

## Definition of z-transform

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad \text{unilateral, multilateral if starting with } -\infty$$

$$X(z) = Z_z\{x(\cdot)\}$$

Alternatively

$$X = Z\{x\}$$

### Example:

if  $x(n) = n + 1$  for  $0 \leq n \leq 2$  then  $X(z) = 1 + 2z^{-1} + 3z^{-2}$

## Shift Theorem

Delay of  $\Delta$  in the time domain corresponds to multiplication by  $z^{-\Delta}$  in the frequency domain.

**Intuition:** when  $x(n) = c$  shifted to the right (or delayed) by  $\Delta$ , all the  $c$  values will be multiplied by an extra  $z^{-\Delta}$ . Here, the **causality** assumption is used. A causal signal is one that is 0 prior to time 0, so when the signal is delayed by  $\Delta$ , the new values inserted are 0.

proof: [https://ccrma.stanford.edu/~jos/filters/Shift\\_Theorem.html](https://ccrma.stanford.edu/~jos/filters/Shift_Theorem.html)

## Convolution theorem

convolution in the time domain is equal to multiplication in the frequency domain.

$$x * y \leftrightarrow X \cdot Y$$

Using Operator notation

$$Z_z\{x * y\} = X(z) \cdot Y(z)$$

Proof: [https://ccrma.stanford.edu/~jos/filters/Convolution\\_Theorem.html](https://ccrma.stanford.edu/~jos/filters/Convolution_Theorem.html)

## Z-Transform of Convolution

The transfer function of a linear time-invariant discrete-time filter is  $H(z) = \frac{Y(z)}{X(z)}$ . Where  $H(z)$  is the z-transfer of the impulse response  $h(n)$  and  $Y, X$  are the z-transfers of  $x(n)$  and  $y(n)$ . This is because  $y(n) = (h * x)(n)$  as we've seen before, and the Z function can be applied to both sides.

## Z-Transform of General Difference Equation

Applying the Z-transform to both sides of the general difference equation gives us the formula:

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 Z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \triangleq \frac{B(z)}{A(z)}$$

Proof: [https://ccrma.stanford.edu/~jos/filters/Z\\_Transform\\_Difference\\_Equations.html](https://ccrma.stanford.edu/~jos/filters/Z_Transform_Difference_Equations.html)

## Factored Form:

Some background covered in Chapter 7 Pole zero analysis, see

[https://ccrma.stanford.edu/~jos/filters/Pole\\_Zero\\_Analysis\\_I.html](https://ccrma.stanford.edu/~jos/filters/Pole_Zero_Analysis_I.html)

## Series and parallel Transfer Functions

1. Transfer function of filters in series multiple: [proof](#)
2. Transfer function of filters in parallel sum [proof](#)

Remember that:

$$y(n) = (h * x)(n)$$

and

$$Y(z) = H(z)X(z)$$

So if a signal  $x$  is being processed in a series with  $H_1$  and  $H_2$  then  $X$  will be multiplied by  $H_z = H_1 \cdot H_2$ . If  $X$  is being processed in parallel then a copy of  $X$  is multiplied individually by each transfer function and summed.

Note that the above suggests that the ordering of filters is commutative since multiplication of the filters is commutative.

## Partial Fraction Expansion

If  $M < N$  (strictly proper transfer function), we can describe the transfer function as:

$$H(z) = \frac{B(z)}{A(z)} = \sum_{i=0}^N \frac{r_i}{1 - p_i z^{-1}}$$

Where  $B(z)$  and  $A(z)$  are the coefficients described in [earlier section](#Z-Transform of General Difference Equation). and  $r_i$  is

$$(1 - p_i)H(z)|_{z=p_i}$$

$r_i$ , or the residue, is the coefficient of the one-pole term  $\frac{1}{1 - p_i z^{-1}}$  as the  $z \rightarrow p_i$

> every strictly proper transfer function (with distinct poles) can be implemented using a parallel bank of two-pole, one-zero filter sections.

See example at: <https://ccrma.stanford.edu/~jos/filters/Example.html>

see complex example: [https://ccrma.stanford.edu/~jos/filters/Complex\\_Example.html](https://ccrma.stanford.edu/~jos/filters/Complex_Example.html)

## Inverse Transform

We have

$$H(z) = \sum_{i=0}^N \frac{r_i}{1 - p_i z^{-1}}$$

So each  $H_i(z) = \frac{r_i}{1 - p_i z^{-1}}$

Which is the geometric series

$$\sum_{n=0}^{\infty} r_i p_i^n z^{-n}$$

Which is the z-transform of  $h_i(n) = r_i p_i^n$ .

So the inverse z-transform of  $H(z)$  is  $h(n)$ :

$$h(n) = \sum_{i=1}^N h_i(n) = \sum_{i=1}^N r_i p_i^n, \quad n = 0, 1, 2, \dots$$

Thus, the [impulse response](#) of every strictly proper [LTI filter](#) (with distinct [poles](#)) can be interpreted as a [linear combination](#) of sampled complex [exponentials](#). Recall that a uniformly sampled [exponential](#) is the same thing as a [geometric sequence](#). Thus  $h$  is a linear combination of  $N$  geometric sequences.

## FIR Part of PFE

If  $M > N$ , then we'll have an FIR part to the PFE.

$$H(z) = \frac{B(z)}{A(z)} = F(z) + \sum_{i=0}^N \frac{r_i}{(1 - p_i z^{-1})}$$

Where  $F(z) = f_0 + f_1 z^{-1} + \dots + f_k z^{-k}$  and  $k = M - N$

There are two main approaches for this scenario, see :

[https://ccrma.stanford.edu/~jos/filters/FIR\\_Part\\_PFE.html](https://ccrma.stanford.edu/~jos/filters/FIR_Part_PFE.html)

## Repeated Poles

Assume there are two identical poles in parallel and in a series:

In a series, they multiply, so we will have a two pole filter with a repeated pole.

In parallel, we will have a one pole filter with a new residue value.

# Dealing with Repeated Poles Analytically:

Assuming there is a pole  $p_i$  repeated  $m_i$  times. A pole with multiplicity  $m_i$  has  $m_i$  coefficients.  $r_{ij}$  is the  $j$ th coefficient associated with pole  $p_i$  then each  $r_{ij}$  is defined by: (note that  $j$  starts from 0):

$$r_{ik} = \frac{1}{(k-1)!(-p_i)^{k-1}} \cdot \frac{d^{k-1}}{d(z^{-1})^{k-1}} (1 - p_i z^{-1})^{m_i} H(z) \Big|_{z=p_i}$$

See example: [https://ccrma.stanford.edu/~jos/filters/Dealing\\_Repeated\\_Poles\\_Analytically.html](https://ccrma.stanford.edu/~jos/filters/Dealing_Repeated_Poles_Analytically.html)

## Impulse Response of Repeated Poles

In the time domain, repeated poles give rise to *polynomial [amplitude envelopes]* ([https://en.wikipedia.org/wiki/Envelope\(waves\)](https://en.wikipedia.org/wiki/Envelope(waves)))\_ on the decaying [exponentials](#) corresponding to the (stable) poles. For example, in the case of a single pole repeated twice, we have:

$$\frac{1}{(1 - pz^{-1})^2} = Z \{ (n+1)p^n \} \leftrightarrow (n+1)p^n$$

proof: [https://ccrma.stanford.edu/~jos/filters/Impulse\\_Response\\_Repeated\\_Poles.html](https://ccrma.stanford.edu/~jos/filters/Impulse_Response_Repeated_Poles.html)

Here,  $p^n$  is the decaying exponential and  $n+1$  is the first order polynomial amplitude envelope. For a pole repeated 3 times, we would have a second order (quadratic) amplitude envelope multiplied by an exponential decay ( $p < 1$ ). Exponential decay will overtake polynomial growth eventually so the impulse response always decays to 0.

Repeated poles give rise to polynomial envelopes on the exponential decay due to the poles. Two different poles yield a convolution (or sum) of two different exponential decays with no polynomial envelope.

Looking at the convolution of the impulse responses of two poles,  $h_1 = p_1^n$  and  $h_2 = p_2^n$ , we get:

$$h(n) = (h_1 * h_2)(n) = \sum_{m=0}^n p_1^m \cdot p_2^{n-m} = p_2^n \sum_{m=0}^n \left( \frac{p_1^m}{p_2^m} \right)$$

if  $p_1 = p_2 = p$  then this yields  $h(n) = p^n \cdot (n+1)$

Else, we [continue the simplification](#) to get:

$$h(n) = \frac{p_1^{n+1} - p_2^{n+1}}{(p_1 - p_2)}$$

## Important examples:

see bottom of [https://ccrma.stanford.edu/~jos/filters/Example\\_General\\_Biquad\\_PFE.html](https://ccrma.stanford.edu/~jos/filters/Example_General_Biquad_PFE.html)

Especially state space filters: [https://ccrma.stanford.edu/~jos/filters/State\\_Space\\_Filters.html](https://ccrma.stanford.edu/~jos/filters/State_Space_Filters.html)

Summary:

[https://ccrma.stanford.edu/~jos/filters/Summary\\_Partial\\_Fraction\\_Expansion.html](https://ccrma.stanford.edu/~jos/filters/Summary_Partial_Fraction_Expansion.html)

## Further Reading

[https://en.wikipedia.org/wiki/Fundamental\\_theorem\\_of\\_algebra](https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra)

Circuits:

- <https://learn.sparkfun.com/tutorials/what-is-a-circuit/all>

Envelopes: [https://en.wikipedia.org/wiki/Envelope\\_\(waves\)](https://en.wikipedia.org/wiki/Envelope_(waves)).

## Questions

Prove the PFE formula mathematically.