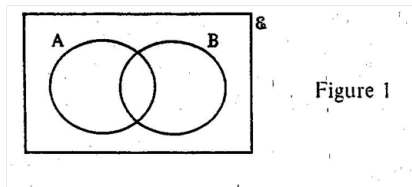


Probability and Statistics

Solutions

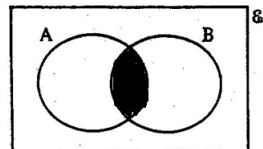
Probability

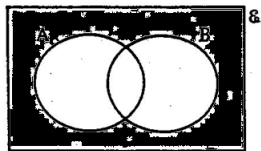
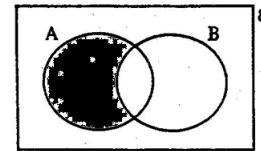
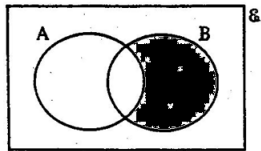
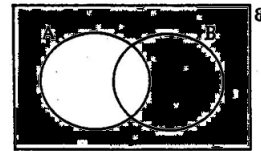
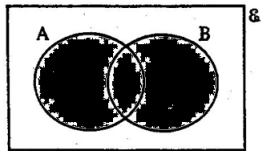
1. Draw six diagrams similar to figure 1 and shade the following sets



- (a) $A \cap B$
- (b) $A \cup B$
- (c) A'
- (d) $A' \cap B$
- (e) $B' \cap A$
- (f) $(B \cup A)'$

Solution 1





2. There are 176 students at a college following a general course in computing. Students on this course can choose to take up to three extra options.
 - 112 take systems support
 - 78 take developing software
 - 81 take networking

41 take developing software and systems support
 34 take networking and developing software
 43 take systems support and networking
 8 take all three extra options

(a) Draw a Venn Diagram to represent this information.

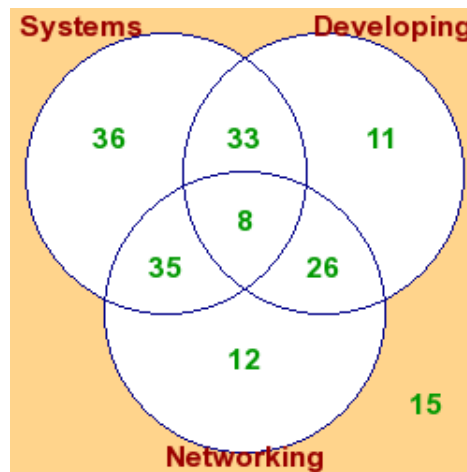
A student from the course is chosen at random. Find the probability that the student takes

- (b) none of the three extra options
 (c) networking only.

Students who want to become technicians take systems support or networking. Given that a randomly chosen student wants to become a technician,

(d) find the probability that this student takes all three extra options

Solution 2

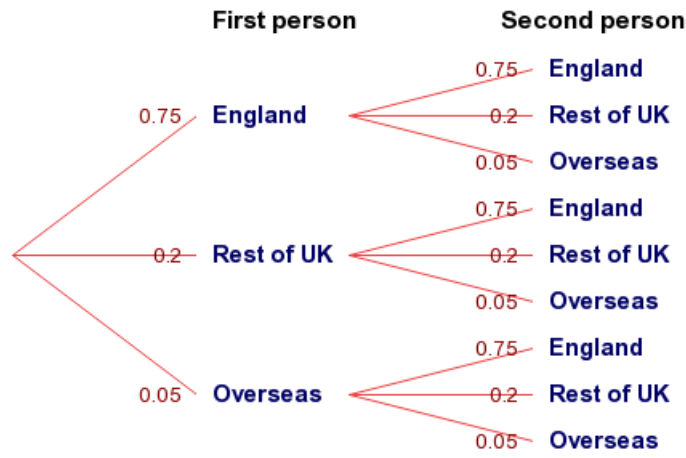


(b) $P = \frac{15}{176}$

(c) $P = \frac{12}{176}$

(d) $P(\text{All 3} \mid \text{Technician}) = \frac{\frac{8}{176}}{\frac{150}{176}} = \frac{8}{150}$

3. In a large town 75% of the population were born in England, 20% in the rest of the UK and 5% abroad. Two people are selected at random.



You may use the above tree diagram in answering this question. Find the probability that

- both these people were born in the rest of the UK.
- at least one of these people was born in England.
- neither of these people was born overseas.
- Find the probability that both these people were born in the rest of the UK given that neither was born overseas.
- 6 people are selected at random. Find the probability that at least one of them was not born in England.
- An interviewer selects n people at random. The interviewer wishes to ensure that the probability that at least one of them was not born in England is more than 80%. Find the least possible value of n .

Solution 3

- $P(\text{both in rest of UK}) = 0.2^2 = 0.04$
 - $P(\text{at least 1 England}) = 0.75 + (0.2 \times 0.75) + (0.05 \times 0.75)$
 - $P(\text{neither overseas}) = (1 - 0.05)^2 = 0.9025$
 -
- A: both in rest of UK
B: neither overseas

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.04}{0.9025} = 0.0443$$

- $P = 1 - 0.75^6 = 0.822$

(f)

$$\begin{aligned}1 - 0.75^n &> 0.8 \\0.75^n &< 0.2 \\n &> \frac{\log 0.2}{\log 0.75} \\&= 5.5945 \\\therefore n &= 6\end{aligned}$$

4. The events A and B are independent such that $P(A) = 0.14$ and $P(B) = 0.23$. Find

(a) $P(A \cap B)$

(b) $P(A \cup B)$

(c) $P(A|B')$

Solution 4

(a) $P(A \cap B) = P(A) \times P(B) = 0.14 \times 0.23 = 0.0322$

(b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.14 + 0.23 - 0.0322 = 0.3378$

(c) $P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{0.14 \times (1 - 0.23)}{1 - 0.23} = 0.14$

5. The discrete random variable X can take only the values 8, 9 or 10. For these values the cumulative distribution function is defined by

$$F(x) = \frac{(x+k)^2}{144} \quad \text{for } x = 8, 9, 10$$

where k is a positive integer

(a) Find k .

(b) Find the probability distribution of X .

Solution 5

(a)

$$\begin{aligned}F(10) &= 1 \\\frac{(10+k)^2}{144} &= 1 \\(10+k)^2 &= 144 \\\therefore k &= 2\end{aligned}$$

(b)

$$P(X = 8) = F(8) = \frac{100}{144}$$

$$P(X = 9) = F(9) - F(8) = \frac{121}{144} - \frac{100}{144} = \frac{21}{144}$$

$$P(X = 10) = F(10) - F(9) = \frac{144}{144} - \frac{121}{144} = \frac{23}{144}$$

x	8	9	10
$P(X = x)$	$\frac{100}{144}$	$\frac{21}{144}$	$\frac{23}{144}$

6. The discrete random variable X has the probability function $f(x)$ defined by

$$f(x) = kx^2 \quad x = 2, 3, 4, 5, 6$$

- (a) Construct a table showing the probability distribution of the random variable X .
- (b) Find the value of k .
- (c) Find $E(X)$ and $Var(X)$.
- (d) Find the mean and variance of the random variable Y where $Y = 3X - 7$

Solution 6

(a)

x	2	3	4	5	6
$f(x)$	$4k$	$9k$	$16k$	$25k$	$36k$

(b)

$$\begin{aligned} \sum f(x) &= 1 \\ (4 + 9 + 16 + 25 + 36)k &= 1 \\ k &= \frac{1}{90} \end{aligned}$$

(c)

$$\begin{aligned}
E(X) &= (8 + 27 + 64 + 125 + 216)k \\
&= 4.89 \\
E(X^2) &= (16 + 81 + 256 + 625 + 1296)k \\
&= 25.27 \\
Var(X) &= E(X^2) - E^2(X) = 1.37
\end{aligned}$$

(d)

$$\begin{aligned}
E(3X - 7) &= 3E(X) - 7 = 7.67 \\
Var(3X - 7) &= 9Var(X) = 12.29
\end{aligned}$$

7. Bob plays 12 squash games. In each game he either wins or loses.

- (a) State, in this context, two conditions needed for a binomial distribution to arise.
- (b) Assuming these conditions are satisfied, define a variable in this context which has a binomial distribution.
- (c) The random variable X has the distribution $B(24, p)$ where $0 < p < 1$. Given that $P(X = 7) = P(X = 6)$ find the value of p .

Solution 7

(a)

- Results are independent
- Probability of winning is constant

(b) Variable: Number of wins

(c)

$$\begin{aligned}
P(X = 7) &= P(X = 6) \\
{}^{24}C_7 p^7 (1-p)^{17} &= {}^{24}C_6 p^6 (1-p)^{18} \\
\frac{24!}{7! \times 17!} p &= \frac{24!}{6! \times 18!} (1-p) \\
\frac{1}{7} p &= \frac{1}{18} (1-p) \\
18p &= 7(1-p) \\
25p &= 7 \\
p &= \frac{7}{25}
\end{aligned}$$

8. The continuous random variable X has probability density function $f(x)$ given by

$$f(x) = \begin{cases} kx^2(7-x) & 0 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Show that $k = \frac{12}{2401}$
 (b) Find $E(X)$
 (c) Find $P(X < 4)$

Solution 8

(a)

$$\begin{aligned} \int_0^7 f(x)dx &= 1 \\ \int_0^7 kx^2(7-x)dx &= 1 \\ k \left[\frac{7}{3}x^3 - \frac{1}{4}x^4 \right]_0^7 &= 1 \\ k \left[\frac{2401}{3} - \frac{2401}{4} \right] &= 1 \\ \therefore k &= \frac{12}{2401} \end{aligned}$$

(b)

$$\begin{aligned} E(X) &= \int_0^7 x.kx^2(7-x)dx \\ &= k \int_0^7 (7x^3 - x^4)dx \\ &= k \left[\frac{7}{4}x^4 - \frac{1}{5}x^5 \right]_0^7 \\ &= \frac{21}{5} \end{aligned}$$

(c)

$$P(X < 4) = \frac{12}{2401} \left[\frac{7}{3}x^3 - \frac{1}{4}x^4 \right]_0^4 = 0.426$$

9. The length of a telephone call made to a company is denoted by the continuous random variable T . It is modelled by the probability density function

$$f(x) = \begin{cases} kt & 0 \leq t \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Show that the value of k is $\frac{1}{18}$
 (b) Find $P(T > 1)$
 (c) Calculate an exact value for $E(T)$ and for $Var(T)$

Solution 9

(a)

$$\begin{aligned} \int ktdx &= 1 \\ \int_0^6 ktdx &= 1 \\ \left[\frac{1}{2}kt^2 \right]_0^6 &= 1 \\ 8k &= 1 \\ k &= \frac{1}{18} \end{aligned}$$

(b)

$$P(T > 1) = \int_1^6 ktdx = \left[\frac{1}{2}kt^2 \right]_1^6 = \frac{35}{36}$$

(c)

$$\begin{aligned} E(T) &= \int_0^6 kt^2dx = \left[\frac{1}{3}kt^3 \right]_0^6 = \frac{4}{1} \\ E(T^2) &= \int_0^6 kt^3dx = \left[\frac{1}{4}kt^4 \right]_0^6 = \frac{!8}{1} \\ Var(T) &= \frac{18}{1} - \left(\frac{4}{1} \right)^2 = \frac{2}{1} \end{aligned}$$

10. The continuous random variable X has cumulative distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4}x^2(5 - x^2) & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

- (a) Find $P(X > 0.6)$
- (b) Find the probability density function $f(x)$ of X
- (c) Calculate $E(X)$ and show that, to 3 decimal places, $Var(X) = 0.057$

Solution 10

(a)

$$\begin{aligned}
 P(X > 0.6) &= 1 - P(X < 0.6) \\
 &= 1 - \frac{1}{4}(0.6)^2(5 - 0.6^2) \\
 &= 0.582
 \end{aligned}$$

(b)

$$f(x) = \frac{dF(x)}{dx} = \frac{10}{4}x - x^3$$

(c)

$$\begin{aligned}
 E(X) &= \int_0^1 x \left(\frac{10}{4}x - x^3 \right) dx \\
 &= \int_0^1 \frac{10}{4}x^2 - x^4 dx \\
 &= \left[\frac{10}{12}x^3 - \frac{4}{20}x^5 \right]_0^1 \\
 &= 0.633 \\
 E(X^2) &= \int_0^1 x^2 \left(\frac{10}{4}x - x^3 \right) dx \\
 &= \int_0^1 \frac{10}{4}x^3 - x^5 dx \\
 &= \left[\frac{10}{16}x^4 - \frac{4}{24}x^6 \right]_0^1 = 0.458 \\
 Var(X) &= E(X^2) - [E(X)]^2 = 0.458 - 0.633^2 = 0.057
 \end{aligned}$$

For the following questions use Standard Normal Distribution tables which can be found at the back of any statistics or probability text book

11. The lifetimes of bulbs used in a lamp are normally distributed. A company X sells bulbs with a mean lifetime of 856 hours and a standard deviation of 58 hours.

- (a) Find the probability of a bulb, from company X , having a lifetime of less than 833 hours
- (b) In a box of 400 bulbs, from company X , find the expected number having a lifetime of less than 833 hours.

A rival company Y sells bulbs with a mean lifetime of 882 hours and 19% of these bulbs have a lifetime of less than 830 hours.

- (c) Find the standard deviation of the lifetimes of bulbs from company Y .

Both companies sell bulbs for the same price.

- (d) State which company you would recommend

Solution 11

- (a)

$$\begin{aligned}
 P(X < 833) &= \Phi\left(\frac{833 - \mu}{\sigma}\right) \\
 &= \Phi\left(\frac{833 - 856}{58}\right) \\
 &= \Phi(-0.397) = 0.345
 \end{aligned}$$

- (b) Expected number = $400 \times 0.345 = 138$

- (c)

$$\begin{aligned}
 P(Y < 830) &= 0.19 \\
 \Phi\left(\frac{830 - 882}{\sigma}\right) &= 0.19 \\
 \frac{830 - 882}{\sigma} &= -0.8778 \\
 \therefore \sigma &= 59.2
 \end{aligned}$$

- (d) Choose company Y because higher mean but smaller standard deviation

12. In large-scale tree-felling operations, a machine cuts down trees, strips off the branches and then cuts the trunks into logs of length X metres for transporting to a sawmill. It may be assumed that values of X are normally distributed with mean 4.3 and standard deviation 0.17.

- (a) Determine $P(X < 4.5)$

- (b) Determine $P(X > 4)$
 (c) Determine $P(4 < X < 4.5)$

Solution 12

(a)

$$S_{xx} = \sum x^2 - \frac{(\sum x)^2}{n} = 211 - \frac{(45)^2}{10} = 8.5$$

$$S_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n} = 1460 - \frac{45 \times 321}{10} = 15.5$$

(b)

$$b = \frac{S_{xy}}{S_{xx}} = \frac{15.5}{8.5} = 1.824$$

$$a = \bar{y} - b\bar{x} = \frac{321}{10} - 1.824 \frac{45}{10} = 23.892$$

$$y = 23.89 + 1.82x$$

- (c) Comment: A typical car will travel 1820 miles every year
 (d)

$$x = 4.5 \quad y = 23.89 + 1.82(4.5) = 32.08$$

mileage = 32080

13. A second hand car dealer has 10 cars for sale. She decides to investigate the link between the age of the cars, x years, and the mileage, y thousand miles. The data collected from the cars are shown in the table below.

Age (years) x	2.5	3.5	5.5	4.5	5.5	5.5	5	4.5	4	4.5
Mileage (1000) y	33	24	39	31	34	32	22	45	21	40

You may assume that

$$\sum x = 45; \quad \sum y = 321; \quad \sum x^2 = 221; \quad \sum xy = 1460;$$

- (a) Find S_{xx} and S_{xy}
 (b) Find the equation of the regression line of y on x in the form $y = a + bx$.
 Give the values of a and b to 2 decimal places
 (c) Give a practical interpretation of the slope b

- (d) Using your answer to part (b), find the mileage predicted by the regression line for a 45 year old car

Solution 13

(a)

$$\begin{aligned} P(x < 4.5) &= \Phi\left(\frac{4.5 - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{4.5 - 4.3}{0.17}\right) \\ &= \Phi(1.176) = 0.881 \end{aligned}$$

(b)

$$\begin{aligned} P(X > 4) &= 1 - P(X < 4) \\ &= 1 - \Phi\left(\frac{4 - \mu}{\sigma}\right) \\ &= 1 - \Phi\left(\frac{4 - 4.3}{0.17}\right) \\ &= 1 - \Phi(-1.765) \\ &= 1 - 0.0392 \\ &= 0.961 \end{aligned}$$

(c)

$$\begin{aligned} P(4 < X < 4.5) &= 0.881 - (1 - 0.961) \\ &= 0.842 \end{aligned}$$

14. A sample of bivariate data was taken and the results were summarised as follows:

$$\begin{aligned} n = 10; \quad \sum x = 771; \quad \sum x^2 = 60379; \quad \sum y = 723; \\ \sum y^2 = 53125; \quad \sum xy = 55905; \end{aligned}$$

Show that the value of the the product moment correlation coefficient is 0181, correct to 3 significant figures

Solution 14

$$\begin{aligned}
S_{xx} &= \sum x^2 - \frac{(\sum x)^2}{n} = 60379 - \frac{(771)^2}{10} = 934.9 \\
S_{yy} &= \sum y^2 - \frac{(\sum y)^2}{n} = 53125 - \frac{(723)^2}{10} = 852.1 \\
S_{xy} &= \sum xy - \frac{(\sum x)(\sum y)}{n} = 55905 - \frac{771 \times 723}{10} = 161.7
\end{aligned}$$

$$r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{161.7}{\sqrt{934.9 \times 852.1}} = 0.181$$