Quantum Mechanics

Stefan Aeschbacher

July 13, 2021

Chapter 1

Dirac notation

A bra $\langle v|$ is an element in a complex vector space. The corresponding ket $|v\rangle$ is an elemen in its dual space. The usual rules of linear algebra are valid:

$$|u\rangle + |v\rangle = |w\rangle \tag{1.1}$$

$$c\,|v\rangle = |u\rangle \,; c \in \mathbb{C} \tag{1.2}$$

Convert between bras and kets:

$$c_1\,|v_1\rangle + c_2\,|v_2\rangle \iff c_1^*\,\langle v_1| + c_2^*\,\langle v_2| \tag{1.3}$$

1.1 Inner product

$$\langle u|v\rangle = \langle v|u\rangle^* \tag{1.4}$$

$$\langle v|v\rangle \geq 0 \tag{1.5}$$

$$\langle v|v\rangle = 0 \iff v = 0 \tag{1.6}$$

Linearity in the second argument and antilinear in the first:

$$\langle u|c_1 v_1 + c_2 v_2\rangle = c_1\,\langle u|v_1\rangle + c_2\,\langle u|v_2\rangle \tag{1.7}$$

$$\langle c_1 u_1 + c_2 u_2|v\rangle = c_1^*\,\langle u_1|v\rangle + c_2^*\,\langle u_2|v\rangle \tag{1.8}$$

$$\tag{1.9}$$

1.2 Outer product

A bra and a ket can be combined in the outer product to create an operator

$$X = |v\rangle\,\langle u| \tag{1.10}$$

$$X\,|\Psi\rangle = |v\rangle\,\langle u|\,|\Psi\rangle = |v\rangle\,\langle u|\Psi\rangle \tag{1.11}$$

$$\tag{1.12}$$

Chapter 2

Postulates

2.1 Postulate 1: state

Postulate 1 *The state of a physical system is described by a state vector that belongs to a complex vector space V, called the state space of the system.*

2.2 Postulate X: time evolution

$$i\hbar \frac{\partial}{\partial t} \ket{\Psi(t)} = \hat{\mathrm{H}}(t) \ket{\Psi(t)} \tag{2.1}$$

Chapter 3

State space

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_3\rangle \tag{3.1}$$

$$|\Psi_1\rangle + |\Psi_2\rangle = |\Psi_2\rangle + |\Psi_1\rangle \tag{3.2}$$

Chapter 4

Operators

4.1 Basic Properties

An operator acting on a ket creates a new ket:

$$\hat{A}\,|\Psi\rangle = |\Psi'\rangle \tag{4.1}$$
$$|\Psi\rangle, |\Psi'\rangle \in V \tag{4.2}$$

Operators are linear

$$\hat{A}(a_1\,|\Psi_1\rangle + a_2\,|\Psi_2\rangle) = (a_1\,\hat{A}\,|\Psi_1\rangle + a_2\,\hat{A}\,|\Psi_2\rangle) \tag{4.3}$$
$$|\Psi\rangle, |\Psi'\rangle \in V; a_1, a_2 \in \mathbb{C} \tag{4.4}$$

Operators are associative and commutative under addition

$$\hat{A} + (\hat{B} + \hat{C}) = (\hat{A} + \hat{B}) + \hat{C} \tag{4.5}$$
$$\hat{A} + \hat{B} = \hat{B} + \hat{A} \tag{4.6}$$

Multiplying operators is interpreted as applying them to kets. It is associative but NOT (in general) commutative.

$$\hat{A}\,\hat{B}\,|\Psi\rangle = \hat{A}(\hat{B}\,|\Psi\rangle) = \hat{A}\,|\Psi'\rangle \tag{4.7}$$
$$\hat{A}(\hat{B}\,\hat{C}) = (\hat{A}\,\hat{B})\,\hat{C} \tag{4.8}$$
$$\hat{A}\,\hat{B} \neq \hat{B}\,\hat{A} \tag{4.9}$$

The lack of commutativeness makes the "commutator" useful

$$[\hat{A}, \hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A} \tag{4.10}$$
$$\tag{4.11}$$

4.2 Hermitian Operators

An operator is called Hermitian (or self-adjoint) if it is it's own conjugate $\hat{A} = \hat{A}^{\dagger}$

$$\hat{A}\ket{A} = \ket{B} \to \bra{A}\hat{A}^{\dagger} = \bra{B} \tag{4.12}$$

$$\hat{A}\ket{A} = \ket{B} \to \bra{A}\hat{A} = \bra{B} \tag{4.13}$$

$$\tag{4.14}$$

A hermitian operator $\hat{A}$ has the following properties

1. $\hat{A}\ket{\lambda} = \lambda\ket{\lambda} \to \lambda \in \mathbb{R}$

2. $\langle\hat{A}\rangle = \bra{\Psi}\hat{A}\ket{\Psi} \in \mathbb{R}$

3. All eigenvectors with different eigenvalues are orthogonal

4.3   Projection Operators

A projection operator is defined

$$[\hat{A}, \hat{B}] = \hat{A}\,\hat{B} - \hat{B}\,\hat{A} \tag{4.15}$$

$$\tag{4.16}$$

Chapter 5

Eigenvectors and Eigenvalues

5.1 Degenerate Eigenvectors

If two eigenvectors have the same eigenvalue:

$$\hat{H} \ket{\lambda_1} = \lambda \ket{\lambda_1} \tag{5.1}$$

$$\hat{H} \ket{\lambda_2} = \lambda \ket{\lambda_2} \tag{5.2}$$

their linear combination is an eigenvector as well:

$$\alpha \, \hat{H} \ket{\lambda_1} = \lambda \alpha \ket{\lambda_1} \tag{5.3}$$

$$\beta \, \hat{H} \ket{\lambda_1} = \lambda \beta \ket{\lambda_1} \tag{5.4}$$

$$\hat{H}[\alpha \ket{\lambda_1} + \beta \ket{\lambda_2}] = \lambda[\alpha \ket{\lambda_1} + \beta \ket{\lambda_2}] \tag{5.5}$$

Therefore it is possible to create two orthogonal eigenvectors for this eigenvalue.

The probability in the degenerate case is the sum of the probabilities for each eigenvector $|\bra{\hat{H}} \ket{\lambda_1}|^2 + |\bra{\hat{H}} \ket{\lambda_2}|^2$