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ORIGINAL ARTICLE

Operational complexity and pumping lemmas

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Abstract
The well-known pumping lemma for regular languages states that, for any regular language L, there is a constant p (depending on L) such that the following holds: If $w \in L$ and $|w| \geq p$, then there are words $x \in V^*$, $y \in V^+$, and $z \in V^*$ such that $w = xyz$ and $xy^tz \in L$ for $t \geq 0$. The minimal pumping constant $\mathrm{mpc}(L)$ of L is the minimal number p for which the conditions of the pumping lemma are satisfied. We investigate the behaviour of mpc with respect to operations, i. e., for an n-ary regularity preserving operation $\circ$, we study the set $g_\circ^{\mathrm{mpc}}(k_1, k_2, \ldots, k_n)$ of all numbers k such that there are regular languages $L_1, L_2, \ldots, L_n$ with $\mathrm{mpc}(L_i) = k_i$ for $1 \leq i \leq n$ and $\mathrm{mpc}(\circ(L_1, L_2, \ldots, L_n) = k$. With respect to Kleene closure, complement, reversal, prefix and suffix-closure, circular shift, union, intersection, set-subtraction, symmetric difference,and concatenation, we determine $g_\circ^{\mathrm{mpc}}(k_1, k_2, \ldots, k_n)$ completely. Furthermore, we give some results with respect to the minimal pumping length where, in addition, $|xy| \leq p$ has to hold.

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1 Introduction

In the last 30 years, the behaviour of complexities of languages under operations was studied very intensively. More precisely, for a complexity measure K on regular languages and a regularity preserving n-ary function f on languages, one was (and is) interested in the set $g_f^K(k_1, k_2, \ldots, k_n)$ of all numbers k such that there are regular languages $L_1, L_2, \ldots, L_n$ with $K(L_i) = k_i$ for $1 \leq i \leq n$ and $K(f(L_1, L_2, \ldots, L_n)) = k$. The sets $g_f^K(k_1, k_2, \ldots, k_n)$, $k_i \geq 0$ for $1 \leq i \leq n$, are called the ranges of f and K.

The investigation started with respect to the state complexity sc where $\mathrm{sc}(L)$ is defined as the minimal number of states of a deterministic finite automata which accepts L, and most of the research on this topic concerns the measure sc. The paper [4] gives a summary of results.

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We mention

$$g_{\cup}^{\mathrm{sc}}(m,n) = \{1, 2, \ldots, mn\} \text{ for } m \geq 2 \text{ and } n \geq 2$$

(see [10]) as an example and note that the ranges are finite for the operations union, complement, intersection, set-subtraction, symmetric difference, product, Kleene star, Kleene-+, reversal, circular closure, right quotient and left quotient.

In 2003, Holzer and Kutrib started the investigation with respect to the nondeterministic state complexity nsc where $\mathrm{nsc}(L)$ is defined as the minimal number of states of a nondeterministic finite automata which accepts L (see [5, 6]). Further results on the behaviour of nsc under operations can be found in [10] and [11]. Again, for all considered operations, the ranges are finite.

In the papers [2] and [3], the measure $\mathrm{Var}(L)$ which is defined as the minimal number of nonterminals in a right linear grammar which generate L was considered. It was shown that often the behaviour of Var is nearly related to the behaviour of nsc, however, the ranges of Var and circular closure or right and left quotient are infinite.

The situation changes drastically, if we consider the parameter asc, which is defined as the minimal number of accepting states of finite automata. In the papers [1] and [9], it was shown that the ranges for asc and intersection are finite, whereas they are infinite for union, complement, set-subtraction, symmetric difference, product, Kleene-*, Kleene-+, reversal, right and left quotient.

In this paper, we consider two parameters associated with classical pumping lemmas: the minimal pumping constant, which is the smallest number p such that all words in L of length $\geq p$ allow a pumping of a non-empty subword, and the minimal pumping length, which is the smallest number q such that all words in L of length $\geq q$ can be written as xyz with $|xy| \leq q$, $y \neq \lambda$, and $xy^tz \in L$ for all $t \geq 0$. We continue the research along the lines mentioned above with respect to these two parameters.

Concerning the minimal pumping constant, we present complete results with respect to union, set-subtraction, symmetric difference, intersection, concatenation, reversal, complement, Kleene closure, prefix and suffix closure, and circular shift. Moreover, we give (often partial) results concerning the minimal pumping length.

We mention that the ranges for the minimal pumping constant are finite for union, product, Kleene star, and reversal, whereas they are infinite for complement, set-subtraction, and symmetric difference, and prefix and suffix closure. With respect to the minimal pumping length, the ranges are finite for union, product, Kleene closure, and prefix-closure, but infinite for complement, subtraction, symmetric difference, and reversal. Thus, with respect to finiteness/infinity of ranges, we have obtained parameters, which are intermediate between state complexity and the measure number of accepting states.

2 Definitions

We start with a notation. By $\mathbb{N}$ and $\mathbb{N}_+$, we denote the set of all non-negative and all positive integers, respectively.

We recall a form of the classical pumping lemma (see for instance [7, 13]).

Lemma 1 *Let L be a regular language over V. Then, there is a constant p (depending on L) such that the following holds: If $w \in L$ and $|w| \geq p$, then there are words $x \in V^*$, $y \in V^+$, and $z \in V^*$ such that $w = xyz$ and $xy^tz \in L$ for $t \geq 0$.*

We say that $w \in L \subseteq V^*$ allows pumping if there are words $x \in V^*$, $y \in V^+$, and $z \in V^*$ such that $w = xyz$ and $xy^tz \in L$ for $t \geq 0$, and we say that y can be pumped in w if $w = xyz$ and $xy^tz \in L$ for $t \geq 0$.

Definition 1 Let L be a regular language over V. We define the minimal pumping constant $\mathrm{mpc}(L)$ as the minimal number p satisfying the conditions of Lemma 1.

In Lemma 1 and consequently also in Definition 1, one is only interested in lengths which allow a pumping somewhere in the word. We now give a stronger version of the pumping lemma where also the position and length of the pumped word matter (see for instance [8, 14, 15]).

Lemma 2 *Let L be a regular language over V. Then, there is a constant p (depending on L) such that the following holds: If $w \in L$ and $|w| \geq p$, then there are words $x \in V^*$, $y \in V^+$, and $z \in V^*$ such that $w = xyz$, $|xy| \leq p$, and $xy^tz \in L$ for $t \geq 0$.*

The additional requirement $|xy| \leq p$ gives the information that a word that can be pumped can be found in the prefix of length p.

Definition 2 Let L be a regular language over V. We define the minimal pumping length $\mathrm{mpl}(L)$ as the minimal number p satisfying the conditions of Lemma 2.

By our knowledge, the minimal pumping constant and the minimal pumping length are not studied, but they occur in exercises of some textbooks (e. g., [14, Exercises 1.55]).

2.1 Examples

We show how to compute the minimal pumping constant and the minimal pumping length of various families of languages.

Example 1 Let n and r be two natural numbers and

$$L_1 = \{a^n\}^*\{b^r\} \text{ and } L_1' = \{b^r\}\{a^n\}^*.$$

Then, we get

$$\mathrm{mpc}(L_1) = \mathrm{mpc}(L_1') = r + 1,$$
$$\mathrm{mpl}(L_1) = \begin{cases} n & \text{for } n > r \\ r+1 & \text{otherwise} \end{cases} \text{ and } \mathrm{mpl}(L_1') = n + r.$$

Since $b^r \in L_1$ and b^r do not allow a pumping (since pumping would produce infinitely many words without an occurrence of a), we get $\mathrm{mpc}(L_1) \geq r + 1$. On the other hand, any word w of length at least $r + 1$ in L_1 has the form $w = (a^n)^s b^r = a^{ns}b^r$ for some $s \geq 1$. Obviously, if we choose $x = \lambda$, $y = a^n$, and $z = a^{n(s-1)}b^r$, we obtain $w = xyz$ and $xy^tz = a^{(t+s-1)n}b^r \in L_1$ for $t \geq 0$ since $t + s - 1 \geq 0$. Therefore, all words of length at least $r + 1$ allow pumping, and we have $\mathrm{mpc}(L_1) = r + 1$.

If we choose a decomposition $w = xyz$ with $y = a^m$ for some m with $0 < m < n$, then xy^tz does not belong to L_1 for some $t \geq 0$. Hence, we have n as the minimal length of a word that can be pumped. Now the statement for $\mathrm{mpl}(L_1)$ follows from the pumping given above for mpc.

Analogously to the above arguments, we can prove that $\mathrm{mpc}(L_1') = r + 1$.

As above, a^n is the word of minimal length that can be pumped. However, then we get $w = b^r(a^n)^s = xyz$ with $x = b^r$, $y = a^n$, and $z = a^{n(s-1)}$ and $|xy| = n + r$. Thus, $\mathrm{mpl}(L_1') = n + r$.

Example 2 If L is a finite non-empty language and $r = \max\{|w| \mid w \in L\}$, then $\mathrm{mpc}(L) = \mathrm{mpl}(L) = r + 1$.

This assertion follows from the facts
– that a pumping of a non-empty subword of a word of length r in L would lead to words longer than r which are not in L and
– that all words properly longer than r are not in L, and therefore the pumping conditions are satisfied for such words (since the premise is false).

Example 3 Let $n \geq 1$ and $K = \{a^n\}^*$. Then, we have $\mathrm{mpc}(K) = 1$ and $\mathrm{mpl}(K) = n$.

Choosing $(a^n)^s = xyz$ with $x = \lambda$, $y = a^n$, and $z = a^{n(s-1)}$ for $s \geq 1$, we see that each non-empty word in K allows pumping. Obviously, the empty word cannot be pumped (since the y has to be non-empty). Thus, $\mathrm{mpc}(K) = 1$.

Let $a^m = xyz \in K$. If $|y| < n$, then there is a $t \geq 0$ such that $xy^tz \notin K$ (because $m = ns$ for some s by $a^m \in K$ and $ms = |xyz| < |xy^2z| = ms + |y| < m(s + 1)$). This gives $\mathrm{mpl}(K) \geq n$. By the argument for mpc, we see that this bound is tight.

Example 4 Let n be an integer with $n \geq 2$. Further, let R be the set of all words over $\{a, b\}$ which end with ba^{n-2}, i.e., $R = \{wba^{n-2} \mid w \in \{a, b\}^*\}$, and R' the set of all words over $\{a, b\}$ which do not end with ba^{n-2}, i.e., R' consists of all words with a length smaller than $n - 1$ and all words w_1w_2 with $w_1, w_2 \in \{a, b\}^*$, $|w_2| = n - 1$, and $w_2 \neq ba^{n-2}$. Then, we have

$$\mathrm{mpc}(R) = \mathrm{mpl}(R') = n \text{ and } \mathrm{mpc}(R) = \mathrm{mpl}(R') = 1.$$

Assume that $ba^{n-2} = xyz$ for some words x, y, z with $y \neq \lambda$. Then, we get $|xz| < n - 1$ and $xz \notin R$. Thus, $ba^{n-1} \in R$ allows no pumping. Hence, $\mathrm{mpc}(R) \geq n$ and $\mathrm{mpl}(R) \geq n$.

If $v = cwba^{n-2}$ with $c \in \{a, b\}$ and $w \in \{a, b\}^*$ is a word of R of length $\geq n$, then c^twba^{n-2} is in R for all $t \geq 0$. Thus, the decomposition $v = xyz$ with $x = \lambda$, $y = c$ and $z = wba^{n-2}$ satisfies all conditions of Lemmas 1 and 2. Therefore, $\mathrm{mpc}(R) = \mathrm{mpl}(R) = n$.

Let $u = cw$ be a word of R' with $c \in \{a, b\}$ and $1 \leq |cw| = k < n - 1$ (note that this case is only possible if $n \geq 3$). We choose $x = \lambda$, $y = c$, and $z = w$. Then, $xy^0z = w \in R'$ since w has a length $< n - 1$. Moreover, $xy^tz = c^tw$ has also a length $< n - 1$ (if $1 \leq t \leq n - k - 1$) or $c^tw = c^{t-n-k}ccc^{n-2-k}w$ (if $t \geq n - k$). In both cases, we get $c^tw \in R'$.

Let cw be a word of R' with $c \in \{a, b\}$ and $|cw| = n - 1$. Then, $cw \neq ba^{n-2}$. Again, we choose $x = \lambda$, $y = c$, and $z = w$. Then, $xy^0z = w$ has a length $< n - 1$. Thus, $xy^0z \in R'$. For $t \geq 1$, we have $xy^tz = c^tw \in R'$ since it does not end with ba^{n-1}.

Let cw be a word in R' with $c \in \{a, b\}$ and $|cw| \geq n$. Then, w has a length at least $n - 1$ and does not end with ba^{n-1}. Obviously, all words c^tw, $t \geq 0$ also do not end with ba^{n-1}. Thus, $c^tw \in R'$ for all t.

Therefore, the first letter can be pumped in all words of R'. Consequently, $\mathrm{mpc}(R') = \mathrm{mpl}(R') = 1$.

Example 5 For $n \geq 1$ and even $k \geq 1$, we set

$$L_{n,k} = \{a^{2s} \mid s \geq 0\} \cup \{a^r \mid r \geq k + 1\} \cup \{b\}^* \cup \{c^{n-1}\},$$

and for $n \geq 1$ and odd $k \geq 1$, we set

$$L'_{n,k} = \{a^{2s+1} \mid s \geq 0\} \cup \{a^r \mid r \geq k + 1\} \cup \{b\}^+ \cup \{c^{n-1}\}.$$

Then, for every $n, k \geq 1$, we have

$$\mathrm{mpc}(L_{n,k}) = n, \mathrm{mpl}(L_{n,k}) = \mathrm{mpc}(L'_{n,k}) = \mathrm{mpl}(L'_{n,k}) = \max\{2, n\}.$$

i) First we prove that $\mathrm{mpc}(L_{n,k}) = n$. Because c^{n-1} cannot be pumped, we obtain $\mathrm{mpc}(L_{n,k}) \geq n$.

All other non-empty words in $L_{n,k}$ can be pumped which can be seen as follows:

- If $w = a^{2s}$ with $2s \geq n \geq 1$, we choose $x = \lambda$, $y = a^2$, and $z_1 = a^{2s-2}$, which satisfies $xyz = w$, $y \neq \lambda$, and $xy^t z = a^{2(t+s-1)} \in L_n$ for $t \geq 0$.
- If $w = a^r$ for some odd $r \geq k+1$, then we choose $x' = \lambda$, $y' = a$ and $z' = a^{r-1}$, which satisfies $x'y'z' = w$, $y' \neq \lambda$, and $x'(y')^t z' = a^{t+r-1} \in L_n$ (if $t = 0$, then $r-1$ is even; and if $t \geq 1$, then $t + r - 1 \geq k + 1$).
- If $w = b^r$ for some $r \geq 1$, then we can pump the first letter of w.

Therefore $\mathrm{mpc}(L_{n,k}) = n$.

ii) We consider $\mathrm{mpl}(L_{n,k})$. Since c^{n-1} cannot be pumped, we have $\mathrm{mpl}(L_{n,k}) \geq n$. In the cases considered above, at most the first two letters of w are involved in the pumping. Moreover, the pumping of a^2 requires $|xy| = |a^2| = 2$. Thus, $\mathrm{mpl}(L_{n,k}) \geq 2$.

iii) We consider $L'_{n,k}$. We cannot pump a, b, and c^{n-1}, but all other words of $L'_{n,k}$ have a length ≥ 2 and can be pumped which can be shown as in i). Thus, $\mathrm{mpc}(L'_{n,k}) = \max\{2, n\}$. Since only the first two letters of a word are involved in the pumping, we also get $\mathrm{mpl}(L'_{n,k}) = \max\{2, n\}$

2.2 Basic facts

By the classical proof of the pumping lemma(s), it is known that the number of states of an automaton accepting L satisfies the conditions of the pumping lemma(s). Looking on the Definitions 1 and 2, for any language L, we obtain

$$\mathrm{mpc}(L) \leq \mathrm{mpl}(L) \leq \mathrm{sc}(L). \tag{1}$$

The following theorem gives a more detailed relation for these parameters.

Theorem 1 *Let m_1, m_2, and m_3 be three natural numbers with $1 \leq m_1 \leq m_2 \leq m_3$. Then, there is a language L such that*

$$\mathrm{mpc}(L) = m_1, \ \mathrm{mpl}(L) = m_2, \ \textit{and} \ \mathrm{sc}(L) = m_3.$$

Proof If $m_2 < m_3$, we set

$$L = \{b^r\}\{a^n\}^* \cup \{c_1\}^* \cup \{c_2\}^* \cup \cdots \cup \{c_{k-1}\}^*$$

with $m_1 = r + 1$, $m_2 = m_1 + n - 1$, and $m_3 = m_2 + k$ (note that $r \geq 0$, $n \geq 1$, and $k \geq 1$).

Since in a word c_i^j, $1 \leq i \leq k$, $j \geq 1$, we can pump the first letter, we get $\mathrm{mpc}(L) = r + 1 = m_1$ and $\mathrm{mpl}(L) = n + r = n + m_1 - 1 = m_2$ by arguments as in Example 1 for L'_1.

The automaton $\mathcal{A}$ with the set

$$\{q_0, q_1, \ldots, q_{r-1}, q_r, q'_1, q'_2, \ldots, q'_{n-2}, q'_{n-1}, q''_1, q''_2, \ldots, q''_{k-2}, q''_{k-1}, q\}$$

of states, the initial state q_0, the set $\{q_r, q_1'', q_2'', \ldots, q_{k-1}''\}$ of accepting states, and the transition function δ defined by

$$\delta(z,x) = \begin{cases} q_{i+1} & \text{for } z = q_i,\ 0 \le i \le r-1\} \text{ and } x = b \\ q_1' & \text{for } z = r \text{ and } x = a \\ q_{j+1}' & \text{for } z = q_j',\ 1 \le j \le n-2, \text{ and } x = a \\ q_r & \text{for } z = q_{n-1}' \text{ and } x = a \\ q_s'' & \text{for } z \in \{q_0, q_s''\} \text{ and } x = c_s,\ 1 \le s \le k-1 \\ q & \text{otherwise} \end{cases}$$

accepts L and is minimal (which can be proved by standard methods). Thus, $\mathrm{sc}(L_1) = n + r + k = m_3$.

If $m_2 = m_3$, we set $K = \{a^{m_2}\}^*\{a^{m_1-1}\}$. Then, we obtain $\mathrm{mpc}(K) = m_1$ and $\mathrm{mpl}(K) = m_2$ by arguments analogous to those used in Examples 1 and 3. The minimal automaton for K is given by m_2 states $q_0, q_1, \ldots, q_{m_2-1}$, the initial state q_0, the only accepting state q_{m_1-1} and the transition $\delta(q_i, a) = q_{i+1 \bmod m_2}$. □

We give some simple facts.

Lemma 3 *i) If L is a non-empty language and $r = \min\{|w| \mid w \in L\}$, then we have* $\mathrm{mpc}(L) \ge r + 1$.

ii) A language L satisfies $\mathrm{mpc}(L) = 0$ *if and only if* $\mathrm{mpl}(L) = 0$ *if and only if L is empty.*

Proof i) Assume that $\mathrm{mpc}(L) \le r$. Let w be a word in L with $|w| = r$. Then, there is a decomposition $w = xyz$ with words x, y, z such that $y \ne \lambda$ and $xy^tz \in L$ for all $t \ge 0$. This leads to the contradiction of $xz = xy^0z \in L$ and $xz \notin L$ since $|xz| < r$.

ii) Obviously, if L is empty, then all words w do not satisfy $w \in L$, and thus the pumping conditions hold for all words (its premise is false). Thus, $\mathrm{mpc}(L) = 0$ and $\mathrm{mpl}(L) = 0$.

Conversely, if L is non-empty, then $\mathrm{mpc}(L)$ is greater than the minimal length of a word in L by i).

Thus, we get $\mathrm{mpc}(L) = 0$ if and only if $L = \emptyset$.

The statement for mpl now follows from $\mathrm{mpc}(L) \le \mathrm{mpl}(L)$. □

Lemma 4 *i) For all languages L,* $\mathrm{mpc}(L) = 1$ *implies $\lambda \in L$.*

ii) Let $L \subseteq V^*$ be a language with $\mathrm{mpl}(L) = 1$. *Then, L is suffix-closed. Moreover, if $w_1aw_2 \in L$ for some $a \in V$ and $w_1, w_2 \in V^*$, then $a^tw_2 \in L$ for all $t \ge 0$.*

Proof i) Let us assume that $\mathrm{mpc}(L) = 1$ and $\lambda \notin L$. Let w be a word of minimal length in L. Then, there are words x, $y \ne \lambda$, and z such that $w = xyz$ and $xy^tz \in L$ for all $t \ge 0$. Therefore, xz is in L ($t = 0$). Because $\lambda \notin L$, $0 < |xz| < |w|$ which contradicts our choice of w.

Thus, $\lambda \in L$ has to hold.

ii) Let $aw \in L$ for some $a \in V$ and $w \in V^*$. Since $\mathrm{mpl}(L) = 1$, there are words x, $y \ne \lambda$, and z such that $aw = xyz$ and $xy^tz \in L$ for all $t \ge 0$. By $\mathrm{mpl}(L) = 1$, we get $x = \lambda$ and $y = a$. Thus, $a^tw \in L$ for all $t \ge 0$. Especially, for $t = 0$, we obtain $w \in L$. Repetitions of this argument prove the suffix closure.

Let $w_1aw_2 \in L$. By the closure under suffixes, we get $aw_2 \in L$. Furthermore, $\mathrm{mpl}(L) = 1$ gives $a^tw_2 \in L$ for all $t \ge 0$. □

We now define the central concept of this paper, the range of the pumping parameters under operations.

Definition 3 Let $\circ$ be a regularity preserving n-ary function on languages and $K \in \{\text{mpc}, \text{mpl}\}$. Then, we define $g_{\circ}^{K}(k_1, k_2, \ldots, k_n)$ as the set of all numbers k such that there are regular languages $L_1, L_2, \ldots, L_n$ with $K(L_i) = k_i$ for $1 \le i \le n$ and $K(\circ(L_1, L_2, \ldots, L_n)) = k$.

3 Behaviour under unary operations

In this section, we study the behaviour of the measures mpc and mpl for regular languages under standard unary language operations: Kleene closure, reversal, complement, prefix closure, suffix closure and cyclic shift.

3.1 Kleene closure

The Kleene closure of a language $L \subseteq V^*$ is the language

$$L^* = \{\lambda\} \cup \{w_1 w_2 \cdots w_m | w_i \in L \text{ for } 1 \le i \le m\}.$$

Theorem 2 *i) We have* $g_*^{\text{mpc}}(n) = \{1\}$ *for every* $n \ge 0$.
ii) We have $g_*^{\text{mpl}}(n) = \begin{cases} \{1, 2, \ldots, n\} & \text{for } n \ge 1; \\ \{1\} & \text{for } n = 0. \end{cases}$

Proof i) We begin by proving the result concerning the minimal pumping constant. Let us consider a language $L \subseteq V^*$. By Lemma 3, we have that $\text{mpc}(L) = 0$ if and only L is empty, which implies $L^* = \{\lambda\}$ thus $\text{mpc}(L^*) = 1$ as showed in Example 2. This proves that $g_*^{\text{mpc}}(0) = \{1\}$. Now suppose that L is non-empty, and let w be a non-empty word in L^*. Then, $w = w_1 w_2 \cdots w_m$ for some $m \ge 1$ and nonempty words $w_i \in L$ for $1 \le i \le m$. By choosing $x = \lambda$, $y = w_1$, and $z = w_2 \cdots w_m$, we get $xy^t z = w_1^t w_2 \cdots w_m \in L^*$ for all $t \ge 0$ (if $m = 1$ and $t = 0$, we obtain the empty word, which is in L^*). Thus, any non-empty word in L^* allows a pumping, which proves that $\text{mpc}(L^*) = 1$.

ii) We now prove the result concerning the minimal pumping length. As in the first part, combining Lemma 3 and the fact that $\emptyset^* = \{\lambda\}$ yields $g_*^{\text{mpl}}(0) = \{1\}$. Let us fix $n \ge 1$. We proceed in two steps to show that $g_*^{\text{mpl}}(n) = \{1, 2, \ldots, n\}$: First, we prove that neither 0 nor any integers greater than n appears in $g_*^{\text{mpl}}(n)$. Then, to show that all the remaining values can be obtained, we build, for every $k \in \{1, 2, \ldots, n\}$, a language $L_{n,k}$ that satisfies $\text{mpl}(L_{n,k}) = n$ and $\text{mpl}(L_{n,k}^*) = k$.

For every $L \subseteq V^*$ that satisfies $\text{mpl}(L) = n$, since $\lambda \in L^*$, we have $\text{mpl}(L^*) \ge 1$ by Lemma 3. This implies $0 \notin g_*^{\text{mpl}}(n)$. We now show that every word $w \in L^*$ that satisfies $|w| \ge n$ allows a pumping. As $w \in L^*$, we have $w = w_1 w_2 \cdots w_m$ with $w_1, w_2, \ldots, w_m \in L$. We differentiate two cases:

- If $|w_1| \ge n$, as $\text{mpl}(L) = n$ there are x, y, z such that $w_1 = xyz$, $y \ne \lambda$, $|xy| \le n$, and $xy^t z \in L$ for all $t \ge 0$. Let $z' = zw_2 \cdots w_m$. Then, we also have that $w = xyz'$ and $xy^t z' = xy^t z w_2 \cdots w_m \in L^*$ for all $t \ge 0$.
- If $|w_1| \le n$, then we choose $x = \lambda$, $y = w_1$ and $z = w_2 \cdots w_m$ to get $w = xyz$, $y = w_1 \ne \lambda$, $|xy| = |w_1| \le n$, and $xy^t z = w_1^t w_2 \cdots w_m \in L^*$ for all $t \ge 0$.

This proves that $\text{mpl}(L^*) \le n$, which implies $g_*^{\text{mpl}}(n) \subseteq \{1, 2, \ldots, n\}$.

For all $k \in \{1, 2, \ldots, n\}$, we design a language $L_{n,k}$ that satisfies $\text{mpl}(L_{n,k}) = n$ and $\text{mpl}(L_{n,k}^*) = k$. If $k = n$, we choose $L_{n,k} = \{a^n\}^*$. Then, $\text{mpl}(L_{n,k}) = n$ by Example 3, and since $L_{n,k}^* = L_{n,k}$ we also have $\text{mpl}(L_{n,k}^*) = n = k$. Now let us suppose that k satisfies

$1 \le k \le n-1$. Then, there exist two positive integers r and s satisfying $k \cdot s + r = n - 1$ and $0 \le r \le k-1$, and we set

$$L_{n,k} = \{a^{kq}b^r \mid 0 \le q \le s\} \cup \{a^k\}.$$

Since $L_{n,k}$ is a finite language, we have $\mathrm{mpl}(L_{n,k}) = ks + r + 1 = n - 1 + 1 = n$ as in Example 2. We now prove that $\mathrm{mpl}(L_{n,k}^*) = k$. First, we have $\mathrm{mpl}(L_{n,k}^*) \ge k$ as the prefix of size $k-1$ of $w = a^k b^r \in L_{n,k} \subseteq L_{n,k}^*$ allows no pumping: Indeed, for every decomposition $w = xyz$ that satisfies $|xy| \le k-1$ and $y \ne \lambda$, we have $y = a^p$ with $1 \le p < k$, hence $xz = a^{k-p}b^r \notin L_{n,k}^*$. To prove that $\mathrm{mpl}(L_{n,k}^*) = k$, it remains to show that for every $w \in L_{n,k}^*$ that satisfies $|w| \ge k$, the prefix of size k of w allows a pumping. Let $w = w_1 w_2 \cdots w_m$ be such a word in $L_{n,k}^*$, with $m \ge 1$ and nonempty words $w_i \in L_{n,k}$ for $1 \le i \le m$. We consider two possible cases.

- If w starts with b, then $w_1 = b^r$. We choose $x = \lambda$, $y = b^r$ and $z = w_2 \cdots w_n$ to get $w = xyz$, $|xy| = r \le k$ and $xy^t z = (b^r)^t w_2 \cdots w_n \in L_{n,k}^*$ for all $t \ge 0$.
- If w starts with a, then either $w_1 = a^k$ or $w_1 = a^{kq}b^r$ with $q \ge 1$. In the former case, we choose $x = \lambda$, $y = w_1 = a^k$, and $z = w_2 \cdots w_m$. In the later case, we choose $x = \lambda$, $y = a^k$ and $z = a^{k(q-1)}b^r w_2 \cdots w_m$. Then, in both cases, we get $|xy| \le k$ and $xy^t z \in L_{n,k}^*$ for all $t \ge 0$. □

3.2 Reversal

The following results concern reversal (defined as follows: we set $\lambda^R = \lambda$; for $a \in V$, we set $a^R = a$, and for $w_1, w_2 \in V^*$, we set $(w_1 w_2)^R = w_2^R w_1^R$, and for $L \subseteq V^*$, we set $L^R = \{w^R \mid w \in L\}$).

Theorem 3 *We have $g_R^{\mathrm{mpc}}(n) = \{n\}$ for all $n \ge 0$.*

Proof Since $\emptyset^R = \emptyset$, Lemma 3 yields $g_R^{\mathrm{mpc}}(0) = \{0\}$.

Let L be a non-empty language with $\mathrm{mpc}(L) = n$. Moreover, let w be a word in L^R with $|w| \ge n$. Then, $w^R \in L$ and $|w^R| \ge n$. Therefore, there are words x, y, z such that $xyz = w^R$, $y \ne \lambda$, and $xy^t z \in L$ for all $t \ge 0$. Thus, $w = (w^R)^R = z^R y^R x^R$, $y^R \ne \lambda$ and $z^R (y^R)^t x^R = (xy^t z)^R \in L^R$. This proves $\mathrm{mpc}(L^R) \le n = \mathrm{mpc}(L)$. Conversely, $\mathrm{mpc}(L) = \mathrm{mpc}((L^R)^R) \le \mathrm{mpc}(L^R) \le \mathrm{mpc}(L)$, thus $\mathrm{mpc}(L^R) = n$ holds. □

Theorem 4 *We have* $g_R^{\mathrm{mpl}}(n) = \begin{cases} \{0\} & \text{for } n = 0; \\ \mathbb{N}_+ & \text{for } n \ge 1. \end{cases}$

Proof By Lemma 3, $\mathrm{mpl}(L^R) = 0$ if and only if $L^R = \emptyset$ if and only if $L = \emptyset$ if and only if $\mathrm{mpl}(L) = 0$. Therefore, we get $0 \notin g_R^{\mathrm{mpl}}(n)$ for $n \ge 1$ and $g_R^{\mathrm{mpl}}(0) = \{0\}$.

Now let $n \ge 1$ and $k \ge 1$ be two integers. We consider

$$\begin{aligned} L_{n,k} &= \{b^{n-1}\}\{a\}^*\{b^{k-1}\} \cup \{b\}^*; \\ L_{n,k}^R &= \{b^{k-1}\}\{a\}^*\{b^{n-1}\} \cup \{b\}^*. \end{aligned}$$

Analogous to the arguments given in Example 1, we get that $\mathrm{mpl}(L_{n,k}) = n$ and $\mathrm{mpl}(L_{n,k}^R) = k$. Thus, $k \in g_R^{\mathrm{mpl}}(n)$ for all $k \ge 1$. □

3.3 Complement

We now turn to the operation complement. Let V be an alphabet. Then, for a non-empty language $L \subseteq V^*$, we set $C_V(L) = V^* \backslash L$.

We begin by showing that complementing a language with respect to a larger alphabet has no impact on the value of mpc and mpl. Let V be an alphabet and U a subalphabet of V, i. e., $V = U \cup W$. Moreover, let L be a language with $L \subseteq U^*$. Then, $C_V(L) = C_U(L) \cup V^*WV^*$. Since pumping of the first letter of a word in V^*WV^* gives a word in V^*WV^*, again, for $K \in \{\text{mpc}, \text{mpl}\}$, we get immediately

$$K(C_V(L)) = K(C_U(L)). \tag{2}$$

Theorem 5 *Let V be an alphabet with at least 2 letters. Then, we have*

$$g_{C_V}^{\text{mpc}}(n) = g_{C_V}^{\text{mpl}}(n) = \begin{cases} \{1\} & \textit{for } n = 0; \\ \mathbb{N}\backslash\{1\} & \textit{for } n = 1; \\ \mathbb{N}_+ & \textit{for } n \geq 2. \end{cases}$$

Proof By (2), it is sufficient to prove the statement for $V = \{a, b\}$. We first focus on some specific cases that can be derived from the previous lemmas, and then we build witness languages for the remaining cases.

i) As stated in Lemma 3, $\text{mpc}(L) = 0$ if and only if $\text{mpl}(L) = 0$ if and only if $L = \emptyset$. Moreover, $\text{mpc}(V^*) = \text{mpl}(V^*) = 1$. Therefore, since $C_V(\emptyset) = V^*$, we have $g_{C_V}^{\text{mpc}}(0) = g_{C_V}^{\text{mpl}}(0) = \{1\}$. Conversely, as $L = V^*$ is the language that satisfies $C_V(L) = \emptyset$, we have $0 \in g_{C_V}^{\text{mpc}}(n)$ if and only if $0 \in g_{C_V}^{\text{mpl}}(n)$ if and only if $n = 1$.

Furthermore, as stated in Lemma 4, every language $L \subseteq V^*$ that satisfies $\text{mpc}(L) = 1$ or $\text{mpl}(L) = 1$ contains the empty word λ. As a consequence, there is no language that satisfies $\text{mpc}(L) = \text{mpc}(C_V(L)) = 1$ or $\text{mpl}(L) = \text{mpl}(C_V(L)) = 1$, which in turn implies $1 \notin g_{C_V}^{\text{mpc}}(1) \cup g_{C_V}^{\text{mpl}}(1)$.

ii) It now remains to show that for every pair of integers $n, k \geq 1$ such that k and n are not both equal to 1, $k \in g_{C_V}^{\text{mpc}}(n) \cap g_{C_V}^{\text{mpl}}(n)$. To this end, we design a language $L_{n,k}$ satisfying $\text{mpc}(L_{n,k}) = \text{mpl}(L_{n,k}) = n$ and $\text{mpc}(C_V(L_{n,k})) = \text{mpl}(C_V(L_{n,k})) = k$. Note that, as in general $C_V(C_V(L)) = L$, it is sufficient to define the witness languages in the cases where $n \geq 2$ and $n \geq k$. In the remaining cases (that is, for $k \geq 2$ and $k > n$), we can then set $L_{n,k} = C_V(L_{k,n})$. We consider three possible cases.

- If $n = 2$ and $k = 1$, we set

$$L_{2,1} = \{a, b\}^* \backslash \{\lambda\}; \quad C_V(L_{2,1}) = \{\lambda\}.$$

  Then, $\text{mpc}(L_{2,1}) = \text{mpl}(L_{2,1}) = 2$ since the words $a, b \in L_{2,1}$ cannot be pumped, yet we can pump the first letter of every other word in $L_{2,1}$. Furthermore, $\text{mpc}(C_V(L_{2,1})) = \text{mpl}(C_V(L_{2,1})) = 1$.
- If $n = 2$ and $k = 2$, we set

$$L_{2,2} = \{a, b\}; \quad C_V(L_{2,2}) = \{a, b\}^* \backslash \{a, b\}.$$

  We have $\text{mpc}(L_{2,2}) = \text{mpl}(L_{2,2}) = 2$ as shown in Example 2. Moreover, $\text{mpc}(C_V(L_{2,2})) = \text{mpl}(C_V(L_{2,2})) = 2$ since we can pump the first two letters of each word in $C_V(L_{2,1})$.
- Finally, suppose that we have $n \geq 3$ and $n \geq k$. We set

$$L_{n,k} = \{a^s b^{n-2} \mid s \geq 1\} \cup \{a^s \mid 0 \leq s < k - 1\}.$$

We begin by showing that $\mathrm{mpc}(L_{n,k}) = \mathrm{mpl}(L_{n,k}) = n$. First, we have $n \leq \mathrm{mpc}(L_{n,k}) \leq \mathrm{mpc}(L_{n,k})$ as the word $w = ab^{n-2} \in L_{n,k}$ allows no pumping: for every decomposition $w = xyz$ satisfying $y \neq \lambda$, either y contains at least one b and $xy^2z \notin L_{n,k}$, or $y = a$ and $xz = b^{n-2} \notin L_{n,k}$. Furthermore, we have $\mathrm{mpc}(L_{n,k}) \leq \mathrm{mpl}(L_{n,k}) \leq n$ as we can pump each word w in $L_{n,k}$ of length at least n: since $n \geq k$, such a word is necessarily of the form $w = a^s b^{n-2}$ with $s \geq 2$, and by setting $x = \lambda$, $y = a$ and $z = a^{s-1}b^{n-2}$ we get that $xy^tz = a^{s+t-1}b^{n-2} \in L_{n,k}$ for all $t \geq 0$.
We now show that $\mathrm{mpc}(C_V(L_{n,k})) = \mathrm{mpl}(C_V(L_{n,k})) = k$. First, we have $k \leq \mathrm{mpc}(C_V(L_{n,k})) \leq \mathrm{mpl}(C_V(L_{n,k}))$ since $w = a^{k-1} \in C_V(L_{n,k})$ allows no pumping: every decomposition $w = xyz$ such that $y \neq \lambda$ yields $xz \in L_{n,k}$ thus $xz \notin C_V(L_{n,k})$. Conversely, we have $\mathrm{mpc}(C_V(L_{n,k})) \leq \mathrm{mpl}(C_V(L_{n,k})) \leq k$: for every $w \in C_V(L_{n,k})$ of length at least $k \geq 2$, if w starts with the letter b we can pump its second letter, and if w starts with an a we can pump its first letter.

□

3.4 Prefix and suffix closures

The prefix and suffix closures of a language $L \subseteq V^*$ are defined as follows:

$$\begin{aligned} \mathrm{Pref}(L) &= \{x \mid xy \in L \text{ for some } y \in V^*\}; \\ \mathrm{Suff}(L) &= \{x \mid yx \in L \text{ for some } y \in V^*\}. \end{aligned}$$

For every $n \in \mathbb{N}$ and $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$, we now determine $g^K_{\mathrm{Pref}}(n)$ and $g^K_{\mathrm{Suff}}(n)$. Remark that Lemma 3 immediately provides a solution for the specific integer 0: Since a language L is empty if and only if $\mathrm{Pref}(L) = \emptyset$ if and only if $\mathrm{Suff}(L) = \emptyset$, we have $g^K_{\mathrm{Pref}}(0) = g^K_{\mathrm{Suff}}(0) = \{0\}$ and $g^K_{\mathrm{Pref}}(n), g^K_{\mathrm{Suff}}(n) \subseteq \mathbb{N}_+$ for all $n \geq 1$. In the following proofs, we focus on the remaining cases.

Theorem 6 *We have*

$$g^{\mathrm{mpc}}_{\mathrm{Pref}}(n) = g^{\mathrm{mpc}}_{\mathrm{Suff}}(n) = \begin{cases} \{0\} & \text{for } n = 0; \\ \mathbb{N}_+ & \text{for } n \geq 1. \end{cases}$$

Proof First, we show that $g^{\mathrm{mpc}}_{\mathrm{Pref}}(n) = \mathbb{N}_+$ for all $n \geq 1$. For all $k \in \mathbb{N}_+$, we design a language $L_{n,k}$ satisfying $\mathrm{mpc}(L'_{n,k}) = n$ and $\mathrm{mpc}(\mathrm{Pref}(L_{n,k})) = k$.

For $k \leq n$, we define

$$\begin{aligned} L_{n,k} &= \{a\}^*\{b\} \cup \{b\}^* \cup \{a^{n-1}, c^{k-1}\}; \\ \mathrm{Pref}(L_{n,k}) &= \{a\}^* \cup \{a\}^*\{b\} \cup \{b\}^* \cup \{c, c^2, \ldots, c^{k-1}\}. \end{aligned}$$

Then, we have $\mathrm{mpc}(L_{n,k}) = n$ since a^{n-1} and c^{k-1} allow no pumping but the first letter of all non-empty words in $\{a\}^*\{b\} \cup \{b\}^*$ can be pumped. Moreover, $\mathrm{mpc}(\mathrm{Pref}(L_{n,k})) = k$ by a similar argument.

For $k > n$, we define

$$\begin{aligned} L_{n,k} &= \{a^{k-1}b\}^* \cup \{b^{n-1}\}; \\ \mathrm{Pref}(L_{n,k}) &= \{a^{k-1}b\}^*\{\lambda, a, a^2, \ldots, a^{k-1}\} \cup \{b, b^2, \ldots, b^{n-1}\}. \end{aligned}$$

Then, $\mathrm{mpc}(L_{n,k}) = n$ since b^{n-1} allows no pumping, yet we can pump the first occurrence of $a^{k-1}b$ in all words of $\{a^{k-1}b\}^+$. Similarly, $\mathrm{mpc}(\mathrm{Pref}(L_{n,k})) = k$ since a^{k-1} allows no pumping, yet in every longer word of $\mathrm{Pref}(L_{n,k})$ we can pump the first occurrence of $a^{k-1}b$.

Thus, any natural number $k \geq 1$ is in $g^{\mathrm{mpc}}_{\mathrm{Pref}}(n)$. Finally, because $\mathrm{Suff}(L) = (\mathrm{Pref}(L^R))^R$, we obtain the statement for $g^{\mathrm{mpc}}_{\mathrm{Suff}}(n)$ from Theorem 3. □

Theorem 7 *We have*

$$g^{\mathrm{mpl}}_{\mathrm{Pref}}(n) = \begin{cases} \{0\} & \text{for } n = 0; \\ \{1, 2, \ldots, n\} & \text{for } n \geq 1. \end{cases}$$

Proof We begin by proving that $g^{\mathrm{mpl}}_{\mathrm{Pref}}(n) \subseteq \{1, 2, \ldots, n\}$ for all $n \geq 1$. Let L be a language satisfying $\mathrm{mpl}(L) = n \geq 1$. Further, let w be a word of length s with $s \geq n$ in $\mathrm{Pref}(L)$. Then, there is a word u such that $wu \in L$ and words x, y, and z such that $xyz = wu$, $|xy| \leq n$, $y \neq \lambda$, and $xy^tz \in L$ for all $t \geq 0$. From these properties, there is a word w' such that $w = xyw'$, $z = w'u$, and $xy^tw'u \in L$ for all $t \geq 0$. Hence, $xy^tw' \in Pref(L)$ for all $t \geq 0$, i. e., the pumping conditions for w are satisfied. This proves that $\mathrm{mpl}(\mathrm{Pref}(L)) \leq n$.

To conclude the proof, remark that we can use the languages $L_{n,k}$ for $k \leq n$ defined in the proof of Theorem 6 to show that every $k \in \{1, 2, \ldots, n\}$ is in $g^{\mathrm{mpl}}_{\mathrm{Pref}}(n)$. □

Theorem 8 *We have*

$$g^{\mathrm{mpl}}_{\mathrm{Suff}}(n) = \begin{cases} \{0\} & \text{for } n = 0; \\ \{1\} & \text{for } n = 1; \\ \mathbb{N}_+ & \text{for } n \geq 2. \end{cases}$$

Proof First, by Lemma 4, for every language L satisfying $\mathrm{mpl}(L) = 1$, we have $\mathrm{Suff}(L) = L$, hence $g^{\mathrm{mpl}}_{\mathrm{Suff}}(1) = \{1\}$. For every $n \geq 2$, we now show that $g^{\mathrm{mpl}}_{\mathrm{Suff}}(n) = \mathbb{N}_+$ by designing a witness language $L_{n,k}$ for every $k \in \mathbb{N}_+$. We proceed in three steps.

i) In the specific case $k = 1$, let $L_{n,1} = \{a^s \mid s \geq n-1\}$. Then, $\mathrm{mpl}(L_{n,1}) = n$ since we cannot pump a subword of a^{n-1}, but we can pump the first letter of all words a^s with $s \geq n$. Furthermore, $\mathrm{Suff}(L_{n,1}) = \{a\}^*$ which implies $\mathrm{mpl}(\mathrm{Suff}(L_{n,1}) = 1$. Hence, $1 \in g^{\mathrm{mpl}}_{\mathrm{Suff}}(n)$ for all $n \geq 2$.

ii) For $n \geq k \geq 2$, let

$$\begin{aligned} L_{n,k} &= \{b\}\{a\}^* \cup \{b\}^* \cup \{a^{n-1}, c^{k-1}\}; \\ \mathrm{Suff}(L_{n,k}) &= \{a\}^* \cup \{b\}\{a\}^* \cup \{b\}^* \cup \{c, c^2, \ldots, c^{k-1}\}. \end{aligned}$$

As in the proof of Theorem 6, we obtain $\mathrm{mpl}(L_{n,k}) = n$ and $\mathrm{mpl}(\mathrm{Suff}(L_{n,k}) = k$.

iii) Finally, for $k > n \geq 2$, let

$$\begin{aligned} L_{n,k} &= \{a^s \mid s \geq n-1\} \cup \{ab\}^+\{c^{k-2}\} \cup \{c\}^+; \\ \mathrm{Suff}(L_{n,k}) &= \{a\}^* \cup \{ab\}^*\{c^{k-2}\} \cup \{ba\}^*\{bc^{k-2}\} \cup \{c\}^*. \end{aligned}$$

Then, $\mathrm{mpl}(L) = n$ since a^{n-1} allows no pump, yet we can pump the first letter in each word in $\{a\}^+ \cup \{c\}^+$ of length at least $n \geq 2$, and the first occurrence of ab in each word in $\{ab\}^+\{c^{k-2}\}$. Furthermore, $\mathrm{mpl}(\mathrm{Suff}(L)) = k$ since bc^{k-2} allows no pumping, yet in every other word of $\mathrm{Suff}(L)$ we can pump the first a, ab, ba or c. □

3.5 Cyclic shift

The last unary operation is cyclic shift, defined as follows. For a language $L \subseteq V^*$,

$$\mathrm{Circ}(L) = \{xy \mid yx \in L,\ x \in V^*,\ y \in V^*\}.$$

Theorem 9 *We have* $g_{\mathrm{Circ}}^{\mathrm{mpc}}(n) = \begin{cases} \{0\} & \textit{for } n = 0; \\ \mathbb{N}_+ \setminus \{2,3\} & \textit{for } n = 1; \\ \mathbb{N}_+ \setminus \{1,3\} & \textit{for } n = 2; \\ \mathbb{N}_+ & \textit{for } n \geq 3. \end{cases}$

Proof We prove the theorem by determining which pairs $(n, k) \in \mathbb{N} \times \mathbb{N}$ can appear as $(\mathrm{mpc}(L), \mathrm{mpc}(\mathrm{Circ}(L)))$ for some language L. Lemma 3 immediately implies that $n = 0$ if and only if $k = 0$. We partition the remaining cases in three parts: First we look at $1 \leq n \leq 2$, $1 \leq k \leq 3$, second we deal with $k \geq 4$ and $1 \leq n \leq k$, and finally we consider $n \geq 3$ and $1 \leq k \leq n$.

i) It is clear that $1 \in g_{\mathrm{Circ}}^{\mathrm{mpc}}(1)$ (witnessed by $L_{1,1} = \{\lambda\}$) and $2 \in g_{\mathrm{Circ}}^{\mathrm{mpc}}(2)$ (witnessed by $L_{2,2} = \{a\}$). To prove that $\{2, 3\} \notin g_{\mathrm{Circ}}^{\mathrm{mpc}}(1)$ and $\{1, 3\} \notin g_{\mathrm{Circ}}^{\mathrm{mpc}}(2)$, we show how to adapt the pumpings of the words of size 1 and 2 from L to $\mathrm{Circ}(L)$.

If $\mathrm{mpc}(L) = 1$, then each word of length 1 in L allows a pumping, thus each word of length 1 in $\mathrm{Circ}(L)$ allows a pumping, hence $\mathrm{mpc}(\mathrm{Circ}(L))) \neq 2$. Moreover, we can show that $\mathrm{mpc}(\mathrm{Circ}(L))) \neq 3$ by applying the same observation to the words of length 2: since $\mathrm{mpc}(L) = 1$ each two-letter word $w = w_1w_2 \in L$ allows a pumping, thus either $\{w_1w_2\}^* \subseteq L$, $\{w_1\}^*\{w_2\} \subseteq L$, or $\{w_1\}\{w_2\}^* \subseteq L$. We can adapt each of these three cases to $\mathrm{Circ}(L)$ to show that the two cyclic shifts w_1w_2 and w_2w_1 of w in $\mathrm{Circ}(L)$ both allow a pumping, and since this is true for every word of length 2 we have $\mathrm{mpc}(\mathrm{Circ}(L))) \neq 3$.

If $\mathrm{mpc}(L) = 2$, we can again adapt the pumpings of the words of length 2 from L to $\mathrm{Circ}(L)$, thus $\mathrm{mpc}(\mathrm{Circ}(L))) \neq 3$. Furthermore, there exists $w \in L$ of length 1 that allows no pumping, thus w allows no pumping in $\mathrm{Circ}(L)$ and $\mathrm{mpc}(\mathrm{Circ}(L))) \neq 1$.

ii) For every $k \geq 4$ and $1 \leq n \leq k$, we design a witness language:

$$\begin{aligned} L_{n,k} &= \{a^{n-1}\} \cup \{b\}^* \cup \{a^2\}^*\{b^{k-3}\}; \\ \mathrm{Circ}(L_{n,k}) &= \{a^{n-1}\} \cup \{b\}^* \cup \{a^s b^{k-3} a^t \mid s + t \text{ is even}\} \cup \{b^s a^{2i} b^t \mid s + t = k - 3\}. \end{aligned}$$

Then, $\mathrm{mpc}(L_{n,k}) = n$ since a^{n-1} allows no pumping, yet we can pump b in each word of $\{b\}^+$, and a^2 in each word of $\{a^2\}^+\{b^{k-3}\}$.

Furthermore, we show that $\mathrm{mpc}(\mathrm{Circ}(L_{n,k})) = k$. First, $ab^{k-3}a$ allows no pumping: for every decomposition $ab^{k-3}a = xyz$ with $y \neq \lambda$, the word xy^2z is not in $\mathrm{Circ}(L_{n,k}))$ as either y contains at least one b thus xy^2z contains more that one a and more than $k - 3$ b's, or y contains a single a thus xy^2z contains at least one b and an odd number of a's. Moreover, we can pump every word $w \in \mathrm{Circ}(L_{n,k})$ satisfying $|w| \geq k$: if $w \in \{b\}^*$ then b can be pumped, and otherwise w contains two consecutive a's that can be pumped.

iii) For every $n \geq 3$ and $1 \leq k \leq n$, we design a witness language:

$$\begin{aligned} L_{n,k} &= \{a^s b^t \mid s + t \geq k - 1\} \cup \{ba^{n-2}\}; \\ \mathrm{Circ}(L_{n,k}) &= \{a^r b^s a^t \mid r + s + t \geq k - 1\} \cup \{b^r a^s b^t \mid r + s + t \geq k - 1\}. \end{aligned}$$

Then, $\mathrm{mpc}(L_{n,k}) = n$ since ba^{n-2} allows no pumping (as it is the only word of $L_{n,k}$ that starts with b and ends with a), yet we can pump the first letter of every word $w \in L_{n,k}$ longer than $n - 1$. Furthermore, $\mathrm{mpc}(\mathrm{Circ}(L_{n,k})) = k$ as $\mathrm{mpc}(\mathrm{Circ}(L_{n,k})) \geq k - 1$ by Lemma 3, yet we can pump the first letter of every word $w \in \mathrm{Circ}(L_{n,k})$ longer than $k - 1$. □

4 Behaviour under binary operations

In this section, we study the behaviour of the measures mpc and mpl for regular languages under standard binary language operations: union, set subtraction, symmetric difference, concatenation and intersection. We sometimes omit proofs that the minimal pumping constant or minimal pumping length of a certain language has some given value if it can be done along lines given in a previous proof.

4.1 Union

We start with union.

Theorem 10 *For $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$, we have*

$$g_{\cup}^{K}(m,n) = g_{\cup}^{K}(n,m) = \begin{cases} \{m\} & \text{for } m \geq 0, n = 0; \\ \{1,2,\ldots,\max\{m,n\}\} & \text{for } m \geq 1, n \geq 1. \end{cases}$$

Proof i) Since union is a commutative operation, $K(L \cup M) = K(M \cup L)$ holds. Now $g_{\cup}^{K}(m,n) = g_{\cup}^{K}(n,m)$ follows by definition.

ii) Since $K(M \cup \emptyset) = K(M)$ we get $g_{\cup}^{K}(m,0) = \{m\}$, and $g_{\cup}^{K}(0,m) = \{m\}$ by i).

iii) We prove that, for any two languages L_m and L_n with $K(L_m) = m \geq 1$ and $K(L_n) = n \geq 1$, we have $K(L_m \cup L_n) \leq \max\{m,n\}$. Let w be a word of $L_m \cup L_n$ and $|w| \geq \max\{m,n\}$. If $w \in L_m$, then $|w| \geq m$. Hence, there are words x, y, z such that $xyz = w$, $y \neq \lambda$, and $xy^t z \in L_m$ for all $t \geq 0$ (and $|xy| \leq m$ in the case $K = \mathrm{mpl}$). Obviously, $xy^t z \in L_m \cup L_n$. Analogously, if $w \notin L_m$, then $w \in L_n$, $|w| \geq n$ and there are words x', y', z' with $x'y'z' = w$, $y' \neq \lambda$, and $x'(y')^t z' \in L_n \subseteq L_m \cup L_n$ (and $|xy| \leq n$ in the case $K = \mathrm{mpl}$). Hence, we have that all words longer than $\max\{m,n\}$ allow pumping. Therefore, $K(L_m \cup L_n) \leq \max\{m,n\}$.

iv) We now prove that all numbers k with $1 \leq k \leq \max\{m,n\}$ occur in $g_{\cup}^{\mathrm{mpc}}(m,n)$. We set $L_n = \{b\}^* \cup \{a^{n-1}\}$, thus $\mathrm{mpc}(L_n) = n$. By part i), without loss of generality, we can assume that $m \geq n$; hence, we also have $m \geq k$. We differentiate three cases:

- If $m \geq k \geq 3$, we use $L_{m,k} = \{a\}^* \cup \{b^{m-1}, ab^{k-2}\}$, thus $L_{m,k} \cup L_n = \{a\}^* \cup \{b\}^* \cup \{ab^{k-2}\}$. Then, we get $\mathrm{mpc}(L_{m,k}) = m$ and $\mathrm{mpc}(L_{m,k} \cup L_n) = k$.
- If $m \geq k = 2$, we use $L_{m,k} = \{a\}^* \cup \{b^{m-1}, c\}$, thus $L_{m,k} \cup L_n = \{a\}^* \cup \{b\}^* \cup \{c\}$. Then, we obtain $\mathrm{mpc}(L_{m,k}) = m$ and $\mathrm{mpc}(L_{m,k} \cup L_n) = 2$.
- If $m \geq k = 1$, we use $L_{m,k} = \{a\}^* \cup \{b^{m-1}\}$, thus $L_{m,k} \cup L_n = \{a\}^* \cup \{b\}^*$. Then, we obtain $\mathrm{mpc}(L_{m,k}) = m$ and $\mathrm{mpc}(L_{m,k} \cup L_n) = 1$.

v) Finally, if L_m and L_n are languages such that $\mathrm{mpc}(L_m) = m \geq 1$ and $\mathrm{mpc}(L_n) = n \geq 1$, then L_m and L_n are not empty. Thus, $L_m \cup L_n$ is non-empty, too. Therefore, $0 \notin g_{\cup}^{\mathrm{mpc}}(m,n)$ for $m \geq 1$ and $n \geq 1$. □

4.2 Set subtraction and symmetric difference

The following results concern subtraction of sets and symmetric difference.

Theorem 11 *We have*

$$g_{\setminus}^{\mathrm{mpc}}(m,n) = \begin{cases} \{0\} & \text{for } m = 0,\ n \geq 0; \\ \{m\} & \text{for } m \geq 0,\ n = 0; \\ \mathbb{N}\setminus\{1\} & \text{for } m \geq 1,\ n = 1; \\ \mathbb{N} & \text{for } m \geq 1,\ n \geq 2. \end{cases}$$

Proof The statements for $m = 0$, respectively $n = 0$, follow from the fact that for every language L, we have $\emptyset\setminus L = \emptyset$, respectively $L\setminus\emptyset = L$.

We now construct witness languages for every $m \geq 1$, $n \geq 1$ and $k \geq 0$.

- If $k = 0$, we set $L_m = \{a^{m-1}\}$ and $L_n = \{a\}^* \cup \{b^{n-1}\}$. We get $\mathrm{mpc}(L_m) = m$, $\mathrm{mpc}(L_n) = n$, and $\mathrm{mpc}(L_m \backslash L_n) = \mathrm{mpc}(\emptyset) = 0$.
- If $n = 1$, by Lemma 4 every language L' satisfying $\mathrm{mpc}(L') = 1$ contains the empty word. Then, for any language L, $L \backslash L'$ does not contain λ. Hence, $\mathrm{mpc}(L \backslash L') \neq 1$. This implies $1 \notin g_{\backslash}^{\mathrm{mpc}}(m, 1)$ for $m \geq 1$.
  For $m \geq 1$ and $k \geq 2$, we consider the languages
  $$L_m = \{a\}^* \cup \{b^{m-1}\} \text{ and } L_{1,k} = \{\lambda\} \cup \{a^r \mid r \geq k\}.$$
  Then, we get $\mathrm{mpc}(L_m) = m$ and $\mathrm{mpc}(L_{1,k}) = 1$ since we can pump each non-empty word w of $L_{1,k}$ (choose $x = z = \lambda$ and $y = w$). Moreover, $L_m \backslash L_{1,k} = \{a, a^2, \ldots, a^{k-1}\}$ which gives $\mathrm{mpc}(L_m \backslash L_{1,k}) = k$ by Example 2.
  Thus, $k \in g_{\backslash}^{\mathrm{mpc}}(m, 1)$ for $m \geq 1$ and $k \geq 2$.
- If $n \geq 2$ and $k \geq 1$ is an even integer, we set
  $$\begin{aligned} L'_m &= \{a\}^* \cup \{b^{m-1}\}; \\ L'_{n,k} &= \{a^{2s} \mid s \geq 0\} \cup \{a^r \mid r \geq k+1\} \cup \{b\}^* \cup \{c^{n-1}\}; \\ L'_m \backslash L'_{n,k} &= \{a, a^3, \ldots, a^{k-1}\}. \end{aligned}$$
  We get $\mathrm{mpc}(L'_m) = m$, $\mathrm{mpc}(L'_{n,k}) = n$, and $\mathrm{mpc}(L'_m \backslash L'_{n,k}) = k$ as in the examples given in Section 2.
- If $n \geq 2$ and $k \geq 1$ is an odd integer, we set $L''_m = L'_m$ and
  $$L''_{n,k} = \{a^{2s+1} \mid s \geq 0\} \cup \{a^r \mid r \geq k+1\} \cup \{b\}^+ \cup \{c^{n-1}\}$$
  As above, we obtain $\mathrm{mpc}(L''_m) = m$, $\mathrm{mpc}(L''_{n,k}) = n$ and $\mathrm{mpc}(L''_m \backslash L''_{n,k}) = k$.

□

Theorem 12 *We have*

$$g_{\backslash}^{\mathrm{mpl}}(m, n) = \begin{cases} \{0\} & \textit{for } m = 0,\ n \geq 0; \\ \{m\} & \textit{for } m \geq 0,\ n = 0; \\ \mathbb{N} & \textit{for } m \geq 1,\ n \geq 2. \end{cases}$$

Proof Let L_m, L_n, L'_m, $L'_{n,k}$ and $L''_{n,k}$ as in the proof of Theorem 11. Then, we have $\mathrm{mpl}(L_m) = m$, $\mathrm{mpl}(L'_{n,k}) = \mathrm{mpl}(L''_{n,k}) = n \geq 2$ (because we pump a^2 in one case) and $\mathrm{mpl}(L_m \backslash L_{n,k}) = \mathrm{mpl}(L_m \backslash L'_{n,k}) = k$. □

Remark that the cases $m \geq 1$, $n = 1$ are still open.

Theorem 13 *We have*

$$g_{\Delta}^{\mathrm{mpc}}(m, n) = \begin{cases} \{n\} & \textit{for } m = 0,\ n \geq 0; \\ \{m\} & \textit{for } m \geq 0,\ n = 0; \\ \mathbb{N} \backslash \{1\} & \textit{for } m = n = 1; \\ \mathbb{N} & \textit{for } m = n > 1; \\ \mathbb{N}_+ & \textit{for } m \geq 1,\ n \geq 1,\ m \neq n. \end{cases}$$

Proof i) First we mention that $L \Delta K = K \Delta L$ and thus $g_{\Delta}^{\mathrm{mpc}}(m, n) = g_{\Delta}^{\mathrm{mpc}}(n, m)$. Hence, we can assume without loss of generality that $m \leq n$.

ii) Because $L \Delta K = \emptyset$ if and only if $L = K$, we obtain $0 \notin g_{\Delta}^{\mathrm{mpc}}(m, n)$ for $m \neq n$ and $0 \in g_{\Delta}^{\mathrm{mpc}}(m, m)$.

Moreover, $L\Delta\emptyset = \emptyset\Delta L$ holds for all L, which implies the relations $g_{\Delta}^{\mathrm{mpc}}(m, 0) = g_{\Delta}^{\mathrm{mpc}}(0, m) = \{m\}$.

These remarks prove all statements concerning 0 in the domain or range of $g_{\Delta}^{\mathrm{mpc}}$, respectively.

iii) Let $k \geq 2$.

If k is even, $m \geq 2$, and $n \geq 2$, then we consider the languages

$$\begin{aligned}
L_m \quad &= \{a\}^* \cup \{bc^{m-2}\} \cup \{d\}^*\{e\}^*;\\
L_{n,k} \quad &= \{a^{2s} \mid s \geq 0\} \cup \{a^r \mid r \geq k+1\} \cup \{b\}^*\{c\}^* \cup \{de^{n-2}\};\\
L_m \Delta L_{n,k} &= \{a, a^3, \ldots, a^{k-1}\} \cup \{b^r c^s \mid r \geq 0, s \geq 0, (r, s) \neq (1, m-2)\}\\
&\quad \cup \{d^r e^s \mid r \geq 0, s \geq 0, (r, s) \neq (1, n-2)\}.
\end{aligned}$$

Let w be a non-empty word in $\{a\}^*$ and $\{d\}^*\{e\}^*$. Then, we can pump the first letter of w. Because bc^{m-2} allows no pumping, we have $\mathrm{mpc}(L_m) = m$. We also obtain $\mathrm{mpc}(L_{n,k}) = n$.

Obviously, since $a^{k-1} \in L_m\Delta L_{n,k}$ is the longest word in $L_m\Delta L_{n,k}$ over $\{a\}$, we get $\mathrm{mpc}(L_m\Delta L_{n,k}) \geq k$. We now prove that all words of length ≥ 2 in $(L_m\Delta L_{n,k}) \cap \{b, c, d, e\}^*$ allow pumping.

Let $w = b^r c^s \in L_m\Delta L_{n,k}$, $r + s \geq 2$, and $m \geq 3$. If $r = 0$ or $s = 0$, we have $w \in \{c\}^*$ or $w \in \{b\}^*$, respectively, and can pump the first letter of w (all words $xy^t z$ also belong to $\{c\}^*$ or $\{b\}^*$, respectively). If $r \geq 1$, $s \geq 1$, and $s \neq m-2$, then we pump the first letter b, which yields $xy^t z \in L_m\Delta L_{n,k}$ for all $t \geq 0$, because $xy^t z$ contains exactly $s \neq m-2$ occurrences of c. If $s = m-2$ and $r \neq 0$, we have $r \geq 2$. If $w = b^r c^{m-2}$ with $r > 2$, we pump the first letter b, which gives that $xy^t z$ contains at least two occurrences of b and thus $xy^t z \in L_m\Delta L_{n,k}$. If $w = b^2 c^{m-2}$, we pump b^2 and obtain that $xy^t z$ contains an even number of occurrences of b and therefore $xy^t z \in L_m\Delta L_{n,k}$.

An analogous proof can be given for $b^r c^s$ and $m = 2$ and for $d^r e^s \in L_m\Delta L_{n,k}$ with $r + s \geq 2$. Hence, we obtain $k = \mathrm{mpc}(L_m\Delta L_{n,k})$.

If $m = 1$ and $n \geq 2$, we consider

$$L'_m = \{a\}^* \cup \{d\}^*\{e\}^* \text{ and } L'_{n,k} = \{a^{2s} \mid s \geq 0\} \cup \{a^r \mid r \geq k+1\} \cup \{de^{n-2}\},$$

which results in

$$L'_m\Delta L'_{n,k} = \{a, a^3, \ldots, a^{k-1}\} \cup \{d^r e^s \mid r \geq 0, s \geq 0, (r, s) \neq (1, n-2)\},$$

and as above we can show that

$$\mathrm{mpc}(L'_m) = 1, \quad \mathrm{mpc}(L'_{n,k}) = n, \quad \text{and} \quad \mathrm{mpc}(L'_m\Delta L'_{n,k}) = k.$$

The modifications for odd $k \geq 2$, $m \geq 1$, and $n \geq 2$ are done by a change from $L_{n,k}$ to a new language $L''_{n,k}$ analogous to the change from $L'_{n,k}$ to $L''_{n,k}$ in the proof of Theorem 11.

If $k \geq 2$ and $m = n = 1$, we consider $L''_m = \{a\}^*$ and $L''_{n,k} = \{\lambda\} \cup \{a^s \mid s \geq k\}$. Then, we have $\mathrm{mpc}(L''_m) = \mathrm{mpc}(L''_{n,k}) = 1$ (because we can pump the first letter of words in L''_m and the whole words a^s with $s \geq k$ of $L''_{n,k}$). Moreover, we have $L''_m\Delta L''_{n,k} = \{a, a^2, \ldots, a^{k-1}\}$, which gives $\mathrm{mpc}(L''_m\Delta L''_{n,k}) = k$ by Example 2.

Hence, all numbers $k \geq 2$ occur in $g_{\Delta}^{\mathrm{mpc}}(m, n)$ for $m \geq 1$ and $n \geq 1$.

iv) Finally, we focus on $k = 1$. First, we note that $1 \notin g_{\Delta}^{\mathrm{mpc}}(1, 1)$. This can be seen as follows: Let L and L' be languages with $\mathrm{mpc}(L) = \mathrm{mpc}(L') = 1$. Then, Lemma 4 implies that $\lambda \in L$ and $\lambda \in L'$, hence $\lambda \notin L\Delta L'$ and $\mathrm{mpc}(L\Delta L') \neq 1$.

Now we prove that $1 \in g_{\Delta}^{\mathrm{mpc}}(m, n)$ for $(m, n) \neq (1, 1)$. Let $1 \leq m \leq n$. We consider the following cases:

– For $m \geq 2$, we choose $L_m = \{a^{n-1}\}^* \cup \{b^{m-1}\}$ and $L_n = \{a^{n-1}, b^{m-1}\}$. We get $\mathrm{mpc}(L_m) = m$, $\mathrm{mpc}(L_n) = n$, and

$$\mathrm{mpc}(L_m \Delta L_n) = \mathrm{mpc}(\{(a^{n-1})^s \mid s \geq 2 \text{ or } s = 0\}) = 1,$$

where the last relation follows since we can pump $(a^{n-1})^2$ in all non-empty words of $L_m \Delta L_n$.
– For $m = 1$ and $n \geq 2$, we choose $L'_m = \{a^{n-1}\}^*$ and $L'_n = \{a^{n-1}\}$, which implies $\mathrm{mpc}(L'_m) = 1$, $\mathrm{mpc}(L'_n) = n$, and, as above,

$$\mathrm{mpc}(L'_m \Delta L'_n) = \mathrm{mpc}(\{(a^{n-1})^s \mid s \geq 2 \text{ or } s = 0\}) = 1.$$

By i) this implies $1 \in g_{\Delta}^{\mathrm{mpc}}(m, n)$ for $m \geq 1, n \geq 1$, and $(m, n) \neq (1, 1)$. □

Theorem 14 *i) For $m \geq 1$ and $n \geq 1$ with $m + n \geq 3$, we have $\{2, 3, \ldots\} \subseteq g_{\Delta}^{\mathrm{mpl}}(m, n)$.*
ii) For $m \geq 0$, we have $g_{\Delta}^{\mathrm{mpl}}(m, 0) = g_{\Delta}^{\mathrm{mpl}}(0, m) = \{m\}$.

Proof i) Taking into consideration that, in all cases $m + n \geq 2$ in the proof of Lemma 13, any pumping satisfies $|xy| \leq 2$, the statement follows.
ii) follows as in the proof of Lemma 13 □

4.3 Concatenation

We now discuss concatenation.

Lemma 5 *For $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$, we have*

$$g_{\cdot}^{K}(m, n) = \begin{cases} \{0\} & \text{for } m = 0, n \geq 0; \\ \{0\} & \text{for } m \geq 0, n = 0; \\ \{1, 2, \ldots, m + n - 1\} & \text{for } m \geq 1, n \geq 1. \end{cases}$$

Proof i) For every language L, we have $L \cdot \emptyset = \emptyset \cdot L = \emptyset$. Therefore, by Lemma 3, for $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$, we immediately get that $g_{\cdot}^{K}(m, 0) = g_{\cdot}^{K}(0, m) = \{0\}$ for all $m \in \mathbb{N}$. Conversely, if two languages L_m and L_n are nonempty, so is $L_m \cdot L_n$, therefore if both $m \geq 1$ and $n \geq 1$ then $0 \notin g_{\cdot}^{K}(m, n)$.

ii) We now show that for $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$, $m \geq 1$ and $n \geq 1$, we have that $k \in g_{\cdot}^{K}(m, n)$ implies $k \leq m+n-1$. We give the proof for mpc, in brackets we give the necessary additions for mpl.

Let L_m and L_n be two languages with $K(L_m) = m$ and $K(L_n) = n$. Let $w = w_m w_n$ such that $w_m \in L_m$, $w_n \in L_n$, and $|w| \geq m+n-1$. If $|w_m| \geq m$, there are words x_m, y_m, z_m such that $x_m y_m z_m = w_m$, $y_m \neq \lambda$, ($|x_m y_m| \leq m \leq m + n - 1$ if $K = \mathrm{mpl}$), and $x_m y_m^t z_m \in L_m$ for all $t \geq 0$. Let $z'_m = z_m w_n$. Then, $x_m y_m z'_m = w$ and $x_m y_m^t z'_m = x_m y_m^t z_m w_n \in L_m \cdot L_n$ for all $t \geq 0$. If $|w_m| \leq m - 1$, then $|w_n| \geq n$. Therefore, there are words x_n, y_n, z_n such that $x_n y_n z_n = w_n$, $y_n \neq \lambda$, ($|x_n y_n| \leq n$ if $K = \mathrm{mpl}$), and $x_n y_n^t z_n \in L_n$ for all $t \geq 0$. Let $x'_n = w_m x_n$. Then, $x'_n y_n z_n = w$, ($|x'_n y_n| = |w_m| + |x_n y_n| \leq m - 1 + n$), and $x'_n y_n^t z_n = w_m x_n y_n^t z_n \in L_m \cdot L_n$ for all $t \geq 0$. Therefore, $K(L_m \cdot L_n) \leq m + n - 1$.

iii) Finally, for every quadruple of integers $m \geq m' \geq 1$ and $n \geq n' \geq 1$, we construct two witness languages $L_{m,m'}$ and $L_{n,n'}$ such that for $K \in \{\mathrm{mpc}, \mathrm{mpl}\}$ we get $K(L_{m,m'}) = m$, $K(L_{n,n'}) = n$ and $K(L_{m,m'} \cdot L_{n,n'}) = m' + n' - 1$. This concludes the proof of the theorem since every integer $k \in \{1, 2, \ldots, m + n - 1\}$ can be decomposed as $k = m' + n' - 1$ for some $m' \leq m$ and $n' \leq n$. Let

$$\begin{aligned}
L_{m,m'} &= \{\lambda, a^{m-1}, c^{m'-1}\} \cup \{a,b,c\}^* \cdot \{b, ba^{m-1}, ca^{m-1}, ac^{m'-1}, bc^{m'-1}\};\\
L_{n,n'} &= \{\lambda, b^{n-1}, c^{n'-1}\} \cup \{a,b,c\}^* \cdot \{a, ab^{n-1}, cb^{n-1}, ac^{n'-1}, bc^{n'-1}\};\\
L_{m,m'} \cdot L_{n,n'} &= \{a,b,c\}^* \cdot \{a,b\} \cdot \{\lambda, c^{m'-1}, c^{n'-1}, c^{m'+n'-2}\}\\
&\quad \cup \{\lambda, c^{m'-1}, c^{n'-1}, c^{m'+n'-2}\}.
\end{aligned}$$

Then, $K(L_{m,m'}) = m$ because λ, a^{m-1} and $c^{m'-1}$ allow no pumping, but we can pump the first letter of each word in $\{a,b,c\}^* \cdot \{b, ba^{m-1}, ca^{m-1}, ac^{m'-1}, bc^{m'-1}\}$. Similarly, $K(L_{n,n'}) = n$ and $K(L_{m,m'} \cdot L_{n,n'}) = m' + n' - 1$. □

4.4 Intersection

Finally, we study the behaviour under intersection.

Theorem 15 *We have*

$$g_{\cap}^{\text{mpc}}(m,n) = g_{\cap}^{\text{mpc}}(n,m) = \begin{cases} \{0\} & \text{for } m = 0 \text{ or } n = 0;\\ \mathbb{N}\backslash\{2\} & \text{for } m = n = 1;\\ \mathbb{N} & \text{for } m \geq 1,\ n \geq 1,\ m+n \geq 3. \end{cases}$$

Proof Since intersection is a commutative operation, for all $m, n \in \mathbb{N}$ we have $g_{\cap}^{\text{mpc}}(m,n) = g_{\cap}^{\text{mpc}}(n,m)$. Without loss of generality, we can assume that $m \geq n$.

i) Let $m \geq 2$, $n \geq 1$, and $k \geq 2$. We consider the languages

$$L_m = \{a^{m-1}\} \cup \{b\}^*\{c\} \text{ and } L_{n,k} = \{d^{n-1}\} \cup \{b^{k-2}c\}^*.$$

We obtain (analogous to the already given proofs)

$$\text{mpc}(L_m) = m,\ \text{mpc}(L_{n,k}) = n,\ \text{and}\ \text{mpc}(L_m \cap L_{n,k}) = \text{mpc}(\{b^{k-2}c\}) = k.$$

ii) For $m \geq 1$, $n \geq 1$ and $k \geq 3$, let $L'_m = \{a^{m-1}\} \cup \{b\}^*\{c\}^*$. Again, we yield

$$\text{mpc}(L'_m) = m,\ \text{mpc}(L_{n,k}) = n,\ \text{and}\ \text{mpc}(L'_m \cap L_{n,k}) = \text{mpc}(\{b^{k-2}c\} = k.$$

iii) For $m \geq 1$ and $n \geq 1$, let $L''_m = \{a^{m-1}\} \cup \{b\}^*$ and $L''_n = \{d^{n-1}\} \cup \{b\}^*$. we get

$$\text{mpc}(L''_m) = m,\ \text{mpc}(L''_n) = n,\ \text{and}\ \text{mpc}(L''_m \cap L''_n) = \text{mpc}(\{b\}^*) = 1,$$

which proves $1 \in g_{\cap}^{\text{mpc}}(m,n)$.

iv) For $m \geq 1$, $n \geq 1$, $L'''_m = \{a^{m-1}\}$, and $L'''_n = \{b^{n-1}\}$, we get $\text{mpc}(L'''_m) = m$, $\text{mpc}(L'''_n) = n$, and $\text{mpc}(L'''_m \cap L'''_n) = \text{mpc}(\emptyset) = 0$.

Now the statement concerning $m \geq 1$, $n \geq 1$ and $m + n \geq 3$ follows from a), c), and d). Furthermore, we have $\mathbb{N}\backslash\{2\} \subseteq g_{\cap}^{\text{mpc}}(1,1)$ by b), c), and d).

Therefore, it remains to show that $2 \notin g_{\cap}^{\text{mpc}}(1,1)$. Assume that there are two languages L and L' with $\text{mpc}(L) = \text{mpc}(L') = 1$ and $\text{mpc}(L \cap L') = 2$. Then, all words in $L \cap L'$ with a length at least 2 allow a pumping and there is a letter $a \in L \cap L'$ which does not allow a pumping. Obviously, $a \in L$ as well as $a \in L'$. Thus, $a^t \in L$ and $a^t \in L'$ for $t \geq 0$ by $\text{mpc}(L) = \text{mpc}(L') = 1$. This implies $a^t \in L \cap L'$ for all $t \geq 0$, i.e., a allows pumping. This contradiction shows that $\text{mpc}(L \cap L') = 2$ is impossible for languages L and L' with $\text{mpc}(L) = \text{mpc}(L') = 1$.

The remaining statement follows from $L \cap \emptyset = \emptyset \cap L = \emptyset$ for all languages L. □

5 Conclusion

In this paper, we have started the study of the behaviour of certain parameters of regular language, which are related to pumping lemmas, under operation. Concerning the minimal pumping constant mpc, we have completely determined the sets $g_\circ^{\mathrm{mpc}}(k_1, k_2, \ldots, k_n)$ for all the operations. However, with respect to the minimal pumping length, some cases are still open: our results for set subtraction and symmetric difference are only partial, and we have no result for circular shift and intersection. It remains to solve the cases left open and the behaviour under further operations as quotients, for example.

In contrast to state complexity, we have obtained infinite ranges for the minimal pumping constant and complement, set-subtraction, symmetric difference, prefix and suffix closure and for the minimal pumping length and complement, set-subtraction, symmetric difference, and reversal. On the other hand, in contrast to the measure number of accepting states, we have only finite ranges with respect to the minimal pumping constant/length and union, product, and Kleene closure. Thus, concerning finiteness/infinity of ranges the minimal pumping constant/length are intermediate between state complexity and the measure number of accepting states.

Furthermore, the behaviours of mpc and mpl differ with respect to finiteness/infinity of ranges. All ranges of minimal pumping constant and reversal are finite, whereas there are infinite ranges for minimal pumping length and reversal. The difference comes from the fact that mpl takes into consideration that the pumping has to be done within a prefix of bounded length.

A little bit more surprising is the fact that with respect to prefix closure the ranges are finite for mpl, whereas infinite ranges exist for mpc.

In many papers, with respect to state complexity and some other parameters discussed in the introduction, only the worst case was considered. More precise, we define $f_\circ^K(k_1, k_2, \ldots, k_n)$ as the maximal number in $g_\circ^K(k_1, k_2, \ldots, k_n)$ if $g_\circ^K(k_1, k_2, \ldots, k_n)$ is finite; otherwise, we set $f_\circ^K(k_1, k_2, \ldots, k_n) = \infty$. Looking on the results of this paper the values $f_\circ^{\mathrm{mpc}}(n)$ or $f_\circ^{\mathrm{mpc}}(m, n)$ were presented for all arguments and all considered operations. Moreover, if we restrict to $m \geq 2$ and $n \geq 2$, then the values $f_\circ^{\mathrm{mpl}}(n)$ or $f_\circ^{\mathrm{mpl}}(m, n)$ were given for Kleene closure, complement, reversal, prefix closure, suffix closure, union, set-subtraction, symmetric difference and product. But for circular shift and intersection, the determination of the corresponding values $f_\circ^{\mathrm{mpl}}(n)$ or $f_\circ^{\mathrm{mpl}}(m, n)$, respectively, remains open.

Moreover, pumping lemmas are known for further classes of languages, e. g., context-free languages, finite index matrix languages. It remains to study the behaviour of the corresponding minimal pumping constant under operations for such classes. Essentially, we expect results which are analogues to those given in this paper.

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