3 Differential Equations Problem Sheet

1. For arbitrary constants $c_1,\ c_2,\ c_3,\ c_4$ find the differential equations satisfied by y when:

   a. $y = c_1 x + \dfrac{2}{c_1}$ Ans: $x\left(y'\right)^2 - yy' + 2 = 0$

   b. $y = \left(c_1 + c_2 x\right) e^{-\lambda x}$ Ans: $y'' + 2\lambda y' + \lambda^2 y = 0$

   c. $y = c_1 \sin \rho x + c_2 \cos \rho x + c_3 \sinh \rho x + c_4 \cosh \rho x$ Ans: $y^{(4)} = \rho^4 y$

2. Solve the following differential equations/I.V.P.'s

   a. $\left(\dfrac{dy}{dx}\right)^3 = y^2 \quad y = 1,\ x = 0$ Ans: $y = \left(\dfrac{x+3}{3}\right)^3$

   b. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2} \quad y = 1,\ x = 0$ Ans: $y = \dfrac{1+x}{1-x}$

   c. $\sqrt{1+x^2}\dfrac{dy}{dx} = xe^{-y} \quad y = 0,\ x = 0$ Ans: $y = \frac{1}{2}\log\left(1+x^2\right)$

   d. $(1-y)^2 \dfrac{dy}{dx} + \left(1+x^2\right) y = 0$ Ans: $x + \dfrac{x^3}{3} = -\log y + 2y - \frac{1}{2}y^2 + c$

   e. $x\dfrac{dy}{dx} + 3y = 8x^5$ Ans: $y = x^5 + \dfrac{c}{x^3}$

   f. $\dfrac{dy}{dx} - 2y \tan x = x^2 \sec^2 x$ when $x = 0$ and $y = 0$ Ans: $y = \frac{x^3}{3} \sec^2 x$

   g. $\sin x\dfrac{dy}{dx} + 2y \cos x = \cos x$ Ans: $y = \dfrac{1}{2} + k\,\mathrm{cosec}^2 x$

   h. $(x+1)\, y' - 2y = 3\,(x+1)^3$ Ans: $y = (3x + c)\,(x+1)^2$

3. Solve the 2nd order equations

a. $\dfrac{d^2y}{dx^2} = 2y^3 + 8y$ where $y = 2$, $y' = -8$ when $x = \frac{\pi}{4}$ Ans: $y = 2\tan\left(\frac{3\pi}{4} - 2x\right)$

b. $\dfrac{d^2y}{dx^2} + 2x\left(\dfrac{dy}{dx}\right)^2 = 0$ where $y = 0$, $y' = 1$ when $x = 0$ Ans: $y = \arctan x$.

4. For each of the following constant coefficient differential equations,

$$y'' + by' + cy = g(x)$$

find the complimentary function and state which function you would use to try and find a Particular Solution by the method of undetermined coefficients.

a. $b = 3$, $c = 2$, $g(x) = e^{5x}$ Ans: C.F: $y = Ae^{-2x} + Be^{-x}$ PS $y = Ce^{5x}$.

b. $b = 1$, $c = -6$, $g(x) = 2e^{2x} + \sin 3x$ Ans: C.F: $y = Ae^{-3x} + Be^{2x}$ PS: $y_1 = Cxe^{2x}$, because 2 is a root of the A.E $y_2 = (D\sin 3x + E\cos 3x)$.

c. $b = 7$, $c = 0$, $g(x) = 4x^2 + x + 2$ Ans: C.F: $y = A + Be^{-7x}$ PS $y = \left(p_2x^2 + p_1x + p_0\right)x$ because 0 is a root of the A.E.

d. $b = 1$, $c = 1$, $g(x) = 2e^{-x}$ Ans: C.F: $y = e^{-x/2}\left(A\sin\frac{\sqrt{3}}{2}x + B\cos\frac{\sqrt{3}}{2}x\right)$ PS $y = Ce^{-x}$.

e. $b = 4$, $c = 4$, $g(x) = 3e^{-2x} + 2e^{3x} + \sin x$ Ans: C.F: $y = e^{-2x}(A + Bx)$ PS $y_1 = Cx^2e^{-2x}$ because -2 is a two fold root of the A.E, $y_2 = De^{3x}$, $y_3 = (E\sin x + F\cos x)$.

5. By converting the Euler equation

$$x^2y''(x) - 2xy'(x) + 2y(x) = 4x^3$$

to a constant coefficient problem show that the solution is given by

$$y(x) = Ax + Bx^2 + 2x^3.$$