In previous chapters we saw that the transfer function for the recursive LTI filter is:

$$H(z) = \frac{g(1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_M z^{-M})}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots + \alpha_N z^{-N}}$$

We know that the polynomials in the numerator and denominator can be factored, giving the form:

$$H(z) = \frac{g(1 - q_1 z^{-1})(1 - q_2 z^{-1})\dots(1 - q_M z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\dots(1 - p_N z^{-1})}$$

Assuming none of the terms in the numerator and denominator cancel out, poles are points where $z$, a complex number, is equal to any of the $p_i$ values. Zeroes are when $z$ is equal to the $q_i$ values. We can think of $z$ values lying on a 2D plane, and the $H$ magnitude, which is a real number being the 3rd dimension that lies above the z-plane. $q_i$ and $p_i$ values would be points where the $H$ magnitude is either infinitely small (zero) or large (resembling a pole).

Also note that the filter order is either $M$ or $N$, or the number of poles or zeros, whichever is greater. **So the order of an LTI filter is the order of its transfer function.**

# Graphical Computation of Amplitude Response

If we take the factored form of the transfer function and set $z$ to $e^{jwt}$, we get the frequency response in factored form:

$$H(z) = \frac{g(1 - q_1 e^{-jwt})(1 - q_2 e^{-jwt})\dots(1 - q_m e^{-jwt})}{(1 - p_1 e^{-jwt})(1 - p_2 e^{-jwt})\dots(1 - p_N e^{-jwt})}$$

The amplitude response $G(w) = |H(e^{jwt})|$ ends up being:

$$G(w) = |g| \frac{|e^{jwt} - q_1||e^{jwt} - q_2|\dots|e^{jwt} - q_M|}{|e^{jwt} - p_1|\dots|e^{jwt} - p_M|}$$

proof: https://ccrma.stanford.edu/~jos/filters/Graphical_Amplitude_Response.html

Consider that the magnitude of vector $u - v$ is a point drawn from the tip of $v$ to $u$.
This leads to the following conclusion:
> The frequency response magnitude (amplitude response) at frequency $w$ is given by the product of lengths of vectors drawn from the zeros to the point $e^{jwt}$ divided by the product of vectors drawn from the poles to the point $e^{jwt}$.

# Computation of Phase Response

$$\Theta(\omega) \triangleq \angle g + (N-M)\omega T + \angle(e^{jwT} - q_1) + \angle(e^{jwT} - q_2) + \cdots + \angle(e^{jwT} - q_M)$$
$$-\angle(e^{jwT} - p_1) - \angle(e^{jwT} - P_2) - \ldots \angle(e^{jwT} - p_N)$$

Proof:https://ccrma.stanford.edu/~jos/filters/Graphical_Phase_Response_Calculation.html

The additional term $(N-M)\omega T$ arises when we consider a $z^{N-M}$ term in the transfer function, with poles/zeros at $z = 0$.

$$H(z) = gz^{N-M}\frac{(z-q_1)(z-q_2)\ldots(z-q_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

## Question:

Can you write a simple function to calculate the phase response graph here?
https://ccrma.stanford.edu/~jos/filters/Graphical_Phase_Response_Calculation.html

# Further reading

analytic function: https://mathworld.wolfram.com/AnalyticFunction.html
Conformal mapping: https://mathworld.wolfram.com/ConformalMapping.html