$y' = y^2 - xy$, Let $y(x)$ be solution through $y(0) = -1$.

a) Estimate $y(1)$ using Euler and step size $h = 0.5$.

b) Is the first step an error or underestimate of the exact solution.

h - step size.

Sol: $y_{n+1} = y_n + h f(x_n, y_n)$

$x_{n+1} = x_n + h$

$f(x, y) = y^2 - xy$

$y'' = 2yy' - xy' - y$

n	x_n	y_n	y'	hf_n
0	0	-1	1	0.5
1	0.5	-0.5	0.5	0.25
2	1	-0.25		

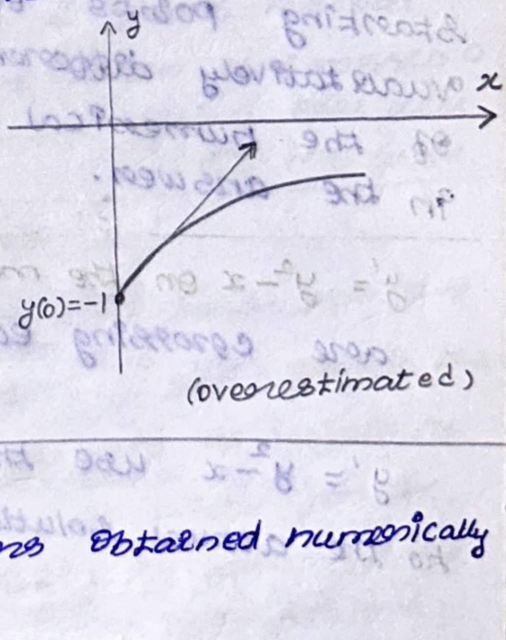
a) $x_0 = 0,\ y_0 = -1$

$y' = f(x, y) = 1 - 0(-1)$
$= 1.$

$y'' = 2(-1)(1) - 0 - (-1)$
$= -2 + 1 = -1 < 0$ (concave)

$$\boxed{y_2(1) \approx y(1) \approx -0.25}$$

b) Our initial app is going to overestimate the solutions



(overestimated)

Reliability

How can we decide whether answers obtained numerically can be trusted?

Here are some heuristic tests.
(Enable oneself to learn themselves)

(Heuristic — Loosely speaking, these tests work in practice, but they are not proved to work always).

Self Consistency:
Solution curves should not cross! If numerically computed solution curves across, a smaller step size

is needed. (Try mathlet "Euler's method" → $\dot{y} = y^2 - x$, step size 1, and starting points (0, 0) and (0, ½).

Convergence as $h \to 0$:

The estimate for $y(t)$ at a fixed later time t should converge to the true value as $h \to 0$. If shrinking h causes the estimate to change a lot, then h is probably not small enough yet. (E.g. try mathlet with $y' = y^2 - x$ with starting point (0, 0) and various step sizes.

Structural stability:

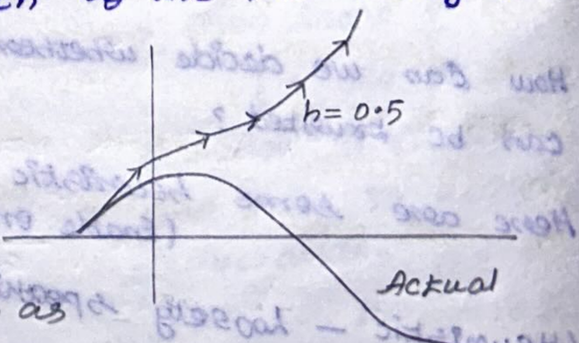
Small changes in the DE's parameters shouldn't change the outcome completely. If small changes in the parameters drastically change the outcome, the answer should be trusted.

Stability:

Small changes in the DE's initial conditions don't change the outcome much. One reason for instability could be a separatrix, a curve such that nearby starting points on different sides lead to qualitatively different outcomes; this is not a fault of the numerical method, but rather an instability in the answer.

$y' = y^2 - x$ on the mathlet. ($h = 1.0, 0.5, 0.25$) Solutions are crossing each other.

$y' = y^2 - x$ use the IC $(-0.98, 0)$. Compare $h = 0.5$ to the actual solution. Which of the reliability failure



As step size ↓, the solution does converges to the actual solution.

However, nearby points, such as $(-1, 0)$ and $(-1, -0.1)$ lead to different qualitative behaviour. So this is not stable.

Change of Variable

Euler's method generally can't be trusted to give reasonable values when (t, y) strays very far from the starting point. In particular, the solution it produces usually deviate from the truth as $t \to \pm\infty$, or in situations in which $y \to \pm\infty$ in finite time. Anything that goes off the screen can't be trusted.

e.g.1: $\dot{y} = y^2 - t$ $(-2, 1) \to$ starting seems to go $+\infty$ in finite time. But Euler's method never produces a value of $+\infty$.

Sol^n: Let $u = \frac{1}{y}$ (To see what's happening). To rewrite the DE in terms of u, substitute $y = 1/u$ and $\dot{y} = -\frac{\dot{u}}{u^2}$:

$$-\frac{\dot{u}}{u^2} = \frac{1}{u^2} - t$$

$$\dot{u} = -1 + tu^2$$

This is equivalent to the original DE, but now, when y is large, u is small & Euler's method can be used to find the time when u crosses 0 which is when y blows up. $\to$ modified equ will do the work.

Runge - Kutta methods

When computing $\int_a^b f(t)\,dt$ numerically, the most primitive method is to use the left Riemann sum: Divide the range of integration in to sub intervals of width h, and estimate the value of $f(t)$ for each t on the subinterval by the value at the left endpoint. More sophisticated methods are the trapezoid rule & Simpson's rule, which have smaller errors.

There are analogous improvements to Euler's method:

Integration	Differential equ	Error
Left riemann sum	Euler's method	$O(h)$
Trapezoid rule	II-order Runge Kutta method (RK2)	$O(h^2)$

Simpson's rule   Fourth-order Runge-Kutta method (RK4) $O(h^4)$

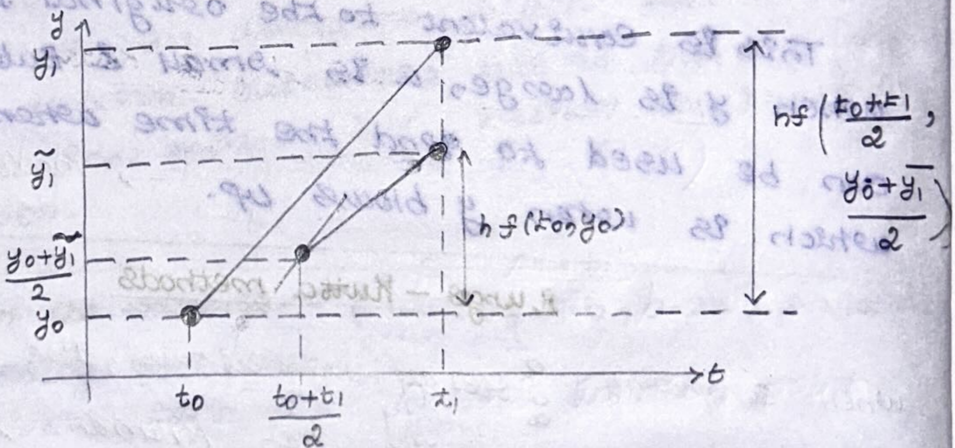
The big O notation $O(h^4)$ means that there is a constant C (depending on the diff eqn but not on h) such that the error is at most Ch^4, assuming that h is small. The error estimates in the table are valid for reasonable diff eqn.

<u>Better methods:</u>

The Runge-Kutta methods evaluate at more points on the interval $[t_0, t_0+h]$ to get a better estimate of what happens to the slope over the course of that interval.

Below is an example of a II order Runge-Kutta method (RK 2). It's also called the midpoint method or the modified Euler method.

<u>Here is how one step of this method goes:</u>



1) Starting from (t_0, y_0) look ahead to see where one step of Euler's method would land, but don't go there! Call this temporary point $(t_1, \tilde{y}_1)$

2) Find the midpoint b/w the starting point & the temporary point: $\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right)$.

3) Use the slope at this midpoint to find y_1

$$y_1 = y_0 + hf\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right)$$

Repeat the steps above using (t_1, y_1) as the starting point.

Here is a summary of eqn:

$$t_1 = t_0 + h$$

$$\tilde{y}_1 = y_0 + hf(t_0, y_0)$$

$$y_1 = y_0 + hf\left(\frac{t_0+t_1}{2}, \frac{y_0+\tilde{y}_1}{2}\right)$$

$$(t_0, y_0) \longrightarrow (t_1, y_1)$$

Even better methods:

The fourth order Runge-Kutta method (RK4) is similar, but more elaborate, averaging several slopes. It's the most commonly used method for solving DEs numerically, some people simply call it the Runge-Kutta method. The mathlet's use (we have been playing use RK4 with a small step size to compute the actual solution to a DE.

Runge-Kutta method also known as Heun's method

Better method - Same idea as Euler's method (Improves slope A_n)

'Improved Euler method' (or) 'modified Euler method', 'RK2' $\rightarrow$ II order method.

meaning error varies with a step size, like some constant (It won't be same as the constant for Euler's method) ~~[illegible]~~.

$$\boxed{e \sim C_2 h^2}$$

Halve the step size - Decrease the errors by a factor of $\frac{1}{4}$.

Hardest step: Evaluation of slope.

RK4 – need 4 evaluations of slope. ($\frac{1}{16}$ reduction factor of error).

Standard method

Inefficient → But accurate. (Runge – Kutta) ஜெர்மன்காரன்.

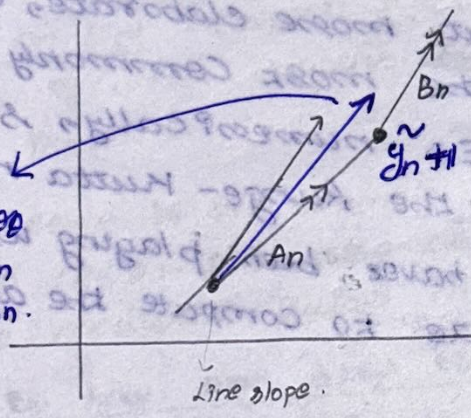
↳ Requires Four slopes

$$\left(\frac{A_n + 2B_n + 2C_n + D_n}{6}\right) \leftarrow \text{super slope}$$

Divide by their coefficients.

<u>RK2:</u>

Taking average of both A_n and B_n.



$B_n \rightarrow$ Next line element's slope.

Draw the exactly same slope $\parallel$ to A_n.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h\,(\text{Blue slope})$$

$$= y_n + h\left(\frac{A_n + B_n}{2}\right)$$

$B_n = ?$ Let take the Euler's solution point be $\tilde{y}_n$

$$\tilde{y}_{n+1} = y_n + hA_n$$

$$B_n = f(x_{n+1}, \tilde{y}_{n+1})$$

We are correcting the slope to get a better slope

$$RK4 \rightarrow \frac{1}{16} \text{ factor}$$

Software

Most software systems have numerical routines that are built in. Many of these numerical routines use variable order methods. You may want to check some out, and play around with them. Make sure that you check the documentation for how to enter the arguments of each method.

Maple:

Euler (ODE, IC, t=b, opts)

Runge Kutta (ODE, IC, t=b, opts)

Matlab

ode45 (ode func, tspan, y0, options)

ode23 (ode func, tspan, y0, options)

ode 113 (ode func, tspan, y0, options)

Python (scipy. integrate)

odeint (func, y0, t, args = ( ))

Each nonempty isocline for $y' = \frac{1}{x+y}$ has

$$m = \frac{1}{x+y} \Rightarrow mx + my = 1$$

$$y = \frac{1 - mx}{m}$$

$$= -x + \frac{1}{m}.$$

$\therefore$ All isoclines are straight lines with a negative slope of -1.

There is a function $y(x)$ defined for all real numbers x that satisfies the diff eqn $y' = e^{x}y$ and the IC $y(0) = 2$.

$\therefore$ The statement that there is a function $g(x)$ defined for all real numbers x that satisfy the diff eqn $y' = e^x y$ is true by the existence & uniqueness theorem for a linear ODE. (Infact, separation of variables leads to the solution $y(x) = \frac{2}{e} e^{e^x}$

$$\frac{dy}{y^3} = dx$$

$$\frac{y^{-2}}{-2} = x + c$$

$x = 0$, $y(0) = 2$

leads to $-\frac{1}{8} = 0 + c$

$$\boxed{c = -\frac{1}{8}}$$

Thus

$$\frac{y^{-2}}{-2} = x - \frac{1}{8}$$

$$-y^{-2} = 2x - \frac{1}{4}$$

$$y = \left(\frac{1}{4} - 2x\right)^{-1/2}$$

As solution tends to $+\infty$ as $x \to \frac{1}{8}$ from the left. Therefore it's not defined for all real numbers

---

Each nonempty isocline for the diff eqn $y' = y^2 - 4$ consists of lines.

Parachute jumper.

$m\dot{v} = Kv^2 - mg$ describes the (-ve velocity

Solu: points downwards) of a parachute jumper of mass m subject to gravity g, and wind resistance from the open parachute of Kv^2, with K a constant

Solu: $m = 100\,kg$, $g = 10\,m/sec^2$, $K = 10\,kg/meter$.

Draw phase line:

$$\dot{v} = \frac{10}{100} \, kg/m.kg \; v^2 - g$$

$$= \frac{1}{10} v^2 - 10$$

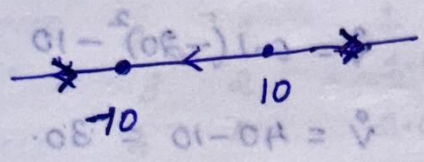
$$\dot{v} = 0.1v^2 - 10$$

critical points

$0.1v^2 = 10$

$v^2 = 100$

$v = \pm 10$



• For $v = 2$

$\dot{v} = 0.1(4) - 10$

$= -9.6$

For $v = 11$

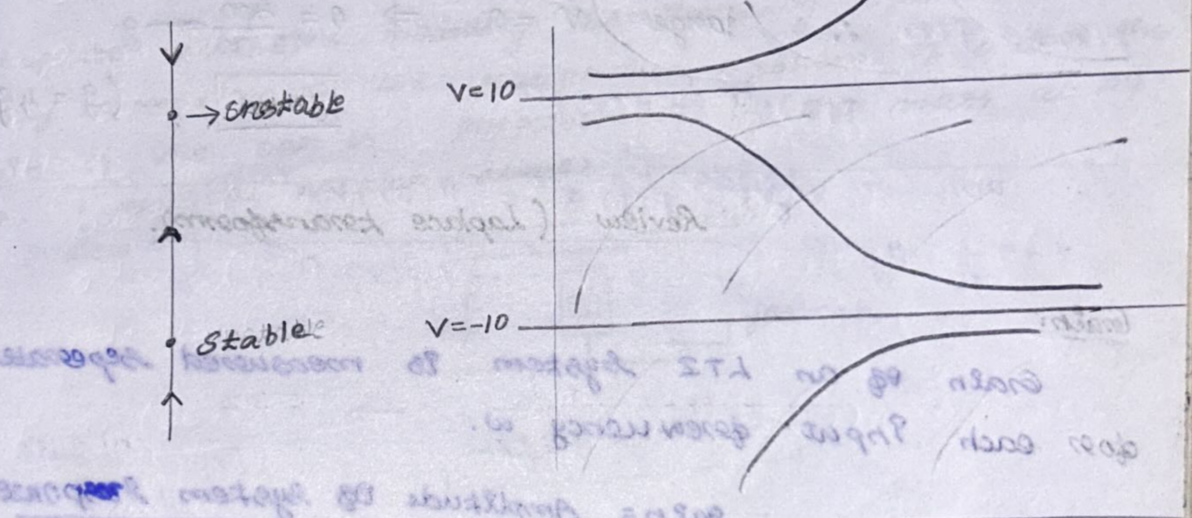
$\dot{v} = 0.1(121) - 10$

$= +ve$

For $v = -11$

$\dot{v} = 0.1(+121) - 10$

$= +2.1$



g-forces

pilots, astronauts and parachute jumpers are all concerned with the g-forces they experience. The g-force is a measurement of the acceleration that causes weight. The unit of measurement is 1g, the force (ordinary) of gravity on earth. When the parachute jumper is in free fall with closed parachute, experiencing only the effect of gravity with no normal force opposing the force due to gravity, the jumper feels weightless. Thus the jumper experiences 0 g's. The g-force experienced by a parachute jumper is the number of g's of acceleration relative to free fall.

g-force.

$t = 0$, $v(0) = -20\,m/s$. Find the time at which the acceleration $\dot{v}$ is largest in absolute value. At that moment, what's the g force experienced by the jumper?

$$-20\,m/s = 0.1v^2 - mg$$

$$-20\,m/s = -mg$$

$$20 = 100g$$

$$g = \frac{20}{100} = 0.2$$

At $t = 0$, $v(0) = -20\,m/s$

$$\dot{v} = 0.1(-20)^2 - g$$

$$0 = 0.1(400) - g$$

$$g = 40$$

$$\dot{v} = 0.1(-20)^2 - 10$$

$$\dot{v} = 40 - 10 = 30.$$

when $t = 0$ and $\dot{v} = 30$

$\therefore$ g = 1. As $t \uparrow$, $v \downarrow$ to the steady state $-10\,m/s$ (from eqn).

$\therefore$ largest $\dot{v} = 0 \Rightarrow 0 = \frac{400}{10} - g$

$\boxed{g = 40}$ $(g = 4g)$

Review (Laplace transform).

Gain:

Gain of an LTI system is measured seperately for each input frequency ω.

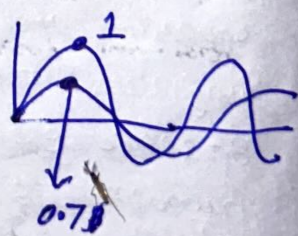
$$\text{gain} = \frac{\text{Amplitude of system response}}{\text{Amplitude of input signal}}.$$

Input - sinusoid of freq ω, In this, the amplitude of the input signal is fixed at 1, so the gain equals the amplitude of the sinusoidal system response.

Amplitude & phase: second order II

$$\ddot{x} + b\dot{x} + kx = b\dot{y} \quad (y = \cos\omega t)$$

$b = 1$, $k = 2$, $\omega = 1$, measure gain



Soln:

gain = 0.78

Maximum gain can a system take = 1 (For any values of b & k)

Resonant frequency

Suggesting a formula

b	K	ω_r
1	1	1
1	2	1.4
1	4	2

$b = 1.0,\ K = 1 \Rightarrow \omega_r = 1$

$b = 1,\ K = 2 \Rightarrow \omega_r = 1.45$

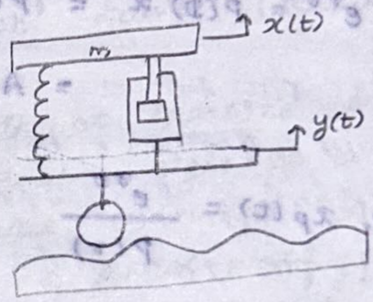
$b = 1,\ K = 4 \Rightarrow \omega_r = 2.05$

b	K	ω_r
2	1	1
3	2	1.4
4	4	2.0

$\therefore \omega_r = \sqrt{K}$ is a good formula

Modelling

Setup the diff eqn modelling the following system. It closely approximates the suspension system of a car. The cabin of the car is represented by the mass at the top. (vertical motion alone)



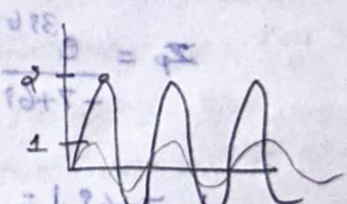
1) Diagram, 2) Named

Two forces:

$$m\ddot{x} = F = -K(x-y) - b(\dot{x}-\dot{y})$$

$$m\ddot{x} + b\dot{x} + Kx = Ky + b\dot{y}$$

Assume $m = 1$, $K = 3$, $b = 1$



maximum gain = 2

Resonant freq = 1.65

$\frac{2}{1} = 2$

At ($\omega = \omega_r$ – maximum gain occurs)

Resonant frequency: The angular frequency at which the gain is maximal is the (resonant angular frequency) ω_r.

$e^{(-1+2i)t}$ as $t \to \infty \Rightarrow r = e^{-t}$, $\theta = 2t$, so it travels counter clockwise. ∴ θ is ↑. Also it move inwards as t ↑.

$$z^2 + 2z + 2 = 0$$

$$\frac{-4 \pm \sqrt{4 - 4(2)}}{2} = \frac{-4 \pm \sqrt{-4}}{2}$$

$$= -2 \pm i$$

$\sqrt{4+1} = \sqrt{5}$

$$\arg\left(\frac{1}{-1}\right) = -\frac{\pi}{4} + \pi$$

$$= \frac{3\pi}{4}$$

Exponential response

$P(D)\, x = f(t)$

$$q(D) e^{rt} = q(r) e^{rt}$$

$$e^{rt} = P(D)\, x = P(D)(A e^{rt})$$

$$= A P(r)\, e^{rt}$$

$$\therefore x_p(t) = \frac{e^{rt}}{p(r)} \longrightarrow \text{ERP}$$

(Exponential Response formula)

Complex replacement

$\ddot{z} + 2\dot{z} + 2z = e^{3it}$ is the " - " of $\ddot{x} + 2\dot{x} + 2x = \cos 3t$

$$p(3i) = (3i)^2 + 2(3i) + 2 = -7 + 6i$$

$$z_p = \frac{e^{3it}}{-7+6i}$$

$$|-7+6i| = \sqrt{85}$$

$$z_p = \frac{e^{3it}}{\sqrt{85}\, e^{i\phi}} = \frac{e^{i(3t-\phi)}}{\sqrt{85}}$$

Real part

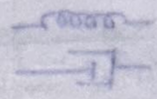
$$x_p = \frac{1}{\sqrt{85}} \cos(3t - \phi)$$

lll^{ly} $\ddot{y}+2\dot{y}+2y=\sin 3t \Rightarrow y_p = \text{Im}\, z_p = \frac{1}{\sqrt{85}}\sin(3t-\phi)$

Solving,

$$p(r) = r^2+2r+2 = (r+1)^2+1$$
$$= -1 \pm i$$

$\therefore$ $e^{-t}\cos t$ and $e^{-t}\sin t \rightarrow$



So our general solution is

$$x = \frac{1}{\sqrt{85}}\cos(3t-\phi) + ae^{-t}\cos t + be^{-t}\sin t$$

RLC circuits and the system function:

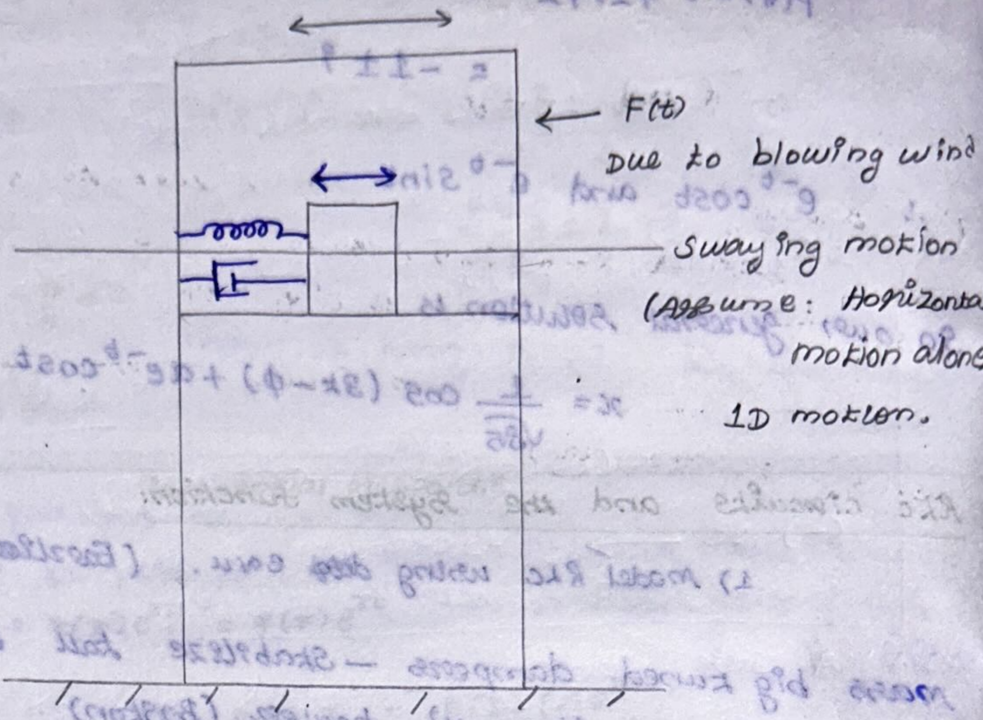
1) model RLC using ~~diff~~ equ. (Earlier we saw)

mass big tuned dampers — stabilize tall buildings such as the 'John Hancock' tower. (Boston)

Two 300-ton weights sit at opposite ends of the 58th floor of the Hancock. Each weight is a box of steel, filled with lead, 17 feet square by 3 feet high. Each weight rests on a steel plate. The steel plate is covered with lubricant so the weight is free to slide. But the weight is attached to the steel frame of the building by means of springs & shock absorbers. When the Hancock sways, the weight tend to remain still, allowing the floor to slide underneath it. Then as the spring & shocks tend to hold, they begin to tug the building back. The effect is like that of a gyroscope, stabilizing the tower. The reason there are two weights, instead of one, is so they can tug in opposite directions when the building twists. The cost of the damper was \$3 million. The dampers are free to move a few feet relative to the floor.

'Robert Campbell, architecture critic for the Boston globe'

modelling a swaying building having a tuned mass Damper



Fact: It turns out that a steady wind won't exert a constant force, but instead it will exert an oscillating sinusoidal force

↳ Due to a fluid mechanics phenomenon 'vortex shedding'

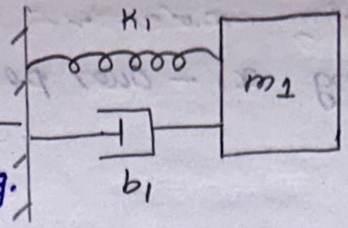
This periodic forcing can cause tall buildings to sway back & forth in the wind. Due to linearization (For small oscillations - the building will act like a linear spring.) → Because it has a stable equilibrium position when it's standing vertically. However, if the building is pushed away from this equilibrium, the metal in the buildings will resist the change & push it back towards the stable equilibrium.

Mass dampers.

(Coupled)

Damper will be connected to the building using springs

Fixed point represents the base of the building.



m_1 - mass of the building

when building sways - Energy will be lost to friction as the metal stretches and compresses. we'll model this dissipation of Energy by a damper.

Assume linear damper (valid as long as the swaying velocity is not too great.

Generally, the swaying motion of the building is not dangerous to the structural integrity of the building. But causes motion sickness to the people.

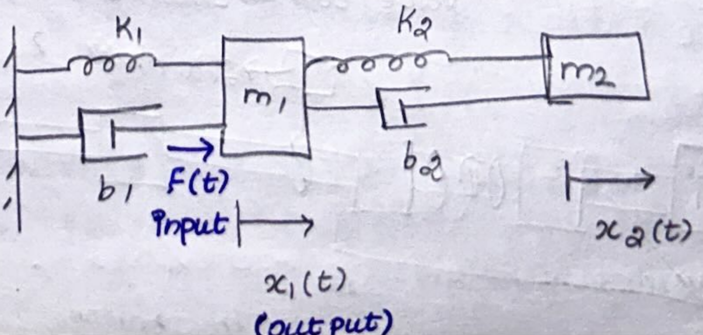
↳ So Engineers usually place a large tuned mass damper. (Not attached directly with the building's floor but sits in a slider - that it allow to move independently of the building.

The tuned mass damper is then coupled to the building by springs on all four sides.

Here we are assuming 1D motion

Damper: For frictional losses.

The oscillation of the tuned mass damper will then affect the swaying motion of the building itself if the tuned mass damper (TMD) is designed correctly. It should reduce the oscillation amplitude of the building.



we are interested in varying movement of the building. So we are choosing x_1 – our primary output of interest.

Equations of motion:

(From Newton's second law)

Mass 1:

$$m_1 \ddot{x}_1 = F(t) - k_1 x_1 - b_1 \dot{x}_1 + k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1)$$

why – in $k_1 x_1$?

(stretched)

Spring will try to comeback ← so (−)

As k_2 & b_2 depends upon both the positions of x_1 and x_2.

why $+ k_2(x_2 - x_1)$ → As the spring is stretched the movement will be rightwards.

mass 2:

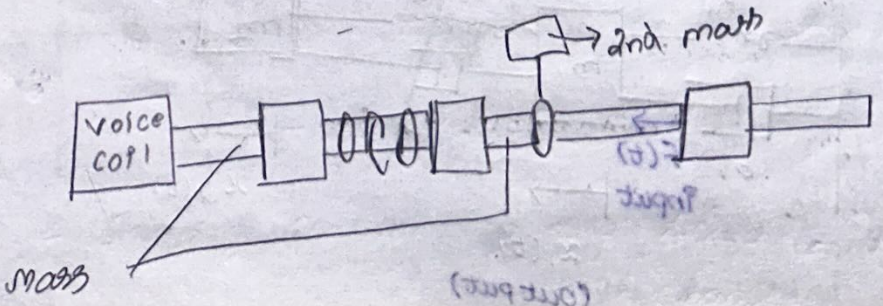
$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1)$$

The second mass only moves to damp out the Energy.

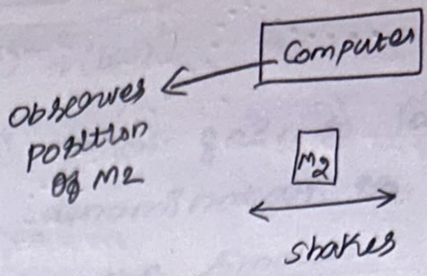
Bode plots of the mascot:

Bode plot of a fourth order system:

voice coil → shakes mascot.



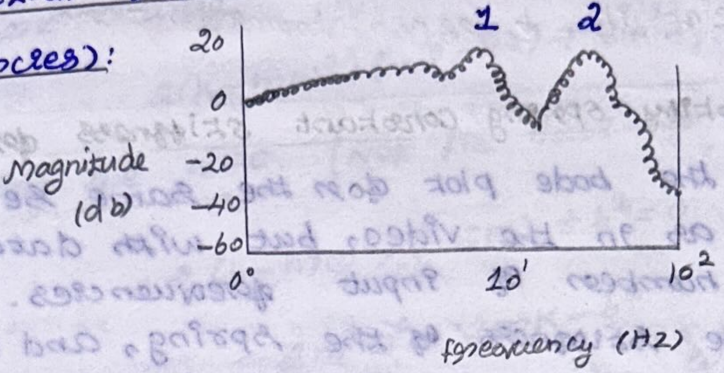
The mass is suspended by two black cubes (air bearings powered by compressors).

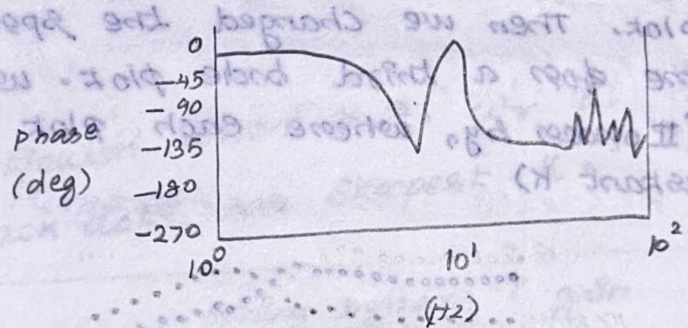


Bode plot:

Log-Log plot.

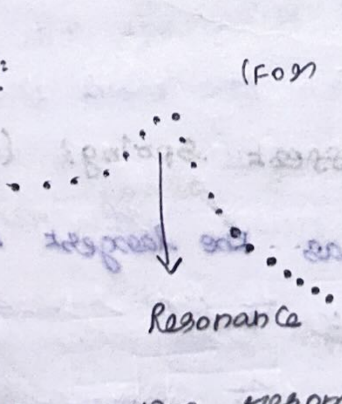
Reproduction of plot (using a larger number of input frequencies):

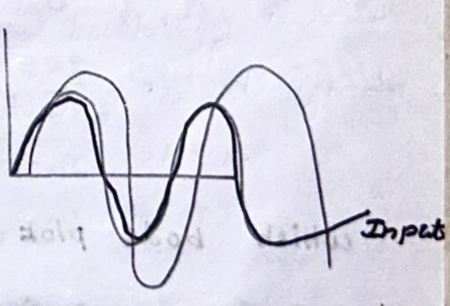




Bode plot has two peaks, that is two different local maxima & one one local minimum b/w them. The local minimum corresponds to the case where mass 1 stops moving, but the second mass is still moving quite a bit. By the end of the course, you will gain a better understanding of how to construct a system with this bode plot.

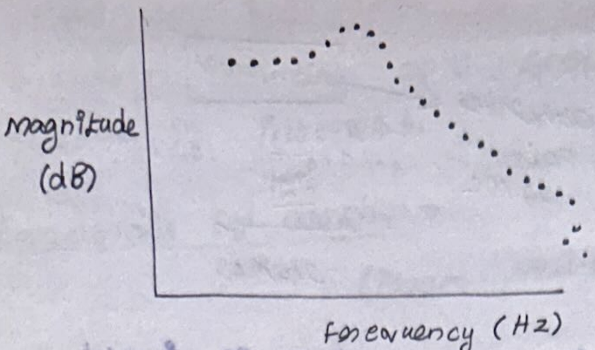
Bode plot: (For ↑ frequency)





There is a phase lag due to computer calibration.

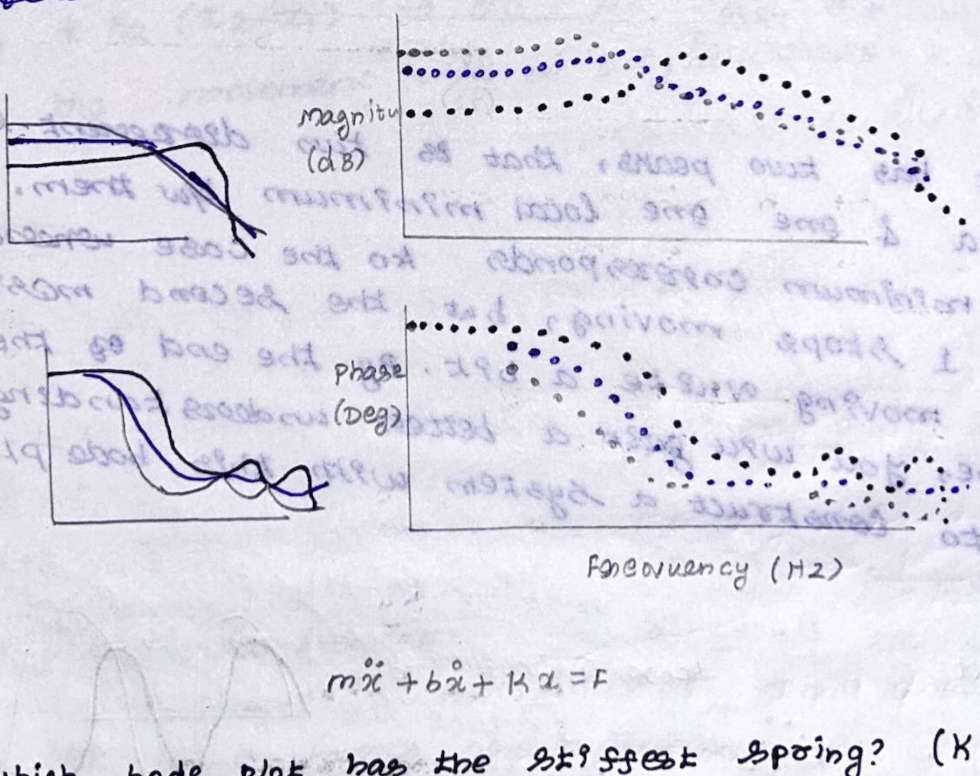
After reaching resonance, again the system response is falling.



As frequency ↑, the amplitude ↑ until it reaches a maximum value, at which point it ↓ again towards zero.

Identify Spring Constant Stiffness from Bode

We recreated the bode plot for the same second order system as in the video, but with data taken over a large number of input frequencies. Then we changed the stiffness of the spring, and made another bode plot. Then we changed the spring constant one more time for a third bode plot. What you see below. (II order sy, where each plot has different constant K).



$$m\ddot{x} + b\dot{x} + Kx = F$$

Which bode plot has the stiffest spring? (K largest?)

The (Black dots) has the largest stiffness spring.

$$m\ddot{x} + b\dot{x} + Kx = F$$

m & damping constant b are fixed. Only K is changed.

$$g(\omega) = \frac{1}{\sqrt{(k-m\omega^2)^2+(b\omega)^2}}$$

The resonant ~~fre~~ occures when gain is largest, (smallest denominator). The denominator is small when $(k-m\omega^2)^2+(b\omega)^2$ is the smallest.

$\therefore$ ~~Diff~~ Differenciating,

$$2(k-m\omega^2)(-2m\omega) + 2b^2\omega = 0$$

Ignoring the $\omega = 0$. (Not the case we are interested)

$$2(k-m\omega^2)(-2m) + b^2 = 0$$

$$2m^2\omega^2 = 2mk - b^2$$

$$\omega = \frac{\sqrt{mk - b^2/2}}{m}$$

In particular when $k\uparrow$, $\omega_r \uparrow$. $\therefore$ The bode plot of black dots has steepest k.

Spring systems & gain

1) Complex gain is defined as the complex number such that exponential system response to input $G(\omega) e^{i\omega t}$ is the signal $e^{i\omega t}$.

2) The gain $g(\omega)$ is the magnitude of the complex gain $G(\omega)$:

$$g(\omega) = |G(\omega)|$$

3) If the system is modeled by

$$P(D)x = Q(D)y$$

$y(t)$ — input & system response $x(t)$

then,

$$G(\omega) = \frac{Q(i\omega)}{P(i\omega)}$$

$$G(\omega) = \frac{bi\omega}{m(i\omega)^2+b(i\omega)+k} = \frac{bi\omega}{(k-m\omega^2)+bi\omega} \quad (m\ddot{x}+b\dot{x}+kx = b\dot{y})$$

Resonance

SHM:

(undamped system, driven through spring. Let's begin,

$$m\ddot{x} + kx = f(t)$$

Let $f(t) = A\cos\omega t$

solu:

$$x_h = C_1\cos(\omega_n t) + C_2\sin(\omega_n t)$$

where, $\omega_n = \sqrt{\frac{k}{m}}$ is the natural frequency of the oscilator.

P.I:

$$x_p = \begin{cases} \frac{A\cos\omega t}{|k-m\omega^2|} = \frac{A}{m}\frac{\cos\omega t}{|\omega_n^2-\omega^2|} & \text{if } \omega < \omega_n \\ \frac{A\cos(\omega t-\pi)}{|k-m\omega^2|} = -\frac{A}{m}\frac{\cos\omega t}{|\omega_n^2-\omega^2|} & \text{if } \omega > \omega_n \\ \frac{At\cos(\omega_n t-\pi/2)}{2m\omega_n} = \frac{A}{m}t\frac{\sin(\omega_n t)}{2\omega_n} & \text{if } \omega = \omega_n \end{cases}$$

The bottom case $(\omega = \omega_n)$ — requires complex replacement.

Resonance:

gain of the system is given by

$$g = g(\omega) = \frac{1}{m|\omega_n^2 - \omega^2|}$$

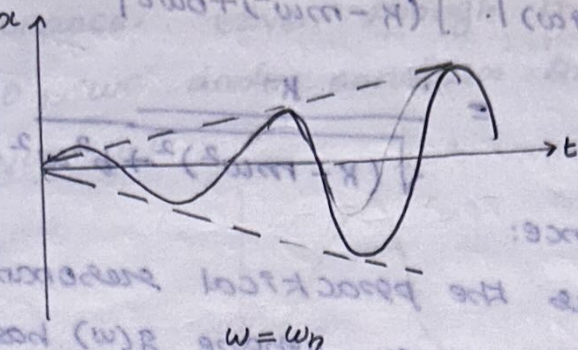
This is a function of ω and the right hand plot below shows its graph.

In this case when the input frequency coincides with the natural frequency ω_n the system is in what is called pure resonance.

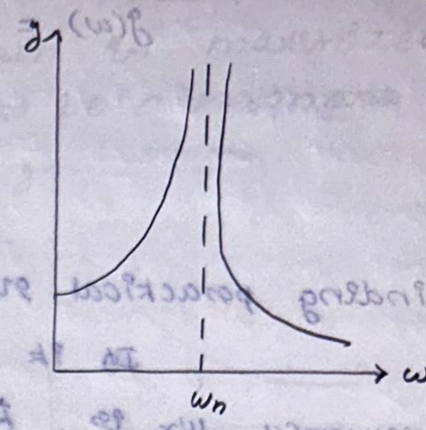
ω_n - Resonant frequency of the system.

when $\omega = \omega_n$, $x_p = \frac{At\sin\omega_n t}{2m\omega_n}$. Notice how the response is oscillatory but not periodic. The factor

t in x_p causes the amplitude to keep growing over time.



Resonant response



undamped amplitude response

Amplitude response & resonance:

The gain $g(\omega)$ is a function of ω. It tells the size of the system's response to the given input frequency. The graph of $g(\omega)$ vs ω is one of the bode plots. In many complex systems, the bode plot can be quite complicated, exhibiting maxima & minima. The maxima occur at frequencies near to some natural frequency of the system, and are called resonant frequencies. In second order systems, there is at most one +ve resonant frequency, which we will often denote by ω_r.

Exam: 7.2:

Consider the spring / mass / dashpot system driven through the spring.

$$m\ddot{x} + b\dot{x} + Kx = KA\cos(\omega t)$$

Input = $A\cos\omega t$

Solu:

Expression for ω_r:

$g(\omega) = |G(\omega)|$. Then ω_r is the value where $g(\omega)$ attain its maximum. The function $g(\omega)$ is rather complicated; are there simpler functions that have minima or maxima at some places.

The gain is

$$g(\omega) = \left|\frac{q(i\omega)}{p(i\omega)}\right| = \left|\frac{k}{(k-m\omega^2)+b\omega i}\right|$$

$$= \frac{k}{\sqrt{(k-m\omega^2)^2+b^2\omega^2}}$$

Finding practical resonance:

If it exits the practical resonance frequency ω_r is the frequency where $g(\omega)$ has a maximum. Now $g(\omega)$ has a maximum where the expression

$$f(\omega) = (k-m\omega^2)^2 + b^2\omega^2 \text{ (is very low)}$$

$$\therefore f'(\omega) = 2(k-m\omega^2)(-2m\omega) + 2b^2\omega = 0$$

$$(-4m\omega)(k-m\omega^2) + 2b^2\omega = 0$$

excluding $\omega = 0$ case:

$$(-4m)(k-m\omega^2) + 2b^2 = 0$$

$$-4mk + 4km\omega^2 = -2b^2$$

$$4km\omega^2 = -2b^2 + 4mk$$

$$\omega_r^2 = \frac{-2b^2+4mk}{4mk}$$

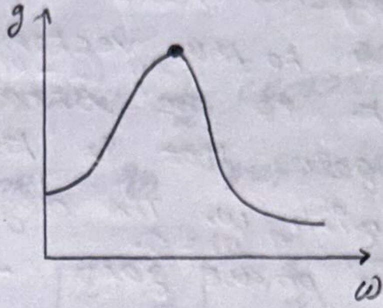
$$\omega_r^2 = -\frac{b^2}{2mk} + 1$$

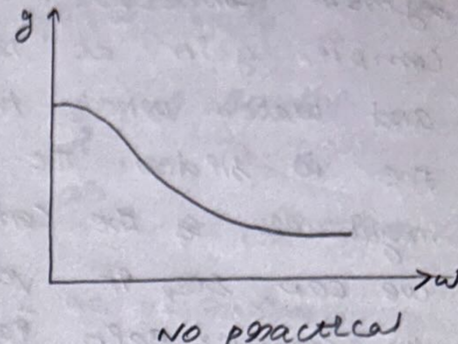
$$\omega_r = \pm\sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$$

$$\therefore \omega_r = \begin{cases} \sqrt{k/m - b^2/2m^2} & \text{when } \omega_n^2 - b^2/2m^2 > 0 \\ \text{No resonant frequence} & \text{otherwise.} \end{cases}$$

Note: In the damped case $b>0$ when it exists, $\omega_r < \omega_n$

Ratio $\frac{b}{m}$ has units $\frac{1}{\text{time}}$; if $\frac{b}{m} > \sqrt{2}\,\omega_n$ then there's no +ve resonant frequency. In this case, we can say that there is no practical resonance. (Even though the gain is maximized at $\omega = 0$, we don't consider this to be a resonant frequency.).





$$\omega_r = \sqrt{\omega_n^2 - b^2/2m^2}$$

$$\ddot{x} + b\dot{x} + Kx = K\cos(\omega t)$$

solu:

Cos ωt - Input. (Mathlet)

$K = 1,\ b = 1,\ \omega_r = 0.75,$

$$g^2(\omega) = \frac{1}{|1-\omega^2+i\omega|^2} = \frac{1}{(1-\omega^2)^2+\omega^2}$$

$$2(1-\omega^2)(-2\omega) + 2\omega = 0.$$

$$\omega = \pm\sqrt{\frac{1}{2}} \approx 0.7.$$

$b = 1.5,\ K = 0.5$

$$g^2(\omega) = \left|\frac{0.5}{0.5-\omega^2+1.5i\omega}\right|^2 = \frac{0.25}{(0.5-\omega^2)^2+1.5^2\omega^2}$$

$$2(0.5-\omega^2)(-2\omega) + 2(1.5)^2\omega = 0$$

$$2\omega^2 = -1.25$$

Eqn. doesn't have a real root. (The gain is maximized when $\omega = 0$)

The Nyquist plot

picture of the complex gain, displaying the trajectory of the complex gain, as a function of ω. An orange line segment connects the origin to the value $G(\omega)$ of the complex gain at the selected value of ω. Set $b=1$ and watch what happens to this vector as you move the ω slider. The length of the strut is the magnitude of the complex gain, that is the gain $g(\omega)$. we can see it vary with ω. The argument of the complex gain is the phase gain $-\phi(\omega)$.

This curve in the Nyquist plot, displays the relationship b/w the gain & the phase lag.

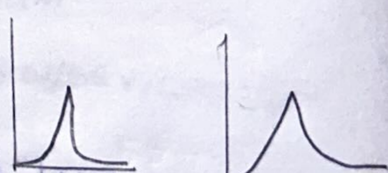
Food for thought..

1) when b is small, the resonant peak is narrow, that is, as soon as ω differs much from ω_r, the gain becomes very small. Make a prediction about how the Nyquist trajectory will be traversed, based on this observation. That is, as ω increases at a steady rate, will $G(\omega)$ move at a steady rate along its trajectory? or will it move faster in some positions than in others?

2) Show that the Nyquist plot, for this system, is gn by a circle of radius $\frac{1}{2}$ with center at the complex number $\frac{1}{2}$ (minus the origin).

when b is too small

Remarks:



when b is small, the gain is very close to zero - everywhere except near the resonant peak. Thus we expect that as we vary ω, the complex gain stays close to zero most of the time, and then traverses very quickly to a large amplitude near resonance.

$$G(\omega) = \frac{ib\omega}{K-\omega^2+ib\omega}$$

$\therefore$ Nyquist plot is a plot of the complex gain.

A circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}+0i$ in the complex plane is represented as

$$|z-\tfrac{1}{2}| = \tfrac{1}{2}$$

$\therefore$ $z = \dfrac{ib\omega}{K-\omega^2+ib\omega}$ satisfies the eqn of circle of radius $\frac{1}{2}$.

$$|z-\tfrac{1}{2}| = \left|\frac{ib\omega}{K-\omega^2+ib\omega} - \frac{1}{2}\right| = \left|\frac{ib\omega}{K-\omega^2+ib\omega} - \frac{1}{2}\frac{K-\omega^2+ib\omega}{K-\omega^2+ib\omega}\right|$$

$$= \left|\frac{1}{2}\frac{(-(K-\omega^2)+ib\omega)}{(K-\omega^2+ib\omega)}\right|$$

$$= \tfrac{1}{2}.$$

$$\ddot{x}+\dot{x}+2x = 2f(t) \quad \text{with input } f(t)$$

Soln:

$b=1,\ K=2.$

The bode plot suggests answers to various questions about the behaviour of this system. Let's enumerate some & then verify them by computation.

- What is $G(0)$, so what are $g(0)$, $\phi(0)$?
- What is the limiting value of $G(\omega)$ as $\omega \to \infty$.
- Does this system exhibit a resonant peak at a +ve value of the input frequency? At what value of ω? What's the maximum gain?

Soln

$$G = \frac{2}{p(i\omega)} = \frac{2}{2-\omega^2+i\omega} \qquad p(s) = s^2+s+2$$

$$g(\omega) = \frac{2}{|P(i\omega)|} = \frac{2}{|2-\omega^2+i\omega|} = \frac{2}{\sqrt{(2-\omega^2)^2+\omega^2}}$$

phase lag:

$$\phi(\omega) = -\operatorname{Arg}\left(\frac{2}{P(i\omega)}\right) = \operatorname{Arg}(P(i\omega))$$

$$= \operatorname{Arg}(2-\omega^2+i\omega)$$

Answering,

$G(0) = 1$, so $g(0) = 1$, $\phi(0) = 0$.

As $\omega \to \infty$, the complex gain is roughly

$$G(\omega) \approx \frac{2}{-\omega^2}$$

$g(\omega) \to 0$ and $\phi(\omega) \to \pi$ as $\omega \to \infty$

· To complete our understanding of the frequency response of this system, we will look for resonant peaks.

From mathlet, $\omega = 1.22$

$g(\omega) \to$ maximum when denominator $|P(i\omega)|$ is minimal or what's the same when ω minimizes the square.

$$|P(i\omega)|^2 = (2-\omega^2)^2 + \omega^2$$

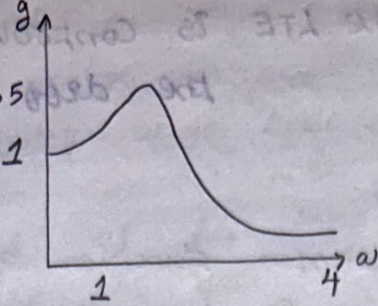
$$2(2-\omega^2)(-2\omega) + 2\omega = \omega(4\omega^2 - 6)$$

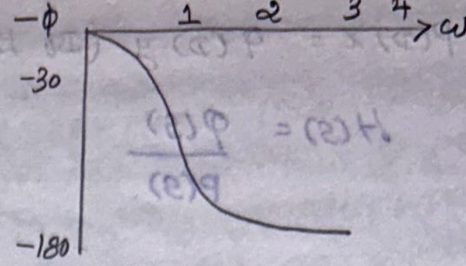
$$\omega = 0, \quad \omega = \pm\sqrt{\frac{3}{2}}$$

$$\omega_r = \sqrt{\frac{3}{2}} \approx 1.2247$$

In good agreement with our observation.

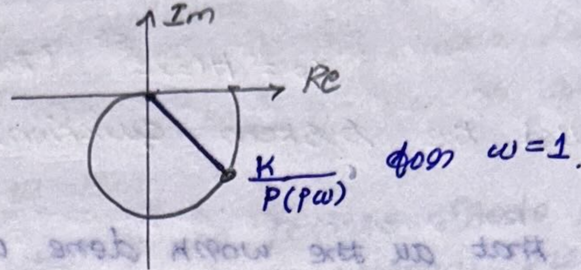
$$g(\omega_r) = g\left(\sqrt{\frac{3}{2}}\right) = 2\sqrt{\frac{4}{7}} \approx 1.5119.$$





Bode phase plot (degrees)

Bode gain plot



Nyquist plot.

The Transfer function.

The complex gain $G(\omega)$ of an LTI system is the factor by which the input signal $e^{i\omega t}$ gets multiplied. For a system modeled by the ~~diff~~ equ.

$$P(D)\,x = Q(D)\,e^{i\omega t}$$

$$\text{complex gain} = \frac{Q(i\omega)}{P(i\omega)} \quad \left(\frac{Q(r)}{P(r)}\right)$$

But these facts encourage the following question: what's the exponential system response to a more general exponential input signal, one of the form

$$e^{st}, \quad s - \text{a complex constant.}$$

This is a reasonable question. For example, the input signal might be a damped sinusoid, something like $e^{-t/10}\cos(\pi t) = \mathrm{Re}\left(\left(e^{(-0.1+\pi i)t}\right)\right)$

The exponential system response of an LTI system to the input signal e^{st}, for s - any complex constant.

$$H(s)\,e^{st}$$

$P(D)x = q(D)y$ (If the LTE is controlled by the diff eqn)

$$H(s) = \frac{q(s)}{p(s)}.$$

The factor $H(s)$ is the transfer function of the LTI system. You can think that it transfers the input e^{st} to the response

$x_p = H(s)e^{st}$ (The transfer function is also called the system function).

Note: Note that all the work done with complex gain was a study of a special case of the transfer function when $s = i\omega$.

$$G(\omega) = H(i\omega)$$

H(S) Comments:

* The T.F will usually be a quotient of one polynomial by another: a rational function. (In more general LTI systems it may be more complicated. But it's always a function of a complex number s, taking on complex values.)

If r is a root of the denominator, $P(r) = 0$, then $H(s)$ is not defined. This corresponds to resonance; the input signal e^{rt} doesn't produce an exponential system response at all.

T.F of $2\ddot{x} + 3\dot{x} + x = 5\dot{y} + 3y$ $\quad r = s$

Soln:

$$H(x) = \frac{q(i\omega)}{P(i\omega)} = \frac{(5\cancel{\omega}+3)}{(2\cancel{\omega^2} + 3\cancel{\omega} + 1)} = \frac{5s+3}{2s^2+3s+1}$$

$$2s^2 + 3s + 1 = 0$$

$$\frac{-3 \pm \sqrt{9-4(2)}}{2(2)} = \frac{-3 \pm 1}{2(2)}$$

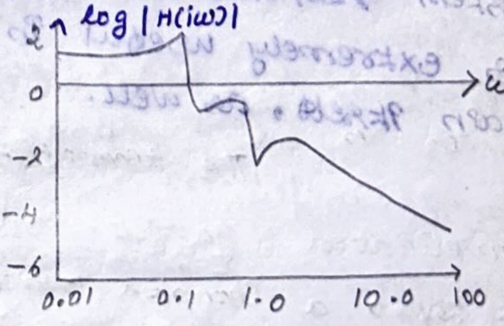
$$s = -\frac{2}{4}, -\frac{4}{4} \Rightarrow -\frac{1}{2}, -1.$$

why bode (Foot notes) → refer word.txt

Hendrick wade bode (1905 – 1982). (One of the fathers of modern control theory. For most of his career, he worked at Bell labs. During world war – II, he devised the feeback linking radar data to anti-aircraft fire. After the war he occupied high administrative positions at Bell labs. In 1967 he retired as vice president in charge of military development & systems Engineering, and took up the position of Gordon mckay professor at Harvard, a post he held till 1974.

Our use of the phrase 'Bode plots' in this course is inaccurate in several response. First of all, in Engineering applications one typically has to cope with a wide range of frequencies. To represent them, one plots the log of the frequency horizontally. Similarly, the gain often spans a wide range of values, and to provide clear visualizations of the gain it is sensible to plot log g(ω) rather than g(ω). So typically the gain bode plot uses log-log scale.

The vertical logarithmic scale is often measured in 'decibels'. The decibel measure of gain g is $10 \log_{10} g^2$, $20 \log_{10} g$ Db. This can also be written which is of interest since the power of a sinusoidal signal is proportional to the square of its amplitude. Use of these units organized at Bell labs in 1920's.



The vertical logarithmic scale – Decibel (measurement)

The phase bode uses log ω horizontally, and −ϕ vertically. (No need for a log vertically, after all, in a sense ϕ is already a logarithm – in the complex gain it occurs as $e^{-i\phi}$.

Even these smooth log-log plots are not quite the thing that bode introduced. He provided quick & efficient ways to sketch these plots, or piecewise linear approximations of them.

Pole diagrams

1) Determine the pole/zero diagram of the system function of an LTI system, and use it to sketch both the modulus of the system function & the amplitude response curve.

2) Identify the resonance frequency, stability & gain of an LTI system from the pole diagram of its transfer function.

3) Use the location of poles & zeros to determine the amplitude response.

Higher goal:

Confront higher order systems without (அச்சமின்றி) fear and appreciate the potential complexity of their frequency response.

Poles of the transfer functions

The transfer function formalism makes our notation neater, and broadens its scope: s rather than iω and it gives information about more general system responses, but this generalization of G(ω) is extremely useful in understanding the complex gain itself, as well.

The transfer function is fairly complicated gadget. You can't really graph it! It takes a complex number as input, so its graph lies over the complex plane. and it produces a complex number at output, which needs another 2d to represent. So we can't graph in 3 dimensions.

Most useful part of the complex gain — Is its magnitude.