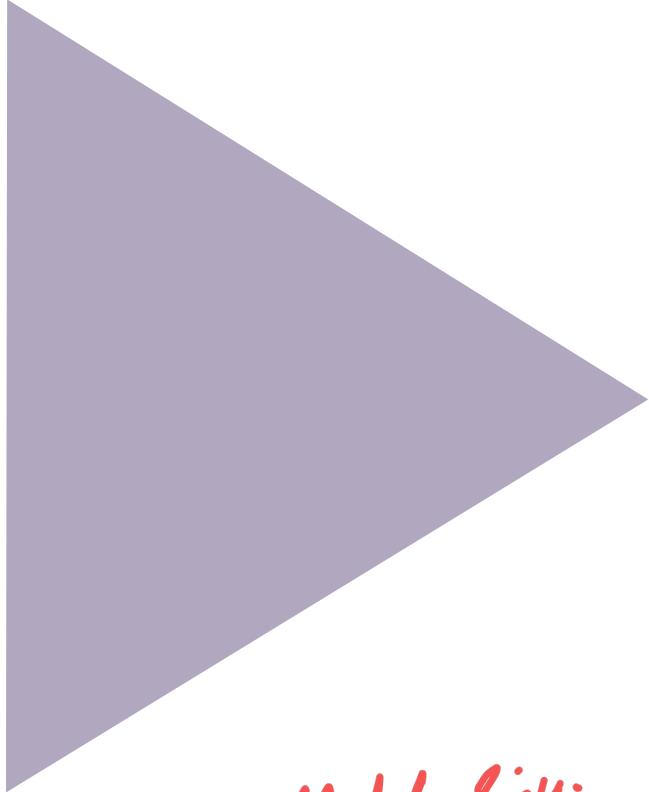

Value at Risk and Expected Shortfall

CQF | INSTITUTE



Risk

Model Risk:

How well are we making assumptions?
↳ Inputs → Outputs

Operational Risk:

Entering wrong data for example

What is Risk?

risk, noun, situation involving exposure to danger.

probability x impact

Origin: Mid 17th century: from French risque (noun), risquer (verb), from Italian risco 'danger' and rischiare 'run into danger'.

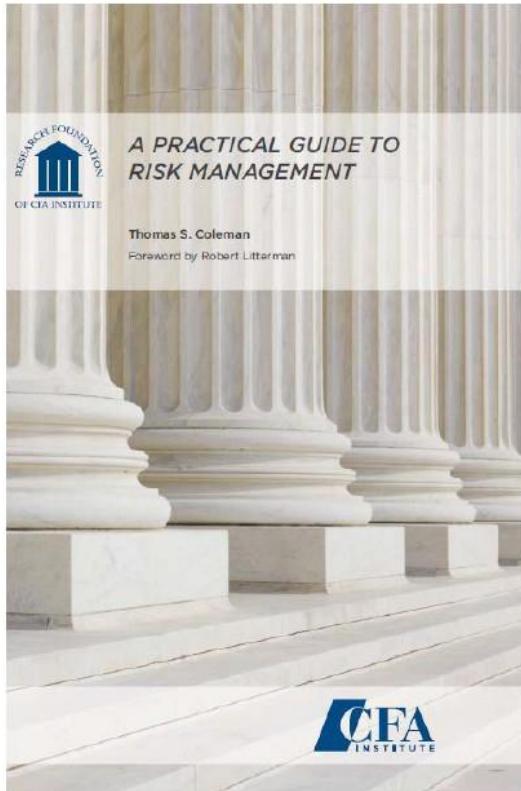
Oxford English Dictionary

PROCESS
→ Holistic approach and response to risk

risk management is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events or to maximize the realization of opportunities.

Hubbard, Douglas (2009). *The Failure of Risk Management: Why It's Broken and How to Fix It*. John Wiley & Sons.

Financial Risk



Coleman, Tom, A Practical Guide to Risk Management (July 27, 2011). CFA Institute Research Foundation M2011-2.

Available at SSRN:

<http://ssrn.com/abstract=2586032>

Risk measurement has three goals:

- **Uncovering “known” risks** faced by the portfolio or the firm. By “known” risks, I mean risks that can be identified and understood with study and analysis because these or similar risks have been experienced in the past by this particular firm or others. Such risks often are not obvious or immediately apparent, possibly because of the size or diversity of a portfolio, but these risks can be uncovered with diligence.
- **Making the known risks easy to see**, understand, and compare—in other words, the effective, simple, and transparent display and reporting of risk. **Value at risk, or VaR, is a popular tool** in this arena, but there are other, complementary, techniques and tools.
- **Trying to understand and uncover the “unknown”** or unanticipated risks—those that may not be easy to understand or anticipate, for example, because the organization or industry has not experienced them before.

Known - unknowns
vs *Unknown - unknowns*

Black - swans

- Overconfidence in ourselves
- Representativeness → Lazy stereotyping
→ Bayesian False Positives
→ Bayesian Posterior Probabilities

The Axioms of Risk

→ Adopted as sensible ways to think about risk

A Risk measure, R , is *coherent* if it satisfies the following
(Artzner et al. 1999)

- **Monotonicity:**

If portfolio $Y \geq X$ always, then $R(Y) \leq R(X)$

- **Subadditivity:** → diversification: by combining assets less risk than standalone

$$R(X + Y) \leq R(X) + R(Y) \quad \hookrightarrow \text{correlation} \leq 1$$

- **Positive Homogeneity:**

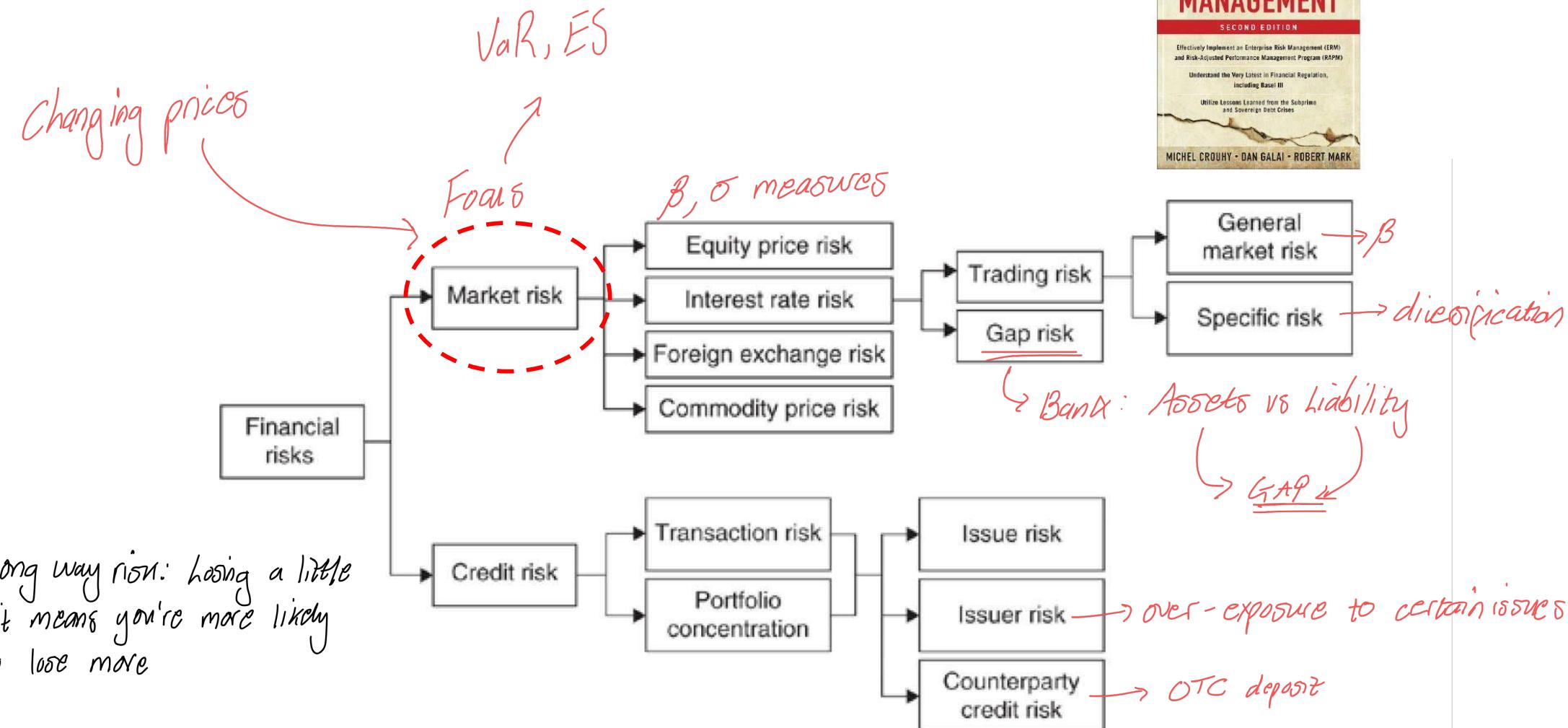
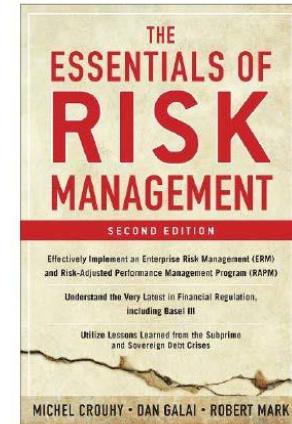
For a constant $h \geq 0$, $R(hX) = h R(X)$

- **Translational Invariance:**

For a risk-free amount n , $R(X + n) = R(X) - n$

Risk free asset reduces total risk by n

Financial Risks



What is Market Risk?

“Market risk encompasses the risk of **financial loss** resulting from **movements in market prices**.”

Market risk is rated on assessment of the following evaluation factors:

The sensitivity of the financial institution's earnings (or the economic value of its capital) to adverse changes in interest rates, FX rates, commodity prices, or equity prices.

https://www.federalreserve.gov/supervisionreg/topics/market_risk_mgmt.htm

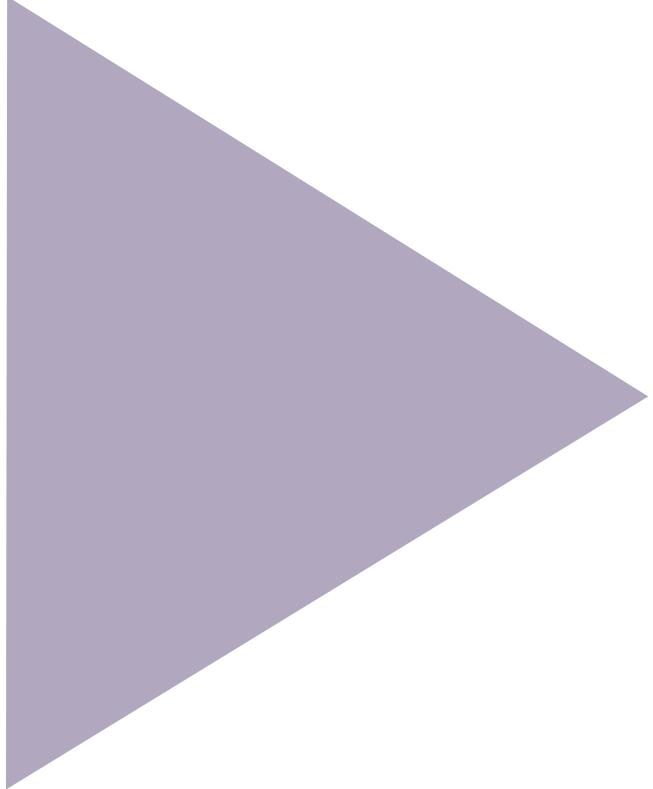
The ability of management to identify, measure, monitor, and control exposure to market risk given the institution's size, complexity, and risk profile.

The nature and complexity of interest rate risk exposure arising from nontrading positions.

The nature and complexity of market risk exposure arising from trading and foreign operations.

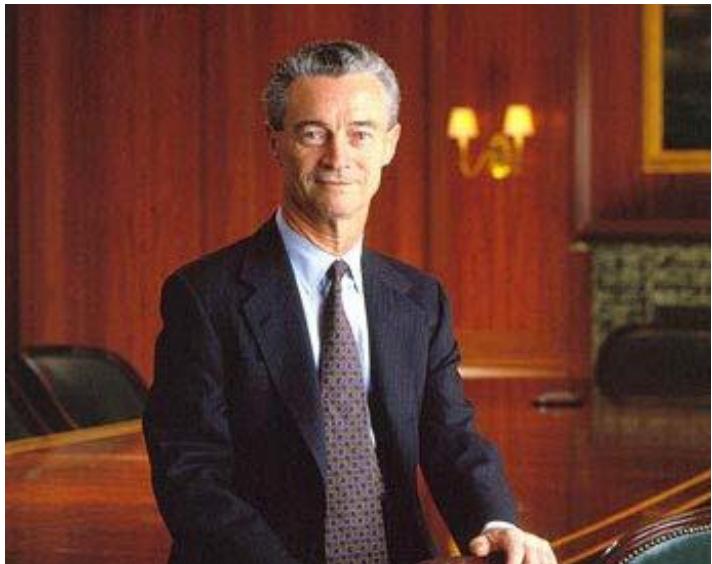
Market Risk Management Supervisory Policy and Guidance

https://www.federalreserve.gov/supervisionreg/topics/market_risk_mgmt.htm



Value at Risk

“Invention” of Value at Risk



JPMorgan is credited with helping to make VaR a widely used measure. The Chairman, Sir Dennis Weatherstone was dissatisfied with the long-risk reports he received every day. These contained a huge amount of detail on the Greek letters for different exposures, but very little that was really useful to top management. He asked for something simpler that focused on the bank's total exposure over the next 24 hours measured across the bank's entire trading portfolio.

At first his subordinates said this was impossible, but eventually they adapted the Markowitz portfolio theory to develop a VaR report. This became known as the 4:15 report because it was placed on the chairman's desk at 4:15 pm every day after the close of trading.

RISK METRICS

John C. Hull, Risk Management and Financial Institutions, Wiley, 2012

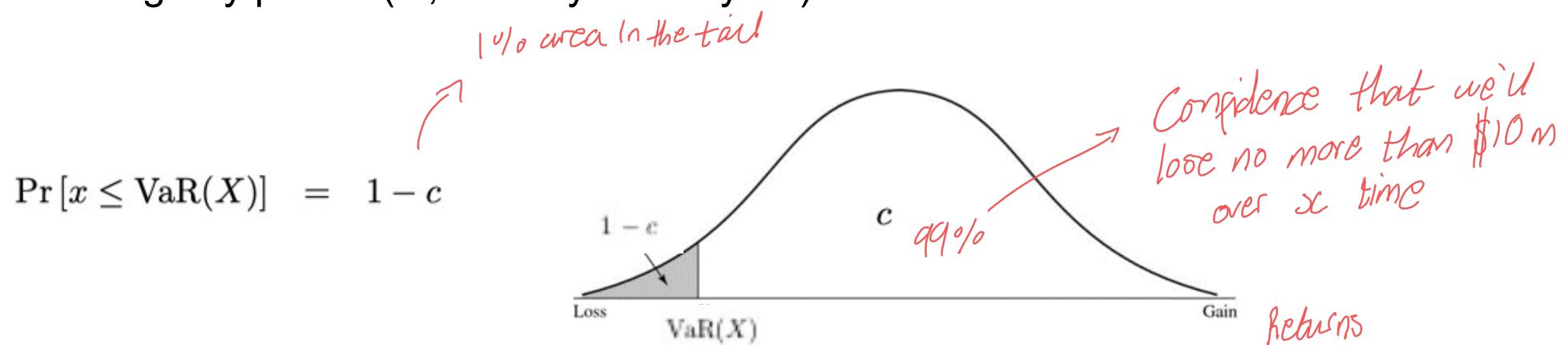
Value at Risk (VaR)

Definition: Value at Risk (VaR)

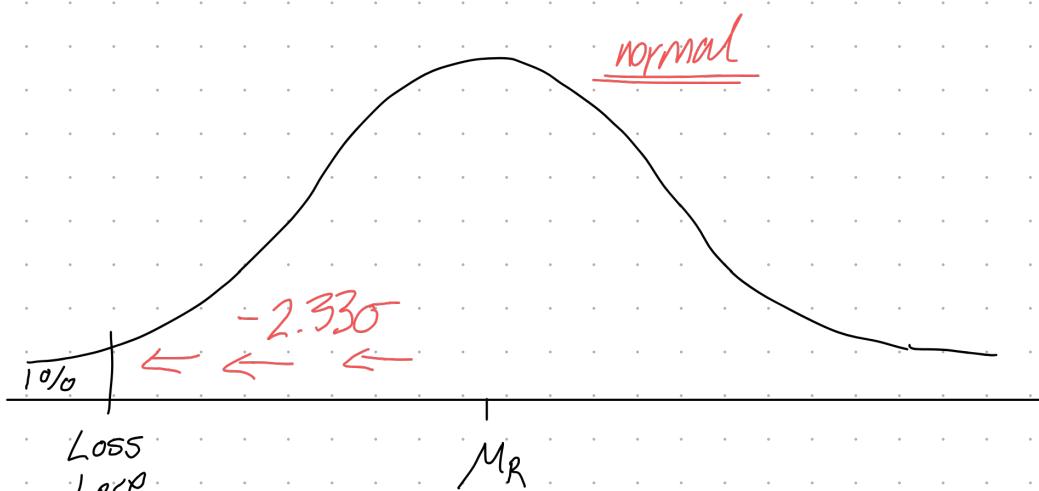
Value-at-risk (VaR) defined as the worst loss that might be expected from holding a security or portfolio over a given period of time (usually 10 days for the purpose of regulatory capital reporting), given a specified level of confidence.

Example:

A position has a **daily** VaR of \$10m at the 99% confidence. We mean that the realised daily losses will be higher than \$10m on only one day within 100 trading day period (ie, 2-3 days each year).



Standard normal CDF:



$$\text{Loss (VaR)} = M_R - 2.330$$

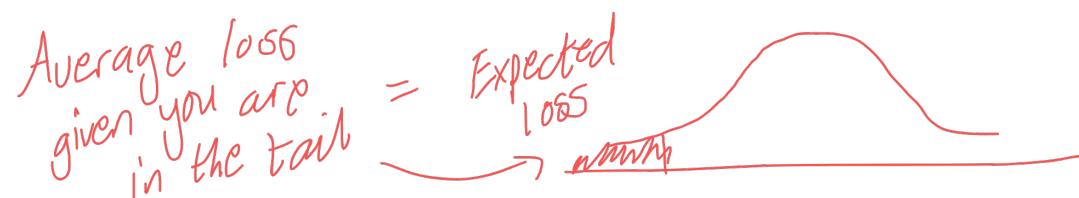
Interpretation of VaR

VaR is **not the answer** to the question: “How much can I lose on my portfolio over a given period of time?”

$$\Pr\left(\phi \leq \frac{\text{VaR}(X) - \mu}{\sigma}\right) = 1 - c$$

Instead, VaR offers a **probability statement** about the potential change in the value of a portfolio resulting from a change in market factors over a specified period of time. Factor values are usually standardised. 99% Standard VaR (loss) is equal to -2.32635 under the assumption of normality.

VaR does not state by how much actual losses are likely to exceed the VaR figure (that is Expected Shortfall’s business). The statement is about how likely (or unlikely) the threshold, eg 99th percentile, will be exceeded.



VaR and time horizons

VaR is often used to manage market risk over a 1-day time horizon. For this purpose, it's necessary to derive VaR from the daily distribution of the portfolio returns.

However, regulators have set a time horizon of 10 days for the purpose of VaR calculations that are used to report regulatory capital requirements. Ideally, this '10-day VaR' would be derived from a corresponding distribution of results over a 10-day horizon. **That is problematic**, however, as it implies the time series of data used for the analysis must be much longer (plus **distribution propagation**).

The simple workaround is multiplying the **daily VaR** by the square root of time.

Why?

Explorations in Asset Returns

Normal continuous
compounded returns
i.i.d.

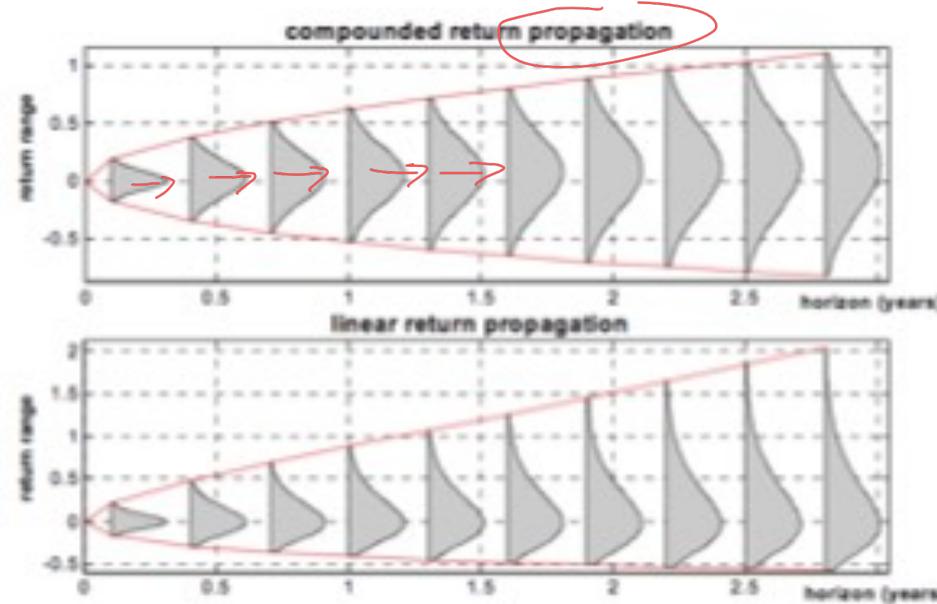
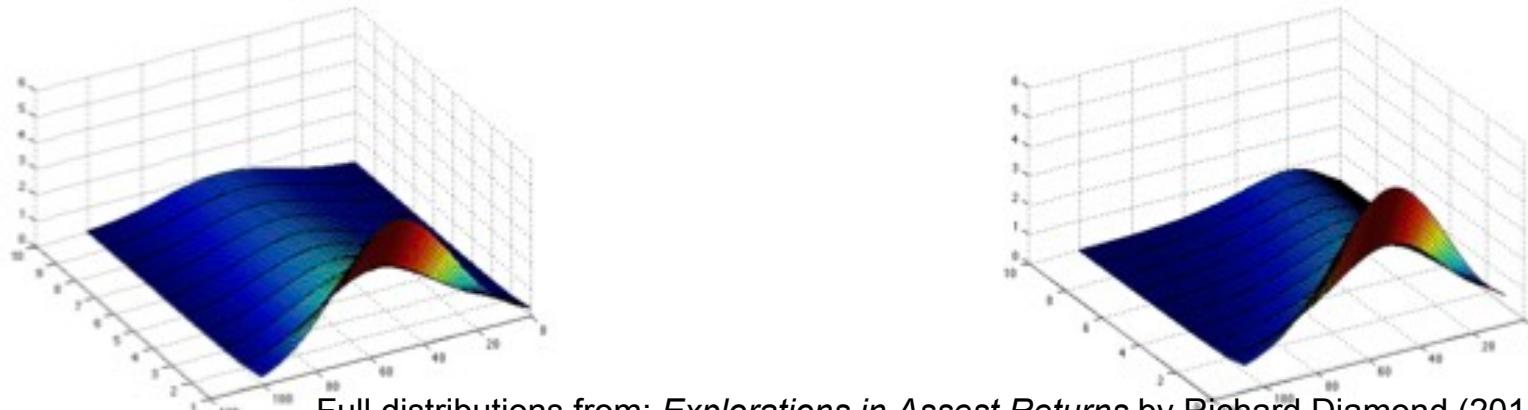


Figure 1: From: Linear versus Compounded Returns by Attilio Meucci (2010).



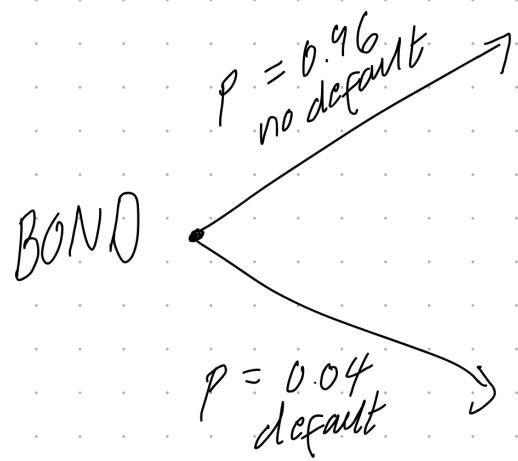
Full distributions from: *Explorations in Asset Returns* by Richard Diamond (2015)

$$\sigma_{2D}^2 = 2\sigma_D^2$$

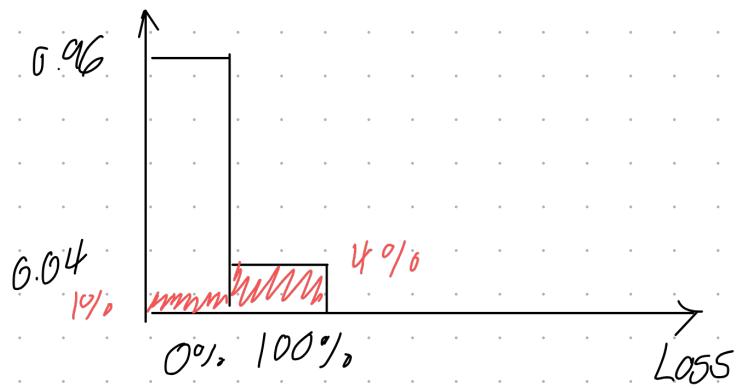
∴

$$\sigma_{2D} = \sqrt{2}\sigma_D$$

$$\sigma_{10D} = \sqrt{10}\sigma_D$$



One bond: What is 5% VaR
of that bond? = 0 %



Two bonds: $\text{VaR}(A) + \text{VaR}(B) = 0$

$\text{VaR}(A+B)$

Expected Shortfall (ES) *Average loss in tail*

Expected Shortfall also sometimes referred to as conditional value at risk, conditional tail expectation, or expected tail loss.

Whereas VaR asks the question: "How bad can things get?" expected shortfall asks: "**If we are in the tail, what is the expected loss?**

Expected shortfall, like VaR, is a function of two parameters: T (the time horizon) and c (the confidence level).

It is the expected loss during time T conditional on the loss being greater than the cth percentile of the loss distribution.

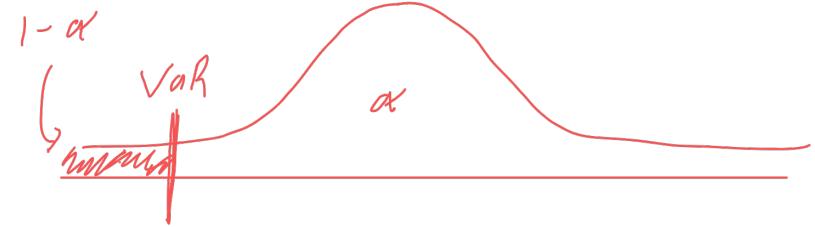
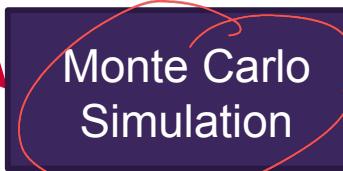
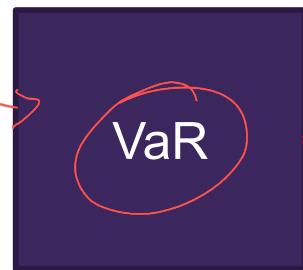
$$\text{ES}_c(X) = \mathbb{E}[X | X \leq \text{VaR}_c(X)]$$

return
c = 99% *return*

Expected shortfall is a coherent risk measure, whereas VaR is not (VaR fails the subadditivity axiom)

Methods for Estimating VaR

$$VaR_{\alpha} = M_R - 2.33 \sigma$$



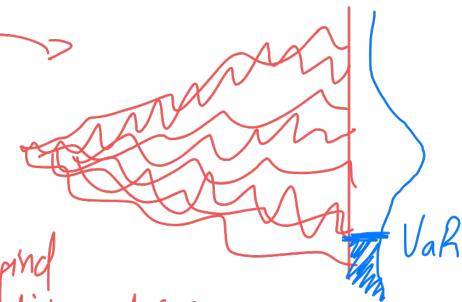
Normal distribution

Rank historical:



5% = VaR 5%

Many sims and find
VaR from dist. returns



Methods for Estimating VaR

First Method: Analytic variance-covariance approach (aka parametric VaR)

Under the analytic variance-covariance approach or ‘delta normal’ approach, we assume the risk factors (portfolio values) are log-normally distributed.

That makes the calculation much simpler, since the normal distribution is completely characterised by its first two moments. We need the mean and the variance of the portfolio return distribution from:

- a. The multivariate distribution of the risk factors
- b. The composition of the portfolio

$$\text{VaR}(X) = \mu + \Phi^{-1}(1 - c) \times \sigma$$

Methods for Estimating VaR

β : stock market movement
 D_w : interest rates

Second Method: historical simulation (aka non-parametric VaR)

The historical simulation approach to VaR calculation is conceptually simple and does not oblige the user to make assumptions about the distribution.

However, at least 2-3 years of historical data are necessary to produce meaningful results.

1. Select a sample of actual daily risk factor changes over a given period of time, say 500 days (ie, two years' worth of trading days).
2. Apply those daily changes to the current value of the risk factors, revaluing the current portfolio as many times as the number of days in the historical sample. Sum these changes across all positions, keeping the days synchronized.
3. Construct the histogram of portfolio values and identify the VaR that isolates the first percentile of the distribution in the left-hand tail (assuming VaR is derived at the 99% confidence level).

Advantages

- The major attraction of historical simulation is that the method is completely nonparametric (i.e., we don't need to worry about setting parameters) and does not depend on any assumptions about the distribution of the risk factors.
- The nonparametric nature of historical simulation also obviates the need to estimate volatilities and correlations. Historical volatilities and correlations are already reflected in the data set, so all we need to calculate are the synchronous risk-factor returns over a given historical period.
- Historical simulation has also no problem accommodating fat tails in distributions, since the historical returns already reflect actual synchronous moves in the market across all risk factors.

Disadvantages

- The main drawback of historical simulation is its complete dependence on a particular set of historical data.
- The underlying assumption is that the past, as captured in this historical data set, is a reliable representation of the future.

Methods for Estimating VaR

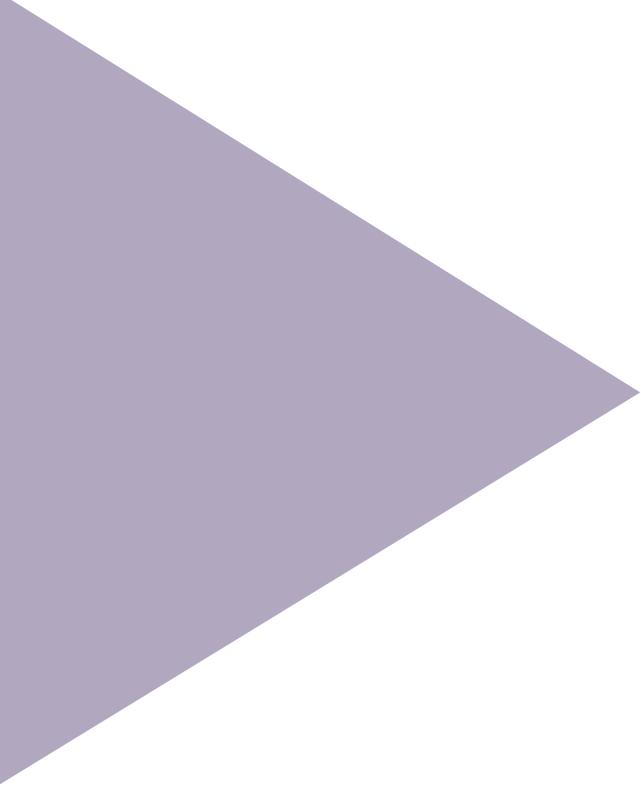
Third Method: Monte Carlo Simulation

Consists of repeatedly simulating the random processes that govern market prices and rates. Each simulation (scenario) generates a possible value for the portfolio at the target horizon (eg, 10 days).

- a. Specify all the relevant riskfactors, their stochastic processes and parameter estimates
- b. Construct price paths. Price paths are constructed using random numbers produced and advancing one step at a time (daily) the numerical solution to the stochastic processes
- c. Value the portfolio for each path (scenario).

The process is repeated a large number of times, say 10,000 times, to generate the distribution, at the risk horizon, of the portfolio return.

VaR at the 99% confidence level is then simply derived as the distance to the mean of the first percentile of the distribution.



VaR Examples

Example 1: VaR for One Asset (parametric method)

Position: USD100,000 in QQQ (Nasdaq 100)

Standard deviation of daily returns: 2.21% = σ_0

99% confidence level: 2.33 standard deviations = $\varphi^{-1}(0.99)$

Assignment:

1. Calculate 1 day VaR for this asset at 99% confidence
2. How would you then scale this up to 10 days? $\sigma_0 \times \sqrt{T} \times 2.33 \times S$
3. List three limitations to the use of VaR as a measurement of market risk

VaR formula for a single asset = Vol x Square Root [T] x CL x N

Vol: Daily volatility of the underlying asset

Square Root [T]: Square root of time (days)

CL: Confidence level (in standard deviations)

N: Notional or underlying amount

Example 1: VaR for One Asset (parametric method)

1 day VaR = $\underline{2.21\%} * \text{Square Root}[1] * \underline{2.33} * \$100,000 = \$5,149$ *1% chance I lose this or more*

10 day VaR = $2.21\% * \text{Square Root}[10] * 2.33 * \$100,000 = \$16,284$

- Note: Complexities arise at portfolio level as we have to take into account correlations between different positions

Limitations:

- Past not always good indicator of the future
- May not cover unusual market conditions (e.g. pandemic, Lehman default)
- Some market risks are not easily captured by VaR e.g.

– Non-linear pay-offs such as options

– Managed rates

Manipulated markets (QE)

– Relative value risks e.g. basis, correlation

Assumptions about liquidity – time horizon may be inappropriate

Assumes continuous, small moves in value (no discontinuity)

Example 2: VaR for One Asset (historical simulation)

Screenshot of Microsoft Excel showing historical simulation data for the FTSE index from June 30, 2010, to August 13, 2010. The data includes daily returns, current price, and calculated VaR and ES values.

Excel Sheet: Lab_VaR_ES_Example_FTSE

Key Data Points:

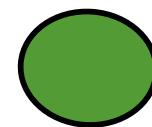
- Current Price:** £6169.74
- total sample size:** 2524
- n=:** 25
- with a probability of 1.00%:** -3.11%
- VaR = -3.11%:** £ 3.11 (for £100 invested)
- ES = -4.17%:** £ 4.17 (for £100 invested)
- there is a 99% probability that losses will not go beyond £191.88 in one day:** £ 191.88 (for £6169.74 invested)
- VaR = -0.19%:** £ 606.78 (for 10d, 99%)

Backtesting Value at Risk

- Backtesting tests validity of VaR model
 - Regulatory requirement if using model to calculate capital
- Compare VaR figures to actual outcomes
- More rigorous approach is to compare VaR to ‘hypothetical’ P&L
 - Based on holding end of day position throughout next day

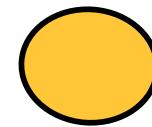
250 days with 1% VaR
≈ 2.5 days or exceptions
if our model works

‘Traffic lights’ – Basel I Approach



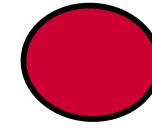
< 5 exceptions pa

→ per annum



5 - 10 exceptions pa

Regulator Judgement

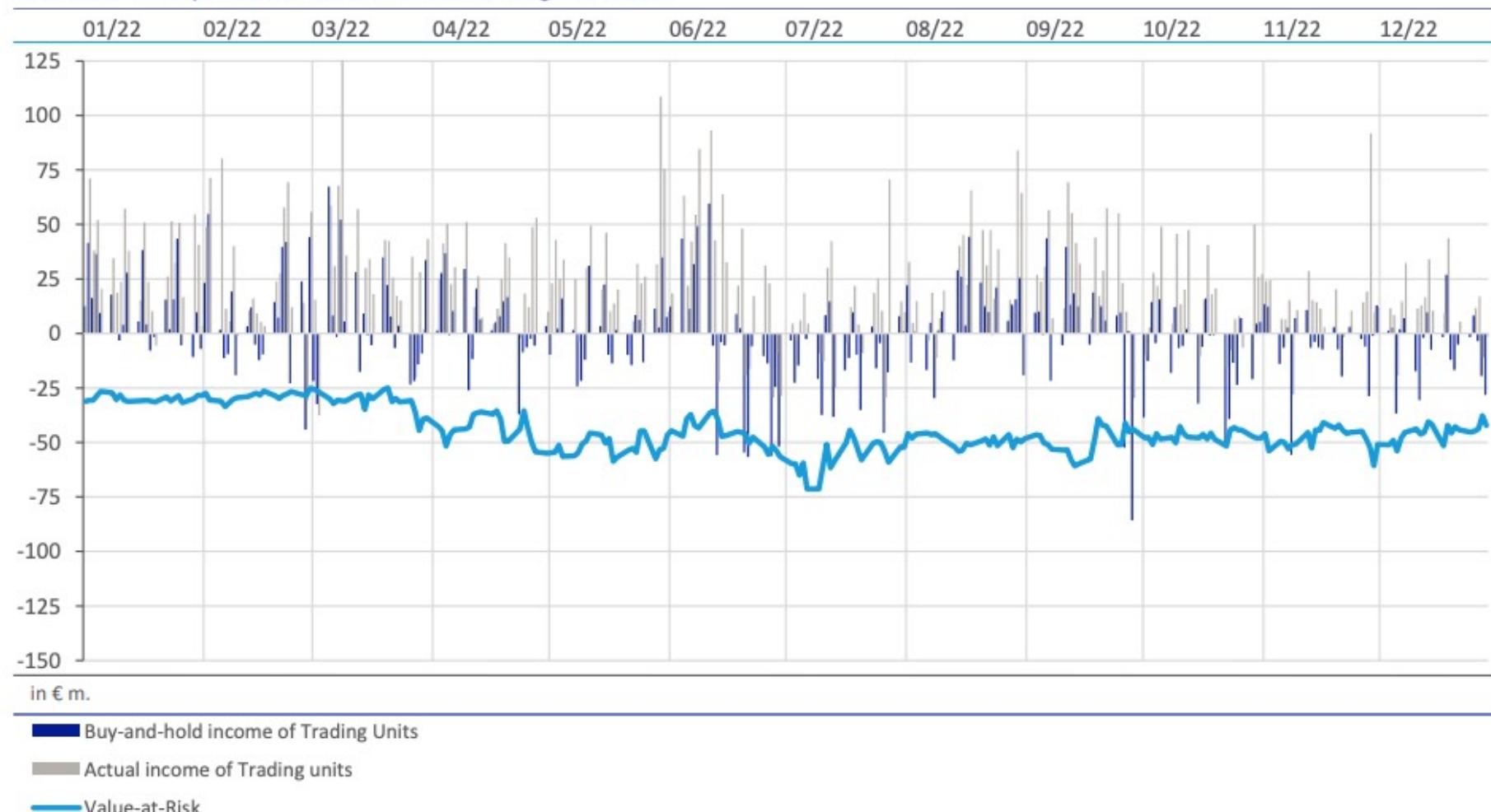


> 10 exceptions pa

PROBLEM

Backtesting Value at Risk: Deutsche Bank

EU MR4 – Comparison of VAR estimates with gains/losses



Source: Deutsche Bank Annual Report 2022

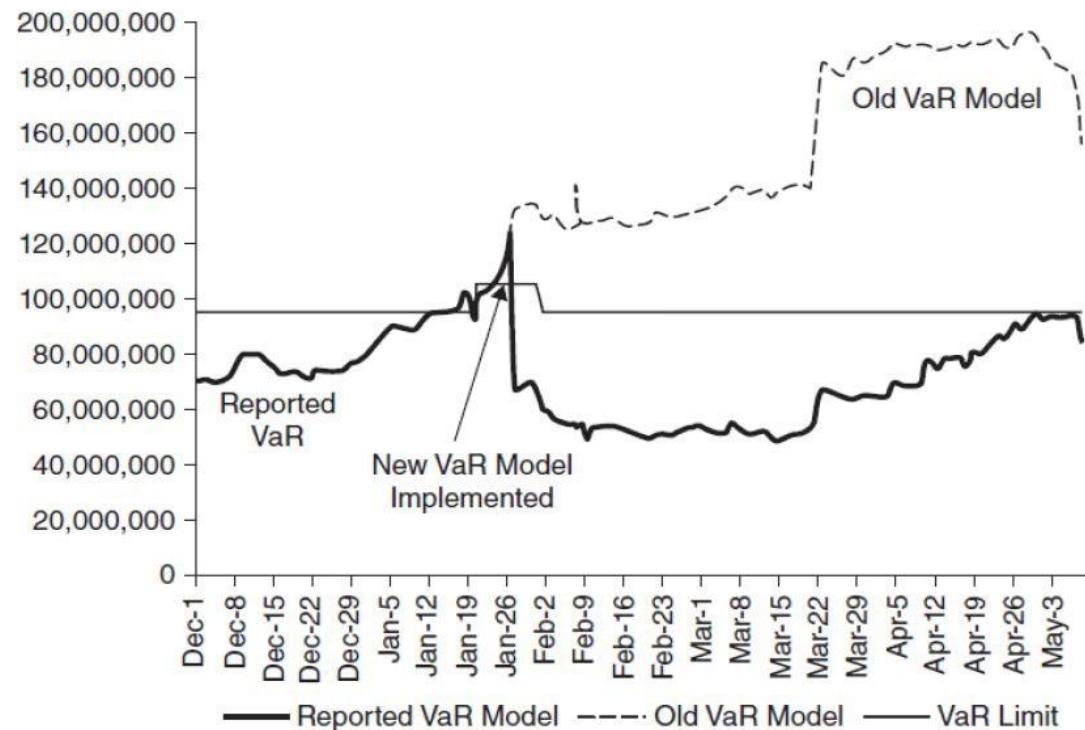
Case Study: The London Whale



'London Whale' Rattles Debt Market

by Wall Street Journal (Published on 6 Apr 2012)
https://www.youtube.com/watch?v=pen_3uYu0s4

Case Study: The London Whale

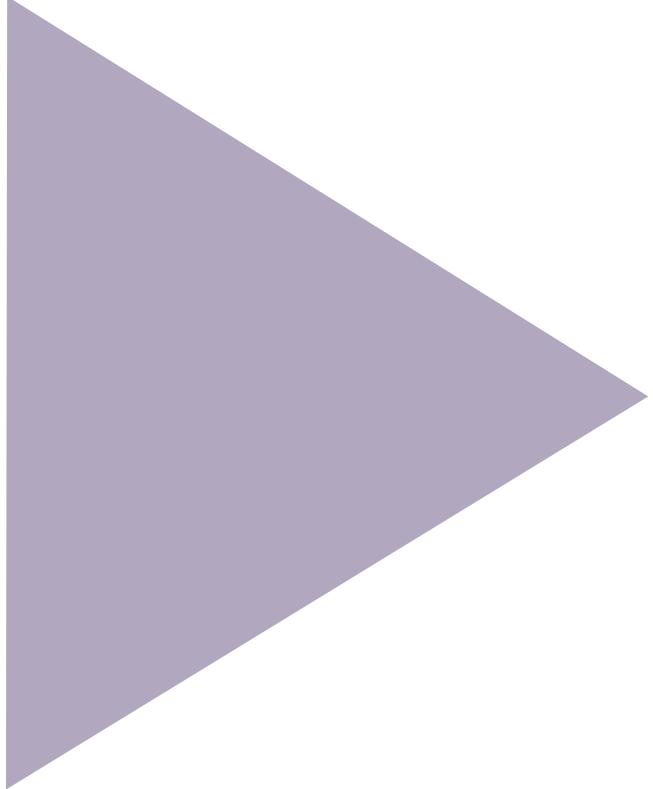


United States Senate Permanent Subcommittee on Investigations, *JPMorgan Chase Whale Trades: A Case History of Derivatives Risks and Abuses*, Hearing, March 15, 2013, Exhibits.

In contrast to JPMorgan Chase's reputation for best-in-class risk management, the whale trades exposed a bank culture in which risk limit breaches were routinely disregarded, risk metrics were frequently criticized or downplayed, and risk evaluation models were targeted by bank personnel seeking to produce artificially lower capital requirements."

During the first half of 2012, JPMorgan Chase lost billions of dollars from exposure to a massive credit derivative portfolio.

The losses were the result of the so-called "London Whale" trades executed by traders in its London office. Initially dismissed by the bank's chief executive as a "tempest in a teapot," the trading losses quickly doubled and then tripled.



Portfolio VaR and ES

Harry Markowitz



Harry Max Markowitz (born August 24, 1927) is an American economist, and a recipient of the 1989 John von Neumann Theory Prize and the 1990 Nobel Memorial Prize in Economic Sciences.

<https://www.thefamouspeople.com/profiles/images/harry-markowitz-1.jpg>

PORFOLIO SELECTION*

HARRY MARKOWITZ
The Rand Corporation

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage. We first consider the rule that the investor does (or should) maximize discounted expected, or anticipated, returns. This rule is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. We next consider the rule that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. This rule has many sound points, both as a maxim for, and hypothesis about, investment behavior. We illustrate geometrically relations between beliefs and choice of portfolio according to the "expected returns—variance of returns" rule.

One type of rule concerning choice of portfolio is that the investor does (or should) maximize the discounted (or capitalized) value of future returns.¹ Since the future is not known with certainty, it must be "expected" or "anticipated" returns which we discount. Variations of this type of rule can be suggested. Following Hicks, we could let "anticipated" returns include an allowance for risk.² Or, we could let the rate at which we capitalize the returns from particular securities vary with risk.

The hypothesis (or maxim) that the investor does (or should) maximize discounted return must be rejected. If we ignore market imperfections the foregoing rule never implies that there is a diversified portfolio which is preferable to all non-diversified portfolios. Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim.

Modern Portfolio Theory

We consider a universe of n assets. Let $x = (x_1, \dots, x_n)$ be the vector of weights in the portfolio. We assume that the portfolio is fully invested meaning that $\sum_{i=1}^n x_i = 1^\top x = 1$. We denote $R = (R_1, \dots, R_n)$ the vector of asset returns where R_i is the return of asset i . The return of the portfolio is then equal to $R(x) = \sum_{i=1}^n x_i R_i$. In a matrix form, we also obtain $R(x) = x^\top R$. Let $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}[(R - \mu)(R - \mu)^\top]$ be the vector of expected returns and the covariance matrix of asset returns. The expected return of the portfolio is:

$$\mu(x) = \mathbb{E}[R(x)] = \mathbb{E}[x^\top R] = x^\top \mathbb{E}[R] = x^\top \mu$$

*Extract from Thierry Roncalli, *Introduction to Risk Parity and Budgeting*, CRC Press, 2014

Modern Portfolio Theory

whereas its variance is equal to:

$$\begin{aligned}\sigma^2(x) &= \mathbb{E} \left[(R(x) - \mu(x)) (R(x) - \mu(x))^{\top} \right] \\ &= \mathbb{E} \left[(x^{\top} R - x^{\top} \mu) (x^{\top} R - x^{\top} \mu)^{\top} \right] \\ &= \mathbb{E} \left[x^{\top} (R - \mu) (R - \mu)^{\top} x \right] \\ &= x^{\top} \mathbb{E} \left[(R - \mu) (R - \mu)^{\top} \right] x \\ &= x^{\top} \Sigma x\end{aligned}$$

Modern Portfolio Theory

We can then formulate the investor's financial problem as follows:

1. Maximizing the expected return of the portfolio under a volatility constraint (σ -problem):

$$\max \mu(x) \quad \text{u.c.} \quad \sigma(x) \leq \sigma^* \quad (1.1)$$

2. Or minimizing the volatility of the portfolio under a return constraint (μ -problem):

$$\min \sigma(x) \quad \text{u.c.} \quad \mu(x) \geq \mu^* \quad (1.2)$$

Portfolio Risk Measures

$$\text{Loss} = -\text{Return}$$

By definition, the loss of the portfolio is $L(x) = -R(x)$ where $R(x)$ is the return of the portfolio. We consider then different risk measures:

- Volatility of the loss

Risk metric / function

$$R(x) = \sigma(L(x)) = \underline{\sigma(x)}$$

vd of returns

The volatility of the loss is the portfolio's volatility.

- Standard deviation-based risk measure

Standard deviation

$$R(x) = \underline{\text{SD}_c(x)} = \mathbb{E}[L(x)] + c \cdot \sigma(L(x)) = -\mu(x) + c \cdot \sigma(x)$$

To obtain this measure, we scale the volatility by factor $c > 0$ and subtract the expected return of the portfolio.

*Extract from Thierry Roncalli, *Introduction to Risk Parity and Budgeting*, CRC Press, 2014, pp. 71

Portfolio Risk Measures

- Value-at-risk

$$\mathcal{R}(x) = \text{VaR}_\alpha(x) = \inf \{\ell : \Pr \{L(x) \leq \ell\} \geq \alpha\}$$

The value-at-risk is the α -quantile of the loss distribution F and we note it $F^{-1}(\alpha)$.

- Expected shortfall (*average loss in the tail*)

$$\mathcal{R}(x) = \text{ES}_\alpha(x) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(x) du$$

The expected shortfall is the average of the VaRs at level α and higher (Acerbi and Tasche, 2002). We note that it is also equal to the expected loss given that the loss is beyond the value-at-risk:

$$\text{ES}_\alpha(x) = \mathbb{E}[L(x) | L(x) \geq \text{VaR}_\alpha(x)]$$

probability weighted losses
in the tail

Portfolio Risk Measures

$$\text{Expected loss} = -\text{Expected return}$$

Let us assume that the asset returns are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$. We have $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. It follows that the standard deviation-based risk measure is:

S.D. risk measure

$$\underline{\text{SD}_c(x)} = -x^\top \mu + c \cdot \underline{\sqrt{x^\top \Sigma x}} \quad \sigma \text{ or volatility} \quad (2.1)$$

For the value-at-risk, we have $\Pr\{L(x) \leq \text{VaR}_\alpha(x)\} = \alpha$. Because $L(x) = -R(x)$ and $\Pr\{R(x) \geq -\text{VaR}_\alpha(x)\} = \alpha$, we deduce that:

$$\Pr\left\{\frac{R(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}} \leq \frac{-\text{VaR}_\alpha(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}}\right\} = 1 - \alpha \quad \alpha = 99\%$$

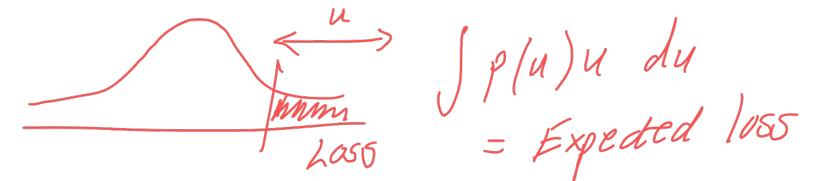
It follows that:

$$\frac{-\text{VaR}_\alpha(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}} = \Phi^{-1}(1 - \alpha)$$

We finally obtain:

$$\boxed{\text{VaR}_\alpha(x) = -x^\top \mu + \Phi^{-1}(\alpha) \sqrt{x^\top \Sigma x}} \quad (2.2)$$

Portfolio Risk Measures



This is a special case of the standard deviation-based risk measure with $c = \Phi^{-1}(\alpha)$. It implies that the value-at-risk is a coherent and convex risk measure if the asset returns are normally distributed. The expression of the expected shortfall is:

$$ES_{\alpha}(x) = \frac{1}{1-\alpha} \int_{-\mu(x)+\sigma(x)\Phi^{-1}(\alpha)}^{\infty} \frac{u}{\sigma(x)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{u+\mu(x)}{\sigma(x)}\right)^2\right) du$$

integrating across
losses

With the variable change $t = \sigma(x)^{-1}(u + \mu(x))$, we obtain:

$$\begin{aligned} ES_{\alpha}(x) &= \frac{1}{1-\alpha} \int_{\Phi^{-1}(\alpha)}^{\infty} (-\mu(x) + \sigma(x)t) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) dt \\ &= -\frac{\mu(x)}{1-\alpha} [\Phi(t)]_{\Phi^{-1}(\alpha)}^{\infty} + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}} \int_{\Phi^{-1}(\alpha)}^{\infty} t \exp\left(-\frac{1}{2}t^2\right) dt \\ &= -\mu(x) + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}} \left[-\exp\left(-\frac{1}{2}t^2\right) \right]_{\Phi^{-1}(\alpha)}^{\infty} \\ &= -\mu(x) + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}} \exp\left(-\frac{1}{2}[\Phi^{-1}(\alpha)]^2\right) \end{aligned}$$

$$t = \frac{u + \mu}{\sigma}$$

$$\begin{aligned} u &= -\mu + \sigma t \\ du &= \sigma dt \end{aligned}$$

standard normal pdf where
 variable $= \varphi^{-1}(x) = -2.33$

Portfolio Risk Measures

The expected shortfall of portfolio x is then:

$$\text{ES}_\alpha(x) = -x^\top \mu + \frac{\sqrt{x^\top \Sigma x}}{(1 - \alpha)} \phi(\Phi^{-1}(\alpha)) \quad (2.3)$$

Like the value-at-risk, it is a standard deviation-based risk measure with $c = \phi(\Phi^{-1}(\alpha)) / (1 - \alpha)$.

↑ 2.33 0.01
Standard normal pdf 2.33

Example: Portfolio VaR and ES

Example We consider three stocks A, B and C. Their current prices are respectively 244, 135 and 315 dollars. We assume that their expected returns are equal to 50 bps, 30 bps and 20 bps on a daily basis, whereas their daily volatilities are 2%, 3% and 1%. The correlation of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.25 & 0.60 & 1.00 \end{pmatrix}$$

*Extract from Thierry Roncalli, *Introduction to Risk Parity and Budgeting*, CRC Press, 2014, pp. 75

Example: Portfolio VaR and ES

We consider Portfolio #1 composed of two stocks A , one stock B and one stock C . The value of the portfolio is then equal to 938 dollars. We deduce that the weights x are $(52.03\%, 14.39\%, 33.58\%)$. Computation gives $\mu(x) = 37.0$ bps and $\sigma(x) = 1.476\%$. In Table 2.1, we report the values taken by the value-at-risk and the expected shortfall for different confidence levels α . Because these risk measures are computed using Formulas (2.2) and (2.3) with the portfolio weights, the values are expressed as a percentage. For example, the value-at-risk of the portfolio for $\alpha = 99\%$ is equal to²:

$$\text{VaR}_{99\%}(x) = -0.370\% + 2.326 \times 1.476\% = 3.06\%$$

2.33 σ

For the expected shortfall, we obtain³:

$$\text{ES}_{99\%}(x) = -0.370\% + \frac{2.667}{\Phi(2.33)} \times 1.476\% = 3.56\%$$

$\Phi(2.33)$
 $1 - \alpha$

Example: Portfolio VaR and ES

TABLE 2.1: Computation of risk measures $\text{VaR}_\alpha(x)$ and $\text{ES}_\alpha(x)$

| Portfolio | $\mathcal{R}(x)$ | α | | | |
|-----------|------------------|----------|-------|-------|-------|
| | | 90% | 95% | 99% | 99.5% |
| #1 | VaR (in %) | 1.52 | 2.06 | 3.06 | 3.43 |
| | | 14.27 | 19.30 | 28.74 | 32.20 |
| | ES (in %) | 2.22 | 2.67 | 3.56 | 3.90 |
| | | 20.83 | 25.09 | 33.44 | 36.58 |

```
# LAB: VAR and ES portfolio
# Thierry Roncalli, Introduction to Risk Parity and Budgeting, CRC Press, 2014
# Example page 75
#
# Developed Alonso Pena, Fitch Learning UK, Jan 2018
```

```
import math
import numpy as np
from scipy.stats import norm
```

```
# STEP 1 input data three stocks A,B,C
x = np.matrix([[0.5203], [0.1439], [0.3358]])
mu = np.matrix([[50/10000],[30/10000], [20/10000]])
vol = np.matrix([[2/100], [3/100], [1/100]])
```

```
rho = np.matrix([[1, 0.5, 0.25],
[0.5, 1, 0.6],
[0.25, 0.6, 1]])
```

```
Sigma = np.matrix([[4.0000e-004, 3.0000e-004, 5.0000e-005],
[3.0000e-004, 9.0000e-004, 1.8000e-004],
[5.0000e-005, 1.8000e-004, 1.0000e-004]])
```

```
alpha=0.99
```

```
# STEP 2 calculations
# compute VAR
VAR_x = -x.T*mu + norm.ppf(alpha)*math.sqrt(x.T*Sigma*x)
```

```
# compute ES
ES_x = -x.T*mu + (math.sqrt(x.T*Sigma*x))/(1-alpha)*norm.pdf(norm.ppf(alpha))
```

```
# STEP 3 output results
print('VAR_x = ', VAR_x )
print('ES_x = ', ES_x )
```

