# Ten Different Ways to Derive Black–Scholes

**In this lecture. . .**

- All the ways we know of to derive either the Black–Scholes equation or the Black–Scholes formulæ

## Introduction

The ten different ways of deriving the Black–Scholes equation or formulæ that follow use different types of mathematics, with different amounts of complexity and mathematical baggage.

Some derivations are useful in that they can be **generalized**, and some are very specific to this one problem. Naturally we will spend more time on those derivations that are most useful or give the most insight.

Some derivations are useful for their **insights**.

In most cases we work within a framework in which the stock path is **continuous**, the returns are **normally distributed**, there aren't any dividends, or transaction costs, etc. **But not always!**

Sometimes we will be deriving the **Black–Scholes Equation** and sometimes the **Black–Scholes formulæ**.

To get the closed-form formulæ (the Black–Scholes *formulæ*) we need to assume that volatility is constant, or perhaps time dependent, but for the derivations of the equations relating the greeks (the Black–Scholes *equation*) the assumptions can be weaker, if we don't mind not finding a closed-form solution.

## 1. Hedging and the partial differential equation

The original derivation of the Black–Scholes partial differential equation was via stochastic calculus, Itô's lemma and a simple hedging argument.

Assume that the underlying follows a lognormal random walk

$$dS = \mu S\ dt + \sigma S\ dX.$$

Use $\Pi$ to denote the value of a portfolio of one long option position and a short position in some quantity $\Delta$ of the underlying:

$$\Pi = V(S,t) - \Delta S. \tag{1}$$

The first term on the right is the option and the second term is the short asset position.

Ask how the value of the portfolio changes from time $t$ to $t + dt$. The change in the portfolio value is due partly to the change in the option value and partly to the change in the underlying:

$$d\Pi = dV - \Delta \, dS.$$

¿From Itô's lemma we have

$$d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta \, dS.$$

The right-hand side of this contains two types of terms, the deterministic and the random. The deterministic terms are those with the $dt$, and the random terms are those with the $dS$. Pretending for the moment that we know $V$ and its derivatives then we know everything about the right-hand side *except for the value of* $dS$, because this is random.

These random terms can be eliminated by choosing

$$\Delta = \frac{\partial V}{\partial S}.$$

After choosing the quantity $\Delta$, we hold a portfolio whose value changes by the amount

$$d\Pi = \left( \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

This change is completely *riskless*. If we have a completely risk-free change $d\Pi$ in the portfolio value $\Pi$ then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free interest-bearing account:

$$d\Pi = r\Pi \, dt.$$

This is an example of the no arbitrage principle.

Putting all of the above together to eliminate $\Pi$ and $\Delta$ in favour of partial derivatives of $V$ gives

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

the Black–Scholes equation.

Solve this quite simple linear diffusion equation with the final condition

$$V(S,T) = \max(S - K, 0)$$

and you will get the Black–Scholes call option formula.

This derivation of the Black–Scholes equation is perhaps the most useful since it is readily generalizable (if not necessarily still analytically tractable) to different underlyings, more complicated models, and exotic contracts.

Because of the relationship between the Black–Scholes equation and the Fokker–Planck equation for the expected value of a function following a random walk, the solution of the BSE can be interpreted as an expectation:

$$V = e^{-r(T-t)} E_t^{\mathbb{Q}}[\max(S_T - K, 0)],$$

Here $E_t^{\mathbb{Q}}[\cdot]$ just means the expected value at time $t$ of the argument, *under the risk-neutral random walk* $dS = rS\,dt + \sigma S\,dX$.

## 2. Parameters as variables

The next derivation is rather novel in that it involves differentiating the option value with respect to the parameters strike, $K$, and expiration, $T$, instead of the more usual differentiation with respect to the variables $S$ and $t$.

This will lead to a partial differential equation that can be solved for the Black–Scholes formulæ. But more importantly, this technique can be used to deduce the dependence of volatility on stock price and time, given the market prices of options as functions of strike and expiration, and is the basis for deterministic volatility models and calibration.

We begin with the call option result from above

$$V = e^{-r(T-t)} E_t^{\mathbb{Q}}[\max(S_T - K, 0)],$$

that the option value is the present value of the risk-neutral expected payoff. This can be written as

$$V(K,T) = e^{-r(T-t^*)} \int_0^\infty \max(S-K,0) p(S^*,t^*;S,T)\, dS$$
$$= e^{-r(T-t^*)} \int_K^\infty (S-K) p(S^*,t^*;S,T)\, dS,$$

where $p(S^*,t^*;S,T)$ is the transition probability density function for the risk-neutral random walk with $S^*$ being today's asset price and $t^*$ today's date. Note that here the arguments of $V$ are the 'variables' strike, $K$, and expiration, $T$.

If we differentiate this with respect to $K$ we get

$$\frac{\partial V}{\partial K} = -e^{-r(T-t^*)} \int_K^\infty p(S^*, t^*; S, T)\, dS.$$

After another differentiation, we arrive at this equation for the probability density function in terms of the option prices

$$p(S^*, t^*; K, T) = e^{r(T-t^*)} \frac{\partial^2 V}{\partial K^2}$$

We also know that the forward equation for the transition probability density function, the Fokker–Planck equation, is

$$\frac{\partial p}{\partial T} = \tfrac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp).$$

Here $\sigma(S,t)$ is evaluated at $t = T$. We also have

$$\frac{\partial V}{\partial T} = -rV + e^{-r(T-t^*)}\int_K^\infty (S-K)\frac{\partial p}{\partial T}\,dS.$$

This can be written as

$$\frac{\partial V}{\partial T} = -rV + e^{-r(T-t^*)}\int_K^\infty \left(\tfrac{1}{2}\frac{\partial^2(\sigma^2 S^2 p)}{\partial S^2} - \frac{\partial(rSp)}{\partial S}\right)(S-K)\,dS.$$

using the forward equation.

Integrating this by parts twice we get

$$\frac{\partial V}{\partial T} = -rV + \tfrac{1}{2}e^{-r(T-t^*)}\sigma^2 K^2 p + re^{-r(T-t^*)} \int_K^\infty Sp\, dS.$$

In this expression $\sigma(S,t)$ has $S = K$ and $t = T$. After some simple manipulations we get

$$\frac{\partial V}{\partial T} = \tfrac{1}{2}\sigma^2 K^2 \frac{\partial^2 V}{\partial K^2} - rK\frac{\partial V}{\partial K}.$$

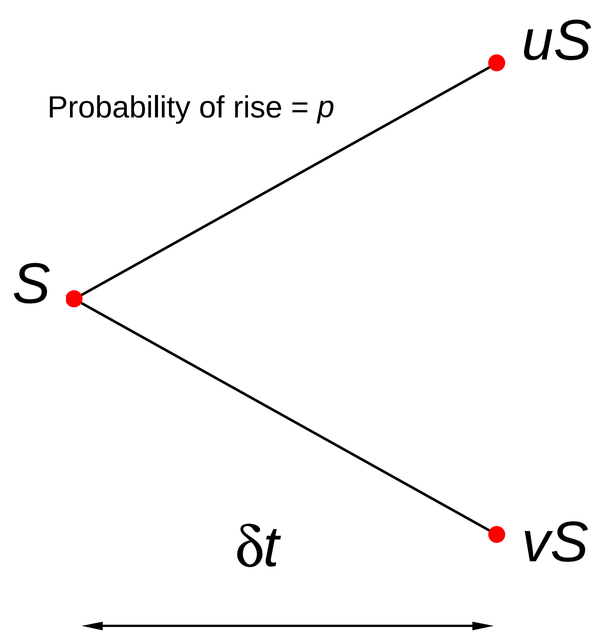
This partial differential equation can now be solved for the Black–Scholes formulæ.

This method is not used in practice for finding these formulæ, but rather, knowing the traded prices of vanillas as a function of $K$ and $T$ we can turn this equation around to find $\sigma$, since the above analysis is still valid even if volatility is stock and time dependent.

## 3. Continuous-time limit of the binomial model

Some of our ten derivations lead to the Black–Scholes partial differential equation, and some to the idea of the option value as the present value of the option payoff under a risk-neutral random walk. The following simple model does both.

In the binomial model the asset starts at $S$ and over a time step $\delta t$ either rises to a value $u \times S$ or falls to a value $v \times S$, with $0 < v < 1 < u$. The probability of a rise is $p$ and so the probability of a fall is $1 - p$.



We choose the three constants $u$, $v$ and $p$ to give the binomial walk the same drift, $\mu$, and volatility, $\sigma$, as the asset we are modelling. This choice is far from unique and here we use the choices that result in the simplest formulæ:

$$u = 1 + \sigma\sqrt{\delta t},$$

$$v = 1 - \sigma\sqrt{\delta t}$$

and

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}.$$

Having defined the behaviour of the asset we are ready to price options.

Suppose that we know the value of the option at the time $t + \delta t$. For example this time may be the expiration of the option. Now construct a portfolio at time $t$ consisting of one option and a short position in a quantity $\Delta$ of the underlying.

At time $t$ this portfolio has value

$$\Pi = V - \Delta S,$$

where the option value $V$ is for the moment unknown.

At time $t+\delta t$ the option takes one of two values, depending on whether the asset rises or falls

$$V^+ \quad \text{or} \quad V^-.$$

At the same time the portfolio of option and stock becomes either

$$V^+ - \Delta u S \quad \text{or} \quad V^- - \Delta v S.$$

Having the freedom to choose $\Delta$, we can make the value of this portfolio the same whether the asset rises or falls. This is ensured if we make

$$V^{+} - \Delta u S = V^{-} - \Delta v S.$$

This means that we should choose

$$\Delta = \frac{V^{+} - V^{-}}{(u - v)S}$$

for hedging.

The portfolio value is then

$$V^{+}-\Delta uS=V^{+}-\frac{u(V^{+}-V^{-})}{(u-v)}=V^{-}-\Delta vS=V^{-}-\frac{v(V^{+}-V^{-})}{(u-v)}.$$

Let's denote this portfolio value by

$$\Pi+\delta\Pi.$$

This just means the original portfolio value plus the change in value.

But we must also have $\delta\Pi = r\Pi\ \delta t$ to avoid arbitrage opportunities. Bringing all of these expressions together to eliminate $\Pi$, and after some rearranging, we get

$$V = \frac{1}{1 + r\ \delta t}\left(p'V^{+} + (1 - p')V^{-}\right),$$

where

$$p' = \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}.$$

This is an equation for $V$ given $V^{+}$, and $V^{-}$, the option values at the next time step, and the parameters $r$ and $\sigma$.

The right-hand side of the equation for $V$ can be interpreted, rather clearly, as the present value of the expected future option value using the probabilities $p'$ for an up move and $1 - p'$ for a down.

Again this is the idea of the option value as the present value of the expected payoff under a risk-neutral random walk. The quantity $p'$ is the risk-neutral probability, and it is this that determines the value of the option not the real probability. By comparing the expressions for $p$ and $p'$ we see that this is equivalent to replacing the real asset drift $\mu$ with the risk-free rate of return $r$.

We can examine the equation for $V$ in the limit as $\delta t \to 0$. We write

$$V = V(S,t), \quad V^+ = V(uS, t + \delta t) \quad \text{and} \quad V^- = V(vS, t + \delta t).$$

Expanding these expressions in Taylor series for small $\delta t$ we find that

$$\Delta \sim \frac{\partial V}{\partial S} \quad \text{as} \quad \delta t \to 0,$$

and the binomial pricing equation for $V$ becomes

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

This is the Black–Scholes equation.

## 4. CAPM

This derivation, originally due to Cox & Rubinstein (1985) starts from the **Capital Asset Pricing Model** in continuous time. In particular it uses the result that there is a linear relationship between the expected return on a financial instrument and the covariance of the asset with the market. The latter term can be thought of as compensation for taking risk.

But the asset and its option are perfectly correlated, so the compensation in excess of the risk-free rate for taking unit amount of risk must be the same for each.

For the stock, the expected return (dividing by $dt$) is $\mu$.

Its risk is $\sigma$.

¿From Itô we have

$$dV = \frac{\partial V}{\partial t}dt + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt + \frac{\partial V}{\partial S}dS.$$

Therefore the expected return on the option is

$$\frac{1}{V}\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}\right)$$

and the risk is

$$\frac{1}{V}\sigma S \frac{\partial V}{\partial S}.$$

Since both the underlying and the option must have the same compensation, in excess of the risk-free rate, for unit risk

$$\frac{\mu - r}{\sigma} = \frac{\frac{1}{V}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S}\right)}{\frac{1}{V}\sigma S \frac{\partial V}{\partial S}}.$$

Now rearrange this. The $\mu$ drops out and we are left with the Black–Scholes equation.

## 5. A diffusion equation

The penultimate derivation of the Black–Scholes partial differential equation is rather unusual in that it uses just pure thought about the nature of Brownian motion and a couple of trivial observations.

It also has a very neat punchline that makes the derivation helpful in other modelling situations.

Stock prices can be modelled as Brownian motion, the stock price plays the role of the position of the 'pollen particle' and time is time.

In mathematical terms Brownian motion is just an example of a diffusion equation.

So let's write down a diffusion equation for the value of an option as a function of space and time, i.e. stock price and time, that's $V(S,t)$.

What's the general linear diffusion equation?

It is

$$\frac{\partial V}{\partial t} + a\frac{\partial^2 V}{\partial S^2} + b\frac{\partial V}{\partial S} + cV = 0.$$

Note the coefficients $a$, $b$ and $c$. At the moment these could be anything.

Now for the two trivial observations.

First, cash in the bank must be a solution of this equation. Financial contracts don't come any simpler than this. So plug $V = e^{rt}$ into this diffusion equation to get

$$re^{rt} + 0 + 0 + ce^{rt} = 0.$$

So $c = -r$.

Second, surely the stock price itself must also be a solution? After all, you could think of it as being a call option with zero strike.

So plug $V = S$ into the general diffusion equation. We find

$$0 + 0 + b + cS = 0.$$

So $b = -cS = rS$.

Putting $b$ and $c$ back into the general diffusion equation we find

$$\frac{\partial V}{\partial t} + a\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

This is the risk-neutral Black–Scholes equation.

Two of the coefficients (those of $V$ and $\partial V/\partial S$) have been pinned down exactly without any modelling at all.

Ok, so it doesn't tell us what the coefficient of the second derivative term is, but even that has a nice interpretation. It means at least a couple of interesting things.

- First, if we do start to move outside the Black–Scholes world then chances are it will be the diffusion coefficient that we must change from its usual $\frac{1}{2}\sigma^2 S^2$ to accommodate new models.

- Second, if we want to fudge our option prices, to massage them into line with traded prices for example, we can only do so by fiddling with this diffusion coefficient, i.e. what we now know to be the volatility. This derivation tells us that our only valid fudge factor is the volatility.

## 6. Black–Scholes for accountants

The final derivation of the Black–Scholes equation requires very little complicated mathematics, and doesn't even need assumptions about Gaussian returns, all we need is for the variance of returns to be finite.

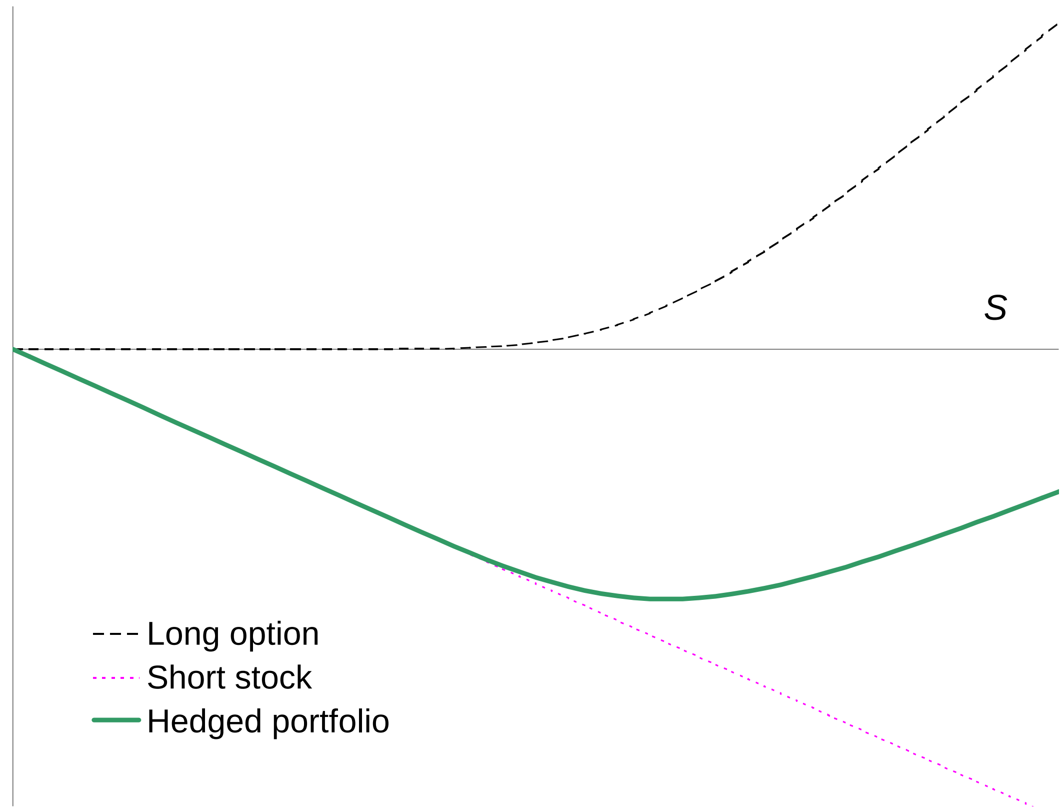
The Black–Scholes analysis requires *continuous* hedging, which is possible in theory but impossible, and even undesirable, in practice. Hence one hedges in some discrete way.

Let's assume that we hedge at equal time periods, $\delta t$. And consider the value changes associated with a delta-hedged option.

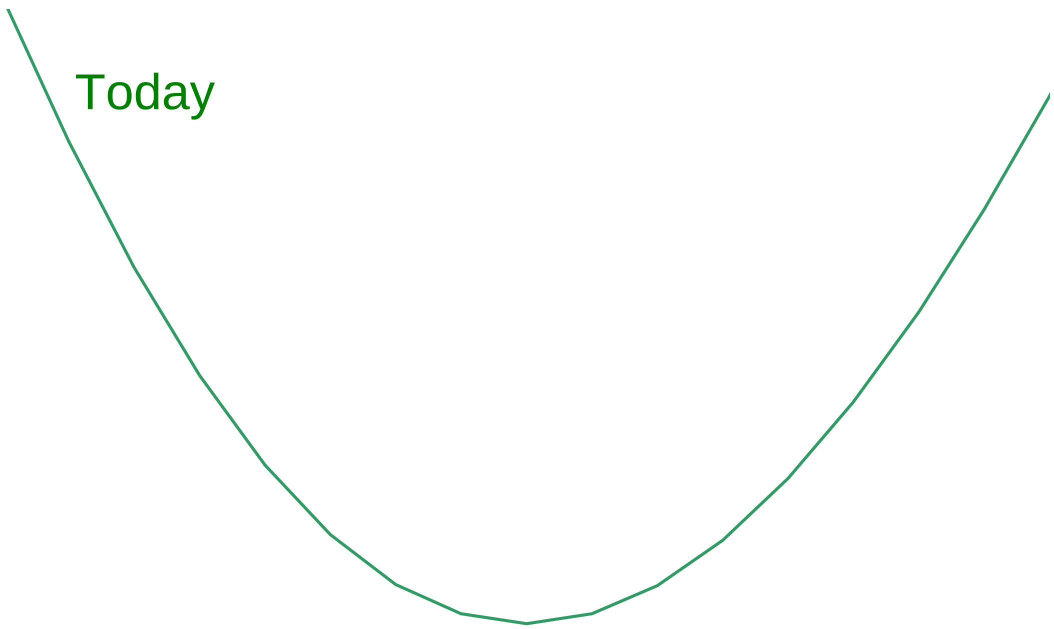
- We start with zero cash
- We buy an option
- We sell some stock short
- Any cash left (positive or negative) is put into a risk-free account.

We start by borrowing some money to buy the option. This option has a delta, and so we sell delta of the underlying stock in order to hedge. This brings in some money. The cash from these transactions is put in the bank. At this point in time our net worth is zero.

Our portfolio has a dependence on $S$ as shown here.



We are only concerned with small movements in the stock over a small time period, so zoom in on the current stock position. Locally the curve is approximately a parabola.



Now think about how our net worth will change from now to a time $\delta t$ later. There are three reasons for our total wealth to change over that period.

1. The option price curve changes

2. There is an interest payment on the money in the bank

3. The stock moves

The option curve falls by the time value, the theta multiplied by the time step:

$$\Theta \times \delta t.$$

To calculate how much interest we received we need to know how much money we put in the bank. This was

$$\Delta \times S$$

from the stock sale and

$$-V$$

from the option purchase. Therefore the interest we receive is

$$r(S\Delta - V)\,\delta t.$$

Finally, look at the money made from the stock move. Since gamma is positive, any stock price move is good for us. The larger the move the better.

The curve in the earlier figure is locally quadratic, a parabola with coefficient $\frac{1}{2}\Gamma$. The stock move over a time period $\delta t$ is proportional to three things:

- the volatility $\sigma$
- the stock price $S$
- the square root of the time step

Multiply these three together, square the result because the curve is parabolic and multiply that by $\frac{1}{2}\Gamma$ and you get the profit made from the stock move as

$$\frac{1}{2}\sigma^2 S^2 \Gamma \, \delta t.$$

Put these three value changes together (ignoring the $\delta t$ term which multiplies all of them) and set the resulting expression equal to zero, to represent no arbitrage, and you get

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma + r(S\Delta - V) = 0,$$

the Black–Scholes equation.

Now there was a bit of cheating here, since the stock price move is really random. What we should have said is that

$$\tfrac{1}{2}\sigma^2 S^2 \Gamma \, \delta t$$

is the profit made from the stock move *on average*. Crucially all we need to know is that the variance of returns is

$$\sigma^2 S^2 \delta t,$$

we don't even need the stock returns to be normally distributed.

There is a difference between the square of the stock prices moves and its average value and this gives rise to hedging error, something that is always seen in practice.

If you hedge discretely, as you must, then Black–Scholes only works on average.

But as you hedge more and more frequently, going to the limit $\delta t = 0$, then the total hedging error tends to zero, so justifying the Black–Scholes model.

# 7. The Martingale Approach

## 7.1. Overview

If we assume that asset prices change due to some deterministic factors and some random factors, then to value a security, we could adopt one of two basic approaches:

1. either we **concentrate on the deterministic part** by removing the randomness, which can be done via a hedging and no-arbitrage argument or by taking an expectation;
2. or we **concentrate on the stochastic part**, by somehow, removing the deterministic part and transforming the stochastic processes we are dealing with into **martingales**.

This is where a technique called **change of measure** comes in handy!

If you recall your experience from the binomial model, probability distribution, or “probability measures” as they are known to probabilists, are not unique.

Indeed, in the binomial model, we started from a real world, with real probabilities (a.k.a. the real or physical measure) and ended up computing the derivative price in the “risk neutral” world using risk neutral probabilities (a.k.a. the “risk neutral” measure).

This is what the change of measure technique does:

- allow us to define new worlds with their own sets of probabilities (probability measure);

- set rules under which we can transition from one world to another, from one measure from another.

In particular, we will look for a world and a measure in which our stochastic process are driftless, i.e. are martingales, so that we can focus on the probabilistic properties of our pricing problem.

## 7.2. Deriving the Black Scholes Formula

**Step 1:** *Discount the Asset Process*

We will start by considering the process of the discounted price process $S^*$ associated with the underlying asset $S$. $S^*$ is defined as

$$S^*(t) = \frac{S(t)}{B(t)} = \frac{S(t)}{e^{rt}}$$

Why consider the discounted price rather than the current price? The reason is the **time value of money**. Because of time value of money, the risk-free rate is already embedded inside the drift of all financial assets. So we want to remove it and consider the underlying asset's dynamics excluding the time value of money effect.

The mathematical implications of such a choice will become obvious a bit later...

For any time $t \in [0, T]$, the discounted asset process evolves according to the GBM

$$dS^* = (\mu - r)S^* dt + \sigma S^* dX, \quad S^*(0) = S_0^* \tag{2}$$

**Note** that the discounted asset price $S^*$ follows a martingale process (under the measure $\mathbb{P}$) iff $\mu = r$.

By extension, if $\mu > r$ (as "should be" the case in financial markets) then $S^*$ is a submartingale and if $\mu < r$, $S^*$ is a supermartingale.

**Step 2:** *Change of Measure*

It is clear that in the general case (i.e. $\mu \neq r$), $S^*$ is not a martingale. However, we would like to be able to, somehow, transform it into one in order to benefit from all the nice properties of martingales.

Our broad objective is therefore to find a measure $\mathbb{Q}$ under which $S^*$ is a martingale.

**Change of measure** is a standard technique in probability theory and is one of the key techniques (alongside Itô's lemma) in the applications of stochastic analysis to asset valuation. In fact, we will use it several times in this and the next valuation techniques.

When we talk about change of measure, we actually refer to two very important results:

1. the **Radon Nikodým** theorem;

2. the **Girsanov** theorem.

But first, a definition...

## Equivalent Measure

If two measures $\mathbb{P}$ and $\mathbb{Q}$ share the same sample space $\Omega$ and if $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for all subset $A$, we say that $\mathbb{Q}$ is **absolutely continuous** with respect to $\mathbb{P}$ and denote this by $\mathbb{Q} << \mathbb{P}$.

The key point is that all impossible events under $\mathbb{P}$ remain impossible under $\mathbb{Q}$. The probability mass of the possible events will be distributed differently under $\mathbb{P}$ and $\mathbb{Q}$.

In short "it is alright to tinker with the probabilities as long as we do not tinker with the (im)possibilities"

If $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$ then the two measures are said to be **equivalent**.

This extremely important result is formalized in the **Radon Nikodym Theorem**.

## The Radon Nikodým Theorem

If the measures $\mathbb{P}$ and $\mathbb{Q}$ share the same null sets, then, there exists a random variable $\Lambda$ such that

$$\mathbb{Q} = \int_A \Lambda d\mathbb{P} \tag{3}$$

where

$$\Lambda = \frac{d\mathbb{Q}}{d\mathbb{P}} \tag{4}$$

is called the **Radon Nikodým derivative**

The measure $\mathbb{Q}$ we are seeking actually belongs to a special subclass of equivalent measures, called the class of **martingale measures**.

<u>**Definition:**</u> **Martingale Measure**
A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ and equivalent to $\mathbb{P}$ is called a martingale measure for $S^*$ if $S^*$ is a martingale under $\mathbb{Q}$.

While the Radon Nicodým theorem can be used to help us change measure once we know the measure we want to change into, it is not helpful in identifying the actual martingale measure $\mathbb{Q}$ we are looking for!

That's where **Girsanov's theorem** comes in.

**Definition: Novikov Condition**

A process $\theta$ satisfies the Novikov condition if

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 ds\right)\right] < \infty$$

**Proposition 1:**

If a process $\theta$ satisfies the Novikov condition, then the process $M^\theta$ defined as

$$M_t^\theta = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0,T]$$

is an (exponential) martingale.

## Girsanov's Theorem

Given a process $\theta$ satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nicodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t + \int_0^t \theta(s) ds, \quad t \in [0, T] \tag{5}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

**Aside:** *Stochastic Exponential*

Sometimes, in the literature, you will encounter the notation

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right)$$

This "curly E" denotes the **stochastic** (or **Doléans**) **exponential**, which is defined as

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

The Girsanov Theorem can also be defined in terms of the stochastic exponential.

**Girsanov's Theorem**(in terms of stochastic exponential)

Given a process $\theta$ satisfying the condition

$$\mathbf{E}\left[\mathcal{E}\left(\int_0^T \theta_s dX_s\right)\right] = 1 \tag{6}$$

we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nicodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\int_0^t \theta_s dX_s\right), \quad t \in [0, T]$$

In this case, the process $X^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t - \int_0^t \theta(s) ds, t \in [0, T] \tag{7}$$

is a standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Observe that condition 6 requires that the stochastic exponential is a martingale. It is therefore equivalent to the Novikov condition in our standard presentation of Girsanov.

Also, note the sign change between 5 and 7, which is due to the difference in sign of the stochastic integral in between the Novikov condition and the stochastic exponential.

*End of aside and back to our problem...*

*What Does Girsanov Do?*

Girsanov effectively extends the Radon Nikodým result by giving an expression for the Radon Nikodým derivative as well as expliciting a correspondence between the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure in terms of their respective Brownian motion.

The process $\theta$ acts as the "key" enabling us to define the measure $\mathbb{Q}$ via the Radon Nikodým derivative and to travel in between the $\mathbb{P}$ measure and the $\mathbb{Q}$ measure via the Brownian motion correspondence.

The catch with Girsanov is that the theorem stops short of identifying the process $\theta$. We therefore need to have a process in mind and this process has to satisfy the Novikov condition if we want to use Girsanov.

As we are going to see next, this is actually not an impediment when dealing with equivalent martingale measures.

So, what Girsanov is telling us is that to define an equivalent measure $\mathbb{Q}$ and be able to use Radon Nikodým, all we need is an appropriate process $\theta$.

In our case, we want to find $\theta$ such that $S^*$ is a $\mathbb{Q}$-martingale. Applying Girsanov for an arbitrary process $\theta$, we see that under $\mathbb{Q}$, the dynamics of $S^*$ is given by

$$\frac{dS^*}{S^*} = (\mu - r)dt + \sigma\left(-\theta dt + dX^{\mathbb{Q}}\right)$$

If we assume that $\theta$ is constant *, then we have

$$\frac{dS^*}{S^*} = (\mu - r - \sigma\theta)dt + \sigma dX^{\mathbb{Q}}$$

Recall that our objective is to find an equivalent martingale measure, i.e. a measure $\mathbb{Q}$ in which $S^*$ is a martingale. One of the requirements for $S^*$ to be a martingale is that its dynamics should be driftless:

$$\mu - r - \sigma\theta = 0$$

i.e.

$$\theta = \frac{\mu - r}{\sigma}$$



*After all why not? This is consistent with our assumptions made with respect to the other parameters of the problem.

Now, $\theta$ satisfies the Novikov condition. Invoking **Girsanov's theorem**, we can define the equivalent martingale measure $\mathbb{Q}$ via the Radon Nikodým derivative as:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} X_t - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} t\right), \quad t \in [0, T]$$

Moreover, the $\mathbb{Q}$-Brownian Motion, $X^{\mathbb{Q}}$, is defined as

$$X_t^{\mathbb{Q}} = X_t + \frac{\mu - r}{\sigma} t, \\ t \in [0, T]$$

and the discounted asset process is effectively a $\mathbb{Q}$-martingale:

$$\frac{dS^*}{S^*} = \sigma dX^{\mathbb{Q}}$$

**Step 3:** *Trading Strategies*

So far we have not said anything about the way we plan to price the derivative.

To price the derivative, we will setup a **self-financing replicating portfolio** using the underlying stock and the bank account to recreate the payoff profile of the derivative. **Self-financing** means that once we have set up the replicating portfolio we cannot add new cash in and we cannot take any cash out. Any purchase of stock will need to be financed by borrowing from the bank and the proceeds of any sale of stock will go to the bank account.

Now, since the portfolio and the derivative do the same thing, then to prevent arbitrage the value $V_t$ of the replicating portfolio has to be equal to the value of the derivative at time $t$.

To formalize things a bit, we will consider a trading strategy $\phi_t = (\phi_t^S, \phi_t^B)$ defined over the time interval $[0, T]$.

The value of the portfolio $V(\phi)$ induced by the strategy given by

$$V_t(\phi) = \phi_t^S S_t + \phi_t^B B_t \quad \forall t \in [0, T]$$

For $\phi$ to be **self-financing** we therefore require that the "no cash in, no cash out" condition

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^S dS_u + \int_0^t \phi_u^B dB_u, \quad \forall t \in [0, T] \tag{8}$$

hold.

**Step 4:** *No-Arbitrage Valuation*

Let $\chi(t, S_t)$ be the time $t$ arbitrage-free value of the derivative we are attempting to price.

We now define the time $t$ discounted value of the replicating portfolio as

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t}, \quad t \in [0, T]$$

To prevent arbitrage, the value of the replicating portfolio must be equal to the value of the derivative:

$$\chi(t, S_t) = V_t, \quad t \in [0, T]$$

and hence

$$\frac{\chi(t, S_t)}{B_t} = V_t^*(\phi), \quad t \in [0, T] \tag{9}$$

In particular, for $t = T$, we have

$$\frac{\chi(T, S_T)}{B_T} = \frac{G(S_T)}{B_T} = V_T^*$$

where $G(\cdot)$ is the payoff function of the derivative.

Taking the conditional expectation under $\mathbb{Q}$,

$$\mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right] = \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{10}$$

Now, how can we link expressions 9 and 10 in order to find the time $t$ value of the derivative maturing at time $T$?

**Answer**: through our self-financing trading strategy!

First note that

$$dB_t^{-1} = -rB_t^{-1}dt$$

By the **Itô Product Rule**,

$$\begin{aligned} dV_t^* &= d(V_t B_t^{-1}) \\ &= V_t dB_t^{-1} + B_t^{-1} dV_t \\ &= \left(\phi_t^S S_t + \phi_t^B B_t\right) dB_t^{-1} \\ &\quad + B_t^{-1} \left(\phi_t^S dS_t + \phi_t^B dB_t\right) \\ &= \phi_t^S \left(B_t^{-1} dS_t + S_t dB_t^{-1}\right) \\ &= \phi_t^S dS_t^* \end{aligned}$$

Integrating, we see that

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t \phi_u^S dS_u^* \\ &= V_0^* + \int_0^t \phi_u^S \sigma S_u^* dX_u^{\mathbb{Q}} \end{aligned} \tag{11}$$

Since

$$\int_0^t \phi_u^S \sigma S_u^* dX_u^{\mathbb{Q}}$$

is an Itô integral, $V^*$ is a martingale. Now, taking the expectation of 11 under $\mathbb{Q}$ we have

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}}\left[V_t^*\right] &= \mathbf{E}^{\mathbb{Q}}\left[V_0^* + \int_0^t \phi_u^S \sigma S_u^* dX_u^{\mathbb{Q}}\right] \\ &= V_0^* \end{aligned}$$

Under $\mathbb{Q}$, not only is $S^*$ a martingale, but so is $V^*$!

Since $V^*$ is a martingale, it becomes obvious that

$$V_t^* = \mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{12}$$

Considering in addition relationships 9 and 10, we have

$$\begin{aligned} \frac{\chi(t, S_t)}{B_t} & \\ &= V_t^*(\phi) \\ &= \mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}G(S_T)|\mathcal{F}_t\right], \quad t \in [0, T] \end{aligned} \tag{13}$$

Equation 13 is the cornerstone of no-arbitrage valuation.

Expressing it as

$$\chi(t, S_t) \;=\; B_t \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} G(S_T) | \mathcal{F}_t\right], \quad t \in [0, T] \tag{14}$$

we see that the value at time $t$ of a derivative maturing at time $T$ is the expected value under the $\mathbb{Q}$ measure of the discounted terminal value of the contract.

**Note** that this equation is also at the heart of Monte-Carlo-based asset valuation.

**Step 5:** *Check that the Strategy is Self-Financing*

We have just seen that under $\mathbb{Q}$, $V^*$ is a martingale. Hence, by the **Martingale Representation Theorem** (see the Martingale I Lecture), there exists a process $\theta$ satisfying technical condition such that

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t \theta_u dX_u^{\mathbb{Q}} \\ &= V_0^* + \int_0^t h_u dS_u^*, \quad t \in [0, T] \end{aligned}$$

where $h_t = \frac{\theta_t}{\sigma S_t^*}$.

Consider a trading strategy $\phi$ defined as:

$$\begin{aligned}\phi_t^S &= h_t \\ \phi_t^B &= V_t^* - h_t S_t^* = B_t^{-1}(V_t - h_t S_t)\end{aligned}$$

From *Step 4*, we already know that $V_T(\phi) = G(T, S_T)$. We will now check that the strategy $\phi$ is self-financing. To do so, we need to go back to the current, or undiscounted, value of the replicating portfolio.

By the Itô Product Rule:

$$\begin{aligned}
dV_t(\phi) &= d(B_t V_t^*) \\
&= B_t dV_t^* + V_t^* dB_t \\
&= B_t h_t dS_t^* + rV_t dt \\
&= B_t h_t \left( B_t^{-1} dS_t^* - r B_t^{-1} S_t dt \right) + rV_t dt \\
&= h_t dS_t + r(V_t - h_t S_t) dt
\end{aligned}$$

which confirms the fact that the portfolio is indeed self financing.

**Note** that we do not know the specifics of the trading strategy, namely how much of the underlying asset to hold. All we know is that a strategy exists and that it is self-financing.

After seeing the importance of the Delta-hedging strategy in both the PDE approach and the Binomial model, it may seem quite strange to be dealing with an approach that do not require a specific knowledge of the strategy.

But this is precisely what the probabilistic approach does. It only checks that some technical conditions are fulfilled in order to guarantee that there exists an “appropriate” replicating strategy, without actually defining it.

Now, all we have to do is evaluate the no-arbitrage pricing equation 13 (obtained at the end of *Step 4*) for the specific derivative we are trying to price.

**Step 6:** *Solve the Black-Scholes Call Option Problem*

We will now solve the Black-Scholes European call option problem. A similar derivation could be done for European put options and for binary options.

In our case, we can make one additional simplification. Since $r$ is constant, we can write 13 as

$$\begin{aligned}\chi(t, S_t) & \\ &= e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}\left[F(S_T)|\mathcal{F}_t\right], \quad t \in [0, T]\end{aligned}$$

Going back to our undiscounted asset price process, $S$, note that under the measure $\mathbb{Q}$, we have

$$\frac{dS}{S} = r dt + \sigma dX^{\mathbb{Q}}$$

Seems familiar? It should be! It is the "Risk-Neutral" GBM.

At time $T$, we have

$$S_T = S_t \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(X_T^{\mathbb{Q}} - X_t^{\mathbb{Q}}\right)\right\}$$

Setting $Y_T = \ln \frac{S_T}{S_t}, \quad \forall t \in [0, T]$, we see that

$$Y_T \sim \mathcal{N}\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

i.e. the log return of the asset over the period $[0, T]$, $Y_T$, is normally distributed with mean

$$\left(r - \frac{1}{2}\sigma^2\right)(T-t)$$

and variance

$$\sigma^2(T-t)$$

Our pricing formula can now be rewritten as

$$V_t^*(\phi) = e^{-r(T-t)} \int_{-\infty}^{\infty} G(S_0 e^y) p(y) dy$$

where $p$ is the PDF of $Y$.

In order to alleviate our computations, we are going to normalize our expression.

First, define

$$\tilde{r} = r - \frac{1}{2}\sigma^2$$
$$\tau = T - t$$

With this notation, we have

$$Y_T \sim \mathcal{N}\left(\tilde{r}\tau, \sigma^2\tau\right)$$

Now, define the standardized normal random variable $Z$ as

$$Z = \frac{Y - \tilde{r}\tau}{\sigma\sqrt{\tau}}$$

(Recall that $Z \sim \mathcal{N}(0, 1)$)

We are now ready to tackle the pricing of a specific derivative.

**Step 7:** *Pricing a Call Option*

The payoff function for a call is given by

$$G(S_T) = \max\left[S_T - E, 0\right]$$

Writing $X(\tau)$ as $\sqrt{\tau}Z$ and substituting into the pricing equation, we get

$$\begin{aligned} & V_t^*(\phi) \\ = \; & e^{-r\tau}\int_{-\infty}^{\infty} \max\left[S_t e^{\tilde{r}\tau+\sigma\sqrt{\tau}z} - E, 0\right]\varphi(z)dz \end{aligned}$$

where $\varphi$ is the standard normal PDF:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

We can get rid of the max by noticing that the integral vanishes when

$$S_t e^{\tilde{r}\tau+\sigma\sqrt{\tau}z} < E$$

i.e. when

$$z < z_0 := \frac{\ln\left(\frac{E}{S_t}\right) - \tilde{r}\tau}{\sigma\sqrt{\tau}}$$

The pricing formula becomes

$$\begin{aligned}
\chi(t, S_t) &= e^{-r\tau}\int_{z_0}^{\infty}\left(S_t e^{\tilde{r}\tau+\sigma\sqrt{\tau}z} - E\right)\varphi(z)dz \\
&= e^{-r\tau}\int_{z_0}^{\infty} S_t e^{\tilde{r}\tau+\sigma\sqrt{\tau}z}\varphi(z)dz \\
&\quad -e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz
\end{aligned}$$

Evaluating the second term on the right-hand side yields:

$$
\begin{aligned}
-e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz &= -Ee^{-r\tau}\int_{z_0}^{\infty}\varphi(z)dz \\
&= -Ee^{-r\tau}P[Z \geq z_0]
\end{aligned}
$$

By symmetry of the normal distribution, this can also be written as

$$-Ee^{-r\tau}P[Z \leq -z_0] = -EN(-z_0)$$

where $N$ is the standard normal CDF.

To evaluate the first term, we need to complete the square in the exponent:

$$
\begin{aligned}
& e^{-r\tau} \int_{z_0}^{\infty} S_t e^{\tilde{r}\tau + \sigma\sqrt{\tau}z} \varphi(z) dz \\
= & \frac{e^{(\tilde{r}-r)\tau} S_t}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{\sigma\sqrt{\tau}z - \frac{1}{2}z^2} dz \\
= & \frac{e^{(\tilde{r}-r)\tau} S_t}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2 + \frac{1}{2}\sigma^2\tau} dz \\
= & \frac{S_t}{\sqrt{2\pi}} \int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2} dz \\
= & S_t P[U \geq z_0]
\end{aligned}
$$

where $U = Z - \sigma\sqrt{\tau}$ so that $U \sim \mathcal{N}(\sigma\sqrt{\tau}, 1)$. Standardizing, we see that the first term is actually equal to

$$S_t N(-z_0 + \sigma\sqrt{\tau})$$

In order to write this in the more familiar form, all we need to do is to define $d_1$ and $d_2$ as

$$\begin{aligned} d_1 &= -z_0 + \sigma\sqrt{\tau} = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= -z_0 = \frac{\ln\left(\frac{S_t}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

and to substitute the first and the second term into the pricing equation:

$$\chi(t, S_t) = S_t N(d_1) - Ee^{-r(T-t)} N(d_2)$$

## To conclude on the martingale approach...

The change of measure technique underpinning the martingale approach is one of the standard techniques in probability theory. It underlines the relationship between different worlds and their probabilities (probability measures).

Although we have lost a direct connection to the hedging argument and to the PDE, we have gained the following understanding:

- $N(d_1)$ and $N(d_2)$ are not actual probabilities in the real world. They are risk-neutral probabilities;

- In the risk-neutral world, $N(d_2)$ is the probability that the option will end up in the money and be exercised.

# 8. The Numéraire Approach

## 8.1. Overview

In the martingale approach we saw that we could define a new universe and a new set of probabilities. The next question is: is there more than two alternative universes (i.e. physical and risk-neutral)? If so, how many and how can we define all of them?

The answer to the first question is yes. To see that let's go back to the end of the martingale derivation, where we could have actually used a second numéraire to get a better understanding of $N(d_1)$. Let's do that now.

## 8.2. Deriving the Black Scholes Formula: Alternative Step 6&7

Recalling equation 14, in the call case we are trying to evaluate

$$
\begin{aligned}
& B_t \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_T-K\right]^{+}|\mathcal{F}_t\right] \\
= \; & B_t \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_t e^{\sigma\left(X_T^{\mathbb{Q}}-X_t\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}-K\right]^{+}|\mathcal{F}_t\right]
\end{aligned}
$$

where the strike price is now denoted by $K$ to avoid any confusion with $\mathbf{E}$, the expectation.

Focusing on the case $t=0$, we can drop the conditional expectation and deal with an unconditional expectation:

$$
\begin{aligned}
& \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}\left[S_T-K\right]^{+}|\mathcal{F}_t\right] \\
= \; & \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} S_T \mathbf{1}_{\{S_T>K\}}\right]-\mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} K \mathbf{1}_{\{S_T>K\}}\right]
\end{aligned}
$$

Tackling the second expectation on the RHS,

$$
\begin{aligned}
& \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} K \mathbf{1}_{\{S_T > K\}}\right] \\
= \;& e^{-rT} K P^{\mathbb{Q}}\left[S_T > K\right] \\
= \;& e^{-rT} K P^{\mathbb{Q}}\left[S_0 e^{\sigma X_T^{\mathbb{Q}} + (r - \frac{1}{2}\sigma^2)T} > K\right] \\
= \;& e^{-rT} K P^{\mathbb{Q}}\left[\ln\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2)T > -\sigma X_T^{\mathbb{Q}}\right] \\
= \;& e^{-rT} K P^{\mathbb{Q}}\left[\frac{\ln\left(\frac{S_0}{K}\right) + (r - \frac{1}{2}\sigma^2\sigma^2)T}{\sigma\sqrt{T}} > \xi\right] \\
= \;& e^{-rT} K N(d_2)
\end{aligned}
$$

where we have emphasized the fact that the probability $P$ is taken with respect to the measure $\mathbb{Q}$ and have defined $\xi = -X_T^{\mathbb{Q}}/\sqrt{T}$. Note that $\xi \sim \mathcal{N}(0, 1)$.

As for the first expectation on the RHS,

$$
\begin{aligned}
& \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} S_T \mathbf{1}_{\{S_T > K\}}\right] \\
= \ & \mathbf{E}^{\mathbb{Q}}\left[S_T^{*} \mathbf{1}_{\{S_T > K\}}\right] \\
= \ & \mathbf{E}^{\mathbb{Q}}\left[S_0 \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \mathbf{1}_{\{S_T > K\}}\right]
\end{aligned}
$$

Unless we can find a trick, it does not seem that this expectation can be computed analytically.

But, as happens frequently in mathematics, a simple trick can be found!

Notice that the

$$\exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \tag{15}$$

term inside the expectation looks reminiscent of the Doléans exponential we introduced in our second formulation of Girsanov's theorem:

$$\mathcal{E}\left(\int_0^t \theta_s dX_s\right) = \exp\left(\int_0^t \theta_s dX_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

Indeed, we can reformulate 15 as

$$\exp\left\{\int_0^T \sigma dX_t^{\mathbb{Q}} - \frac{1}{2}\int_0^T \sigma^2 dt\right\}$$

and check that this is the Doléans exponential with $\theta = \sigma$!

This is an important observation, because it means that we could get rid of this bothersome term in our expectation by defining a new measure and changing measure via Girsanov's theorem.

All we do is check that

$$\mathbf{E}^{\mathbb{Q}}\left[\exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}\right] = 1$$

i.e. that 15 is an exponential martingale.

(check left to the reader!)

Concretely, e will now define a new probability measure $\bar{\mathbb{Q}}$ via the Radon Nikodym derivative

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}$$

Note that under the $\bar{\mathbb{Q}}$ measure,

$$X_t^{\bar{\mathbb{Q}}} = X_t^{\mathbb{Q}} - \sigma t, \qquad t \in [0, T]$$

is a Brownian motion and that

$$S_T^* = S_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\} \tag{16}$$

Therefore,

$$
\begin{aligned}
& \mathbf{E}^{\mathbb{Q}}\left[S_T^* \mathbf{1}_{\{S_T>K\}}\right] \\
= \; & \mathbf{E}^{\mathbb{Q}}\left[S_0 \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\} \mathbf{1}_{\{S_T>K\}}\right] \\
= \; & S_0 \int \frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} \mathbf{1}_{\{S_T>K\}} d\mathbb{Q} \\
= \; & S_0 P^{\bar{\mathbb{Q}}}\left[S_T > K\right] \\
= \; & S_0 P^{\bar{\mathbb{Q}}}\left[S_T^* > K B_T^{-1}\right] \\
= \; & S_0 P^{\bar{\mathbb{Q}}}\left[S_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\} > K e^{-rT}\right] \\
= \; & S_0 P^{\bar{\mathbb{Q}}}\left[\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T > -\sigma X_T^{\bar{\mathbb{Q}}}\right] \\
= \; & S_0 P^{\bar{\mathbb{Q}}}\left[\frac{\ln\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} > \xi\right] \\
= \; & S_0 N(d_1)
\end{aligned}
$$

where we have emphasized the fact that the probability $P$ is taken with respect to the measure $\bar{\mathbb{Q}}$ and have defined $\xi = -X_T^{\mathbb{Q}}/\sqrt{T}$.

## 8.3. The Numéraire pair

So, in the Black-Scholes model we could have three alternative universes to do our pricing:

1. the "physical" universe where we do not discount our processes and the probability measure is $\mathbb{P}$;

2. the "risk-neutral" universe where we discount our processes by the risk-free asset and the probability measure is $\mathbb{Q}$;

3. the "stock growth" universe where we discount our processes by the stock price and the probability measure is $\bar{\mathbb{Q}}$;

So it seems that each universe can be characterized based on a pairing of a discounting process and a probability measure.

This is indeed the basic idea underlying the numéraire method. By picking a discounting asset or portfolio of asset, you can define one (or several) equivalent martingale measures. Each pair (discounting asset, measure) in turns define a valuation framework.

In economics a **numéraire** is a base asset in units of which the value of all other assets are expressed. The commonly accepted numéraires in modern economics are currencies.

By picking one asset to discount all the assets in the market, we make of the discounting asset our numéraire. This is why this pricing approach is called the numéraire approach.

**Note** that in order to be a numéraire, an asset or portfolio must always have strictly positive value.

**To conclude on the change of numéraire...**

The change of numéraire is a generalization of the martingale approach which emphasizes the link between discounting process and martingale measure.

The numéraire approach is now widely adopted, especially in the pricing of interest rate securities where bonds of various maturities $T$ are used as numéraire assets to define various $T$-forward measure.

# 9. The Numéraire Approach

## 9.1. Overview

This approach starts from an entirely different perspective. Instead of trying to either hedge or replicate the option a priori, we define a new strategy called a **stop-loss start-gain** strategy.

It turns out that although this strategy is not self-replicating, it enables us to price a call option consistently with Black-Scholes and shed new light on the pricing exercise.

However, this approach is a bit tricky as it requires an extension of Itô's formula called the Tanaka-Meyer formula, a fairly advanced stochastic calculus result.

## 9.2. The Investment Strategy

We will define the following **stop-loss start-gain** investment strategy:

- if $S_t > B_t K$, i.e. if the stock price is above the present value of the strike, hold 1 share financed through borrowing;

- if $S_t \leq B_t K$, i.e. if the stock price is below the present value of the strike, liquidate the share and pay back the loan.

Mathematically, if we denote by $a_t$ the stock holdings at time $t$ and by $b_t$ the holdings in the bank account, this strategy translates into

$$\begin{aligned} a_t &= \mathbf{1}_{\{S_t > KB_t\}} \\ b_t &= -a_t = -\mathbf{1}_{\{S_t > KB_t\}} \end{aligned}$$

The value $Y_t$ of the portfolio at time $t$ is therefore equal to

$$\begin{aligned} Y_t &= a_t S_t + b_t B_t \\ &= \mathbf{1}_{\{S_t > K B_t\}} (S_t - B_t) \\ &= (S_t - B_t)^+ \end{aligned}$$

which implies that at time $T$ this strategy replicates the payoff of a call.

**Note** that at time 0, the portfolio may not cost anything if $S_0 < B_0 K$. If the strategy is self-financing, this would create an arbitrage opportunity. Therefore, we must first check that the "stop-loss start-gain" strategy is not self-financing.

Define the discounted stock price $S_t^*$ as $S_t^* = \frac{S_t}{B_t}$. The budget equation up to time $t$ is

$$\frac{Y_t}{B_t} = \frac{Y_0}{B_0} + \int_0^t a_u dS_u^* \tag{17}$$

and thus, by definition of $a_t$ and $Y_t$,

$$(S_t^* - K)^+ = (S_0^* - K)^+ + \int_0^t \mathbf{1}_{\{S_t^* > K\}} dS_u^* \tag{18}$$

For the portfolio to be self-financing we would need this equation to hold.

To check that the strategy is not self-financing, we face a problem: we cannot use Itô on the function $F(x) = (x - K)^+$.

Why is that? Because Itô requires functions to be "smooth enough" which means to be continuous with continuous first and second order derivatives.

Since $F$ does not have a first derivative at 0, we cannot use Itô. Fortunately, $F$ is convex, so we can use an extension of Itô called the Tanaka-Meyer formula to help us prove that 18 does not hold.

## 9.3. A Brief Overview of Local Time and the Tanaka-Meyer Formula

The Tanaka-Meyer formula can itself be seen from three perspectives:

1. as a result to deal with reflecting Brownian motions;
2. as an extension of Itô's formula;
3. as a tool in the study of local times.

Given our familiarity with Itô, we will start from the second perspective.

Let's consider the function $F(X(t)) = |X_t|$ where $X_t$ is a Brownian motion.

As a function, $F(x) = |x|$ is continuous and strictly convex. But its first derivative has a singularity at 0 and is therefore not continuous for all $x$. As a result, we cannot use Itô's formula directly.

Some calculations would in fact show that

$$|X_T| = |X_0| + \int_0^T \mathrm{sgn}(X_s)dX_s + \Lambda_T(0) \tag{19}$$

where $\Lambda_T(0)$ is called the **local time** of $X$ at 0 through time $T$.

This is the first Tanaka-Meyer formula.

The concept of local time is byproduct of the Tanaka-Meyer formula.

The local time $\Lambda_T(\alpha)$ of a stochastic process $Y$ at 0 through time $T$ is defined as the total amount of time between time 0 and time $T$ during which $Y = \alpha$ and it is used as part of an "adjustment" factor for the lack of smoothness of the function

Quite intuitively, a local time $\Lambda_T(\alpha)$ has the following properties

- is positive, since it represents a measure of time;

- increasing, since the time spent at $\alpha$ can only increase as $T$ increases, and;

- continuous.

A generalization of the Tanaka-Meyer formula, called the Meyer-Itô formula extends the scope of Itô's formula to functions that may not be continuous with continuous first and second derivatives.

A not too rigorous and quite narrow statement of the powerful Meyer-Itô formula would be that if $F$ is the difference of two convex functions and $S$ is a diffusion process, then

$$F(S_T) = F(S_0) + \int_0^T F'(S_t)dS_t + \frac{1}{2}\int_{-\infty}^{\infty} \mu(da)\Lambda_T(a) \qquad (20)$$

where $\mu$ is a "generalized" version of the second derivative of $f$.

## 9.4. Back to our Problem...

By the Tanaka-Meyer formula

$$(S_t^* - K)^+ = (S_0^* - K)^+ + \int_0^t \mathbf{1}_{\{S_t^* > K\}} dS_u^* + \Lambda_t(K) \qquad (21)$$

where $\Lambda_t(K)$ is the local time at $K$ up to time $t$.

Clearly, comparing equations 18 and 21, we see that the portfolio is not self-financing if $\Lambda_t(K) \geq 0$ and if there is a positive probability that $\Lambda_t(K) > 0$.

Taking the risk-neutral expectation (i.e. the expectation under the measure $\mathbb{Q}$ defined by taking the bank account as numéraire) at time 0, we would have

$$\mathbb{E}^{\mathbb{Q}}\left[(S_t^* - K)^+\right] = (S_0^* - K)^+ + \mathbb{E}^{\mathbb{Q}}\left[\Lambda_t(K)\right] \tag{22}$$

To get to this equality, we have used the fact that under $\mathbb{Q}$, $S_t^*$ is a martingale and therefore the integral $\int_0^t \mathbf{1}_{\{S_t^* > K\}} dS_u^*$ is also a $\mathbb{Q}$-martingale.

Since the function $g(x) = (x - K)^+$ is strictly in intervals containing $K$, we can use Jensen's inequality to deduce that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(S_t^* - K)^+\right] &> \left(\mathbb{E}^{\mathbb{Q}}\left[S_t^* - K\right]\right)^+ \\
&= \left(\mathbb{E}^{\mathbb{Q}}\left[S_t^*\right] - K\right)^+ \\
&= \left[(S_0^* - K)^+\right] \\
&\geq 0
\end{aligned}$$

and hence, substituting in 22

$$\mathbb{E}^{\mathbb{Q}}\left[\Lambda_t(K)\right] > 0 \tag{23}$$

Since, by definition of local times we know that $\Lambda_t(K) \geq 0$, then we conclude that $\mathbb{Q}\left[\Lambda_t(K) \geq 0 > 0\right] > 0$ and thus $\mathbb{P}\left[\Lambda_t(K) \geq 0 > 0\right] > 0$. Hence, the portfolio is not self-financing.

The next step is to show that the expectation 22, once evaluated at time $T$, is the Black-Scholes formula.

From the fundamental pricing equation 14,

$$C_0 = B_0 \mathbf{E}^{\mathbb{Q}} \left[ (S_T^* - K)^+ \right] = \mathbf{E}^{\mathbb{Q}} \left[ (S_T - K)^+ \right]$$

and hence

$$C_0 = \left( S_0 - e^{-rT} K \right)^+ + e^{-rT} \mathbf{E}^{\mathbb{Q}} \left[ \Lambda_T(K) \right] \tag{24}$$

This pricing formula has a very nice interpretation

- $\left( S_0 - e^{-rT} K \right)^+$ is the intrinsic value of the option;
- $e^{-rT} \mathbf{E}^{\mathbb{Q}} \left[ \Lambda_T(K) \right]$ is the time value of the option. We can further observe that this amount is the external financing cost required in a "stop-loss start-gain strategy".

The rest of this section is a rather cumbersome clean-up of the pricing formula 24 to obtain the Black-Scholes formula.

By Girsanov's theorem, we know that under $\mathbb{Q}$, $S_t^*$ is given by

$$S_t^* = S_0^* \exp\left(\sigma X_t^{\mathbb{Q}} - \frac{\sigma^2 t}{2}\right) \tag{25}$$

and hence has the transition density function

$$\psi(S_t^*, t, S_0^*, 0) = \frac{1}{S_t^* \sigma\sqrt{t}} \phi\left(\frac{\ln\left(\frac{S_0^*}{S_t^*}\right) - \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) \tag{26}$$

where $\phi$ is the standard normal PDF.

Applying one of the local time theorems, we get

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T k(S_t^*)\sigma^2(S_t^*)^2 dt\right] = \mathbb{E}^{\mathbb{Q}}\left[2\int_{-\inf}^{+\inf} k(x)\Lambda_T(x)dx\right] \tag{27}$$

where $k(x)$ is some function.

Substituting the transition density function on the left-hand side and using Fubini's theorem to switch the order of integration on the right-hand side, we get

$$\begin{aligned} & \int_{-\inf}^{+\inf} k(x)\int_0^T \sigma^2 x^2 \psi(x,t;S_0^*)dtdx \\ = & \int_{-\int}^{+\int} 2k(x)\mathbb{E}^{\mathbb{Q}}\left[\Lambda_T(x)\right]dx \end{aligned} \tag{28}$$

Choosing $k(x) = \mathbf{1}_{\{x \in A\}}$ with $A \in \mathcal{F}$, the filtration on the probability space, then 28 becomes

$$\int_A \int_0^T \sigma^2 x^2 \psi(x,t;S_0^*)dtdx = \int_A \mathbb{E}^{\mathbb{Q}}\left[\Lambda_T(x)\right] dx$$

Since the integrands are non-negative and the equality is satisfied for all $A \in \mathcal{F}$, we can drop the integrals to get

$$\int_0^T \sigma^2 x^2 \psi(x,t;S_0^*)dt = \mathbb{E}^{\mathbb{Q}}\left[\Lambda_T(x)\right] \tag{29}$$

Substituting the definition of $\psi$ in the 29, we get a formula for the time value of the option

$$\mathbb{E}^{\mathbb{Q}}\left[\Lambda_T(x)\right] = \frac{\sigma K}{2}\int_0^T \frac{1}{\sqrt{t}}\phi\left(\frac{\ln\left(\frac{S_0^*}{S_t^*}\right) - \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) dt \tag{30}$$

Now that we have a formula for the time value of the option, we can substitute it into the pricing equation 24,

$$\begin{aligned} C_0 &= \left(S_0 - e^{-rT}K\right)^+ \\ &\quad e^{-rT}\frac{\sigma K}{2}\int_0^T \frac{1}{\sqrt{t}}\phi\left(\frac{\ln\left(\frac{S_0^*}{S_t^*}\right) - \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}\right) dt \end{aligned} \tag{31}$$

We now change variable by introducing $\nu = \frac{\sqrt{t}\sigma}{\sqrt{T}}$ to get

$$\begin{aligned} C_0 &= \left(S_0 - e^{-rT}K\right)^+ \\ & e^{-rT}K\sqrt{T}\int_0^\sigma \phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{\nu^2 T}{2}}{\nu\sqrt{t}}\right) d\nu \end{aligned} \tag{32}$$

which, quite incredibly, is the Black-Scholes formula... but not expressed in the usual way! To be precise, it is the integral of Vega, i.e.

$$\frac{\partial C}{\partial \sigma} = e^{-rT}K\sqrt{T}\phi\left(\frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{t}}\right) dt$$

integrated from 0 to $\sigma$, together with the boundary condition $C_0 = \left(S_0 - e^{-rT}K\right)^+$ at $\sigma = 0$ to ensure that $C_0$ is the price of a European call option.

**To conclude on the local time approach...**

Through the use of the Tanaka-Meyer formula, the local time approach shows how a stop-loss start-gain strategy produces a pricing formula with two components: the intrinsic value and the time value of the option.

The time value, in particular, represents the external financing cost of the strategy and reflects the amount of time the option is expected to stay at-the-money.

# 10. The Representative Investor

## 10.1. Overview

This approach is purely economic. The plan here is first to derive a general valuation equation for any asset and then to use it do derive the Black-Scholes formula

Here we will take the perspective of a "representative" investor who tries to value the call in consistent way with both the asset market and his/her risk preferences.

## 10.2. Utility and Utility Function

Economists have noticed that people do not react linearly with a change in their wealth. The wealthier a person becomes, and the less any additional  seems to matter. Conversely, the poorer a person becomes and the more any  seems to matter. Economists concluded that wealth was actually not a very good measure of economic satisfaction and they proposed the concept of **utility**.

The concept of utility is meant to evaluate the economic degree of “satisfaction” that an economic agent would derive from owning wealth (utility from terminal wealth) or from spending wealth to consume (utility from consumption).

A **utility function** $U(w)$ is a function mapping wealth $w$ into utility. A utility function must be

- increasing (i.e. $\frac{dU}{dw} > 0$) as people prefer more wealth to less wealth;

- concave (i.e. $\frac{d^2U}{dw^2} < 0$) as a person's utility rises less an less as their wealth increases.

In this section, we will be using one of the most common utility functions: the power utility function defined as

$$U(w) = \frac{w^{1+\gamma}}{1+\gamma}$$

where $\gamma < 0$ is the (constant relative) risk aversion of the investor.

## 10.3. A General Pricing Equation

We start in an economy with $n$ assets $S_1, \ldots, S_n$. The vector of asset prices at time $t$ is denoted by $S_t$ and the vector of portfolio weights is denoted by $a$.

The initial wealth of the investor, $w_0$, is fully invested in the portfolio, resulting in a first budget constraint:

$$w_0 = a^* S_0 \tag{33}$$

where * denotes the transposition.

In addition, the portfolio is self-financing so at any time t,

$$w_t = a^* S_t \tag{34}$$

The investor's objective is to chose a portfolio allocation which maximizes the expected utility of terminal wealth,

$$\mathbf{E}\left[U(w_T)\right] \tag{35}$$

subject to the budget equation 33.

We define the Lagrange function for the Lagrange multiplier $\lambda$

$$\mathcal{L}(a,\lambda) = \mathbf{E}\left[U(w_T)\right] - \lambda(a^* S_0 - w_0) \tag{36}$$

Taking the derivative with respect to $a$, we get the first order condition

$$S_0 = \lambda^{-1}\mathbf{E}\left[U'(w_T)S_T\right] \tag{37}$$

where $'$ denotes the first derivative.

In particular, for the risk-free asset, this equation can be written as

$$e^{-rT} = \lambda^{-1}\mathbf{E}\left[U'(w_T)\right] \tag{38}$$

Combining the last two equation to get rid of $\lambda$, we obtain the general valuation equation

$$S_0 = e^{-rT}\mathbf{E}\left[\frac{U'(w_T)}{\mathbf{E}\left[U'(w_T)\right]}S_T\right] \tag{39}$$

## 10.4. Derivation of the Black-Scholes Equation

Let's change a bit the rules of the game. We now have only 1 stock whose price is given by the usual GBM

$$S_T = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma X_T\right) \tag{40}$$

and the investor's wealth is also given by the usual GBM

$$w_T = w_0 \exp\left(\left(\mu_w - \frac{\sigma_w^2}{2}\right)T + \sigma_w X_{w,T}\right) \tag{41}$$

The two Brownian motions are correlated with correlation coefficient $\rho$.

In this framework, we see that

$$\frac{U'(w_T)}{\mathbf{E}\left[U'(w_T)\right]} = \exp\left(-\frac{1}{2}\gamma^2\sigma_w^2 T + \gamma\sigma_w X_{w,T}\right) \tag{42}$$

The stock must satisfy the valuation equation (otherwise the market is not in equilibrium), i.e.

$$
\begin{aligned}
S_0 &= S_0 e^{-rT}\mathbf{E}\left[\exp\left(\left(\mu - \frac{\sigma^2+\gamma^2\sigma_w^2}{2}\right)T + \sigma X_t + \gamma\sigma_w X_{x,T}\right)\right] \\
&= S_0 \exp\left(\mu + \sigma\gamma\sigma_w\rho - r\right)
\end{aligned}
$$

and thus we must have

$$
\mu = r - \sigma\gamma\sigma_w\rho
$$

What about a call on the stock? Once again, we can use the general valuation equation to deduce that the value of the call, $C_0$,

$$
\begin{aligned}
C_0 &= e^{-rT}\mathbf{E}\left[\exp\left(-\frac{1}{2}\gamma^2\sigma_w^2 T + \gamma\sigma_w X_{w,T}\right)(S_T - K)^+\right] \\
&= S_0 e^{-rT}\mathbf{E}\left[\exp\left(-\frac{1}{2}\left(\sigma^2 + 2\gamma\rho\sigma\sigma_w\right)T + \sigma X T + \gamma\sigma_w X_{w,T}\right)\mathbf{1}_{\{S_T \geq K\}}\right] \\
&\quad -K e^{-rT}\mathbf{E}\left[\exp\left(-\frac{1}{2}\gamma^2\sigma_w^2 T + \gamma\sigma_w X_{w,T}\right)\mathbf{1}_{\{S_T \geq K\}}\right]
\end{aligned}
$$

We will now tidy up this expression through a highly opportunistic use of Girsanov's Theorem. Notice that

$$\exp\left(-\frac{1}{2}\left(\sigma^2 + 2\gamma\rho\sigma\sigma_w\right)T + \sigma XT + \gamma\sigma_w X_{w,T}\right) \tag{43}$$

and

$$\exp\left(-\frac{1}{2}\gamma^2\sigma_w^2 T + \gamma\sigma_w X_{w,T}\right) \tag{44}$$

are two exponential martingales satisfying the Novikov condition. Hence, applying Girsanov, we can define two new measures $\mathbb{Q}$ and $\bar{\mathbb{Q}}$ via their Radon-Nikodým derivatives

$$\frac{d\bar{\mathbb{Q}}}{\mathbb{P}} = \exp\left(-\frac{1}{2}\left(\sigma^2 + 2\gamma\rho\sigma\sigma_w\right)T + \sigma XT + \gamma\sigma_w X_{w,T}\right) \tag{45}$$

and

$$\frac{d\mathbb{Q}}{\mathbb{P}} = \exp\left(-\frac{1}{2}\gamma^2\sigma_w^2 T + \gamma\sigma_w X_{w,T}\right) \tag{46}$$

We can therefore rewrite the call price as

$$C_0 = S_0 e^{-rT} \bar{\mathbb{Q}} [S_T \geq K] - K e^{-rT} \mathbb{Q} [S_T \geq K]$$

Finally, taking into consideration the Girsanov theorem and the equation 43, we see that

$$S_T = S_0 \exp\left(rT + \frac{\sigma^2}{2} T + \sigma X_T^{\bar{\mathbb{Q}}}\right) \tag{47}$$

where $X_T^{\bar{\mathbb{Q}}}$ is a Brownian motion under $\bar{\mathbb{Q}}$ and

$$S_T = S_0 \exp\left(rT - \frac{\sigma^2}{2} T + \sigma X_T^{\mathbb{Q}}\right) \tag{48}$$

where $X_T^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. The conclusion follows from here using similar arguments as in the martingale approach.