14.310x: Data Analysis for Social Scientists
Functions and Moments of a Random Variable & Intro to Regressions

Welcome to your fourth homework assignment! You will have about one week to work through the assignment. We encourage you to get an early start, particularly if you still feel you need more experience using R. We have provided this PDF copy of the assignment so that you can print and work through the assignment offline. You can also go online directly to complete the assignment. If you choose to work on the assignment using this PDF, please go back to the online platform to submit your answers based on the output produced.

Good luck!

Question 1
Assume that the random variable X has a PDF given by $f_X(x) = 1\ for\ 0 < x < 1$. What is the PDF of the random variable $Y = X^2$?

- o $f_Y(y) = \sqrt{y}\ for\ 0 < y < 1$
- o $f_Y(y) = \frac{1}{2\sqrt{y}}\ for\ 0 < y < 1$
- o $f_Y(y) = \frac{1}{2\sqrt{y}}\ for - 1 < y < 1$
- o $f_Y(y) = \frac{1}{2}y^{-\frac{3}{2}}\ for - 1 < y < 1$

Question 2
Suppose X has the geometric PMF $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x\ for\ x = 0,1,2, \ldots$. What is the probability distribution of $Y = \frac{X}{X+1}$, its PMF? *Note that both X and Y are discrete random variables?*

- o $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{1-y}{y}}\ for\ y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{x}{x+1}, \ldots$
- o $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{(1-y)}{y}}\ for\ y = 0,1,2, \ldots$
- o $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}\ for\ y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{x}{x+1}, \ldots$
- o $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}\ for\ y = 0, 1,2, \ldots$

Question 3
Suppose the random variable X has a PDF given by $f_X(x) = \begin{cases} \frac{x-1}{2}, if\ 1 < x < 3 \\ 0, otherwise \end{cases}$. Which of the following is the monotone function $u(x)$ such that the random variable $u(X)$ has a uniform distribution between 0 and 1?

- o The monotone function does not exist.

- $\frac{(x-1)^2}{4}$
- $\frac{x(x-2)}{4}$
- $\frac{x-1}{2}$

Question 4
We have N i.i.d random variables from the uniform distribution between 0 and 1. If N=1, what is the probability that the nth order statistic is less than or equal to the value x? (In other words, what is Pr $(X_1^n \leq x)$?)

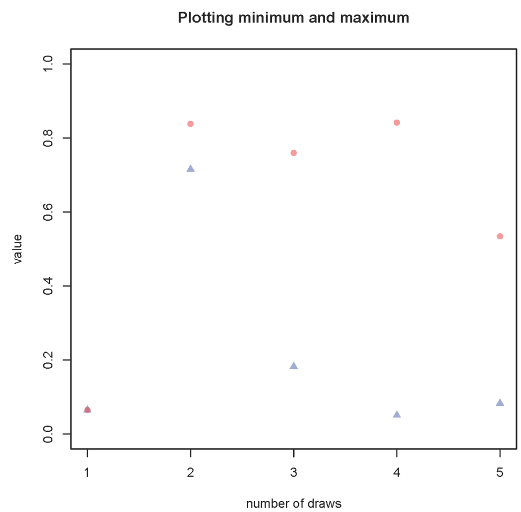
Questions 5-9 will be based on the following code:

```
#Creating a random draw of 1000 numbers
u <- runif(1000)
```

Question 5
Based on the above draw, is it possible to create from the above vector a random draw of a uniform distribution between 2 and 5?
- Yes
- No

Question 6
What is the PDF of the minimum of the draw?
- It is given by $f_{y^1}(y) = 999y^{998}$
- It is given by $f_{y^1}(y) = 1000(1-y)^{999}$
- It is given by $f_{y^1}(y) = 999(1-y)^{1000}$
- It is given by $f_{y^1}(y) = 999(1-y)^{998}$

Question 7
The following plot shows the maximum and the minimum of a uniform distribution by changing the number of draws.



A student is claiming that this plot is wrong since both the maximum and the minimum should show a monotonous relationship with the number of draws. Is this student's statement **True or False?**

- o True
- o False

Question 8

Use $F_X^{-1}(x)$ to transform $U \sim \mathcal{U}[0,1]$ into variable X which follows the original distribution of $\mathcal{N}(0,1)$. What is the command in R that allows you to do this?

- o `pnorm`
- o `rnorm`
- o `dnorm`
- o `qnorm`

Question 9

Suppose that the PDF $f_X(x)$ of a random variable X is an even function. Note: $f_X(x)$ is an even function if $f_X(x) = f_X(-x)$.

Is it true that the random variables X and -X are identically distributed?

- o True
- o False

Question 10

A couple decides to continue to have children until a daughter is born. What is the expected number of children this couple will have if the probability that a daughter is born is given by p?

- o The expected number of children is given by $\frac{1}{p} - 1$
- o The expected number of children is given by $\frac{1-p}{p}$

- The expected number of children is given by $\frac{p}{p^3}$
- The expected number of children is given by $\frac{1}{p}$

Question 11
For each of the following expressions, find $\mathbb{E}[X]$.

$$f_X(x) = ax^{a-1}, 0 < x < 1, a > 0$$

$$f_X(x) = \frac{1}{n}, x = 1,2, \dots, n; n > 0$$

$$f_X(x) = \frac{3}{2}(x-1)^2, 0 < x < 2$$

Question 12
Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform (0,1) random variable. What is $\mathbb{E}[Y]$?

- This is given by $\frac{X}{n}$
- This is given by $\frac{n}{2}$
- This is given by $\frac{n}{3}$
- This is given by n

Question 13
As in Question 12, suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform(0,1) random variables. What is $Var(Y)$?

- This is given by $\frac{n^2}{18} + \frac{n}{12}$
- This is given by $\frac{n^2}{18} + \frac{n}{6}$
- This is given by $\frac{n^2}{12} + \frac{n}{6}$
- This is given by $\frac{n^2}{6} + \frac{n}{12}$

Question 14
Assume that $y = \alpha + \beta X + U$, where $\beta = \frac{\rho_{XY}\sigma_Y}{\sigma_X}$ and $\alpha = \mu_Y - \beta\mu_X$ where μ_Y is the mean of Y and μ_X is the mean of X. What is the expected value of U?

- The expected value of U is α

- The expected value of U is μ_Y
- The expected value of U is 0
- The expected value of U is $\alpha + \beta\mu_X$

Question 15

Assume that $Y = \alpha + \beta X + U$, where $\beta = \frac{\rho_{XY}\sigma_Y}{\sigma_X}$ and $\alpha = \mu_Y - \beta\mu_X$. What is $cov(X, U)$? (Select all that apply).

- ☐ We have that $cov(X, U) = \rho_{XU}\sigma_X\sigma_U$
- ☐ We have that $cov(X, U) = var(X)$
- ☐ We have that $cov(X, U) = \sigma_X\sigma_U$
- ☐ We have that $cov(X, U) = 0$
- ☐ We have that $cov(X, U) = \sigma_X\sigma_Y$

Homework due Oct 5, 2020 19:30 EDT **Past Due**

Question 1

0.0/1.0 point (graded)

Assume that the random variable X has a PDF given by $f_X(x) = 1$ for $0 < x < 1$. What is the PDF of the random variable $Y = X^2$?

- ◯ $f_Y(y) = \sqrt{y}$ for $0 < y < 1$
- ◯ $f_Y(y) = \frac{1}{2\sqrt{y}}$ for $0 < y < 1$ ✔
- ◯ $f_Y(y) = \frac{1}{2\sqrt{y}}$ for $-1 < y < 1$
- ◯ $f_Y(y) = \frac{1}{2}y^{-\frac{3}{2}}$ for $-1 < y < 1$

Explanation

We know that if $0 < x < 1$ then it should also be the case that $0 \leq y \leq 1$. We also have that: $Pr(Y \leq y) = Pr(X^2 \leq y) = Pr(X \leq \sqrt{y}) = \int_0^{\sqrt{y}} dx = \sqrt{y}$. To get the pdf we take the derivative with respect to y, and then we have that $f_y(y) = \frac{1}{2\sqrt{y}}$ for $0 < y < 1$.

Suppose X has the geometric PMF $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$ for $x = 0, 1, 2, \cdots$. What is the probability distribution of $Y = \frac{X}{X+1}$, its PMF? *Note that both X and Y are discrete random variables.*

- $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{1-y}{y}}$ for $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{x}{x+1}, \cdots$
- $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{1-y}{y}}$ for $y = 0, 1, 2, \cdots$
- $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$ for $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{x}{x+1}, \cdots$ ✔
- $f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$ for $y = 0, 1, 2, \cdots$

Explanation

We have that:

$$Pr\left(Y=y\right)=Pr\left(\frac{X}{X+1}=y\right)=Pr\left(X=\frac{y}{1-y}\right)=\frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$$

Since $x=0,1,2,\cdots$, then $y=0,\frac{1}{2},\frac{2}{3},\frac{3}{4},\cdots,\frac{x}{x+1},\cdots$

Show answer

Submit You have used 0 of 2 attempts

Answers are displayed within the problem

Question 3

0 points possible (ungraded)

Suppose the random variable X has a PDF given by $f_X\left(x\right)=\begin{cases}\frac{x-1}{2}, \text{ if } 1<x<3\\ 0 \text{ otherwise}\end{cases}$

Which of the following is the monotone function $u\left(x\right)$ such that the random variable $u\left(X\right)$ has a uniform distribution between 0 and 1?

- The monotone function does not exist.
- $\frac{(x-1)^2}{4}$ ✔
- $\frac{x(x-2)}{4}$
- $\frac{x-1}{2}$

Explanation

This question asks us to find a monotone function $u(x)$ such that the random variable $u(X)$ has a $U[0,1]$ distribution. Since X is continuous, we can transform X by its own CDF. To do this, we integrate the PDF, $f_X(x) = \frac{x-1}{2}$:

$$u(X) = F_X(x) = \int_1^x f_X(x)\,dx = \int_1^x \frac{x-1}{2}dx = \frac{x(x-2)}{4}\Big|_1^x = \frac{(x-1)^2}{4} for 1 < x < 3$$

Show answer

Submit You have used 0 of 2 attempts

Answers are displayed within the problem

Answers are displayed within the problem

Question 4

0.0/1.0 point (graded)

We have N i.i.d random variables from the uniform distribution between 0 and 1. If $N = 1$, what is the probability that the n^{th} order statistic is less than or equal to the value x? (In other words, what is $Pr\left(X_1^{(n)} \leq x\right)$?)

Answer: x

Explanation

Since this is a random draw of just one number, then we know that $Pr\left(X_1^{(1)} \leq x\right) = x$.

Show answer

Submit You have used 0 of 2 attempts

Answers are displayed within the problem

Questions 5-9 will be based on the following code:

```
#Creating a random draw of 1000 numbers
u <- runif(1000)
```

Question 5

0.0/1.0 point (graded)

Based on the above draw, is it possible to create from the above vector a random draw of a uniform distribution between 2 and 5?

- ◯ Yes ✔
- ◯ No

Explanation

We need to apply $F_X^{-1}(u)$ to the random draw that we have created. If X is distributed between 2 and 5, then we know that $F_X(x) = \frac{x-2}{3}$, thus we can construct the random draw of X as $5 - 2(1 - u)$.

Show answer

Question 6

0.0/1.0 point (graded)

What is the PDF of the minimum of the draw?

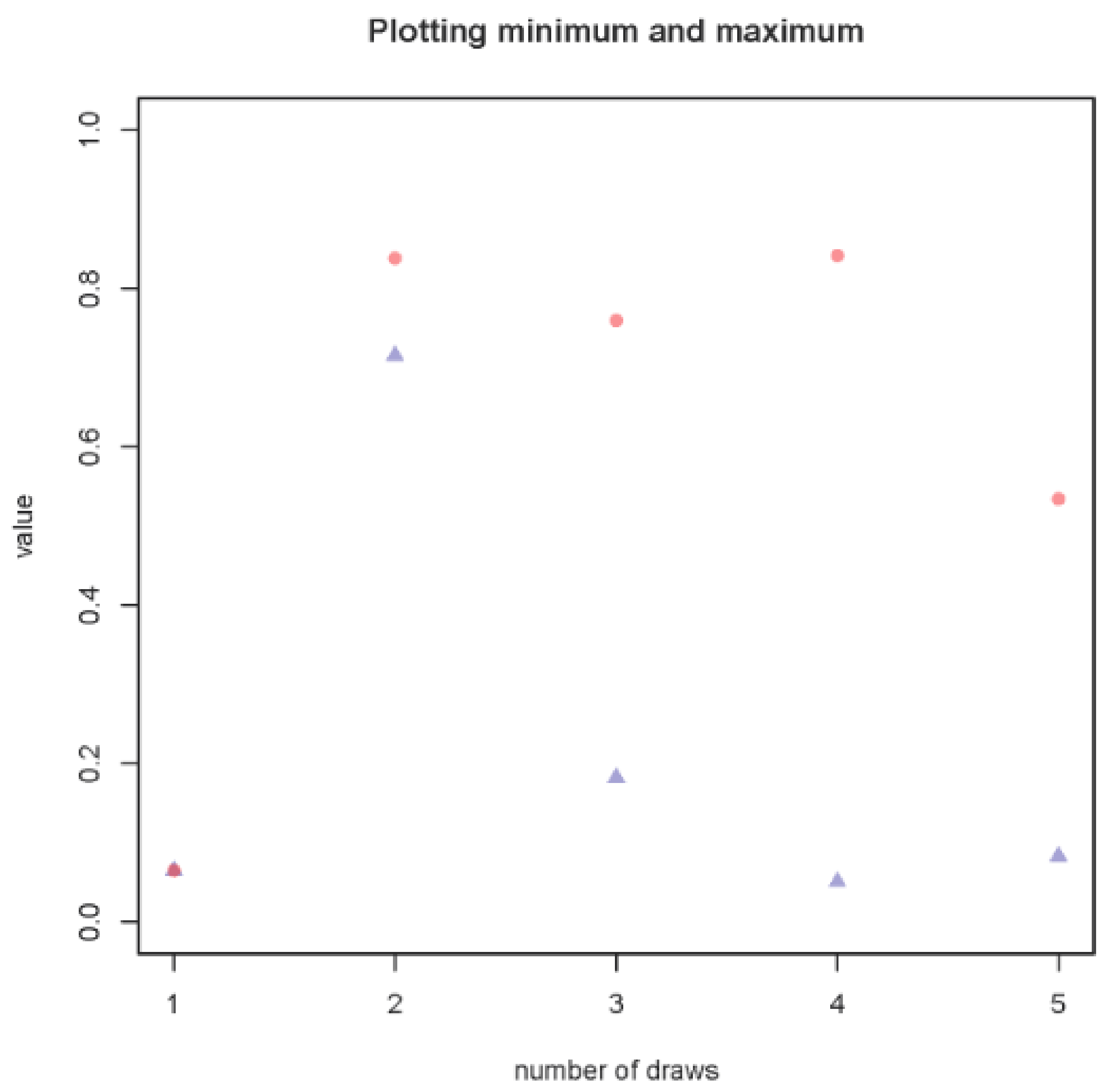
- It is given by $f_{y^{(1)}}(y) = 999y^{998}$
- It is given by $f_{y^{(1)}}(y) = 1000(1-y)^{999}$ ✔
- It is given by $f_{y^{(1)}}(y) = 999(1-y)^{1000}$
- It is given by $f_{y^{(1)}}(y) = 999(1-y)^{998}$

Explanation

In R, the code creates a vector of 1000 draws from the uniform distribution between 0 and 1. We know that the minimum corresponds to the first order statistic and that it's PDF is given by: $f_{y^{(1)}}(y) = n(1-F_X(y))^{n-1} f_X(y)$. If we substitute for $n = 1000$ and $F_X(y) = y$ we obtain the answer.

Show answer

The following plot shows the maximum and the minimum of a uniform distribution by changing the number of draws.



A student is claiming that this plot is wrong since both the maximum and the minimum should show a monotonous relationship with the number of draws. Is this student's statement **True or False**?

- ◯ True
- ◯ False ✔

Explanation

The statement is false. Even though it is true that for the maximum it is more likely to have higher values when the number of draws increases, there is still a chance that this is not the case.

Show answer

Submit You have used 0 of 1 attempt

Answers are displayed within the problem

Question 8

0.0/1.0 point (graded)

Use $F_X^{-1}(x)$ to transform $U \sim \mathcal{U}[0, 1]$ into variable X which follows the original distribution of $\mathcal{N}(0, 1)$. What is the

- ◯ `pnorm`
- ◯ `rnorm`
- ◯ `dnorm`
- ◯ `qnorm` ✔

Explanation

This problem asks you to use one of the R commands to look for the inverse of the CDF. `dnorm` returns the value of the probability density function, $f_x(X)$. `pnorm` returns the value of CDF at a particular point. `rnorm` allows us to generate n draws from the $\mathcal{N}(mean, sd^2)$ variable. Finally, `qnorm` returns the value for which the CDF is equal to p. Therefore, `qnorm` is correct.

Show answer

Submit You have used 0 of 2 attempts

Answers are displayed within the problem

Homework due Oct 5, 2020 19:30 EDT **Past Due**

Question 9

0.0/1.0 point (graded)

Suppose that the PDF $f_X(x)$ of a random variable X is an even function. **Note:** $f_X(x)$ is an even function if $f_X(x) = f_X(-x)$.

Is it true that the random variables X and $-X$ are identically distributed?

- ◯ True ✔
- ◯ False

Explanation

This statement is true. The proof is the following $Y = -X$, and $g^{-1}(y) = -y$. Therefore, for every y:

$$f_Y(y) = f_X(g^{-1}(y)) = \left|\frac{d}{dy} g^{-1}(y)\right| = f_X(-y)|-1| = f_X(y)$$

Show answer

Question 10

0.0/1.0 point (graded)

A couple decides to continue to have children until a daughter is born. What is the expected number of children this couple will have if the probability that a daughter is born is given by p?

- The expected number of children is given by $\frac{1}{p} - 1$
- The expected number of children is given by $\frac{1-p}{p}$
- The expected number of children is given by $\frac{p}{p^3}$
- The expected number of children is given by $\frac{1}{p}$ ✔

Explanation

If X is the number of children until and by including the first daughter then $P\left(X=k\right)=\left(1-p\right)^{k-1}p$. Thus X is a geometric random variable and we have that as saw in the lecture:

$$\mathbb{E}\left[X\right]=\sum_{k=0}^{\infty}k(1-p)^{k-1}p=\frac{1}{p}$$

Question 11

0.0/1.0 point (graded)

For each of the following expressions, find $\mathbb{E}\left[X\right]$.

$f_X\left(x\right) = ax^{a-1}, 0 < x < 1, a > 0$

Answer: a/(a+1)

$f_X\left(x\right) = \frac{1}{n}, x = 1, 2, \cdots, n; n > 0$

Answer: (n+1)/2

$f_X\left(x\right) = \frac{3}{2}(x-1)^2, 0 < x < 2$

Answer: 1

Explanation

Explanation

In this case we have that:

If $f_X(x) = ax^{a-1}, 0 < x < 1, a > 0$ then:

$$\mathbb{E}[X] = \int_0^1 ax^a dx = \frac{a}{a+1} x^{a+1}|_0^1 = \frac{a}{a+1}$$

If $f_X(x) = \frac{1}{n}, x = 1, 2, \cdots, n$, for integer $n > 0$ then:

$$\mathbb{E}[X] = \sum_{x=1}^{n} \frac{1}{n} x = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \frac{(n+1)n}{2} = \frac{n+1}{2}$$

If $f_X(x) = \frac{3}{2}(x-1)^2, 0 < x < 2$ then:

$$\mathbb{E}[X] = \int_0^2 x \frac{3}{2}(x-1)^2 dx = \frac{(x-1)^3(3x+1)}{8}|_0^2 = \frac{7}{8} + \frac{1}{8} = 1$$

Show answer

Submit

You have used 0 of 2 attempts

Question 12

0.0/1.0 point (graded)

Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform(0,1) random variable. What is $\mathbb{E}[Y]$?

- This is given by $\frac{X}{n}$
- This is given by $\frac{n}{2}$ ✔
- This is given by $\frac{n}{3}$
- This is given by n

Explanation

In general, we have that since Y is binomial with probability success X then $\mathbb{E}[Y|X] = nX$. Using this, and the law of iterated expectations, we have that:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[nX].$$

Since X is a uniform variable between 0 and 1, then we know that $\mathbb{E}[nX] = \frac{n}{2}$.

Question 13

0.0/1.0 point (graded)

As in Question 12, suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a uniform(0,1) random variables. What is $Var(Y)$?

- This is given by $\frac{n^2}{18} + \frac{n}{12}$
- This is given by $\frac{n^2}{18} + \frac{n}{6}$
- This is given by $\frac{n^2}{12} + \frac{n}{6}$ ✔
- This is given by $\frac{n^2}{6} + \frac{n}{12}$

Explanation

Here we use the law of total probability. We have that:

$$Var(Y) = Var(\mathbb{E}[Y|X]) + \mathbb{E}[Var(Y|X)] = Var(nX) + \mathbb{E}[nX(1-X)] = \frac{n^2}{12} + \frac{n}{6}$$

Question 14

0.0/1.0 point (graded)

Assume that $Y = \alpha + \beta X + U$, where $\beta = \frac{\rho_{XY}\sigma_Y}{\sigma_X}$ and $\alpha = \mu_Y - \beta\mu_X$ where μ_Y is the mean of Y and μ_X is the mean of X. What is the expected value of U?

- ○ The expected value of U is α
- ○ The expected value of U is μ_Y
- ○ The expected value of U is 0. ✔
- ○ The expected value of U is $\alpha + \beta\mu_X$.

Explanation

In the case we have that:

$$\mathbb{E}\left[U\right] = \mathbb{E}\left[Y - \alpha - \beta X\right]$$

$$= \mu_Y - \alpha - \beta\mu_X$$

$$= \mu_Y - \mu_Y + \beta\mu_X - \beta\mu_X = 0$$

Assume that $Y = \alpha + \beta X + U$, where $\beta = \frac{\rho_{XY} \cdot \sigma_Y}{\sigma_X}$ and $\alpha = \mu_Y - \beta\mu_X$. What is $cov(X, U)$? (Select all that apply).

- [ ] We have that $cov(X, U) = \rho_{XU}\sigma_X\sigma_U$ ✔
- [ ] We have that $cov(X, U) = var(X)$
- [ ] We have that $cov(X, U) = \sigma_X\sigma_U$
- [ ] We have that $cov(X, U) = 0$ ✔
- [ ] We have that $cov(X, U) = \sigma_X\sigma_Y$

Explanation

By definition we know that $\rho_{XU} = \frac{cov(X,U)}{\sigma_X\sigma_U}$. In this case we also have that:

$$cov(X, U) = cov(X, Y - \alpha - \beta X)$$

$$= cov(X, Y) - \beta var(X)$$

$$= cov(X, Y) - \left(\frac{\rho_{XY}\sigma_Y}{\sigma_X}\right) var(X)$$