Understanding Volatility

In this lecture. . .

- The many types of volatility

- Vega, and why it is dangerous

- What the market prices of options tells us about volatility

- The term structure of volatility

- Volatility skews and smiles

- Volatility arbitrage: Should you hedge using implied or actual volatility?

By the end of this lecture you will

- know about all of the most important kinds of volatility
- be able to relate implied volatility to actual volatility
- understand simple volatility arbitrage strategies

Introduction

Volatility is the most important parameter determining the value of an option, yet it is also the hardest to measure.

Why?

And why?

For this reason people often refer to different kinds of volatility, some of these are abstract quantities, some are statistical and some are measures of the market's assessment of volatility.

The different types of volatility

- Actual/Local
- Historical/Realized
- Implied
- Forward

Let's look at each in turn.

Actual/Local Volatility

This is the measure of the amount of randomness in an asset return at any particular time. The 'noise.'

It's the σ in this

$$dS = \cdots + \sigma S dX.$$

It is very difficult to measure, but is supposed to be an input into all option pricing models.

In particular, the actual (or 'local') goes into the Black–Scholes *Equation*.

- There is no 'timescale' associated with actual volatility, it is a quantity that exists at an instant.

Example: The actual volatility is now 20%. . . now it is 22%. . . now it is 24%. . .

Questions (to be answered in later lectures!):

1. Do we know what it is **now**?

2. Do we know what it will be in the future? (Up to expiration.)

3. Can it be modelled?

4. What if it's random?

Historical/Realized Volatility

A measure of the amount of randomness over some period in the past.

The period is always specified, and so is the mathematical method for its calculation.

Sometimes used as an estimate for what volatility will be in the future.

- There are two 'timescales' associated with historical/realized volatility: one short, and one long.

Example 1: The 60-day volatility using daily returns.

Perhaps of interest if you are pricing a 60-day option, which you are hedging daily.

Example 2: I sold a 30-day option for a 25% volatility, I hedged it every day.

Did I make money?

Implied Volatility

The implied volatility is the volatility which when input into the Black-Scholes option pricing formulae gives the market price of the option.

It is often described as the market's view of the future actual volatility over the lifetime of the particular option.

However, it is also influenced by other effects such as supply and demand.

- There is one 'timescale' associated with implied volatility: expiration.

Note: Use of implied volatility does not mean we believe in the Black–Scholes model! It's a way of 'levelling the playing field.'

Example: A stock is at 100, a call has strike 100, expiration in one year, interest rates 5% and the option market price is \$10.45.

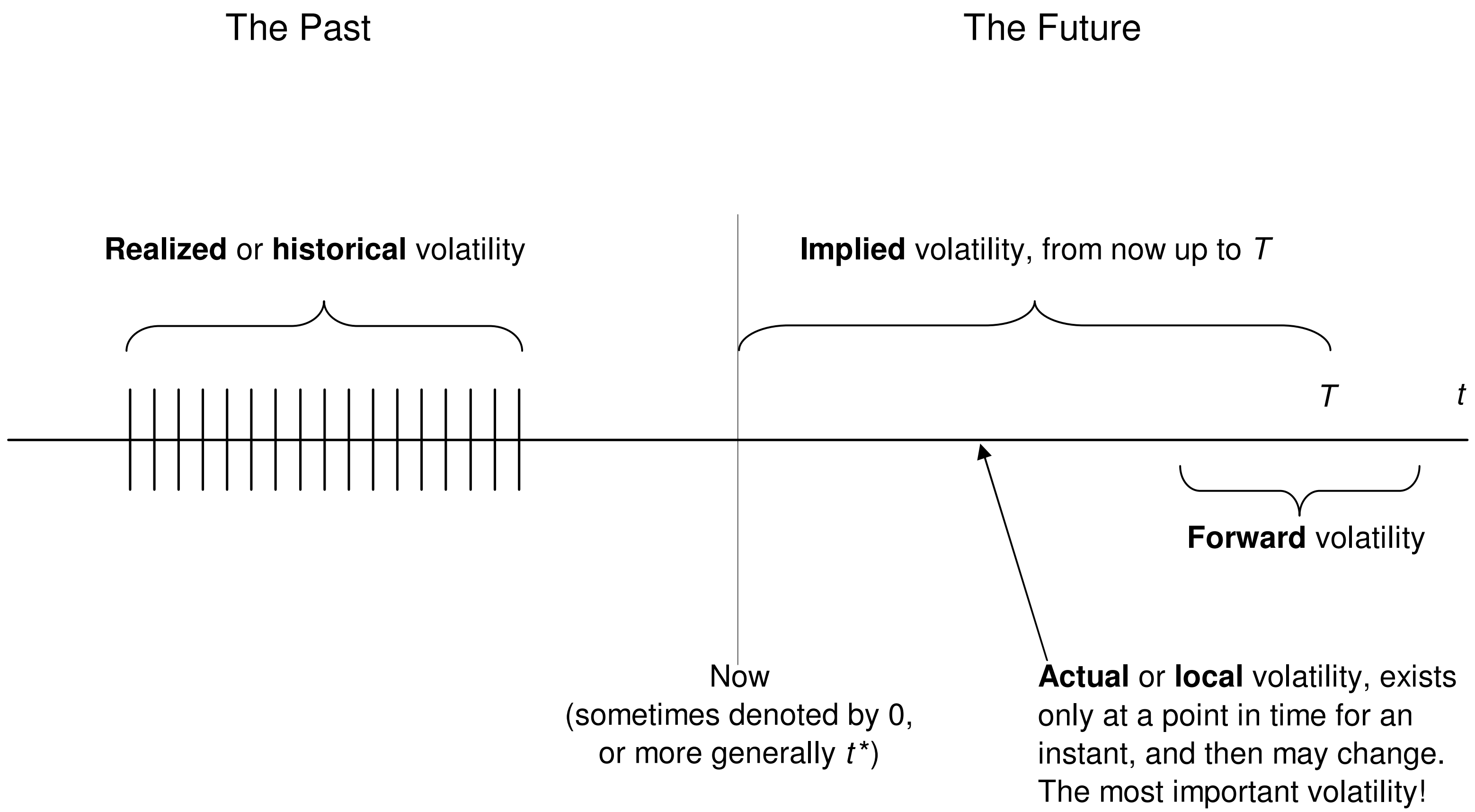
What volatility are traders using?

Forward Volatility

The adjective 'Forward' can be applied to many forms of volatility, and refers to the volatility (whether actual or implied) over some period in the future.

- Forward volatility is associated with either a time period, or a future instant.

Please. . . always know which volatility you, and others, are talking about. Do not get them confused!


The Past
The Future
Realized or **historical** volatility
Implied volatility, from now up to *T*
T
t
Forward volatility
Now
(sometimes denoted by 0,
or more generally *t**)
Actual or **local** volatility, exists
only at a point in time for an
instant, and then may change.
The most important volatility!

More about implied volatility

In the Black–Scholes world of constant volatility, the value of a European call option is simply

$$V(S,t;\sigma,r;E,T) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

We have given the function V six arguments, all but σ are easy to measure. If we know σ then we can calculate the option price.

How do we know what volatility to put into the formula?

For vanilla options things usually work the other way around! We use the price of traded options to give us some ‘information’ about volatility!

A trader can see on his screen that a certain call option with four months until expiry and a strike of 100 is trading at 6.51 with the underlying at 101.5 and a short-term interest rate of 6%.

Can we use this information in some way?

Turn the relationship between volatility and an option price on its head, if we can see the price at which the option is trading, we can ask

"What volatility must I use to get the correct market price?"

This is called the **implied volatility**.

The implied volatility is the volatility of the underlying which when substituted into the Black-Scholes formula gives a theoretical price equal to the market price.

In a sense it is the market's view of volatility over the life of the option.

Implied volatility is the only volatility we can ‘see’ without having to use statistical methods.

It should not be confused with actual volatility. Actual volatility exists even in the absence of options.

Implied volatility has many interpretations and uses.

Important: Interpretation and use of implied volatility

Buy side/Hedge funds: Forecast their own volatility and hope it is different from implied. (If they are the same then there are no arbitrage opportunities!) Buy/sell options and make a profit if they are right.

Sell side/Investment banks: Use implied volatility to tell them how to price exotics (and then add on a profit margin!) Sell exotics and hedge so as to lock in the profit margin. (This is the subject of **calibration** or **fitting**, more shortly.)

This can be summed up as "Do you believe implied volatility contains information?"

Vega

An option value can change even when the underlying doesn't move

Implied volatility can change. A market that is panicking will have a high volatility.

Because implied volatility can change we would like to know how sensitive our options are to that change.

Vega is the sensitivity of an option value to volatility:

$$\frac{\partial V}{\partial \sigma}.$$

(Although whether that means sensitivity to actual or implied volatility can sometimes be very vague.)

Vega is the sensitivity to a *parameter*, and not sensitivity to a *variable*.

This makes it different from the other 'Greeks,' delta, gamma and theta. The importance of this will be seen later.

For a call or put in a Black–Scholes world

$$S\sqrt{T-t}\,N'(d_1).$$

Do you see the inconsistency here?

The Black–Scholes world assumes that volatility is constant, but vega measures change in value if that constant changes!

Despite the worries, why is vega important?

- **Hedging:** you can hedge one option with another to reduce sensitivity to volatility. If one option has a vega of 37.5 and the other has a vega of 75 then long two of the first and short the second will be vega neutral. The sell side will use this to reduce risk when they sell exotic contracts (for more than they are worth).

- **Risk management:** If you buy an option to speculate on stock direction then the option's implied volatility only matters when you buy the option and when you sell it (if you don't hold it to expiration). However, in the meantime your profit will depend on the market value of the option, and hence on its implied volatility.

Examples: Suppose the following contracts are available to us:

	Delta	Gamma	Vega
Stock	1	0	0
Option A	0.4	0.026	27
Option B	0.6	0.018	36

Imagine that Option B is OTC/exotic, you are going to sell it for more than it is worth. Construct a portfolio that is delta and vega neutral. (Why?)

How this might work: Sell OTC Option B to make a profit, but be delta and vega neutral.

Buy a quantity A of option A, -1 of option B, and C of the stock.

Delta neutral: $C + 0.4A + 0.6 \times (-1) = 0$.

Vega neutral: $27A + 36 \times (-1) = 0$.

The solution is

$$C = 0.06667, \quad \text{and} \quad A = 1.3333.$$

Dangers of vega

Vega is a **bastard greek**.

A bastard greek is the rate of change of the value of an option to a parameter, as opposed to the classical greeks which are rates of change with respect to variables. Classical greeks are delta, gamma, theta, etc. Bastard greeks are vega, rho, etc.

Bastard greeks are illegitimate because they usually involve differentiating a formula, such as Black–Scholes, with respect to a parameter that *has been assumed constant in the derivation of the formula.* They are therefore internally inconsistent and lead to risk-management errors. For example, a vega may be calculated as zero leading one to think there is no volatility risk at precisely the point at which there is a great deal of volatility risk.

Important question: What happens to the value of an option and the greeks when we increase/decrease volatility?

Some Rules of Thumb:
How the value and greeks change with varying volatility

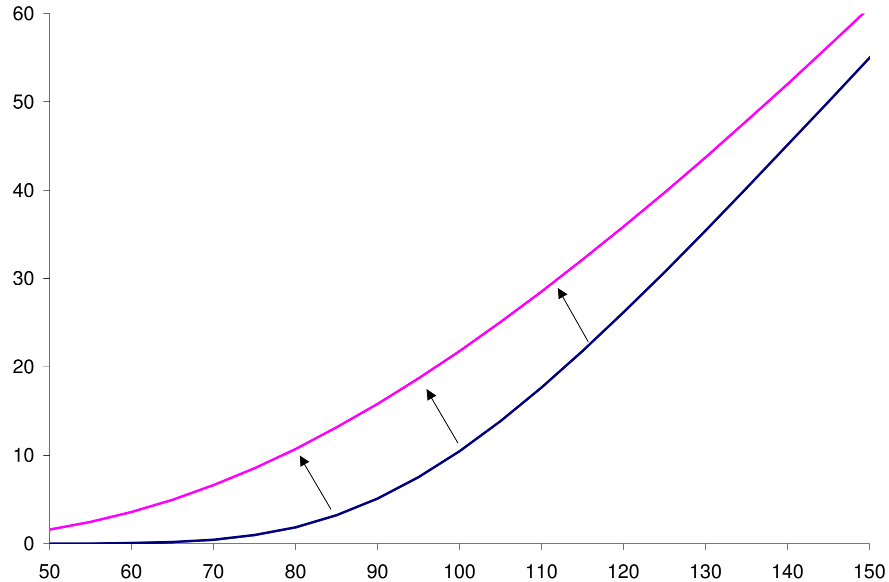
Why is this important?

- If you are speculating with options you will need to know how the option value changes when the market's perception of future volatility changes

- If you are hedging then your hedge ratios may need to be corrected

- As part of risk management

Rule of Thumb 1: Increasing volatility will increase option value if Gamma is positive.



Rule of Thumb 2: Doubling volatility is like quadrupling time to expiry.

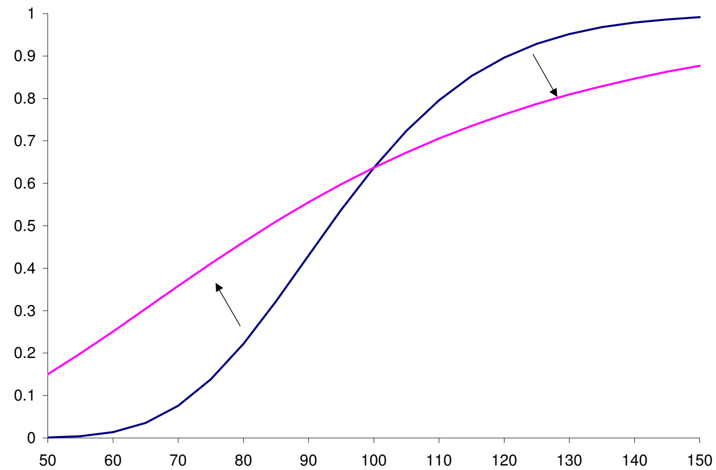
Volatility and time to expiry always appear in this combination:

$$\sigma\sqrt{T}.$$

Rule of Thumb 3: Increasing volatility 'smoothes' out curvature in the asset direction.

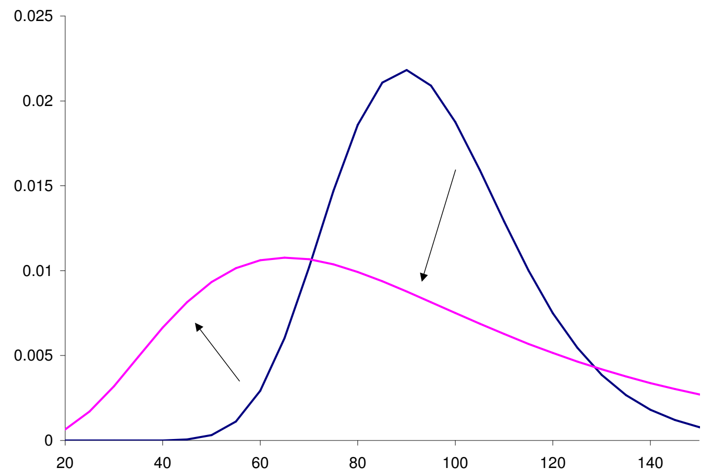
This is true for price, or for the Greeks.

For example, Delta versus underlying asset.



Rule of Thumb 4: Increasing volatility makes Gamma less 'severe'

When volatility is low Gamma can get very large.



We are now going to move away from the world of known, constant volatility.

Some problems with Black–Scholes

- Transaction costs on the underlying, hedging costs
- Illiquidity
- Uncertainty in volatility
- Uncertainty in dividends
- Feedback, market manipulation
- Market jumps, crashes, discontinuous asset paths
- Discrete price effect (especially in the teenies)
- Discrete hedging effects
- **Supply and demand**

Supply and demand

Suppose your car is worth $20,000 and annual insurance against a crash is $1,000. What is the probability of you crashing?

Is it 5% per annum?

Car insurance has to cover: Loss due to crash (in an actuarial sense); Business costs; Oh, and Profit!

So all we can say is: Probability of crash ≤ 0.05. (And we can't even be sure of that, think of supermarket 'loss leaders.')

When someone sells a deep OTM put the premium has to cover: Replication costs; Probability of crash; Fat tails; Market frictions; Supply; Demand; Profit margin.

Classic Black–Scholes only has a parameter measuring replication costs, the volatility.

So we cannot really back out a perfectly meaningful volatility from a price.

This is why can be helpful to distinguish between price and value.

- Is there any useful information contained within *implied* volatility concerning the future behavior of *actual* volatility?

One is usually taught to think of the implied volatility as the market's view of the future value of volatility.

(People will often say the nonsensical 'the market is always right.')

Yes and no.

If the 'market' does have a view on the future of volatility then it will be seen in the implied volatility. But the market also has views on the direction of the underlying, and also responds to supply and demand.

Two different approaches

Now we are going to take **two** different directions.

First, we will assume that **the market is right!** We assume that there is perfect information about actual volatility in implied volatility. Then we just need some maths to relate the two. This is **calibration** or **fitting**. Calibration is making our theoretical option values the same as the market prices. We present here a simple no-arbitrage model.

Second, we will assume that **the market is wrong!** We will assume we have a better idea of volatility than the market (implied). This is **volatility arbitrage**. If we are right we will make money.

The Market is Right!: Calibration

- We are now going to **assume** that **the market knows precisely what actual volatility is going to be in the future and has incorporated this information into the market value of exchange-traded contracts**

The problem we have is that there is not a single implied volatility! Each option series (expiration and strike) has a different value. How do we handle this?

First case: Time-dependent implied volatility

Below are the market prices of European call options with three, six, nine and 12 months until expiry.

All have strike prices of 110 and the underlying asset is currently 108.5. The short-term interest rate over this period is 4%.

Expiry	Value
3 month	4.74
6 months	6.72
9 months	8.22
12 months	9.63

Expiry	Impl. Vol.
3 months	22.8%
6 months	20.9%
9 months	19.7%
12 months	19.1%

Clearly these prices cannot be correct if actual volatility is constant for the whole year. What is to be done?

Goal: reconcile a *term*-dependent implied volatility with the Black–Scholes model.

The simplest adjustment we can make to the Black–Scholes world to accommodate these prices (without any serious effect on the theoretical framework) is to assume

- a *time*-dependent, deterministic actual volatility.

Let's assume that actual volatility is a function of time. And because the distinction actual and implied volatility is about to become very important we shall add a suitable subscript:

$$\sigma_a(t).$$

- Believe it or not, the Black–Scholes formulæ are still valid when volatility is time dependent provided we use

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma_a(\tau)^2 d\tau}$$

in place of σ i.e. now use

$$d_1 = \frac{\log(S/E) + r(T-t) + \frac{1}{2}\int_t^T \sigma_a(\tau)^2 d\tau}{\sqrt{\int_t^T \sigma_a(\tau)^2 d\tau}},$$

and the obvious expression for d_2.

Whoa!

Where did I get this from?

What does it mean?

Is this obvious?

Let's step back a bit. . .

If σ_a is constant then (given a lot of assumptions) we have the Black–Scholes *equation*:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_a^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

And for vanilla calls and puts this has nice, closed-form solutions, the Black–Scholes *formulæ*.

$$C(S,t) = \cdots,$$

$$P(S,t) = \cdots.$$

If $\sigma_a(t)$ is a function of time then (given a lot of assumptions) we *still* have the Black–Scholes *equation*:

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma_a(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

(To check this just go back over the derivation of the Black–Scholes partial differential equation and ask where did we assume constant volatility. The answer is that we didn't!)

And for vanilla calls and puts this has nice, closed-form solutions, the Black–Scholes *formulæ*... **but with a special expression where you'd expect to see volatility**.

(Aside: If $\sigma_a(S,t)$ is a function of both asset price and time then the Black–Scholes *equation* is still valid but the *formulæ* are not!)

In the case of time-dependent volatility $\sigma_a(t)$ I said that "for vanilla calls and puts this has nice, closed-form solutions, the Black–Scholes *formulæ*. . . **but with a special expression where you'd expect to see volatility**."

I've already shown you the "special expression"? But can we figure it out from common sense? Multiple choice question. . .

Which seems most plausible: If actual volatility varies with time then the value of a vanilla option is a function of. . .

1. The maximum volatility over the life of the option?

2. The minimum volatility over the life of the option?

3. The average volatility over the life of the option?

Correct answer:

3. **The value of an option depends on the 'average' volatility between now and expiration.**

But what does 'average' mean?

Remember that volatility is a measure of standard deviation. And since you aren't allowed to add up standard deviations it would be strange to calculate the average by summing standard deviations. . .

. . . but you can add variances!

At any time the variance is just $\sigma_a(t)^2$.

The sum of variances over the life of the option, from now t to expiration T, is

$$\int_t^T \sigma_a(\tau)^2 d\tau.$$

So the average *variance* is

$$\frac{1}{T-t}\int_t^T \sigma_a(\tau)^2 d\tau.$$

And therefore the equivalent 'average' *volatility* is

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma_a(\tau)^2 d\tau}.$$

But this is not a mathematical proof that using the Black–Scholes formulæ with this new volatility is correct!

Two ways of rigorously showing this:

1. Transform the Black–Scholes equation into simpler heat equation (as before) and change time variable.

2. Substitution!

Implied volatility

If $\sigma_a(t)$ tells us how the *actual* volatility varies with *calendar time*, then

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma_a(\tau)^2 d\tau}$$

tells us how *implied* volatility should behave with *expiration*. After all, it is the volatility you put into the Black–Scholes equations.

However, in practice we know the implied volatilities but not actual. . . so the formula works the other way around!

Assuming that the market knows best, all we need to do to ensure consistent pricing is to make

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma_a(\tau)^2 d\tau} = \text{ implied volatilities.}$$

This is called **calibration** or **fitting**.

We make sure that parameter(s) (here σ_a) is (are) chosen so that theoretical prices are the same as market prices.

(Yes, parameters can be time dependent.)

Traditionally/scientifically we would say we know the parameter, $\sigma_a(t)$, and have to find the answer, effectively implied volatility, $\sigma_i(T)$.

$$\sigma_a(t) \quad \Rightarrow \quad \sigma_i(T).$$

But in finance we often know the answer (the market value) and have to find the question (the parameter):

$$\sigma_a(t) \quad \Leftarrow \quad \sigma_i(T).$$

This is an **inverse** problem.

In this case, the inverse problem is an **integral equation**.

We do this 'fitting' at time t^*. If we write $\sigma_i(T; t^*)$ to mean the implied volatility measured at time t^* of a European option expiring at time T then the solution of the inverse problem is

$$\sigma_a(t) = \sqrt{\sigma_i(t; t^*)^2 + 2(t - t^*)\sigma_i(t; t^*)\frac{\partial \sigma_i(t; t^*)}{\partial t}}.$$

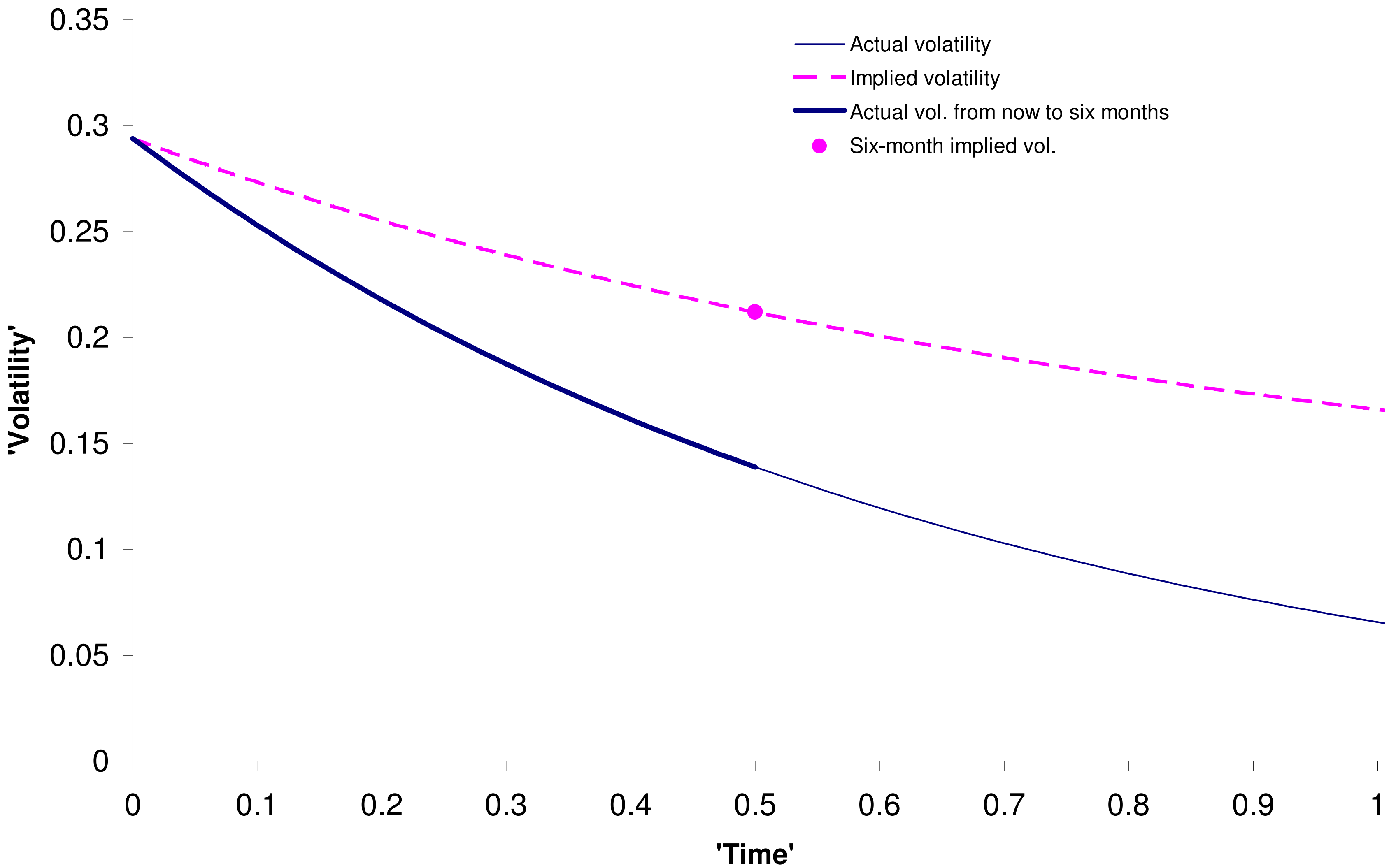
This is the solution of the integral equation.

In words. . .

A term structure in implied volatility can be made consistent with the Black–Scholes option pricing model provided we have a term structure for actual volatility.

The relationship between implied and actual volatility is:

The square of the implied volatility of an option expiring at time T is the average square of actual volatility from now until time T.


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
'Volatility'
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
'Time'
Actual volatility
Implied volatility
Actual vol. from now to six months
Six-month implied vol.

But the data is not differentiable (or even continuous)

In practice implied volatilities are only known at a finite number of expiries.

Example:

One-month implied volatility is 30%.

Two-month implied volatility is 25%.

Three-month implied volatility is 26%.

Construct an actual volatility function of time that is consistent with these.

There is no unique solution.

Why?

We must make some assumptions about the time dependence of σ_a.

For simplicity let us assume that $\sigma_a(t)$ is piecewise constant.

For t less than one month the answer is simple:

$$\sigma_a(t) = 0.3.$$

Use σ_1 to denote this constant value over the first period.

For t greater than one month but less than two months, this is harder.

It must be chosen to be consistent with two-month implied volatility.

But two-month implied volatility depends on $\sigma_a(t)$ from t zero up to two months.

The implied variance is the time-weighted average of actual variance:

$$\frac{2}{12} \times 0.25^2 = \frac{1}{12} \times 0.3^2 + \frac{1}{12} \times \sigma_2^2.$$

Solve this to get

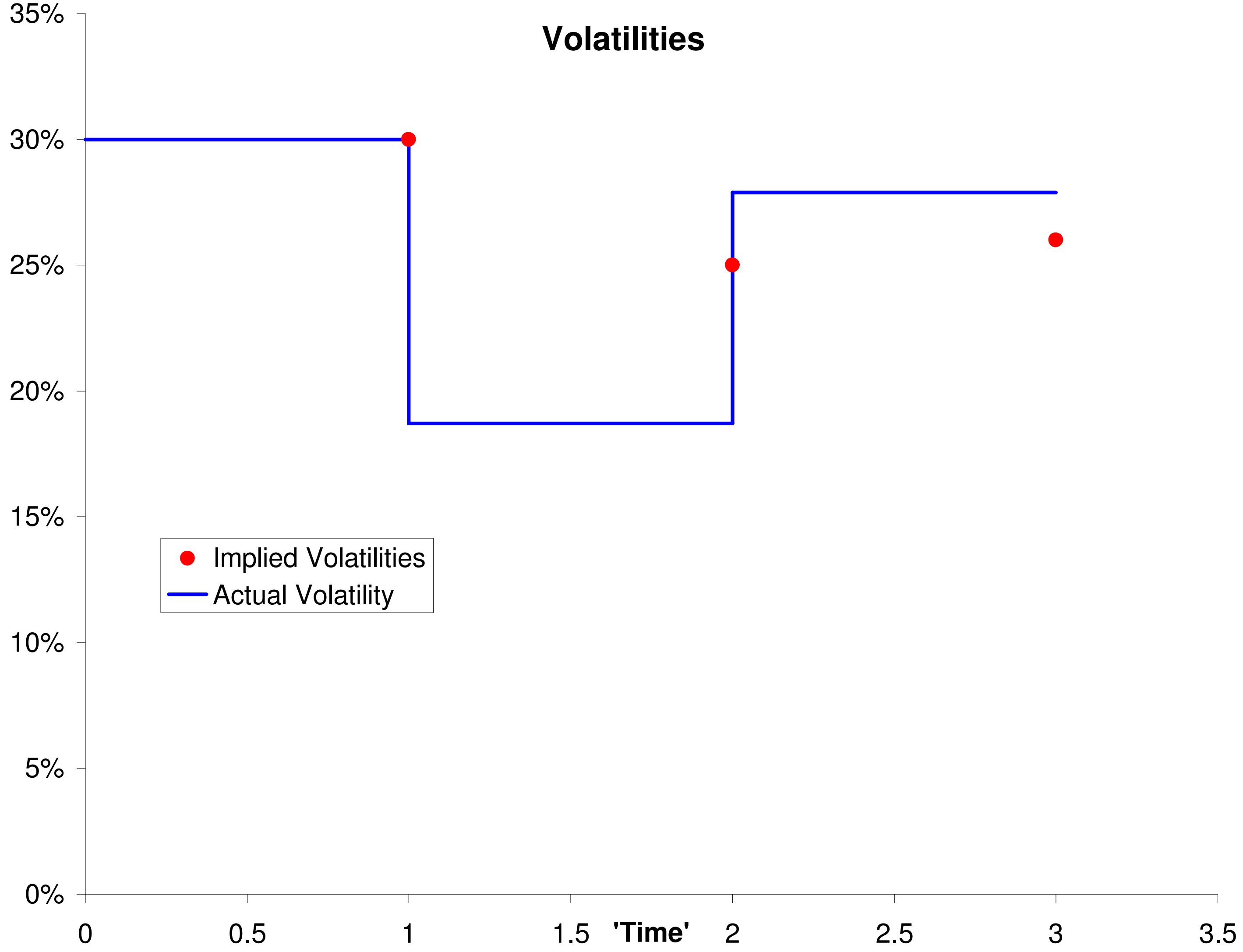
$$\sigma_2 = 0.187.$$

Finally, to get σ_3, which is the actual volatility during the third month, we must have

$$\frac{3}{12} \times 0.26^2 = \frac{1}{12} \times 0.3^2 + \frac{1}{12} \times 0.187^2 + \frac{1}{12} \times \sigma_3^2.$$

The solution is

$$\sigma_3 = 0.279.$$


Volatilities
35%
30%
25%
20%
15%
10%
5%
0%
Implied Volatilities
Actual Volatility
0
0.5
1
1.5
'Time'
2
2.5
3
3.5

If we have implied volatility for expiries T_i and we assume the volatility curve to be piecewise constant then

$$\sigma_a(t) = \sqrt{\frac{(T_i - t^*)\sigma_i(T_i; t^*)^2 - (T_{i-1} - t^*)\sigma_i(T_{i-1}; t^*)^2}{T_i - T_{i-1}}}$$

$$\text{for} \quad T_{i-1} < t < T_i$$

That's term structure dealt with... now strike structure.

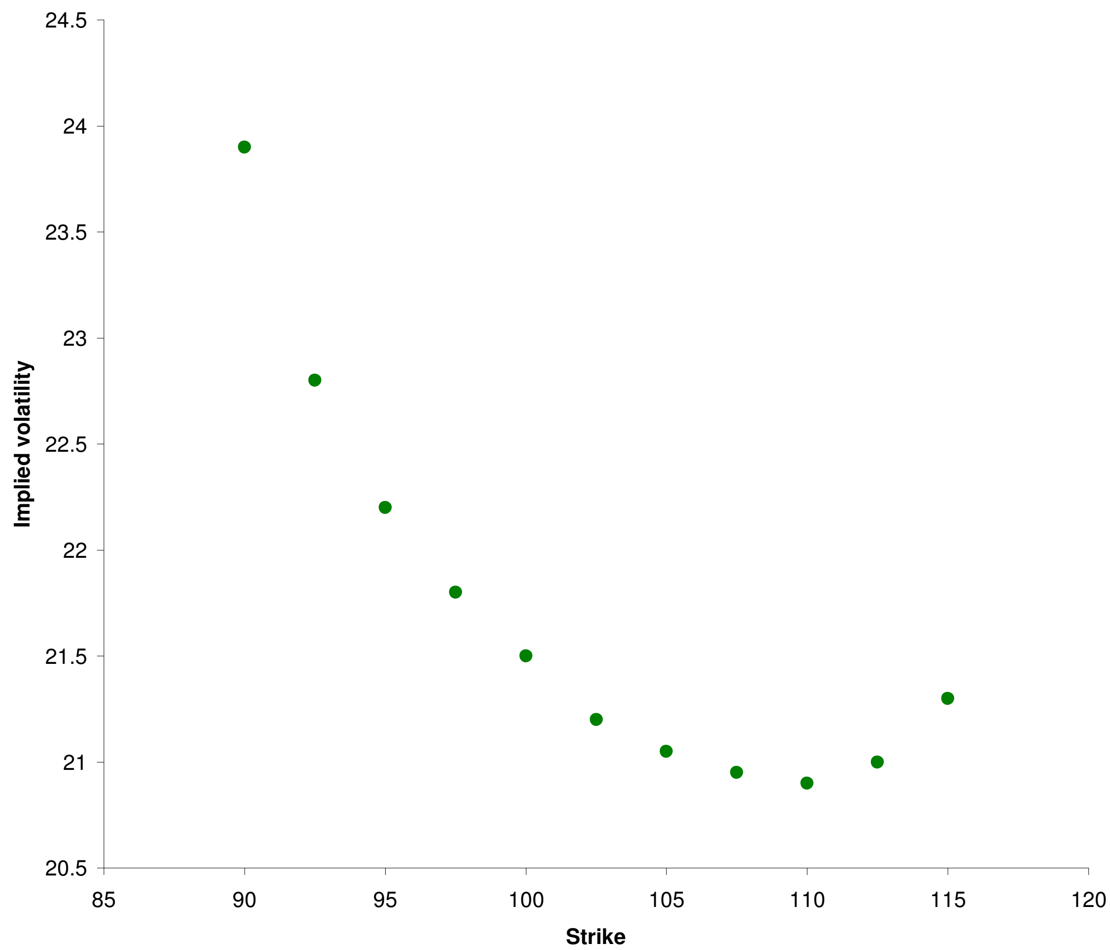
Volatility smiles and skews

Continuing with the example above, suppose that there is also a European call option struck at 100 with an expiry of six months and a price of 12.84.

This corresponds to a volatility of 21.5% in the Black–Scholes equation. Now we have two conflicting volatilities up to the six-month expiry, **20.9%** and **21.5%**.

Clearly we cannot adjust the time dependence of the volatility in any way that is consistent with *both* of these values. What else can we do?

Before we answer this, look at a few more examples. Concentrating on the same example, suppose that there are call options traded with an expiry of six months and strikes of 90, 92.5, 95, 97.5, 100, 102.5, 105, 107.5 and 110.



Implied volatilities against strike price.

The shape of this implied volatility *versus* strike curve is called the **smile**.

In some markets it shows considerable asymmetry, a **skew**, and sometimes it is upside down in a **frown**.

The general shape tends to persist for a long time in each underlying.

Calibrating volatility that varies with expiration and strike

If we managed to accommodate implied volatility that varied with expiry by making actual volatility time dependent perhaps we can accommodate implied volatility that varies with strike as well(?)

We now have implied volatility being a function of two variables, expiration and strike, $\sigma_i(E,T)$, we again ask the question, what is the simplest change to the Black–Scholes model that will result in such an implied volatility?

Just as we made actual volatility a function of t to make implied volatility a function of T, we now make actual volatility a function of t and S!

$$\sigma_a(S,t) \quad \Rightarrow \quad \sigma_i(E,T).$$

But again in finance we often know the answer (the market value) and have to find the question (the parameter). So calibration in practice means

$$\sigma_a(S,t) \quad \Leftarrow \quad \sigma_i(E,T).$$

This is another inverse problem.

But it's mathematically much harder! More in another lecture.

That's enough about calibration for now!

The Market is Wrong!: Volatility arbitrage

And now, how to exploit your volatility forecast! We are *not* going to calibrate, instead we are going to try to make money at the market's expense, assuming we have a better prediction for actual volatility!

We will see a strategy involving buying/selling options, and delta hedging.

Suppose that you believe an option is mispriced. . . how can you profit from this?

Remember that if you are delta hedging then you are only exposed to volatility and not market direction.

So you can interpret a 'mispriced' option as one for which your estimate of volatility differs from the implied volatility.

You have a forecast of volatility, and so does the market (implied).

Black–Scholes tells you all about how to hedge when there is just one volatility, now there are two!

So which delta do you choose? Delta based on actual or implied volatility?

Hedging with actual volatility or implied volatility

This is one of those questions that people always ask, and one that no one seems to know the full answer to. (The only people who know the full details have taken the CQF!)

Scenario: Implied volatility for an option is 20%, but we believe that actual volatility is 30%

Question: How can we make money if our forecast is correct?

Answer: Buy the option and delta hedge.

But what delta do we use?

$$\Delta = N(d_1)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

and

$$d_1 = \frac{\ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

We can all agree on S, E, $T-t$ and r, but not on σ.

So should we use $\sigma = 0.2$ or 0.3?

Vote now!

We have

$$\sigma_a = \text{ actual volatility, 30\%}$$

and

$$\sigma_i = \text{ implied volatility, 20\%.}$$

Case 1: Hedge with actual volatility, σ_a

By hedging with actual volatility we are replicating a short position in a *correctly priced* option.

The payoffs for our long option and our short replicated option will exactly cancel.

The profit we make will be exactly the difference in the Black–Scholes prices of an option with 30% volatility and one with 20% volatility.

(Assuming that the Black–Scholes assumptions hold.)

If $V(S, t; \sigma)$ is the Black–Scholes formula then the guaranteed profit is

$$V(S, t; \sigma_a) - V(S, t; \sigma_i).$$

But how is this guaranteed profit realized?

Let's do the math.

On a mark-to-market basis.

Superscript 'a' means actual and 'i' means implied, these can be applied to deltas and option values. For example, Δ^a is the delta using the actual volatility in the formula. V^i is the theoretical option value using the implied volatility in the formula.

The model

$$dS = \mu S\,dt + \sigma_a S\,dX.$$

Set up a portfolio by buying the option for V^i and hedge with Δ^a of the stock.

Today:

Option	V^i
Stock	$-\Delta^a\,S$
Cash	$-V^i + \Delta^a\,S$

Tomorrow:

Option	$V^i + dV^i$
Stock	$-\Delta^a\, S - \Delta^a\, dS$
Cash	$(-V^i + \Delta^a\, S)(1 + r\; dt)$

Therefore we have made, mark to market,

$$dV^i - \Delta^a \, dS - r(V^i - \Delta^a \, S) \, dt.$$

Because the option would be correctly valued at V^a then we have

$$dV^a - \Delta^a \, dS - r(V^a - \Delta^a \, S) \, dt = 0.$$

So the mark-to-market profit over one time step is

$$dV^i - dV^a + r(V^a - \Delta^a \, S) \, dt - r(V^i - \Delta^a \, S) \, dt$$
$$= dV^i - dV^a - r(V^i - V^a) \, dt = e^{rt} \, d\left(e^{-rt}(V^i - V^a)\right).$$

That is the profit from time t to $t + dt$.

The present value of that profit at time t_0 is

$$e^{-r(t-t_0)} e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right) = e^{rt_0}\, d\left(e^{-rt}(V^i - V^a)\right).$$

So the total profit from t_0 to expiration is

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i.$$

As we said.

We can also write that one time step mark-to-market profit (using Itô's lemma) as

$$\Theta^i\,dt + \Delta^i\,dS + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i\,dt - \Delta^a\,dS - r(V^i - \Delta^a S)\,dt$$

$$= \Theta^i\,dt + \mu S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\sigma_a^2 S^2 \Gamma^i\,dt - r(V^i - \Delta^a S)\,dt + (\Delta^i - \Delta^a)\sigma_a S\,dX$$

$$= (\Delta^i - \Delta^a)\sigma_a S\,dX + (\mu - r)S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i\,dt$$

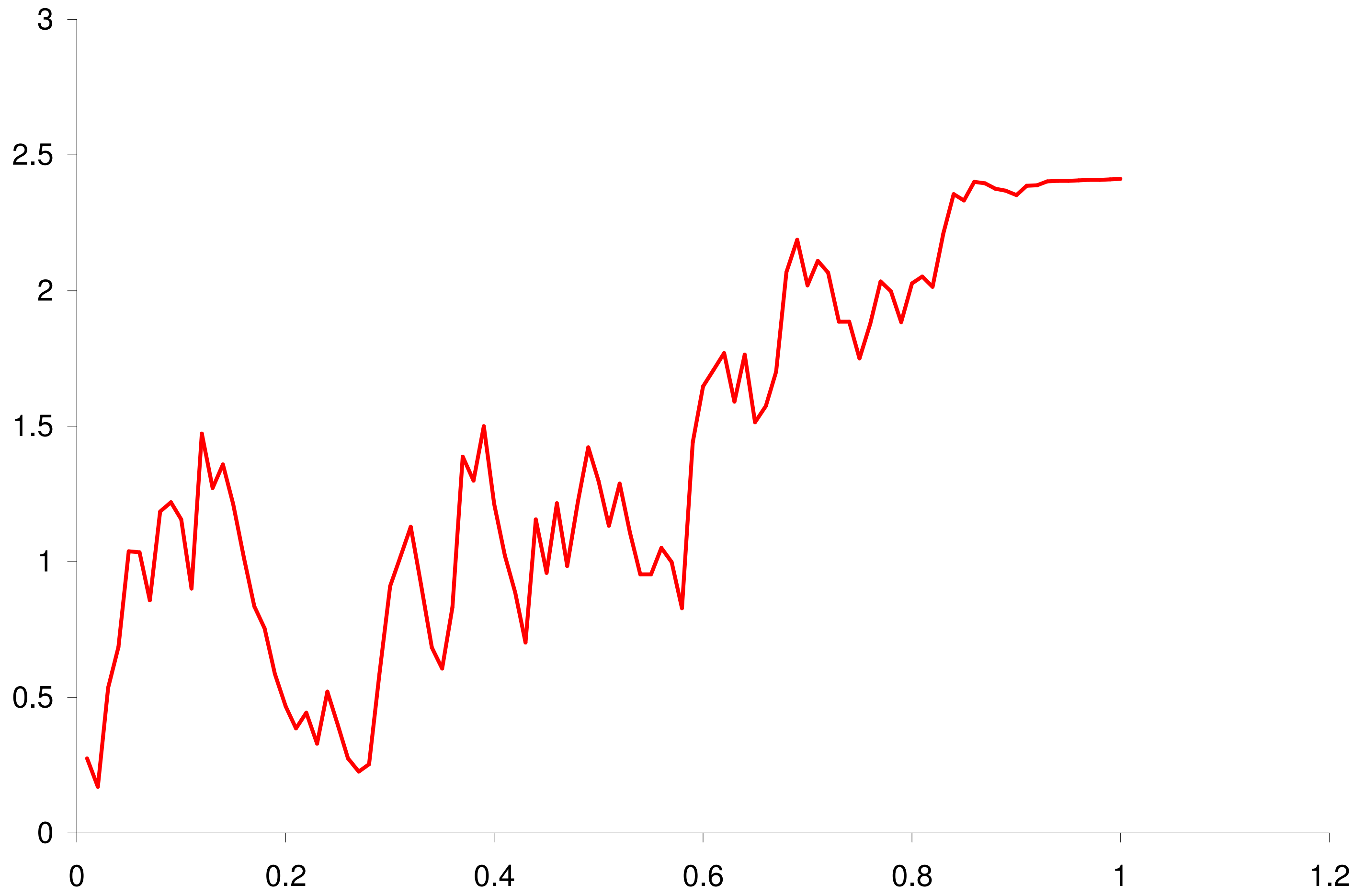
(using Black–Scholes with $\sigma = \sigma_i$)

...

$$= \frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt + \left(\Delta^i - \Delta^a\right)\left((\mu - r)S \, dt + \sigma_a S \, dX\right).$$

Conclusion: The final profit is guaranteed (the difference between the theoretical option values with the two volatilities) but how that is achieved is random.

On a mark-to-market basis you could lose before you gain.



3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2

When S changes, so will V. But these changes do not cancel each other out.

The fluctuation in the portfolio mark-to-market is random. It may even go negative.

Although the path of the profit is random, the final profit is simply the difference between the option valued using actual volatility and that using implied volatility.

This is a known quantity.

An analogy, a bond: Guaranteed outcome, but may lose on a mark-to-market basis in the meantime. May be difficult to exit.

Case 2: Hedge with implied volatility, σ_i

By hedging with implied volatility we are balancing the random fluctuations in the mark-to-market option value with the fluctuations in the stock price.

The evolution of the portfolio value is 'deterministic.'

Let's see how this works.

Buy the option today, hedge using the implied delta, and put any cash in the bank earning r.

The mark-to-market profit from today to tomorrow is

$$dV^i - \Delta^i \, dS - r(V^i - \Delta^i S) \, dt$$

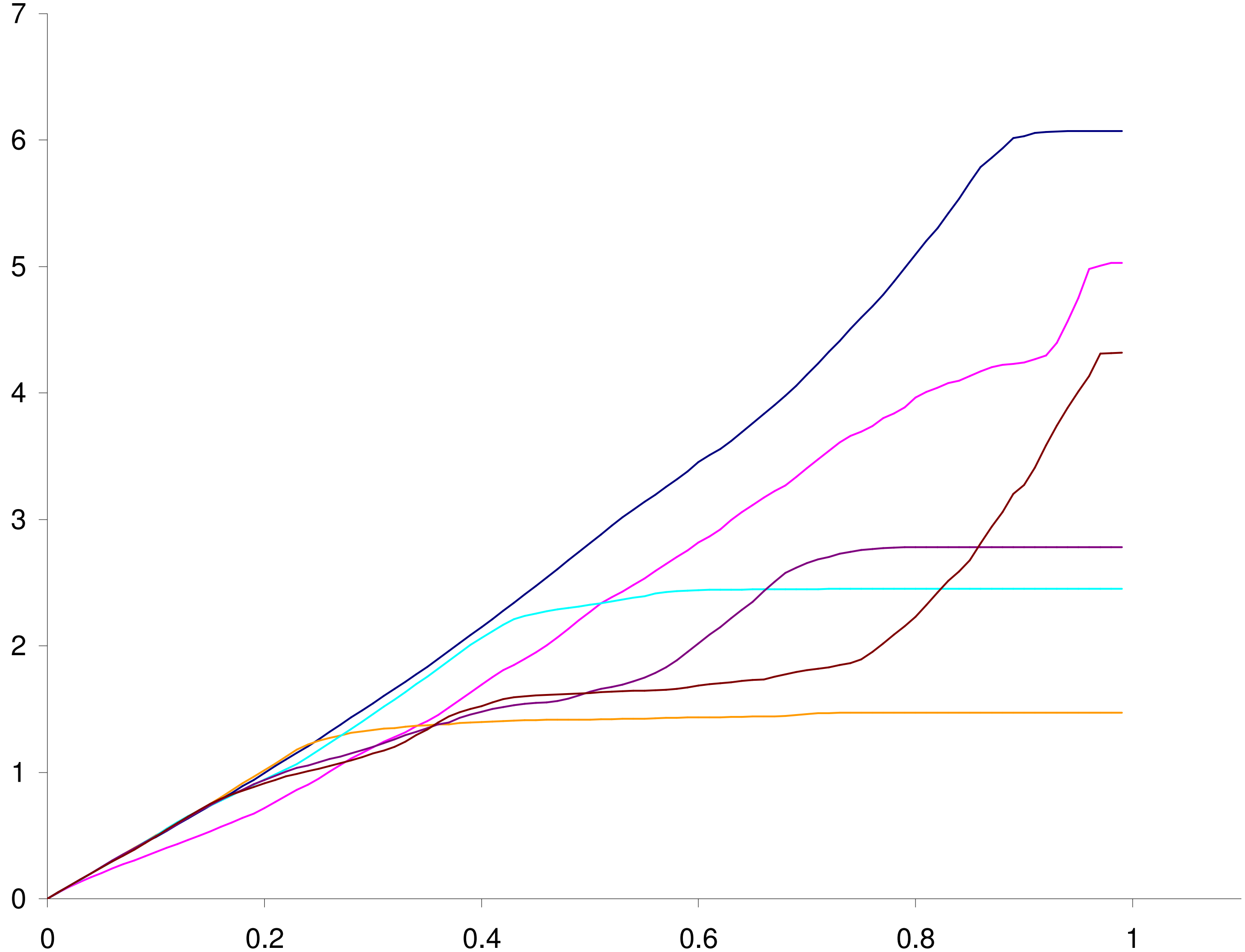
$$= \Theta^i \, dt + \frac{1}{2}\sigma_a^2 S^2 \Gamma^i \, dt - r(V^i - \Delta^i S) \, dt$$

$$= \frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right) S^2 \Gamma^i \, dt.$$

Add up the present value of all of these profits to get a total profit of

$$\frac{1}{2}\left(\sigma_a^2 - \sigma_i^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^i \, dt.$$

This is always positive, but path dependent.



7
6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1

An analogy, money in the bank: Can access money at any time, easy to exit. Always increasing in value. End result uncertain.

Summary

Please take away the following important ideas

- there are many different kinds of volatility to which people refer

- you can relate some of these different volatilities if you make big assumptions, this is calibration

- options can be used for making a profit from volatility models