Definition of z-transform

$$X(z) \triangleq \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{unilateral, multilateral if starting with } -\infty$$

$$X(z) = Z_z\{x(.)\}$$

$$\text{Alternatively}$$

$$X = Z\{x\}$$

Example:

if $x(n) = n + 1$ for $0 \leq n \leq_2$ then $X(z) = 1 + 2z^{-1} + 3z^{-2}$

Shift Theorem

Delay of Δ in the time domain corresponds to multiplication by z^{Δ} in the frequency domain. **Intuition**: when $x(n) = c$ shifted to the right (or delayed) by Δ, all the c values will be multiplied by z^{Δ} instead of $z^{n-\Delta}$. Here, the **causality** assumption is used. A causal signal is one that is 0 prior to time 0, so when the signal is delayed by Δ, the new values inserted are 0.

proof: https://ccrma.stanford.edu/~jos/filters/Shift_Theorem.html

Convolution theorem

convolution in the time domain is equal to multiplication in the frequency domain.

$$x * y \leftrightarrow X \cdot Y$$

Using Operator notation

$$Z_z\{x * y\} = X(z) \cdot Y(z)$$

Proof: https://ccrma.stanford.edu/~jos/filters/Convolution_Theorem.html

Z-Transform of Convolution

The transfer function of a linear time-invariant discrete-time filter is $H(z) = \frac{Y(z)}{X(z)}$. Where $H(z)$ is the z-transfer of the impulse response $h(n)$ and Y,X are the z-transfers of $x(n)$ and $y(n)$. This is because $y(n) = (h * x)(n)$ as we've seen before, and the Z function can be applied to both sides.

Z-Transform of General Difference Equation

Applying the Z-transform to both sides of the general difference equation gives us the formula:

$$H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 Z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_n z^{-N}} \triangleq \frac{B(z)}{A(z)}$$

Proof: https://ccrma.stanford.edu/~jos/filters/Z_Transform_Difference_Equations.html

Factored Form:

Will be covered in Chapter 7 Pole zero analysis, see https://ccrma.stanford.edu/~jos/filters/Pole_Zero_Analysis_I.html

Series and parallel Transfer Functions

1. Transfer function of filters in series multiple: proof
2. Transfer function of filters in parallel sum proof
   Remember that:

   $$y(n) = (h * x)(n)$$

   and

   $$Y(z) = H(z)X(z)$$

   So if a signal x is being processed in a series with H_1 and H_2 then X will be multiplied by $H_z = H_1 \cdot H_2$. If X is being processed in parallel then a copy of X is multiplied individually by each transfer function and summed.

Note that the above suggests that the ordering of filters is commutative since multiplication of the filters is commutative.

Partial Fraction Expansion

If $M < N$ (strictly proper transfer function), we can describe the transfer function as:

$$H(z) = \frac{B(z)}{A(z)} = \sum_{i=0}^{N} \frac{r_i}{1 - p_i z^{-1}}$$

Where $B(z)$ and $A(z)$ are the coefficients described in [earlier section](#Z-Transform of General Difference Equation). and r_i is

$$(1 - p_i)H(z)|_{z=p_i}$$

r_i, or the residue, is the coefficient of the one-pole term $\frac{1}{1-pz^{-1}}$ as the $z \to p_i$

 > every strictly proper transfer function (with distinct poles) can be implemented using a parallel bank of two-pole, one-zero filter sections.

See example at: https://ccrma.stanford.edu/~jos/filters/Example.html
see complex example: https://ccrma.stanford.edu/~jos/filters/Complex_Example.html

Inverse Transform

We have

$$H(z) = \sum_{i=0}^{N} \frac{r_i}{1 - p_i z^{-1}}$$

So each $H_i(z) = \frac{r_i}{1-p_i z^{-1}}$
Which is the geometric series

$$\sum_{n=0}^{\infty} r_i p_i^n z^{-n}$$

Which is the z-transform of $h_i(n) = r_i p_i^n$.

So the inverse z-transform of $H(z)$ is $h(n)$:

$$h(n) = \sum_{i=1}^{N} h_i(n) = \sum_{i=1}^{N} r_i p_i^n, \quad n = 0, 1, 2, \ldots$$

> Thus, the impulse response of every strictly proper LTI filter (with distinct poles) can be interpreted as a *linear combination of sampled complex exponentials*. Recall that a uniformly sampled exponential is the same thing as a *geometric sequence*. Thus h is a linear combination of N geometric sequences.

FIR Part of PFE

If M>N, then we'll have an FIR part to the PFE.

$$H(z) = \frac{B(z)}{A(z)} = F(z) + \sum_{i=0}^{N} \frac{r_i}{(1 - p_i z^{-1})}$$

Where $F(z) = f_0 + f_1 z^{-1} + \cdots + f_2 z^{-k}$ and $k = M - N$

There are two main approaches for this scenario, see :
https://ccrma.stanford.edu/~jos/filters/FIR_Part_PFE.html

Repeated Poles

Assume there are two identical poles in parallel and in a series:
In a series, they multiply, so we will have a two pole filter with a repeated pole.
In parallel, we will have a one pole filter with a new residue value.

Dealing with Repeated Poles Analytically:

Assuming there is a pole p_i repeated m_i times. A pole with multiplicity m_i has m_i coefficients. r_{ij} is the jth coefficient associated with pole p_i then each r_{ij} is defined by: (note that j/k starts from 0):

$$r_{ik} = \frac{1}{(k-1)!(-p_i)^{k-1}} \cdot \frac{d^{k-1}}{d(z^{-1})^{k-1}} (1 - p_i z^{-1})^{mi} H(z)|_{z=p_i}$$

See example: https://ccrma.stanford.edu/~jos/filters/Dealing_Repeated_Poles_Analytically.html

Impulse Response of Repeated Poles

In the time domain, repeated poles give rise to *polynomial [amplitude envelopes] (https://en.wikipedia.org/wiki/Envelope*(waves))_ on the decaying exponentials corresponding to the (stable) poles. For example, in the case of a single pole repeated twice, we have:

$$\frac{1}{(1 - pz^{-1})^2} = \mathcal{Z}\{(n+1)p^n\} \leftrightarrow (n+1)p^n$$

proof: https://ccrma.stanford.edu/~jos/filters/Impulse_Response_Repeated_Poles.html

Here, p^n is the decaying exponential and $n+1$ is the first order polynomial amplitude envelope. For a pole repeated 3 times, we would have a second order (quadratic) amplitude envelope multiplied by an exponential decay ($p < 1$). Exponential decay will overtake polynomial growth eventually so the impulse response always decays to 0.

> Repeated poles give rise to polynomial envelopes on the exponential decay due to the poles. Two different poles yield a convolution (or sum) of two different exponential decays with no polynomial envelope.

Looking at the convolution of the impulse responses of two poles, $h_1 = p_1^n$ and $h_2 = p_2^n$, we get:

$$h(n) = (h_1 * h_2)(n) = \sum_{m=0}^{n} p_1^m \cdot p_2^{n-m} = p_2^n \sum_{m=0}^{n} \left(\frac{p_1^m}{p_2^m}\right)$$

if $p_1 = p_2 = p$ then this yields $h(n) = p^n \cdot (n+1)$
Else, we continue the simplification to get:

$$h(n) = \frac{p_1^{n+1} - p_2^{n+1}}{(p_1 - p_2)}$$

Important examples:

see bottom of https://ccrma.stanford.edu/~jos/filters/Example_General_Biquad_PFE.html
Especially state space filters: https://ccrma.stanford.edu/~jos/filters/State_Space_Filters.html

Further Reading

https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra
Circuits:
- https://learn.sparkfun.com/tutorials/what-is-a-circuit/all
Envelopes: https://en.wikipedia.org/wiki/Envelope_(waves)

Questions

Prove the PFE formula mathematically.