Definition of z-transform

$$X(z) \triangleq \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{unilateral, multilateral if starting with } -\infty$$

$$X(z) = Z_z\{x(.)\}$$

$$\text{Alternatively}$$

$$X = Z\{x\}$$

Example:

if $x(n) = n + 1$ for $0 \leq n \leq_2$ then $X(z) = 1 + 2z^{-1} + 3z^{-2}$

Shift Theorem

Delay of Δ in the time domain corresponds to multiplication by z^{Δ} in the frequency domain. **Intuition**: when $x(n) = c$ shifted to the right (or delayed) by Δ, all the c values will be multiplied by z^{Δ} instead of $z^{n-\Delta}$. Here, the **causality** assumption is used. A causal signal is one that is 0 prior to time 0, so when the signal is delayed by Δ, the new values inserted are 0.

proof: https://ccrma.stanford.edu/~jos/filters/Shift_Theorem.html

Convolution theorem

convolution in the time domain is equal to multiplication in the frequency domain.

$$x * y \leftrightarrow X \cdot Y$$

Using Operator notation

$$Z_z\{x * y\} = X(z) \cdot Y(z)$$

Proof: https://ccrma.stanford.edu/~jos/filters/Convolution_Theorem.html

Z-Transform of Convolution

The transfer function of a linear time-invariant discrete-time filter is $H(z) = \frac{Y(z)}{X(z)}$. Where $H(z)$ is the z-transfer of the impulse response $h(n)$ and Y,X are the z-transfers of $x(n)$ and $y(n)$. This is because $y(n) = (h * x)(n)$ as we've seen before, and the Z function can be applied to both sides.

Z-Transform of General Difference Equation

Applying the Z-transform to both sides of the general difference equation gives us the formula:

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 Z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_n z^{-N}} \triangleq \frac{B(z)}{A(z)}$$

$$\boxed{H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \triangleq \frac{B(z)}{A(z)}.}$$

Proof: https://ccrma.stanford.edu/~jos/filters/Z_Transform_Difference_Equations.html

Factored Form:

Will be covered in Chapter 7 Pole zero analysis, see
https://ccrma.stanford.edu/~jos/filters/Pole_Zero_Analysis_I.html

Series and parallel Transfer Functions

1. Transfer function of filters in series multiple: proof
2. Transfer function of filters in parallel sum proof
   Remember that:

$$y(n) = (h * x)(n)$$

   and

$$Y(z) = H(z)X(z)$$

   So if a signal x is being processed in a series with H_1 and H_2 then X will be multiplied by $H_z = H_1 \cdot H_2$. If X is being processed in parallel then a copy of X is multiplied individually by each transfer function and summed.

Note that the above suggests that the ordering of filters is commutative since multiplication of the filters is commutative.

Jump to chapter 7 before continuing

Further Reading

https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra
Circuits:
- https://learn.sparkfun.com/tutorials/what-is-a-circuit/all