Differential Equations

To solve: $\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0$ $\quad V(S) \quad \sigma \in \mathbb{R}$

B.C. 1: $\lim_{S\to\infty} V(S) = 0$

B.C. 2: For $E, S \in \mathbb{R}$ $\quad V(S^*) = E - S^*$

Note: This ODE is Cauchy-Euler $\therefore \exists$ solution: $V(S) = S^\lambda$ and $aS^2y'' + bSy' + cy = 0$

Auxillary Equation: $\frac{1}{2}\sigma^2\lambda^2 + \left(r - \frac{1}{2}\sigma^2\right)\lambda - r = 0$

A.E: $\lambda^2 + \left(\frac{2r}{\sigma^2} - 1\right)\lambda - \frac{2r}{\sigma^2} = 0 \implies (\lambda - 1)\left(\lambda + \frac{2r}{\sigma^2}\right) = 0 \implies \lambda = 1$

$\lambda = -\frac{2r}{\sigma^2}$

G.S: $V(S) = AS + BS^{-\frac{2r}{\sigma^2}}$ then find A and B...

$$V(S) = AS + BS^{-2r/\sigma^2}$$

① $\lim_{S \to \infty} V(S) = 0$

$$V(S) = \cancel{AS} + \cancel{\frac{B}{S^{2r/\sigma^2}}}$$

$= \infty$

$S \to \infty$

$= 0$ when $S \to \infty$

$\therefore$ For B.C. to hold $A = 0$ when $S \to \infty$

$$\therefore V(S) = BS^{-2r/\sigma^2}$$

② $V(S^*) = E - S^*$

$$V(S^*) = BS^{*^{-2r/\sigma^2}} = E - S^* \Rightarrow B = \frac{E - S^*}{S^{*^{-2r/\sigma^2}}} = (E - S^*)\, S^{*^{2r/\sigma^2}}$$

Partial Differential Equations

Black-scholes equation : $\dfrac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0$ $\quad V = V(S,t)$

"diffusion type equation", backward equation and linear parabolic P.D.E

$\exists$ solution of the form $V(S,t) = f(t)\,g(S)$ seperable solution

$\dot{} \equiv \dfrac{d}{dt}$ $\quad {}' \equiv \dfrac{d}{dS}$ some shorthand notation $\dfrac{\partial V}{\partial t} = \dot{f}g$; $\dfrac{\partial V}{\partial S} = fg'$; $\dfrac{\partial^2 V}{\partial S^2} = fg''$

$\dot{f}g + \frac{1}{2}\sigma^2 S^2 f g'' + rSfg' - rfg = 0$ divide through by fg

$$\frac{\dot{f}g}{fg} + \frac{1}{2}\sigma^2 S^2 \frac{fg''}{fg} + rS\frac{fg'}{fg} - \frac{rfg}{fg} = 0 \implies \frac{\dot{f}}{f} + \frac{1}{2}\sigma^2 S^2 \frac{g''}{g} + rS\frac{g'}{g} - r = 0$$

$$\underbrace{-\frac{\dot{f}}{f}}_{t \text{ independent of } S} = \underbrace{\frac{1}{2}\sigma^2 S^2 \frac{g''}{g} + rS\frac{g'}{g} - r}_{\text{independent of } t \text{, dependent on } S}$$

= Some function independent of both t and S, i.e a const. C

$$-\frac{\dot{f}}{f} = \frac{1}{2}\sigma^2 S^2 \frac{g''}{g} + rS\frac{g'}{g} - r = C$$

① $\dot{f} = -Cf$ ② $\frac{1}{2}\sigma^2 S^2 g'' + rSg' - (c+r)g = 0$

equating each side to C
② is cauchy-Euler eqn.

① $\frac{df}{f} = -C\,dt \Rightarrow \int \frac{df}{f} = \int -c\,dt = \log f(t) = -ct + K = f(t) = Ae^{-ct}$

② $\exists$ solution of form $V(S) = S^m$

A.E: $\frac{1}{2}\sigma^2 m^2 + \left(r - \frac{1}{2}\sigma^2\right)m - (c+r) = 0$ multiply through by $\frac{2}{\sigma^2}$

$$m^2 + \left(\frac{2r}{\sigma^2} - 1\right)m - \frac{2}{\sigma^2}(c+r) = 0$$

$$m_\pm = -\frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right) \pm \frac{1}{2}\sqrt{\left(\frac{2r}{\sigma^2} - 1\right)^2 + \frac{8}{\sigma^2}(c+r)}$$

Multiple cases of $m_\pm$ on following page to get to G.S

Case 1: $m_+ \neq m_- \in \mathbb{R}$ 2 real distinct roots $\therefore$ $g(S) = BS^{m_+} + CS^{m_-}$

$$V(S,t) = f(t)g(S) = Ae^{-ct}\left(BS^{m_+} + CS^{m_-}\right)$$
$$= e^{-ct}\left(\alpha S^{m_+} + \beta S^{m_-}\right)$$

Case 2: $m_+ = m_- = m$ repeated root $\therefore$ $S^m, \ S^m \log S$

$$\therefore \ g(S) = S^m\left(a + b\log S\right)$$

Case 3: $m_\pm = p \pm iq$ $\therefore$ $S^p\left[c\cos(q\log S) + d\sin(q\log S)\right]$