CQF Module 2
An Introduction to Portfolio Theory
Solutions

1. Denote by w_A^G and w_B^G the weights of the global minimum variance portfolio invested respectively in assets A and B.

   (a) The standard deviation of the portfolio return is given (see slide 22) by

   $$\sigma_\Pi(w_A, w_B) = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho_{AB}w_A w_B \sigma_A\sigma_B} \tag{1}$$

   The budget equation $w_A + w_B = 1$ tells us that the investor's wealth must be fully invested in the portfolio. This equation implies that $w_B = 1 - w_A$. Substituting in equation (1), we can now express the standard deviation of the portfolio return as a sole function of w_A:

   $$\sigma_\Pi(w_A) = \sqrt{w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2\rho_{AB}w_A(1 - w_A)\sigma_A\sigma_B}$$

   Developing and factoring, this expression yields

   $$\sigma_\Pi(w_A) = \sqrt{w_A^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2w_A\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2}$$

   From this relation, we deduce an equation for the **variance** $\sigma_\Pi^2(w_A)$ of the portfolio return as a sole function of w_A:

   $$\sigma_\Pi^2(w_A) = w_A^2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2w_A\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2$$

   and we note that $\sigma_\Pi^2(w_A)$ is a quadratic function of w_A.

   (b) To derive the weight w_A^G of the global minimum variance portfolio invested in A, we need to solve the unconstrained optimization problem

   $$\min_{w_A} \sigma_\Pi^2(w_A) \tag{2}$$

   Differentiating the objective function $\sigma_\Pi^2(w_A)$ yields the first order (necessary) condition

   $$\left.\frac{d\sigma_\Pi^2(w_A)}{dw_A}\right|_{w_A^G} = 0$$

   which implies that

   $$2w_A^G(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2\sigma_B(\rho_{AB}\sigma_A - \sigma_B) = 0$$

   and in turns results in the candidate solution

   $$w_A^G = \frac{\sigma_B(\sigma_B - \rho_{AB}\sigma_A)}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B} \tag{3}$$

Before concluding, we need to check that the candidate solution w_A^G defined in equation (3) actually yields a minimum point for the function σ_Π^2. By the second order (sufficient) condition we need to have

$$\left.\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\right|_{w_A^G} > 0$$

for a minimum to be reached at w_A^G

Here,

$$\left.\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\right|_{w_A^G} = 2(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)$$

Because $-1 \leq \rho_{AB} \leq 1$, we observe that

$$2(\sigma_A - \sigma_B)^2 \leq \left.\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\right|_{w_A^G} \leq 2(\sigma_A + \sigma_B)^2$$

which implies that $\left.\frac{d^2\sigma_\Pi^2(w_A)}{dw_A^2}\right|_{w_A^G} > 0$ as long as either

- $\sigma_A \neq \sigma_B$, or;
- $\rho_{AB} > -1$

2. Denote by w_A^t and w_B^t the weights of the tangency portfolio invested respectively in assets A and B.

(a) The slope S of the tangency line is equal to the Sharpe ratio:

$$S = \frac{\mu_t - r_f}{\sigma_t} \tag{4}$$

where r_f is the risk-free return and the return μ_t of the tangency portfolio and the standard deviation σ_t of the tangency portfolio are respectively given by

$$\mu_t = w_A^t\mu_A + w_B^t\mu_B$$

and

$$\sigma_t = \sqrt{(w_A^t)^2\sigma_A^2 + (w_B^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(w_B^t)\sigma_A\sigma_B}$$

Substituting into equation (5), we find a functional form $S(w_A^t, w_B^t)$ for the slope of the tangency line:

$$S(w_A^t, w_B^t) = \frac{w_A^t\mu_A + w_B^t\mu_B - r_f}{\sqrt{(w_A^t)^2\sigma_A^2 + (w_B^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(w_B^t)\sigma_A\sigma_B}} \tag{5}$$

(b) Because the tangency portfolio is fully invested in risky assets, the budget equation $w_A^t + w_B^t = 1$ applies. Substituting the budget equation into equation (5), we can express the slope of the tangency line as a sole function $S(w_A^t)$ of the weight w_A^t invested in asset A:

$$\begin{aligned} S(w_A^t) &= \frac{w_A^t\mu_A + (1-w_A^t)\mu_B - r_f}{\sqrt{(w_A^t)^2\sigma_A^2 + (1-w_A^t)^2\sigma_B^2 + 2\rho_{AB}(w_A^t)(1-w_A^t)\sigma_A\sigma_B}} \\ &= \frac{w_A^t(\mu_A-\mu_B) + \mu_B - r_f}{\sqrt{(w_A^t)^2(\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A - \sigma_B) + \sigma_B^2}} \end{aligned} \tag{6}$$

(c) As long as $\mu_B > r_f$ or $\mu_A > r_f$, the slope of the tangency line will be positive. In this case, rather than finding w_A^t as the maximizer of $S(w_A^t)$, we could instead find w_A^t as the maximizer of $S^2(w_A^t)$ by solving

$$\max_{w_A^t} S^2(w_A^t)$$

with

$$S^2(w_A^t) = \frac{\left(w_A^t(\mu_A-\mu_B)+\mu_B-r_f\right)^2}{(w_A^t)^2(\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A-\sigma_B)+\sigma_B^2}$$

Denote by

$$f(w_A^t) := \left(w_A^t(\mu_A-\mu_B)+\mu_B-r_f\right)^2$$

and by

$$g(w_A^t) := (w_A^t)^2(\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B) + 2(w_A^t)\sigma_B(\rho_{AB}\sigma_A-\sigma_B)+\sigma_B^2 = \sigma_t^2(w_A)$$

so that

$$S^2(w_A^t) = \frac{f(w_A)}{g(w_A)}$$

Considering the first order (necessary) condition associated with this optimization problem, we are looking for w_A^t such that

$$\frac{dS^2(w_A^t)}{dw_A^t} = 0$$

i.e. such that

$$\frac{dS^2(w_A^t)}{dw_A^t} = \frac{f'(w_A)g(w_A) - f(w_A)g'(w_A)}{g^2(w_A)} = 0$$

where

$$\begin{aligned} f'(w_A^t) &= \frac{df(w_A^t)}{dw_A^t} \\ &= 2(\mu_A-\mu_B)\left(w_A^t(\mu_A-\mu_B)+\mu_B-r_f\right) \end{aligned}$$

and

$$
\begin{aligned}
g'(w_A) &= \frac{dg(w_A^t)}{dw_A^t} \\
&= 2(w_A^t)(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) + 2\sigma_B(\rho_{AB}\sigma_A - \sigma_B)
\end{aligned}
$$

After some rather tedious calculations (substituting, simplifying as much as possible and concentrating exclusively on the numerator since the denominator is positive), we finally get

$$
w_A^t := \frac{\sigma_B\left((\mu_B - r_f)\rho_{AB}\sigma_A - (\mu_A - r_f)\sigma_B\right)}{-(\mu_a - r_f)\sigma_B^2 - (\mu_B - r_f)\sigma_A^2 + (\mu_A + \mu_B - 2r_f)\rho_{AB}\sigma_A\sigma_B} \tag{7}
$$

Checking the second order (sufficient) condition for a maximization, that is

$$
\frac{dS^2(w_A^t)}{d{w_A^t}^2} < 0
$$

is equally unpleasant.

We will develop a much more efficient approach to this problem using matrix algebra in the next lecture on "Fundamentals of Optimization and Application to Portfolio Selection".

3. In the Market Model, we relate the return on an asset to the return on a representative index, M. We write the return on the i $^{\text{th}}$ asset as

$$
R_i = \alpha_i + \beta_i R_M + \epsilon_i.
$$

α is a measure of the return of the stock independent of the market as a whole. β is the ratio of the expected risk premium of the asset to the expected risk premium of the market. It is a measure of the asset's sensitivity to changes in the market index.

We use historical data to measure these parameters, and perform a linear regression on a plot of R_i's and R_M to find β. Since the β terms have a tendency to revert to their means, high β's will over-predict (and low β's will under-predict) future true β's. An adjustment must be made to account for this mean reversion. It is also important to account for the changing underlying characteristics of the company that issued the stock.