3 Differential Equations Problem Sheet

1. For arbitrary constants $c_1,\ c_2,\ c_3,\ c_4$ find the differential equations satisfied by y when:

a. $y = c_1 x + \dfrac{2}{c_1}$

$$\tfrac{dy}{dx} = c_1 \therefore y = \tfrac{dy}{dx}x + \tfrac{2}{dy/dx} \longrightarrow x\left(y'\right)^2 - yy' + 2 = 0$$

b. $y = (c_1 + c_2 x)\, e^{-\lambda x}$

$$\begin{aligned}
\tfrac{dy}{dx} &= -\lambda\left(c_1 + c_2 x\right) e^{-\lambda x} + c_2 e^{-\lambda x} = -\lambda y e^{-\lambda x} + c_2 e^{-\lambda x}\\
\tfrac{d^2y}{dx^2} &= -\lambda \tfrac{dy}{dx} - c_2 \lambda e^{-\lambda x} = -\lambda \tfrac{dy}{dx} - \lambda^2 y - \lambda \tfrac{dy}{dx} \therefore y'' + 2\lambda y' + \lambda^2 y = 0
\end{aligned}$$

c. $y = c_1 \sin\rho x + c_2 \cos\rho x + c_3 \sinh\rho x + c_4 \cosh\rho x$

$$\begin{aligned}
\frac{d^4}{dx^4}\sin\rho x &= \rho^4 \sin\rho x;\ \frac{d^4}{dx^4}\cos\rho x = \rho^4\cos\rho x;\\
\frac{d^4}{dx^4}\sinh\rho x &= \rho^4 \sinh\rho x;\ ;\ \frac{d^4}{dx^4}\cosh\rho x = \rho^4\cosh\rho x\\
\frac{d^4 y}{dx^4} &= \rho^4 y
\end{aligned}$$

2. Solve the following differential equations/I.V.P.'s

a. $\left(\dfrac{dy}{dx}\right)^3 = y^2 \quad y = 1,\ x = 0$

$$\begin{aligned}
\frac{dy}{dx} &= y^{2/3} \longrightarrow dx = y^{-2/3}dy\\
\int_0^x ds &= \int_0^x y^{-2/3}dy \longrightarrow x = 3y^{1/3}\Big|_0^x = 3y^{1/3}(x) - 3y^{1/3}(0)\\
\frac{x}{3} &= y^{1/3} - 1 \therefore y = \left(\frac{x+3}{3}\right)^3
\end{aligned}$$

b. $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2} \quad y = 1,\ x = 0$

$$\begin{aligned}
\int \tfrac{dy}{1+y^2} &= \int \tfrac{dx}{1+x^2} \longrightarrow \arctan y = \arctan x + c \text{ and use } \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}\\
y &= \frac{x+C}{1-Cx};\ \text{I.C } y(0) = 1 \Longrightarrow C = 1 \therefore y = \frac{1+x}{1-x}
\end{aligned}$$

c. $\sqrt{1+x^2}\dfrac{dy}{dx} = xe^{-y} \quad y=0,\ x=0 \qquad\qquad$ Ans: $y=\frac{1}{2}\log\left(1+x^2\right)$

$$\int_0^x e^y dy = \int_0^x \tfrac{s}{\sqrt{1+s^2}} ds$$

The right hand integral is done by substitution $u = 1+s^2 \longrightarrow du = 2sds$

$$\begin{aligned} e^y - 1 &= \sqrt{1+s^2}\Big|_0^x = \sqrt{1+x^2}-1 \\ e^y &= \sqrt{1+x^2} \longrightarrow y = \frac{1}{2}\log\left(1+x^2\right) \end{aligned}$$

d. $(1-y)^2\dfrac{dy}{dx} + \left(1+x^2\right)y = 0 \qquad\qquad$ Ans: $x+\dfrac{x^3}{3} = -\log y + 2y - \frac{1}{2}y^2 + c$

$$\begin{aligned} \int \tfrac{(1-y)^2}{y} dy &= -\int \left(1+x^2\right)dx \\ \int \left(\tfrac{1}{y}+y-2\right)dy &= -\int\left(1+x^2\right)dx \\ \log y + \tfrac{y^2}{2} - 2y &= -x - \tfrac{x^3}{3} + c \end{aligned}$$

which is an implicit solution.

e. $x\dfrac{dy}{dx} + 3y = 8x^5 \qquad\qquad$ Ans: $y = x^5 + \dfrac{c}{x^3}$

$$\frac{dy}{dx} + \tfrac{3}{x}y = 8x^4$$

linear equation with IF: $e^{3\int 1/x dx} = x^3$

$$\begin{aligned} x^3\frac{dy}{dx} + 3x^2y &= 8x^7 \\ \frac{d}{dx}\left(yx^3\right) &= 8x^7 \longrightarrow \int d\left(yx^3\right) = 8\int x^7 dx \\ yx^3 &= x^8 + c \longrightarrow y = x^5 + c/x^3. \end{aligned}$$

f. $\dfrac{dy}{dx} - 2y\tan x = x^2\sec^2 x$ when $x=0$ and $y=0$

So comparing with standard form we have $P = -2\tan x$, so

$$\text{I.F } R(x) = e^{-2\int \tan x dx} = e^{-2\ln\sec x} = e^{\ln(\sec x)^{-2}} = (\sec x)^{-2}.$$

Note: apart from the few basic integrals, you need not worry about remembering others - always consult a list of integrals in a book. So the differential equation is multiplied by the I.F

$$\begin{aligned} (\sec x)^{-2}\left(y' - 2y\tan x\right) &= x^2\sec^2 x\,(\sec x)^{-2} \\ y\,(\sec x)^{-2} &= \int x^2 dx \longrightarrow y = \frac{x^3}{3}\sec^2 x + c \end{aligned}$$

the initial condition gives $c = 0$, so the particular solution becomes $y = \frac{x^3}{3}\sec^2 x$

1. **g.** $\sin x\dfrac{dy}{dx} + 2y\cos x = \cos x$

$$\frac{dy}{dx} + 2y\cot x = \cot x$$

which is a linear equation with IF: $e^{2\int \cot x} = e^{2\log \sin x} = \sin^2 x$

$$\begin{aligned}
\sin^2 x\tfrac{dy}{dx} + 2\left(\sin x\cos x\right)y &= \sin x\cos x \\
\frac{d}{dx}\left(y\sin^2 x\right) &= \sin x\cos x \\
\int d\left(y\sin^2 x\right) &= \int \sin x\cos x dx
\end{aligned}$$

The right hand integral is solved by writing $I = \int \sin x\cos x dx$ and solving by parts to give $I = \frac{1}{2}\sin^2 x$

$$\begin{aligned}
y\sin^2 x &= \tfrac{1}{2}\sin^2 x + c \\
y &= \tfrac{1}{2} + c\csc^2 x
\end{aligned}$$

h. $(x+1)\,y' - 2y = 3\,(x+1)^3$ Ans: $y = (3x+c)\,(x+1)^2$

start by putting in standard form, divide through by $(x+1)$ to express as a linear equation

$$y' - \tfrac{2}{(x+1)}y = 3\,(x+1)^2$$

so $P(x) = -\frac{2}{(x+1)}$, hence I.F

$$R(x) = \exp\left(-\int \tfrac{2}{(x+1)}dx\right) = \exp\left(\ln(x+1)^{-2}\right) = \tfrac{1}{(x+1)^2}$$

multiply DE through by $R(x)$

$$y'\tfrac{1}{(x+1)^2} - \tfrac{2}{(x+1)^3}y = 3$$

$$\begin{aligned}
\frac{d}{dx}\left(y\,(x+1)^{-2}\right) &= 3 \\
y\,(x+1)^{-2} &= 3\int dx + c \\
&= 3x + c \\
y &= (3x+c)\,(x+1)^2
\end{aligned}$$

3. Solve the 2nd order equations

a. $\dfrac{d^2y}{dx^2} = 2y^3 + 8y$ where $y = 2,\ \ y' = -8$ when $x = \frac{\pi}{4}$
Put $p = y' \longrightarrow p' = y''$

$$y'' = \tfrac{dp}{dx} = \tfrac{dy}{dx}\tfrac{dp}{dy} = p\tfrac{dp}{dy}$$

$$\begin{aligned}
p\tfrac{dp}{dy} &= 2y^3 + 8y \text{ which is variable separable} \\
\tfrac{1}{2}p^2 &= \tfrac{1}{2}\left(\tfrac{dy}{dx}\right)^2 = \tfrac{y^4}{2} + 4y^2 + c \\
y &= 2,\ \ y' = -8 \Longrightarrow c = 8 \\
\tfrac{dy}{dx} &= \sqrt{y^4 + 8y^2 + 16} = \sqrt{\left(y^2+4\right)^2} = -\left(y^2+4\right)
\end{aligned}$$

we have taken the negative sign to satisfy the IC $y'(2) = -8$

$$\begin{aligned}
\int dx &= -\int \tfrac{dy}{(y^2+4)} \\
x &= -\tfrac{1}{2}\arctan\left(y/2\right) + d
\end{aligned}$$

using the IC $y\left(\frac{\pi}{4}\right) = 2$ gives $d = 3\pi/8$, so the PS becomes

$$y = 2\tan\left(\frac{3\pi}{4} - 2x\right)$$

b. $\dfrac{d^2y}{dx^2} + 2x\left(\dfrac{dy}{dx}\right)^2 = 0$ where $y = 0,\ \ y' = 1$ when $x = 0.$

$$p = y';\ p' = y''$$

the ODE becomes $\frac{dp}{dx} = -2xp^2 \longrightarrow \int p^{-2}dp = -2\int x dx$

$$\tfrac{1}{p} = x^2 + c : y' = 1,\ x = 0 \Longrightarrow c = 1$$

$$\frac{1}{dy/dx} = x^2 + 1 \longrightarrow \frac{dy}{dx} = \frac{1}{x^2+1} \longrightarrow \int dy = \int \tfrac{dx}{x^2+1}$$

$$y = \arctan x + d : y(0) = 0 \Longrightarrow d = 0$$

therefore the PS is $y = \arctan x$

4. For each of the following constant coefficient differential equations,

$$y'' + by' + cy = g(x)$$

find the complimentary function and state which function you would use to try and find a Particular Solution by the method of undetermined coefficients.

a. $b = 3, \quad c = 2, \quad g(x) = e^{5x}$ Ans: C.F: $y = Ae^{-2x} + Be^{-x}$ PS $y = Ce^{5x}$.

b. $b = 1, \quad c = -6, \quad g(x) = 2e^{2x} + \sin 3x$ Ans: C.F: $y = Ae^{-3x} + Be^{2x}$ PS: $y_1 = Cxe^{2x}$, because 2 is a root of the A.E $y_2 = (D \sin 3x + E \cos 3x)$.

c. $b = 7, \quad c = 0, \quad g(x) = 4x^2 + x + 2$ Ans: C.F: $y = A + Be^{-7x}$ PS $y = \left(p_2 x^2 + p_1 x + p_0\right) x$ because 0 is a root of the A.E.

d. $b = 1, \ c = 1, \ g(x) = 2e^{-x}$ Ans: C.F: $y = e^{-x/2}\left(A \sin \frac{\sqrt{3}}{2} x + B \cos \frac{\sqrt{3}}{2} x\right)$ PS $y = Ce^{-x}$.

e. $b = 4, \quad c = 4, \quad g(x) = 3e^{-2x} + 2e^{3x} + \sin x$ Ans: C.F: $y = e^{-2x}(A + Bx)$ PS $y_1 = Cx^2 e^{-2x}$ because -2 is a two fold root of the A.E, $y_2 = De^{3x}, \quad y_3 = (E \sin x + F \cos x)$.

5. By converting the Euler equation

$$x^2 y''(x) - 2xy'(x) + 2y(x) = 4x^3$$

to a constant coefficient problem show that the solution is given by

$$y(x) = Ax + Bx^2 + 2x^3.$$

The change of variable is $t = \log x$, with the derivatives represented as

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dt} \\ \frac{d^2y}{dx^2} &= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right). \end{aligned}$$

The ODE becomes

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{3t}$$

A.E $\lambda^2 - 3\lambda + 2 = 0 \longrightarrow y_c = Ae^t + Be^{2t}$.

For the PI look for a solution of the form $y_p = Ce^{3t}$: substitute in ODE

$$(9C - 9C + 2C)\, e^{3t} = 4e^{3t} \Longrightarrow C = 2$$

General Solution $y(t) = Ae^t + Be^{2t} + 2e^{3t} \longrightarrow$

$$y(x) = Ae^x + Be^{2x} + 2x^3$$