$$n^2 + (n-1)^2 + (n-2)^2 + \cdots + 1^2 \approx \frac{1}{3} n^3$$

$$\int_1^n x^2 \, dx = \frac{1}{3} x^3$$

Cost for any right hand augmented is n^2

Cost of augmented Column: n^2 (For each right hand side column).

Row exchange - Allowed.

Permutations: Need to do for row exchanges.

3x3 → 6 permutation matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No exchange}$$

P_{12} (exchanges 1 and 2) $= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$P_{13} P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_{13} P_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$P^{-1} = P^T$

How many 4x4 permutations?

→ 24 permutation matrices

For Row exchange:

$n!$ different ways to permute the rows of an $n \times n$ matrix.

Recitation

LUD - decomposition:

$A = \begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix}$ when it exists? For which real numbers a and b does it?

Solu:

* we will eliminate A_{21} entry first

$$\begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix} \xrightarrow[E_{21}]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ b & b & a \end{pmatrix} \quad R_2 \to R_2 - aR_1$$

$$\Big\downarrow E_{31}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{pmatrix} \quad R_3 \to R_3 - bR_1$$

a to be non-zero to be a pivot

Else - Row exchange.

Assume: A is non zero.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{pmatrix} \xrightarrow[E_{32}]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix} \quad R_3 \to R_3 - \frac{b}{a} R_2$$

$= b - a\left(\frac{b}{a}\right)$
$= 0$

$\therefore \quad E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{pmatrix}$

$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{a} & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix}$

Find $\boxed{A = LU}$

$$E_{32}\, E_{31}\, E_{21}\, A = U$$

$$A = LU \qquad L = (E_{21})^{-1} (E_{31})^{-1} (E_{32})^{-1}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{b}{a} & 1 \end{pmatrix}$$

Inverse of Elementary matrix (Lower triangular matrix) change the minus sign to plus alone.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \frac{b}{a} & 1 \end{pmatrix} \rightarrow \text{component absorption}$$

If a is there in any of three elementary matrix → Reflects in L.

$\boxed{\text{Exists only when } a \neq 0}$

"singular matrices can have LU decomposition"

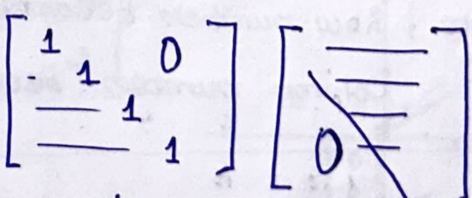
To account for row exchanges in Gaussian elimination, we include a permutation matrix P in the factorization

$$PA = LU.$$

Lecture : 6 Permutations & Transpose

Permutations P: matrix executes row exchange.

LU =



L

Matlab doesn't not only checks whether the pivot is non-zero but also whether the pivot is big enough. (close to zero & numerically bad will be avoided).

With row exchanges:

$PA = LU$ → For any invertible A.

Row exchanging permutation matrix.

Permutations:

P = Identity matrix with reordered rows.

$n!$ → possibilities for row exchange.

$(n \times n)$

P → Invertible P^{-1}

$$\boxed{P^{-1} = P^T}$$

$$\boxed{P^T P = I}$$

Transpose:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

(3×2) (2×3)

$$(A^T)_{ij} = A_{ji}$$

[since, Row number becomes column number
Column number becomes row number]

Special matrices: Symmetric matrices:

$$A^T = A$$

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix} \rightarrow \text{Symmetric matrix example.}$$

How to get Symmetric matrix:

$R^T R$ is always symmetric.

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}$$

$(R^T)^T = R$

why:

$$(R^T R)^T \xrightarrow[\text{reversed}]{\text{order gets}} R^T \cdot R$$

Vector spaces & Subspaces

Space of vectors (Bunch)

* we are able to add vectors & multiply by scalars.
* Able to take linear combinations.

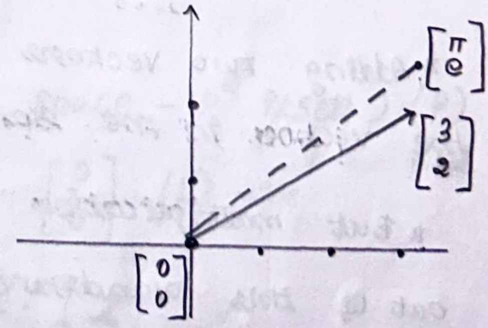
Example: R^2

$R^2 \rightarrow$ All 2-dimensional real vectors

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}$$

we can add:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



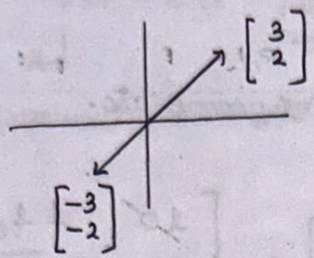
whole plane is R^2

$R^2 \rightarrow$ The whole x-y plane.

'vector space' $\rightarrow$ It has all real vectors.

Every vector space has zero vector in it.

$$\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}-3\\-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$



$R^3 \rightarrow$ All 3-dimensional vectors (Real) [3-real components]

$$\begin{bmatrix}3\\2\\0\end{bmatrix}.$$

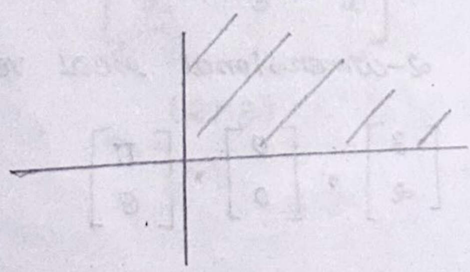
$R^n \rightarrow$ All vectors with n- components
'column vectors, Real components'

Key:
we can add & multiply by any numbers.

we have few rules:

Not a vector Space

1 quadrant alone



* Adding two vectors in the same quadrant gives the vector in the same quadrant.

* But multiplication by a scalar can force me out of this quadrant.

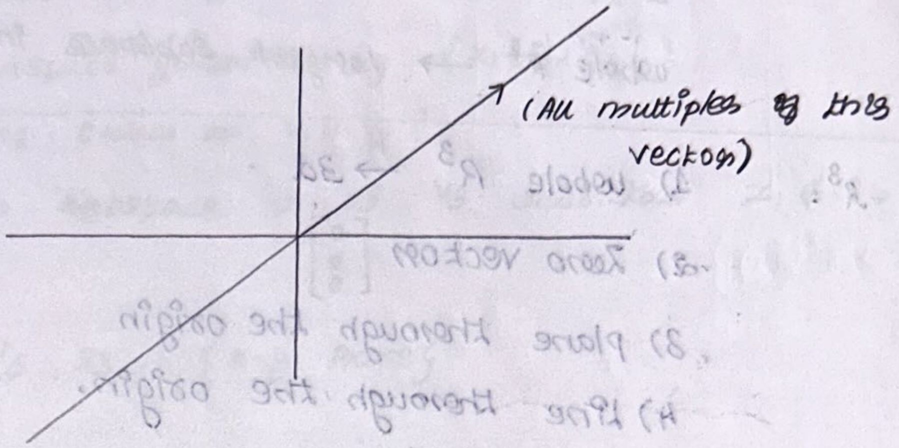
* Multiply by -1 will force me to go to IV quadrant.

'Not closed under scalar multiplication'

↳ (All real numbers)

Subspace: Vector space inside a vector space.

A vector space inside R^2 (subspace).



$(-)\begin{pmatrix} + \\ + \end{pmatrix} \rightarrow$ IV quadrant

$(+)\begin{pmatrix} + \\ + \end{pmatrix} \rightarrow$ I quadrant

$\begin{pmatrix} + \\ + \end{pmatrix} + \begin{pmatrix} + \\ + \end{pmatrix} \rightarrow$ I quadrant

'Every subspace has zero space in it'

Subspace of R^2 line in R^2 must go through zero vector. [Allowed to multiply by zero]

Possible subspaces of R^2:

2d → 1) All of R^2 (whole space – R^2 itself) (P) plane

1d → ② Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (L) – line

0d → ③ Zero vector alone. (Z)

why $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a subspace:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{still there}$$

$$100 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \text{still in the subspace.}$$

'1-point space' $\rightarrow$ Tiniest subspace

whole R^2 $\rightarrow$ Largest subspace in R^2.

R^3:
1) whole R^3 $\rightarrow$ 3d
2) Zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
3) plane through the origin
4) Line through the origin.

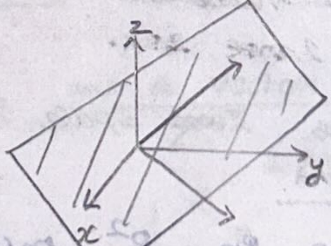
How do they come out of matrices:

Take $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \xrightarrow{\text{subspace}}$ 1) $\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ (Itself)

2) columns in R^3 & all their linear combination form a subspace. (column space)

$C(A)$ — column space; A — matrix

3)



Through zero vector

[Plane] $\rightarrow$ combination of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ & $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$

If two vectors are same, they are on a line.

(Not - plane).

Subspaces of 3-d space (Recitation)

$$x_1 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

1) Find V_1 = Subspace generated by x_1

V_2 = " " by x_2

Describe $V_1 \cap V_2$

2) V_3 = Subspace generated by $\{x_1, x_2\}$

Is V_3 equal to $V_1 \cup V_2$?

Find a subspace S of V_3 such that $x_1 \notin S$, $x_2 \notin S$

3) what's $V_3 \cap \{x\text{-}y \text{ plane}\}$

Solu:

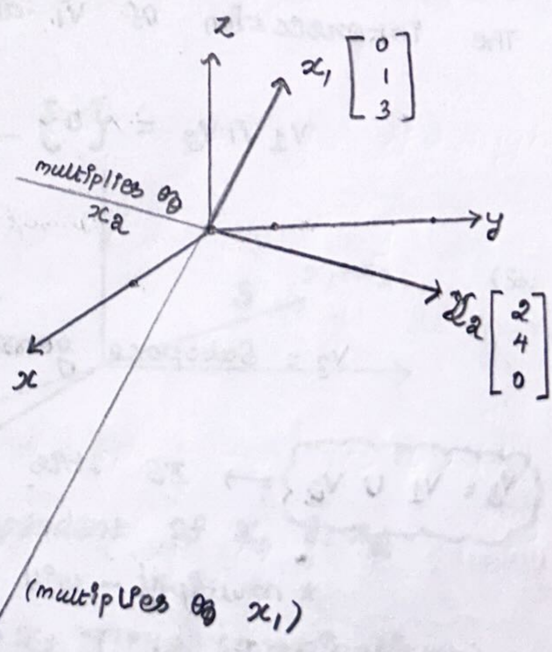
Set to be linear space: (Subspace)

1) Sum of any two elements from that set need to give the answers that also need to be in the same set.

2) Any multiple of the elements of the set need to be in the same set.

Answers

1) Subspace generated by x_1:

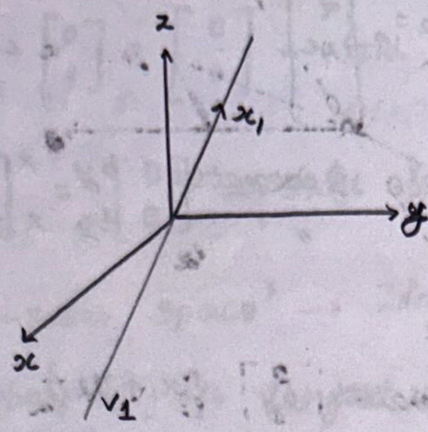


condition: 1 : mul by any scalar (Line)

condition: 2: sum of any two elements in the line will give the element in the line.

∴ That line is the least subspace of x_1 going through the zero vector.

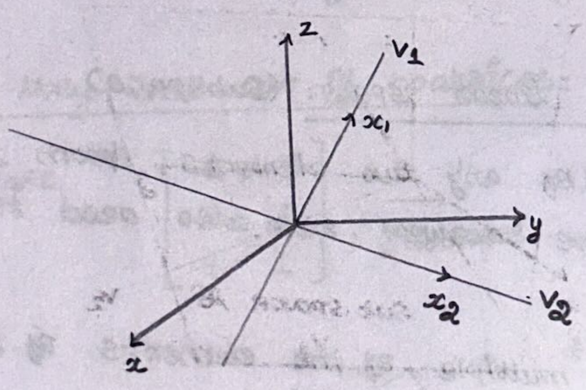
Subspace of x_1:



'Line - perfect set V_1'

V_1 - subspace generated by x_1.

Subspace of x_2:



x_1 and x_2 are not parallel.

The intersection of V_1 and V_2 is the origin.

$V_1 \cap V_2 = \{0\}$ → set with only one element (Also a subspace).

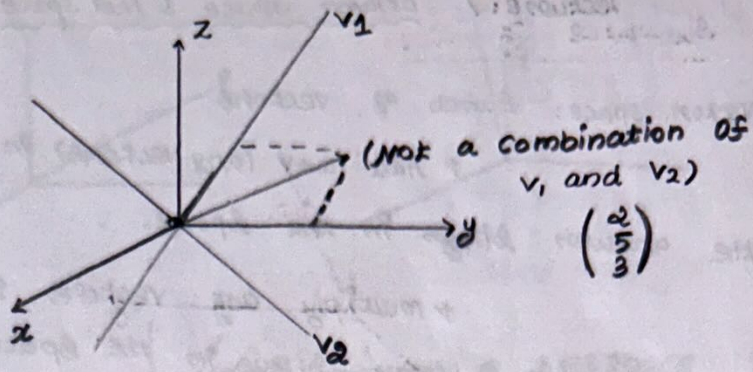
2)

V_3 = Subspace generated by $\{x_1, x_2\}$

$V_3 = V_1 \cup V_2$ → Is this subspace?

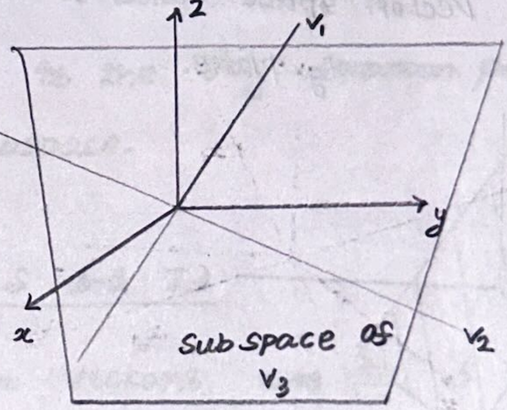
* multiply - will still on the line.

* Sum - $x_1 + x_2 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$ like combinations.



So, $x_1 + x_2 = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \notin V_1 \cup V_2$ $(V_3 \notin V_1 \cup V_2)$

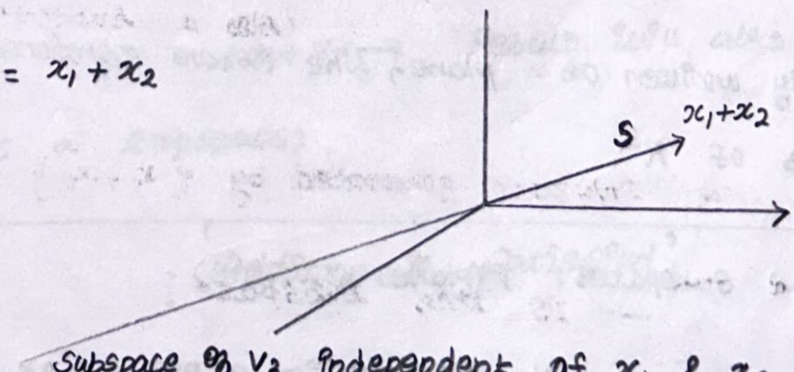
<u>Subspace of V_3:</u> 'Entire plane spend by v_1 & v_2'



Find a subspace S of V_3 such that $x_1 \notin S$, $x_2 \notin S$.

<u>Solu:</u>

$V_3 = x_1 + x_2$



Subspace of V_3 independent of x_1 & x_2.

It may be as $2x_1 + x_2$ (like combinations)

3) $V_3 \cap \{xy\text{-plane}\} = V_2$ $\left[\because \begin{array}{l} z \text{ plane} \rightarrow 0 \\ V_2 \text{ lies in } V_3 \text{ and} \\ z \text{ coordinate is zero} \end{array} \right]$

Lecture: 7 column space & Null space

Vector space: Bunch of vectors

* Add u+v (any vectors) in the space and the answer stays in the space.

* multiply any vectors in the space by a constant = Result stays in the space

$cv + dw$ → stays in the space.

→ Linear combinations

c, d – constants

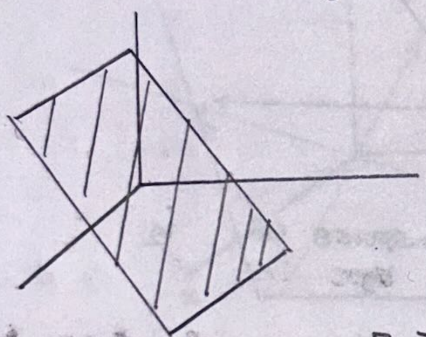
v, w – vectors.

---

eg: vector space: R^3 (Entire – 3d space)

Subspace: vector space inside a vector space.

Eg: plane.



Plane through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a subspace

Line through origin is a subspace.

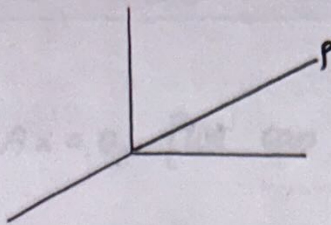
correctly written as: plane, Line (Above mentioned) are subspaces of R^3.

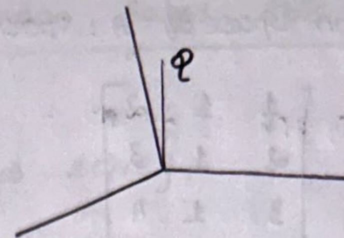
---

2 subspaces: P (plane), L (Line)

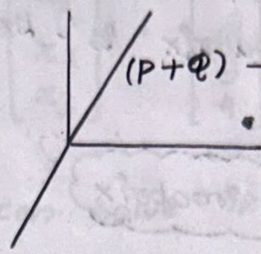
union: $P \cup L$ → all vectors in P or L or both

$P \cup L$ → This is clearly not a subspace.





$P \cup Q$



$(P+Q) \rightarrow$ Not a subspace

(proof).

$P \cap Q \rightarrow$ All vectors in both P and L

$\boxed{P \cap Q \rightarrow \text{This is a subspace.}}$

Above (origin) is the only common point which itself is a subspace.

Subspaces (Say S and T)

* If two vectors are in S

  $u+v \rightarrow$ Result will also be in S

(S is a subspace)

* If two vectors are in T

  $u+v \rightarrow$ Result will also be in T.

(T is a subspace)

'Addition is satisfied'

multiplication:

$7u \rightarrow$ If u is in S then $7u$ will also be in S.

$\therefore S \cap T \rightarrow$ smallest subspace

Column Space of A :

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Column space of this matrix is the sub space of R^4. (4-dimensional)

$\therefore$ A (4×3) matrix

C(A) → column space of A. [which is the subspace of R^4]

C(A) → Linear combinations of the column vectors
It gives a subspace (vector space)

How big is that space:

Is the column space (subspace) fill the entire 4-d space ?

Answer: NO

Does $Ax = b$ always have a solution?

Answer: NO

Then which b's are O.K ?

$$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

we have 4 linear eqn with three unknowns.

$$Ax = b$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

For some R.H.S (b) we can solve this

eqn:

* $AX = 0$ [we can always solve]

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

↳ we can solve it.

one possible solution: $x_1 = 1, x_2 = x_3 = 0.$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$x_1 = x_3 = 0, \; x_2 = 1.$

I can solve $AX = b$, only when the right hand side is exactly the same as that of any column vector of A. or its combinations [b is in the column space or its combinations]

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$ → Are these columns independant?

No 3rd column is the sum of first two columns.

The combinations of these three columns won't cover entire 3-dimensional subspace of R^4.

∴ By column 3, we don't get any new combinations.

Column 1, Column 2 → Forms 2-dimensional subspace of R^4.

So we can throw column 1 or 3:

By throwing 3, we can form the same subspace by C_1 and C_2 (or) C_3 and C_2 (or) C_1 and C_3. By convention,

C_1 → pivot (It's OK - So we won't throw it)

Conclusion: Column space of matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ is a 2-dimensional subspace of R^4.

Null space

* Totally different subspace.

Nullspace of A:

* Contains all solutions $(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ solves the equ $AX = 0$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In R^3

↳ In R^4 ↳ In R^3

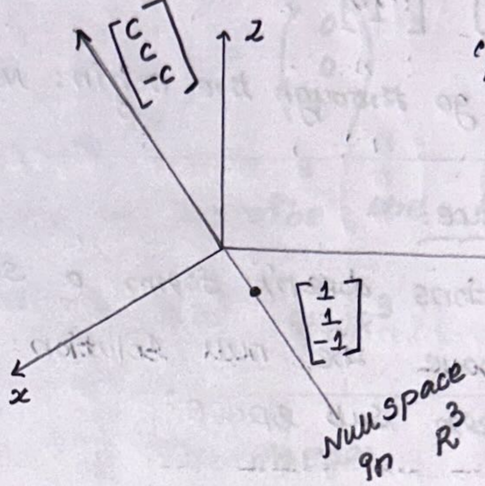
* One Solution: $x_1 = x_2 = x_3 = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

∴ Any multiple of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Draw the null space:



'passes through origin'

'Null space - line in R^3'

How?

Requirements: check $cu + dv \rightarrow$ stay in the subspace

If $AV = 0$, $AW = 0$

$A(V+W)$ must be zero. [$A(V+W)$ also in null space]

$AV + AW = 0 \rightarrow$ Distributive law.

If $AV = 0$, then $A(7V) = 0$ $\therefore$ $7AV = 7(0) = 0$

Other solutions?

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

The solutions of this don't form a sub space.
(Because: zero vector is not a solution) $\rightarrow$ So can't

be a vector space.

* one Solution: $x_1 = 1,\ x_2 = x_3 = 0$

Are they any other solutions?

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

Line doesn't go through the origin: Not a Subspace.

In column space:

Solutions doesn't form a subspace

'They don't have the null solution: Not passing through the zero sub space'

In null space:

If three columns of A are independent,..

then
$$\begin{bmatrix}1 & 4 & 1\\2 & 5 & 7\\3 & 6 & 9\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

The only solution is $x_1 = x_2 = x_3 = 0$.

In above example,

(solutions)

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, c\begin{bmatrix}1\\1\\-1\end{bmatrix}$$ forms a bunch of vectors

They form a line passing through the origin.

(forms a Subspace)

Vector Subspaces

which are subspaces of $R^3 = \left\{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right\}$

1) $b_1 + b_2 - b_3 = 0$

2) $b_1 b_2 - b_3 = 0$

3) $\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix}$

4) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

<u>solu</u>:

1) $b_1 + b_2 - b_3 = 0$ → Linear.

$$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b_1, b_2, b_3 → describe the null space.

'subspace' of R^3

---

2) $b_1 b_2 - b_3 = 0$ → Non-linear

say: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the subspace — then it's

linear combinations must also be in the subspace.
(Also a null solution)

$$2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$\therefore$ $2(2) - 2 = 0$

$\boxed{2 \neq 0}$ → Not satisfied

'Not a subspace' of R^3.

---

3) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

linear combination of other two

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \left[\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right] + c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \left(c_1 + \frac{1}{2}\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \left(c_2 + \frac{1}{2}\right)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

we have $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, when $c_1 = c_2 = 0$

and other combinations are available.

The subspace is the plane created by the two vectors.

subspace

'vector space' of R^3

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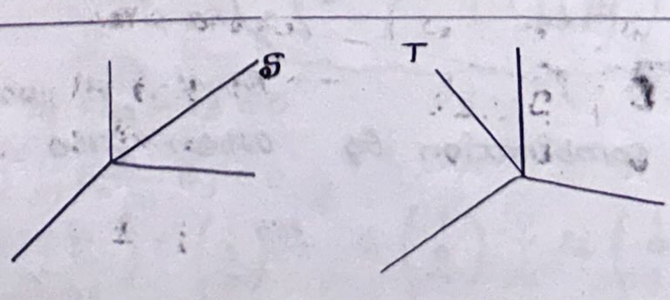
4) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

'Linearly independent'

Null solution → Is not possible.

$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ → Not possible.

'Not a subspace'



$S+T \rightarrow$ plane

$S \cup T \rightarrow$ combination of points on the two lines.

Lecture : 8

Solving $Ax=0$, pivot variables, special solutions

$$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

* C_2 is the multiple of C_1 $\rightarrow$ Not independent.
* $R_1 + R_2 = R_3$ $\rightarrow$ Not independent

Execute elimination: "Rectangular case: continue even pivot $= 0$"

Goal: Not changing the null space.
(changing (not) — the solutions).

$$A=\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow$$

pivot $= 0$ $\rightarrow$ Row exchange $\rightarrow$ These also zero.

"So dependent on earlier ones"

Go on:

$$u = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ Second pivot

$\rightarrow$ Echelon form (staircase)

Not upper triangular.

No. of pivot = Rank of A = 2

Back-substitution:

"How to describe solutions?"

2 pivot columns
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns → Free columns.

we can assign anything for x_2 and x_4.

(say) $x = \begin{bmatrix} \\ 1 \\ \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$2x_3 + 4x_4 = 0$$

$x_2 = 1$, $x_4 = 0$ say (systematic)

Now ⇒ $2x_3 = 0$, $x_3 = 0$ | $x_4 = 0$

$x_1 + 2 = 0$, $\boxed{x_1 = -2}$ | $x_2 = 1$.

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{Null space solution.}$$
[Any multiple]

↳ line in 4-dimensional space

Other choice:

$$x = c\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$x_2 = 0$, $x_4 = 1$ | $x_3 = -2$, $x_1 = -2$

Special solutions: (our convenience)

$$x = c\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \quad d\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$

what's special: numbers assigned to the free variables.

Solving $Ux = 0$

Taking all the combinations of the special solutions.

No. of Free variables = No. of special solutions

Let,

n = number of pivot variable

r = rank of a matrix

no. of free variables = $n - r$

One more step: (Reduced - row echelon)

$$A = \begin{bmatrix}1 & 2 & 2 & 2\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix} \rightarrow$$ Reduced - row echleon (Above and below pivots → zero)

R_3 is a combination of R_1 and R_2

$$= \begin{bmatrix}1 & 2 & 0 & -2\\0 & 0 & 2 & 4\\0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 0 & -2\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

$R_2 \rightarrow \frac{R_2}{2}$

'Null space exactly all the combinations of the special solution'

Matlab: rref (A)

reduced row echelon form of A.

$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ pivot rows

pivot columns

Identity matrix sitting in pivot rows & columns

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ Third row is zero - Really 2 rows - 3rd one is the combination of other two

Now back substitution:

$$x_1 + 2x_2 - 2x_4 = 0$$

$$x_3 + 2x_4 = 0$$

Note: The solutions to $Ax = 0$, $Ux = 0$, $Rx = 0$ all the same.

A → original one

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ → Identity part of the matrix. $\begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$ → Free part of matrix

pivot columns   Free columns:

0 0   0 0 → No value.

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

→ r pivot rows
→ Typical form of R.

↑ pivot columns   ↑ n−r free columns.

Null space matrix: (N)

Matrix columns = special solutions.

$RN = 0$

↳ Null solution

$$N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{matrix} \rightarrow \text{Free variables} \\ \rightarrow \text{pivot variables} \end{matrix}$$

$$Rx = 0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0$$

$$I\, x_{pivot} = -F\, x_{free}$$

---

say,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

column 3 is the combination of 1 and 2.

2nd row is the combination of 1.

soln:

To find: row reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{matrix}$$

Row exchange

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow R_4 = R_4 - 2R_2$$

$$u = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now reducing to the ...

$$\text{Rank} = 2$$

$$\begin{array}{c} \text{pivot column} \qquad \text{Free column} \\ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

no. of pivot columns of A = no. of pivot columns of A^T.

$r = 2$

No. of free column $= n - r$
$= 3 - 2 = 1$

<u>special case:</u>

$$x_1 + 2x_2 + 3x_3 = 0$$
$$2x_2 + 2x_3 = 0$$

$x = \begin{bmatrix} \\ \\ 1 \end{bmatrix} \rightarrow$ Set the free variable as 1.

Sub $x_3 = 1$

$x_2 = -1$

$x_1 = -2(-1) - 3(1)$

$\boxed{x_1 = -1}$

$$x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

whole null space: line $\rightarrow c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Basis : $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (vector) Null space [Combination]

Null vector

$$x = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

why the free variable is not zero?

$$\begin{bmatrix} \\ \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_3 &= 0 \\ x_2 &= 0 \\ x_1 &= 0 \end{aligned}} \rightarrow \text{No progress}$$

Keep going to R:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_1 = R_1 - R_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{2}$$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = C \begin{bmatrix} -F \\ I \end{bmatrix}$$

Null space matrix N is the guy whose columns are the special solutions (Free variables has special values 1 and pivot variables $-F$)

$$x_1 + x_3 = 0$$

$$x_2 + x_3 = 0$$

$$X = \begin{bmatrix} \\ \\ 1 \end{bmatrix} \qquad \begin{aligned} x_3 &= 1 \\ x_2 &= -1 \\ x_1 &= -1 \end{aligned}$$

$$X = C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Null space.

'Recitation'

The set S of points $P(x, y, z)$. s.t $x - 5y + 2z = 9$

is a ______ in R^3, It is ______ to the

______ S_0 of $P(x, y, z)$

Sol: $x - 5y + 2z = 0$

All points of the S have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solu:

$Ax = 0 \rightarrow$ homogeneous

$Ax = b \rightarrow$ Non homogeneous

$x - 5y + 2z = 0$

Answer:

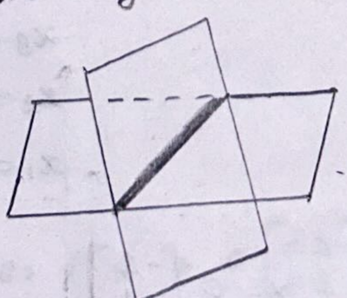
$x - 5y + 2z = 0$ is a plane in R^3. It is ______ to the plane S_0 of $P(x, y, z)$

$\therefore$ $x - 5y + 2z = 0 \rightarrow$ Form a plane in R^3

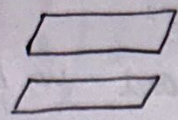
Now what's the relation b/w S and S_0:

* Are they intersecting?
* Are they parallel? (Not intersecting)

Intersecting:



* All points in S, satisfies eqn of S
* " S_0, " " S_0
* All points in the intersection solves both the plane.



→ No points in common.

$$\begin{cases} x - 5y + 2z = 9 \rightarrow S \\ x - 5y + 2z = 0 \rightarrow S_0 \end{cases}$$

The combination of x, y, z can't produce 9 and 0 at the same time for the same eqn.

'So S and S_0 are parallel'

'Not intersecting'

ii) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix}}^{P_0} + c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

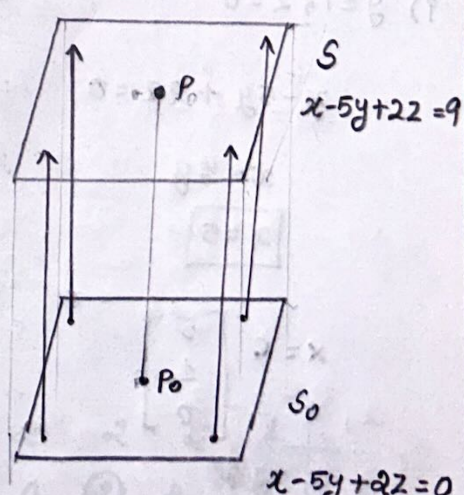
Plug $c_1 = c_2 = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix}}_{P_0 \text{ is in } S}$$

$x - 5y + 2z = 9 \quad (y = z = 0)$

$\boxed{x = 9}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix} \rightarrow P_0.$$



P_0 in $S_0 \rightarrow (0, 0, 0)$

P_0 in $S \rightarrow (9, 0, 0)$

we can obtain any points in S by:

Going to any point in S_0 and up by the vector $\begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$.

$$\text{Any point in } S = P_0 + \begin{pmatrix} \text{Any point} \\ \text{in } S_0 \end{pmatrix}$$

$$P_0 = \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$$

---

we can write it as,

$$\begin{bmatrix} 1 & -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

Row reduction:

$$\begin{bmatrix} 1 & -5 & 2 \end{bmatrix}$$

pivot   Free

* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\longrightarrow$ special cases

i) $y = 1, z = 0$

$x - 5y + 2z = 0$

$x = 5y$

$\boxed{x = 5}$

$$x = c \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$z = 1, y = 0$

$x = -2$

$$x = d \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$