Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}.\,(x - x_0)^n$$

geometric series

$$\sum_{k=0}^{\infty} ar^k \text{→generator term}$$
$$\frac{a}{1-r} \text{ for |r|<1}$$

Euler's number

$$e = \lim_{\delta->0} (1+\delta)^{1/\delta}$$

Think of this as the rate for constant growth. See further reading.

Eulers Identity

$$e^{j\theta} = cos(\theta) + jsin(\theta)$$

Continuous

$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

discrete

$$e^{j(\omega nT+\phi)} = \cos(\omega nT + \phi) + j\sin(\omega nT + \phi)$$

complex sinusoid:

$$e^{j(\omega * n * T + \phi)}$$

proof:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \ldots \text{ using taylor series}$$

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2} + \frac{(j\theta)^3}{3!} + \ldots \\ &= 1 + j\theta - \frac{\theta^2}{2} - \frac{j\theta^3}{3!} + \ldots \end{aligned}$$

$$re\{e^{j\theta}\} = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!}$$
$$im\{e^{j\theta}\} = j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{5!}$$

$$\left.\frac{d^n}{d\theta^n}\cos(\theta)\right|_{\theta=0} = \begin{cases} (-1)^{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$\left.\frac{d^n}{d\theta^n}\sin(\theta)\right|_{\theta=0} = \begin{cases} (-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

Plugging into the general Maclaurin series gives

$$\begin{aligned} \cos(\theta) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\theta^n \\ &= \sum_{\substack{n\geq 0 \\ n \text{ even}}}^{\infty} \frac{(-1)^{n/2}}{n!}\theta^n \\ \sin(\theta) &= \sum_{\substack{n\geq 0 \\ n \text{ odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n!}\theta^n. \end{aligned}$$

Separating the Maclaurin expansion for $e^{j\theta}$ into its even and odd terms (real and imaginary parts) gives

$$\begin{aligned} e^{j\theta} \triangleq \sum_{n=0}^{\infty} \frac{(j\theta)^n}{n!} &= \sum_{\substack{n\geq 0 \\ n \text{ even}}}^{\infty} \frac{(-1)^{n/2}}{n!}\theta^n + j\sum_{\substack{n\geq 0 \\ n \text{ odd}}}^{\infty} \frac{(-1)^{(n-1)/2}}{n!}\theta^n \\ &= \cos(\theta) + j\sin(\theta) \end{aligned}$$

Mth Roots

Any real or complex number z can be represented with $re^{j\theta}$, which can represent any point in a 2D space. Here, theta would be $\tan^{-1}\left(\frac{a}{b}\right)$ where a and b are the real and imaginary coordinates.

$$1 = e^{2j\pi k} = \cos 2\pi\theta + j * 0$$
$$z = re^{j\theta}e^{2j\pi k}$$
$$z^{1/M} = r^{\left(\frac{1}{M}\right)}e^{j\frac{\theta}{M}}e^{\left(\frac{2\pi jk}{M}\right)}$$

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The formula above  has M unique answers for every positive integer k<M.
$r^\left( \frac{1}{M} \right)$ will grow into $r$ and $(e^\frac{j\theta+2\pi
k}{M})^M$ will rotate to the correct angle for every integer value of k.
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Roots of Unity

$$1^{k/M} = e^{2\pi j \frac{k}{M}}$$

These are M equally spaced values on the unit circle. Different k values correspond to the complex sinusoids that are used in DFT for analysis of different frequencies

De Moivre Theorem

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

Easy to prove using euler's identity.
This establishes that integer powers (only integer powers?) of $\cos(x) + i\sin(x)$ line up on the unit circle.

complex sinusoids

This is $Ae^{j(\omega t+\phi)}$ for continuous and $Ae^{j(\omega nT+\phi)}$ for discrete cases.

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2}$$

The above equations show that $\cos$ and $\sin$ are less fundimental than $e^{j\theta}$. Note that using real linear operations on complex sinusoids will treat the real and imaginary parts independently. Another takeaway is that a real sinusoid is the equal combination of a positive and negative frequency components. (this comes up when taking fourier transforms?)

Phasor

$\mathcal{A} \triangleq Ae^{j\phi}$ is the complex amplitude/phase of the complex sinusoids. It will off set the phase by ϕ and change amplitude from 1 to A. This is also known as the phasor of the sinusoid.

General LTI filter effects

in general, an LTI filter can only chnage the amplitudes and phases of the frequencies in a signal. Any LTI filter is completely characterized by it's relative gain $\frac{A_1}{A_2}$ and phase $\phi_1 - \phi_2$

change at each frequency

$$
\begin{aligned}
y(n) &= (\mathit{Complex\ Filter\ Gain})\ \mathit{times}\ (\mathit{Input\ Circular\ Motion} \\
&\quad \mathit{with\ Radius}\ A,\ \mathit{Phase}\ \phi) \\
&= \left[G(\omega)e^{j\Theta(\omega)}\right]\left[Ae^{j(\omega nT+\phi)}\right] \\
&= [G(\omega)A]\, e^{j[\omega nT+\phi+\Theta(\omega)]} \\
&= \mathit{Circular\ Motion\ with\ Radius}\ [G(\omega)A]\ \mathit{and\ Phase}\ [\phi+\Theta(\omega)].
\end{aligned}
$$

further reading

https://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/
https://www.dsprelated.com/freebooks/mdft/Sinusoids_Exponentials.html