Quantum Mechanics

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Contents

Chapter 1

Dirac notation

A bra $\langle v|$ is an element in a complex vector space. The corresponding ket $|v\rangle$ is an elemen in its dual space. The usual rules of linear algebra are valid:

$$|u\rangle + |v\rangle = |w\rangle \tag{1.1}$$
$$c\,|v\rangle = |u\rangle\,; c \in \mathbb{C} \tag{1.2}$$

Convert between bras and kets:

$$c_1\,|v_1\rangle + c_2\,|v_2\rangle \iff c_1^*\,\langle v_1| + c_2^*\,\langle v_2| \tag{1.3}$$

1.1 Inner product

$$\langle u|v\rangle = \langle v|u\rangle^* \tag{1.4}$$
$$\langle v|v\rangle \geq 0 \tag{1.5}$$
$$\langle v|v\rangle = 0 \iff v = 0 \tag{1.6}$$

Linearity in the second argument and antilinear in the first:

$$\langle u|c_1v_1 + c_2v_2\rangle = c_1\,\langle u|v_1\rangle + c_2\,\langle u|v_2\rangle \tag{1.7}$$
$$\langle c_1u_1 + c_2u_2|v\rangle = c_1^*\,\langle u_1|v\rangle + c_2^*\,\langle u_2|v\rangle \tag{1.8}$$
$$\tag{1.9}$$

For $v, u \in \mathbb{C}^n$ as vectors ($\langle v|$ is a row vector, $|u\rangle$ is a column vector)

$$\langle v|u\rangle = \sum_n v_i^* u_i \tag{1.10}$$

For functions $f, g \in \mathbb{C}$ as vectors with $x \in [0, L]$

$$\langle f|g\rangle = \int_0^L f^*(x)g(x)dx \tag{1.11}$$

For a set of basis vectors $\{e_i\}$ (kronecker delta)

$$\langle e_i|e_j\rangle = \delta_{ij} \tag{1.12}$$

Write a vector as a linear combination of basis vectors

$$|v\rangle = \sum_{i=0}^{n} v_i |e_i\rangle = \sum_{i=0}^{n} |e_i\rangle \langle e_i| |v\rangle \tag{1.13}$$

$$\langle e_i|v\rangle = v_i \tag{1.14}$$

1.2 Outer product

A bra and a ket can be combined in the outer product to create an operator

$$X = |v\rangle \langle u| \tag{1.15}$$

$$X |\Psi\rangle = |v\rangle \langle u| |\Psi\rangle = |v\rangle \langle u|\Psi\rangle \tag{1.16}$$

$$\tag{1.17}$$

Chapter 2

Postulates

2.1 Postulate 1: state

Postulate 1 *The state of a physical system is described by a state vector that belongs to a complex vector space V, called the state space of the system.*

2.2 Postulate X: time evolution

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{\mathrm{H}}(t)\left|\Psi(t)\right\rangle \tag{2.1}$$

Chapter 3

State space

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_3\rangle \tag{3.1}$$

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_2\rangle + |\Psi_1\rangle \tag{3.2}$$

Chapter 4

Operators

4.1 Basic Properties

An operator acting on a ket creates a new ket:

$$\hat{\mathrm{A}}\,|\Psi\rangle = |\Psi'\rangle \tag{4.1}$$

$$|\Psi\rangle\,, |\Psi'\rangle \in V \tag{4.2}$$

Operators are linear

$$\hat{\mathrm{A}}(a_1\,|\Psi_1\rangle + a_2\,|\Psi_2\rangle) = (a_1\,\hat{\mathrm{A}}\,|\Psi_1\rangle + a_2\,\hat{\mathrm{A}}\,|\Psi_2\rangle) \tag{4.3}$$

$$|\Psi\rangle\,, |\Psi'\rangle \in V; a_1, a_2 \in \mathbb{C} \tag{4.4}$$

Operators are associative and commutative under addition

$$\hat{\mathrm{A}} + (\hat{\mathrm{B}} + \hat{\mathrm{C}}) = (\hat{\mathrm{A}} + \hat{\mathrm{B}}) + \hat{\mathrm{C}} \tag{4.5}$$

$$\hat{\mathrm{A}} + \hat{\mathrm{B}} = \hat{\mathrm{B}} + \hat{\mathrm{A}} \tag{4.6}$$

Multiplying operators is interpreted as applying them to kets. It is associative but NOT (in general) commutative.

$$\hat{\mathrm{A}}\,\hat{\mathrm{B}}\,|\Psi\rangle = \hat{\mathrm{A}}(\hat{\mathrm{B}}\,|\Psi\rangle) = \hat{\mathrm{A}}\,|\Psi'\rangle \tag{4.7}$$

$$\hat{\mathrm{A}}(\hat{\mathrm{B}}\,\hat{\mathrm{C}}) = (\hat{\mathrm{A}}\,\hat{\mathrm{B}})\,\hat{\mathrm{C}} \tag{4.8}$$

$$\hat{\mathrm{A}}\,\hat{\mathrm{B}} \neq \hat{\mathrm{B}}\,\hat{\mathrm{A}} \tag{4.9}$$

The lack of commutativeness makes the ”commutator” useful

$$[\hat{\mathrm{A}}, \hat{\mathrm{B}}] = \hat{\mathrm{A}}\,\hat{\mathrm{B}} - \hat{\mathrm{B}}\,\hat{\mathrm{A}} \tag{4.10}$$

$$\tag{4.11}$$

The inverse $\hat{\mathrm{A}}^{-1}$ of an operator is defined by

$$\hat{\mathrm{A}}^{-1}\,\hat{\mathrm{A}} = \hat{\mathrm{A}}\,\hat{\mathrm{A}}^{-1} = \mathbb{1} \tag{4.12}$$

4.2 Hermitian Operators

An operator is called Hermitian (or self-adjoint) if it is it's own hermitian conjugate $\hat{\mathrm{A}} = \hat{\mathrm{A}}^\dagger$

$$\hat{\mathrm{A}}\,|A\rangle = |B\rangle \rightarrow \langle A|\,\hat{\mathrm{A}}^\dagger = \langle B| \tag{4.13}$$
$$\hat{\mathrm{A}}\,|A\rangle = |B\rangle \rightarrow \langle A|\,\hat{\mathrm{A}} = \langle B| \tag{4.14}$$
$$\tag{4.15}$$

A hermitian operator $\hat{\mathrm{A}}$ has the following properties

1. $\hat{\mathrm{A}}\,|\lambda\rangle = \lambda\,|\lambda\rangle \rightarrow \lambda \in \mathbb{R}$
2. $\langle\hat{\mathrm{A}}\rangle = \langle\Psi|\,\hat{\mathrm{A}}\,|\Psi\rangle \in \mathbb{R}$
3. All eigenvectors with different eigenvalues are orthogonal

4.3 Projection Operators

A projection operator is defined

$$\tag{4.16}$$

4.4 Unitary operators

A unitary operator is defined by

$$\hat{\mathrm{U}}^{-1} = \hat{\mathrm{U}}^\dagger \tag{4.17}$$

This leads to (see also (4.12))

$$\hat{\mathrm{U}}^\dagger\,\hat{\mathrm{U}} = \hat{\mathrm{U}}\,\hat{\mathrm{U}}^\dagger = \mathbb{1} \tag{4.18}$$

The product of two unitary operators ($\hat{\mathrm{U}}^{-1} = \hat{\mathrm{U}}^\dagger; \hat{\mathrm{V}}^{-1} = \hat{\mathrm{V}}^\dagger$) is as well unitary

$$(\hat{\mathrm{U}}\,\hat{\mathrm{V}})^\dagger(\hat{\mathrm{U}}\,\hat{\mathrm{V}}) = \mathbb{1} \tag{4.19}$$
$$(\hat{\mathrm{U}}\,\hat{\mathrm{V}})(\hat{\mathrm{U}}\,\hat{\mathrm{V}})^\dagger = \mathbb{1} \tag{4.20}$$

The eigenvalues of a unitary operator have magnitude 1

$$\hat{\mathrm{U}}\,|\lambda\rangle = \lambda\,|\lambda\rangle \Rightarrow |\lambda|^2 = 1 \tag{4.21}$$
$$|\lambda\| = 1 \Rightarrow \lambda = e^{i\phi_\lambda}; \phi_\lambda \in \mathbb{R} \tag{4.22}$$

The eigenvectors are orthogonal $\langle\mu|\lambda\rangle = 0$.

Unitary transformations conserve the scalar product of two kets and the norm of a ket.

$$|\Psi_1'\rangle = \hat{\mathrm{U}}\,|\Psi_1'\rangle; |\Psi_2'\rangle = \hat{\mathrm{U}}\,|\Psi_2'\rangle \tag{4.23}$$
$$\langle\Psi_1'|\Psi_2'\rangle = \langle\Psi_1|\,\hat{\mathrm{U}}^\dagger\,\hat{\mathrm{U}}\,|\Psi_1\rangle = \langle\Psi_1|\Psi_2\rangle \tag{4.24}$$
$$\langle\Psi_1'|\Psi_1'\rangle = \langle\Psi_1|\Psi_1\rangle \tag{4.25}$$
$$\tag{4.26}$$

See also: Prof.M. Unitary Operators and TM Lecture 4

Chapter 5

Eigenvectors and Eigenvalues

5.1 Degenerate Eigenvectors

If two eigenvectors have the same eigenvalue:

$$\hat{\mathrm{H}}\,|\lambda_1\rangle = \lambda\,|\lambda_1\rangle \tag{5.1}$$

$$\hat{\mathrm{H}}\,|\lambda_2\rangle = \lambda\,|\lambda_2\rangle \tag{5.2}$$

their linear combination is an eigenvector as well:

$$\alpha\,\hat{\mathrm{H}}\,|\lambda_1\rangle = \lambda\alpha\,|\lambda_1\rangle \tag{5.3}$$

$$\beta\,\hat{\mathrm{H}}\,|\lambda_1\rangle = \lambda\beta\,|\lambda_1\rangle \tag{5.4}$$

$$\hat{\mathrm{H}}[\alpha\,|\lambda_1\rangle + \beta\,|\lambda_2\rangle] = \lambda[\alpha\,|\lambda_1\rangle + \beta\,|\lambda_2\rangle] \tag{5.5}$$

Therefore it is possible to create two orthogonal eigenvectors for this eigenvalue.

The probability in the degenerate case is the sum of the probabilities for each eigenvector $|\,\langle\hat{\mathrm{H}}|\,|\lambda_1\rangle\,|^2 + |\,\langle\hat{\mathrm{H}}|\,|\lambda_2\rangle\,|^2$

Chapter 6

Uncertanity

6.1 Probabilities

When Ψ is represented in a basis u

$$\hat{\mathrm{A}}\,|u_n\rangle = \lambda_n\,|u_n\rangle \tag{6.1}$$

$$|\Psi\rangle = \sum_n c_n\,|u_n\rangle \tag{6.2}$$

$$c_n = \langle u_n|\Psi\rangle \tag{6.3}$$

$|c_n|^2$ is the probability to get the eigenvalue λ_n as a result.

6.2 Expectation value and RMS

Expectation value of $\hat{\mathrm{A}}$ in state Ψ

$$\langle\hat{\mathrm{A}}\rangle_\Psi = \langle\Psi|\,\hat{\mathrm{A}}\,|\Psi\rangle \tag{6.4}$$

Root mean square deviation

$$\Delta\,\hat{\mathrm{A}} = \sqrt{\langle\hat{\sigma}_A^2\rangle_\Psi} \tag{6.5}$$

$$\hat{\sigma}_A = \hat{\mathrm{A}} - \langle\hat{\mathrm{A}}\rangle_\Psi \tag{6.6}$$

$$\Delta\,\hat{\mathrm{A}} = \sqrt{\langle\hat{\mathrm{A}}^2\rangle_\Psi - \langle\hat{\mathrm{A}}\rangle_\Psi^2} \tag{6.7}$$

6.3 Uncertanity Principle

$$\Delta\,\hat{\mathrm{A}}\,\Delta\,\hat{\mathrm{B}} \geq \frac{1}{2}|\,\langle[\hat{\mathrm{A}}\,\hat{\mathrm{B}}]\rangle\,| \tag{6.8}$$

6.4 Position and Momentun

$$[\hat{\mathrm{x}}, \hat{\mathrm{p}}] = i\hbar \tag{6.9}$$

$$\Delta\,\hat{\mathrm{x}}\,\Delta\,\hat{\mathrm{p}} \geq \frac{\hbar}{2} \tag{6.10}$$