Martingales

Note: UCL
e-mail
from NS
about
Itô Integral

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game. We encounter Martingales through three distinct, but closely connected ideas:

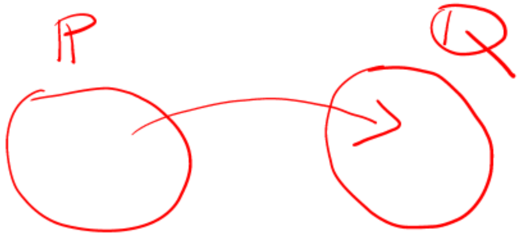
Given: $(\Omega, \mathcal{F}, \mathbb{P})$

1. *Martingales* as a class of stochastic process; focus

2. *Exponential martingales*, which are a specific and extremely useful example of a martingale; finish off & then use extensively in Mod J

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

$\mathbb{P}, \mathbb{Q}$



Discrete Time Martingales

$0 \quad 1 \quad 2 \quad \cdots \quad T$

A discrete time stochastic process $\{M_t : \underbrace{t = 0, \ldots, T}_{\mathbb{T}}\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \{0, \ldots, T\}$ is a **martingale** if $\underbrace{\mathbb{E}\,|M_t| < \infty}_{\text{integrability condition}}$ and

Main def$^{\underline{n}}$

$$\mathbb{E}_t[M_{t+1}|\mathcal{F}_t] = M_t \tag{1}$$

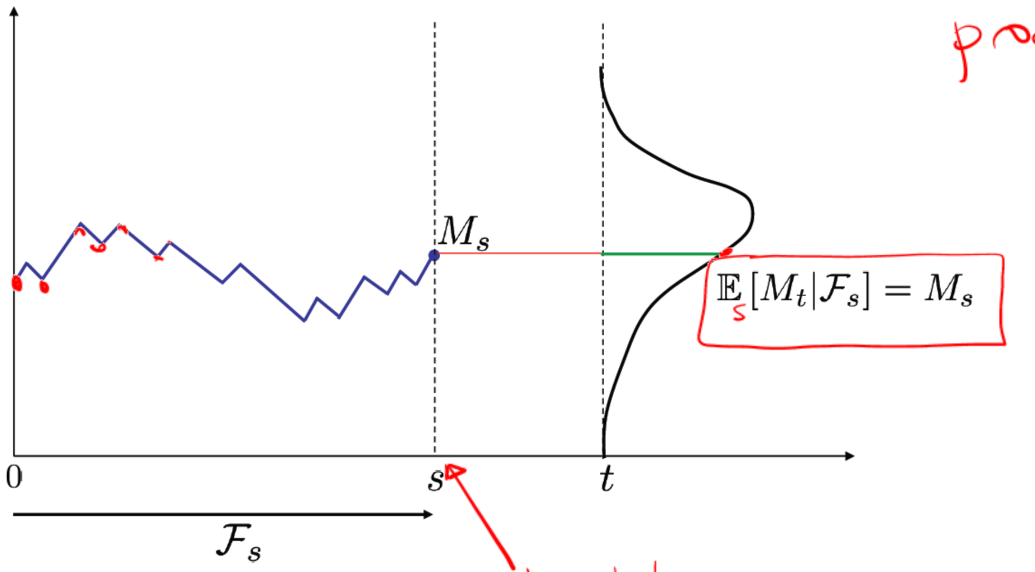
Coin example $\mathbb{E}[S_6 | \underbrace{R_1, \ldots, R_5}] = S_5$

constant mean process

$\nearrow +1$

$\searrow -1$

$\mathbb{E}[R_i] = \frac{1}{2}(+1) + \frac{1}{2}(-1)$



$t > s$

weigh ave

The first equation represents a standard integrability condition. finiteness

$$\mathbb{E}\left[\mathbb{E}[X \mid G] \mid F\right] = \mathbb{E}[X \mid F]$$

The second equation tells you that the expected value of M at time $t+1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**. process with zero mean.

no filtration

If we take expectation on both sides of eqn. 1, then

$$\boxed{\mathbb{E}[M_{t+1}] = \mathbb{E}[M_t]} = \mathbb{E}[M_{t-1}] = \dots \mathbb{E}[M_0]$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

In addition, Doob and Meyer have developed a powerful theory centred around martingales.

$\mathbb{E}[M_{t+1}] \to M_{t+1} \to M_t \to M_{t-1}$

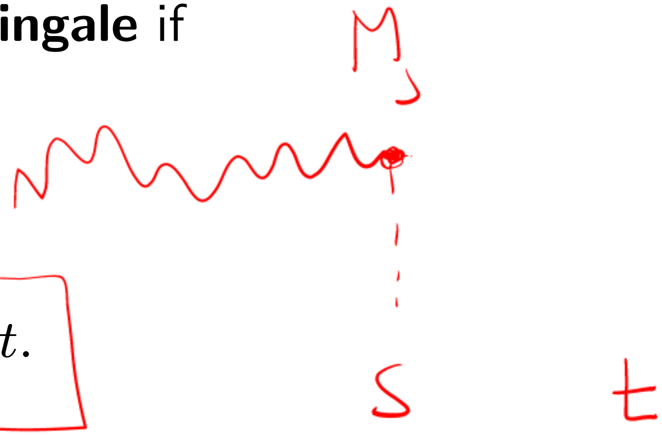
$= M_0$

Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

Cts time stoch. process $\left\{ M_t : t \in \mathbb{R}^+ \right\}$

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if



1. $$\mathbb{E}\,|M_t| < \infty$$

and

ii. $$\boxed{\underset{s}{\mathbb{E}}\,[M_t|\mathcal{F}_s] = M_s, \quad 0 \le s \le t.}$$

Lévy's Martingale Characterisation: Let X_t, $t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. X_t is a Brownian motion iff the following conditions are satisfied:

$$\mathbb{E}\left[X_t - X_s + X_s \mid \mathcal{F}_s\right]$$

$$\underbrace{\mathbb{E}\left[X_t - X_s \mid \mathcal{F}_s\right]}_{=0} + \underbrace{\mathbb{E}\left(X_s \mid \mathcal{F}_s\right)}_{X_s} = X_s$$

$$\boxed{\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] = X_s}$$

starting point.

1. $X_0 = 0$ a.s.;

cts.

2. the sample paths $t \mapsto X_t$ are continuous a.s.;

3. X_t is a martingale with respect to the filtration $\mathcal{F}_t$;

$$\mathbb{E}\left[|X_t|^2\right] = t$$

4. $\overbrace{|X_t|^2}$ $- t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

Quadratic Variation

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process X_t satisfying:

A standard B.M.

1. $X_0 = 0$ a.s.;

$N(0, dt)$

mean

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $(X_{t_4} - X_{t_3})$, $(X_{t_2} - X_{t_1})$ are independent;

$(X_{t_1} - X_{t_0})$

variance

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t - s|)$.

mean

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

Itô Integrals and Martingales

$\mathbb{E}[X_t] = 0$

$\mathbb{V}[X_t] = \mathbb{E}[X^2_t] = dt$

Next, we explore the link between Itô integration and martingales.

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

Do Itô I on $Y(t)$ & $\int_0^T$

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

$$\mathbb{E}[X^2(T)] = T + \mathbb{E}\left[\int_0^T 2X(t)dX(t)\right]$$

$= 0$

$\mathbb{E}[M_t | \mathcal{F}_s] = M_s$

Now, the quadratic variation property of Brownian motions implies that

$$\boxed{\mathbb{E}[X^2(T)] = T}$$

this is a mart

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)dX(t)\right] = 0$$

Itô integral

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale $\therefore$ $\mathbb{E}$ value $= 0$

finite expectation

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T g(t, X_t)dX_t$$

is a martingale.

So, all Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

Mod. J **Martingale Representation Theorem:** If M_t is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$M_T = M_0 + \int_0^T g(t, X_t) dX_t$$

Example Using only Itô and the fact that Itô integrals are martingales, we will show that

$$\boxed{\mathbb{E}\left[X^2(T)\right] = T.}$$

Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma,

$$\begin{aligned} X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\ &= \int_0^T dt + 2\int_0^T X_t dX_t \end{aligned}$$

since $X_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\cancel{\int_0^T X_t dX_t}\right]$$

$= 0$

Now,

$$\int_0^T X_t dX_t$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$

Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\mathbb{E}\left[X^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

another integral

$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right] dt$$

Take as a definition

This is due to an analysis result known as $\underbrace{\textbf{Fubini's Theorem}}_{\text{Analysis}}$.

$$\int\int \ldots dx\,dy \longrightarrow \int\int \ldots dy\,dx$$

Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x)\,dx$$

as $N \to \infty$

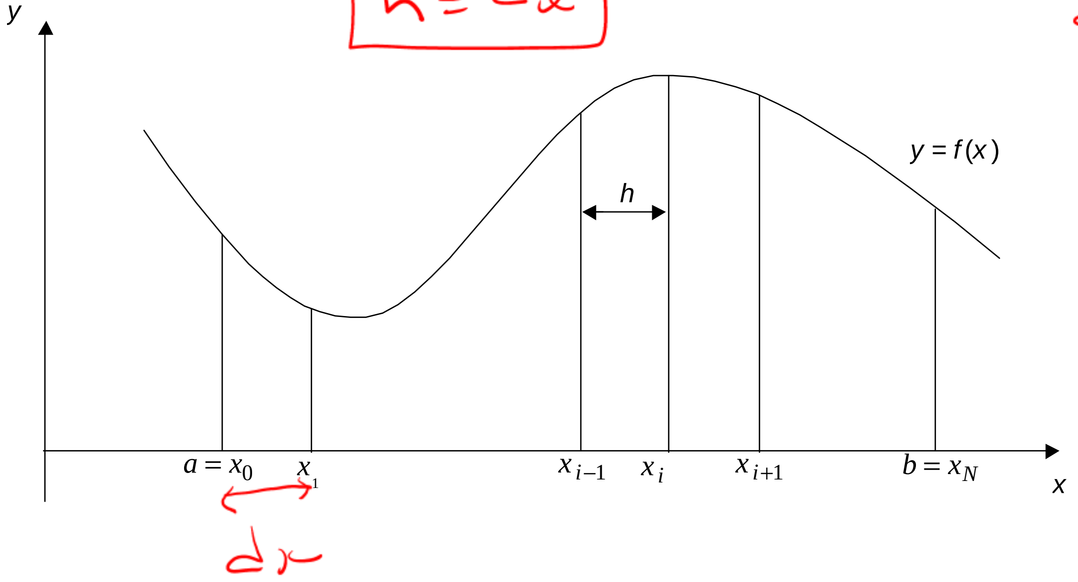
$dx \to 0$

$\boxed{h = dx}$

$dx = x_{i+1} - x_i$

$(x_i - x_{i-1})$

$dx = \dfrac{b-a}{N}$



which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against x.

Assuming f is a "well behaved" function on $[a, b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a, b]$ into N intervals with end points $x_0 = a < x_1 < x_2 < \quad ...... \quad < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are N intervals and $N + 1$ points x_i.

Discretising x gives

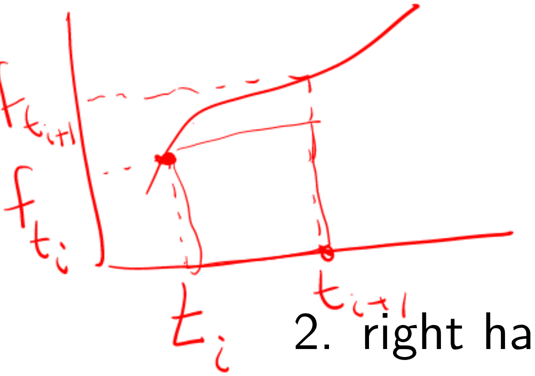
$$x_i = a + idx$$

Now consider the definite integral

$$\int_0^T f(t)\,dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;



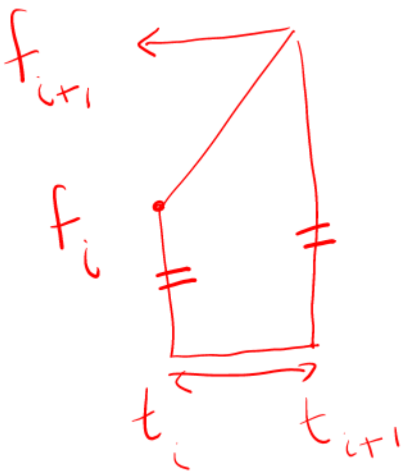
$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)\underbrace{(t_{i+1}-t_i)}_{dt}$$

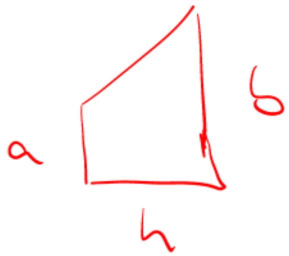
2. right hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})\underbrace{(t_{i+1}-t_i)}_{dt}$$

3. trapezium rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}\left(f(t_i)+f(t_{i+1})\right)\underbrace{(t_{i+1}-t_i)}$$





$\left(\frac{a+b}{2}\right)h$

$F = X^3 \quad dF = ?$ Itô I

$\frac{dF}{dX} = 3X^2$

$\frac{d^2F}{dX^2} = 6X$

$dF = \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2F}{dX^2} dt$

$= 3X^2 dX + 3X dt$

$= \underbrace{3X}\, dt + 3X^2 dX$

drift $= 3X \neq 0 \Rightarrow$ NOT A MARTINGALE

4. midpoint rule

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}\left(t_i + t_{i+1}\right)\right)\left(t_{i+1} - t_i\right)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

$$\int_0^T f(t, X)\,dX = \int_0^T f(t, X(t))\,dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

i.

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_i, X_i\right)\underbrace{\left(X_{i+1} - X_i\right)}_{dX},$$

similar to left hand rule

Itô Integral

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

right had rule.

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

Very Important: In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$I = \overset{\mathbb{E}}{} \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} \mathbb{E}_{t_i}\left[f(t_i, X_i)(X_{i+1} - X_i)\right] = 0$$

is special. This definition results in the **Itô Integral**.

It is special because it is **non-anticipatory**; given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $\underbrace{X_{i+1} - X_i}_{dX}$ term.

Compare this to a definition such as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ — there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f(t_{i+1}, X_{i+1})$.

$(a-b)^3$

The main thing to note about Itô integrals is that I is a random variable (unlike the deterministic case). Additionally, since I is essentially the limit of a sum of normal random variables, then by the CLT I is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = \underbrace{X(T)^3 - X(0)^3 - 3\int_0^T X(t)\,dt}.$$

Show that the result also can be found by writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} X_i{}^2 (X_{i+1} - X_i)$$

def$^{\underline{n}}$ of Itô integral

Hint: use $\underbrace{3b^2(a-b)} = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3.$

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\,dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i^2 \underbrace{(X_{i+1} - X_i)}_{a-b}$$

brown $\quad b^2$

Now note the hint:

Trick :

$$3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$$

This will be given

hence

$$3b^2(a-b)$$

$$\begin{aligned} &\equiv 3X_i^2(X_{i+1}-X_i) \\ &= X_{i+1}^3 - X_i^3 - 3X_i(X_{i+1}-X_i)^2 - (X_{i+1}-X_i)^3, \end{aligned}$$

so that

$$(X_1^3 + X_2^3 + \cdots + X_N^3)$$

$$\sum_{i=0}^{N-1} 3X_i^2(X_{i+1}-X_i) =$$

$$(X_0^3 + \cdots + X_{N-1}^3)$$

$$\left(\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3\right) - \sum_{i=0}^{N-1} 3X_i \underbrace{(X_{i+1}-X_i)^2}_{dt}$$

$$-\sum_{i=0}^{N-1} (X_{i+1}-X_i)^{3=2+1}$$

$$\to 0$$

$$= X_N^3 - X_0^3 \to X_0^3$$

$$\hookrightarrow X_T^3$$

$$dt$$

$$\int_0^T 3X\,dt$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= X(T)^3 - X(0)^3 .$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = (X_{i+1} - X_i)^2 \cdot (X_{i+1} - X_i)$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0.$

Hence putting together gives

$$X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt$$

which is consistent with Itô's lemma.

The other important property that the Itô integral has is that it is a martingale. We know that

$$X_{i+1} - X_i$$

is a martingale; i.e. in the context

$$\mathbb{E}\left[X_{i+1} - X_i\right] = 0.$$

Since

$$\mathbb{E}\left[\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\left(X_{i+1} - X_i\right)\right] =$$
$$\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\mathbb{E}\left[X_{i+1} - X_i\right] = 0$$

Thus

$$\mathbb{E}\left[\int_0^T f\left(t, X\left(t\right)\right) dX\left(t\right)\right] = 0.$$

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

Exercise We know from Itô's lemma that

$$4\int_0^T X^3(t)\,dX(t) = X^4(T) - X^4(0) - 6\int_0^T X^2(t)\,dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4\int_0^T X^3 dX = \lim_{N\to\infty} 4\sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)$$

(**Hint**: use $4b^3(a-b) = a^4 - b^4 - 4b(a-b)^3 - 6b^2(a-b)^2 - (a-b)^4$.)

Try also $\int X^4$

think of the appropriate result / hint to use.

Recap on the stochastic integration formula derived in M1C3

Proving that a Continuous Time Stochastic Process is a Martingale

Y_t $\quad dY_t = f\,dt + g\,dX_t$

Consider a stochastic process $Y(t)$ solving the following SDE:

diff. form

$$(*) \qquad dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0 \quad \text{I.C.}$$

How can we tell whether $Y(t)$ is a martingale? $\boxed{dY = g\,dX}$

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}_s[Y_t|\mathcal{F}_s] = Y_s, \quad 0 \le s \le t$$

Let's start by integrating the SDE $(*)$ between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

$(*)$ written in integral

$\mathbb{E}[Y_t | F_s] = Y_s$

Taking the expectation conditional on the filtration at time s, we get

$$\mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}\left[Y(s) + \int_s^t f(Y_u,u)du + \int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right]$$

$Y = X^3$

$\mathbb{E}$

$$= Y(s) + \mathbb{E}\left[\int_s^t f(Y_u,u)du|\mathcal{F}_s\right]$$

$\to$ Itô Integral

i.e $\mathbb{E}(\int s)$

where the last line follows from the fact that a Itô integral is a martingale, $\therefore$

$$\mathbb{E}\left[\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] = \int_s^s g(Y_u,u)dX(u) = 0.$$

So, $Y(t)$ is a martingale iff

In Taylor $dx^2 << 1$
i.e neglect.

$$\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0$$

This condition is satisfied only if $f(Y_t,t) = 0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

B.M.

$$dY(t) = g(Y_t,t)dX(t), \qquad Y(0) = Y_0$$

$Y_t = X_t + t$

$$\boxed{3\int_0^t X^2 dX}$$

Let $F = XY$ where X, Y are stochastic processes

Let's do "Itô"

$$F(X+dX, Y+dY) = F(X,Y) + \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dX^2 + \frac{1}{2}\frac{\partial^2 F}{\partial Y^2} dY^2 + \frac{\partial^2 F}{\partial X \partial Y} dX\, dY$$

dF

Look at $F = XY$: $\underbrace{\frac{\partial F}{\partial X} = Y}$; $\frac{\partial F}{\partial Y} = X$; $\frac{\partial^2 F}{\partial X^2} = 0 = \frac{\partial^2 F}{\partial Y^2}$; $\frac{\partial^2 F}{\partial X \partial Y} = 1 = \frac{\partial^2 F}{\partial Y \partial X}$

$$dF = Y\,dX + X\,dY + \frac{1}{2} \times 0 + \frac{1}{2} \times 0 + 1 \times dX\,dY$$

$$d(XY) = Y\,dX + X\,dY + \underbrace{dX\,dY}_{\text{extra term not seen in regular calculus}}$$

Itô Product Rule

Exponential Martingales

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

I.C

t

$$dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$.

Qu : How should we choose $\underline{f(t)}$ if we want the process $Z(t)$ to be a martingale?

$$dY^2 = f^2 \cancel{dt^2} + 2fg\, \cancel{dt\,dX} + g^2 \cancel{dX^2}$$

$\cancel{dt^2} \to 0$, $\cancel{dt\,dX} \to 0$, $\cancel{dX^2} \to dt$

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

diff. in Z

$$
\begin{aligned}
dZ(t) &= \frac{dZ}{dY} dY(t) + \frac{1}{2}\frac{d^2 Z}{dY^2} \overset{g^2 dt}{dY^2(t)} \\
&= \frac{dZ}{dY}\left(f(t)dt + g(t)dX(t)\right) + \frac{1}{2}\frac{d^2 Z}{dY^2} g^2(t)dt \\
&= e^{Y(t)} \underbrace{\left(f(t) + \frac{1}{2}g^2(t)\right)}_{\text{drift}} dt + e^{Y(t)} g(t)dX(t) \\
&= Z(t)\left[\cancel{\left(f(t) + \frac{1}{2}g^2(t)\right)} dt + g(t)dX(t)\right]
\end{aligned}
$$

$\frac{dZ}{dY} = e^Y \qquad = \frac{d^2 Z}{dY^2}$

zero

$Z(t)$ is a martingale if and only if it is a driftless process.

$$\boxed{dZ = g_t \, dX_t}$$

Therefore for $Z(t)$ to be a martingale we must have

drift must $= 0$

$$\boxed{f(t) + \frac{1}{2}g^2(t) = 0}$$

This is only possible if

$$\boxed{f(t) = -\frac{1}{2}g^2(t)}$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

for e^Y to be a Martingale

implying that

$\int_0^T$

$$Y(T) = \overbrace{Y_0}^{\text{I.C}} - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of $Z(t)$:

$$\boxed{dZ(t) = Z(t)g(t)dX(t).}$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

$Z_t = e^{Y_t}$

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

$Z_T = e^{Y_T}$

to give

$$\boxed{Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}}$$

$Z_0 = e^{Y_0}$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

→ exp Martingales

change measure from $\mathbb{P}$ to $\mathbb{Q}$

Key Condition (Novikov Condition)

A trading strategy $(\phi, \psi) = \{(\phi_t, \psi_t); t \in [0, ..., T]\}$ is a previsible process in that $\phi_t \in \mathcal{F}_t$ → revisited in Mod 3

A stochastic process Y_t satisfies the *Novikov condition* if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_s^2 ds\right)\right] < \infty$$

where is a γ_t previsible process.

Recap: Δ hedged approach as done Paul

Introduce: Bi. Model using replication

ϕ no. of shares

ψ no. of bonds

Key Fact

Given a process γ_t satisfying the Novikov condition, then the process M_t^γ defined as we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$M_t^\gamma = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.

In our earlier example $\gamma_t = -g(t)$; $M_t^\gamma = Z(t)$.

Key Fact (Girsanov's Theorem)

Given a process θ_t satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X_t^{\mathbb{Q}}$ defined as

$$\boxed{X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} + \int_0^t \gamma_s dX_s}$$

as is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

$$X^{\mathbb{P}} \longrightarrow X^{\mathbb{Q}}$$