

Introduction to Numerical Methods

In this lecture...

- The justification for pricing by Monte Carlo simulation
- Grids and discretization of derivatives
- The explicit finite-difference method

By the end of this lecture you will be able to

- ✓ • implement the Monte Carlo method for simulating asset paths and pricing options

- implement the explicit finite-difference method for pricing options

v. powerful numerical
techniques for
solving PDEs

- Stability Analysis

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Introduction

More often than not we must solve option-pricing problems by numerical means.

It is rare to be able to find closed-form solutions for prices unless both the contract and the model are very simple.

The most useful numerical techniques are Monte Carlo simulations and finite-difference methods.

Monte Carlo Simulations

$$P.V. \ E^{\mathbb{Q}} [\text{Payoff}(S)]$$

Relationship between derivative values and simulations

Theory says:

$\mathbb{R} \subset \mathbb{Q}$

- The fair value of an option is the present value of the expected payoff at expiry under a *risk-neutral* random walk for the underlying.

(risk-neutral measure)

The risk-neutral random walk for S is

Write SDE below

- $\longrightarrow dS = rS dt + \sigma S dX. \quad (*)$

This is simply our usual lognormal random walk but with the risk-free rate instead of the real growth rate.

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Justification:

- Binomial method (PW)
- Black–Scholes Equation similar to backward Kolmogorov equation (RA)
- Martingale theory (SL)

We can therefore write

• option value = $e^{-r(T-t)} E[\text{payoff}(S)]$

Handwritten notes:
PV (above $e^{-r(T-t)}$)
 \mathbb{E}^Q (below E)
comes from (*) (above E)

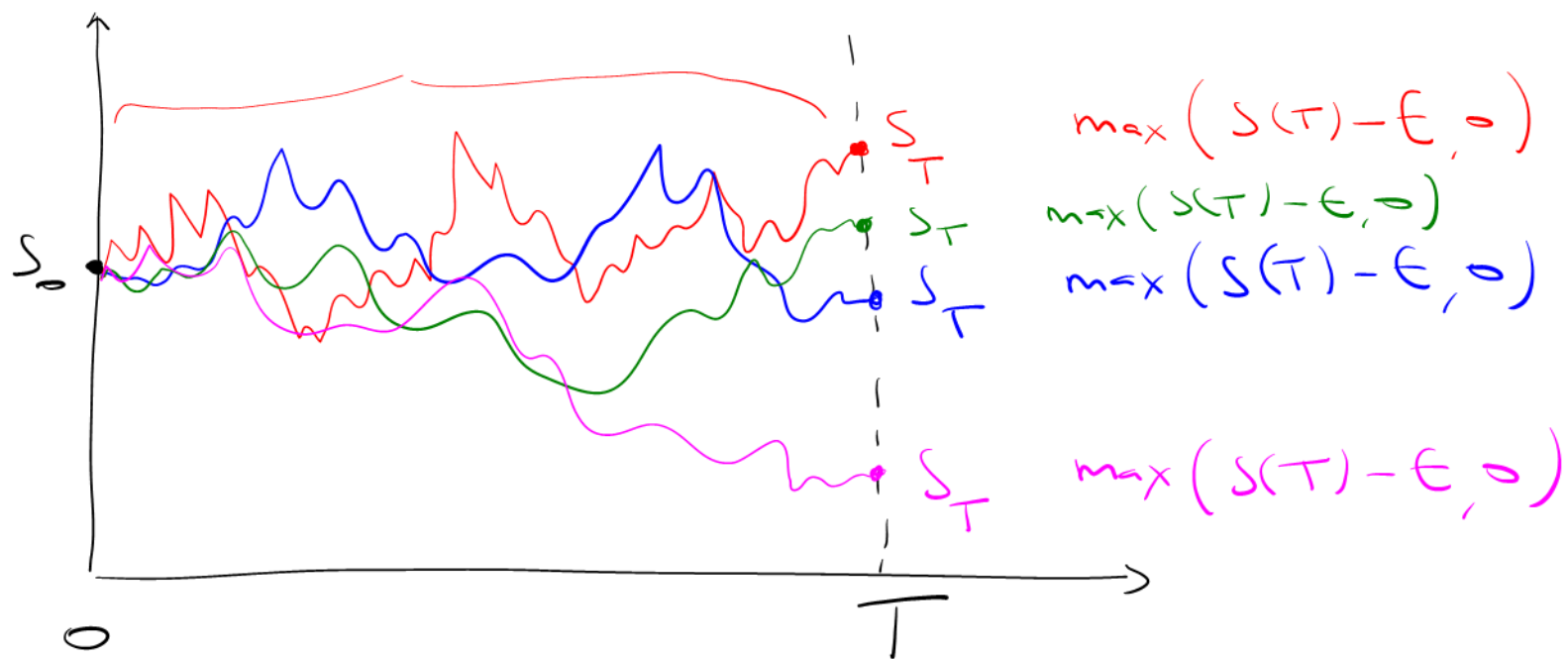
provided that the expectation is with respect to the risk-neutral random walk, not the *real* one.

The algorithm: Monte Carlo Recipe

1. Simulate the risk-neutral random walk starting at today's value of the asset S_0 over the required time horizon. This gives one realization of the underlying price path.



2. For this realization calculate the option payoff.
3. Perform many more such realizations over the time horizon.
4. Calculate the average payoff over all realizations.
5. Take the present value of this average, this is the option value.



$$\text{Average} = \frac{1}{n} \sum_{i=1}^n \max(S^{(i)}(T) - E, 0)$$

$$\text{Option price } C = e^{-r(T-t)} \underbrace{\frac{1}{n} \sum_{i=1}^n \max(S^{(i)}(T) - E, 0)}_{\mathbb{E}^Q[\text{Payoff}(S)]}$$

How do we simulate the asset?

Two ways:

1. **If** the s.d.e. for the asset path is integrable **and** the contract is not path dependent (or American) **then** simulate in 'one giant leap'

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\phi\sqrt{T}}$$

2. **Otherwise** you will have to simulate time step by time step, the entire path

$$S_{i+1} = S_i (1 + r\delta t + \sigma\phi\sqrt{\delta t}) \leftarrow$$

An advantage of method 2. is then being able to price all sorts of complex products.

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One giant leap: A method that works in special cases

For the lognormal random walk we are lucky that we can find a simple, and exact, time stepping algorithm.

We can write the risk-neutral stochastic differential equation for S in the form

$$d(\log S) = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dX.$$

This can be integrated exactly to give

$$\mu, \sigma, r \in \mathbb{R}$$

$$S(t) = S(0) \exp \left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \int_0^t dX \right).$$

i.e.

H0 IV & then integration

- $\left(S(T) = S(0) \exp \left(\left(r - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T} \phi \right) \right)$ M1L5

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Because this expression is exact and simple it is the best time stepping algorithm to use... but only if we have a payoff that only depends on the final asset value, i.e. is European and path independent.

We can then simulate the final asset price in one giant leap, using a time step of T if both of these are true

- the s.d.e. is integrable and

$$\text{If } r = r(t)$$

- the contract is European and not path dependent

$$S_{t+\delta t} = S_t e^{(r - \frac{1}{2}\sigma^2)\delta t + \sigma\phi\sqrt{\delta t}}$$

Simulating the entire path: A method that always works

Price paths are simulated using a discrete version of the stochastic differential equation for S .

An obvious choice is to use

GBM (risk-neutral)
discrete

$$\delta S = rS \delta t + \sigma S \sqrt{\delta t} \phi,$$
$$S_{i+1} - S_i = rS_i \delta t + \sigma S_i \phi \sqrt{\delta t}$$

where ϕ is from a standardized Normal distribution.

- This way of simulating the time series is called the **Euler method**. This method has an error of $O(\delta t)$.

Errors

There are two (at least) sources of error in the Monte Carlo method:

$$dS = rS dt + \sigma S \phi \sqrt{dt}$$

- If the size of the time step is δt then we may introduce errors of $O(\delta t)$ by virtue of the discrete approximation to continuous events

$$dX \sim O(\sqrt{dt})$$

$$dX \sim N(0, dt)$$

- Because we are only simulating a finite number of an infinite number of possible paths, the error due to using N realizations of the asset price paths is $O(N^{-1/2})$.

$$\text{Error} = \frac{\sigma}{\sqrt{n}}$$

n - no. of simulations

Generating Normal variables

- **Quick 'n' dirty:** A useful distribution that is easy to implement on a spreadsheet, and is fast, is the following *approximation* to the Normal distribution:

$$\frac{12}{n} \left[\sum_{i=1}^n \psi_i - \frac{n}{2} \right] \quad \left(\sum_{i=1}^{12} \psi_i \right) - 6,$$

$\sim N(0, 1)$

where the ψ_i are independent random variables, drawn from a uniform distribution over zero to one.

There are other methods such as **Box-Muller** more later.

Other issues

- Greeks



requires more work

$$\Delta = \frac{V(S+\delta S, t) - V(S, t)}{\delta S}$$

- Early exercise (and other decisions)

↳ free boundary.

$$\Delta = \frac{V(S+\delta S, t) - V(S-\delta S, t)}{\delta S}$$

Advantages of Monte Carlo simulations

- ✓ • The mathematics that you need to perform a Monte Carlo simulation can be very basic

- Correlations can be easily modeled, and it is easy to price options on many assets (high-dimensional contracts)

$$dx_i dx_j = \rho_{ij} \quad dS_i = \mu_i S_i dt + \sigma_i S_i dx_i \quad i=1, \dots, n$$

- It is computationally quite efficient in high dimensions

- There is plenty of software available, at the very least there are spreadsheet functions that will suffice for most of the time

- To get a better accuracy, just run more simulations

$$\varepsilon \sim O\left(\frac{1}{\sqrt{n}}\right) \quad \begin{matrix} n \rightarrow \infty \\ \varepsilon \rightarrow 0 \end{matrix}$$

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- The effort in getting *some* answer is very low

hard to make mistakes

- The models can often be changed without much work

flexible

- Complex path dependency can often be easily incorporated

Exotic

- Many contracts can be priced at the same time

- People accept the technique, and will believe your answers

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Disadvantages of Monte Carlo simulations

- The method is very slow, you need a lot of simulations to get an accurate answer *especially in low dim's*
- Finding the greeks can be hard *∴ we need several option prices*
- The method does not cope well with early exercise *American*

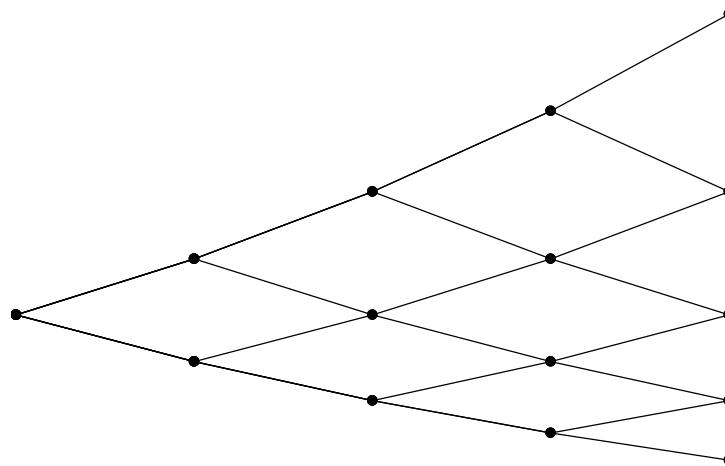
Finite Difference Methods

Monte Carlo simulations can be very slow to converge to the answer, and they do not give us the greeks without further effort.

There is a method that is very similar to the binomial tree method which is the method of choice for certain types of problem.

Grids

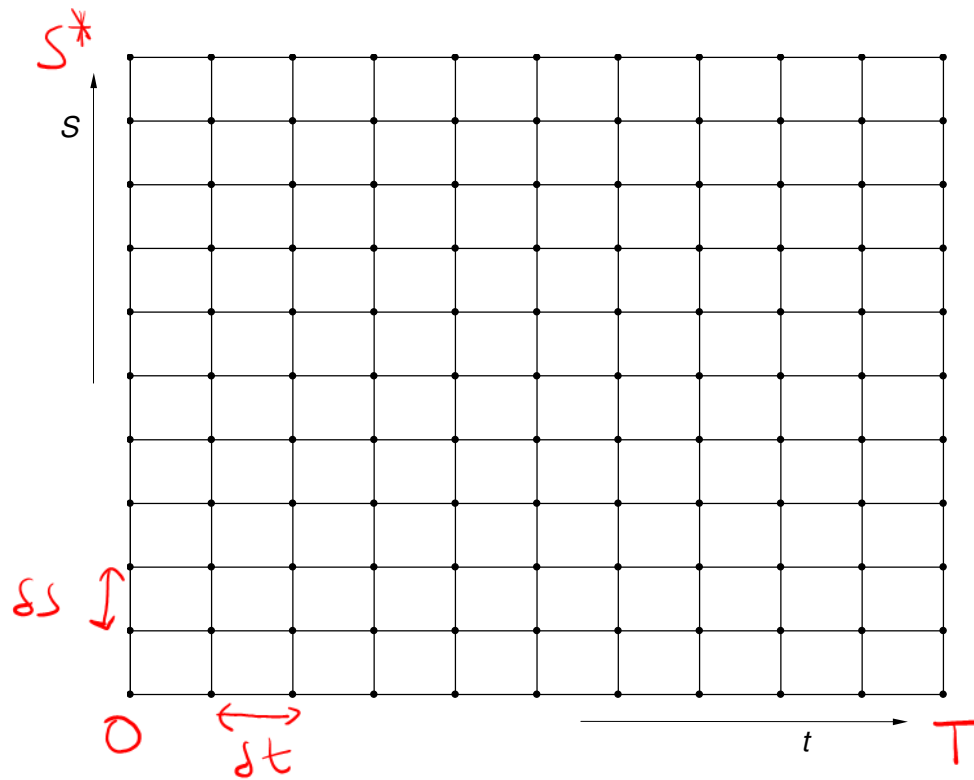
Recall the shape of the binomial tree. . .



The shape of the tree is determined by the asset volatility.

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$S^* \approx \lim_{S \rightarrow \infty}$
 $S^* \approx 1E$ very common



The finite-difference grid.

once $\delta t, \delta S$ defined they are fixed

The finite-difference grid usually has equal time steps and equal S steps.

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Differentiation using the grid

Replace each term in PDE using T.S.E

Notation: time step δt and asset step δS . The grid is made up of the points at asset values

$\tau = T - t$ $t = T$ $\tau = 0$ I, K max. no. of
 $\frac{\partial}{\partial t} \equiv \frac{d\tau}{dt} \frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \tau}$ steps in $S := i \delta S$
stock and
time
respectively. $t := T - k \delta t$

$$\delta S = \frac{S^*}{I}$$

$$\delta t = \frac{T}{K}$$

Time to expir.

$$0 \longleftarrow T$$

$$\frac{\partial v}{\partial t} \rightarrow -\frac{\partial v}{\partial \tau}$$

where $0 \leq i \leq I$ and $0 \leq k \leq K$.

We will be solving for the asset value going from zero up to the asset value $I \delta S$.

without truncation K is decreasing from K

The Black-Scholes equation is to be solved for $0 \leq S < \infty$ so that $I \delta S$ is our approximation to infinity.

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etc.

$$V(S, t) = V_i^k = V(i \delta S, T - k \delta t).$$

time

S time to expiry

stock

- $$V_{k+1}$$


$$V(t) = V_{i-1}$$

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Approximating θ

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0$$

The definition of the first time derivative of V is simply

$$\frac{\partial V}{\partial t} = \lim_{h \rightarrow 0} \frac{V(S, t) - V(S, t - h)}{h = \delta t}$$

backward in time

$V(S, t + \delta t)$ " V_i^{k+1}

$V(S, t)$ " V_i^k

It follows naturally that we can approximate the time derivative from our grid of values using

$$\frac{\partial V}{\partial t}(S, t) \approx \frac{V_i^k - V_i^{k+1}}{\delta t}$$

1st order accurate,

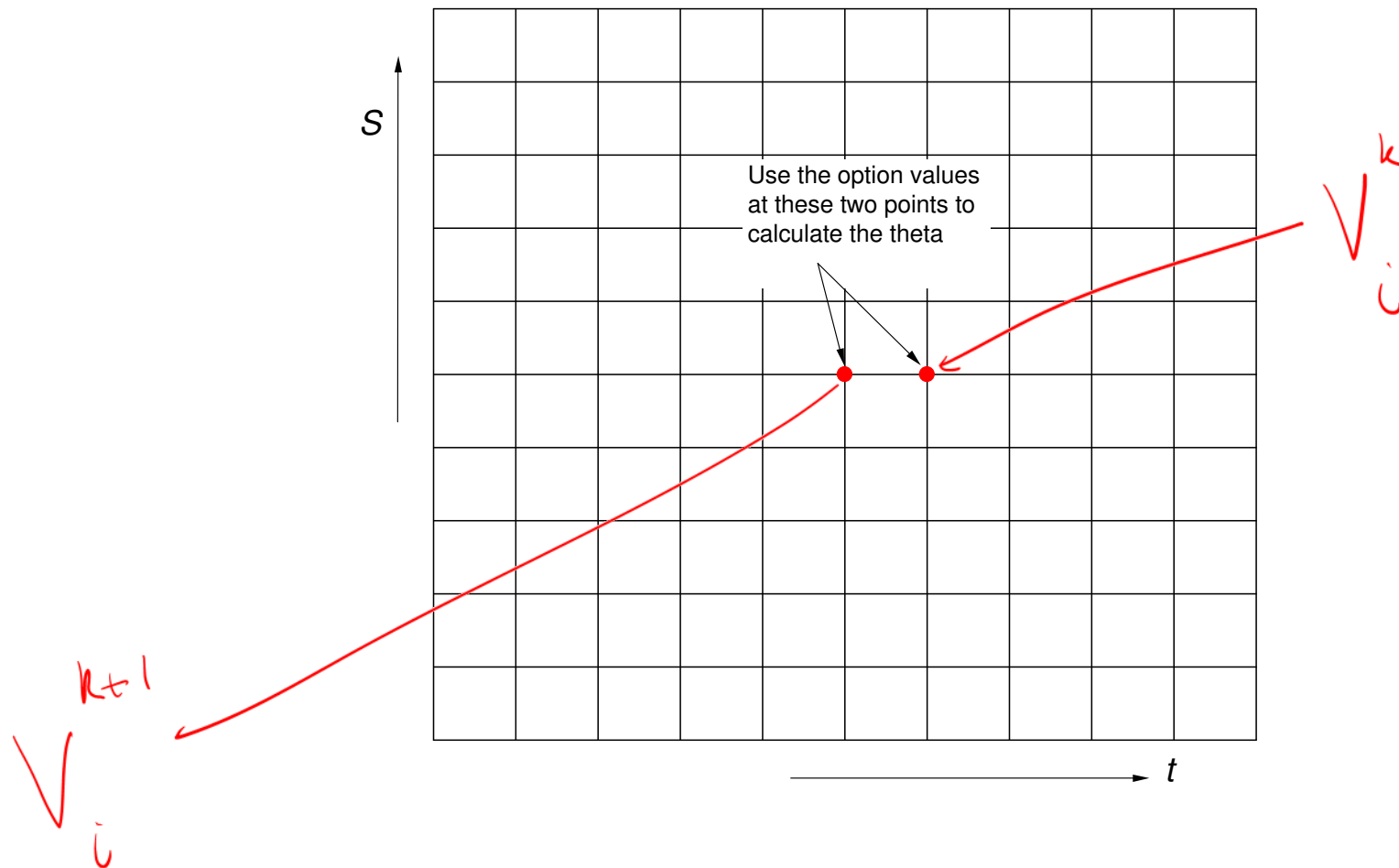
This is our approximation to the option's theta.

$$V(t + \delta t, S) = V(t, S) + \frac{\partial V}{\partial t} \delta t + O(\delta t^2)$$

$\delta(\delta t)$

$$\frac{V(t + \delta t, S) - V(t, S)}{\delta t} = \frac{\partial V}{\partial t}$$

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Approximating the theta.

How accurate is this approximation?

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We can expand the option value at asset value S and time $t - \delta t$ in a Taylor series about the point S, t as follows.

$$V(S, t - \delta t) = V(S, t) - \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

In terms of values at grid points this is just

$$V_i^k = V_i^{k+1} + \delta t \frac{\partial V}{\partial t}(S, t) + O(\delta t^2).$$

Which, upon rearranging, is

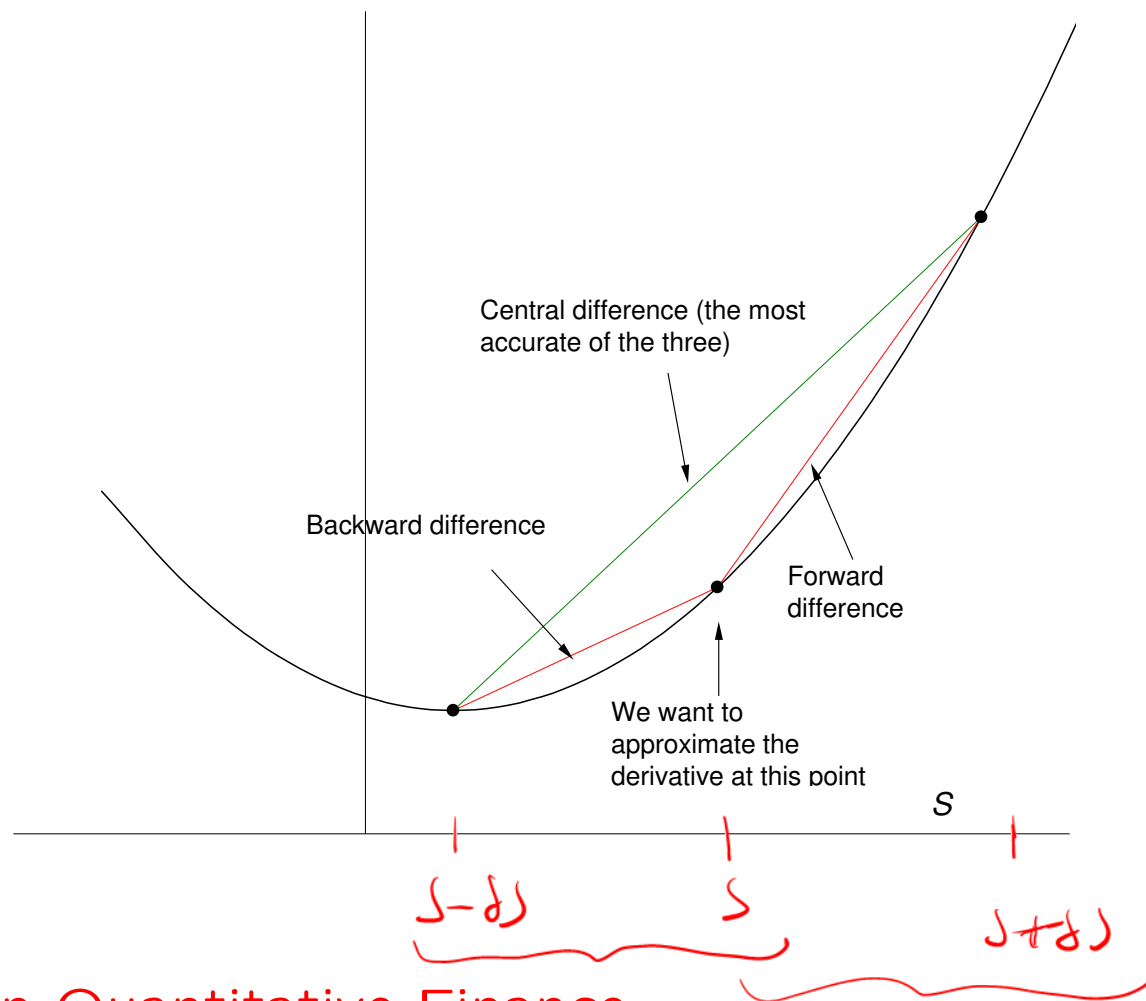
$$\frac{\partial V}{\partial t}(S, t) = \frac{V_i^k - V_i^{k+1}}{\delta t} - O(\delta t).$$

- The error is $O(\delta t)$.

Approximating Δ

$$\frac{\partial V}{\partial S}$$

Examine a cross section of the grid at one of the time steps.



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These three approximations are

Computations	$\frac{V_{i+1}^k - V_i^k}{\delta S},$	fwd
Fluid	$\frac{V_i^k - V_{i-1}^k}{\delta S}$	bwd
Dynamics	and $\frac{V_{i+1}^k - V_{i-1}^k}{2 \delta S}.$	central

These are called a **forward difference**, a **backward difference** and a **central difference** respectively.

[One of these approximations is better than the others.]

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From a Taylor series expansion of the option value about the point $S + \delta S, t$ we have

$$\textcircled{1} \quad V(S + \delta S, t) = V(S, t) + \delta S \frac{\partial V}{\partial S}(S, t) + \frac{1}{2} \delta S^2 \frac{\partial^2 V}{\partial S^2}(S, t) + \frac{1}{3!} \frac{\partial^3 V}{\partial S^3}(\delta S^3) + o(\delta S^4)$$

Similarly,

$$\textcircled{2} \quad V(S - \delta S, t) = V(S, t) - \delta S \frac{\partial V}{\partial S}(S, t) + \frac{1}{2} \delta S^2 \frac{\partial^2 V}{\partial S^2}(S, t) - \frac{1}{3!} \frac{\partial^3 V}{\partial S^3}(\delta S^3) + o(\delta S^4)$$

$$\textcircled{1} - \textcircled{2}: V(S + \delta S) - V(S - \delta S) = 2 \delta S \frac{\partial V}{\partial S} + o(\delta S^3) \quad \text{Rearrange:} \quad \frac{V(S + \delta S) - V(S - \delta S)}{2 \delta S} + o(\delta S^2)$$

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2 \delta S}$$

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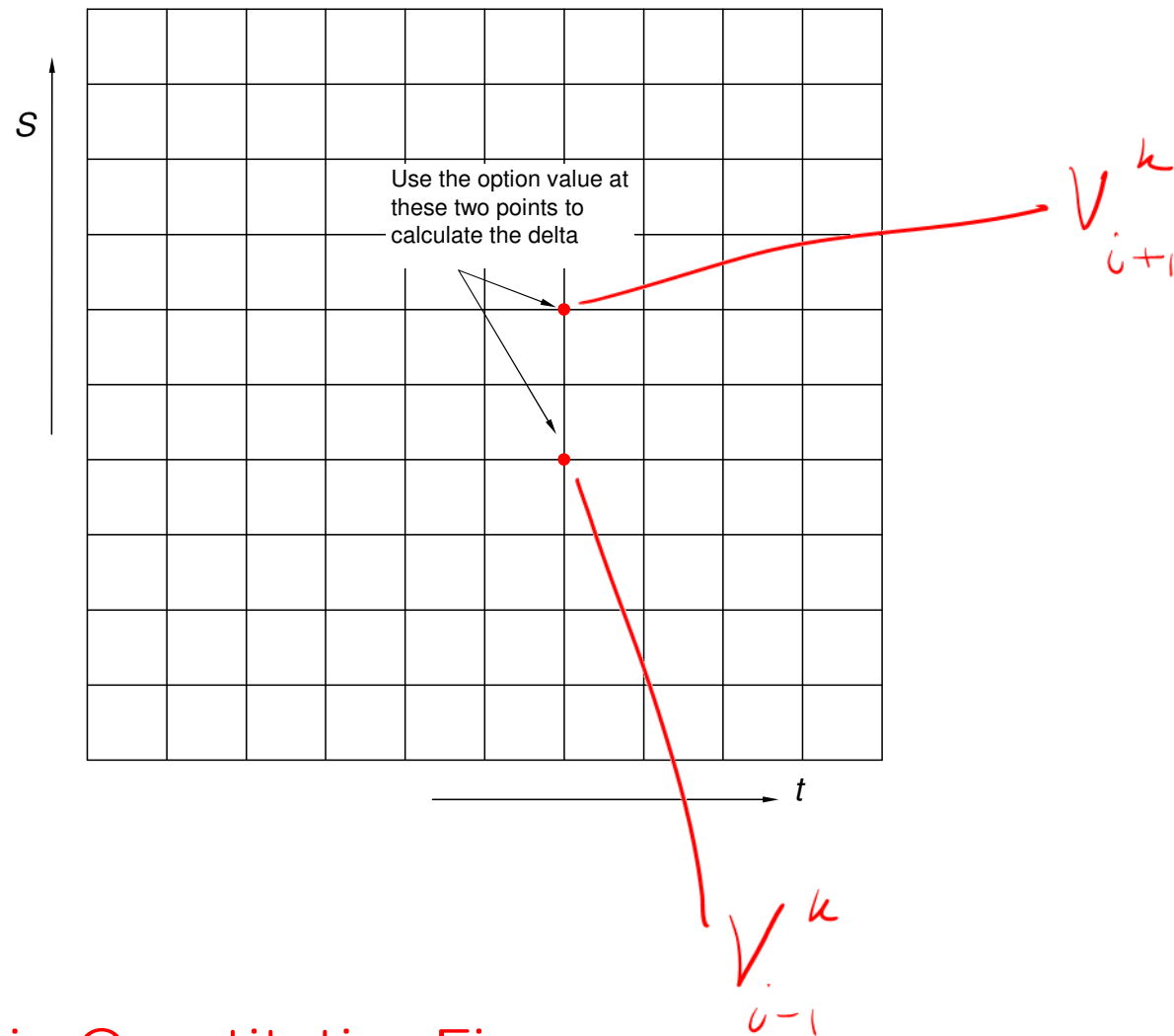
From these we get

$$\checkmark \frac{\partial V}{\partial S}(S, t) = \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} + O(\delta S^2).$$

2nd order
accurate

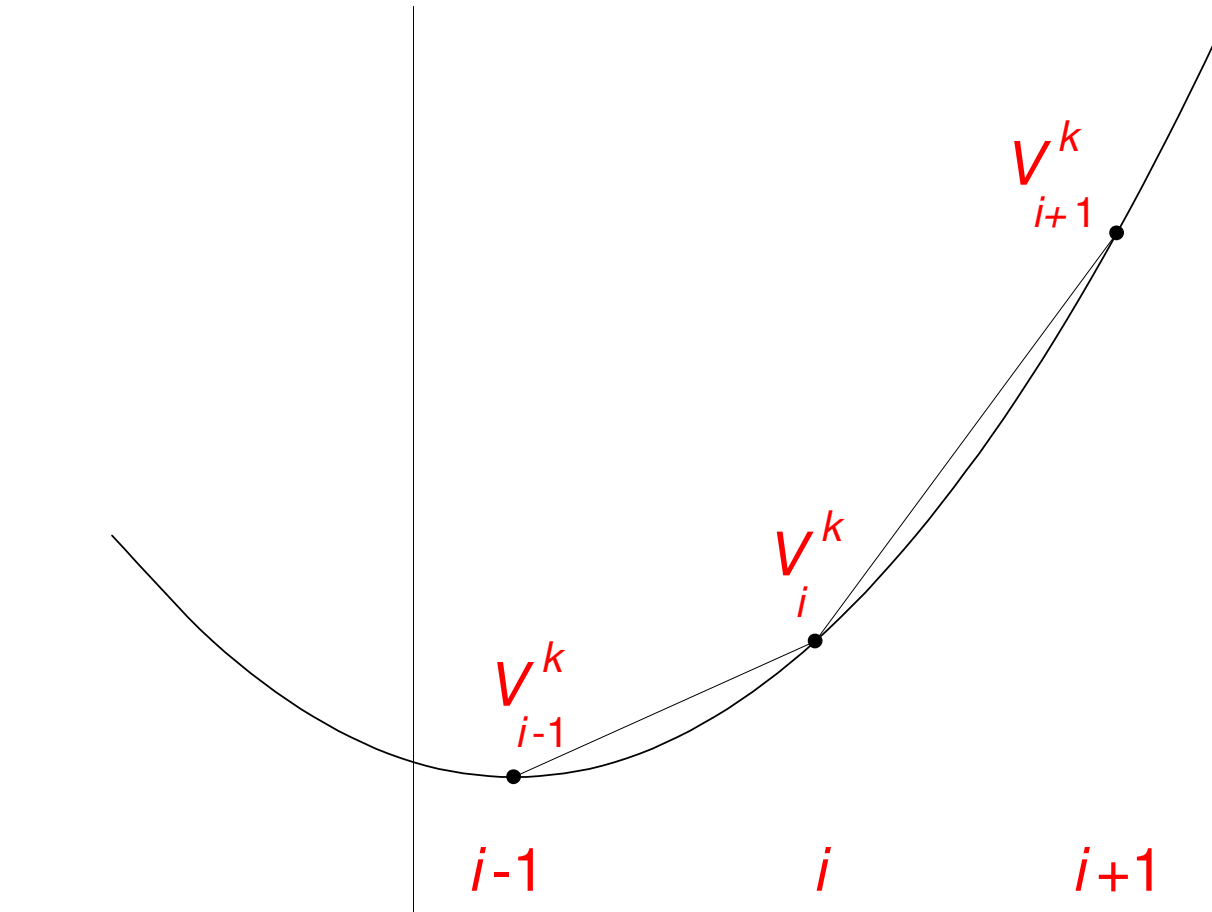
- The central difference has an error of $O(\delta S^2)$, the error in the forward and backward differences are both much larger, $O(\delta S)$.

The central difference calculated at S requires knowledge of the option value at $S + \delta S$ and $S - \delta S$.



Approximating Γ

Gamma is the sensitivity of the delta to the underlying.




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Calculate the delta half way between i and $i + 1$, and the delta half way between $i - 1$ and i ... and difference them!

$$\text{Forward difference} = \frac{V_{i+1}^k - V_i^k}{\delta S}.$$

$$\text{Backward difference} = \frac{V_i^k - V_{i-1}^k}{\delta S}.$$


① + ②: $V(S + \delta S) + V(S - \delta S) = 2V(S, t) + \frac{\partial^2 V}{\partial S^2} \delta S^2 + O(\delta S^4)$



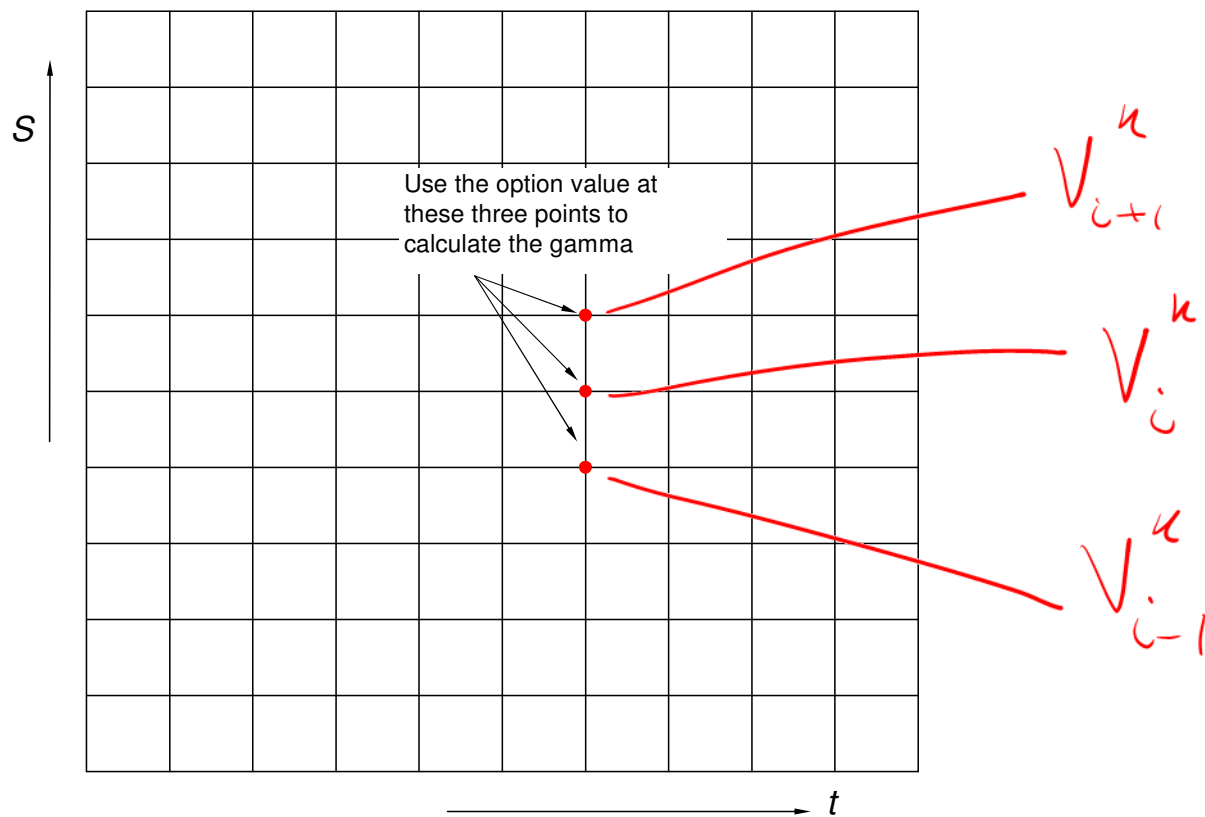
rearrange to get $\frac{\partial^2 V}{\partial S^2} = \frac{V(S - \delta S, t) - 2V(S, t) + V(S + \delta S, t)}{\delta S^2} + O(\delta S^2)$

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Therefore the natural approximation for the gamma is

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2}(S, t) &\approx \frac{\frac{V_{i+1}^k - V_i^k}{\delta S} - \frac{V_i^k - V_{i-1}^k}{\delta S}}{\delta S} \\ &= \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}.\end{aligned}$$


The error in this approximation is also $O(\delta S^2)$.



To summarise

$$\frac{\partial V}{\partial t} \approx \frac{V_i^k - V_{i+1}^{k+1}}{\delta t}; \quad \frac{\partial V}{\partial S} \approx \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i-1}^k - 2V_i^k + V_{i+1}^k}{\delta S^2}$$

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Final conditions and payoffs

We know that at expiry the option value is just the payoff function. At expiry we have

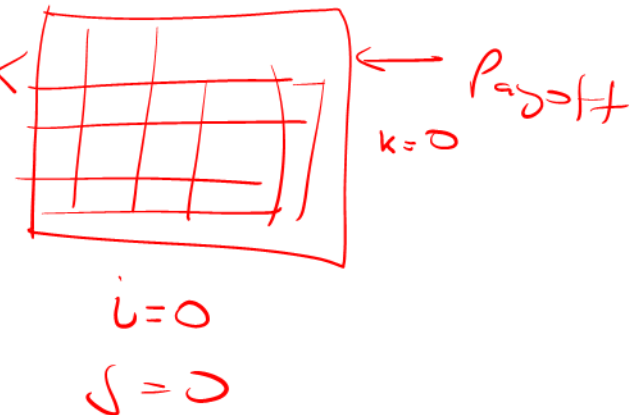
$$V(S, t) := V_i^k$$

$$V(S, T) = \text{Payoff}(S)$$

or, in our finite-difference notation,

At expiry $k=0$

$$V_i^0 = \text{Payoff}(i \delta S).$$



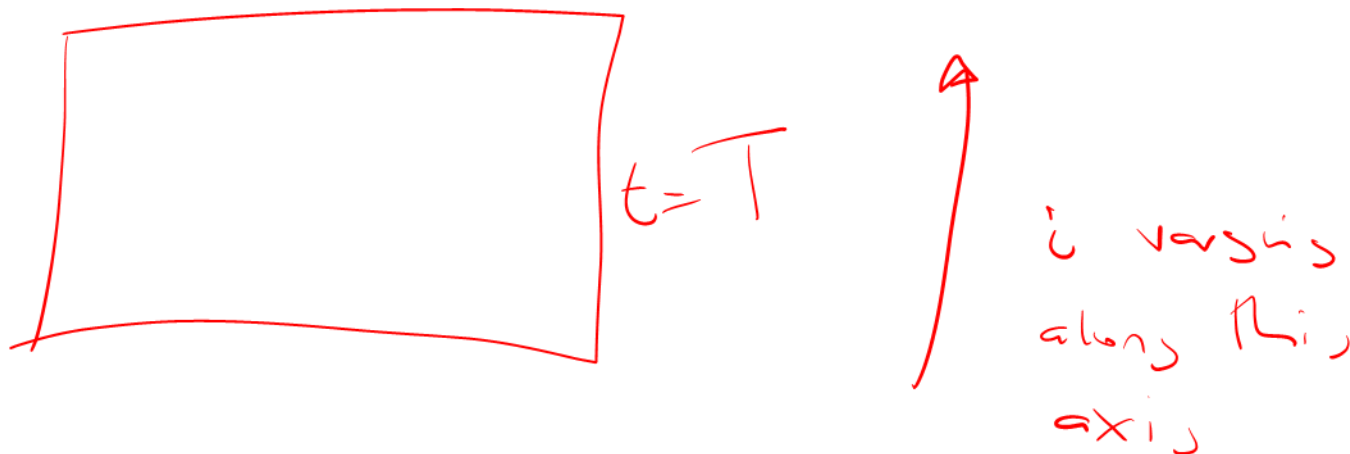
The right-hand side is a known function.

For example, if we are pricing a call option we have

in F.D. notation $k = K$ $S(T)$

$$V_i^0 = \max(\overbrace{i \delta S - E}^{S(T)}, 0).$$

This final condition will get our finite-difference scheme started.



The explicit finite-difference method

The Black-Scholes equation is

$$\sigma(S, t)$$
$$r(t) \quad D(S, t)$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.$$

Write this as

$$\frac{\partial V}{\partial t} + a(S, t) \frac{\partial^2 V}{\partial S^2} + b(S, t) \frac{\partial V}{\partial S} + c(S, t) V = 0$$

Handwritten annotations above the equation:

- $\frac{1}{2}\sigma^2 S^2$ above $a(S, t)$
- $(r-D)S$ above $b(S, t)$
- $-r$ above $c(S, t)$

so that we can examine more general problems.

Using the above approximations

$$\frac{\frac{\partial V}{\partial t}}{\delta t} + a_i^k \left(\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\cancel{\delta S^2}} \right) + b_i^k \left(\frac{V_{i+1}^k - V_{i-1}^k}{2\cancel{\delta S}} \right)$$

$$+ \overset{\checkmark}{c_i^k} V_i^k = O(\delta t, \delta S^2).$$

$-r = 0$ & is 1st order accurate in time, 2nd order accurate in stock price

This can be rearranged...

In the above there is only one option price that is unknown

$$V_i^{k+1}$$

What coeffs. of $V_{i-1}^k, V_i^k, V_{i+1}^k$

$$V_i^{k+1} = V_i^k + \frac{1}{2} \sigma^2 \left(V_{i+1}^k - 2V_i^k + V_{i-1}^k \right) \delta t + \frac{(r-D)\delta t}{2} (V_{i+1}^k - V_{i-1}^k) - r V_i^k \delta t$$

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Difference
Eq.

$$V_i^{k+1} = \dots \overset{(\quad)}{a_i} V_{i+1}^k + \dots \overset{(\quad)}{b_i} V_i^k + \dots \overset{(\quad)}{c_i} V_{i-1}^k.$$

This is an equation for V_i^{k+1} given three option values at time k .

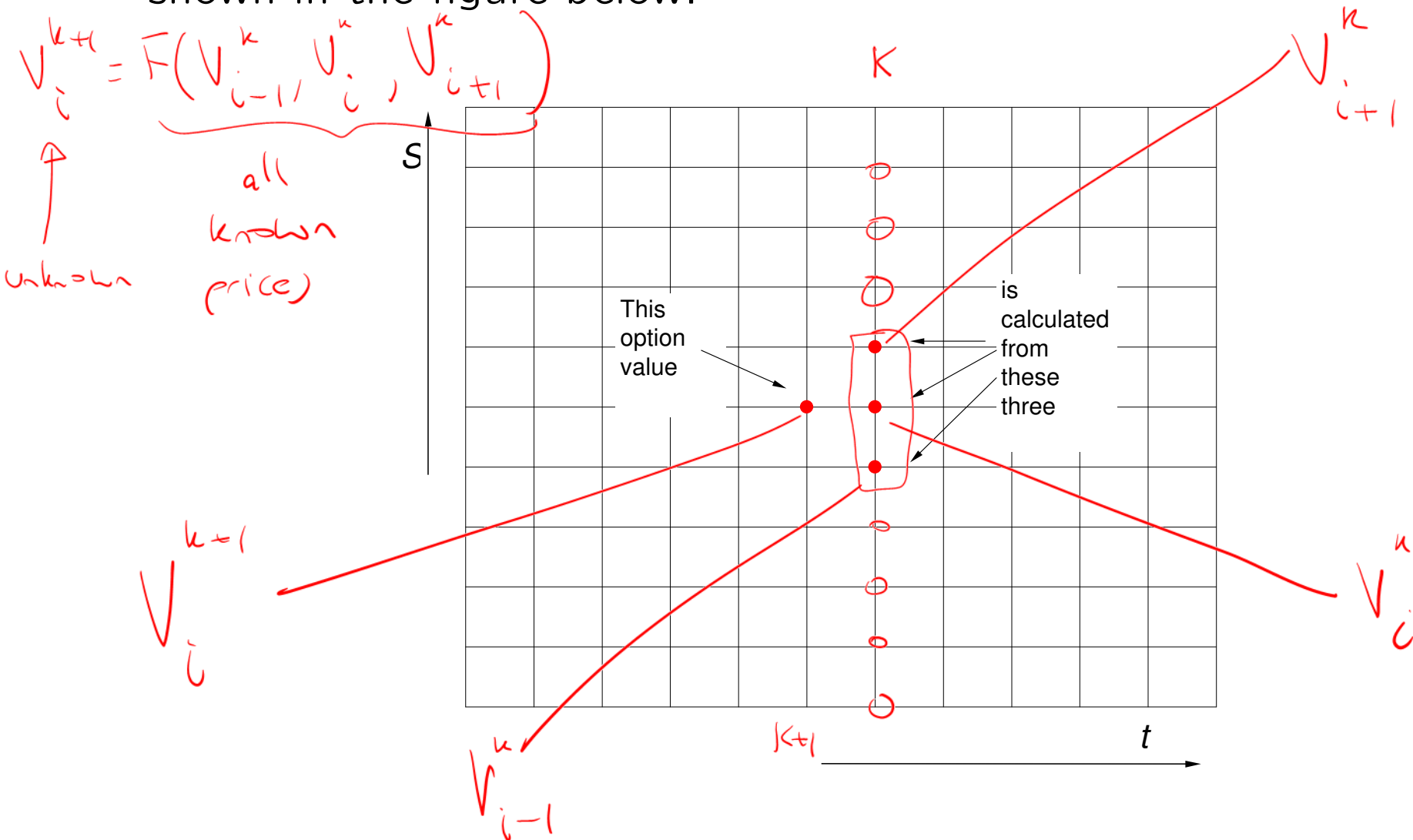
$$S=0 \Rightarrow i=0 \quad Q_0=0=\phi_0 \quad D_0=1-r\Delta t$$

(That's why this is called the **explicit finite-difference method**.)

each new option price at $k+1$ is
obtained from 3 option prices at k .

$y = f(x)$ explicit
 $g(x, y) = 0$ implicit

The relationship between the option values in the algorithm is shown in the figure below.

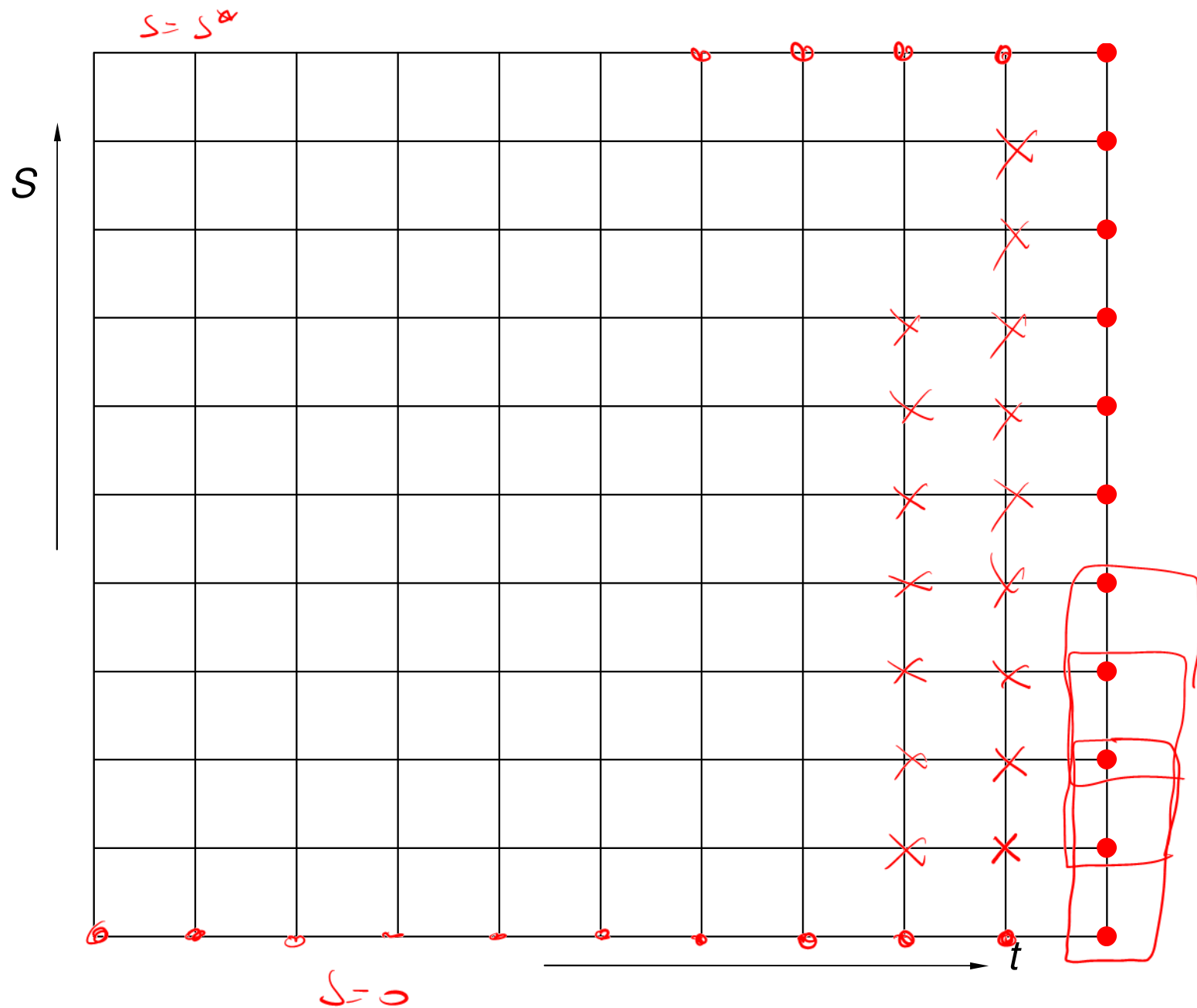


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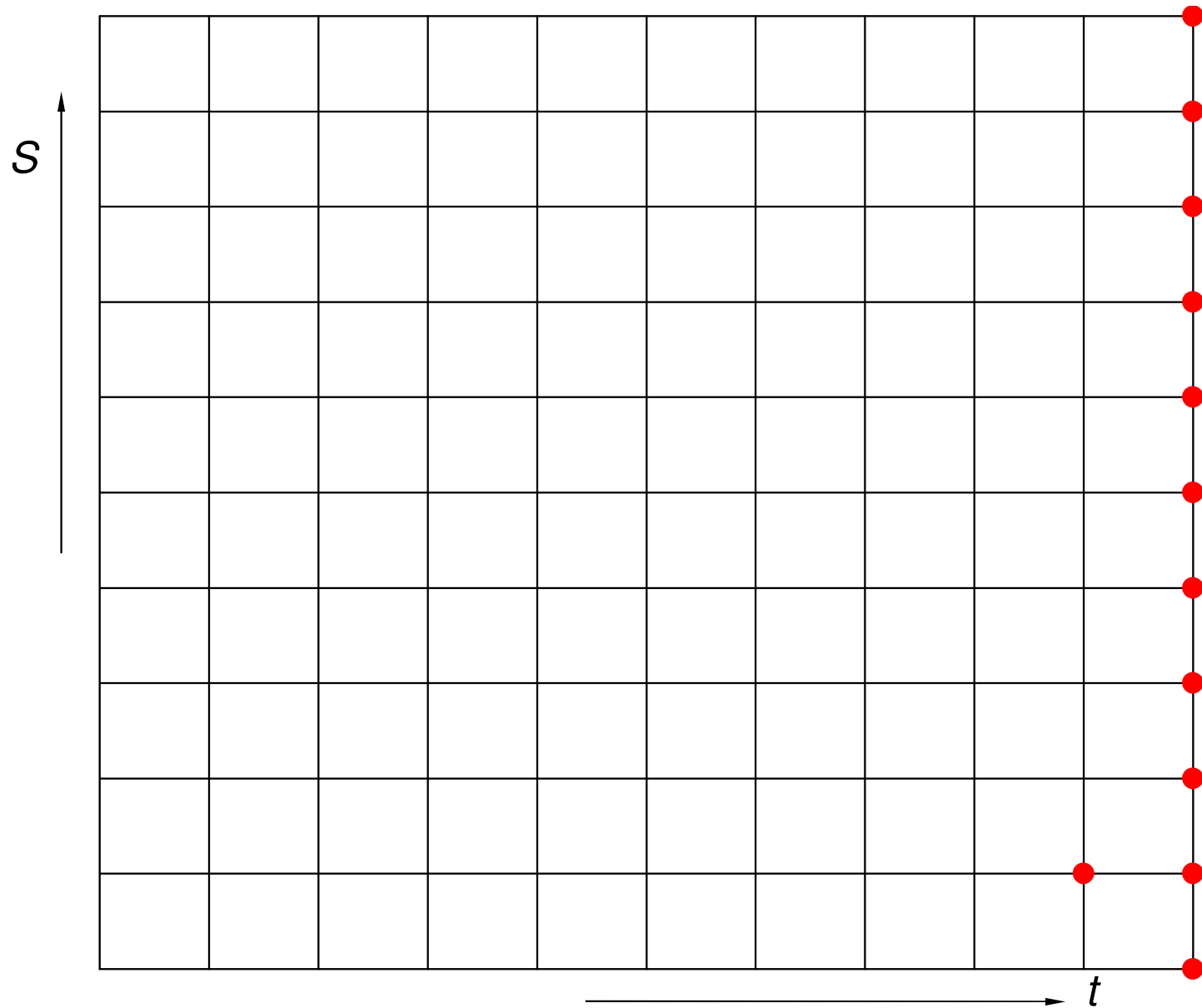
Points to note:

- The time derivative uses the option values at 'times' k and $k + 1$, whereas the other terms all use values at k .
- The gamma term is a central difference, in practice one never uses anything else.
- The delta term uses a central difference. There are often times when a one-sided derivative is better. We'll see examples later.

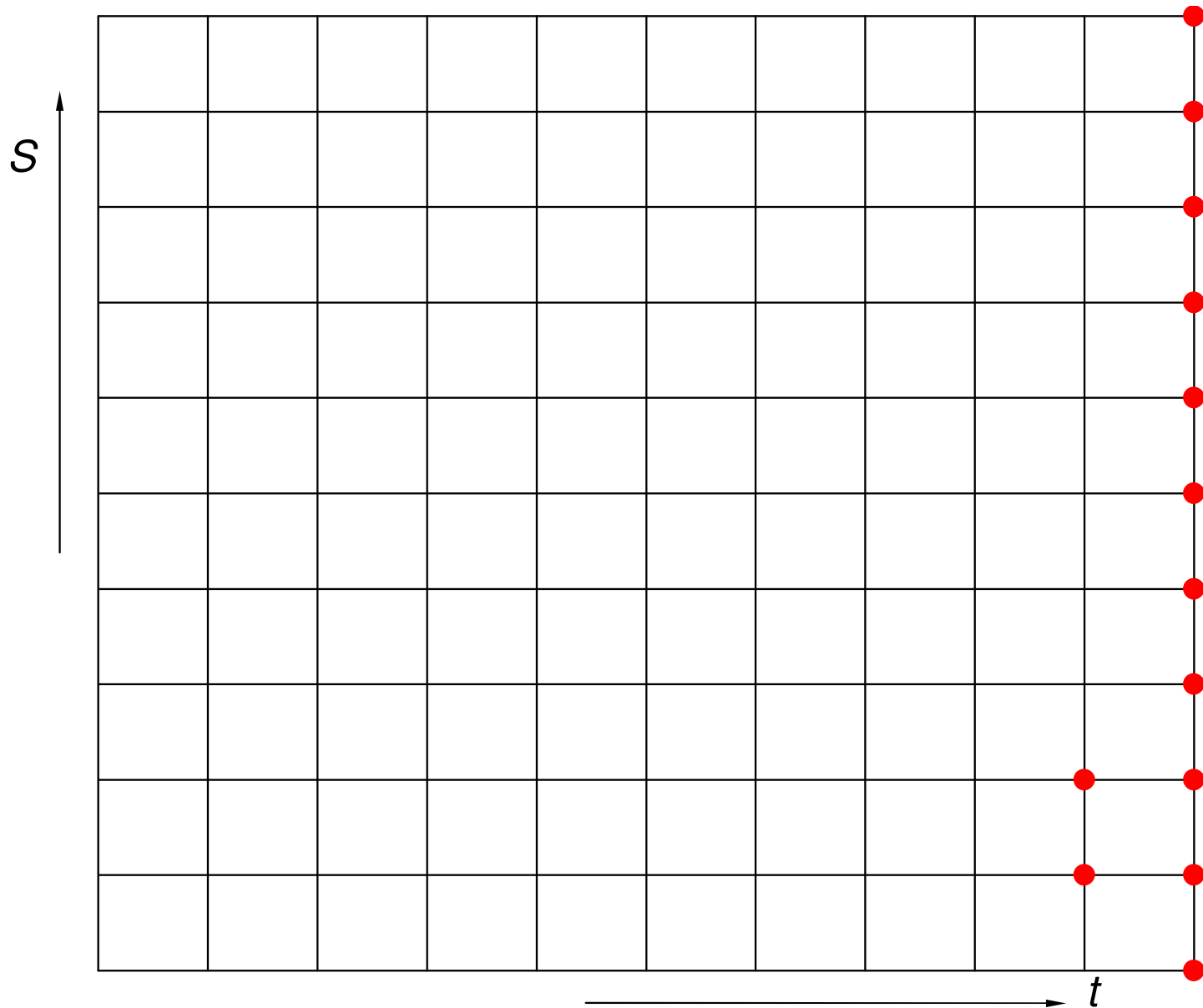
- The asset- and time-dependent functions a , b and c have been valued at $\underbrace{S_i = i \delta S}_{\text{stock}}$ and $\underbrace{t = T - k \delta t}_{\text{time}}$ with the obvious notation.
- The error in the equation is $O(\delta t, \delta S^2)$.



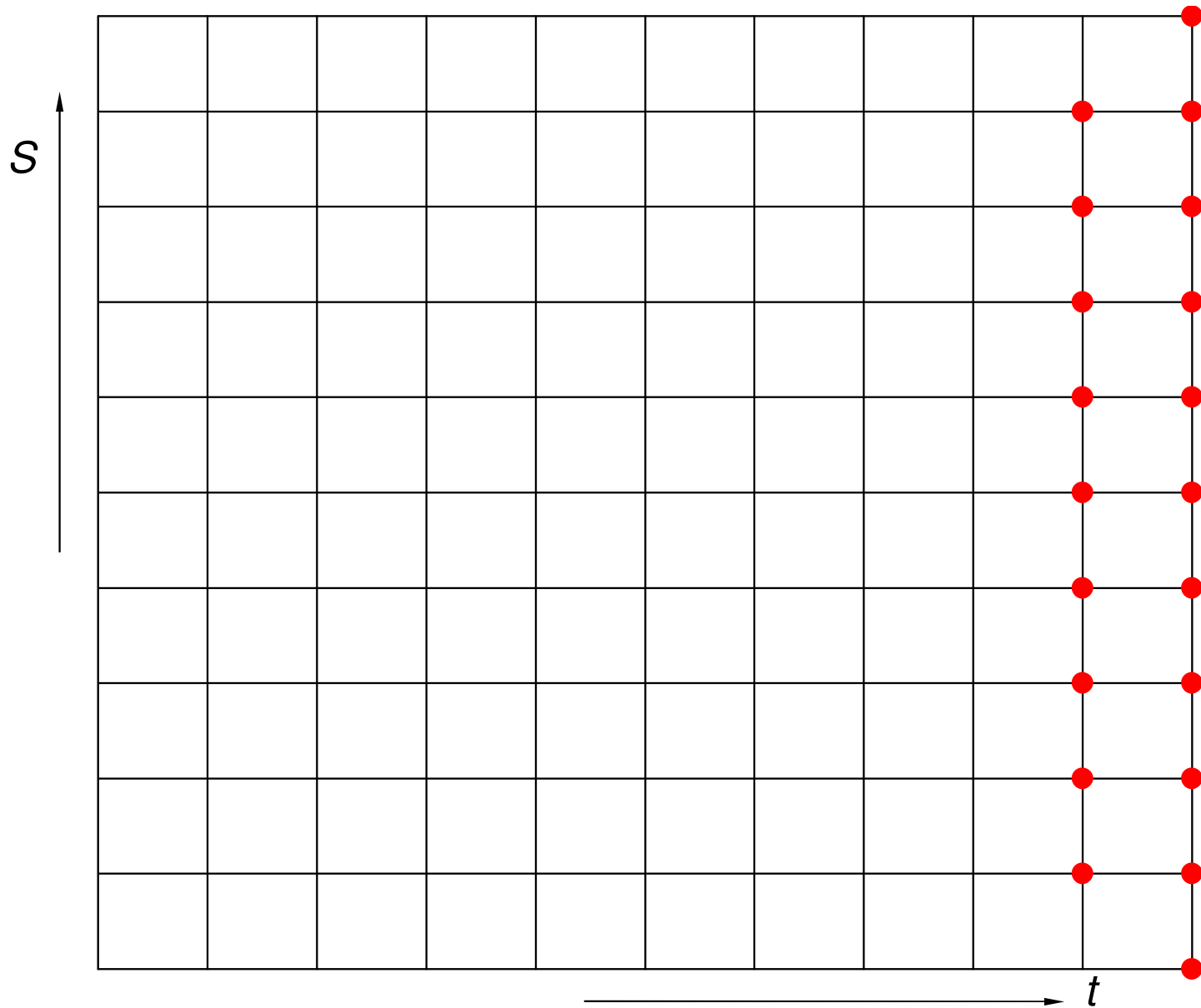
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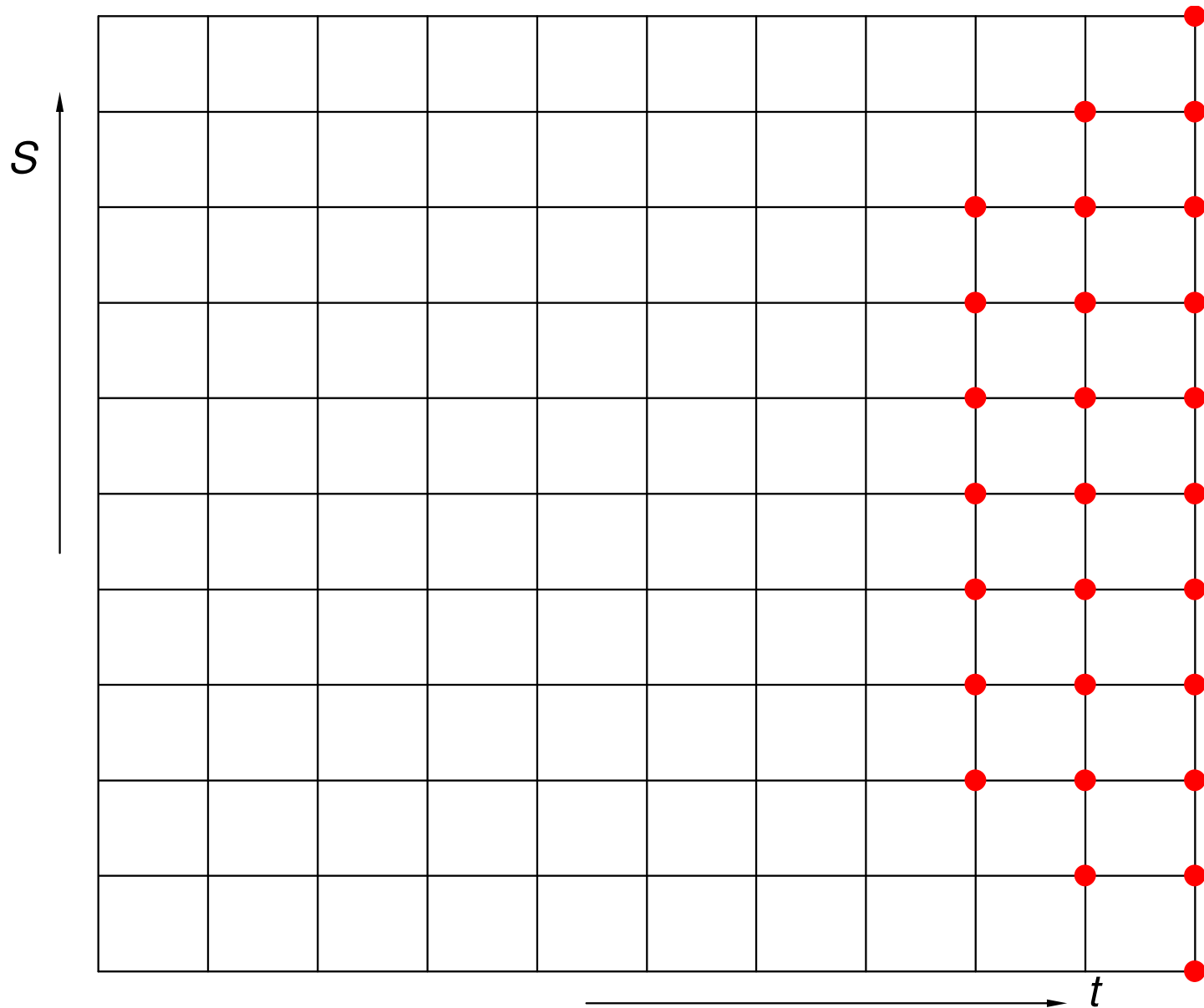
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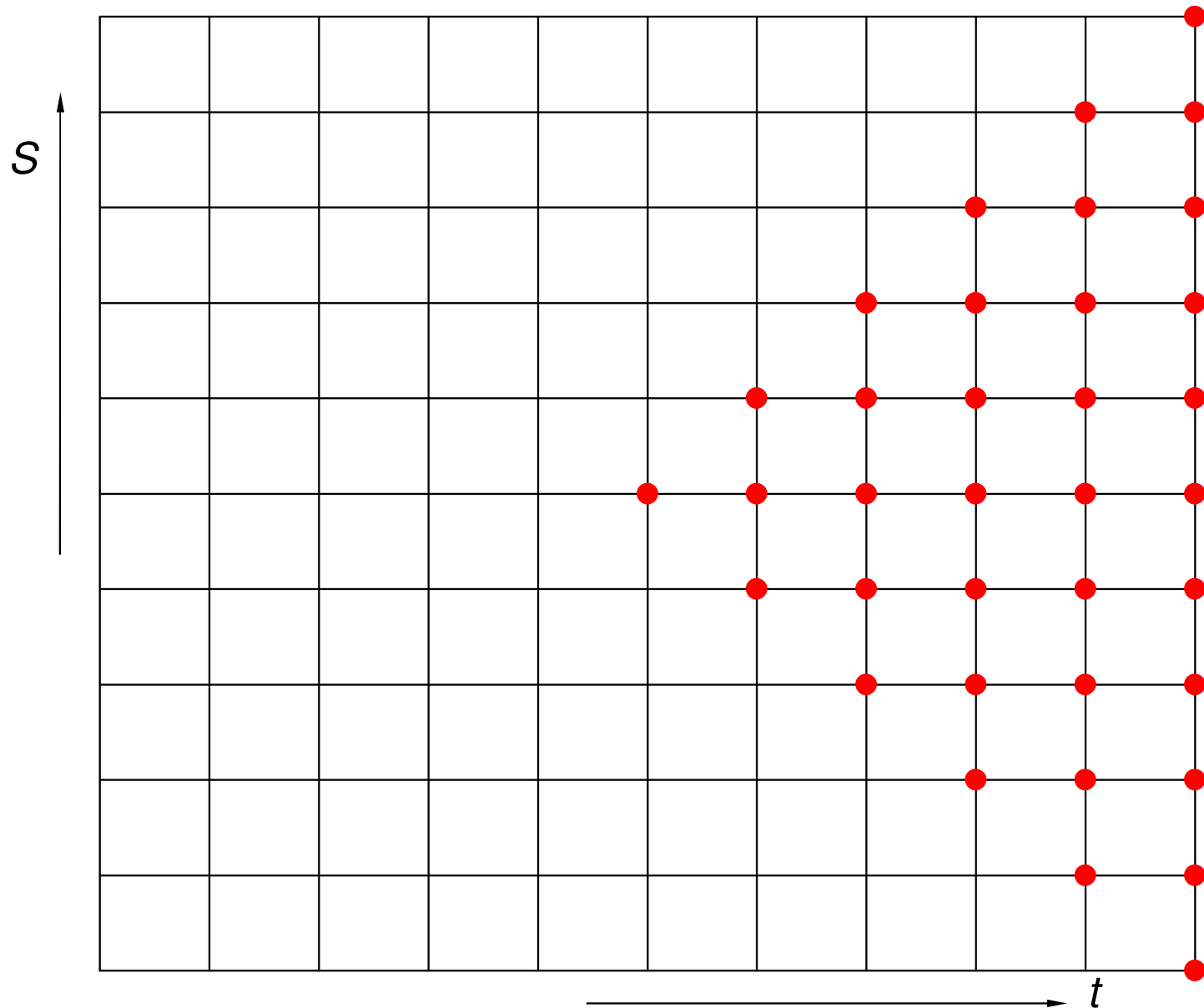
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Boundary conditions

We must specify the option values at the extremes of the region, at $S = 0$ and at $S = I \delta S$. They will depend on our option.

Example 1: Call option at $S = 0$

At $S = 0$ we know that the value is always zero, therefore

$$V_0^k = 0.$$

Example 2: Call option for large S

For large S the call value asymptotes to $S - Ee^{-r(T-t)}$. Thus

$$V_I^k = I \delta S - Ee^{-rk \delta t}.$$

Example 3: Put option at $S = 0$

At $S = 0$ $V = Ee^{-r(T-t)}$. I.e.

$$V_0^k = Ee^{-rk\delta t}.$$

Example 4: Put option for large S

The put option becomes worthless for large S and so

$$V_I^k = 0.$$

Example 5*: General condition at $S = 0$

A useful boundary condition to apply at $S = 0$ is that the diffusion and drift terms 'switch off.'

$$\rightarrow \frac{\partial V}{\partial t}(0, t) - rV(0, t) = 0$$

$\frac{V^k - V^{k+1}}{\delta t} = rV^k$

F.D form

$$\underline{V_0^{k+1} = (1 - r\delta t)V_0^k}$$

BS-E

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r - D)S \frac{\partial V}{\partial S} - rV = 0$$

$\rightarrow 0$

$$\frac{\partial V}{\partial t} = rV$$

Example 6*: General condition at infinity

When the option has a payoff that is linear in the underlying for large S then

When $S \rightarrow \infty$ $\Delta = ?$ Δ is insensitive to small changes

$$\Delta(S, t) := \Delta(t) \quad \frac{\partial^2 V}{\partial S^2}(S, t) \rightarrow 0 \quad \text{as } S \rightarrow \infty.$$

$$P = \frac{\partial}{\partial S} \Delta(t) = 0 \quad P \approx \frac{V_{i-1}^k - 2V_i^k + V_{i+1}^k}{\Delta S^2} \approx 0$$

The finite-difference representation is

$$V_I^k = 2V_{I-1}^k - V_{I-2}^k.$$

at $i = I$ ΔS^2

$$V_{I+1}^k = 2V_I^k - V_{I-1}^k$$

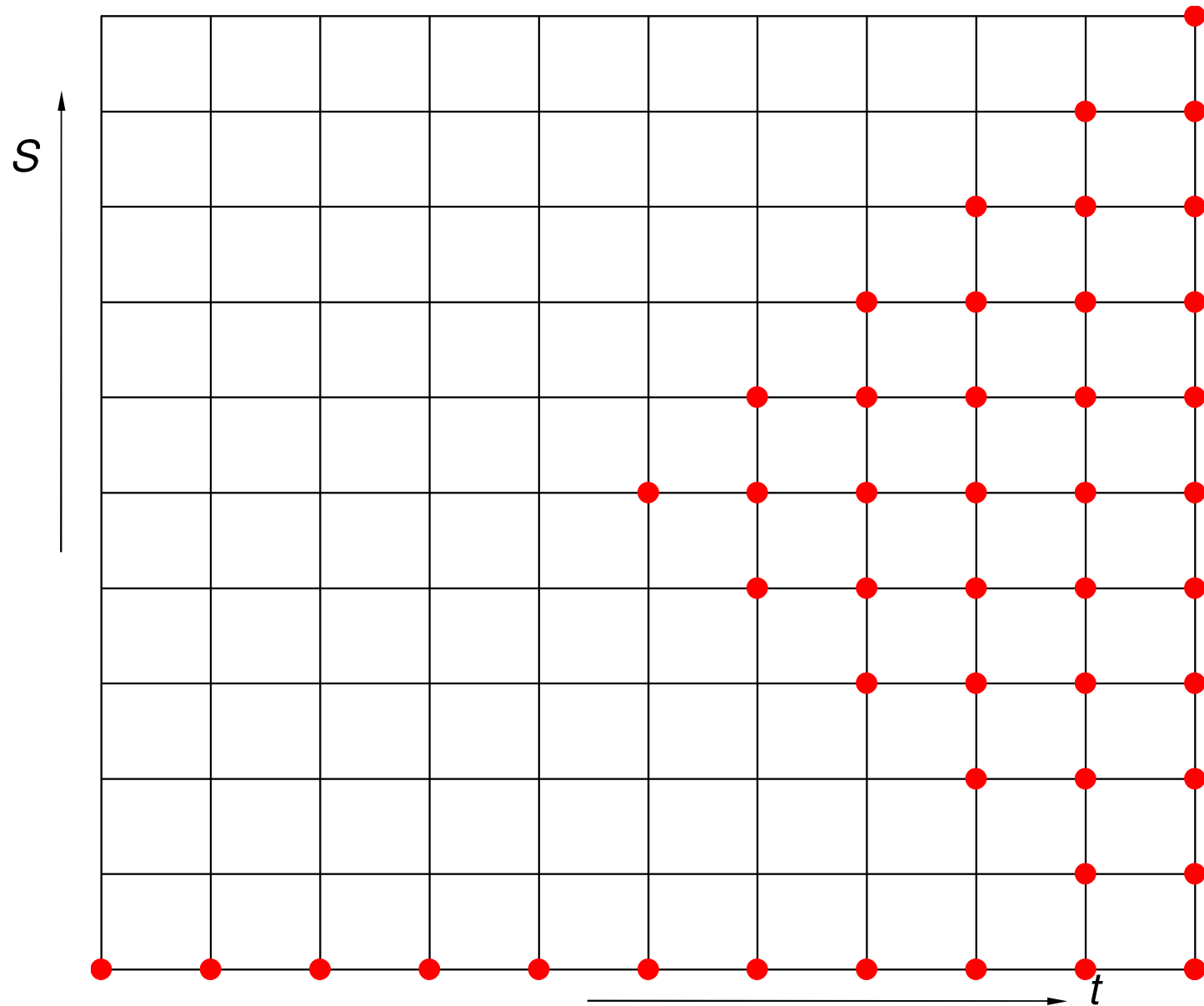
$$V_i^{k+1} = a_i V_{i-1}^k + b_i V_i^k + c_i V_{i+1}^k$$

when $i = I \Rightarrow S = S^*$

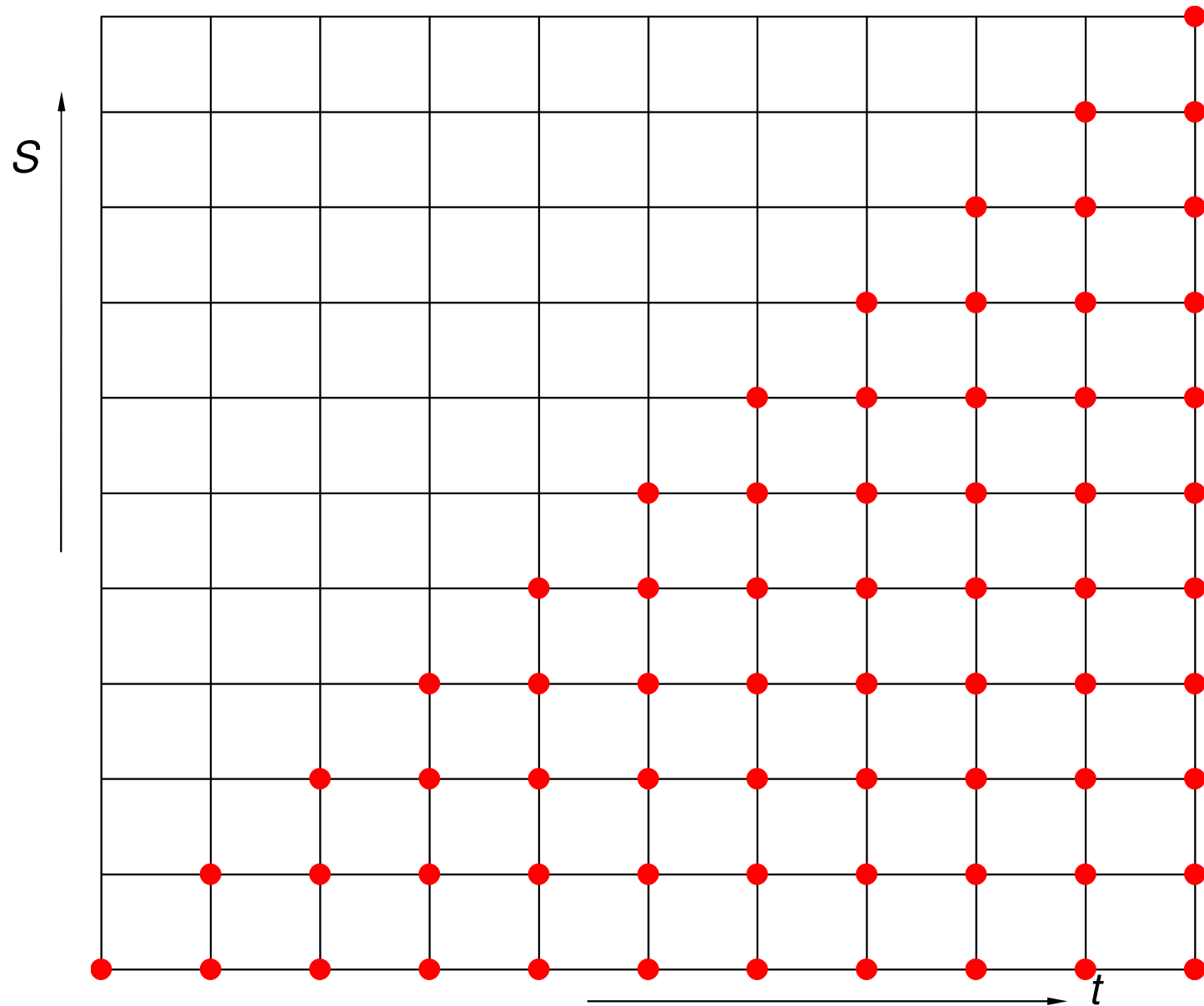
$$\rightarrow V_I^{k+1} = \dots + c_I V_{I+1}^k$$

V_{I+1} is not defined

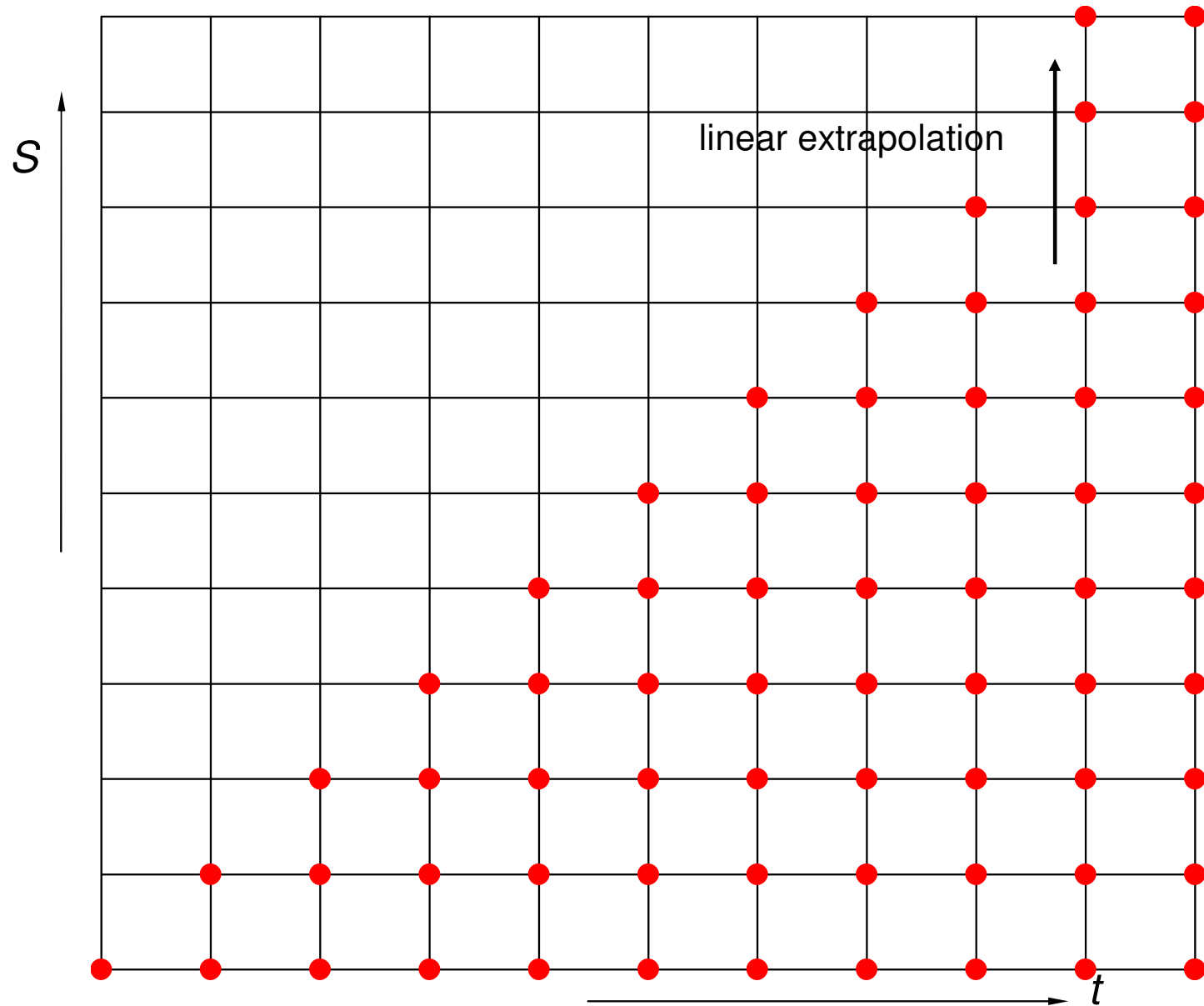
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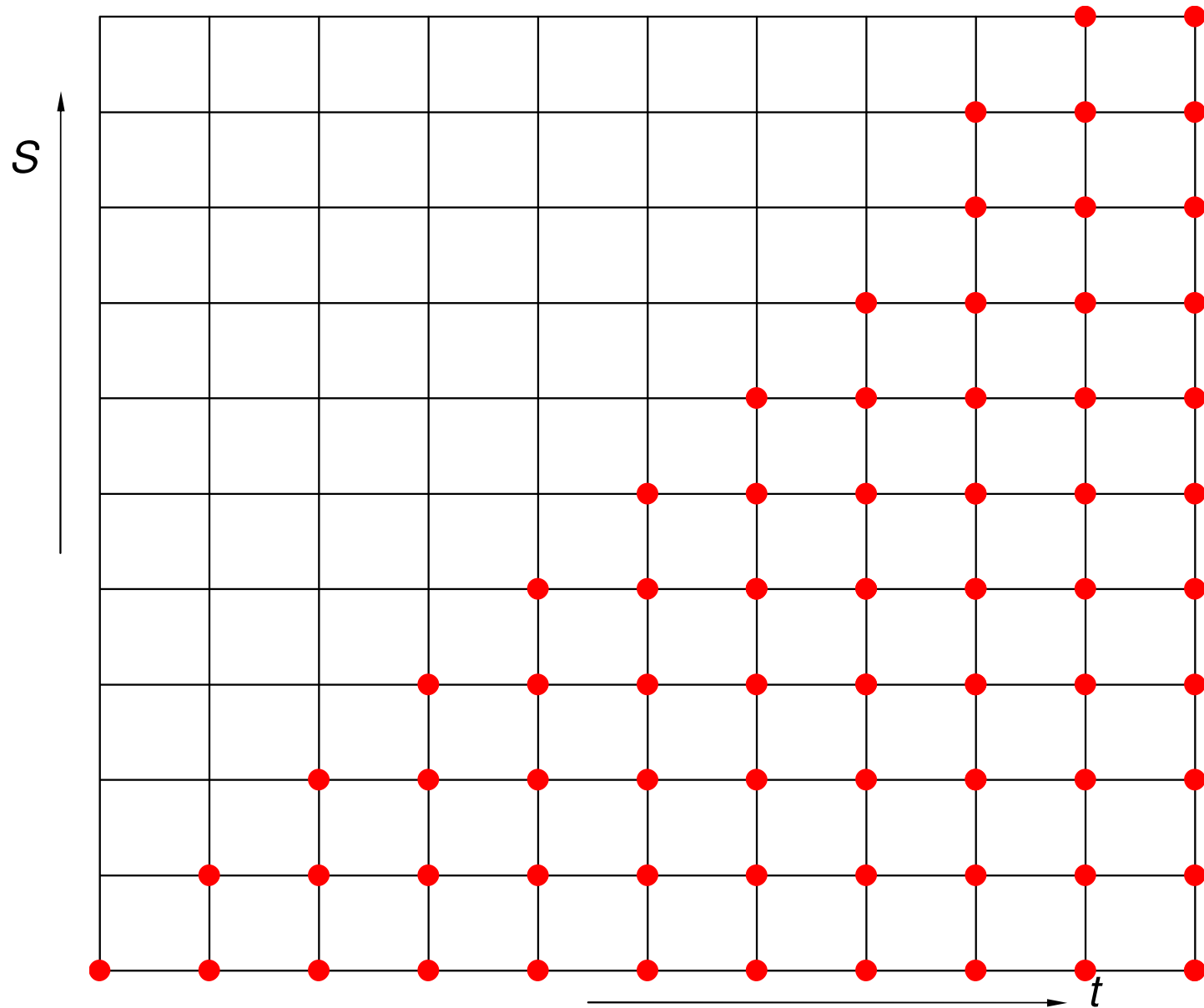
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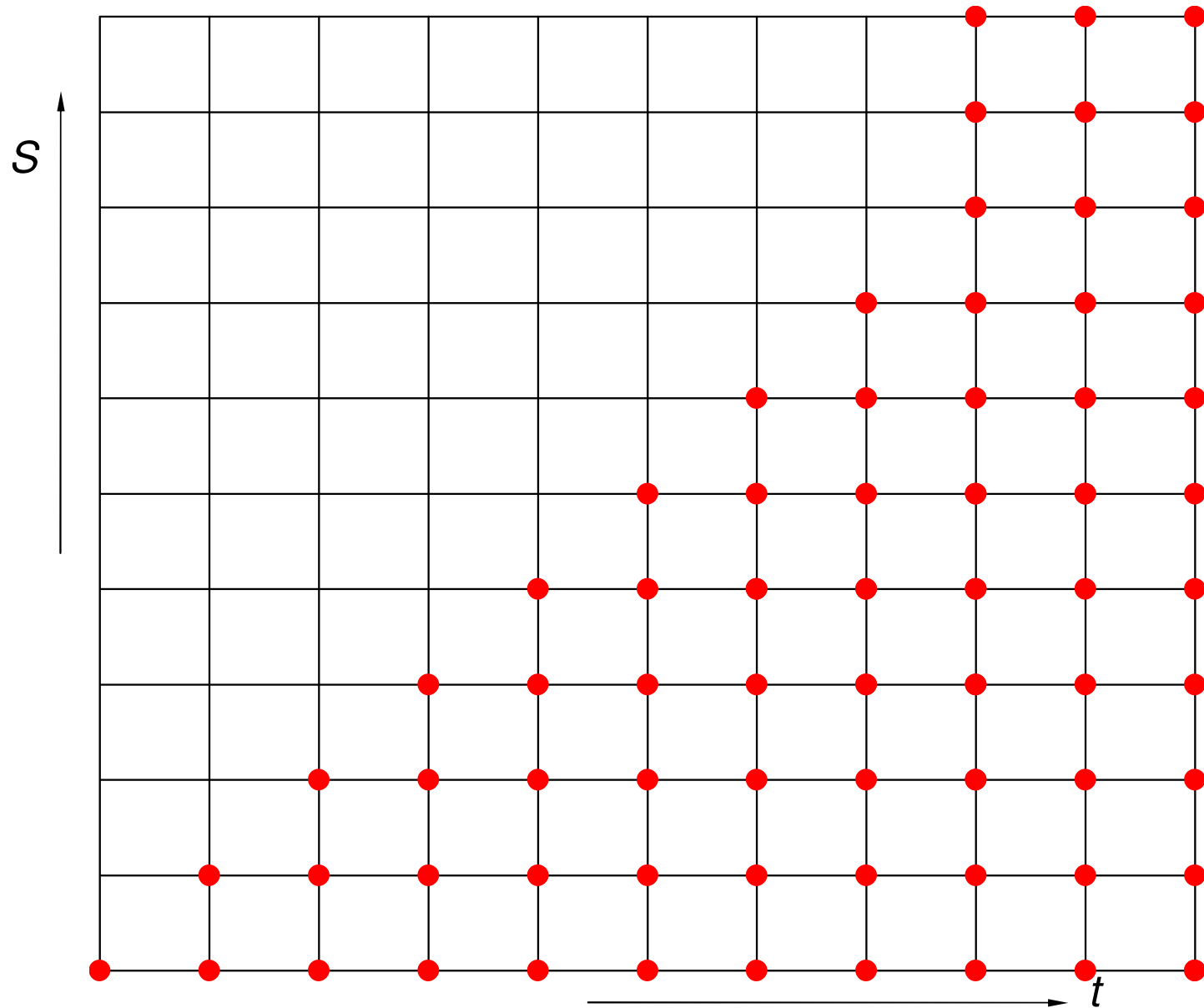
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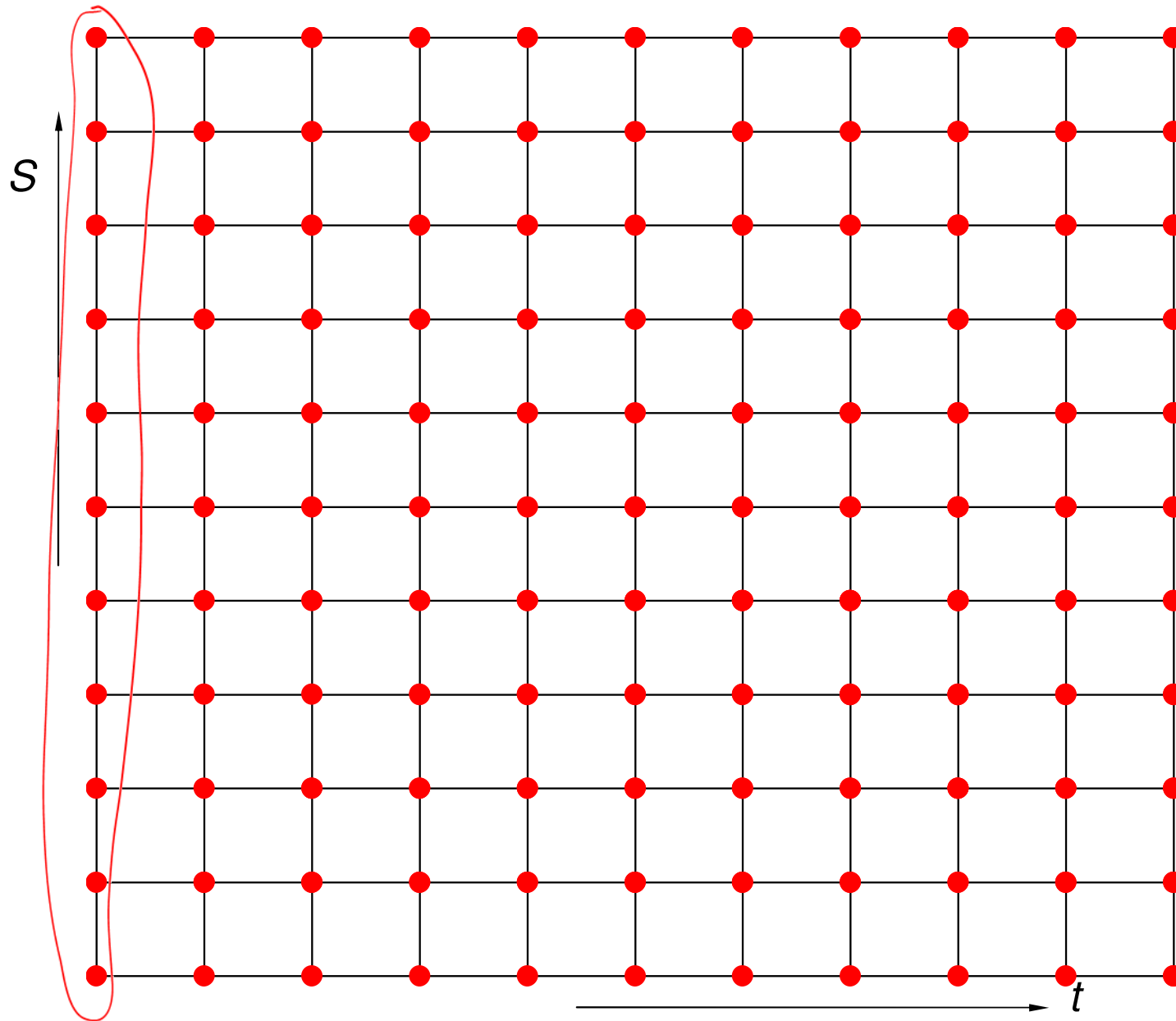
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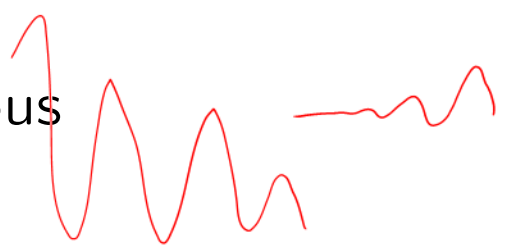


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Other issues

- Greeks *for free*
- Early exercise (and other decisions) *Good for early exercise*

The advantages of the explicit method

- It is ^{relatively} ~~very~~ easy to program and hard to make mistakes
- When it does go ^{solution "blows up"} unstable it is usually obvious 
- It copes well with coefficients that are asset and/or time dependent ^{parameters \rightarrow f.m.s}
- it copes very well with early exercise
- It can be used for modern option-pricing models (with caution)

The disadvantages of the explicit method

[• There are restrictions on the time step] stability

- It is slower than Monte Carlo in high dimensions

• It is a nightmare for $\text{dim}^2 > 3$

$\Delta t \sim O(N^{-2})$ strict condition

Summary

60% M.C

30% F.D.M

Laplace/
Fourier.

Please take away the following important ideas

10% Trees & transform methods,

- There are two main numerical methods for pricing derivatives
- Monte Carlo methods exploit the relationship between option prices and expectations
- The finite-difference method solved a discretized version of the Black-Scholes equation

$$V = PV \mathbb{E}^Q [\text{Payoff}]$$

$$y = f(x)$$

$$\delta t < \frac{1}{\sigma^2 N^2}$$
$$4 \delta t < \frac{1}{\sigma^2 N^2}$$

$K \rightarrow N$ time
 $I \rightarrow N$

Stability Analysis

By expressing $S = n\delta S$ and $t = m\delta t$, we will obtain a difference equation for the Black-Scholes equation.

step

$$V(S, t) = V(n\delta S, m\delta t) = V_n^m.$$

$T-$

$$\delta S = \frac{S^*}{N} \text{ where } S^* \gg E \text{ is a suitably large value of } S ;$$

$$\delta t = \frac{T}{M}. \text{ Taking } N \text{ and } M \text{ steps for } S \text{ and } t \text{ respectively,}$$

so

$$\begin{aligned} S &:= n\delta S & 0 \leq n \leq N \\ t &:= m\delta t & 0 \leq m \leq M. \end{aligned}$$

$T-$

B.J.E :
$$\frac{V_n^m - V_n^{m+1}}{\delta t} + \frac{1}{2}n^2\sigma^2 (V_{n-1}^m - 2V_n^m + V_{n+1}^m) +$$

$$\frac{1}{2}(r - D)n (V_{n+1}^m - V_{n-1}^m) - rV_n^m = 0$$

and rearrange to obtain a *forward marching* scheme in time

$$\begin{aligned} V_n^{m+1} &= V_n^m + \delta t \left(\frac{1}{2}n^2\sigma^2 (V_{n-1}^m - 2V_n^m + V_{n+1}^m) \right) \\ &\quad + \delta t \left(\frac{1}{2}(r - D)n (V_{n+1}^m - V_{n-1}^m) - rV_n^m \right) \\ &\equiv F(V_{n-1}^m, V_n^m, V_{n+1}^m) \end{aligned}$$

rearrange

Now for the RHS collect coefficients of each variable term V , to get

B.J.C. with a difference eqⁿ

$$V_n^{m+1} = \alpha_n V_{n-1}^m + \beta_n V_n^m + \gamma_n V_{n+1}^m \quad (1)$$

where

$\alpha_0 = 0$ $\beta_0 = 1 - r\delta t$ $\gamma_0 = 0$

$$\begin{aligned} \alpha_n &= \frac{1}{2} (n^2 \sigma^2 - n(r - D)) \delta t, \\ \beta_n &= 1 - (r + n^2 \sigma^2) \delta t, \\ \gamma_n &= \frac{1}{2} (n^2 \sigma^2 + n(r - D)) \delta t \end{aligned} \quad (2)$$

stability

Fourier Stability (Von Neumann's) Method instability

A method is called step-wise unstable if for a fixed grid (i.e. δt , δS constant) there exists an initial perturbation which "blows up" as $t \rightarrow \infty$, i.e. as we march in time. Here in a forward marching scheme. The question we wish to answer is "do small errors propagate along the grid and grow exponentially?"

Assume an initial disturbance which is proportional to $\exp(in\omega)$. We therefore study the propagation of perturbations created at any given point in time.

$$i = \sqrt{-1} \in \mathbb{C}$$

If \hat{V}_n^m is an approximation to the exact solution V_n^m then

$$\hat{V}_n^m = V_n^m + E_n^m$$

exact

where E_n^m is the associated error. Then E_n^m also satisfies the difference equation (1) to give

$$E_n^{m+1} = \alpha_n E_{n-1}^m + \beta_n E_n^m + \gamma_n E_{n+1}^m.$$

Put

$$E_n^m = \bar{a}^m \exp(in\omega)$$

amplitude

(3)

which is oscillatory of amplitude \bar{a} and frequency ω . Substituting (3) into (1) gives

$$\bar{a}^{m+1} e^{i(n+1)\omega} = \alpha_n \bar{a}^m e^{in\omega} + \beta_n \bar{a}^m e^{in\omega} + \gamma_n \bar{a}^m e^{i(n+1)\omega}$$

which becomes (after cancellation) :

$$\bar{a} = \alpha_n e^{-i\omega} + \beta_n + \gamma_n e^{i\omega}.$$

Now stability criteria arises from the balancing of the time dependency and diffusion terms, so that

$$\left(H \right) \sim -\frac{1}{2} \sigma^2 S^2 \left[\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] = 0$$

From (2) we take the following contributions

$$\left[\alpha_n = \frac{1}{2} n^2 \sigma^2 \delta t, \beta_n = 1 - n^2 \sigma^2 \delta t, \gamma_n = \frac{1}{2} n^2 \sigma^2 \delta t \right]$$

dependencies

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\bar{a} = \frac{1}{2} n^2 \sigma^2 \delta t (e^{i\omega} + e^{-i\omega}) + 1 - n^2 \sigma^2 \delta t$$

$$= n^2 \sigma^2 \delta t (\cos \omega - 1) + 1.$$

we have

$$\bar{a} = 1 - 2n^2 \sigma^2 \sin^2 \frac{\omega}{2} \delta t$$

$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$$

For stability \bar{a} must be bounded, i.e. $|\bar{a}| < 1$

$$\begin{aligned} -2\sin^2 x &= \cos 2x - 1 \\ \text{with } x &= \frac{\omega}{2} \\ -2\sin^2 \frac{\omega}{2} &= \cos \omega - 1 \end{aligned}$$

$$\left| 1 - 2n^2 \sigma^2 \sin^2 \frac{\omega}{2} \delta t \right| < 1$$

which upon simplifying we find is

$$|1 - 2N^2 \sigma^2 \delta t| < 1$$

$$\delta t < \frac{1}{\sigma^2 N^2}$$

$$n = N$$

$$|\sin^2 \frac{\omega}{2}| = 1$$

(4)

so $\delta t \sim O(N^{-2})$.

$$N^2 \sigma^2 \delta t < 1$$

$$\therefore \delta t < \frac{1}{\sigma^2 N^2}$$

$$\begin{aligned} -1 &< 1 - 2N^2 \sigma^2 \delta t < 1 \\ -2 &< -2N^2 \sigma^2 \delta t < 0 \\ 1 &> N^2 \sigma^2 \delta t > 0 \end{aligned}$$

$$\begin{aligned} N \delta J &= J^a \\ \delta J &= \frac{J}{N} \\ N &= \frac{J}{\delta J} \end{aligned}$$

Stability condition

$$\delta t < \frac{1}{\sigma^2 N^2}$$

Explicit scheme is

conditionally stable

$$N \rightarrow 2N \quad 4 \delta t < \frac{1}{\sigma^2 N^2}$$