

Quantum Mechanics

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Chapter 1

Dirac notation

A bra $\langle v|$ is an element in a complex vector space. The corresponding ket $|v\rangle$ is an element in its dual space. The usual rules of linear algebra are valid:

$$|u\rangle + |v\rangle = |w\rangle \quad (1.1)$$

$$c|v\rangle = |u\rangle ; c \in \mathbb{C} \quad (1.2)$$

Convert between bras and kets:

$$c_1|v_1\rangle + c_2|v_2\rangle \iff c_1^*\langle v_1| + c_2^*\langle v_2| \quad (1.3)$$

1.1 Inner product

$$\langle u|v\rangle = \langle v|u\rangle^* \quad (1.4)$$

$$\langle v|v\rangle \geq 0 \quad (1.5)$$

$$\langle v|v\rangle = 0 \iff v = 0 \quad (1.6)$$

Linearity in the second argument and antilinear in the first:

$$\langle u|c_1v_1 + c_2v_2\rangle = c_1\langle u|v_1\rangle + c_2\langle u|v_2\rangle \quad (1.7)$$

$$\langle c_1u_1 + c_2u_2|v\rangle = c_1^*\langle u_1|v\rangle + c_2^*\langle u_2|v\rangle \quad (1.8)$$

$$(1.9)$$

For $v, u \in \mathbb{C}^n$ as vectors ($\langle v|$ is a row vector, $|u\rangle$ is a column vector)

$$\langle v|u\rangle = \sum_n v_i^* u_i \quad (1.10)$$

For functions $f, g \in \mathbb{C}$ as vectors with $x \in [0, L]$

$$\langle f|g\rangle = \int_0^L f^*(x)g(x)dx \quad (1.11)$$

For a set of basis vectors $\{e_i\}$ (kronecker delta)

$$\langle e_i|e_j\rangle = \delta_{ij} \quad (1.12)$$

Write a vector as a linear combination of basis vectors

$$|v\rangle = \sum_{i=0}^n v_i |e_i\rangle = \sum_{i=0}^n |e_i\rangle \langle e_i| |v\rangle \quad (1.13)$$

$$\langle e_i|v\rangle = v_i \quad (1.14)$$

1.2 Outer product

A bra and a ket can be combined in the outer product to create an operator

$$X = |v\rangle \langle u| \quad (1.15)$$

$$X |\Psi\rangle = |v\rangle \langle u| |\Psi\rangle = |v\rangle \langle u|\Psi\rangle \quad (1.16)$$

$$(1.17)$$

Chapter 2

Postulates

2.1 Postulate 1: state

Postulate 1 *The state of a physical system is described by a state vector that belongs to a complex vector space V , called the state space of the system.*

2.2 Postulate X: time evolution

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle \quad (2.1)$$

Chapter 3

State space

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_3\rangle \tag{3.1}$$

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_2\rangle + |\Psi_1\rangle \tag{3.2}$$

Chapter 4

Operators

4.1 Basic Properties

An operator acting on a ket creates a new ket:

$$\hat{A} |\Psi\rangle = |\Psi'\rangle \quad (4.1)$$

$$|\Psi\rangle, |\Psi'\rangle \in V \quad (4.2)$$

Operators are linear

$$\hat{A}(a_1 |\Psi_1\rangle + a_2 |\Psi_2\rangle) = (a_1 \hat{A} |\Psi_1\rangle + a_2 \hat{A} |\Psi_2\rangle) \quad (4.3)$$

$$|\Psi\rangle, |\Psi'\rangle \in V; a_1, a_2 \in \mathbb{C} \quad (4.4)$$

Operators are associative and commutative under addition

$$\hat{A} + (\hat{B} + \hat{C}) = (\hat{A} + \hat{B}) + \hat{C} \quad (4.5)$$

$$\hat{A} + \hat{B} = \hat{B} + \hat{A} \quad (4.6)$$

Multiplying operators is interpreted as applying them to kets. It is associative but NOT (in general) commutative.

$$\hat{A} \hat{B} |\Psi\rangle = \hat{A}(\hat{B} |\Psi\rangle) = \hat{A} |\Psi'\rangle \quad (4.7)$$

$$\hat{A}(\hat{B} \hat{C}) = (\hat{A} \hat{B}) \hat{C} \quad (4.8)$$

$$\hat{A} \hat{B} \neq \hat{B} \hat{A} \quad (4.9)$$

The lack of commutativity makes the "commutator" useful

$$[\hat{A}, \hat{B}] = \hat{A} \hat{B} - \hat{B} \hat{A} \quad (4.10)$$

$$(4.11)$$

The inverse \hat{A}^{-1} of an operator is defined by

$$\hat{A}^{-1} \hat{A} = \hat{A} \hat{A}^{-1} = \mathbb{1} \quad (4.12)$$

4.2 Hermitian Operators

An operator is called Hermitian (or self-adjoint) if it is its own hermitian conjugate $\hat{A} = \hat{A}^\dagger$

$$\hat{A} |A\rangle = |B\rangle \rightarrow \langle A| \hat{A}^\dagger = \langle B| \quad (4.13)$$

$$\hat{A} |A\rangle = |B\rangle \rightarrow \langle A| \hat{A} = \langle B| \quad (4.14)$$

$$(4.15)$$

A hermitian operator \hat{A} has the following properties

1. $\hat{A} |\lambda\rangle = \lambda |\lambda\rangle \rightarrow \lambda \in \mathbb{R}$
2. $\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle \in \mathbb{R}$
3. All eigenvectors with different eigenvalues are orthogonal

The Hermitian conjugate of a product is

$$(\hat{A} \hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger \quad (4.16)$$

4.3 Projection Operators

A projection operator is defined

$$(4.17)$$

4.4 Unitary operators

A unitary operator is defined by

$$\hat{U}^{-1} = \hat{U}^\dagger \quad (4.18)$$

This leads to (see also (4.12))

$$\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \mathbf{1} \quad (4.19)$$

The product of two unitary operators ($\hat{U}^{-1} = \hat{U}^\dagger$; $\hat{V}^{-1} = \hat{V}^\dagger$) is as well unitary

$$(\hat{U} \hat{V})^\dagger (\hat{U} \hat{V}) = \mathbf{1} \quad (4.20)$$

$$(\hat{U} \hat{V})(\hat{U} \hat{V})^\dagger = \mathbf{1} \quad (4.21)$$

The eigenvalues of a unitary operator have magnitude 1

$$\hat{U} |\lambda\rangle = \lambda |\lambda\rangle \Rightarrow |\lambda|^2 = 1 \quad (4.22)$$

$$|\lambda| = 1 \Rightarrow \lambda = e^{i\phi_\lambda}; \phi_\lambda \in \mathbb{R} \quad (4.23)$$

The eigenvectors are orthogonal $\langle \mu | \lambda \rangle = 0$.

Unitary transformations conserve the scalar product of two kets and the norm of a ket.

$$|\Psi'_1\rangle = \hat{U} |\Psi_1\rangle; |\Psi'_2\rangle = \hat{U} |\Psi_2\rangle \quad (4.24)$$

$$\langle \Psi'_1 | \Psi'_2 \rangle = \langle \Psi_1 | \hat{U}^\dagger \hat{U} | \Psi_2 \rangle = \langle \Psi_1 | \Psi_2 \rangle \quad (4.25)$$

$$\langle \Psi'_1 | \Psi'_1 \rangle = \langle \Psi_1 | \Psi_1 \rangle \quad (4.26)$$

$$(4.27)$$

The unitary transformation of an operator is

$$\hat{A}' = \hat{U} \hat{A} \hat{U}^\dagger \quad (4.28)$$

Which has the following properties

1. if \hat{A} is Hermitian, so is \hat{A}'
2. $(\hat{A}')^n = (\hat{A}^n)'$
3. $\hat{A} |\alpha\rangle = \alpha |\alpha\rangle \Rightarrow \hat{A}' |\alpha'\rangle = \alpha |\alpha'\rangle; |\alpha'\rangle = \hat{U} |\alpha\rangle$

An infinitesimal unitary operator is

$$\hat{U}(\epsilon) = \mathbb{1} - i\epsilon \hat{A}; \epsilon \in \mathbb{R}; \hat{A} = \hat{A}^\dagger \quad (4.29)$$

A common form of unitary operators is

$$\hat{U} = e^{i\hat{A}}; \hat{A} = \hat{A}^\dagger \quad (4.30)$$

$$\text{e.g. the translation operator } \hat{T}(\alpha) = e^{i\alpha \frac{\hat{p}}{\hbar}} \quad (4.31)$$

See also: [Prof.M. Unitary Operators](#) and [TM Lecture 4](#)

Chapter 5

Eigenvectors and Eigenvalues

5.1 Degenerate Eigenvectors

If two eigenvectors have the same eigenvalue:

$$\hat{H} |\lambda_1\rangle = \lambda |\lambda_1\rangle \quad (5.1)$$

$$\hat{H} |\lambda_2\rangle = \lambda |\lambda_2\rangle \quad (5.2)$$

their linear combination is an eigenvector as well:

$$\alpha \hat{H} |\lambda_1\rangle = \lambda \alpha |\lambda_1\rangle \quad (5.3)$$

$$\beta \hat{H} |\lambda_1\rangle = \lambda \beta |\lambda_1\rangle \quad (5.4)$$

$$\hat{H}[\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle] = \lambda[\alpha |\lambda_1\rangle + \beta |\lambda_2\rangle] \quad (5.5)$$

Therefore it is possible to create two orthogonal eigenvectors for this eigenvalue.

The probability in the degenerate case is the sum of the probabilities for each eigenvector $|\langle \hat{H} | \lambda_1 \rangle|^2 + |\langle \hat{H} | \lambda_2 \rangle|^2$

Chapter 6

Uncertainty

6.1 Probabilities

When Ψ is represented in a basis u

$$\hat{A} |u_n\rangle = \lambda_n |u_n\rangle \quad (6.1)$$

$$|\Psi\rangle = \sum_n c_n |u_n\rangle \quad (6.2)$$

$$c_n = \langle u_n | \Psi \rangle \quad (6.3)$$

$|c_n|^2$ is the probability to get the eigenvalue λ_n as a result.

6.2 Expectation value and RMS

Expectation value of \hat{A} in state Ψ

$$\langle \hat{A} \rangle_\Psi = \langle \Psi | \hat{A} | \Psi \rangle \quad (6.4)$$

Root mean square deviation

$$\Delta \hat{A} = \sqrt{\langle \hat{\sigma}_A^2 \rangle_\Psi} \quad (6.5)$$

$$\hat{\sigma}_A = \hat{A} - \langle \hat{A} \rangle_\Psi \quad (6.6)$$

$$\Delta \hat{A} = \sqrt{\langle \hat{A}^2 \rangle_\Psi - \langle \hat{A} \rangle_\Psi^2} \quad (6.7)$$

6.3 Uncertainty Principle

$$\Delta \hat{A} \Delta \hat{B} \geq \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle | \quad (6.8)$$

6.4 Position and Momentum

$$[\hat{x}, \hat{p}] = i\hbar \quad (6.9)$$

$$\Delta \hat{x} \Delta \hat{p} \geq \frac{\hbar}{2} \quad (6.10)$$