The frequency response yields phase and gain shift as a function of frequency.
We have seen that

$$y(z) = H(z)X(z)$$

If we set $z = e^{j\omega T}$ where ω is a real radian frequency and T is $\frac{1}{f_s}$ where f_s is the sampling rate, we get the *DTFT* or discrete time fourier transform.

The bilateral *DTFT* where T is normalized to 1 is

$$X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega Tn}$$

Where x is causal, we get the unilateral DTFT

$$\mathrm{DTFT}_w(x) = \sum_{n=0}^{\infty} x(n)e^{-jwn}$$

So the spectrum of the output is equal to the spectrum of the input times the spectrum of the impulse response. So

> The frequency response of an LTI is equal to the transfer function evaluated on the unit circle in the z plane i.e $H(e^{jwT})$
> This means that the frequency reponse is the DTFT of the fourier transform

$$H(e^{jwT}) = DTFT_{\omega T}(h)$$

We typically restrict ω to $[-\pi, \pi]$

The frequency response can also be represented in polar form as:

$$H(e^{j\omega T} = G(\omega)e^{j\phi(\omega)})$$

Where $G(\omega) \triangleq |H(e^{-j\omega T})$ is the magnitude and $\phi(\omega) \triangleq \angle H(e^{-j\omega T})$

This implies:

$$|Y(e^{jwT})| = g(\omega)|X(e^{jwT})|$$

$$\angle(Y(e^{jwT}) = \phi(\omega) + \angle X(e^{jwT})$$

$$Y(e^{j\omega T}) = |Y(e^{j\omega T})| \cdot e^{\angle Y(e^{j\omega T})}$$

if $a = re\{H(e^{jwt})\}$ and $b = im\{H(e^{jwt})\}$ then

$$G(\omega) = \sqrt{a^2 + b^2}$$
$$\phi(\omega) = \tan^{-1}\left(\frac{b}{a}\right)$$
$$a = G(\omega)\cos(\phi(\omega))$$
$$b = G(\omega)\sin(\phi(\omega))$$

Seperating numerator and denominator

$$H(z) = \frac{B(z)}{A(z)}$$

Gives:

$$g(\omega) = |\frac{B(e^{j\omega T})}{A(e^{j\omega T})}|$$

$$\phi(\omega) = \angle B(e^{j\omega T}) - \angle A(e^{jwt})$$

DFT Definition

$$DFT_{\omega_k}T(x) = \sum_{n=0}^{N_s-1} x(n)e^{-j\omega_k nT}$$

Where $\omega \triangleq \frac{2\pi f_s k}{N_s}$ and $f_s = \frac{1}{T}$

This might? reduce to:

$$DFT_{wk} = \sum_{n=0}^{N_s-1} x(n)e^{-j2\pi k/N_s} \quad k = 0, 1, 2, \ldots N_s - 1$$

The inverse is defined as:

$$h(n) = IDFT_n(H) \triangleq \frac{1}{N_s}\sum_{k=0}^{N_s-1} H(k)e^{j2\pi k/N_s}$$

The choice of N_s is typically $\frac{N_s > 7}{(1 - R_{max})}$, see https://ccrma.stanford.edu/~jos/filters/Frequency_Response_Matlab.html

Phase Delay

The phase delay $\phi(\omega)$ gives the radian phase shift added to the phase of each sinusoidal component of the input signal. $P(\omega)$ is defined as

$$P(\omega) \triangleq -\frac{\phi(\omega)}{\omega}$$

which gives the time delay in seconds for each sinusoidal component.
If an input $x(n) = \cos(\omega nT)$ is given to $H(e^{j\phi T}) = G(\omega)e^{-j\phi(\omega)}$
The output is

$$Y(n) = G(\omega)\cos(\omega(nT - p(\omega)))$$

Group Delay

$$D(\omega) = -\frac{d(\phi(\omega))}{d(\omega)}$$

This would be identical to phase delay when if the phase response is linear.
The "group" in the name refers to the fact that it specifies the delay experienced by a narrow band of sinusoidal components around the frequency ω where the phase response is linear.

Derivation: https://ccrma.stanford.edu/~jos/filters/Derivation_Group_Delay_Modulation.html