Mathematics Primer Exercises

1 Calculus Problem Sheet

1. Consider two functions $f(x) = 9x + 2$ and $g(x) = \frac{x}{9} - \frac{2}{9}$. Show that they are functions of one another.

2. Obtain the inverse of the function $f(x) = x^{1/3} + 2$.

3. Calculate the following limits:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} \qquad \lim_{x\to 1}\frac{x^2-x}{2x^2+5x-7} \qquad \lim_{x\to -25}\frac{\sqrt{x}+5}{x-25} \qquad \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$
$$\lim_{h\to -2}\frac{h^3+8}{h+2} \qquad \lim_{t\to 1}\frac{(1/t)-1}{t-1} \qquad \lim_{x\to\sqrt{2}}\left(x^2+3\right)(x-4)$$

4. Using the definition of the derivative, show that for

$$\begin{aligned} y &= 2x+1, \quad y' = 2 \\ f(x) &= \frac{1}{x-2}, \quad f'(x) = -\frac{1}{(x-2)^2} \\ g(x) &= |x-5|, \quad \text{no derivative exists at } x = 5 \end{aligned}$$

5. Differentiate the following functions y, to obtain $\dfrac{dy}{dx}$ where :

$$y = \left(x^2-4x+2\right)^5 \qquad y = \frac{1}{\left(4x^2+6x-7\right)^3} \qquad y^4+3y-4x^3 = 5x+1 \qquad y = \ln\sqrt[3]{(2x+5)^2}$$
$$y = \cos(4-3x) \qquad y = x^2\exp(x) \qquad y = \frac{3x^2-x+2}{4x^2+5}$$

6. Calculate the following

$$\int \sqrt{x}\left(x^2-4x+2\right)dx \qquad \int_4^1 \left(3\sqrt{x}+1\right)\left(\sqrt{x}-2\right)\,dx \qquad \int_{-1}^{-2}\frac{2s-7}{s^3}\,ds$$
$$\int_3^2 \frac{x^2-1}{x-1}\,dx \qquad \int_{-1}^{5}|2x-3|\,dx \qquad \int \frac{5x-12}{x(x-4)}dx$$

7. By using suitable substitutions (change of variable), evaluate the following

$$\int \left(3-x^4\right)^3 x^3\,dx \qquad \int \frac{x^2+x}{\left(4-3x^2-2x^3\right)^4}\,dx \qquad \int \frac{\left(\sqrt{u}+3\right)^4}{\sqrt{u}}\,du$$
$$\int \left(1+\frac{1}{u}\right)^{-3}\left(\frac{1}{u^2}\right)du \qquad \int x\exp\left(x^2\right)dx \qquad \int \sin x\exp(\cos x)\,dx$$

8. If $f(x,y) = (x-y)\sin(3x+2y)$, determine $f_x,\ f_y,\ f_{xx},\ f_{yy},\ f_{xy},\ f_{yx}$. Now evaluate these expressions at $(0, \pi/3)$.

9. Show that $z = \ln\left((x-a)^2 + (y-b)^2\right)$ satisfies

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

except at (a, b).

10. Obtain Taylor series expansions for the following functions about the given point x_0. If no point is given, then expand about the point 0 (in which case you can use standard Taylor series expansions)

$$f(x) = x^2 \sin x \qquad f(x) = \cos x;\ x_0 = \pi/3 \qquad f(x) = \exp x;\ x_0 = -3$$
$$f(x) = \frac{1}{1-4x} \qquad f(x) = \frac{3}{2x+5} \qquad f(x) = \frac{x^2+1}{x-1}$$

11. If $U(x,y,z) = 2x^2 - yz + xz^2$, where $x = 2\sin t$, $y = t^2 - t + 1$, $z = 3\exp(-t)$,

find $\dfrac{dU}{dt}$ at $t = 0$.

12. Given $w = f(x,y)$; $x = r\cos\theta$, $y = r\sin\theta$; show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$