

The frequency response yields phase and gain shift as a function of frequency.
We have seen that

$$y(z) = H(z)X(z)$$

If we set $z = e^{j\omega T}$ where ω is a real radian frequency and T is $\frac{1}{f_s}$ where f_s is the sampling rate, we get the *DTFT* or discrete time fourier transform.

The bilateral *DTFT* where T is normalized to 1 is

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega Tn}$$

Where x is causal, we get the unilateral DTFT

$$\text{DTFT}_w(x) = \sum_{n=0}^{\infty} x(n)e^{-j\omega n}$$

So the spectrum of the output is equal to the spectrum of the input times the spectrum of the impulse response. So

The frequency response of an LTI is equal to the transfer function evaluated on the unit circle in the z plane i.e $H(e^{j\omega T})$

This means that the frequency response is the DTFT of the impulse response

$$H(e^{j\omega T}) = \text{DTFT}_{\omega T}(h)$$

We typically restrict ω to $[-\pi, \pi]$

The frequency response can also be represented in polar form as:

$$H(e^{j\omega T}) = G(\omega)e^{j\phi(\omega)}$$

Where $G(\omega) \triangleq |H(e^{j\omega T})|$ is the magnitude and $\phi(\omega) \triangleq \angle H(e^{j\omega T})$

This implies:

$$|Y(e^{j\omega T})| = g(\omega)|X(e^{j\omega T})|$$

$$\angle(Y(e^{j\omega T})) = \phi(\omega) + \angle X(e^{j\omega T})$$

$$Y(e^{j\omega T}) = |Y(e^{j\omega T})| \cdot e^{j\angle Y(e^{j\omega T})}$$

if $a = \text{re}\{H(e^{j\omega T})\}$ and $b = \text{im}\{H(e^{j\omega T})\}$ then

$$G(\omega) = \sqrt{a^2 + b^2}$$

$$\phi(\omega) = \tan^{-1} \left(\frac{b}{a} \right)$$

$$a = G(\omega) \cos(\phi(\omega))$$

$$b = G(\omega) \sin(\phi(\omega))$$

Seperating numerator and denominator

$$H(z) = \frac{B(z)}{A(z)}$$

Gives:

$$g(\omega) = \left| \frac{B(e^{j\omega T})}{A(e^{j\omega T})} \right|$$

$$\phi(\omega) = \angle B(e^{j\omega T}) - \angle A(e^{j\omega T})$$

DFT Definition

$$DFT_{\omega_k} T(x) = \sum_{n=0}^{N_s-1} x(n) e^{-j\omega_k n T}$$

Where $\omega \triangleq \frac{2\pi f_s k}{N_s}$ and $f_s = \frac{1}{T}$

This might? reduce to:

$$DFT_{\omega_k} = \sum_{n=0}^{N_s-1} x(n) e^{-j2\pi k n / N_s} \quad k = 0, 1, 2, \dots, N_s - 1$$

The inverse is defined as:

$$h(n) = IDFT_n(H) \triangleq \frac{1}{N_s} \sum_{k=0}^{N_s-1} H(k) e^{j2\pi k n / N_s}$$

The choice of N_s is typically $\frac{N_s > 7}{(1-R_{max})}$, see

https://ccrma.stanford.edu/~jos/filters/Frequency_Response_Matlab.html

Phase Delay

The phase delay $\phi(\omega)$ gives the radian phase shift added to the phase of each sinusoidal component of the input signal. $P(\omega)$ is defined as

$$P(\omega) \triangleq -\frac{\phi(\omega)}{\omega}$$

which gives the time delay in seconds for each sinusoidal component.

If an input $x(n) = \cos(\omega nT)$ is given to $H(e^{j\phi T}) = G(\omega)e^{-j\phi(\omega)}$

The output is

$$Y(n) = G(\omega) \cos(\omega(nT - p(\omega)))$$

Group Delay

$$D(\omega) = -\frac{d(\phi(\omega))}{d(\omega)}$$

This would be identical to phase delay when if the phase response is linear.

The "group" in the name refers to the fact that it specifies the delay experienced by a narrow band of sinusoidal components around the frequency ω where the phase response is linear.

Derivation (wtf?):

https://ccrma.stanford.edu/~jos/filters/Derivation_Group_Delay_Modulation.html