{n-1}}$$

How many terms:

$x^2 = \sum_q x_q^2 + \sum_{q,s;\, q \neq s} x_q x_s$

$(a+b+c+d)^2 \rightarrow n \times n = n^2$ terms

$= (a^2 + b^2 + c^2 + \dots) + (ab + bc + cd + \dots)$

n terms $\quad$ $n^2 - n$ terms

~~$E[x^2] = E[x_q^2]$~~ +

$E[x^2] = \sum E[x_q^2] + \sum E[x_q x_s]$

$= n\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)(n^2 - n)$

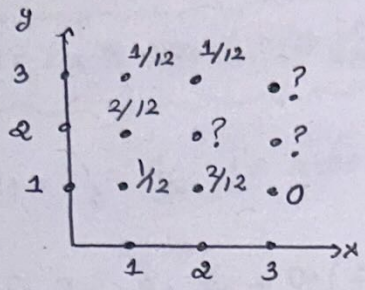
$= 1 + \left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right) n(n-1) = 2$

$$\boxed{E[x^2] = 2}$$

$$Var(x) = 2 - 1$$

$$\boxed{Var(x) = 1} = \text{mean}$$

JPMF – Drill



? → unknowns

★ $P_X(1) = \sum_y P_{X,Y}(1, y)$

$= 1/12 + 2/12 + 1/12$

$= 4/12 = 1/3$

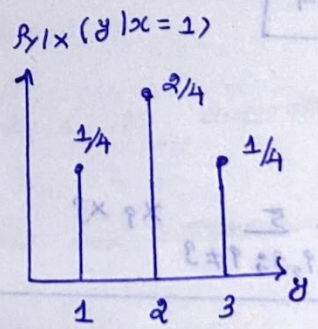
★ Sketch $P_{Y|X}(y|x=1)$

Solu: universe

3	1/12
2	2/12
1	1/12

$x = 1$

$P_{Y|X}(y|x=1)$



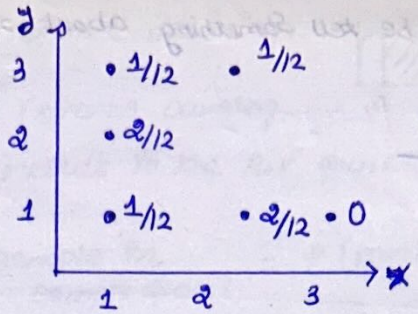
$$\boxed{P_{Y|X}(1|x=1) = \frac{1/12}{4/12} = 1/4}$$

$$= \frac{P_{X,Y}(1,1)}{P_Y(x)}$$

c) $E[y|x=1] = 1(1/4) + 2(2/4) + 3(1/4)$ → 'Average'

$= 1/4 + 4/4 + 3/4 = 2$

d) Can you make X [choose unspecified prob], make Y independent,



<u>condition:</u>

* $x=3 \longrightarrow y=1$ doesn't happen ($y=2$ or $y=3$)
(Tells some info)

* $x=1 \rightarrow y=1$, $\rightarrow y=2$, $\rightarrow y=3$

"No matter what we give to unspecified – the gn is dependent"

As $x=3$, gives info $y \neq 1$.

Q) $P_{X,Y}(2,2) \longrightarrow B = \{X \leq 2, Y \leq 2\}$

2	2/12	?
1	1/12	2/12
	1	2

$$\boxed{P_{X,Y}(2,2 \mid B) = ?}$$

"If two random variables are independent & condition one of them: (conditioning x) → the relative frequency of y is the same"

$$P_{X,Y}(x,y) = P_X(x)\, P_Y(y)$$

	2/12	?
	1/12	2/12

Twice, Twice, Twice, Twice

$P_{X,Y}(2,2) = 4/12$

If $y=1$ → $x \rightarrow 1/12$, → $x \rightarrow 2/12$) Twice

If $y=2$ → $2/12$, → $4/12$) Twice

gives no info (new)

∴ → only for $4/12$ → relative freq is same.
(No new info)

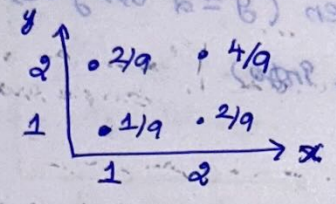
For others → we have dependent.

∴ other than $\frac{4}{12}$ → y will be tell something about x

f) $P_{x,y}(2,2 \mid B) = \dfrac{P_{x,y}(2,2)}{P_B(b)}$

$= \dfrac{4/12}{9/12}$

$= 4/9$



John → 18 minutes without traffic lights.

5 traffic lights — 1/3 prob of red. (Independent)

Each traffic light adds 1 min in travel time.

i) PMF = ?

In a signal

$$P_{R_i}(r) = \begin{cases} 2/3, & r = 0 \\ 1/3, & r = 1 \\ 0, & \text{else} \end{cases}$$

∴ time spends at i th red light : R_i : Bernoulli (success or lose)

$p = 1/3$. The PMF for R_i is :

1/3 chance of waiting at each signal.

---

* Probability density function
* Cumulative distribution func
* Normal random variables

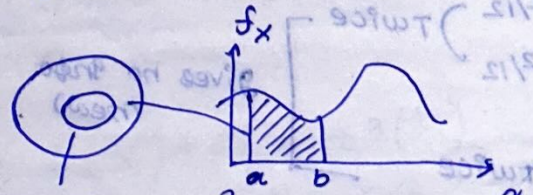
Unit - II - Lecture - 8

General R.V

Continuous R.V

A continuous r.v is described by a pdf f_X.

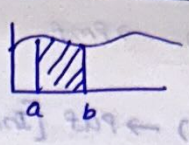
(Any real number)



Event $\{a \le X \le b\}$

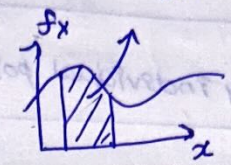
Sample point $\xrightarrow{\text{function}}$ (Real no. on Real line)

Find: Particular events occurity.



∴ Go to the Ω (events causing the particular result in the RoV function)

∴ P (sample in sample space) = P (particular event in RV)



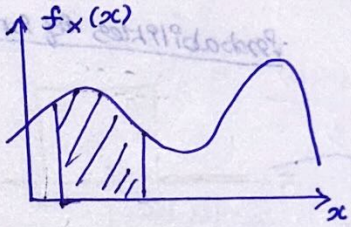
$$\boxed{P_a(a \le x \le b) = \int_a^b f_X(x)\,dx}$$

'work on original sample space eraplte eg f_X'

Discrete: we used prob eg previous stage RV for present RV

Continuous RV

'Just work on the real axis line'



'using integration'

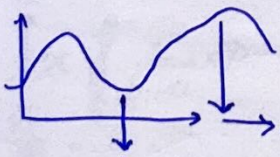
In discrete → pmf

In continuous → pdf. → Discrete o/p [1, 2, 3, ..]

→ Not discrete.

'Dealing with Area' — 'Spread all over the real axis not discrete points'

* Deal with densities: how thick is the mass sitting here.



$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

Area under the curve = 1.

→ uniform density

prob = $\boxed{l \times b}$

Discrete: $\sum_{x} P_X(x) \rightarrow$ Pmf $\qquad x \in [a,b]$

continuous: $\int_a^b f_X(x) \rightarrow$ Pdf [Integrate]

$\boxed{\int_a^a = 0} \rightarrow$ Area $= 0$

$\boxed{P(X=a)=0}$ → 'Any individual point has zero probability'

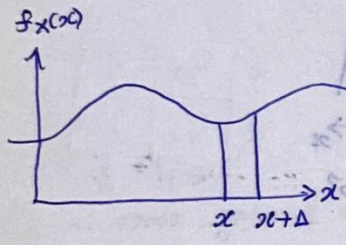
Note: density tells something different.

∴ density → used to find prob.

As prob ≥ 0, density ≥ 0

$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1} \rightarrow$ 1 unit of total prob spread over all space.

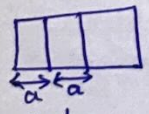
Probabilities of small intervals



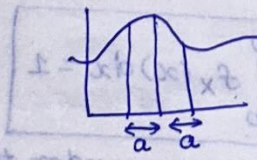
$$P(x \leq X \leq x+\Delta) = \int_x^{x+\Delta} f_X(x)\,dx = \underbrace{f_X(x)}_{\text{Height}} \cdot \underbrace{\Delta}_{\text{Base}}$$

As $\Delta \rightarrow 0$ (small number):

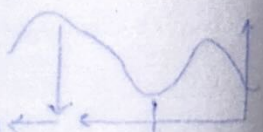
* Density not going to change much
* Area under the curve = base × height.



Density same in both a



Different densities.



$$f_X(x) = \frac{P(x \leq X \leq x+\delta)}{\delta}$$ Prob/unit length

'Density: Prob/unit length' → Can be more than 1.

(As long as area of the curve = 1)

Nice sets

$$P(X \in B) = \int_B f_X(x)\,dx \quad \text{for 'nice' sets } B.$$

'union of intervals' — $\int$ interval 1 + $\int$ interval 2

pdf: complete description of any statistical info.

Expectation

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \longrightarrow \text{Central of gravity.}$$

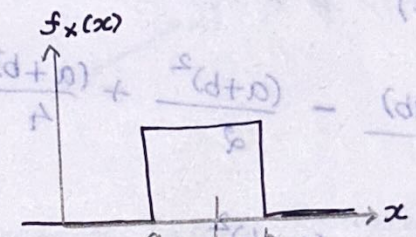
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

→ Translated version of discrete case

$$\text{var}(X) = \sigma_X^2 = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\,dx.$$

$$= E[X^2] - (E[X])^2$$

Continuous R.V (uniform)



→ whole x's in lines $[a, b]$ equally likely

any x → has zero prob (Area = 0)

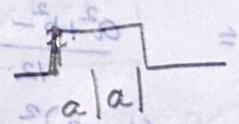
$f_X(x) = ?$

$$\therefore \int_b^a f_X(x)\,dx = 1$$

Uniform

base $= b - a$

height $= \dfrac{1}{b-a}$

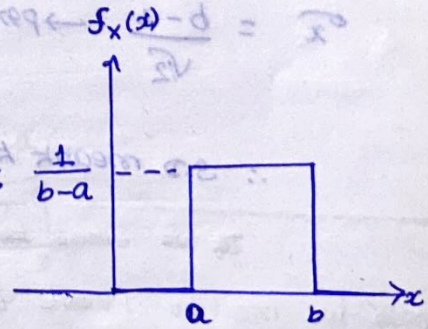


Same prob in interval

Then only

$$\int_a^b x \int_a^b \frac{1}{b-a}(x)\,b-a = 1$$

$$\therefore \frac{1}{b-a}$$



$$f_X(x) = \begin{cases} 0, & \text{otherwise} \\ \dfrac{1}{b-a}, & a \le x \le b \end{cases}$$

$$E[X] = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\,\frac{(b-a)(b+a)}{2}$$

$$= \frac{a+b}{2}$$

$$\sigma_X^2 = \int_a^b (x - E[X])^2 f_X(x)\,dx$$

$$= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \frac{1}{b-a}\,dx$$

$$= \int_a^b \left(x^2 - 2x\left(\frac{a+b}{2}\right) + \left(\frac{a+b}{2}\right)^2\right)\frac{1}{b-a}\,dx$$

$$= \left(\frac{x^3}{3}\right)_a^b \frac{1}{b-a} - \left(\frac{x^2}{2}\right)_a^b \left(\frac{a+b}{b-a}\right) + \left(\frac{a+b}{2}\right)^2 \left(\frac{1}{b-a}\right)(b-a)$$

$$= \frac{(b-a)(b^2+a^2+ab)}{3(b-a)} - \frac{(a+b)(b-a)(a+b)}{2(b-a)} + \frac{(a+b)^2}{4}$$

$$= \frac{(a^2+b^2+ab)}{3} - \frac{(a+b)^2}{2} + \frac{(a+b)^2}{4}$$

$$= \frac{a^2+b^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(b-a)^2}{12}$$

$\sigma_X = \frac{b-a}{\sqrt{12}}$ → proportional to the width of our interval.

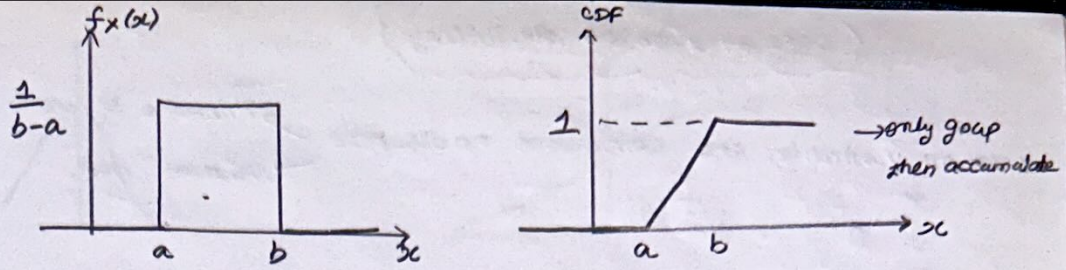
∴ S.D means to say how wide is our distribution.

If we want to say something is true for every R.V → we need to prove for both discrete & continuous.

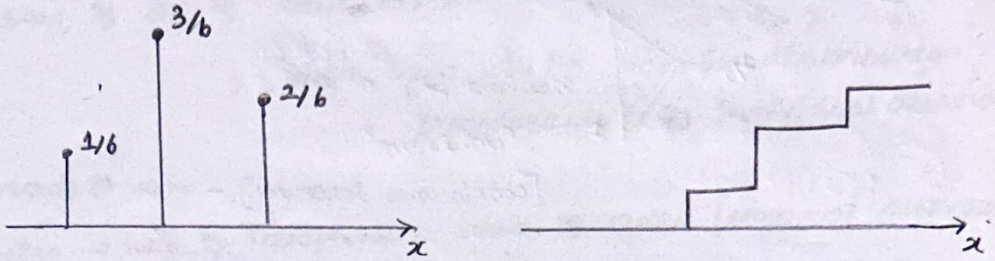
Is there any way to unite them?

Cumulative distribution function

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$



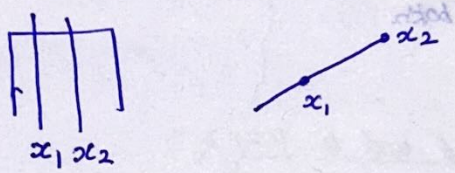
Continuous



Discrete

Continuous case → No jump masses

Continuous case:



As distance ↑, Area Increasing → Linear relation.
(Area up to that point)

Find density – when CDF is gn

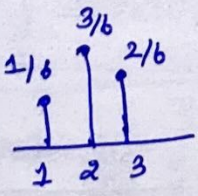
'Derivative of CDF'

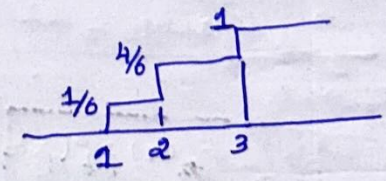
$$\boxed{\frac{dF_X(x)}{dx} = f_X(x)}$$

Exception: undefined at corners. [Ambiguous] 0 or 1

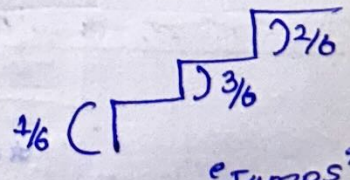
changing single point: doesn't affect. → 'Don't care'

Discrete





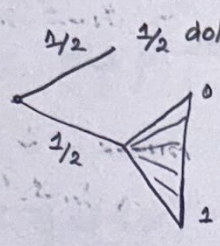
'Jumps associated with point masses'



'Jumps'

(CDFs → gives us flexibility)

* Not all variable are continuous or discrete → mixture eg two
→ Neither nor.



reward b/w 0 and 1
'uniform'
[continuous interval] - wheel eg donture

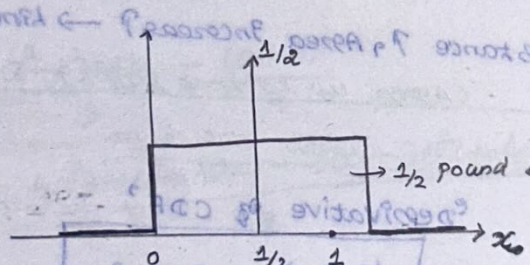
Continuous → No (since 1/2 dollars)

Discrete → No (∴ It has uniform cont. interval)

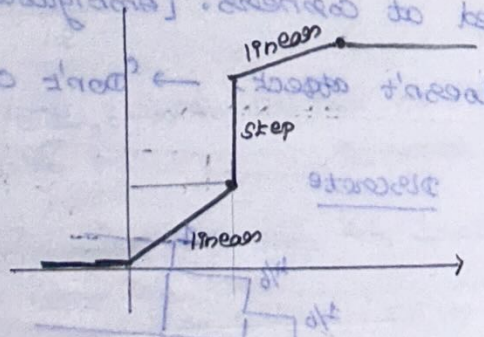
↓

mixture eg both
(mixed RV)

Combination eg pdf and a PDF



CDF



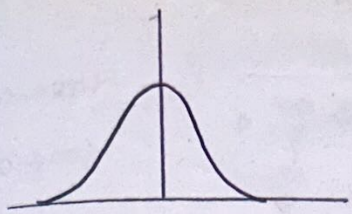
→ 'mixed R.V'

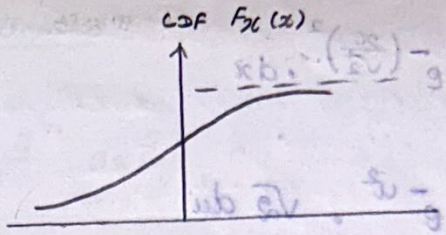
$F_X(x) = P(X \le x)$

Guassian (normal) PDF

* most important in probability theory.

• Standard normal $N(0,1)$: $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$





phenomenon: quantity (lot & lot of random contribution). The r.v is the sum of lot of little r.v s.

"Then sum will be normal distribution irrespective of individual distribution"

eg: Noise → lots of independent basis of noise [Normal distribution]

Special case:

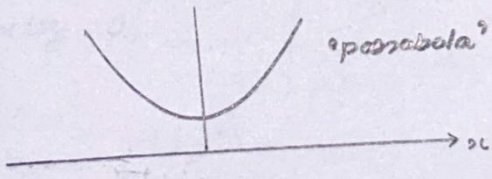
Binomial distribution: Sum of Bernoulli r.v

↓

Large no. of point fields

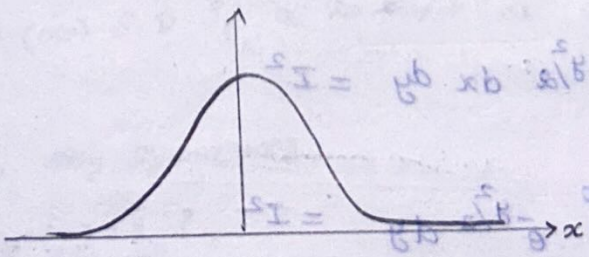
It will also look like a normal

$x^2/2$



↓

Negative exponent $\quad e^0 = 1, \; e^{-\infty} = 1$



$$\int_b^a f_X(x)\,dx = 1$$

$\boxed{\text{Factor: } \frac{1}{\sqrt{2\pi}} \; ?}$

$u = -x^2/2$

$du = -\frac{2x}{2}dx$

$$\boxed{\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt}$$

$\int e^{-x^2/2}\,dx \Rightarrow du = -x\,dx$

$u = \frac{x}{\sqrt{2}}$

$\boxed{du = \frac{1}{\sqrt{2}}dx}$ $\quad \boxed{dx = \sqrt{2}\,du}$

$$= \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}}\right)^2} \cdot dx$$

$$= \int_{-\infty}^{\infty} e^{-u^2} \cdot \sqrt{2}\, du$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{2e^{-u^2}}{\sqrt{\pi}}\, du$$

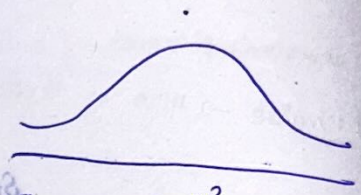
$\hookrightarrow$ Gauss error function

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \operatorname{erf}(u)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$= \sqrt{2\pi}$$

$\therefore$ To cancel this we divide by $\sqrt{2\pi}$



$$e^{-x^2}$$

$$= \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

$$= 2\int_{0}^{\infty} e^{-x^2}\, dx$$

(or)

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = I$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2-y^2}{2}}\, dx\, dy = I^2$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2} \cdot e^{-y^2/2}\, dx\, dy = I^2$$

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = I^2$$

$$I \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = I^2$$

$$\boxed{\int_{-\infty}^{\infty} e^{-y^2/2}\, dy = I}$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2-y^2}{2}}\, dx\, dy$$

Polar coordinates

$\theta\ (0 \to 2\pi)$

$r\ (0 \to \infty)$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{x^2-y^2}{2}}\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2/2}\, r\, d\theta\, dr \longrightarrow \text{See Integral calculus.}$$

$x^2+y^2=r^2$

$$= 2\pi \int_0^{\infty} e^{-r^2/2}\, r\, dr$$

$$= 2\pi \left(\frac{-e^{-r^2/2}}{2\cdot 1/2}\right)\Bigg|_0^{\infty}$$

$$= 2\pi\,(0+1)$$

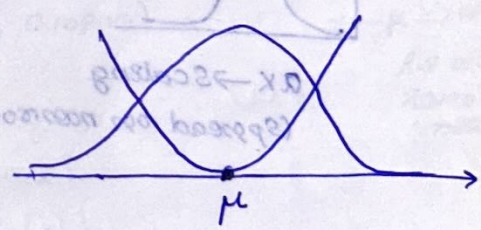
$$I^2 = 2\pi$$

$$\boxed{I = \sqrt{2\pi}}$$

$E[X]$ = symmetry $= 0$

$Var(X) = 1$

"centered at μ"



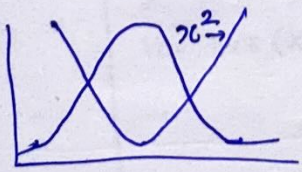
$\therefore\ E[X] = \mu$ & $Var(X) = \sigma^2$

(variance as (or) S.D as $\propto$ to σ)

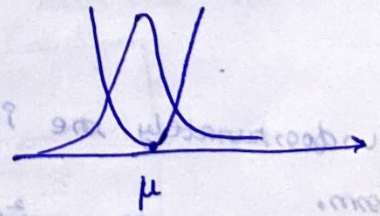
when the width of my density (reduced)

In $E[X]=0$, why symmetry

$Var(X) = 1$?



Scaling factor $= 1$



"Fall off much faster"

$\therefore\ Var(X) = 1 \rightarrow$ Here,

$$\frac{(x-\mu)^2}{\sigma^2}$$

Small σ directly density. (Falls very fast)

stretches or widens my parabola.

$\therefore$ widespread of curve (variance) is directly prop to σ^2

Properties

* Behave nicely in linear functions

$$y = ax + b$$

Then $E[y] = a\,E[x] + b$

$= a\mu + b$

Variance:

$Var(y) = a^2\, Var(x)$

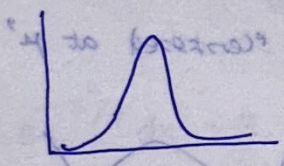
$= a^2 \sigma^2$

If x is normal, y also normal. (Linear)

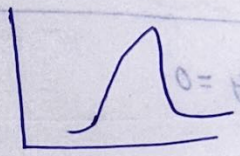
$y \sim N(a\mu + b, a^2\sigma^2)$

Exp variance.

Intuitive:



$ax \rightarrow$ scaling (spread or narrow)



$ax + b$ ↳ move the function.

Linear transformation: Bell shape is still a bell shape, Normals remains Normal under this.

Calculate probability

Unfortunately the integral expression is not in closed form.

$$\int_{-\infty}^{x} e^{-x^2/2}\, dx$$

Value of integral : giving by closed formula

↓

Not available

Solution:- Tabulate the results of all x's upto some precision.

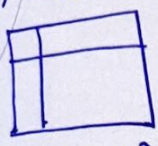
"we are working the mean & variance from standard normal"

* $X \sim (\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(\ )$

* If $X \sim N(2, 16)$

$$P(X \leq 3) = P\left(\frac{X-2}{4} \leq \frac{3-2}{4}\right) = CDF(0.25)$$

$N(0,1) \rightarrow$ standard normal



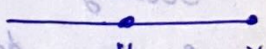
"Table"

"Standardising the R.V"

$\frac{X-\mu}{\sigma} \longrightarrow$ Now r.v with zero mean

$\therefore X \sim (\mu, \sigma^2)$

$\downarrow$ mean, $\downarrow$ variance



$\therefore$ make the mean 0 (standard normal has 0 mean)

$X - \mu \Rightarrow$ Now R.V with zero mean.

$X - \mu$

$\therefore$ make the variance σ

$\frac{X-\mu}{\sigma}$ (scale). $\rightsquigarrow N(0, 1)$

$\downarrow$ mean, $\downarrow$ unit variance.

$\downarrow$

$$\boxed{\begin{aligned} &\text{since} \\ &\text{Variance}(X) = (X - \text{mean})^2 = \sigma^2 \\ &= \left(\frac{X - \text{mean}}{\sigma}\right)^2 = 1 \end{aligned}}$$

$X - \mu \rightarrow$ shifting to $\mu = 0$

Scaling to 1.

* If $X \sim N(2, 16)$

$$P(X \le 3) = P\left(\frac{X-2}{4} \le \frac{3-2}{4}\right)$$

$$= P\left(\underbrace{\frac{X-2}{4}}_{N(0,1)} \le 0.25\right) = CDF(0.25)$$

up to 0.25 we are adding the probability.

From table:

$$P\left(\frac{X-2}{4} \le 0.25\right) = 0.5987$$

Is Before normalizing, after normalizing → Is the prob same?

yes!

'scaling – scales the entire curve'

'shifting' doesn't effect area of curve.

$P_X(x)$ $\qquad$ $f_X(x)$

$F_X(x)$

$E[X], Var(X)$

$P_{X,Y}(x,y)$ $\qquad$ $f_{X,Y}(x,y)$

$P_{X|Y}(x|y)$ $\qquad$ $f_{X|Y}(x|y)$

Discrete $\qquad$ Common $\qquad$ Continuous.

Re citation- Calc CDF

$$f_Z(z) = \begin{cases} \gamma(1+z^2) & -2 < z < 1 \\ 0 & \text{, else} \end{cases}$$

Solu:

$\gamma = ?$ $\qquad$ $\int_{-\infty}^{\infty} f_Z(z) = 1$

$$\int_{-\infty}^{\infty} \gamma(1+z^2)\, dz = 1$$

$$\int_{-2}^{1} \gamma(1+z^2)\, dz = 1$$

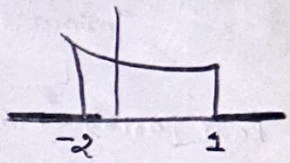
$$\gamma\left(z+\frac{z^3}{3}\right)_{-2}^{1} = 1$$

$$\gamma\left(1+\frac{1}{3}-\left(-2-\frac{8}{3}\right)\right) = 1$$

$$\gamma = \frac{1}{\left(\frac{4}{3}+\frac{14}{3}\right)} \Rightarrow \boxed{\gamma = 1/6}$$

CDF:

$$F_Z(z) = \int_{-\infty}^{z} f_Z(z)\, dz = P(Z \le z)$$



$$\boxed{\begin{aligned} P(-\infty < Z < -2) &= 0 \\ P(2 < Z < \infty) &= 0 \end{aligned}}$$

$-2 \le z \le 1$: [Find upto any point (upto) at this interval]

$$F_Z(z) = \int_{-2}^{z} \frac{1}{6}(1+y^2)\, dy$$

$$= \frac{1}{6}\left[y+\frac{1}{3}y^3\right]_{-2}^{z} = \frac{1}{6}\left[z+\frac{1}{3}z^3-\left((-2)+\frac{1}{3}(-2)^3\right)\right]$$

$$= \frac{1}{6}\left[z+\frac{1}{3}z^3+\frac{14}{3}\right]$$

$$\boxed{F_Z(z) = \begin{cases} 0, & z < -2 \\ 1/6\left[z+\frac{1}{3}z^3+\frac{14}{3}\right], & -2 \le z \le 1 \\ 1, & z > 1 \end{cases}}$$

A mixed distribution.

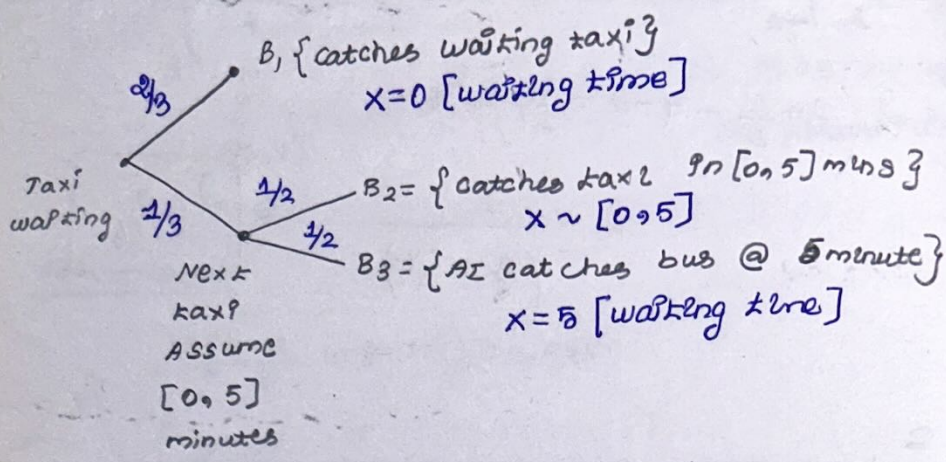
Taxi & bus stand are at same location. A1 goes there at a given time and if a taxi is waiting (it happens with prob 2/3). — See pdf.

$P(\text{taxi}) = 2/3$

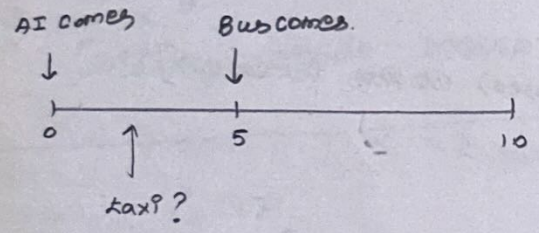
Taxi → [0, 10]

Bus → Exactly 5 minutes.

Solu:



∴ taxi [0, 10] → only he choose if it comes before bus [0, 5]



$F_X(x) = ?$

$E(x) = ?$

'B_1, B_2, B_3' are a partition.

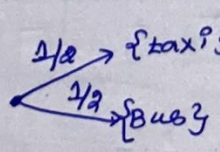
1) Next taxi [0, 10] → uniform distribution.

[0, 5] = 1/2 the mass.

∴ He missed the waiting taxi [condition]

1/2 of taxi

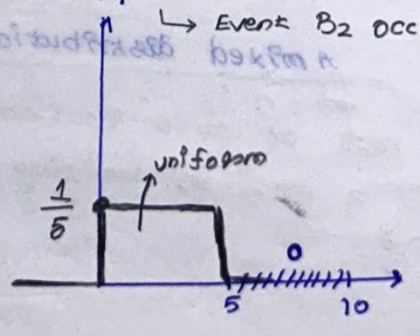
1/2 of Bus.



1/2 chance of taxi coming (It may come - may not come) with in 5 minutes.

$f_{X|B_2}(x)$ ↳ Event B_2 occured

X: waiting time.



$$E[X] = P(B_1)\,E[X|B_1] + P(B_2)\,E[X|B_2] + P(B_3)\,[X|B_3]$$

$$= \frac{2}{3}(0) + \left(\frac{1}{3}\cdot\frac{1}{2}\right)\cdot\frac{5}{2} + \left(\frac{1}{3}\cdot\frac{1}{2}\right)5.$$

$\hookrightarrow$ uniform: $\frac{b-a}{2}$ (symmetry)

$$= \frac{5}{12} + \frac{5}{6}$$

$$= \frac{15}{12} = \frac{5}{4} = 1.25.$$

[$E[X]$: expectancy each event not occuring with same prob?

$\sum_{\text{all events}}$ point · prob $\times E[X]$

C.D.F:

$F_X(x) = P(X \le x)$

If $x < 0 \rightarrow 0$

$x > 5 \rightarrow 1$

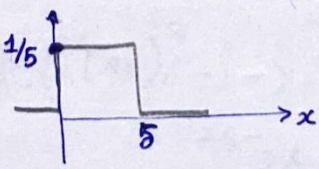
At $x=0$: $2/3$ prob no taxi

use total prob theorem: $[0 \le x \le 5]$

$$= P(B_1)\,P(X = x|B_1) + P(B_2)\,P(X \le x|B_2) + P(B_3) + P(X \le x|B_3)$$

$$= \left(\frac{2}{3}\cdot 1\right) + \left(\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{5}\cdot x\right) + \left(\frac{1}{2}\cdot\frac{1}{3}\cdot 0\right)$$

If B_1 happens $X = 0$ (exactly) $\therefore 1.$



assuming $0 \le x < 5$

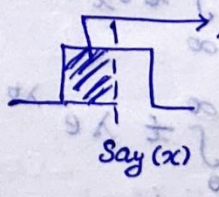
(At 5 minute he will be in something-travelling)

Area = 1

$$\begin{cases} 0 & \text{if } x < 0 \\ \frac{2}{3} + \frac{x}{30} & \text{if } 0 \le x < 5. \\ 1 & \text{if } x > 5 \end{cases}$$

So Anywhere b/w 0 and 5 we don't know width

$\therefore$ Up to x $P(X \le x|B_2) = \frac{1}{5}\times x$ [uniform]



Area $= \frac{1}{5}\times x$

$$= \frac{2}{3} + \frac{x}{30} \longrightarrow 0 \le x \le 5$$

At $x=5$

$$\frac{2}{3} + \frac{x}{30} + \frac{1}{6}\,(\bullet) = \frac{2}{3} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} + \frac{2}{6} = 1$$

only one point (close)

$$= \frac{2}{3} + \frac{x}{30} + \frac{5}{6}$$ (Incase $0 \le x \le 1$.

$\therefore$ up to $0 \le x < 5 \rightarrow$ we don't take 5 $\rightarrow \frac{2}{3} + \frac{30}{x}$

$\therefore \quad 0 \le x \le 5 \rightarrow \frac{2}{3} + \frac{30}{x} + \frac{1}{6}$

$\therefore$ At $x = 5 \rightarrow P(B) = 1$

<u>Note</u>: If he takes something b/w 0 to 5 (excluding)

'he wont go in bus'

'So take $0 \le x < 5$

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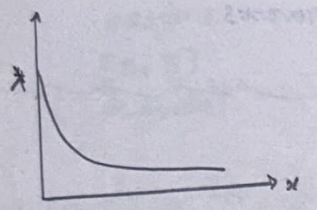
3. Recitation



$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$F_X(x) = P(X < x)$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \ge 0 \end{cases}$$

$$F_X(x) = \int_0^x f_X(x)\, dt$$

$$= \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= \lambda \left(\frac{e^{-\lambda t}}{-\lambda} \right)_0^x$$

$$= \lambda \left(\frac{e^{-\lambda x}}{-\lambda} - \frac{e^0}{-\lambda} \right)$$

$$= \left(-e^{-\lambda x} + 1 \right)$$

$$= 1 - e^{-\lambda x}$$

(b) $E[X] = \int_{-\infty}^{\infty} t\, f_X(t)\, dt$ (take $x = t \rightarrow$ just t)

$$= \int_0^{\infty} t\, \lambda e^{-\lambda t}\, dt$$

'Integration by parts'

$\int u\,dv = uv - \int v\,du$

$u = x \quad dv = e^{-\lambda x}$

$du = dx \quad v = \frac{e^{-\lambda x}}{-\lambda}$

$$= \lambda \left(x \left(\frac{e^{-\lambda x}}{-\lambda} \right) - \int \left(\frac{e^{-\lambda x}}{-\lambda} \right) dx \right)$$