Further Finite Difference Methods

In this lecture

- Fully implicit method
- Crank-Nicolson method
- Numerical Linear Algebra
- Douglas Schemes

By the end of this lecture you will

- know several more ways of solving parabolic partial differential equations numerically

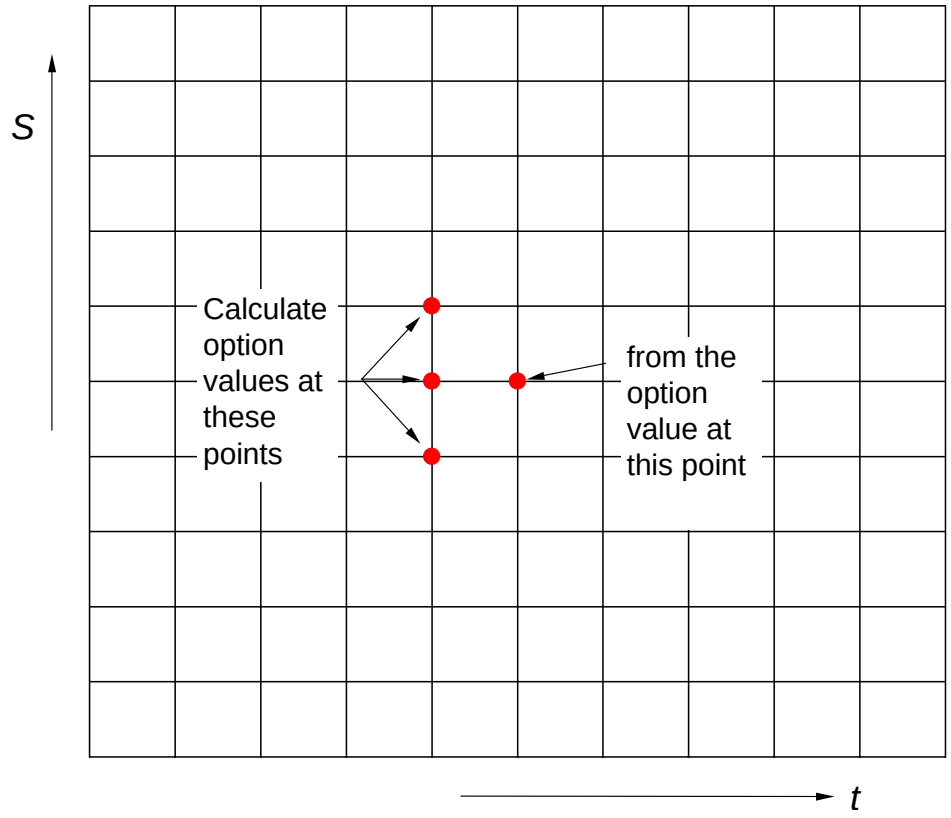
Introduction

There are many more ways of solving parabolic partial differential equations than the explicit method.

The more advanced methods are usually more complicated to program but have advantages in terms of stability and speed.

Implicit finite-difference methods

The **fully implicit method** uses the points shown in the figure below to calculate the option value.



The relationship between the option values on the mesh is simply

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + a_i^{k+1}\left(\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2}\right)$$

$$+b_i^{k+1}\left(\frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\delta S}\right) + c_i^{k+1}V_i^{k+1} = 0.$$

The method is accurate to $O(\delta t, \delta S^2)$.

This can be written as

$$A_i^{k+1} V_{i-1}^{k+1} + (1 + B_i^{k+1}) V_i^{k+1} + C_i^{k+1} V_{i+1}^{k+1} = V_i^k \qquad (1)$$

where

$$A_i^{k+1} = -\nu_1 a_i^{k+1} + \tfrac{1}{2}\nu_2 b_i^{k+1},$$

$$B_i^{k+1} = 2\nu_1 a_i^{k+1} - \delta t\, c_i^{k+1}$$

and

$$C_i^{k+1} = -\nu_1 a_i^{k+1} - \tfrac{1}{2}\nu_2 b_i^{k+1}$$

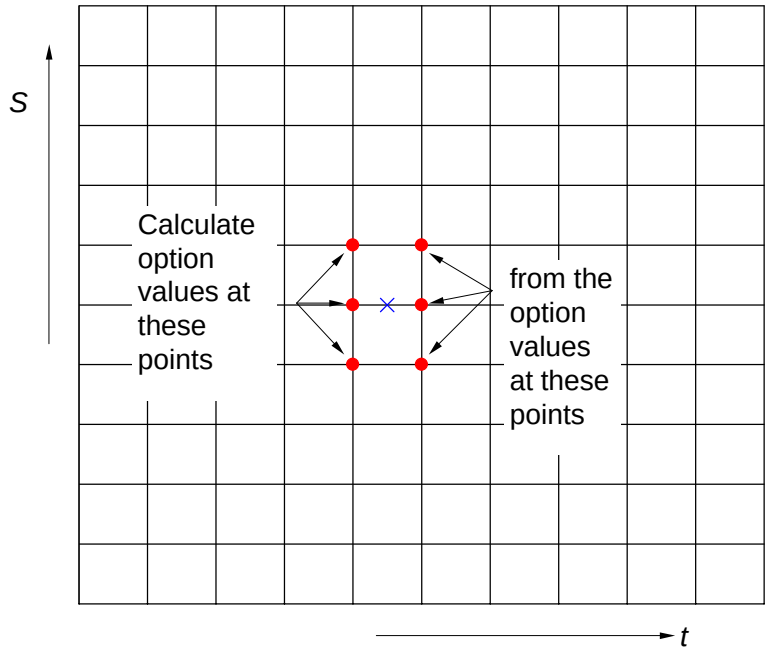
where

$$\nu_1 = \frac{\delta t}{\delta S^2} \quad \text{and} \quad \nu_2 = \frac{\delta t}{\delta S}.$$

Equation (1) does not hold for $i = 0$ or $i = I$, the boundary conditions supply the two remaining equations.

The Crank–Nicolson method

The **Crank–Nicolson method** can be thought of as an average of the explicit method and the fully implicit method. It uses the six points shown in the figure below.



The Crank–Nicolson scheme is

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + \frac{a_i^{k+1}}{2}\left(\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2}\right) + \frac{a_i^k}{2}\left(\frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}\right)$$

$$+ \frac{b_i^{k+1}}{2}\left(\frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\,\delta S}\right) + \frac{b_i^k}{2}\left(\frac{V_{i+1}^k - V_{i-1}^k}{2\,\delta S}\right)$$

$$+ \tfrac{1}{2} c_i^{k+1} V_i^{k+1} + \tfrac{1}{2} c_i^k V_i^k = O(\delta t^2, \delta S^2).$$

This can be written as

$$-A_i^{k+1}V_{i-1}^{k+1} + (1 - B_i^{k+1})V_i^{k+1} - C_i^{k+1}V_{i+1}^{k+1}$$
$$= A_i^k V_{i-1}^k + (1 + B_i^k)V_i^k + C_i^k V_{i+1}^k,$$

where

$$A_i^k = \tfrac{1}{2}\nu_1 a_i^k - \tfrac{1}{4}\nu_2 b_i^k, B_i^k = -\nu_1 a_i^k + \tfrac{1}{2}\delta t\, c_i^k, C_i^k = \tfrac{1}{2}\nu_1 a_i^k + \tfrac{1}{4}\nu_2 b_i^k.$$

The beauty of this method lies in its stability and accuracy. The error in the method is $O(\delta t^2, \delta S^2)$.

The Crank–Nicolson method can be written in the matrix form

$$\begin{pmatrix} -A_1^{k+1} & 1-B_1^{k+1} & -C_1^{k+1} & 0 & . & . & . \\ 0 & -A_2^{k+1} & 1-B_2^{k+1} & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & 1-B_{I-2}^{k+1} & -C_{I-2}^{k+1} & 0 \\ . & . & . & 0 & -A_{I-1}^{k+1} & 1-B_{I-1}^{k+1} & -C_{I-1}^{k+1} \end{pmatrix} \begin{pmatrix} V_0^{k+1} \\ V_1^{k+1} \\ . \\ . \\ . \\ V_{I-1}^{k+1} \\ V_I^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} A_1^k & 1+B_1^k & C_1^k & 0 & . & . & . \\ 0 & A_2^k & 1+B_2^k & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & 1+B_{I-2}^k & C_{I-2}^k & 0 \\ . & . & . & 0 & A_{I-1}^k & 1+B_{I-1}^k & C_{I-1}^k \end{pmatrix} \begin{pmatrix} V_0^k \\ V_1^k \\ . \\ . \\ . \\ V_{I-1}^k \\ V_I^k \end{pmatrix}$$

The two matrices have $I-1$ rows and $I+1$ columns. This is a representation of $I-1$ equations in $I+1$ unknowns. Our aim is to write this as

$$\mathbf{M}_L^{k+1}\mathbf{v}^{k+1} + \mathbf{r}^k = \mathbf{M}_R^k\mathbf{v}^k,$$

for known *square* matrices $\mathbf{M}_L^{k+1}$ and $\mathbf{M}_R^k$, and a known vector $\mathbf{r}^k$ and where details of the boundary conditions have been fully incorporated.

Example:

Sometimes we know that our option has a particular value on the boundary $i = 0$, or on $i = I$.

For example, if we have a European put we know that $V(0, t) = Ee^{-r(T-t)}$.

This translates to knowing that $V_0^{k+1} = Ee^{-r(k+1)\delta t}$.

We can write

$$\begin{pmatrix} -A_1^{k+1} & 1-B_1^{k+1} & -C_1^{k+1} & 0 & . & . & . \\ 0 & -A_2^{k+1} & 1-B_2^{k+1} & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & 1-B_{I-2}^{k+1} & -C_{I-2}^{k+1} & 0 \\ . & . & . & 0 & -A_{I-1}^{k+1} & 1-B_{I-1}^{k+1} & -C_{I-1}^{k+1} \end{pmatrix} \begin{pmatrix} V_0^{k+1} \\ V_1^{k+1} \\ . \\ . \\ . \\ V_{I-1}^{k+1} \\ V_I^{k+1} \end{pmatrix}$$

as

$$\begin{pmatrix} 1-B_1^{k+1} & -C_1^{k+1} & 0 & . & . & \\ -A_2^{k+1} & 1-B_2^{k+1} & . & . & . & \\ & 0 & . & . & . & 0 \\ & . & . & . & 1-B_{I-2}^{k+1} & -C_{I-2}^{k+1} \\ & . & . & 0 & -A_{I-1}^{k+1} & 1-B_{I-1}^{k+1} \end{pmatrix} \begin{pmatrix} V_1^{k+1} \\ . \\ . \\ . \\ \\ V_{I-1}^{k+1} \end{pmatrix} + \begin{pmatrix} A_1^{k+1}V_0^{k+1} \\ 0 \\ 0 \\ . \\ . \\ 0 \\ . \end{pmatrix}$$

$$= \mathbf{M}_L^{k+1}\mathbf{v}^{k+1} + \mathbf{r}^k.$$

The matrix $\mathbf{M}_L$ is square and of size $I-1$.

The matrix equation

Remembering that we know $\mathbf{v}^k$, the matrix multiplication and vector addition on the right-hand side is simple enough to do.

But how do we then find $\mathbf{v}^{k+1}$? In principle, the matrix $\mathbf{M}_L^{k+1}$ could be inverted to give

$$\mathbf{v}^{k+1} = (\mathbf{M}_L^{k+1})^{-1}(\mathbf{M}_R^k\mathbf{v}^k - \mathbf{r}^k),$$

except that matrix inversion is very time consuming, and from a computational point of view extremely inefficient.

There are two much better ways for solving this system, called **LU decomposition** and **successive over-relaxation**.

LU decomposition

The matrix $\mathbf{M}_L^{k+1}$ is special in that it is **tridiagonal**.

We can decompose the matrix into the product of two other matrices, one having non-zero elements along the diagonal and the subdiagonal and the other having non-zero elements along the diagonal and the superdiagonal.

Call these two matrices **L** and **U** respectively.

$$\begin{pmatrix} 1+B_1 & C_1 & 0 & . & . & . & 0 \\ A_2 & 1+B_2 & C_2 & 0 & . & . & . \\ 0 & A_3 & 1+B_3 & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & 1+B_{I-3} & C_{I-3} & 0 \\ . & . & . & 0 & A_{I-2} & 1+B_{I-2} & C_{I-2} \\ . & . & . & . & 0 & A_{I-1} & 1+B_{I-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ l_2 & 1 & 0 & 0 & . & . & . \\ 0 & l_3 & 1 & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & 1 & 0 & 0 \\ . & . & . & 0 & l_{I-2} & 1 & 0 \\ . & . & . & . & 0 & l_{I-1} & 1 \end{pmatrix} \begin{pmatrix} d_1 & u_1 & 0 & . & . & . & 0 \\ 0 & d_2 & u_2 & 0 & . & . & . \\ 0 & 0 & d_3 & . & . & . & . \\ . & 0 & . & . & . & 0 & . \\ . & . & . & . & d_{I-3} & u_{I-3} & 0 \\ . & . & . & 0 & 0 & d_{I-2} & u_{I-2} \\ . & . & . & . & 0 & 0 & d_{I-1} \end{pmatrix}$$

i.e.

$$\mathbf{M = LU}$$

It is not difficult to show that

$$d_1 = 1 + B_1$$

and then

$l_i d_{i-1} = A_i, \quad u_{i-1} = C_{i-1}$ and $d_i = 1 + B_i - l_i u_{i-1}$ for $2 \leq i \leq I-1$.

Now we exploit the decomposition to solve the original matrix equation. This equation is of the form

$$\mathbf{Mv} = \mathbf{q}$$

where we want to find $\mathbf{v}$.

We can write

$$\mathbf{LUv} = \mathbf{q}. \tag{2}$$

The vector $\mathbf{q}$ contains both the old option value array, at time step k, and details of the boundary conditions.

Solve Equation (2) in two steps. First find $\mathbf{w}$ such that

$$\mathbf{Lw} = \mathbf{q}$$

and then $\mathbf{v}$ such that

$$\mathbf{Uv} = \mathbf{w}.$$

And then we are done.

The first step gives

$$w_1 = q_1$$

and

$$w_i = q_i - l_i w_{i-1} \quad \text{for} \quad 2 \leq i \leq I-1.$$

Finally

$$v_{I-1} = \frac{w_{I-1}}{d_{I-1}}$$

and

$$v_i = \frac{w_i - u_i v_{i+1}}{d_i} \quad \text{for} \quad I-2 \geq i \geq 1.$$

Successive over-relaxation, SOR

We now come to an example of an ‘indirect method.’

In this method we solve the equations iteratively.

Suppose that the matrix **M** in the matrix equation

$$\mathbf{Mv} = \mathbf{q}$$

has entries M_{ij} then the system of equations can be written as

$$\begin{aligned} M_{11}v_1 + M_{12}v_2 + \cdots + M_{1N}v_N &= q_1 \\ M_{21}v_1 + M_{22}v_2 + \cdots + M_{2N}v_N &= q_2 \\ &\cdots \\ M_{N1}v_1 + M_{N2}v_2 + \cdots + M_{NN}v_N &= q_N \end{aligned}$$

where N is the number of equations, the size of the matrix.

Rewrite this as

$$
\begin{aligned}
M_{11}v_1 &= q_1 - (M_{12}v_2 + \cdots + M_{1N}v_N) \\
M_{22}v_2 &= q_2 - (M_{21}v_1 + \cdots + M_{2N}v_N) \\
&\cdots \\
M_{NN}v_N &= q_N - (M_{N1}v_1 + \cdots)
\end{aligned}
$$

This system is easily solved *iteratively* using

$$
\begin{aligned}
v_1^{n+1} &= \tfrac{1}{M_{11}}\Big(q_1 - (M_{12}v_2^n + \cdots + M_{1N}v_N^n)\Big) \\
v_2^{n+1} &= \tfrac{1}{M_{22}}\Big(q_2 - (M_{21}v_1^n + \cdots + M_{2N}v_N^n)\Big) \\
&\cdots \\
v_N^{n+1} &= \tfrac{1}{M_{NN}}\Big(q_N - (M_{N1}v_1^n + \cdots)\Big)
\end{aligned}
$$

where the superscript n denotes the level of the iteration *and not the time step*.

This iteration is started from some initial guess $\mathbf{v}^0$.

This iterative method is called the **Jacobi method**.

When we implement the Jacobi method in practice we find some of the values of v_i^{n+1} before others.

In the **Gauss–Seidel** method we use the updated values as soon as they are calculated.

This method can be written as

$$v_i^{n+1} = \frac{1}{M_{ii}} \left(q_i - \sum_{j=1}^{i-1} M_{ij} v_j^{n+1} - \sum_{j=i}^{N} M_{ij} v_j^n \right).$$

Observe that there are some terms on the right-hand side with the superscript $n + 1$.

These are values of v that were calculated earlier but at the same level of iteration.

When the matrix **M** has come from a finite-difference discretization of a parabolic equation (and that includes almost all finance problems) the above iterative methods usually converge to the correct solution *from one side*.

This means that the corrections $v_i^{n+1} - v_i^n$ stay of the same sign as n increases.

This is exploited in the **successive over-relaxation** or **SOR** method to speed up convergence.

The method can be written as

$$v_i^{n+1} = v_i^n + \frac{\omega}{M_{ii}} \left(q_i - \sum_{j=1}^{i-1} M_{ij} v_j^{n+1} - \sum_{j=i}^{N} M_{ij} v_j^n \right).$$

Again, the new values for v_i are used as soon as they are obtained.

But now the factor ω, called the **acceleration** or **over-relaxation parameter** is included.

This parameter, which must lie between 1 and 2, speeds up the convergence to the true solution.

Other methods

The questions that arise in any method are

- What is the error in the method in terms of δt and δS?
- What are the restrictions on the time step and/or asset step?
- Can we solve the resulting difference equations quickly?
- Is the method flexible enough to cope with changes in coefficients, boundary conditions etc.?

Douglas schemes

This is a method that manages to have a local truncation error of $O(\delta S^4, \delta t^2)$ for the same computational effort as the Crank–Nicolson scheme.

The basic diffusion equation is

$$\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial S^2} = 0.$$

The explicit method applied to this equation is just

$$\frac{V_i^{k+1} - V_i^k}{\delta t} = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2}.$$

and the fully implicit is similarly

$$\frac{V_i^{k+1} - V_i^k}{\delta t} = \frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2}.$$

The Crank–Nicolson scheme is an average of these two methods. What about a *weighted* average?

This leads to the θ method.

Take a weighted average of the explicit and implicit methods to get...

$$\frac{V_i^{k+1} - V_i^k}{\delta t} = \theta \left(\frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\delta S^2} \right) + (1-\theta) \left(\frac{V_{i+1}^{k} - 2V_i^{k} + V_{i-1}^{k}}{\delta S^2} \right).$$

When $\theta = \frac{1}{2}$ we are back to the Crank–Nicolson method.

For a general value of θ the local truncation error is

$$O\left(\tfrac{1}{2}\delta t + \tfrac{1}{12}\delta S^2 - \theta \delta t, \delta S^4, \delta t^2\right).$$

When $\theta = 0$, $\frac{1}{2}$ or 1 we get the results we have seen so far.

Three time-level methods

Numerical schemes are not restricted to the use of just two time levels.

We can construct many algorithms using three or more time levels.

Again, we would do this if it gave us a better local truncation error or had better convergence properties.

Two-factor models

We are going to refer to the general two-factor equation

$$\frac{\partial V}{\partial t} + a(S,r,t)\frac{\partial^2 V}{\partial S^2} + b(S,r,t)\frac{\partial V}{\partial S} + c(S,r,t)V$$

$$+d(S,r,t)\frac{\partial^2 V}{\partial r^2} + e(S,r,t)\frac{\partial^2 V}{\partial S \partial r} + f(S,r,t)\frac{\partial V}{\partial r} = 0. \qquad (3)$$

It will be quite helpful if we think of solving the two-factor convertible bond problem.

For the general two-factor problem (3) to be parabolic we need

$$e(S,r,t)^2 < 4a(S,r,t)\, d(S,r,t).$$

As in the one-factor world, the variables must be discretized.

That is, we solve on a three-dimensional grid with

$$S = i\,\delta S, \quad r = j\,\delta r, \quad \text{and} \quad t = T - k\,\delta t.$$

Expiry is $t = T$ or $k = 0$.

The indices range from zero to I and J for i and j respectively.

We will assume that the interest rate model is only specified on $r \geq 0$.

This may not be the case, some simple interest rate models such as Vasicek, are defined over negative r as well.

The contract value is written as

$$V(S, r, t) = V_{ij}^{k}.$$

Whatever the problem to be solved, we must impose certain conditions on the solution.

First of all, we must specify the final condition.

This is the payoff function, telling us the value of the contract at the expiration of the contract.

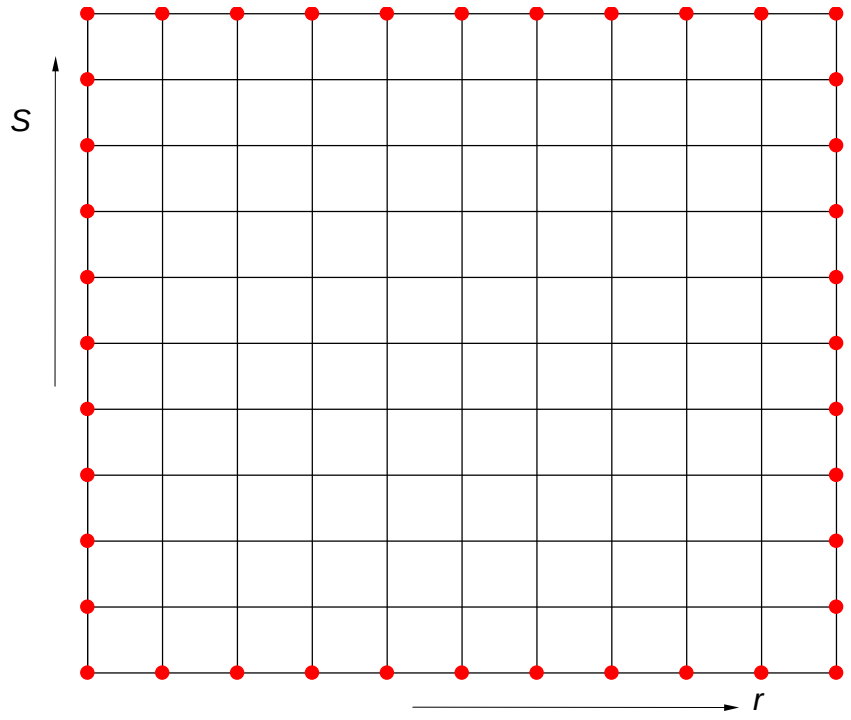
Suppose that we are pricing a long-dated warrant with a call payoff.

The final condition for this problem is then

$$V(S, r, T) = V_{ij}^0 = \max(S - E, 0).$$

Boundary conditions must be imposed at all the grid points marked with a dot. The boundary conditions will depend on the contract.

Remember that there is also a time axis coming out of the page, and not drawn in this figure.



The explicit method

The one-factor explicit method can be extended to two-factors with very little effort.

In fact, the ease of programming make it a very good method for those new to the subject.

We will use symmetric central differences for all derivatives in (1).

This is the best way to approximate the second derivatives but may not be the best for the first derivatives.

We have seen how to use central differences for all of the terms with the exception of the second derivative with respect to both S and r,

$$\frac{\partial^2 V}{\partial S \partial r}.$$

We can approximate this by

$$\frac{\partial \left(\frac{\partial V}{\partial r}\right)}{\partial S} \approx \frac{\frac{\partial V}{\partial r}(S+\delta S, r, t) - \frac{\partial V}{\partial r}(S-\delta S, r, t)}{2\,\delta S}.$$

But

$$\frac{\partial V}{\partial r}(S+\delta S, r, t) \approx \frac{V^k_{i+1,j+1} - V^k_{i+1,j-1}}{2\,\delta r}.$$

This suggests that a suitable discretization might be

$$\frac{\frac{V^k_{i+1,j+1}-V^k_{i+1,j-1}}{2\,\delta r} - \frac{V^k_{i-1,j+1}-V^k_{i-1,j-1}}{2\,\delta r}}{2\,\delta S}$$

$$= \frac{V^k_{i+1,j+1} - V^k_{i+1,j-1} - V^k_{i-1,j+1} + V^k_{i-1,j-1}}{4\,\delta S\,\delta r}.$$

This is particularly good since, not only is the error of the same error as in the other derivative approximations but also it preserves the property that

$$\frac{\partial^2 V}{\partial S \partial r} = \frac{\partial^2 V}{\partial r \partial S}.$$

The resulting explicit difference scheme is

$$\frac{V_{ij}^k - V_{ij}^{k+1}}{\delta t} + a_{ij}^k \left(\frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2} \right)$$

$$+ b_{ij}^k \left(\frac{V_{i+1,j}^k - V_{i-1,j}^k}{2\,\delta S} \right) + c_{ij}^k V_{ij}^k$$

$$+ d_{ij}^k \left(\frac{V_{i,j+1}^k - 2V_{ij}^k + V_{i,j-1}^k}{\delta r^2} \right)$$

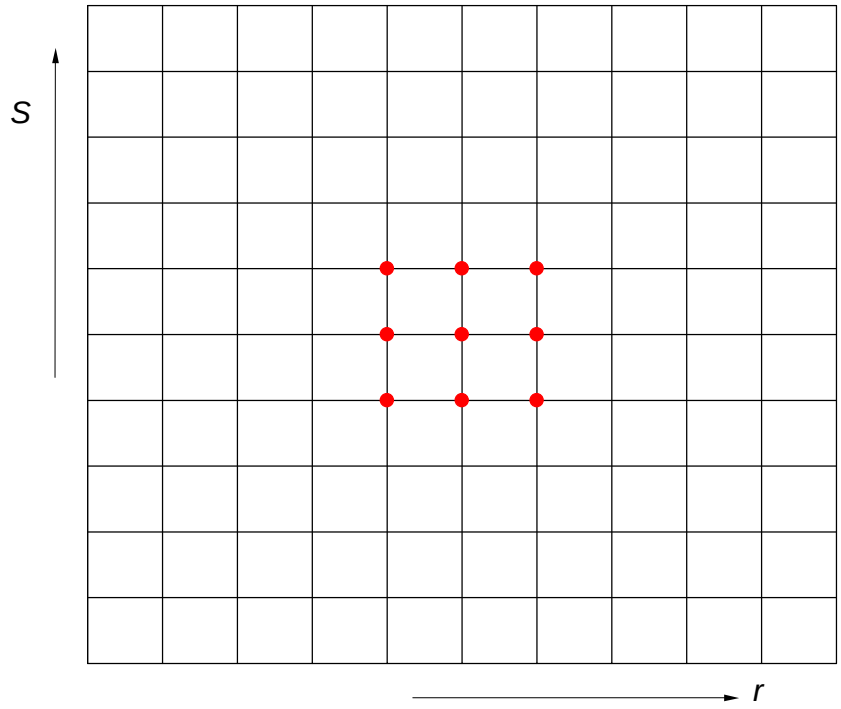
$$e_{ij}^k \left(\frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k - V_{i-1,j+1}^k + V_{i-1,j-1}^k}{4\,\delta S\,\delta r} \right)$$

$$+ f_{ij}^k \left(\frac{V_{i,j+1}^k - V_{i,j-1}^k}{2\,\delta r} \right) = O(\delta t, \delta S^2, \delta r^2).$$

We could rewrite this in the form

$$V_{ij}^{k+1} = \ldots,$$

where the right-hand side is a linear function of the nine option values shown schematically below.



The coefficients of these nine values at time step k are related to a, b etc.

It would not be very helpful to write the difference equation in this form, since the actual implementation is usually more transparent than this.

Note that in general all nine points (i, j), $(i \pm 1, j)$, $(i, j \pm 1)$, $(i \pm 1, j \pm 1)$ are used in the scheme.

If there is no cross derivative term then only the five points (i, j), $(i \pm 1, j \pm 1)$ are used.

This simplifies some of the methods.

Please take away the following important ideas

- There are many more kinds of finite-difference method than explicit
- Each method has its pros and cons
- Finite difference can be used for higher dimensional problems