Delaunay triangulations on hyperbolic surfaces

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Astonishing Workshop
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Nancy, France

Outline

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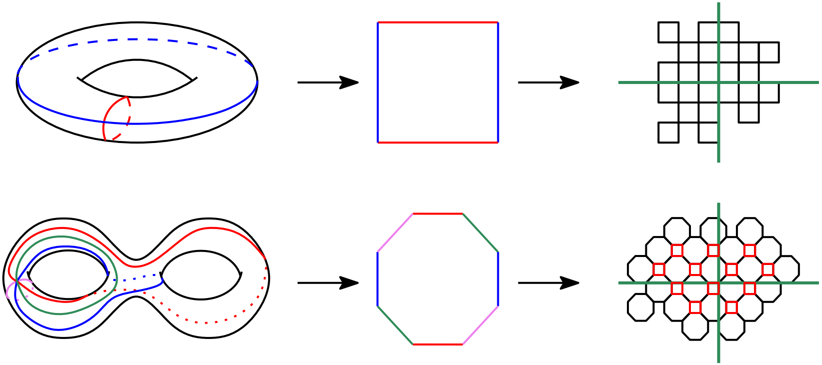
Motivation

Periodic triangulations in the Euclidean plane



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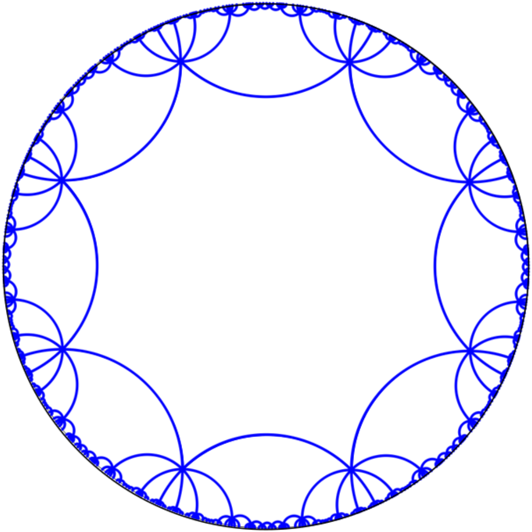
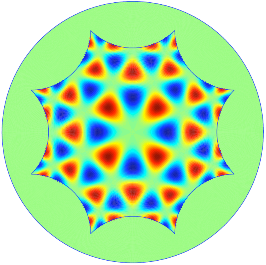


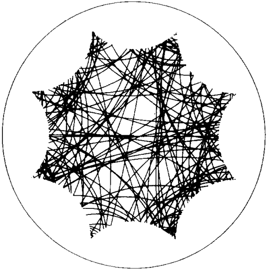
Motivation

Applications

[Sausset, Tarjus, Viot]



[Chossat, Faye, Faugeras]

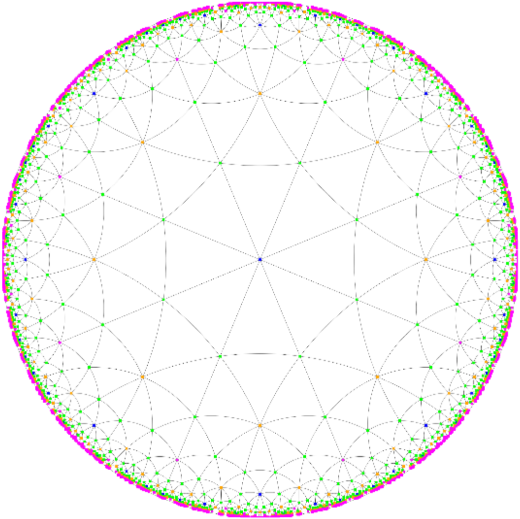


[Balazs, Voros]

Motivation

Beautiful groups

- Fuchsian groups
- finitely presented groups
- triangle groups
- ...



State of the art

Closed Euclidean manifolds

- Algorithms 2D [Mazón, Recio], 3D [Dolbilin, Huson], *d*D [Caroli, Teillaud, DCG'16]
- Software (square/cubic flat torus) 2D [Kruithof], 3D [Caroli, Teillaud]

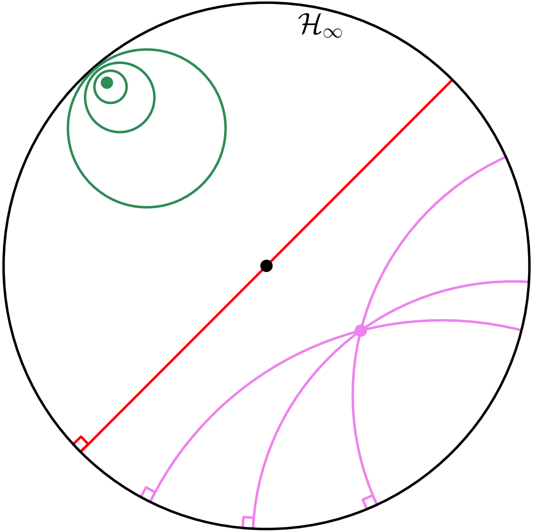
CGAL

Closed hyperbolic manifolds

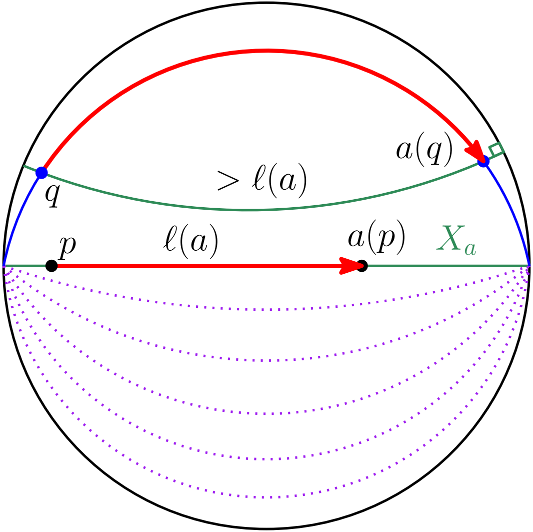
- Algorithms 2D, genus 2 [Bogdanov, Teillaud, Vegter, SoCG'16]
- Software (Bolza surface) [Iordanov, Teillaud, SoCG'17]

CGAL

Poincaré model of the hyperbolic plane $\mathbb{H}^2$



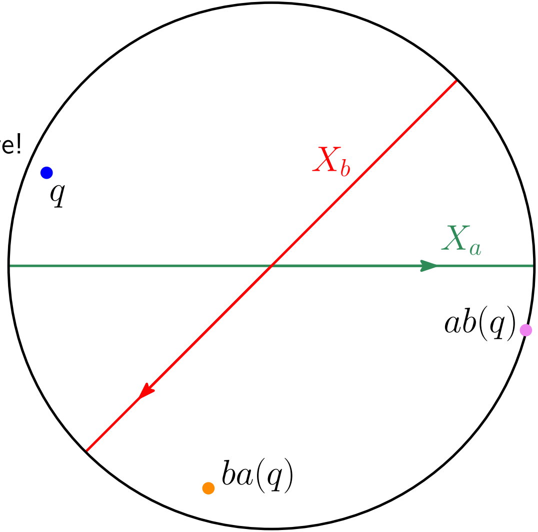
Hyperbolic translations



special case
axis = diameter

Hyperbolic translations

non-commutative!

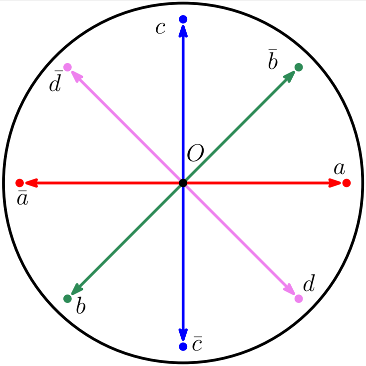


Bolza surface

What is it?

- Closed, compact, orientable surface of genus 2.
- Constant negative curvature $\longrightarrow$ locally hyperbolic metric.
- The most symmetric of all genus-2 surfaces.

Bolza surface



Fuchsian group $\mathcal{G}$ with finite presentation

$$\mathcal{G} = \left\langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \right\rangle$$

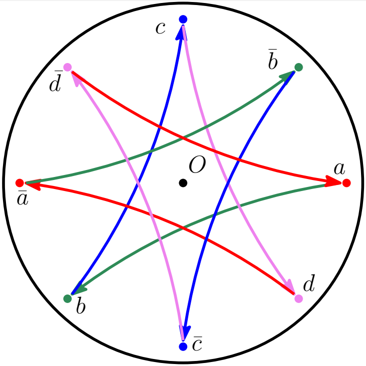
$\mathcal{G}$ contains only translations (and $\mathbb{1}$)

Bolza surface

$$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$$

with projection map $\pi_{\mathcal{M}} : \mathbb{H}^2 \to \mathcal{M}$

Bolza surface



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Bolza surface

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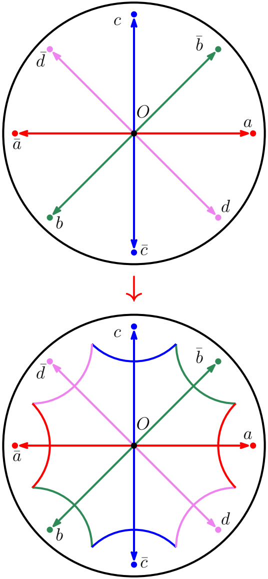
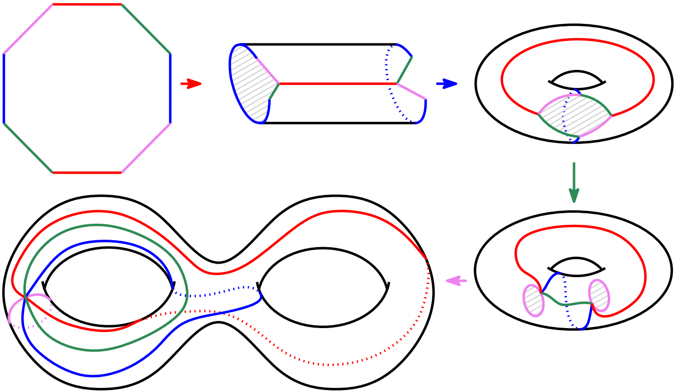
$$\mathcal{A} = \left[a, \overline{b}, c, \overline{d}, \overline{a}, b, \overline{c}, d\right] = [g_0, g_1, ..., g_7]$$

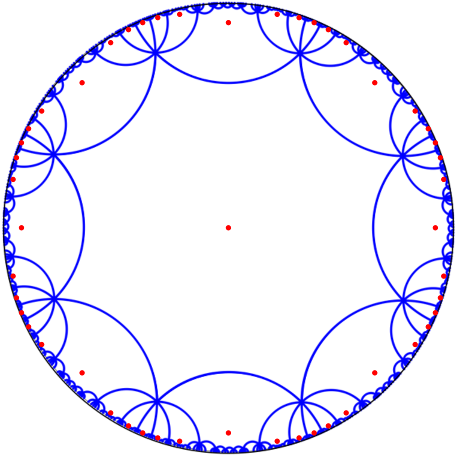
$$g_k = \begin{bmatrix} \alpha & \beta_k \\ \overline{\beta}_k & \overline{\alpha} \end{bmatrix}, \quad g_k(z) = \frac{\alpha z + \beta_k}{\overline{\beta}_k z + \overline{\alpha}}, \quad \alpha = 1 + \sqrt{2}, \quad \beta_k = e^{ik\pi/4}\sqrt{2\alpha}$$

Bolza surface



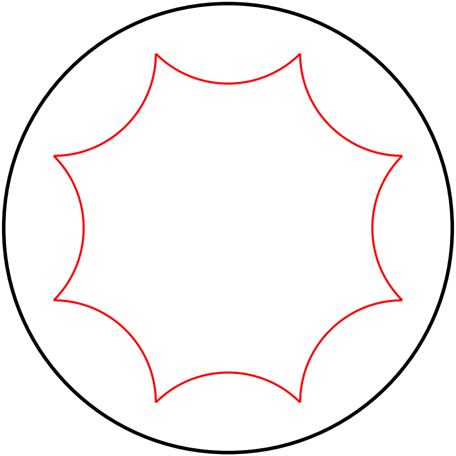


Hyperbolic octagon



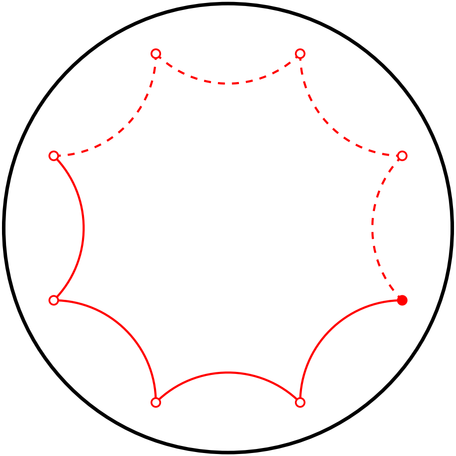
Voronoi diagram of $\mathcal{G}O$

Hyperbolic octagon



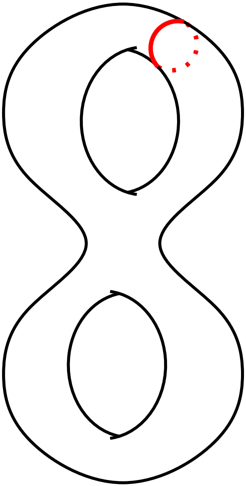
Fundamental domain $\mathcal{D}_O$ = Dirichlet region of O

Hyperbolic octagon



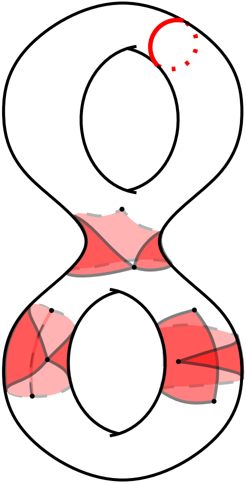
"Original" domain $\mathcal{D}$: contains exactly one point of each orbit

Criterion



Systole $\mathrm{sys}(\mathcal{M})$ = minimum length of a non-contractible loop on $\mathcal{M}$

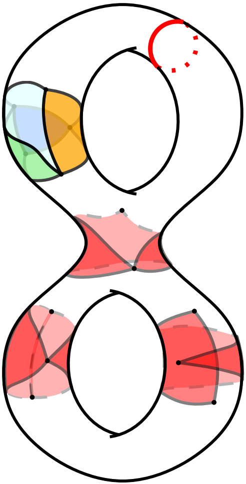
Criterion



Systole $\mathrm{sys}(\mathcal{M})$ = minimum length of a non-contractible loop on $\mathcal{M}$

$\pi_{\mathcal{M}}(\,DT_{\mathbb{H}}\,(\mathcal{G}S)\,)$

Criterion



Systole $\text{sys}(\mathcal{M}) =$ minimum length of a non-contractible loop on $\mathcal{M}$

S set of points in $\mathbb{H}^2$

$\delta_S =$ diameter of largest disks in $\mathbb{H}^2$ not containing any point of $\mathcal{G}S$

$\delta_S < \frac{1}{2}\,\text{sys}(\mathcal{M})$

$\Longrightarrow \pi_{\mathcal{M}}(\,DT_{\mathbb{H}}(\mathcal{G}S)\,) = DT_{\mathcal{M}}(S)$ is a simplicial complex

$\Longrightarrow$ The usual incremental algorithm can be used [Bowyer]

How can we satisfy $\delta_S < \frac{1}{2}\,\mathrm{sys}\,(\mathcal{M})$?

Two ways:

1. Covering spaces
   - effect: increase the systole
   - take copies of the fundamental domain with input points
   - $32 <$ number of sheets ≤ 128
   - new: $34 <$ number of sheets [Ebbens, 2017]

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   - effect: artificially satisfy the condition in the 1-cover
   - set of points given for the Bolza surface
   - more appealing computationally

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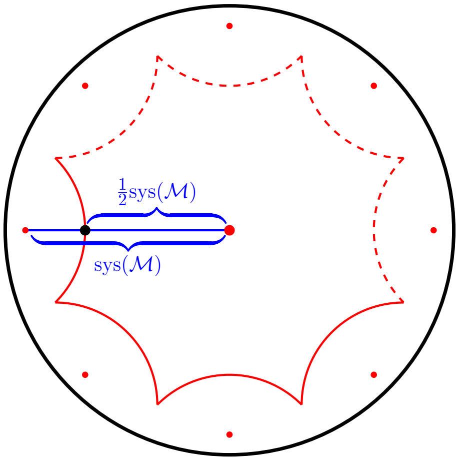
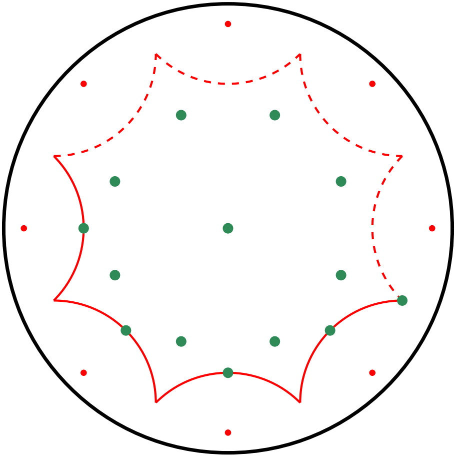
We adopt the second approach.

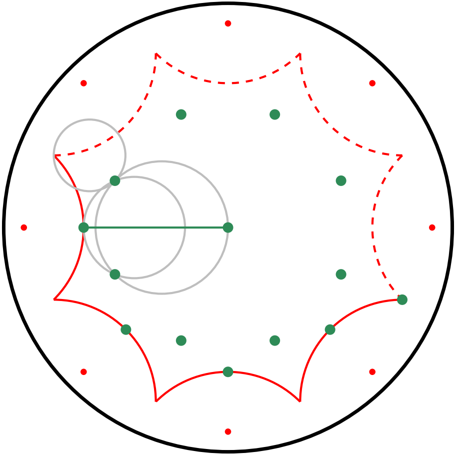
Systole on the octagon



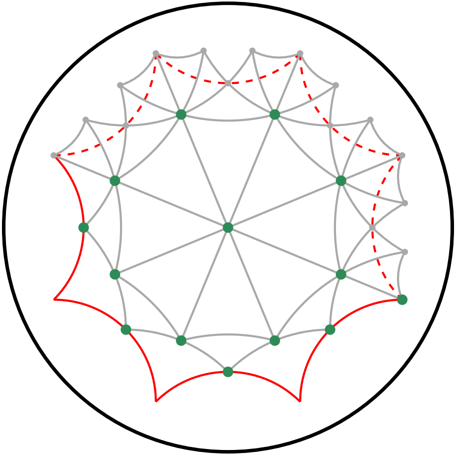
Set of dummy points



Set of dummy points vs. criterion



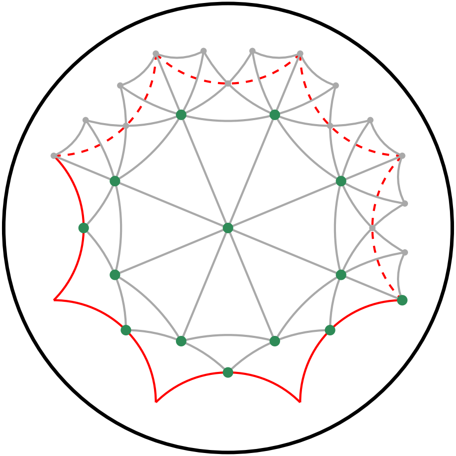
Delaunay triangulation of the dummy points



Delaunay triangulation of the Bolza surface

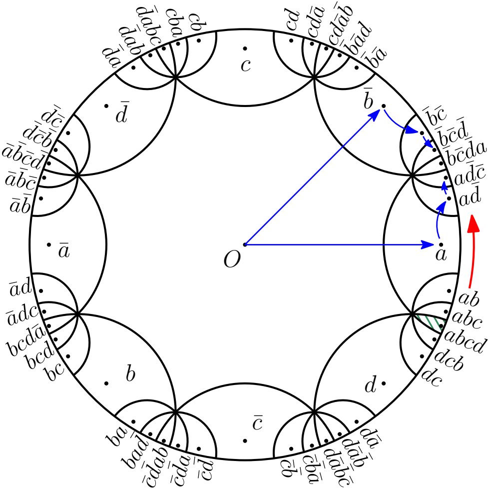
Algorithm:

1. initialize with dummy points
2. insert points in S
3. remove dummy points



Outline

Notation



$g(O)$, $g \in \mathcal{G}$, denoted as g

$$\mathcal{D}_g = g(\mathcal{D}_O),\ g \in \mathcal{G}$$

$$\mathcal{N} = \{g \in \mathcal{G} \mid \mathcal{D}_g \cap \mathcal{D}_O \neq \emptyset\}$$

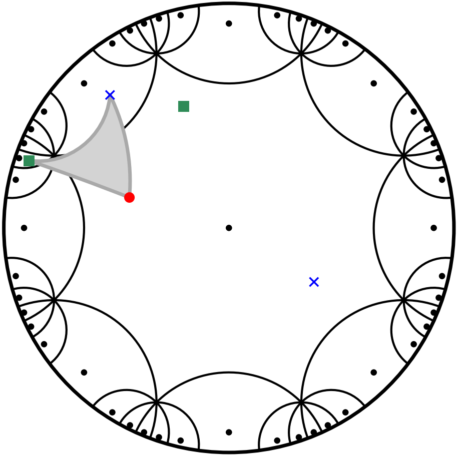
$$\mathcal{D}_{\mathcal{N}} = \bigcup_{g \in \mathcal{N}} \mathcal{D}_g$$

Property of $DT_{\mathbb{H}}(\mathcal{G}S)$

$S \subset \mathcal{D}$ input point set
s.t. criterion $\delta_S < \frac{1}{2}\,\mathrm{sys}(\mathcal{M})$ holds

σ face of $DT_{\mathbb{H}}(\mathcal{G}S)$ with at least one vertex in $\mathcal{D}$

$\longrightarrow$ σ is contained in $\mathcal{D}_{\mathcal{N}}$



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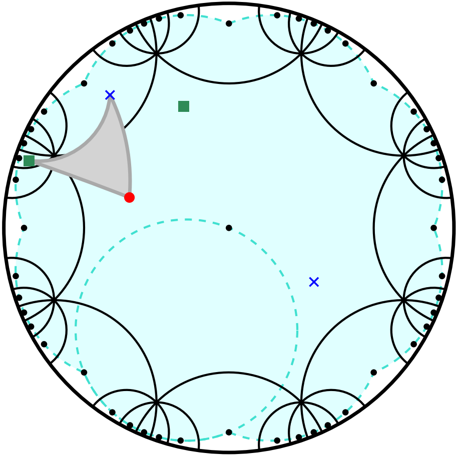
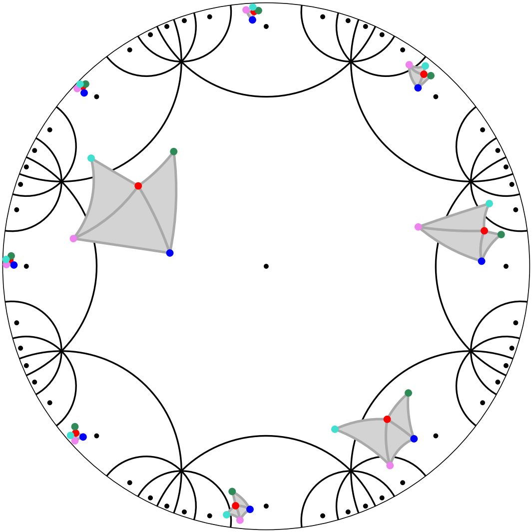


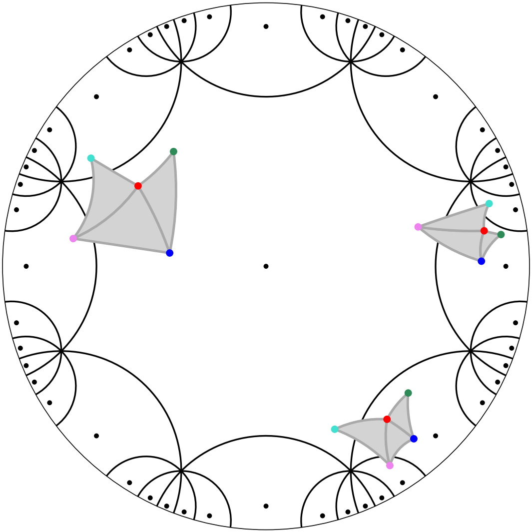
Canonical representative of a face

Each face of $DT_{\mathcal{M}}(S)$ has infinitely many pre-images in $DT_{\mathbb{H}}(\mathcal{G}S)$



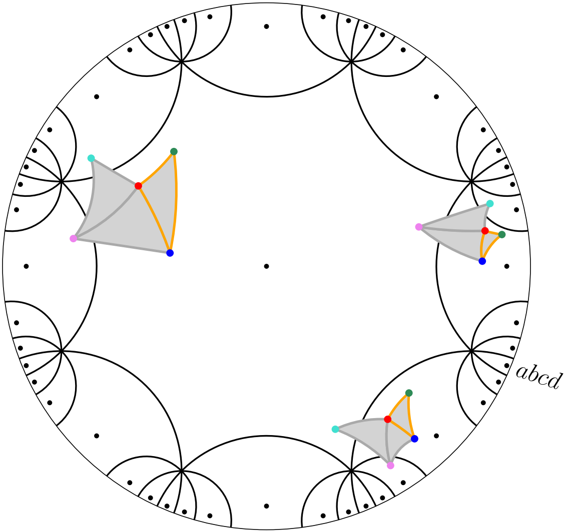
Canonical representative of a face

at least one pre-image with at least one vertex in $\mathcal{D}$



Canonical representative of a face

Case: face with 3 vertices in $\mathcal{D}$



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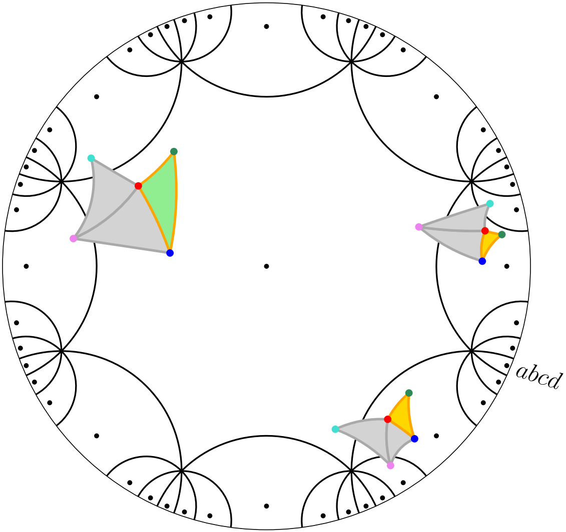

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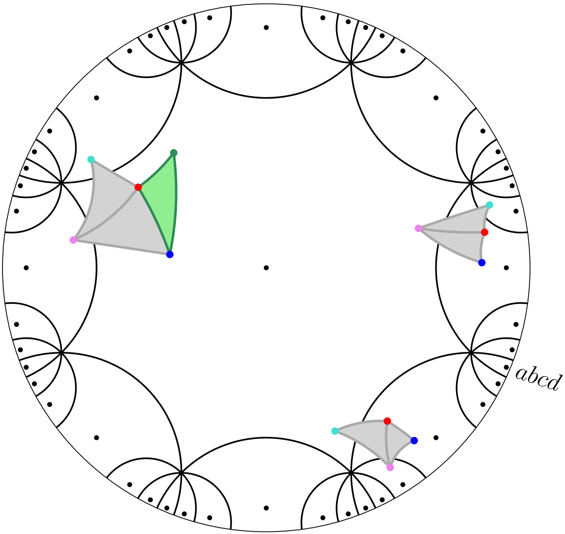
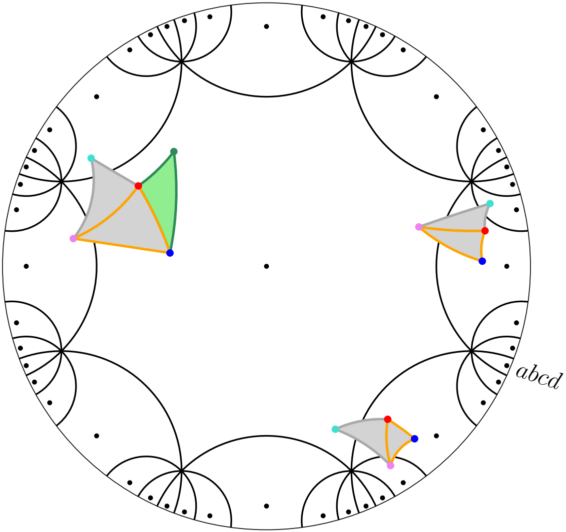


Canonical representative of a face

Case: face with 2 vertices in $\mathcal{D}$



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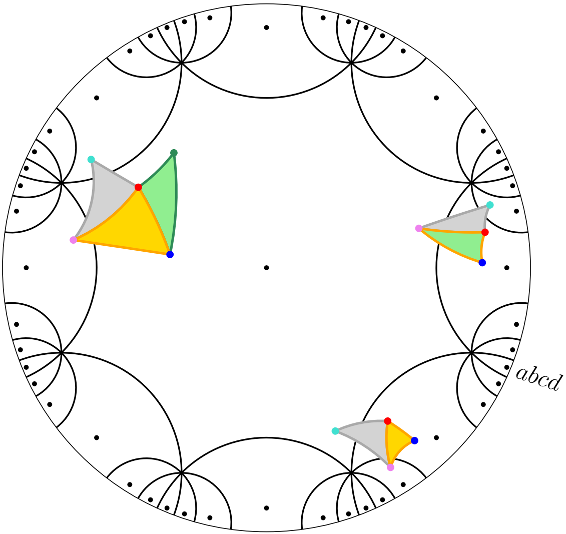

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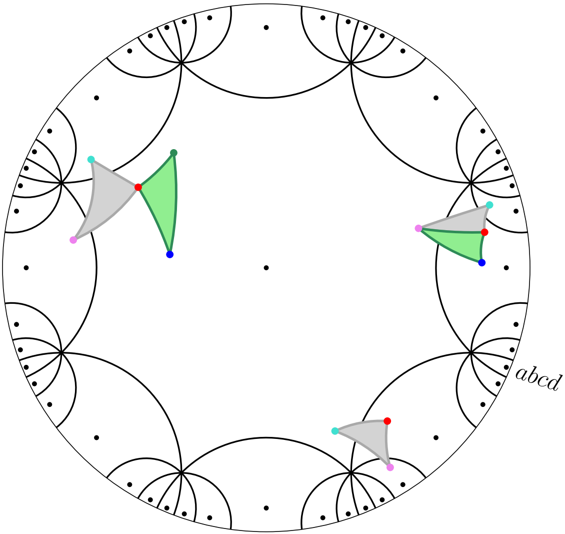
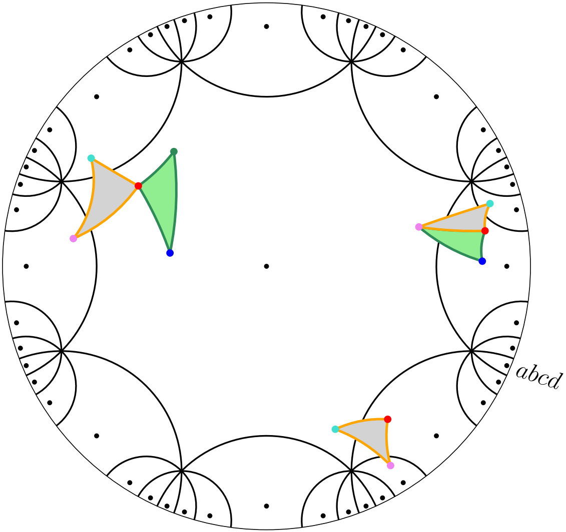


Canonical representative of a face

Case: face with 1 vertex in $\mathcal{D}$



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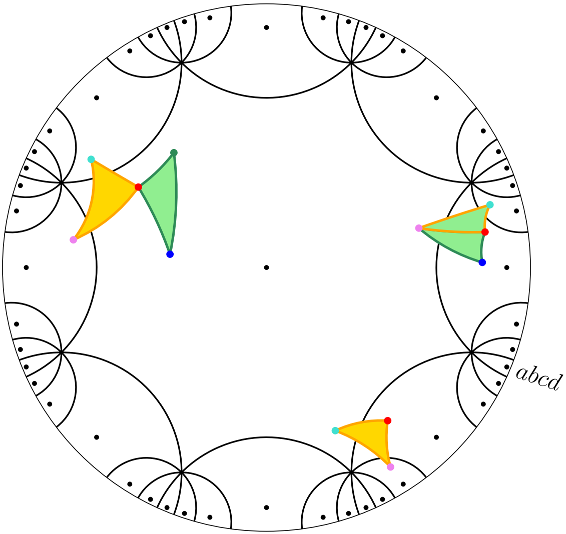

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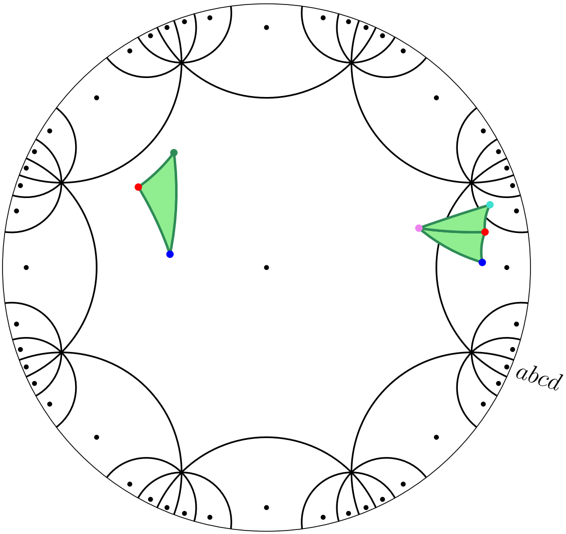
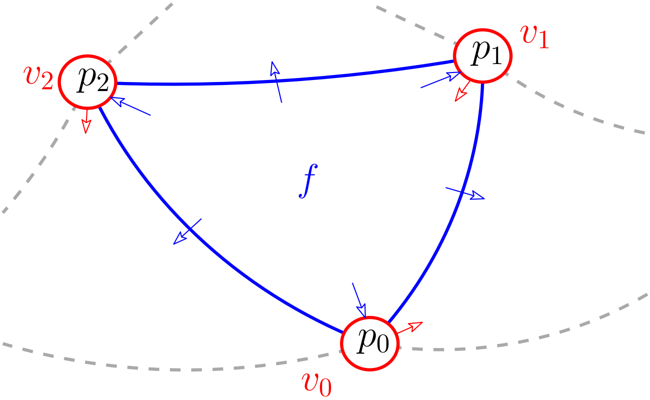
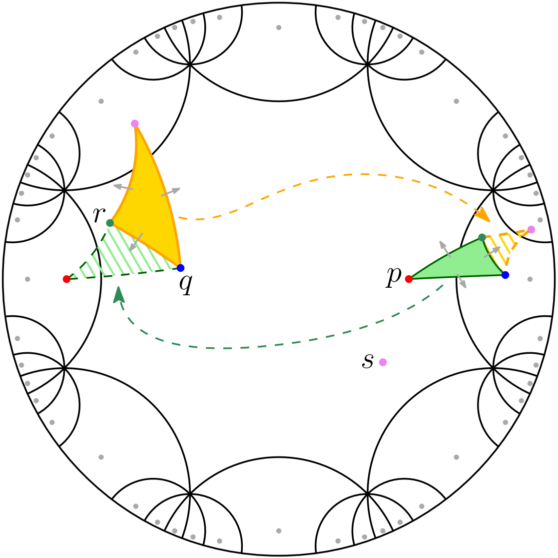


CGAL Triangulations



Face of $DT_{\mathcal{M}}(S)$



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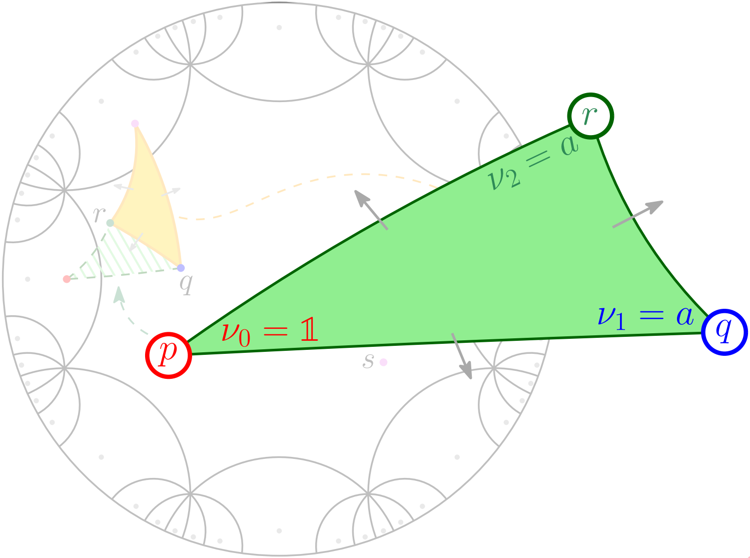
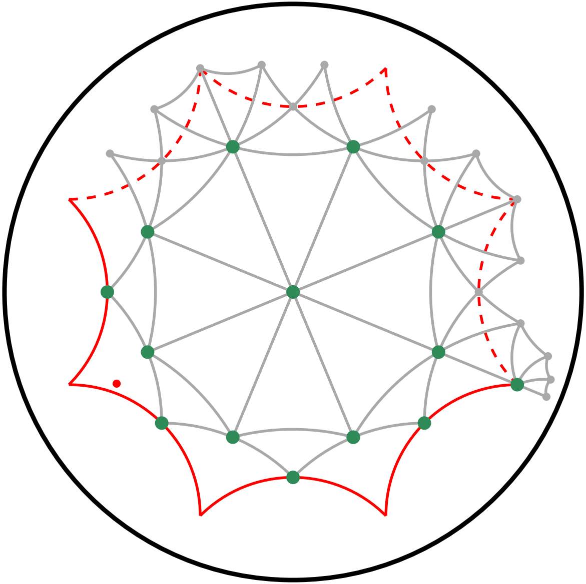


Point Location



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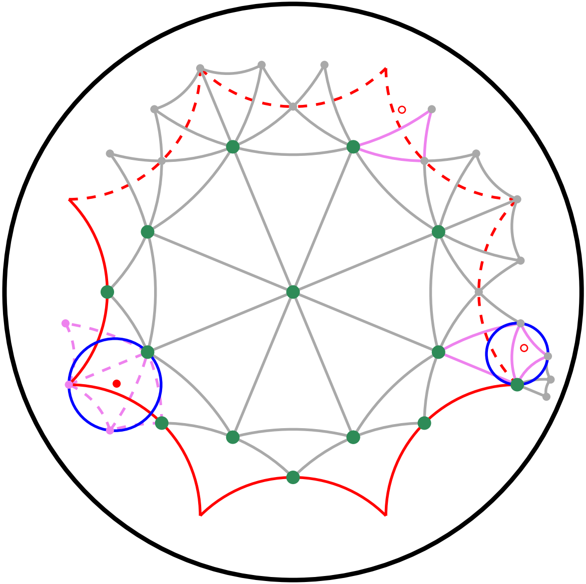

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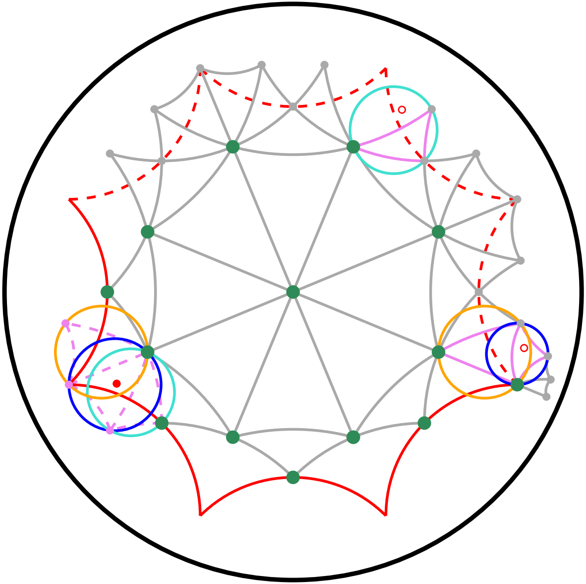

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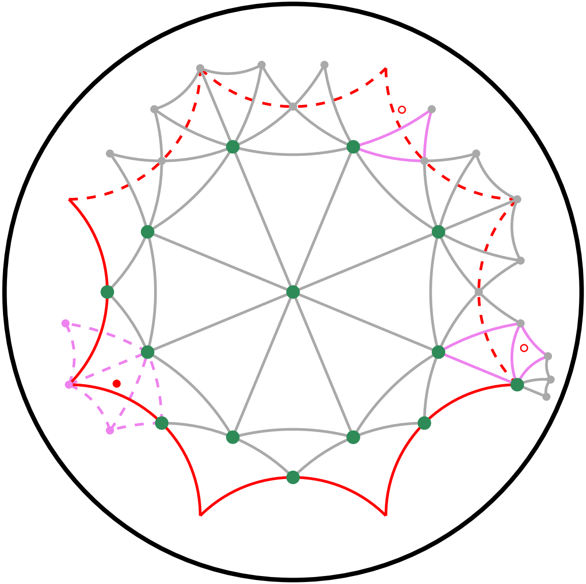
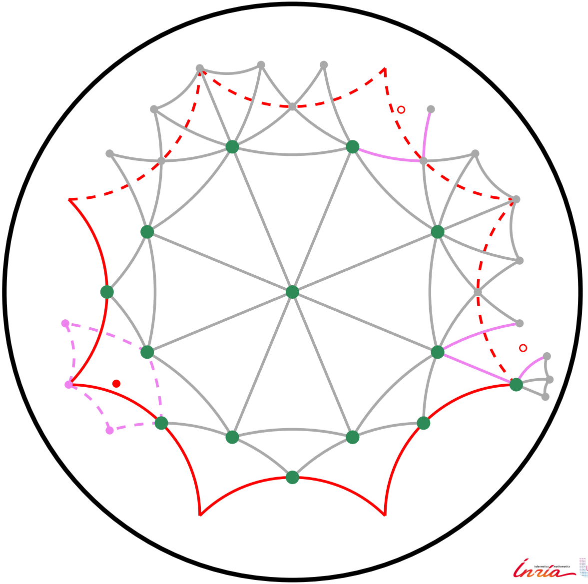


Point Insertion

"hole" = topological disk



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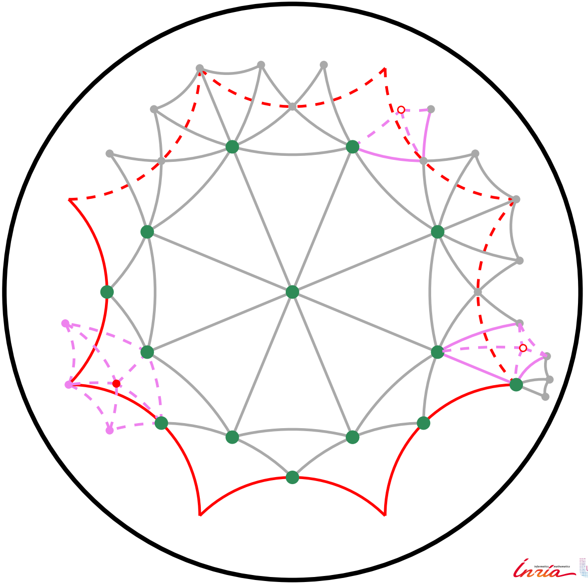
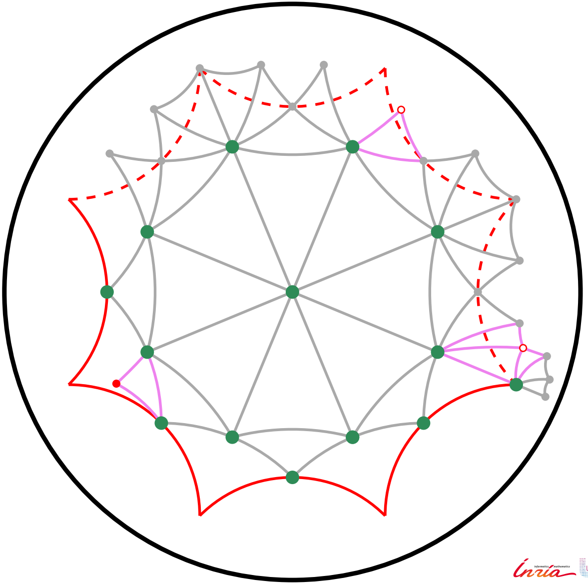


Point Insertion

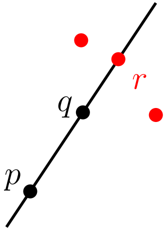
Computations
on translations

Dehn's algorithm
(slightly modified)

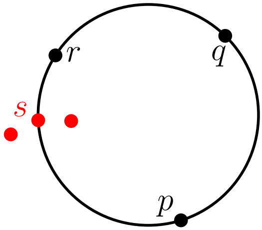


Predicates

$$Orientation(p, q, r) = \text{sign}\begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$



$$InCircle(p, q, r, s) = \text{sign}\begin{vmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{vmatrix}$$



Predicates

Suppose that the points in S are rational.

Input of the predicates can be images of these points under $\nu \in \mathcal{N}$.

$$g_k(z) = \frac{\alpha z + e^{ik\pi/4}\sqrt{2\alpha}}{e^{-ik\pi/4}\sqrt{2\alpha}z + \alpha}, \quad \alpha = 1 + \sqrt{2}, \quad k = 0, 1, ..., 7$$

- the *Orientation* predicate has algebraic degree at most 20
- the *InCircle* predicate has algebraic degree at most 72

Point coordinates represented with `CORE::Expr`
$\longrightarrow$ (filtered) exact evaluation of predicates

Demo

Time to see the code in action!

Experiments

Fully dynamic implementation

1 million random points

- CGAL Euclidean DT (double) ~ 1 sec.
- CGAL Euclidean DT (CORE::Expr) ~ 13 sec.
- Hyperbolic periodic DT (CORE::Expr) ~ 34 sec.

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Predicates

- 0.76% calls to predicates involving translations in $\mathcal{N}$
- responsible for 36% of total time spent in predicates

Dummy points can be removed after insertion of 17–72 random points.

Outline

Future work

The goal is to

generalize

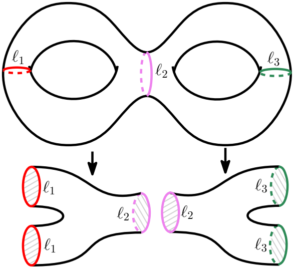
Future work

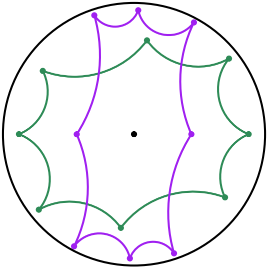
What:

- Algorithm for more general genus-2 surfaces
- Algorithm for surfaces of higher genus

How:

- Pants decomposition & F-N coordinates [Maskit, 2001]
- Octagonal fundamental domain [Aigon-Dupuy et al., 2005]





Future work

Issues:

- Surface representation
- Fundamental domain – Dirichlet or not?
- Generalize property of Delaunay triangles [IT17]
- Condition on something else rather than the systole?
- Canonical representative
- Choice of dummy points

The End

Thank you!

Source code and Maple sheets available online:

`https://members.loria.fr/Monique.Teillaud/DT_Bolza_SoCG17/`