

Mathematics Primer Exercises

1 Calculus Problem Sheet

1. Consider two functions $f(x) = 9x + 2$ and $g(x) = \frac{x}{9} - \frac{2}{9}$. Show that they are inverse functions of one another.

This simply requires showing $f(g(x)) = g(f(x)) = x$

$$\begin{aligned} f(g(x)) &= 9\left(\frac{x}{9} - \frac{2}{9}\right) + 2 = x - 2 + 2 = x \\ g(f(x)) &= \frac{9x + 2}{9} - \frac{2}{9} = x + \frac{2}{9} - \frac{2}{9} = x \end{aligned}$$

2. Obtain the inverse of the function $f(x) = x^{1/3} + 2$.

$$\begin{aligned} y &= x^{1/3} + 2 \longrightarrow x^{1/3} = y - 2 \\ x &= (y - 2)^3 = g(y) \\ \therefore f^{-1}(x) &= g(x) = (x - 2)^3 \end{aligned}$$

3. Calculate the following limits:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) \longrightarrow 4$$

$$\lim_{x \rightarrow 1} \frac{x^2 - x}{2x^2 + 5x - 7} = \lim_{x \rightarrow 1} \frac{x(x - 1)}{(2x + 7)(x - 1)} = \lim_{x \rightarrow 1} \frac{x}{(2x + 7)} \longrightarrow \frac{1}{9}$$

$$\lim_{x \rightarrow -25} \frac{\sqrt{x} + 5}{x - 25} = \lim_{x \rightarrow -25} \frac{\sqrt{x} + 5}{(\sqrt{x} - 5)(\sqrt{x} + 5)} = \lim_{x \rightarrow -25} \frac{\sqrt{x} + 5}{(\sqrt{x} - 5)(\sqrt{x} + 5)} \longrightarrow \infty$$

$$\lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h} : \text{ see primer lecture notes}$$

$$\lim_{h \rightarrow -2} \frac{h^3 + 8}{h + 2} = \lim_{h \rightarrow -2} \frac{(h + 2)^3 - 6h(h + 2)}{h + 2} = \lim_{h \rightarrow -2} (h + 2)^2 - 6h \longrightarrow 12$$

$$\lim_{t \rightarrow 1} \frac{(1/t) - 1}{t - 1} = \lim_{t \rightarrow 1} \frac{(1 - t)/t}{t - 1} = \lim_{t \rightarrow 1} \frac{-(t - 1)/t}{t - 1} \lim_{t \rightarrow 1} \frac{-1}{t} \longrightarrow -1$$

$$\lim_{x \rightarrow \sqrt{2}} (x^2 + 3)(x - 4) = \lim_{x \rightarrow \sqrt{2}} (x^2 + 3) \lim_{x \rightarrow \sqrt{2}} (x - 4) \longrightarrow 5(\sqrt{2} - 4)$$

4. Using the definition of the derivative, show that for

$$\begin{aligned}
 y &= 2x + 1, \quad y' = 2 \\
 y' &= \lim_{h \rightarrow 0} \frac{[2(x+h) + 1] - (2x + 1)}{h} = \lim_{h \rightarrow 0} \frac{2x + 2h + 1 - 2x - 1}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = \lim_{h \rightarrow 0} 2 = 2 \\
 f(x) &= \frac{1}{x-2}, \quad f'(x) = -\frac{1}{(x-2)^2} \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{x-2+h} - \frac{1}{x-2}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{(x-2) - (x-2+h)}{(x-2+h)(x-2)} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{-h}{(x-2+h)(x-2)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{(x-2+h)(x-2)} = -\frac{1}{(x-2)^2}
 \end{aligned}$$

$$\begin{aligned}
 g(x) &= |x-5|, \quad \text{no derivative exists at } x=5 \\
 g'_+(x) &= \lim_{h \rightarrow 0^+} \frac{g(x+h) - g(x)}{h} \longrightarrow g'_+(5) = \lim_{h \rightarrow 0^+} \frac{g(5+h) - g(5)}{h} = \lim_{h \rightarrow 0^+} \frac{|5+h-5| - |0|}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{|h|}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \\
 g'_-(x) &= \lim_{h \rightarrow 0^-} \frac{g(x+h) - g(x)}{h} \longrightarrow g'_-(5) = \lim_{h \rightarrow 0^-} \frac{g(5+h) - g(5)}{h} = \lim_{h \rightarrow 0^-} \frac{|5+h-5| - |0|}{h} \\
 &= \lim_{h \rightarrow 0^-} \frac{|h|}{h} = \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \\
 g'_+(x) &= g'_-(x) \therefore \text{the derivative does not exist at } x=5.
 \end{aligned}$$

5. Differentiate the following functions y , to obtain $\frac{dy}{dx}$:

We know the Chain Rule:

$$y = f(u) \quad \text{where } u = F(x) \quad \text{then} \quad \frac{dy}{dx} = \frac{df}{du} \frac{du}{dx}$$

$$\begin{aligned}
 y &= (x^2 - 4x + 2)^5 \\
 \frac{dy}{dx} &= 5u^4 \cdot (2x - 4) = 5(x^2 - 4x + 2)^4 (2x - 4)
 \end{aligned}$$

$$\begin{aligned}
 y &= \frac{1}{(4x^2 + 6x - 7)^3} = (4x^2 + 6x - 7)^{-3} \\
 \frac{dy}{dx} &= -6(4x^2 + 6x - 7)^{-4} (4x + 3)
 \end{aligned}$$

$$y^4 + 3y - 4x^3 = 5x + 1$$

implicit differentiation:

$$\frac{dy}{dx} = \frac{5 + 12x^2}{4y^3 + 3}$$

$$y = \ln \sqrt[3]{(2x+5)^2} = \ln (2x+5)^{2/3} = \frac{2}{3} \ln (2x+5)$$

$$\frac{dy}{dx} = \frac{4}{3(2x+5)}$$

$$y = \cos(4-3x) : \frac{dy}{dx} = 3 \sin(4-3x)$$

Now use the product rule:

$$y = x^2 \exp(x) : \frac{dy}{dx} = x e^x (2+x)$$

Next problem requires the quotient rule

$$y = \frac{3x^2 - x + 2}{4x^2 + 5}$$

$$y' = \frac{(4x^2 + 5)(6x - 1) - 8x(3x^2 - x + 2)}{(4x^2 + 5)^2}$$

6. Calculate the following

$$\int \sqrt{x} (x^2 - 4x + 2) dx = \int (x^{5/2} - 4x^{3/2} + 2x^{1/2}) dx$$

$$= \frac{2}{7} x^{7/2} - \frac{8}{5} x^{5/2} + \frac{4}{3} x^{3/2} + C$$

$$\int_4^1 (3\sqrt{x} + 1)(\sqrt{x} - 2) dx =$$

$$\int_4^1 (3x - 5x^{1/2} - 2) dx = \left[\frac{3}{2} x^2 - \frac{10}{3} x^{3/2} - 2x \right]_4^1 = \frac{41}{6}$$

$$\int_{-1}^{-2} \frac{(2s-7)}{s^3} ds$$

$$= \int_{-1}^{-2} \frac{2}{s^2} + \frac{7}{2s^3} ds = \left[-\frac{2}{s} \right]_{-1}^{-2} + \left[\frac{7}{2s^2} \right]_{-1}^{-2} = -\frac{29}{8}$$

$$\int_3^2 \frac{(x^2 - 1)}{(x - 1)} dx =$$

$$\int_3^2 (x + 1) dx = \left[\frac{x^2}{2} + x \right]_3^2 = -\frac{7}{2}$$

$$\int_{-1}^5 |2x - 3| dx = \int_{-1}^{3/2} (3 - 2x) dx + \int_{3/2}^5 (2x - 3) dx$$

$$= [3x - x^2]_{-1}^{3/2} + [x^2 - 3x]_{3/2}^5 = \frac{37}{2}$$

$$\begin{aligned}
\int \frac{5x-12}{x(x-4)} dx & : \quad \frac{5x-12}{x(x-4)} \equiv \frac{3}{x} + \frac{2}{(x-4)} \\
& = \int \frac{3}{x} + \int \frac{2}{(x-4)} = 3 \ln x + 2 \ln (x-4) + C \\
& = \ln x^3 (x-4)^2 + C
\end{aligned}$$

7. By using suitable substitutions (change of variable), evaluate the following

$$\int (3-x^4)^3 x^3 dx$$

let $z = 3 - x^4$ so that $dz = -4x^3 dx$ and the integral becomes

$$-\frac{1}{4} \int z^3 dz = -\frac{1}{16} (3-x^4)^4 + c$$

$$\int \frac{(x^2+x)}{(4-3x^2-2x^3)^4} dx$$

put $z = 4 - 3x^2 - 2x^3$ so that $dz = -6(x^2+x) dx$ to give

$$-\frac{1}{6} \int \frac{1}{z^4} dz = \frac{1}{18(4-3x^2-2x^3)^3} + c$$

$$\int \frac{(\sqrt{u}+3)^4}{\sqrt{u}} du$$

let $z = \sqrt{u} + 3 =$ so that $dz = \frac{1}{2\sqrt{u}} du$ and we have

$$2 \int z^4 dz = 2 \frac{z^5}{5} + c = \frac{2(\sqrt{u}+3)^5}{5} + c$$

$$\int \left(1 + \frac{1}{u}\right)^{-3} \left(\frac{1}{u^2}\right) du$$

let $z = 1 + u^{-1}$ so that $dz = -\frac{1}{u^2} du$ and

$$-\int z^{-3} dz = \frac{1}{2} \left(1 + \frac{1}{u}\right)^{-2}$$

$$\int x e^{x^2} dx$$

let $z = x^2$ so that $dz = 2x dx$, so

$$\frac{1}{2} \int e^z dz = \frac{1}{2} e^z + c = \frac{1}{2} e^{x^2} + c$$

$$\int (\sin x) e^{\cos x} dx$$

let $z = \cos x$ so that $dz = -(\sin x) dx$

$$-\int e^z dz = -e^z + c = -e^{\cos x} + c$$

8. If $f(x, y) = (x - y) \sin(3x + 2y)$, determine f_x , f_y , f_{xx} , f_{yy} , f_{xy} , f_{yx} .

Now evaluate these expressions at $(0, \pi/3)$.

$$f_y = 2(x - y) \cos(3x + 2y) - \sin(3x + 2y)$$

$$f_x = \sin(3x + 2y) + 3(x - y) \cos(3x + 2y)$$

$$f_{xx} = 6 \cos(3x + 2y) - 9(x - y) \sin(3x + 2y)$$

$$f_{yy} = -4 \cos(3x + 2y) - 4(x - y) \sin(3x + 2y)$$

$$f_{xy} = f_{yx} = -\cos(3x + 2y) - 6(x - y) \sin(3x + 2y)$$

Now evaluate these expressions at $x = 0; y = \pi/3$.

We use $\sin(2\pi/3) = \sqrt{3}/2$; $\cos(2\pi/3) = -1/2$

$$\begin{aligned} f_x\left(0, \frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} + \frac{\pi}{2}; \quad f_y\left(0, \frac{\pi}{3}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \\ f_{xx}\left(0, \frac{\pi}{3}\right) &= -3 + 3\pi \frac{\sqrt{3}}{2}; \quad f_{yy}\left(0, \frac{\pi}{3}\right) = 2 + \frac{2\pi}{\sqrt{3}} \\ f_{xy}\left(0, \frac{\pi}{3}\right) &= \frac{1}{2} + \pi\sqrt{3} \end{aligned}$$

9. Show that $z = \ln\left(z = \ln\left((x - a)^2 + (y - b)^2\right)\right)$ satisfies

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

except at (a, b) . Write $z = \ln u$; $u = (x - a)^2 + (y - b)^2$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{2(x - a)}{u}; \quad \frac{\partial z}{\partial y} = \frac{2(y - b)}{u} \\ \frac{\partial^2 z}{\partial x^2} &= \frac{2u - 4(x - a)^2}{u^2}; \quad \frac{\partial^2 z}{\partial y^2} = \frac{2u - 4(y - b)^2}{u^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} &= \frac{2u - 4(x - a)^2 + 2u - 4(y - b)^2}{u^2} \\ &= \frac{4\left((x - a)^2 + (y - b)^2\right) - 4(x - a)^2 - 4(y - b)^2}{u^2} = 0 \end{aligned}$$

10. Obtain Taylor series expansions for the following functions about the given point x_0 . If no point is given, then expand about the point 0 (in which case you can use standard Taylor series expansions)

$$\begin{aligned} f(x) &= x^2 \sin x = x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} \end{aligned}$$

$$f(x) = \cos x; \quad x_0 = \pi/3$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$f(\pi/3) = 1/2; \quad f'(\pi/3) = -\sqrt{3}/2; \quad f''(\pi/3) = -1/2; \quad f'''(\pi/3) = \sqrt{3}/2; \quad f^{(4)}(\pi/3) = 1/2$$

The Taylor series expansion about $x = \pi/3$ is thus $f(x) = \cos x =$

$$\frac{1}{2} - \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3} \right) - \frac{1}{4} \left(x - \frac{\pi}{3} \right)^2 + \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{3} \right)^3 + \frac{1}{48} \left(x - \frac{\pi}{3} \right)^4 + \dots + \frac{f^{(n)}(\pi/3)}{n!} \left(x - \frac{\pi}{3} \right)^n + \dots$$

$$f(x) = \exp x; \quad x_0 = -3: \quad \text{put } u = x + 3 \text{ and expand } e^u \text{ about } u = 0$$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \longrightarrow e^{x+3} = \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$$

$$e^x = e^{-3} \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!} = e^{-3} \left(1 + (x+3) + \frac{1}{2!} (x+3)^2 + \frac{1}{3!} (x+3)^3 + \dots \right)$$

$$f(x) = \frac{1}{1-4x} = (1-4x)^{-1}$$

$$= 1 + 4x + 16x^2 + 64x^3 + \dots + 4^n x^n = \sum_{n=0}^{\infty} 4^n x^n$$

this is a convergent series with $|x| < 1/4$.

$$f(x) = \frac{3}{2x+5} = 3(2x+5)^{-1} = 3 \times 5^{-1} \left(1 + \frac{2}{5}x \right)$$

$$\left(1 + \frac{2}{5}x \right) = 1 - \frac{2}{5}x + \frac{4}{25}x^2 - \frac{8}{125}x^3 + \dots + (-1)^n \left(\frac{2}{5} \right)^n x^n$$

$$f(x) = \frac{3}{5} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{5} \right)^n x^n : \left| \frac{2}{5}x \right| < 1 \iff |x| < 5/2$$

$$f(x) = \frac{x^2+1}{x-1} = -(x^2+1)(1-x)^{-1} = -(x^2+1) \sum_{n=0}^{\infty} x^n = -\sum_{n=0}^{\infty} x^{n+2} - \sum_{n=0}^{\infty} x^{n+1}$$

$$= -1 - x - 2 \sum_{n=2}^{\infty} x^n$$

11. If $U(x, y, z) = 2x^2 - yz + xz^2$, where $x = 2 \sin t$, $y = t^2 - t + 1$, $z = 3 \exp(-t)$,

find $\frac{dU}{dt}$ at $t = 0$.

$$\begin{aligned}\frac{dx}{dt} &= 2 \cos t; \quad \frac{dy}{dt} = 2t - 1; \quad \frac{dz}{dt} = -3e^{-t} \\ \frac{\partial U}{\partial x} &= 4x + z^2; \quad \frac{\partial U}{\partial y} = -z; \quad \frac{\partial U}{\partial z} = -y + 2xz\end{aligned}$$

$$\begin{aligned}\frac{dU}{dt} &= \frac{dx}{dt} \frac{\partial U}{\partial x} + \frac{dy}{dt} \frac{\partial U}{\partial y} + \frac{dz}{dt} \frac{\partial U}{\partial z} = (2 \cos t)(4x + z^2) + (2t - 1)(-z) + (-3e^{-t})(-y + 2xz) \\ &= (2 \cos t)(4 \sin t + 9e^{-2t}) + (2t - 1)(-3e^{-t}) + (-3e^{-t})(-t^2 + t - 1 + 12e^{-t} \sin t) \\ \left. \frac{dU}{dt} \right|_{t=0} &= (2)(9) + (-1)(-3) + (-3)(-1) = 24\end{aligned}$$

12. Given $w = f(x, y)$; $x = r \cos \theta$, $y = r \sin \theta$; show that

$$\left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 = \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2$$

$$\begin{aligned}x_r &= \cos \theta; \quad x_\theta = -r \sin \theta \\ y_r &= \sin \theta; \quad y_\theta = r \cos \theta\end{aligned}$$

Now use chain rule II

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial w}{\partial x} + \sin \theta \frac{\partial w}{\partial y} \\ \left(\frac{\partial w}{\partial r} \right)^2 &= \cos^2 \theta \left(\frac{\partial w}{\partial x} \right)^2 + \sin^2 \theta \left(\frac{\partial w}{\partial y} \right)^2 + \sin 2\theta \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (1)\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial w}{\partial x} + r \cos \theta \frac{\partial w}{\partial y} \\ \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 &= \sin^2 \theta \left(\frac{\partial w}{\partial x} \right)^2 + \cos^2 \theta \left(\frac{\partial w}{\partial y} \right)^2 - \sin 2\theta \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad (2)\end{aligned}$$

$$(1) + (2) \text{ gives } \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 =$$

$$\begin{aligned}&\left(\frac{\partial w}{\partial x} \right)^2 (\sin^2 \theta + \cos^2 \theta) + \left(\frac{\partial w}{\partial y} \right)^2 (\sin^2 \theta + \cos^2 \theta) \\ &\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2\end{aligned}$$