# Martingales

**Martingales** are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game. We encounter Martingales through three distinct, but closely connected ideas:

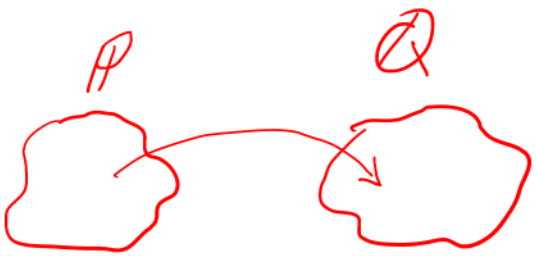
Sample space: $(\Omega, \mathcal{F}, \mathbb{P})$

1. *Martingales* as a class of stochastic process; focus on this today

2. *Exponential martingales*, which are a specific and extremely useful example of a martingale; finish with this, use extensively in mod. 3

module 3

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.



A constant mean process

0 1 2 . . . . . . T

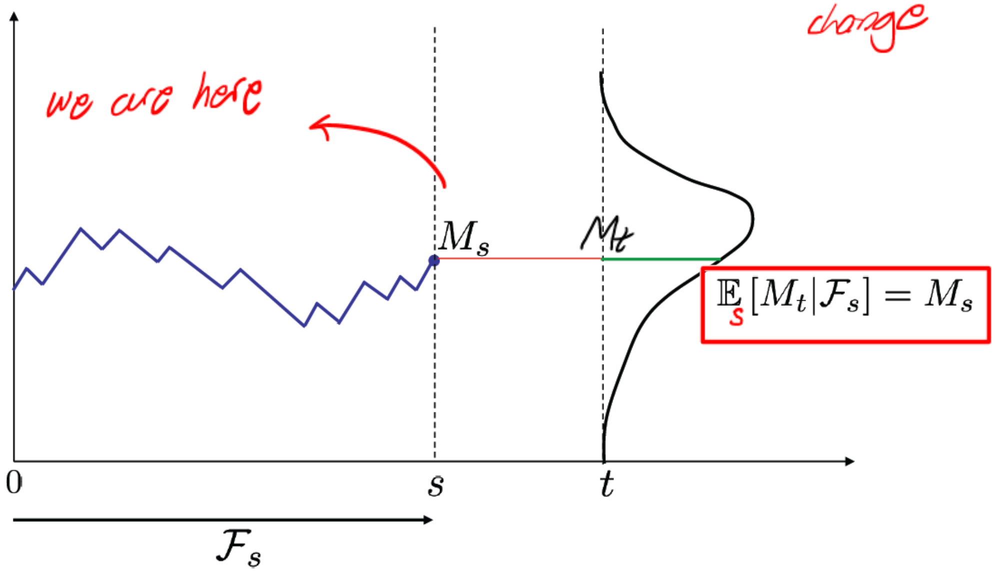
# Discrete Time Martingales

A discrete time stochastic process $\{M_t : t = 0, \ldots, T\}$ such that $M_t$ is $\mathcal{F}_t$-measurable for $\underbrace{\mathbb{T}}_{} = \{0, \ldots, T\}$ is a **martingale** if $\mathbb{E}\,|M_t| < \infty$ and

$$\mathbb{E}\left[M_{t+1}|\mathcal{F}_t\right] = M_t \tag{1}$$

On average M in the future is conditional upon everything we know up to that point

- Coin example: $\mathbb{E}[S_6 \mid R_1, \ldots, R_5] = S_5$
- Mean of a martingale $\mathbb{E}$ doesn't change




$\Rightarrow$ this is the martingale property

The first equation represents a standard integrability condition. "finiteness"

The second equation tells you that the expected value of $M$ at time $t+1$ conditional on all the information available up to time $t$ is the value of $M$ at time $t$. In short, a Martingale is a **driftless process**. process with zero mean always constant

If we take expectation on both sides of eqn. 1, then

Expectation remains the same throughout ∴ martingale

$$\mathbb{E}[M_{t+1}] = \mathbb{E}[M_t] = \mathbb{E}[M_{t-1}] = \mathbb{E}[M_0]$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

In addition, Doob and Meyer have developed a powerful theory centred around martingales.

## Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

Continuous time stochastic process: $\left\{M_t : t \in \mathbb{R}^+\right\}$

such that $M_t$ is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\,|M_t| < \infty$$

and

$$\mathbb{E}\,[M_t|\mathcal{F}_s] = M_s, \quad 0 \le s \le t.$$

**Lévy's Martingale Characterisation:** Let $X_t,\ t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. $X_t$ is a Brownian motion iff the following conditions are satisfied:

$$\mathbb{E}[X_t - X_s + X_s \mid \mathcal{F}_s]$$

starting point

1. $X_0 = 0$ a.s.;

$$\underbrace{\mathbb{E}[X_t - X_s \mid \mathcal{F}_s]}_{=0} + \underbrace{\mathbb{E}[X_s \mid \mathcal{F}_s]}_{=X_s} = X_s$$

continuous

2. the sample paths $t \mapsto X_t$ are continuous a.s.;

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$$

3. $X_t$ is a martingale with respect to the filtration $\mathcal{F}_t$;

$$\mathbb{E}[|X_t|^2] = t$$

4. $|X_t|^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t$. Quadratic Variation

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process $X_t$ satisfying:

1. $X_0 = 0$ a.s.;

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$, $X_{t_2} - X_{t_1}$ are independent;

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t - s|)$.

variance

mean

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

$\mathbb{E}[X_t] = 0$

$\mathbb{V}[X_t] = \mathbb{E}[X_t^2] = dt$

# Itô Integrals and Martingales

Next, we explore the link between Itô integration and martingales.

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

Do Itô I on $Y(t)$ and $\int_0^T$

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

$$\mathbb{E}[X^2(T)] = T + \mathbb{E}\left[\int_0^T 2X(t)dX(t)\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[X^2(T)] = T$$

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)dX(t)\right] = 0$$

Itô integral

Anything we integrate w.r.t a B.M. process

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

**The Itô integral is a martingale** All Itô integrals are martingales

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T g(t, X_t)dX_t$$

is a martingale.

So, Itô integrals are martingales. ∴ $\mathbb{E}$ value of Itô integral $=0$

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

**Martingale Representation Theorem:** If $M_t$ is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$M_T = \underbrace{M_0 + \int_0^T g(t, X_t) dX_t}_{\text{Itô integral representation}}$$

**Example** Using only Itô and the fact that Itô integrals are martingales, we will show that

$$\mathbb{E}\left[X^2(T)\right] = T.$$

Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma, Itô I

$$\begin{aligned} X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\ &= \int_0^T dt + 2\int_0^T X_t dX_t \end{aligned}$$

since $X_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\underbrace{\cancel{\int_0^T X_t dX_t}}_{=0}\right]$$

Now,

$$\int_0^T X_t dX_t$$

use the fact that an Itô integral is a martingale

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$

$\mathbb{E}$ value $= 0$
allows us to simlify

Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\boxed{\mathbb{E}\left[X^2(T)\right] = T}$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

move in

$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right] dt$$

This is due to an analysis result known as $\underbrace{\textbf{Fubini's Theorem}}_{\text{Take as a definition}}$.

# Itô Integral

Recall the usual Riemann definition of a definite integral

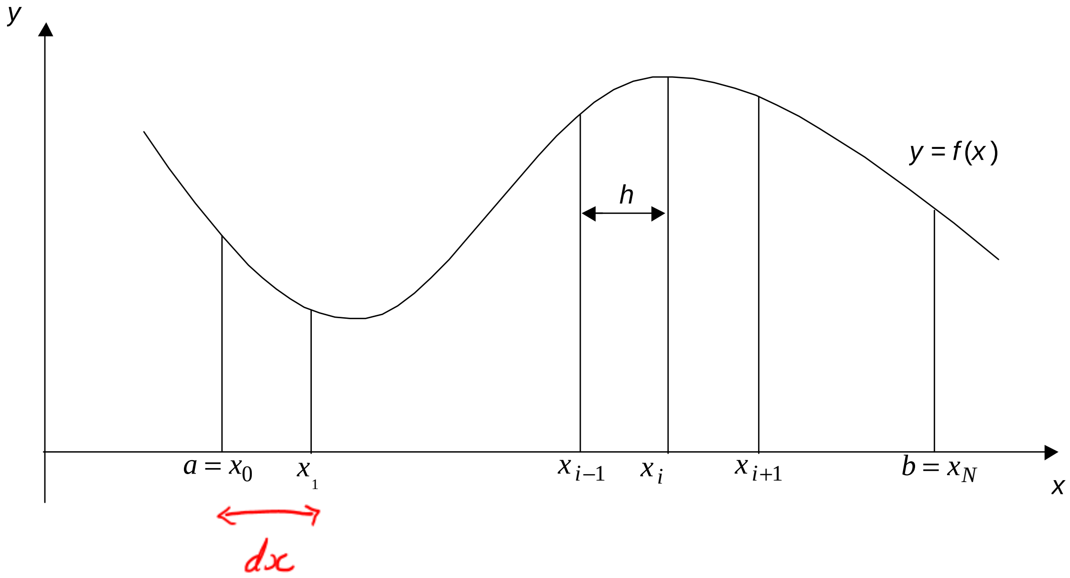
$$\int_a^b f(x)\,dx$$

$h = dx$

as $N \to \infty$
$dx \to 0$

$dx = \dfrac{b-a}{N}$

$dx = x_{i+1} - x_i$




which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against $x$.

Assuming $f$ is a "well behaved" function on $[a, b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a, b]$ into $N$ intervals with end points $x_0 = a < x_1 < x_2 < \ \ldots\ldots \ < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are $N$ intervals and $N + 1$ points $x_i$.

Discretising $x$ gives

$$x_i = a + idx$$

Now consider the definite integral

$$\int_0^T f(t)\, dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;

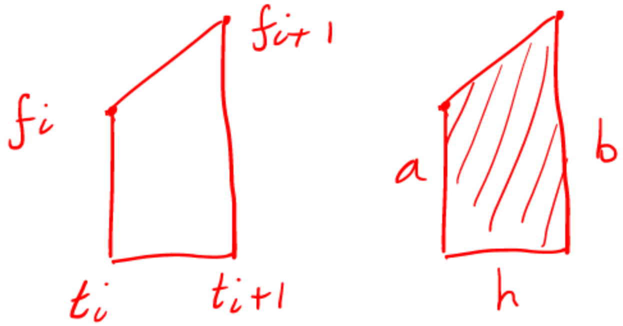
$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)\underbrace{(t_{i+1} - t_i)}_{dt}$$

2. right hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})\underbrace{(t_{i+1} - t_i)}_{dt}$$

3. trapezium rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}\left(f(t_i) + f(t_{i+1})\right)(t_{i+1} - t_i)$$




$$A = \left(\frac{a+b}{2}\right)h$$

Find $\frac{dF}{dX}$ when $F = X^3$ Use Itô I

$$dF = \frac{\partial F}{\partial X} dX + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dt$$

$$\frac{\partial F}{\partial X} = 3X^2 \qquad \frac{\partial^2 F}{\partial X^2} = 6X$$

$$dF = 3X^2 dX + 3X dt$$

$$dF = \underbrace{3X dt}_{} + 3X^2 dX$$

drift $= 3X \neq 0$ $\therefore$ not a martingale

4. midpoint rule

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

$$\int_0^T f(t, X)\,dX = \int_0^T f(t, X(t))\,dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)\,\underbrace{(X_{i+1} - X_i)}_{dx},$$

similar to left hand rule

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1}) \underbrace{(X_{i+1} - X_i)}_{dx},$$ right hand rule

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right) \underbrace{(X_{i+1} - X_i)}_{dx},$$ mid point rule

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

**Very Important:** In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$I = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$ Ito Integral = LH rule

is special. This definition results in the **Itô Integral**.

we know this

It is special because it is **non-anticipatory**; given that we are at time $t_i$ we know $X_i = X(t_i)$ and therefore we know $\underbrace{f(t_i, X_i)}_{\text{and this}}$. The only uncertainty is in the $\underbrace{X_{i+1} - X_i}_{dx \to 0}$ term.

uncertain of this but it $\to 0$

Compare this to a definition such as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

which is **anticipatory**; given that at time $t_i$ we know $X_i$ but are uncertain about the future value of $X_{i+1}$. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ – there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to <u>anticipate</u> the future value of $X_{i+1}$ so that we may evaluate $f(t_{i+1}, X_{i+1})$.

The main thing to note about Itô integrals is that $I$ is a random variable (unlike the deterministic case). Additionally, since $I$ is essentially the limit of a sum of normal random variables, then by the CLT $I$ is also normally distributed, and can be characterized by its mean and variance.

**Example**: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\,dt.$$

Show that the result also can be found by writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} X_i{}^2 \left(X_{i+1} - X_i\right)$$ $\longleftarrow$ definition of Itô integral

Hint: use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$. $\longrightarrow$ Binomial expansion using Pascal's triangle

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\,dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3\underset{\overset{\uparrow}{b^2}}{X_i^2}\underbrace{\left(X_{i+1} - X_i\right)}_{a-b}$$

Now note the hint:

$$3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$$

hence

$$3b^2(a-b)$$

$$\begin{aligned} &\equiv 3X_i^2(X_{i+1}-X_i) \\ &= X_{i+1}^3 - X_i^3 - 3X_i(X_{i+1}-X_i)^2 - (X_{i+1}-X_i)^3 , \end{aligned}$$

so that

$$(X_1^2 + X_2^2 + \ldots + X_N^3)$$

$$\sum_{i=0}^{N-1} 3X_i^2(X_{i+1}-X_i) =$$

$$(X_0^3 + \ldots + X_{N-1}^3)$$

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 - \sum_{i=0}^{N-1} 3X_i\underbrace{(X_{i+1}-X_i)^2}_{dt}$$

$$-\sum_{i=0}^{N-1} (X_{i+1} \cancel{-} X_i)^3 \simeq 2+1$$

$$= 0$$

$$= X_N^3 - X_0^3$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= X(T)^3 - X(0)^3 \,.$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = (X_{i+1} - X_i)^2 \cdot (X_{i+1} - X_i)$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0$.

Hence putting together gives

$$\boxed{X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt}$$

which is consistent with Itô's lemma.

The other important property that the Itô integral has is that it is a martingale. We know that

$$X_{i+1} - X_i$$

is a martingale; i.e. in the context

martingale ∴ $\mathbb{E}$ value is always constant, in this case 0

$$\mathbb{E}\left[X_{i+1} - X_i\right] = 0.$$

Since

$$\mathbb{E}\left[\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\left(X_{i+1} - X_i\right)\right] = \sum_{i=0}^{N-1} f\left(t_i, X_i\right)\mathbb{E}\left[X_{i+1} - X_i\right] = 0$$

Thus

$$\mathbb{E}\left[\int_0^T f\left(t, X\left(t\right)\right) dX\left(t\right)\right] = 0.$$

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

**Exercise** We know from Itô's lemma that

$$4\int_0^T X^3(t)\,dX(t) = X^4(T) - X^4(0) - 6\int_0^T X^2(t)\,dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4\int_0^T X^3 dX = \lim_{N\to\infty} 4\sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)$$

**Hint**: use $4b^3(a-b) = a^4 - b^4 - 4b(a-b)^3 - 6b^2(a-b)^2 - (a-b)^4$.

Recap on the stochastic integration formula derived in MIL3

# Proving that a Continuous Time Stochastic Process is a Martingale

$dY_t = f\,dt + g\,dX_t$

Consider a stochastic process $Y(t)$ solving the following SDE:

SDE Form : $$dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0$$ init conditional

How can we tell whether $Y(t)$ is a martingale?

$\boxed{dY = g\,dX}$ Stochastic process is a martingale if driftless

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}[Y_t | \mathcal{F}_s] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE between $s$ and $t$ to get an exact form for $Y(t)$:

Integral Form : $$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

Taking the expectation conditional on the filtration at time $s$, we get

$$\begin{aligned}\mathbb{E}[Y_t|\mathcal{F}_s] &= \mathbb{E}\left[Y(s) + \int_s^t f(Y_u,u)du + \underbrace{\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s}_{\mathbb{E}[\text{Itô Integral}]=0}\right] \\ &= Y(s) + \mathbb{E}\left[\int_s^t f(Y_u,u)du|\mathcal{F}_s\right]\end{aligned}$$

where the last line follows from the fact that a Itô integral is a martingale, $\therefore$

$$\mathbb{E}\left[\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] = \int_s^s g(Y_u,u)dX(u) = 0.$$

So, $Y(t)$ is a martingale iff

$$\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0$$

$\therefore$ martingale if deterministic part (drift) is 0

This condition is satisfied only if $f(Y_t,t) = 0$ for all $t$. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$\boxed{dY(t) = g(Y_t,t)dX(t), \qquad Y(0) = Y_0}$$

Let $F = XY$ where $X, Y$ are stochastic processes

Let's do "Itô":

$$F(X+dX, Y+dY) = F(X,Y) + \frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{1}{2}\frac{\partial^2 F}{\partial X^2} dX^2$$
$$+ \frac{1}{2}\frac{\partial^2 F}{\partial Y^2} dY^2 + \frac{\partial^2 F}{\partial X \partial Y} dX\,dY$$

Look at $F = XY$: $\frac{dF}{dX} = Y$; $\frac{dF}{dY} = X$; $\frac{\partial^2 F}{\partial X^2} = 0 = \frac{\partial^2 F}{\partial Y^2}$; $\frac{\partial^2 F}{\partial X \partial Y} = 1 = \frac{\partial^2 F}{\partial Y \partial X}$

$$dF = Y\,dX + X\,dY + \frac{1}{2}(0) + \frac{1}{2}(0) + 1\,(dX\,dY)$$

$$\boxed{d(XY) = Y\,dX + X\,dY + \underbrace{dX\,dY}_{\text{extra term not seen in regular calculus}}}$$

Itô Product Rule

# Exponential Martingales

Let's start with a motivating example.

Consider the stochastic process $Y(t)$ satisfying the SDE

$$t \rightarrow dY(t) = f(t)dt + g(t)dX(t), \qquad \underbrace{Y(0) = Y_0}_{\rightarrow I.C.} \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$.

Q: How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

$$dY^2 = f^2 \underbrace{\cancel{dt^2}}_{=0} + 2fg \underbrace{\cancel{dt\,dX}}_{=0} + g^2 \underbrace{\cancel{dX^2}}_{=dt} \Rightarrow dY^2 = g^2 dt$$

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

differential in Z:

$$dZ(t) = \frac{dZ}{dY}dY(t) + \frac{1}{2}\frac{d^2Z}{dY^2}\overbrace{dY^2(t)}^{g^2 dt}$$

$\frac{dZ}{dY} = \frac{d^2 t}{dY^2} = e^Y$

$$= \frac{dZ}{dY}\left(f(t)dt + g(t)dX(t)\right) + \frac{1}{2}\frac{d^2Z}{dY^2}g^2(t)dt$$

$$= e^{Y(t)}\underbrace{\left(f(t) + \frac{1}{2}g^2(t)\right)}_{\text{drift}} dt + e^{Y(t)}g(t)dX(t)$$

$$= Z(t)\left[\left(f(t) + \frac{1}{2}g^2(t)\right) dt + g(t)dX(t)\right]$$

$=0$

$Z(t)$ is a martingale if and only if it is a driftless process.

$dZ = g_t dX_t$

if Z is to be a martingale :: only random

Therefore for $Z(t)$ to be a martingale we must have

drift must $=0$ :

$$f(t) + \frac{1}{2}g^2(t) = 0$$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t)$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

for $e^Y$ to be a martingale

implying that

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of $Z(t)$:

$$\boxed{dZ(t) = Z(t)g(t)dX(t).}$$

$Z_t = e^{Y_t}$

$Z_T = e^{y_t}$

$Z_0 = e^{Y_0}$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

to give

$$Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

↳ exp Martingales: used to change probability measure from $\mathbb{P}$ to $\mathbb{Q}$ (real to risk-neutral)

# Key Condition (Novikov Condition)

A trading strategy $(\phi, \psi) = \{(\phi_t, \psi_t) \,; t \in [0, ..., T]\}$ is a previsible process in that $\phi_t \in \mathcal{F}_{t-}$.

A stochastic process $Y_t$ satisfies the *Novikov condition* if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_s^2 ds\right)\right] < \infty$$

where is a $\gamma_t$ previsible process.

From Riaz extra Binomial notes:

Recap: $\Delta$ hedged approach as done by Paul

Introduce: Binomial Model using replication

$\phi$: no. of shares

$\psi$: no. of bonds

# Key Fact

Given a process $\gamma_t$ satisfying the Novikov condition, then the process $M_t^\gamma$ defined as we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$M_t^\gamma = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.

In our earlier example $\gamma_t = -g(t)$; $M_t^\gamma = Z(t)$.

# Key Fact (Girsanov's Theorem)

Given a process $\theta_t$ satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X_t^{\mathbb{Q}}$ defined as

$$\boxed{X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} + \int_0^t \gamma_s dX_s}$$

as is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

$X^{\mathbb{P}} \longrightarrow X^{\mathbb{Q}}$