Chapter 5

Question-1

$f(x , y)=K(5x + 2y + 4);$ $x = -1 , 1 , 2;$ $y = 1 , 2 , 3$

Find K

$y \backslash x$	-1	1	2	$\sum_x$
1	K	$11K$	$16K$	$28K$
2	$3K$	$13K$	$18K$	$34K$
3	$5K$	$15K$	$20K$	$40K$
$\sum_y$	$9K$	$39K$	$54K$	$102K$

$f(x,y) = K(5x+2y+4) = 1;$ $x=-1,1,2;$ $y=1,2,3$

$102K = 1 \longrightarrow \boxed{K = \frac{1}{102}}$

Find the marginal for x, y

$P_X(x)$

$P_X(x) = \sum_y (f(x,y))$

$= \frac{1}{102} \sum_{y=1}^{3} (5x+2y+4)$

$(5x+6)+(5x+8)+(5x+10)$

$\boxed{P_X(x) = \frac{1}{102}(15x+24); \quad x=-1,1,2}$

$P_Y(y)$

$P_Y(y) = \sum_x (f(x,y))$

$= \frac{1}{102} \sum_{x=-1}^{2} (5x+2y+4)$

$(2y-1)+(2y+9)+(2y+14)$

$\boxed{P_Y(y) = \frac{1}{102}(6y+22); \quad y=1,2,3}$

$y \backslash x$	-1	1	2	$\sum_x$
1	$1/102$	$11/102$	$16/102$	$28/102$
2	$3/102$	$13/102$	$18/102$	$34/102$
3	$5/102$	$15/102$	$20/102$	$40/102$
$\sum_y$	$9/102$	$39/102$	$54/102$	1

Find $E(X), E(Y)$

$$E(X)=\sum x \cdot f(x)$$

$$E(X)=\left(-1\times\frac{9}{102}\right)+\left(1\times\frac{39}{102}\right)+\left(2\times\frac{54}{102}\right)$$

$$\boxed{E(X)=\frac{23}{17}=1.3529}$$

$$E(y)=\sum y \cdot f(y)$$

$$E(y)=\left(1\times\frac{28}{102}\right)+\left(2\times\frac{34}{102}\right)+\left(3\times\frac{40}{102}\right)$$

$$\boxed{E(y)=\frac{36}{17}=2.1176}$$

Find $E(X^2), E(y^2)$

$$E(X^2)=\sum x^2 \cdot f(x)$$

$$E(x^2)=\left(-1^2\times\frac{9}{102}\right)+\left(1^2\times\frac{39}{102}\right)+\left(2^2\times\frac{54}{102}\right)$$

$$\boxed{E(x^2)=\frac{44}{17}=2.5882}$$

$$E(y^2)=\sum y^2 \cdot f(y)$$

$$E(y^2)=\left(1^2\times\frac{28}{102}\right)+\left(2^2\times\frac{34}{102}\right)+\left(3^2\times\frac{40}{102}\right)$$

$$\boxed{E(y^2)=\frac{262}{51}=5.1372}$$

Find $Var(X)$, $Var(y)$

$$Var(x)=E(x^2)-(E(x))^2$$

$$\boxed{Var(X)=\frac{44}{17}-\left(\frac{23}{17}\right)^2=\frac{219}{289}=0.7577}$$

$$Var(y)=E(y^2)-(E(y))^2$$

$$\boxed{Var(y)=\frac{262}{51}-\left(\frac{36}{17}\right)^2=\frac{566}{867}=0.65282}$$

Find $E(Xy)$ $\longrightarrow$ $E(xy)=\sum_x\sum_y x\cdot y\,(f(x,y))$

$$E(Xy)=\left(-1\times1\times\frac{1}{102}\right)+\left(-1\times2\times\frac{3}{102}\right)+\left(-1\times3\times\frac{5}{102}\right)+\left(1\times1\times\frac{11}{102}\right)+\left(1\times2\times\frac{13}{102}\right)+\left(1\times3\times\frac{15}{102}\right)+\left(2\times1\times\frac{16}{102}\right)$$
$$+\left(2\times2\times\frac{18}{102}\right)+\left(2\times3\times\frac{20}{102}\right)=\boxed{E(xy)=\frac{142}{51}=2.784}$$

Find $Cov(x,y)$

$$Cov(x,y)=E(xy)-E(x)\cdot E(y)$$
$$=\frac{142}{51}-\left(\frac{23}{17}\times\frac{36}{17}\right)$$

$$\boxed{Cov(x,y)=-\frac{70}{867}=-0.0807}$$

Find $\rho(x,y)$

$$\rho=\frac{cov(x,y)}{\sqrt{var(x)\cdot var(y)}}=\frac{-\frac{70}{867}}{\sqrt{\frac{219}{289}\times\frac{566}{867}}}$$

$$\boxed{\rho=-0.11479}$$

Question-2

$f(x, y)=K(3x + 2y + 5);$ $x = 1, 2;$ $y = 2, 3$

find k

$f(x,y)=K(3x+2y+5)=1;$

$$\boxed{k=\frac{1}{58}}$$

$x=1,2;$
$y=2,3$

y \ x	1	2	Σ_x
2	12K	15K	27K
3	14K	17K	31K
Σ_y	26K	32K	58K

marginal

$P_X(X)$

$P_x(x)=\sum_y (f(x,y))$

$=(3x+9)+(3x+11)$

$P_x(x)=\frac{1}{58}(6x+20);$ $x=1,2;$

$P_Y(y)$

$P_Y(y)=\sum_x (f(x,y))$

$=(2y+8)+(2y+11)$

$P_Y(y)=\frac{1}{58}(4y+19);$ $y=2,3$

y \ x	1	2	Σ_x
2	12/58	15/58	27/58
3	14/58	17/58	31/58
Σ_y	26/58	32/58	1

$E(x)$

$E(x)=\sum x \cdot f(x)$

$E(x)=\left(1\times\frac{26}{58}\right)+\left(2\times\frac{32}{58}\right)=\boxed{\frac{45}{29}=1.5517}$

$E(y)$

$E(y)=\sum y \cdot f(y)$

$E(y)=\left(2\times\frac{27}{58}\right)+\left(3\times\frac{31}{58}\right)=\boxed{\frac{147}{58}=2.534}$

$E(x^2)$

$E(x^2)=\sum x^2 \cdot f(x)$

$E(x^2)=\left(1^2\times\frac{26}{58}\right)+\left(2^2\times\frac{32}{58}\right)=\boxed{\frac{77}{29}=2.65517}$

$E(y^2)$

$E(y^2)=\sum y^2 \cdot f(y)$

$E(y^2)=\left(2^2\times\frac{27}{58}\right)+\left(3^2\times\frac{31}{58}\right)=\boxed{\frac{387}{58}=6.672}$

Var(X)

$$Var(X) = E(X^2) - (E(X))^2$$

$$Var(X) = \frac{77}{29} - \left(\frac{45}{29}\right)^2 = \boxed{\frac{208}{841} = 0.2473}$$

Var(y)

$$Var(y) = E(y^2) - (E(y))^2$$

$$Var(y) = \frac{387}{58} - \left(\frac{147}{58}\right)^2 = \boxed{\frac{837}{3364} = 0.2488}$$

find E(xy)

$$E(xy) = \sum_x \sum_y x \cdot y (f(x,y))$$

$$E(xy) = \left(1 \times 2 \times \frac{12}{58}\right) + \left(1 \times 3 \times \frac{14}{58}\right) + \left(2 \times 2 \times \frac{15}{58}\right) + \left(2 \times 3 \times \frac{17}{58}\right)$$

$$\boxed{E(xy) = \frac{114}{29} = 3,9310}$$

COV(X, Y)

$$Cov(x,y) = E(xy) - E(x) \cdot E(y)$$

$$= \frac{114}{29} - \left(\frac{45}{29} \times \frac{147}{58}\right)$$

$$Cov(x,y) = \frac{3825}{1682} = \boxed{-0.001783}$$

$\rho(x, y)$

$-1 < \rho < 1$

$$\rho = \frac{cov(x,y)}{\sqrt{Var(x) \cdot Var(y)}}$$

$$= \frac{-0.001783}{\sqrt{\frac{208}{841} \times \frac{837}{3364}}}$$

$$\boxed{\rho = -0.007189}$$

Question-3

f(x , y)=K(4x + 2y + 5); **x = -1 , 1 , 2;** **y = 1 , 3 , 5**

Find (k)

$f(x,y) = K(4x + 2y + 5) = 1$

$123K = 1 \Longrightarrow \boxed{K = \frac{1}{123}}$

y \ x	-1	1	2	Σ_x
1	3/123	11/123	15/123	29/123
3	7/123	15/123	19/123	41/123
5	11/123	19/123	23/123	53/123
Σ_y	21/123	45/123	57/123	1

Marginal

X

$P_X(x) = \sum_y f(x,y)$

$= (4x+7) + (4x+11) + (4x+15)$

$\boxed{P_X(x) = \frac{1}{123}(12x+33);\ x=-1,1,2}$

$E(x) = \sum_x x \cdot f(x)$

$= \frac{-21 \times 45 \times 57 \times 2}{123}$

$\boxed{E(x) = \frac{46}{41} = 1.1219}$

$E(x^2) = \sum_x x^2 \cdot f(x)$

$= \frac{21 + 45 + 57 \times 4}{123}$

$\boxed{E(x^2) = \frac{294}{123} = 2.3902}$

$Var(x) = E(x^2) - (E(x))^2$

$\boxed{Var(x) = \frac{294}{123} - \left(\frac{46}{41}\right)^2 = \frac{1902}{1681} = 1.1314}$

y

$P_Y(y) = \sum_x f(x,y)$

$= (2y+1) + (2y+9) + (2y+13)$

$\boxed{P_Y(y) = \frac{1}{123}(6y+23);\ y=1,3,5}$

$E(y) = \sum_y y \cdot f(y)$

$= \frac{29 + 3 \times 41 + 5 \times 53}{123}$

$\boxed{E(y) = \frac{409}{123} = 3.3252}$

$E(y^2) = \sum_y y^2 \cdot f(y)$

$= \frac{29 + 9 \times 41 + 25 \times 53}{123}$

$\boxed{E(y^2) = \frac{1723}{123} = 14.0081}$

$Var(y) = E(y^2) - (E(y))^2$

$\boxed{Var(y) = \frac{1723}{123} - \left(\frac{409}{123}\right)^2 = 2.9511}$

$$E(XY) = \sum_x \sum_y x \cdot y\,(f(x,y))$$

$$E(XY) = \left(\frac{-3}{123}\right) + \left(\frac{-21}{123}\right) + \left(\frac{-55}{123}\right) + \left(\frac{11}{123}\right) + \left(\frac{45}{123}\right) + \left(\frac{95}{123}\right) + \left(\frac{30}{123}\right) + \left(\frac{114}{123}\right) + \left(\frac{230}{123}\right)$$

$$\boxed{E(XY) = \frac{446}{123} = 3.6260}$$

$$\mathrm{Cov}(X,Y)$$

$$\mathrm{Cov}(x,y) = E(XY) - (E(x) \cdot E(y))$$

$$\mathrm{Cov}(x,y) = \frac{446}{123} - \left(\frac{46}{41} \times \frac{409}{123}\right)$$

$$\boxed{\mathrm{Cov}(x,y) = -0.1047}$$

$$\rho(X,Y)$$

$$\rho(x,y) = \frac{\mathrm{Cov}(x,y)}{\sqrt{\mathrm{Var}(x) \cdot \mathrm{Var}(y)}}$$

$$\rho(x,y) = \frac{-0.1047}{\sqrt{1.1314 \times 2.9511}}$$

$$\boxed{\rho(x,y) = -0.0573}$$

Question-4

f(x , y)=K(4x + 5y + 3);   1 < x < 2 ;   1 < y < 3

Find K $\longrightarrow \int_y \int_x f(x,y)\,dx\,dy \longrightarrow \int_1^3 \int_1^2 K(4x+5y+3)\,dx\,dy$

$\frac{99}{2} - \frac{23}{2} = 38$ $38K = 1 \longrightarrow \boxed{K = \frac{1}{38}}$

Marginal

$P_X(x)$	$P_Y(y)$ 32
$P_X(x) = \int_y f(x,y)\,dy$	$P_Y(y) = \int_x f(x,y)\,dx$
$\frac{1}{38}\int_1^3 (4x+5y+3)\,dy$	$P_Y(y) = \int_1^2 \frac{1}{38}(4x+5y+3)\,dx$
$P_X(x) = \frac{1}{38}(8x+26)$; $1 < x < 2$	$P_Y(y) = \frac{1}{38}(5y+9)$; $1 < y < 3$

$E(X)$	$E(y)$
$E(X) = \int_x x.P_X(x)\,dx$	$E(y) = \int_y y\,P_Y(y)\,dy$
$E(X) = \frac{1}{38}\int_1^2 x.(8x+26)\,dx$	$E(y) = \frac{1}{38}\int_1^3 y.(5y+9)\,dy$
$\boxed{E(X) = \frac{173}{114} = 1.5175}$	$\boxed{E(y) = \frac{119}{57} = 2.0877}$
$E(x^2) = \int x^2.P_X(x)\,dx$	$E(y^2) = \int_y y^2.P_Y(y)\,dy$
$E(x^2) = \frac{1}{38}\int_1^2 x^2.(8x+26)\,dx$	$E(y^2) = \frac{1}{38}\int_1^3 y^2.(5y+9)\,dy$
$\boxed{E(x^2) = \frac{136}{57} = 2.3859}$	$\boxed{E(y^2) = \frac{89}{19} = 4.6842}$

Var(x)

$$Var(x) = E(x^2) - (E(x))^2$$

$$Var(x) = \frac{136}{57} - \left(\frac{173}{114}\right)^2$$

$$\boxed{Var(x) = 0.0830}$$

Var(y)

$$Var(y) = E(y^2) - (E(y))^2$$

$$Var(y) = \frac{89}{19} - \left(\frac{119}{57}\right)^2$$

$$\boxed{Var(y) = 0.3256}$$

$$E(xy) = \int_x\int_y x.y\, f(x,y)\, dy\, dx$$

$$\frac{1}{38}\int_1^2 x \int_1^3 y(4x+5y+3)\,dy\,dx \Rightarrow \frac{1}{38}\int_1^2 x \left| 2xy^2 + \frac{5}{3}y^3 + \frac{3}{2}y^2 \right|_1^3 dx$$

$$\frac{1}{38}\int_1^2 x\left(4x + \frac{166}{3}\right) dx \Rightarrow \frac{1}{38}\left(\frac{4x^3}{3} + \frac{166}{6}x^2\right|_1^2$$

$$\boxed{E(xy) = \frac{19}{6} = 3.1667}$$

Cov(x,y)

$$Cov(x,y) = E(xy) - (E(x) \cdot E(y))$$

$$Cov(x,y) = \frac{19}{6} - \left(\frac{173}{114} \times \frac{119}{57}\right)$$

$$\boxed{Cov(x,y) = -\frac{5}{3249} = -0.00154}$$

$\rho(x,y)$

$$\rho(x,y) = \frac{Cov(x,y)}{\sqrt{Var(x) \cdot Var(y)}}$$

$$\rho(x,y) = \frac{-\frac{5}{3249}}{\sqrt{0.0830 \times 0.3256}}$$

$$\boxed{\rho(x,y) = -0.00936}$$

Question-5

f(x , y)=K(4x + 5y + 3); 1 < x < 3 ; 2 < y < 4

Find $K \longrightarrow \int_1^3 \int_2^4 f(x,y)\,dy\,dx \longrightarrow \int_1^3 \int_2^4 k(4x+5y+3)\,dy\,dx$

$k \int_1^3 \int_2^4 (4xy+\frac{5y^2}{2}+3y)\,d\ d \Longrightarrow k\int_1^3 (8x+36)\,dx$

$k\ \left|4x^2+36x\right|_1^3 \Longrightarrow 104 \qquad 104k=1 \longrightarrow \boxed{k=\frac{1}{104}}$

marginal

$P_x(x)=\int_y f(x,y)\,dy$

$P_x(x)=\int_2^4 \frac{1}{104}(4x+5y+3)\,dy$

$\frac{1}{104}\left(4xy+\frac{5y^2}{2}+3y\right|_2^4$

$\boxed{P_x(x)=\frac{1}{104}(8x+36);\quad 1<x<3}$

$P_Y(y)=\int_x f(x,y)\,dx$

$P_Y(y)=\int_1^3 \frac{1}{104}(4x+5y+3)\,dx$

$=\frac{1}{104}\left(2x^2+5xy+3x\right|_1^3$

$\boxed{P_Y(y)=\frac{1}{104}(10y+22);\quad 2<y<4}$

$E(x)=\int_x x.(P_x(x))\,dx$

$\frac{1}{104}\int_1^3 x(8x+36)\,dx$

$\boxed{E(x)=\frac{80}{39}=2.0513}$

$E(y)=\int_y y.(P_Y(y))\,dy$

$\frac{1}{104}\int_2^4 y(8x+36)\,dy$

$\boxed{E(y)=\frac{239}{78}=3.0641}$

$E(x^2)=\int x^2.P_x(x)\,dx$

$E(x^2)=\frac{1}{104}\int_1^2 x^2.(8x+26)\,dx$

$\boxed{E(x^2)=\frac{59}{13}=4.5384}$

$E(y^2)=\int_y y^2.P_Y(y)\,dy$

$E(y^2)=\frac{1}{104}\int_1^3 y^2.(10y+22)\,dy$

$\boxed{E(y^2)=\frac{379}{39}=9.7179}$

Var(X)

$$Var(x)=E(x^2)-(E(x))^2$$

$$Var(x)=\frac{59}{13}-\left(\frac{80}{39}\right)^2$$

$$\boxed{Var(x)=\frac{503}{1521}=0.3307}$$

Var(y)

$$Var(y)=E(y^2)-(E(y))^2$$

$$Var(y)=\frac{379}{39}-\left(\frac{239}{78}\right)^2$$

$$\boxed{Var(y)=\frac{2003}{6084}=0.3292}$$

$$E(xy)=\iint_{x\,y} x.y\, f(x,y)\, dy\, dx \Longrightarrow \frac{1}{104}\iint_{x\,y} x.y\,(4x+5y+3)\, dy\, dx$$

$$E(xy)=\int_x x\left|2xy^2+\frac{5y^3}{3}+\frac{3y^2}{2}\right|_{2}^{4} dx \Longrightarrow \int_1^3 x\left(24x+\frac{334}{3}\right)dx$$

$$\boxed{E(xy)=\int_1^3 x\left(24x+\frac{334}{3}\right)dx=6.2820}$$

Cov(x,y)

$$Cov(x,y)=E(xy)-(E(x).E(y))$$

$$Cov(x,y)=6.2820-\left(\frac{80}{39}\times\frac{239}{78}\right)$$

$$\boxed{Cov(x,y)=-0.00334}$$

$\rho(x,y)$

$$\rho(x,y)=\frac{Cov(x,y)}{\sqrt{Var(x)\cdot Var(y)}}$$

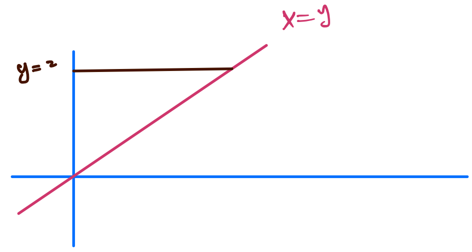
$$\rho(x,y)=\frac{-0.00334}{\sqrt{\frac{2003}{6084}\times\frac{503}{1521}}}$$

$$\boxed{\rho(x,y)=-0.010122}$$

Question-6

f(x , y)=K(3x + 2y + 5); 0 < x <y < 2

$$\int_0^2\int_{y=x}^2 f(x,y)\,dy\,dx \qquad \int_0^2\int_0^{x=y} f(x,y)\,dx\,dy$$



$$\int_0^2\int_0^{x=y} K(3x+2y+5)\,dx\,dy \Rightarrow K\int_0^2 \frac{3x^2}{2}+2xy+5x\Big|_0^{x=y} dy$$

$$K\int_0^2 \frac{3y^2}{2}+2y^2+5y\,dy \Longrightarrow K\left(\frac{3y^3}{6}+\frac{2y^3}{3}+\frac{5y^2}{2}\right|_0^2$$

$$K\left(4+\frac{16}{3}+10\right)=\frac{58}{3}K=1 \qquad \boxed{K=\frac{3}{58}}$$

marginal

$$P_X(x)=\int_y f(x,y)\,dy \qquad P_Y(y)=\int_x f(x,y)\,dx$$

$$\frac{3}{58}\int_{y=x}^2 (3x+2y+5)\,dy \Rightarrow (3xy+y^2+5y)\Big|_{y=x}^2$$

$$\frac{3}{58}\left(6x+4+10\right)-\left(3x^2+x^2+5x\right)$$

$$\boxed{P_X(x)=\frac{3}{58}\left(-4x^2+x+14\right) \quad 0<x<2}$$

$$\frac{3}{58}\int_0^{x=y} (3x+2y+5)\,dx \quad \left(\frac{3x^2}{2}+2xy+5x\right)\Big|_0^{x=y}$$

$$\boxed{P_Y(y)=\frac{3}{58}\left(\frac{3y^2}{2}+2y^2+5y\right) \quad 0<y<2}$$

$$E(X) = \int_x X \cdot (P_x(X))\, dx$$

$$\frac{3}{58}\int_0^2 X(-4X^2+X+14)\, dX$$

$$\boxed{E(X) = \frac{22}{29} = 0.7586}$$

$$E(y) = \int_y y \cdot (P_Y(y))\, dy$$

$$\frac{3}{58}\int_0^2 y\left(\frac{3y^2}{2}+2y^2+5y\right) dy$$

$$\boxed{E(y) = \frac{41}{29} = 1.4138}$$

$$E(X^2) = \int_x X^2 \cdot (P_x(X))\, dx$$

$$\frac{3}{58}\int_0^2 X^2(-4X^2+X+14)\, dX$$

$$\boxed{E(X^2) = \frac{118}{145} = 0.8138}$$

$$E(y^2) = \int_y y^2 \cdot (P_Y(y))\, dy$$

$$\frac{3}{58}\int_0^2 y^2\left(\frac{3y^2}{2}+2y^2+5y\right) dy$$

$$\boxed{E(y^2) = \frac{318}{145} = 2.1931}$$

Var(X)

$$Var(X) = E(X^2) - (E(X))^2$$

$$\boxed{Var(X) = \frac{118}{145} - \left(\frac{22}{29}\right)^2 = \frac{1002}{4205} = 0.2383}$$

Var(y)

$$Var(y) = E(y^2) - (E(y))^2$$

$$\boxed{Var(y) = \frac{318}{145} - \left(\frac{41}{29}\right)^2 = \frac{817}{4205} = 0.1943}$$

$$E(Xy) = \int_x\int_y X \cdot y\, f(X,y)\, dy\, dX$$

$$= \frac{3}{58}\int_0^2\int_0^{x=y} X \cdot y(3X+2y+5)\, dx\, dy$$

$$\boxed{E(X,y) = \frac{171}{145} = 1.1793}$$

$$Cov(X,y) = E(Xy) - (E(X) \cdot E(y))$$

$$Cov(X,y) = \frac{449}{4205} = 0.1068$$

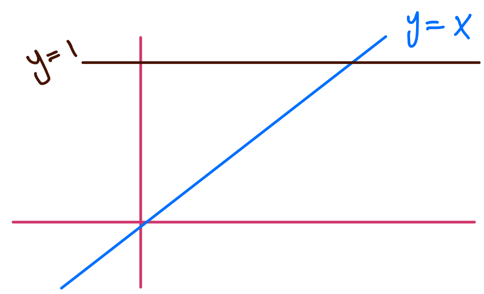
$$\rho(X,y) = \frac{Cov(X,y)}{\sqrt{Var(X) \cdot Var(y)}}$$

$$\boxed{\rho(X,y) = \frac{0.1068}{\sqrt{\frac{817}{4205} \times \frac{1002}{4205}}} = 0.49625}$$

Question-7

f(x , y)=K(x^2 + y^2); **0 < x < y < 1**

$\int_0^1\int_{y=x}^1 f(x,y)\,dy\,dx$	$\int_0^1\int_0^{x=y} f(x,y)\,dx\,dy$



$$\int_0^1\int_0^{x=y}(x^2+y^2)\,dx\,dy=1 \Longrightarrow \frac{1}{3}K=1 \Longrightarrow \boxed{K=3}$$

$$P_X(x)=\int_y f(x,y)\,dy = 3\int_{y=x}^{1}(x^2+y^2)\,dy$$

$$P_X(x)=3\left(\frac{-4x^3}{3}+x^2+\frac{1}{3}\right); \quad 0<x<1$$

$$P_Y(y)=\int_x f(x,y)\,dx$$

$$P_Y(y)=3\left(\frac{y^3}{3}+y^3\right); \quad 0<y<1$$

$$E(x)=\int_x x.(P_X(x))\,dx$$

$$\boxed{E(x)=0.45}$$

$$E(y)=\int_y y.(P_Y(y))\,dy$$

$$\boxed{E(y)=0.8}$$

$$E(x^2)=\int_x x^2.(P_X(x))\,dx$$

$$\boxed{E(x^2)=0.2667}$$

$$E(y^2)=\int_y y^2.(P_Y(y))\,dy$$

$$\boxed{E(y^2)=0.6667}$$

Var(x)

$$Var(x)=E(x^2)-(E(x))^2$$

$$Var(x)=(0.2667)-(0.45)^2$$

$$\boxed{Var(x)=0.0642}$$

Var(y)

$$Var(y)=E(y^2)-(E(y))^2$$

$$Var(y)=0.6667-(0.8)^2$$

$$\boxed{Var(y)=0.0267}$$

$$E(XY) = \int_y \int_x x \cdot y \, f(x,y) \, dx\,dy$$

$$E(x,y) = 3 \int_0^1 y \int_0^{x=y} x(x^2+y^2)\,dx\,dy$$

$$\boxed{E(x,y) = 0.375}$$

$$Cov(x,y) = E(xy) - (E(x) \cdot E(y))$$

$$Cov(x,y) = (0.375) - (0.45 \times 0.8)$$

$$\boxed{Cov(x,y) = \frac{3}{200} = 0.015}$$

$$\rho(X,Y) = \frac{cov(x,y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

$$\boxed{\rho(X,Y) = \frac{0.015}{\sqrt{0.0642 \times 0.0267}} = 0.3623}$$

8.3 The random process $X(t)$ is given by

$$X(t) = Y\cos(2\pi t) \quad t \geq 0$$

where $\boxed{Y \text{ is a random variable}}$ that is uniformly distributed between 0 and 2. Find the expected value and autocorrelation function of $X(t)$.

$U(0,2)$ $\quad \boxed{E(y) = \frac{2+0}{2} = \boxed{1}}$ $\quad \boxed{Var(y) = \frac{(2-0)^2}{12} = \boxed{\frac{1}{3}}}$ $\quad \boxed{E(y^2) = \frac{1}{3} + 1 = \frac{4}{3}}$

(a) The mean of Y:

$$E(X(t)) = E[Y\cos(2\pi t)] \Rightarrow = E[Y]\cos(2\pi t)$$

$$\boxed{E(X(t)) = \cos(2\pi t)}$$

(b) The autocorrelation

$$R_{XX}(t, t+\tau) = E(X(t) \cdot X(t+\tau))$$

$$= E(Y\cos(2\pi t)\, Y\cos(2\pi t + 2\pi\tau))$$

$$= E(Y^2)\cos(2\pi t)\cos(2\pi t + 2\pi\tau)$$

$$= \frac{4}{3}\cos(2\pi t)\cos(2\pi t + 2\pi\tau)$$

$$\cos A \cdot \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\boxed{R_{XX}(t, t+\tau) = \frac{2}{3}(\cos(2\pi\tau) + \cos(4\pi t + 2\pi\tau))}$$

8.4 The sample function $X(t)$ of a stationary random process $Y(t)$ is given by

$$X(t) = Y(t)\sin(wt + \Theta)$$

where w is a constant, $Y(t)$ and Θ are statistically independent, and Θ is uniformly distributed between 0 and 2π. Find the autocorrelation function of $X(t)$ in terms of $R_{YY}(\tau)$.

$U(0,2)$ $\boxed{E(y) = \frac{0+2\pi}{2} = \pi}$ $\boxed{Var(y) = \frac{(2\pi-0)^2}{12} = \frac{\pi^2}{3}}$ $\boxed{f_\theta(\theta) = \frac{1}{2\pi}}$

$$R_{XX}(t,t+\tau) = E(X(t) \cdot X(t+\tau))$$

$$= E(Y(t)\sin(Wt+\theta) \cdot Y(t+\tau)\sin(Wt+W\tau+\theta))$$

$$E(Y(t) \cdot Y(t+\tau)) = R_{YY}(\tau)$$

$$R_{YY}(\tau)E(\sin(Wt+\theta)\sin(Wt+W\tau+\theta)) \Rightarrow \sin A \sin B = \tfrac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\tfrac{1}{2}R_{YY}(\tau)\,E(\cos(W\tau) - \cos(4Wt+2W\tau+2\theta))$$

$$\underline{\tfrac{1}{2}R_{YY}(\tau)\cos(W\tau)} - \underline{E(\cos(4Wt+2W\tau+2\theta))}$$

$$E(\cos(4Wt+2W\tau+2\theta)) = \frac{1}{2\pi}\int_0^{2\pi}\cos(4Wt+2W\tau+2\theta)\,d\theta$$

$$\frac{1}{4\pi}\left(\sin(4Wt+2W\tau+2\theta)\right)\Big|_0^{2\pi} = 0$$

$$\tfrac{1}{2}R_{YY}(\tau)\cos(W\tau) - 0$$

$$\boxed{R_{XX}(t,t+\tau) = \tfrac{1}{2}R_{YY}(\tau)\cos(Wt)}$$

8.7 Assume that Y is a random variable that is uniformly distributed between 0 and 2. If we define the random process $X(t) = Y\cos(2\pi t)$, $t \geq 0$, find the autocovariance function of $X(t)$.

$U(0,2)$ $\quad E(y) = \frac{2+0}{2} = \boxed{1}$ $\quad Var(y) = \frac{(2-0)^2}{12} = \boxed{\frac{1}{3}}$ $\quad E(y^2) = \frac{1}{3} + 1 = \frac{4}{3}$

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - \mu_X(t) \cdot \mu_X(t+\tau)$$

$$\mu_X(t) = \cos(2\pi t)$$

$$\mu_X(t)\mu_X(t+\tau) = \cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau)$$

$$R_{XX}(t, t-\tau) = E(X(t) \cdot X(t+\tau))$$

$$= E(Y^2)\cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau)$$

$$= \frac{4}{3}\cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau)$$

$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - \mu_X(t) \cdot \mu_X(t+\tau)$$

$$\frac{4}{3}\cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau) - \cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau)$$

$$\frac{1}{3}\cos(2\pi t) \cdot \cos(2\pi t + 2\pi\tau) \Longrightarrow \cos A \cdot \cos B \text{ law}$$

$$\frac{1}{6} \cos(2\pi\tau) + \cos(4\pi t + 2\pi\tau)$$

8.8 A random process $X(t)$ is given by

$$X(t) = A\cos(t) + (B+1)\sin(t) \qquad -\infty < t < \infty$$

where A and B are independent random variables with $E[A] = E[B] = 0$ and $\underline{E[A^2] = E[B^2] = 1}$. Find the autocovariance function of $X(t)$.

$$M_X = E(X(t)) = E\left(A\cos(t) + (B+1)\cdot\sin(t)\right)$$
$$= E(A)\cos(t) + E(B\sin(t)) + \sin(t)$$
$$= \sin(t)$$
$$M_X(t)\cdot M_X(t+\tau) = \boxed{\sin(t)\cdot\sin(t+\tau)}$$

$$R_{XX}(t,t+\tau) = E(X(t)\cdot X(t+\tau))$$

$$E((A\cos(t) + B\sin(t) + \sin(t))\cdot(A\cos(t,t+\tau) + B\sin(t,t+\tau) + \sin(t,t+\tau)))$$

$$E(A^2)\cos(t)\cos(t,t+\tau) + E(B^2)\sin(t)\sin(t,t+\tau) + \sin(t)\sin(t,t+\tau)$$

$$C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - M_X(t)\cdot M_X(t+\tau)$$

$$\cos(t)\cos(t,t+\tau) + \sin(t)\sin(t,t+\tau) + \underline{\sin(t)\sin(t,t+\tau)} - \underline{\sin(t)\cdot\sin(t+\tau)}$$

$$\cos(t)\cos(t,t+\tau) + \sin(t)\sin(t,t+\tau) = \cos(t-s)$$

$$\boxed{C_{XX}(t,t+\tau) = \cos(\tau)} \qquad \tau = t-s$$

8.9 Determine the missing elements of the following autocovariance matrix of a zero-mean wide-sense stationary random process $X(t)$, where the missing elements are denoted by xx.

$$\mathbf{C_{XX}} = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 \\ 0.8 & 1 & 0.6 & 0.4 \\ 0.4 & 0.6 & 1 & 0.6 \\ 0.2 & 0.4 & 0.6 & 1 \end{bmatrix}$$

8.11 Given that the autocorrelation function of $X(t)$ is given by $R_{XX}(\tau) = e^{-2|\tau|}$, and the random process $Y(t)$ is defined as follows:

$$Y(t) = \int_0^t X^2(u)du$$

The expected value of $Y(t)$

$$Y(t) = \int_0^t X^2(y)\,dy$$

$$E(Y(t)) = \int_0^t E(X^2(y))\,dy$$

$$= \int_0^t R_{XX}(0)\,dy = t - 0$$

$$\boxed{E(Y(t)) = t}$$

8.10 The random process $X(t)$ is defined as follows:

$$X(t) = A + e^{-B|t|}$$

where A and B are independent random variables. A is uniformly distributed over the range $-1 \le a \le 1$, and B is uniformly distributed over the range $0 \le b \le 2$. Find the following:

a. the mean of $X(t)$

e^{2x}

b. the autocorrelation function of $X(t)$

$E(A) = \frac{-1+1}{2} = 0$	$E(B) = \frac{2}{2} = 1$
$Var(A) = \frac{(1-(-1))^2}{12} = \frac{1}{3}$	$Var(B) = \frac{2^2}{12} = \frac{1}{3}$
$E(A^2) = \frac{1}{3}$	$E(B^2) = \frac{4}{3}$

a. the mean of $X(t)$

$$E(X(t)) = E(A + e^{-B|t|}) = E(A) + E(e^{-B|t|})$$

$$= 0 + \frac{1}{2}\int_0^2 e^{-B|t|}\, dB = \frac{1}{2}\left(-\frac{e^{-B|t|}}{|t|}\right)_0^2$$

$$= \frac{1}{2}\left(-\frac{e^{-2|t|}}{|t|} - \left(-\frac{1}{|t|}\right)\right) \Longrightarrow \frac{1}{2}\left(\frac{1}{|t|} - \frac{e^{-2|t|}}{|t|}\right)$$

b. the autocorrelation function of $X(t)$

$$R_{xx}(t, t+\tau) = E(X(t) \cdot X(t, t+\tau)) = E\left[(A + e^{-B|t|}) \cdot (A + e^{-B|t+\tau|})\right]$$

$$E(\underbrace{A^2}_{\downarrow} + \underbrace{A + e^{-B|t|}}_{0} + \underbrace{e^{-B(|t|+|t+\tau)}}_{\swarrow} + \underbrace{A + e^{-B|t+\tau|}}_{0})$$

$$\boxed{\frac{1}{3} + \frac{1 - e^{-2(|t|+|t+\tau)}}{2(|t|+|t+\tau)}}$$

8.13 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(wt) + B\sin(wt)$$

$$Y(t) = B\cos(wt) - A\sin(wt)$$

where w is a constant, and A and B zero-mean and uncorrelated random variables with variances $\sigma_A^2 = \sigma_B^2 = \sigma^2$. Find the crosscorrelation function $\boxed{R_{XY}(t, t+\tau).}$

$$R_{XY}(t, t+\tau) = E\left(X(t) \cdot Y(t+\tau)\right)$$

$$E\left(\left(A\cos(wt) + B\sin(wt)\right)\left(B\cos(wt+w\tau) - A\sin(wt+w\tau)\right)\right)$$

$$E(AB)\cos(wt)\cdot\cos(wt+w\tau) - E(A^2)\cos(wt)\cdot\sin(wt+w\tau)$$
$$+ E(B^2)\sin(wt)\cdot\cos(wt+w\tau) - E(BA) = \sin(wt)\sin(wt+w\tau)$$

According to the question A and b are uncorrelated
So, The Cov(A , B) is zero and Mean of A and B is zero
E(AB) = 0

$$E(B^2)\sin(wt)\cdot\cos(wt+w\tau) - E(A^2)\cos(wt)\cdot\sin(wt+w\tau)$$

$$\sigma^2\underbrace{\left(\sin(wt)\cdot\cos(wt+w\tau) - \cos(wt)\cdot\sin(wt+w\tau)\right)}_{\sin(A-B)}$$

$$= \sigma^2\sin(wt - wt + w\tau)$$
$$= \sigma^2\sin(-w\tau)$$

$$\boxed{R_{XY}(t, t+\tau) = \sigma^2\sin(w\tau) \longrightarrow \sin \text{ is odd fun}}$$

8.14 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$\boxed{\begin{aligned} X(t) &= A\cos(wt+\Theta) \\ Y(t) &= B\sin(wt+\Theta) \end{aligned}}$$

a. Find the autocorrelation function $R_{XX}(t, t+\tau)$, and show that $X(t)$ is a wide-sense stationary process.

$$R_{XX}(t,t+\tau) = E(X(t)\,X(t+\tau))$$

$$= E(A\cos(Wt+\theta)\cdot A\cos(Wt+W\tau+\theta))$$

$$A^2\, E(\cos(Wt+\theta)\cdot\cos(Wt+W\tau+\theta))$$

$$\frac{A^2}{2}\, E(\underbrace{\cos(-W\tau)}+\underbrace{\cos(2Wt+W\tau+2\theta)})$$

$$\rightarrow A^2 \frac{1}{2\pi}\int_0^{2\pi}\cos(2Wt+W\tau+2\theta)\,d\theta$$

$$\frac{A^2}{4\pi}(\sin(2Wt+W\tau+2\theta))\Big|_0^{2\pi} = 0$$

$$\rightarrow \frac{A^2}{2}\cos(W\tau) \Longrightarrow$$

$R_{xx}(t,t+\tau)$ is a function of τ

$X(t)$ is Wide-Sense Stationary WSS

b. Find the autocorrelation function $R_{YY}(t, t+\tau)$, and show that $Y(t)$ is a wide-sense stationary process.

$$R_{YY}(t, t+\tau) = E(Y(t)\,Y(t+\tau))$$

$$= E(B\sin(Wt+\theta)\cdot B\sin(Wt+W\tau+\theta))$$

$$= \frac{B^2}{2} E(\underbrace{\cos(-W\tau)}, - \underbrace{\cos(2Wt+W\tau+2\theta)})$$

$$\rightarrow \frac{1}{2\pi} \cdot \int_0^{2\pi} \cos(2Wt+W\tau+2\theta)\, d\theta$$

$$\frac{1}{4\pi}(\sin(2Wt+W\tau+2\theta))_0^{2\pi} \Longrightarrow 0$$

$$\rightarrow \frac{B^2}{2}\cos(W\tau)$$

$R_{YY}(t,t+\tau)$ is a function of τ

$Y(t)$ is Wide-Sense Stationary WSS

c. Find the crosscorrelation function $R_{XY}(t, t+\tau)$, and show that $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

$$R_{XY}(t, t+\tau) = E(X(t)\,Y(t+\tau))$$

$$= E(A\cos(Wt+\theta) \cdot B\sin(Wt+W\tau+\theta))$$

$$= \frac{AB}{2} E(\sin(2Wt+W\tau+2\theta) - \sin(-W\tau))$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(2Wt+W\tau+2\theta)\, d\theta$$

$$-\frac{1}{4\pi}\left(\cos(2Wt+W\tau+2\theta)\right)_0^{2\pi} \Longrightarrow 0$$

$$\frac{AB}{2}\sin(W\tau)$$

$R_{XY}(t,t+\tau)$ is a function of τ

$X(t)$ and $Y(t)$ are Jointly Wide-Sense Stationary

8.15 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$X(t) = A\cos(w_1 t + \Theta)$ $\quad Y(t) = B\sin(w_2 t + \Phi)$

where w_1, w_2, A, and B are constants, and Θ and Φ are statistically independent random variables, each of which is uniformly distributed between 0 and 2π

a. Find the crosscorrelation function $R_{XY}(t, t+\tau)$, and show that $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

$$R_{XY}(t, t+\tau) = E(X(t)Y(t+\tau))$$

$$= E(A\cos(wt+\theta) \cdot B\sin(W_2 t + W_2\tau + \phi))$$

$$\frac{AB}{2} E(\sin(wt+\theta+W_2t+W_2\tau+\phi) - \sin(wt+\theta-W_2t-W_2\tau-\phi))$$

$$\to \frac{1}{2\pi} \int_0^{2\pi}\int_0^{2\pi} \sin(wt+\theta+W_2t+W_2\tau+\phi)\, d\theta\, d\phi = 0$$

$R_{XY}(t,t+\tau) = 0$ $X(t)$ and $Y(t)$ are Jointly Wide-Sense Stationary

b. If $\Theta = \Phi$, show that $X(t)$ and $Y(t)$ are not jointly wide-sense stationary.

$$\frac{AB}{2} E(\sin(wt+\theta+W_2t+W_2\tau+\theta) - \sin(wt+\theta-W_2t-W_2\tau-\theta))$$

$$\frac{AB}{2} E(\sin(wt+W_2t+W_2\tau+\theta) - \sin(wt-W_2t-W_2\tau))$$

$$\to \frac{1}{2\pi}\int_0^{2\pi} \sin(wt+W_2t+W_2\tau+\theta)\, d\theta = 0$$

$$-\frac{AB}{2}\sin(wt-W_2t-W_2\tau))$$

$R_{XY}(t,t+\tau)$ is not a function of τ alone

$X(t)$ and $Y(t)$ are not Wide-Sense Stationary

c. If $\Theta = \Phi$, under what condition are $X(t)$ and $Y(t)$ jointly wide-sense stationary?

It will be Jointly Wide-Sense Stationary $\boxed{\text{When } W_1 = W_2}$

8.16 Explain why the following matrices can or cannot be valid autocorrelation matrices of a zero-mean wide-sense stationary random process $X(t)$.

a

$$G = \begin{bmatrix} 1 & 1.2 & 0.4 & 1 \\ 1.2 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 1.3 \\ 1 & 0.9 & 1.3 & 1 \end{bmatrix}$$

G cannot be the autocorrelation matrix of a wide-sense stationary process.
RXX (τ) < RXX (0) for all τ ≠ 0.

b

$$H = \begin{bmatrix} 2 & 1.2 & 0.4 & 1 \\ 1.2 & 2 & 0.6 & 0.9 \\ 0.4 & 0.6 & 2 & 1.3 \\ 1 & 0.9 & 1.3 & 2 \end{bmatrix}$$

H is a symmetric matrix and the diagonal elements that are supposed to be the value of RXX 0() = value that is the largest value in the matrix
H can be the autocorrelation matrix of a wide-sense stationary process.

c

$$K = \begin{bmatrix} 1 & 0.7 & 0.4 & 0.8 \\ 0.5 & 1 & 0.6 & 0.9 \\ 0.4 & 0.6 & 1 & 0.3 \\ 0.1 & 0.9 & 0.3 & 1 \end{bmatrix}$$

K is not a symmetric matrix means that it cannot be the autocorrelation function of a wide-sense stationary process.

8.19 A random process $Y(t)$ is given by

$$Y(t) = A\cos(wt + \phi)$$

where A, w, and ϕ are independent random variables. Assume that A has a mean of 3 and a variance of 9, ϕ is uniformly distributed between $-\pi$ and π, and w is uniformly distributed between -6 and 6. Determine if the process is stationary in the wide sense.

$E(A^2) = 18$, $f(\phi) = \frac{1}{2\pi}$, $f(w) = \frac{1}{12}$

$$R_{YY}(t, t+\tau) = E(Y(t)\,Y(t+\tau))$$

$$E(A\cos(wt+\phi) \cdot A\cos(wt + w\tau + \phi))$$

$$\frac{18}{2}\left(E(\cos(-w\tau)) + E(\cos(2wt + w\tau + 2\phi))\right)$$

$$\rightarrow \frac{1}{12 \times 2\pi}\int_{-6}^{6}\int_{-\pi}^{\pi}\cos(2wt + w\tau + 2\phi)\,d\phi\,dw = 0$$

$$\rightarrow \frac{1}{12}\int_{-6}^{6}\cos(w\tau)\,dw \Longrightarrow \frac{1}{12}\left(\frac{\sin(w\tau)}{\tau}\right)_{-6}^{6}$$

$$\frac{1}{12}\left(\frac{\sin(6\tau)}{\tau} - \frac{\sin(-6\tau)}{\tau}\right)$$

$$\frac{1}{12}\left(\frac{\sin(6\tau)}{\tau} + \frac{\sin(6\tau)}{\tau}\right) \Longrightarrow \frac{\sin(6\tau)}{6\tau}$$

$$9\left(\frac{\sin(6\tau)}{6\tau}\right) = \frac{3\sin(6\tau)}{2\tau}$$

$R_{YY}(t, t+\tau)$ is a function of τ

$Y(t)$ is Wide-Sense Stationary WSS

8.20 A random process $X(t)$ is given by

$$X(t) = A\cos(t) + (B+1)\sin(t) \quad -\infty < t < \infty$$

where A and B are independent random variables with $\underline{E[A] = E[B] = 0}$ and $\underline{E[A^2] = E[B^2] = 1}$. Is $X(t)$ wide-sense stationary?

$R_{XX}(t, t+\tau) = E(X(t)\, X(t+\tau))$

$= E((A\cos(t) + (B+1)\sin(t)).(A\cos(t+\tau) + (B+1)\sin(t+\tau)))$

$E(A) = 0, \quad E(B) = 0$

$E(A^2)\cos(t).\cos(t+\tau) + E(B^2)\sin(t).\sin(t+\tau) + \sin(t).\sin(t+\tau)$

$E(A^2) = 1 \quad , E(B^2) = 1$

$\cos(t).\cos(t+\tau) + \sin(t).\sin(t+\tau) + \underline{\sin(t).\sin(t+\tau)}$

$= \cos(\tau) + \sin(t).\sin(t+\tau)$

$R_{YY}(t, t+\tau)$ is not a Wide-Sense Stationary

8.21 A random process has the autocorrelation function

$$R_{XX}(\tau) = \frac{16\tau^2 + 28}{\tau^2 + 1}$$

Find the mean-square value, the mean value, and the variance of the process.

$\frac{\text{number}}{\infty} = 0$

$$R_{xx}(\tau) = \frac{16\tau^2+28}{\tau^2+1} = \frac{16(\tau^2+1)+12}{\tau^2+1} = \frac{16(\tau^2+1)}{\tau^2+1} + \frac{12}{\tau^2+1} = \boxed{16+\frac{12}{\tau^2+1}}$$

$$E(x(t)) = \pm\sqrt{\lim_{t\to\infty} R_{xx}(\tau)} = \pm\sqrt{16+\frac{12}{\tau^2+1}} = \sqrt{16} = \boxed{\pm 4}$$

$$E(x^2(t)) = R_{xx}(0) = 16 + \frac{12}{0+1} = \boxed{28}$$

$$\sigma^2 = E(x^2(t)) - (E(x(t)))^2 = 28 - 4^2 = \boxed{12}$$

8.23 An ergodic random process $X(t)$ has the autocorrelation function

$$R_{XX}(\tau) = 36 + \frac{4}{1+\tau^2}$$

Determine the mean value, mean-square value, and variance of $X(t)$.

$$E(x(t)) = \sqrt{\lim_{\tau\to 0} R_{xx}(\tau)} = \pm\sqrt{36} = \boxed{\pm 6}$$

$$E(x^2(t)) = R_{xx}(0) = 36 + 4 = \boxed{40}$$

$$\sigma^2 = E(x^2(t)) - (E(x(t)))^2 = 40 - 6^2 = \boxed{4}$$

8.26 A random process $Y(t)$ is given by

$$Y(t) = A\cos(wt + \phi)$$

where w is a constant, and A and ϕ are independent random variables. The random variable A has a mean of 3 and a variance of 9, and ϕ is uniformly distributed between $-\pi$ and π. Determine if the process is a mean-ergodic process.

ensemble average of $Y(t)$

$$f_\Phi(\phi) = \frac{1}{2\pi}$$

$$E(Y(t)) = E(A\cos(wt+\phi)) = E(A)\cos(wt+\phi)$$

$$\frac{3}{2\pi}\int_{-\pi}^{\pi}\cos(wt+\phi)\,d\phi \Rightarrow \frac{3}{2\pi}\left(\sin(wt+\phi)\right)_{-\pi}^{\pi}$$

$$\frac{3}{2\pi}\left(\sin(wt+\pi) - \sin(wt-\pi)\right) = 0$$

---

time average of $Y(t)$

$$\overline{Y(t)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(wt+\phi)\,dt$$

$$\lim_{T\to\infty}\frac{A}{2T}\left(\frac{\sin(wt+\phi)}{w}\right)_{-T}^{T} = \lim_{T\to\infty}\frac{A}{2Tw}\left(\sin(wT+\phi) - \sin(-wT+\phi)\right)$$

$$\lim_{T\to\infty}\frac{A}{2}\cdot 2\left(\frac{\sin(wt)}{Tw}\right)\cos(\phi)$$

$$\lim_{T\to\infty}\left(\frac{\sin(wt)}{Tw}\right)A\cdot\cos(\phi) = 0 \qquad (-1 \le \sin \le 1)$$

The ensemble average of Y(t) is equal to its time average
So, the process is a mean-ergodic process.

8.27 A random process $X(t)$ is given by

where A is a random variable with a finite mean of μ_A and finite variance σ_A^2. Determine if $X(t)$ is a mean-ergodic process.

$$X(t) = A$$

ensemble average of $X(t)$

$$E(X(t)) = E(A) = \mu_A$$

The time average of $X(t)$

$$\overline{X(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\,dt = (AT + AT)\Big|_{-T}^{T}$$

$$\lim_{T\to\infty} \frac{2AT}{2T} = \frac{\cancel{2}A\cancel{T}}{\cancel{2}\cancel{T}} = A$$

$$\boxed{A \neq \mu_A}$$

The ensemble average of X(t) is not equal to its time average
So, the process is not a mean-ergodic process.

8.28 Assume that $V(t)$ and $W(t)$ are both zero-mean wide-sense stationary random processes and let the random process $M(t)$ be defined as follows:

a. If $V(t)$ and $W(t)$ are jointly wide-sense stationary, determine the following in terms of those of $V(t)$ and $W(t)$:

$$M(t) = V(t) + W(t)$$

1. the autocorrelation function of $M(t)$
2. the power spectral density of $M(t)$

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autocorrelation function of $M(t)$

$$R_{MM}(t,t+\tau)=E(M(t).M(t+\tau))$$

$$=E\left((V(t)+M(t)).V(t+\tau)+M(t+\tau)\right)$$

$$E(\underline{V(t)V(t+\tau)}+\underline{V(t)M(t+\tau)}+\underline{M(t)V(t+\tau)}+\underline{M(t)M(t+\tau)})$$

$$=R_{VV}(\tau)+R_{VM}(\tau)+R_{MV}(\tau)+R_{MM}(\tau)$$

$$\boxed{=R_{MM}(\tau)}$$

power spectral density of $M(t)$

$$S_{MM}(w)=\int_{-\infty}^{\infty} R_{MM}(\tau)\, e^{-jwt}\, d\tau$$

$$S_{MM}(w)=S_{VV}(W)+S_{VM}(W)+S_{MV}(W)+S_{MM}(W)$$

b. If $V(t)$ and $W(t)$ are orthogonal, determine the following in terms of those of $V(t)$ and $W(t)$:

1 the autocorrelation function of $M(t)$

2. the power spectral density of $M(t)$

$V(t)$ and $W(t)$ are orthogonal $\longrightarrow$ $\boxed{R_{MV}(\tau)=R_{VM}(\tau)=0}$

$$R_{MM}(t,t+\tau)=R_{VV}(\tau)+R_{MM}(\tau)=R_{MM}(\tau)$$

$$S_{MM}(W)=\int_{-\infty}^{\infty} R_{MM}(\tau).e^{-jW\tau}\, d\tau=S_{VV}(W)+S_{MM}(W)$$

8.30 A random process $X(t)$ has a power spectral density given by

$$S_{XX}(w) = \begin{cases} 4 - \dfrac{w^2}{9} & |w| \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Determine (a) the average power and (b) the autocorrelation function of the process.

$$E(X^2(t)) = \frac{1}{2\pi}\int_{-6}^{6}\left(4 - \frac{w^2}{9}\right)dw = \frac{16}{\pi}$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-6}^{6}\left(4 - \frac{w^2}{9}\right)\cdot e^{jw\tau}\,dw$$

$$= \frac{1}{2\pi}\left(\int_{-6}^{6}(4e^{jw\tau})\,dw - \frac{1}{9}\int_{-6}^{6} w^2 \cdot e^{jw\tau}\,dw\right.$$

$$\frac{4\sin(6\tau)}{\pi\tau} - \frac{1}{18\pi}\underline{\int_{-6}^{6} w^2 \cdot e^{jw\tau}\,dw}$$

$$- \frac{72}{\tau}\sin(6\tau) - \frac{2}{j\tau}\underline{\int_{-6}^{6} w e^{jwt}\,dw}$$

$$\frac{12}{j\tau}\cos(6\tau) + \frac{2j}{\tau^2}\sin(6\tau)$$

Next page

$$\frac{1}{2\pi}\int_{-6}^{6}(4e^{jw\tau})\,dw - \frac{1}{18\pi}\int_{-6}^{6} w^2 e^{jw\tau}\,dw$$

$$\frac{4\sin(6\tau)}{\pi\tau} - \frac{1}{18\pi}\left(\frac{72}{\tau}\sin(6\tau) + \frac{24}{\tau^2}\cos(6\tau) - \frac{4}{\tau^3}\sin(6\tau)\right)$$

$$\boxed{R_{XX}(\tau) = \frac{2}{9\pi\tau^3}\left(\sin(6\tau) - \cos\tau(6\tau)\right)}$$

8.36 Two random processes $X(t)$ and $Y(t)$ are defined as follows:

$$X(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

$$Y(t) = B\cos(w_0 t) - A\sin(w_0 t)$$

where w_0 is a constant, and A and B are zero-mean and uncorrelated random variables with variances $\sigma_A^2 = \sigma_B^2 = \sigma^2$. Find the cross-power spectral density of $X(t)$ and $Y(t)$, $S_{XY}(w)$. (Note that $S_{XY}(w)$ is the Fourier transform of the crosscorrelation function $R_{XY}(\tau)$.)

$$R_{XY}(\tau) = E(A\cos(w_0 t) + B\sin(w_0 t)\,B\cos(w_0 t + w\tau) - A\sin(w_0 t + w\tau))$$

$$-E(A^2)\cos(w_0 t)\sin(w_0 t + w\tau) + E(B^2)\cos(w_0 t + w\tau) - \sin(w_0 t)$$

$$\sigma^2\left(\cos(w_0 t + w\tau) - \sin(w_0 t) - \cos(w_0 t)\sin(w_0 t + w\tau)\right)$$

$$S_{XY}(w) = j\sigma^2\pi\left(\delta(w - w_0) - \delta(w + w_0)\right)$$

8.38 Two jointly stationary random processes $X(t)$ and $Y(t)$ have the crosscorrelation function given by:

Determine the following: $$R_{XY}(\tau) = 2e^{-2\tau} \quad \tau \geq 0$$

a. the cross-power spectral density $S_{XY}(w)$

b. the cross-power spectral density $S_{YX}(w)$

a

$$S_{xy}(W) = \int_0^{\infty} 2e^{-2\tau} \cdot e^{-jw\tau}\, dt$$

$$= 2\int_0^{\infty} e^{-\tau(2+jw)}\, dt \Rightarrow 2\left(\frac{-e^{-\tau(2+jw)}}{2+jw}\right)\Bigg|_0^{\infty} = \boxed{\frac{2}{2+jw}}$$

b

$$S_{YX}(w) = \int_{-\infty}^{\infty} R_{YX}(-\tau) \cdot e^{-jwt}\, dt = \int_{\infty}^{\infty} R_{YX}(u) \cdot e^{jwu}\, dt = S_{XY}(-w)$$

$$= \frac{2}{2-jw}$$

8.44 Find the power spectral density of a random sequence $X[n]$ whose autocorrelation function is given by $R_{XX}[m] = a^m$, $m = 0, 1, 2, \ldots$, where $|a| < 1$.

$$S_{XX}(\Omega) = \sum_{-\infty}^{\infty} R_{XX}[m]\, e^{-jm\Omega}$$

$$= \sum_0^{\infty} a^m \cdot e^{-jm\Omega} = \sum_0^{\infty} \left(ae^{-j\Omega}\right)^m$$

$$\boxed{= \frac{1}{1-ae^{-j\Omega}}}$$

Chapter 10

Section 10.7: Discrete-Time Markov Chains

10.36 Determine the missing elements denoted by x in the following transition probability matrix:

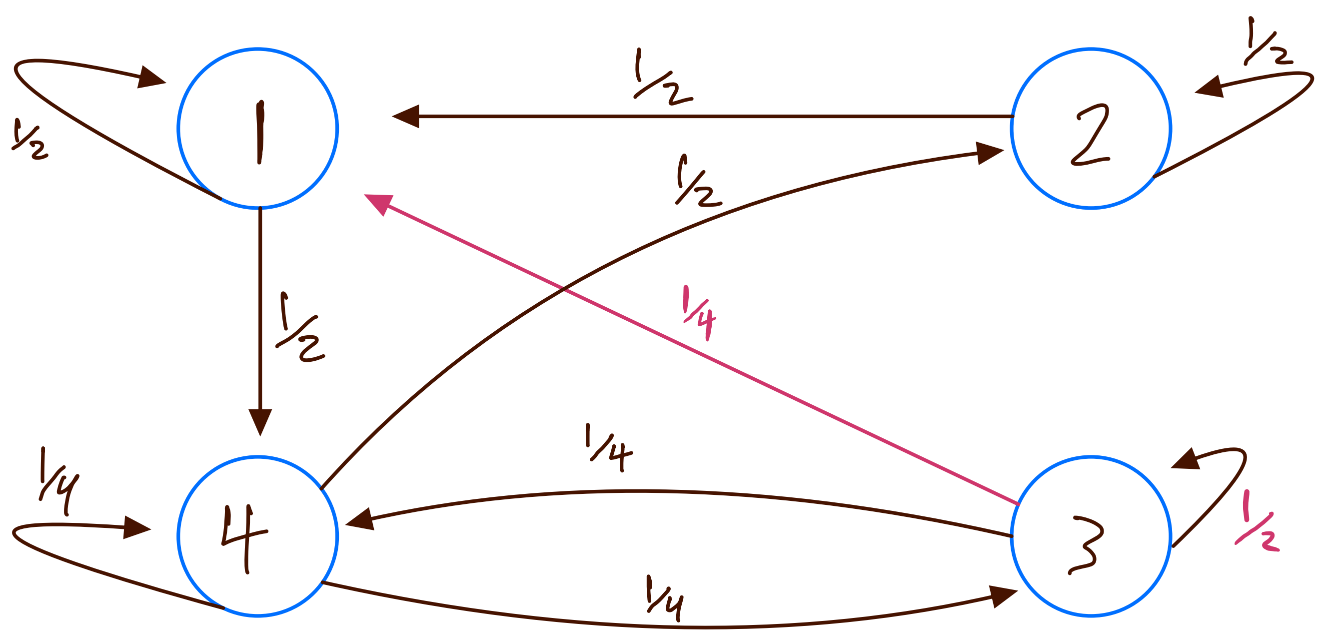
$$P = \begin{bmatrix} x & 1/3 & 1/3 & 1/3 \\ 1/10 & x & 1/5 & 2/5 \\ x & x & x & 1 \\ 3/5 & 2/5 & x & x \end{bmatrix}$$

sum $\rightarrow$ 1

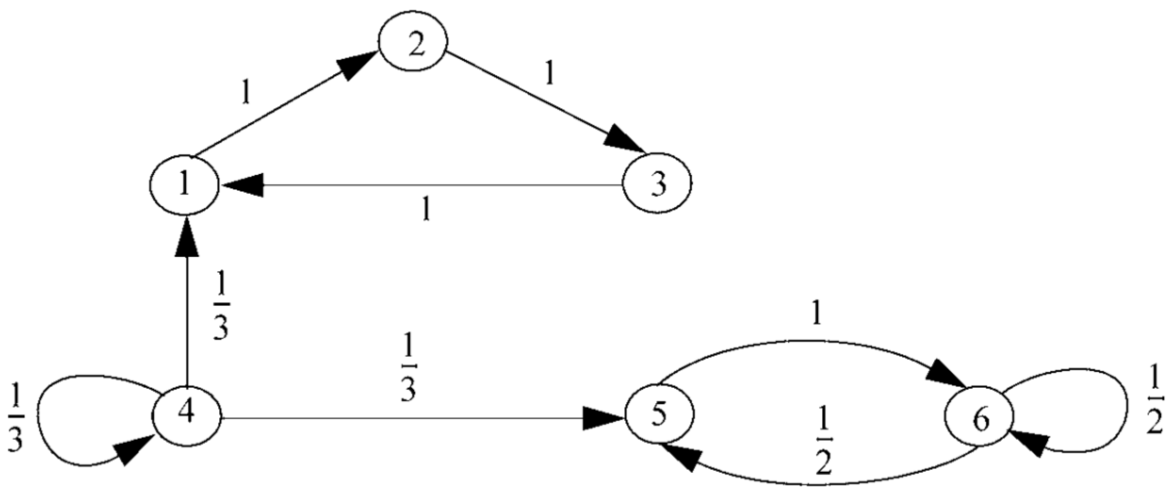
$$P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/10 & 3/10 & 1/5 & 2/5 \\ 0 & 0 & 0 & 1 \\ 3/5 & 2/5 & 0 & 0 \end{bmatrix}$$

10.37 Draw the state-transition diagram for the Markov chain with the following transition probability matrix.

$$P = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \end{bmatrix}$$



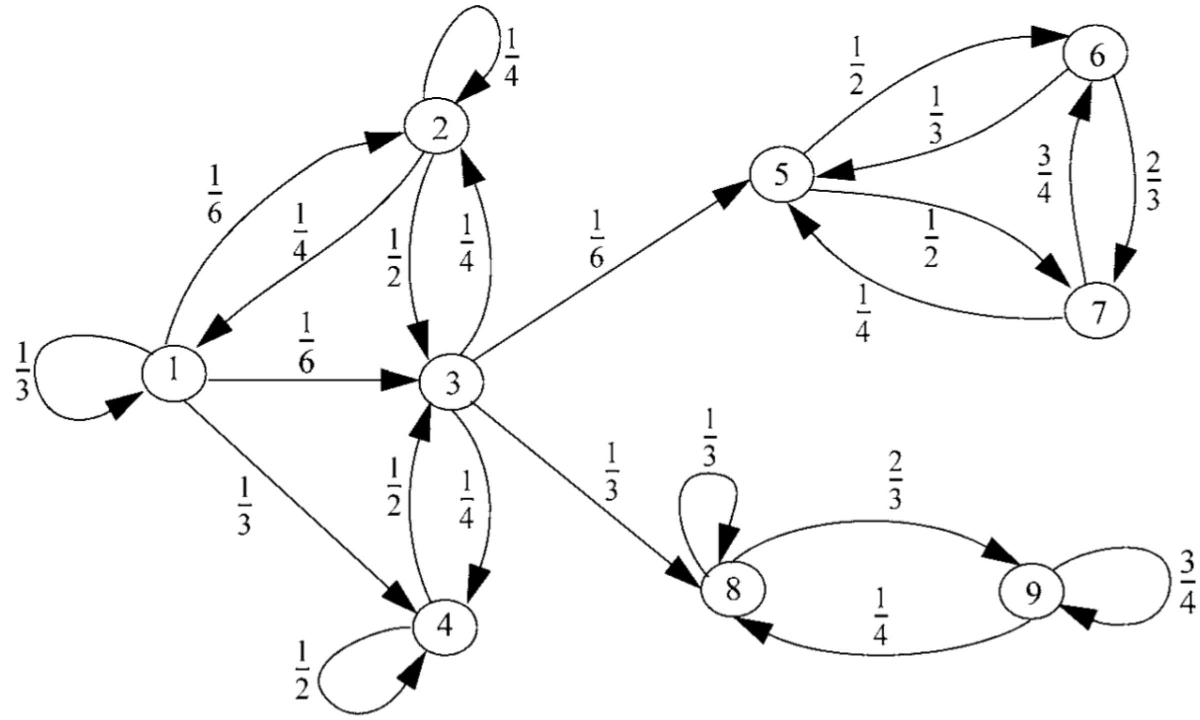
10.38 Consider a Markov chain with the following state-transition diagram.



a. Give the transition probability matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

10.39 Consider the Markov chain with the following state-transition diagram.



$$
\begin{bmatrix}
1/3 & 1/6 & 1/6 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 0 & 1/4 & 1/6 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 0 & 2/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 3/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4
\end{bmatrix}
$$

10.42 Find the limiting-state probabilities associated with the following transition probability matrix:

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

$0.4\pi_1 + 0.3\pi_2 + 0.3\pi_3 = \pi_1$ ①

$0.3\pi_1 + 0.4\pi_2 + 0.3\pi_3 = \pi_2$ ②

$0.3\pi_1 + 0.3\pi_2 + 0.4\pi_3 = \pi_3$ ③

$-0.6\pi_1 + 0.3\pi_2 + 0.3\pi_3 = 0$ ✓

$0.3\pi_1 - 0.6\pi_2 + 0.3\pi_3 = 0$ ✓

$0.3\pi_1 + 0.3\pi_2 - 0.6\pi_3 = 0$ X

$\boxed{\pi_1 + \pi_2 + \pi_3 = 1}$

$-0.6\pi_1 + 0.3\pi_2 = -0.3$

$0.3\pi_1 - 0.6\pi_2 = -0.3$

Method 2: Matrix

$\pi P = \pi$

$A^{-1} \times B$

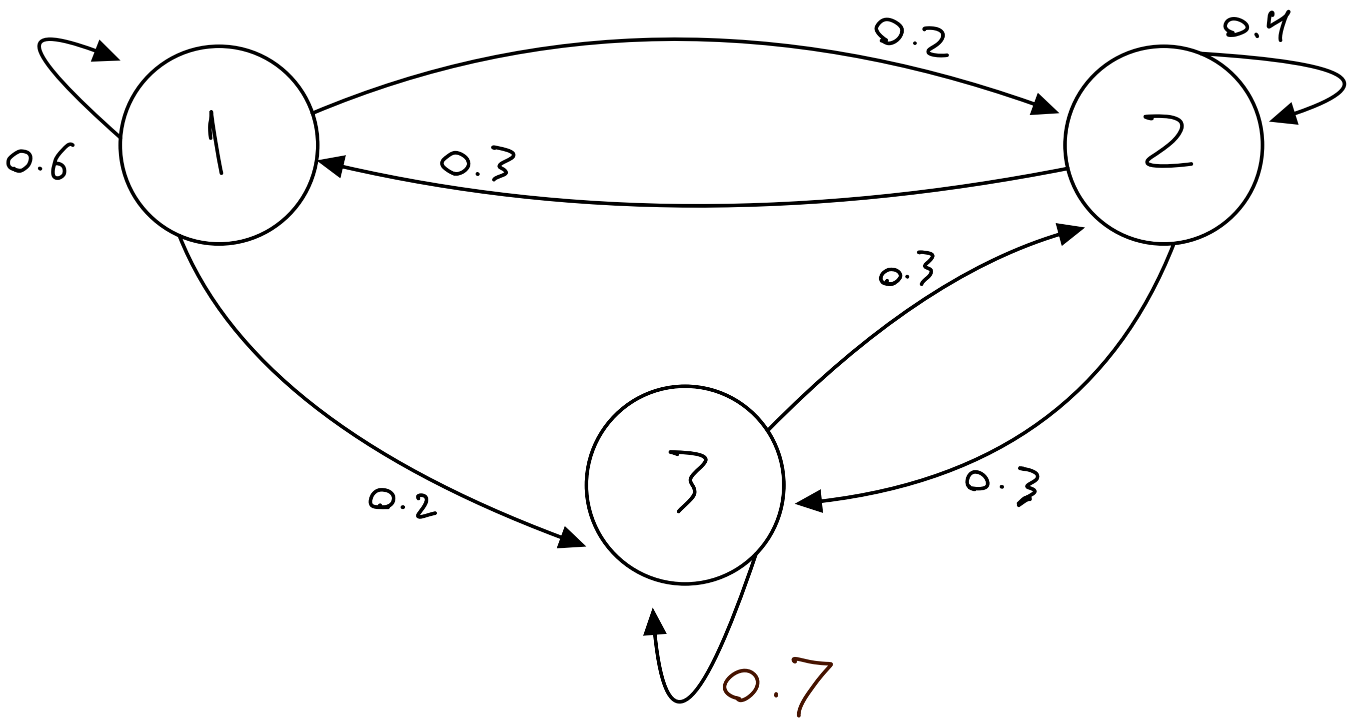
$$\begin{vmatrix} -0.9 & 0 \\ 0 & -0.9 \end{vmatrix}^{-1} \times \begin{vmatrix} -0.3 \\ -0.3 \end{vmatrix} = \begin{vmatrix} \frac{1}{3} \\ \frac{1}{3} \end{vmatrix}$$

$$\boxed{\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}}$$

10.43 Consider the following transition probability matrix:

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

a. Give the state-transition diagram.



b. Given that the process is currently in state 1, what is the probability that it will be in state 2 at the end of the third transition?

$P_{12}(3)$

$$\begin{vmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0 & 0.3 & 0.7 \end{vmatrix} \rightarrow \begin{vmatrix} 0.42 & 0.26 & 0.32 \\ 0.3 & 0.31 & 0.39 \\ 0.09 & 0.33 & 0.58 \end{vmatrix}^{2} \rightarrow \begin{vmatrix} 0.33 & 0.284 & 0.386 \\ 0.273 & 0.301 & 0.426 \\ 0.153 & 0.324 & 0.523 \end{vmatrix}^{3}$$

$$\begin{array}{c c} & 2 \\ 1 & \begin{vmatrix} 0.33 & \boxed{0.284} & 0.386 \\ 0.273 & 0.301 & 0.426 \\ 0.153 & 0.324 & 0.523 \end{vmatrix}^{3} \end{array} \longrightarrow \boxed{P_{12}(3) = 0{,}284}$$

c. Given that the process is currently in state 1, what is the probability that the first time it enters state 3 is the fourth transition?

$f_{13}(4)$

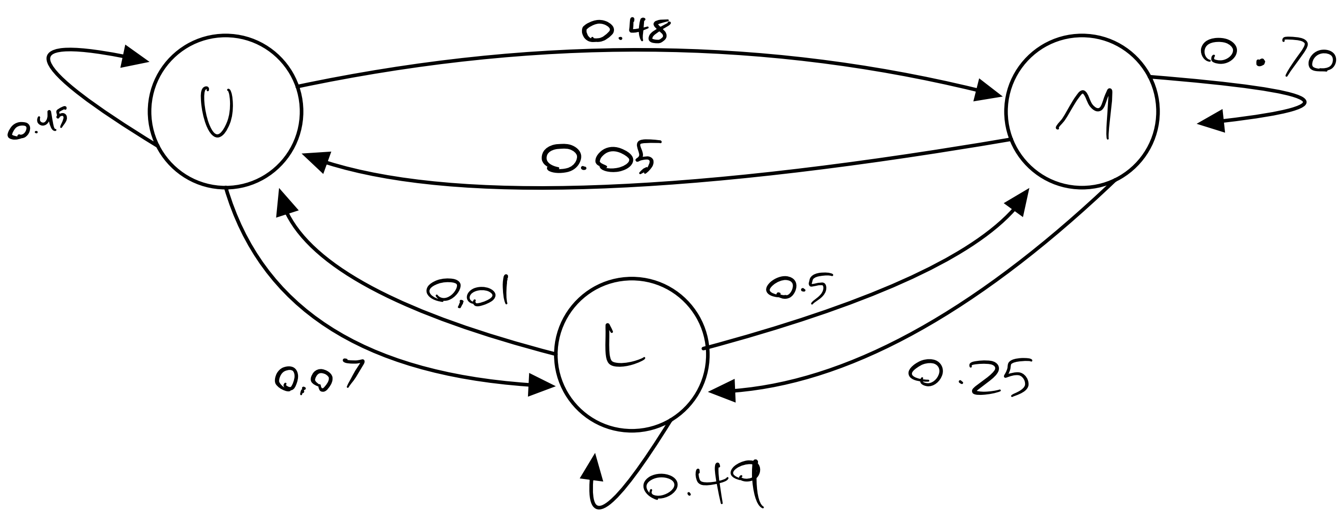
1 — 1 — 1 — 1 — 3	1 — 2 — 1 — 1 — 3
1 — 1 — 1 — 2 — 3	1 — 2 — 2 — 1 — 3
1 — 1 — 2 — 2 — 3	1 — 2 — 1 — 2 — 3
1 — 2 — 2 — 2 — 3	1 — 1 — 2 — 1 — 3

$$
\begin{aligned}
f_{13}(4) = &\left((0.6)^3 \times 0.2\right) + \left(0.6^2 \times 0.2 \times 0.3\right) + \left(0.6 \times 0.2 \times 0.3 \times 0.4\right) \\
&+ \left(0.6 \times 0.2 \times 0.3 \times 0.2\right) + \left(0.2 \times 0.3 \times 0.6 \times 0.2\right) + \left(0.2 \times 0.4 \times 0.3 \times 0.2\right) \\
&+ \left(0.2 \times 0.3 \times 0.2 \times 0.3\right) + \left(0.2 \times 0.4^2 \times 0.3\right)
\end{aligned}
$$

$$\boxed{f_{13}(4) = 0.1116}$$

10.44 Consider the following social mobility problem. Studies indicate that people in a society can be classified as belonging to the upper class (state 1), middle class (state 2), and lower class (state 3). Membership in any class is inherited in the following probabilistic manner. Given that a person is raised in an upper-class family, he will have an upper-class family with probability 0.45, a middle-class family with probability 0.48, and a lower-class family with probability 0.07. Given that a person is raised in a middle-class family, he will have an upper-class family with probability 0.05, a middle-class family with probability 0.70, and a lower-class family with probability 0.25. Finally, given that a person is raised in a lower-class family, he will have an upper-class family with probability 0.01, a middle-class family with probability 0.50, and a lower-class family with probability 0.49. Determine the following:

a. The state-transition diagram of the process.



b. The transition probability matrix of the process.

$$P = \begin{vmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{vmatrix}$$

c. The limiting-state probabilities. Interpret what they mean to the layper-

$$\pi_1 = 0.45\pi_1 + 0.05\pi_2 + 0.01\pi_3$$

$$\pi_2 = 0.48\pi_1 + 0.70\pi_2 + 0.50\pi_3 \quad ①$$

$$1 = \pi_1 + \pi_2 + \pi_3 \quad ②$$

$\rightarrow$ $-0,55 + 0,05 = -0.01$

$0.48 - 0,30 = -0.50$

$\pi P = \pi$

$$\begin{vmatrix} -0.56 & 0,04 \\ -0.02 & -0.8 \end{vmatrix}^{-1} \times \begin{vmatrix} -0.01 \\ -0.50 \end{vmatrix} = \begin{vmatrix} 35/561 \\ \frac{1399}{2244} \end{vmatrix}$$

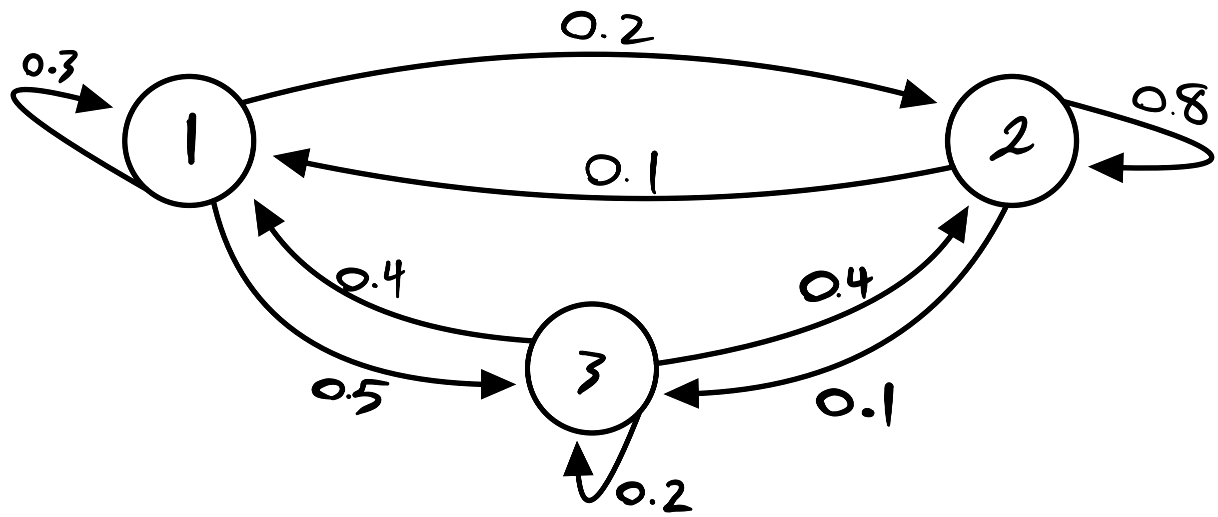
$\pi_1 = 35/561$

$\pi_2 = \frac{1399}{2244}$

$\rightarrow \pi_3 = 1 - \left(\frac{35}{561} + \frac{1399}{2244}\right) = \frac{235}{748} = 0,31417112$

10.45 A taxi driver conducts his business in three different towns 1, 2, and 3. On any given day, when he is in town 1, the probability that the next passenger he picks up is going to town 1 is 0.3, the probability that the next passenger he picks up is going to town 2 is 0.2, and the probability that the next passenger he picks up is going to town 3 is 0.5. When he is in town 2, the probability that the next passenger he picks up is going to town 1 is 0.1, the probability that the next passenger he picks up is going to town 2 is 0.8, and the probability that the next passenger he picks up is going to town 3 is 0.1. When he is in town 3, the probability that the next passenger he picks up is going to town 1 is 0.4, the probability that the next passenger he picks up is going to town 2 is 0.4, and the probability that the next passenger he picks up is going to town 3 is 0.2.

a. Determine the state-transition diagram for the process.



b. Give the transition probability matrix for the process.

$$P = \begin{vmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{vmatrix}$$

c. What are the limiting-state probabilities?

$$\pi_1 = 0.3\pi_1 + 0.1\pi_2 + 0.4\pi_3 \longrightarrow = -0.7\pi_1 + 0.1\pi_2 = -0.4$$
$$1 = \pi_1 + \pi_2 + \pi_3$$
$$\pi_2 = 0.2\pi_1 + 0.8\pi_2 + 0.4\pi_3 \longrightarrow = 0.2\pi_1 - 0.2\pi_2 = -0.4$$

$$= A^{-1} \times P$$

$$\begin{vmatrix} -1.1 & -0.3 \\ -0.2 & -0.6 \end{vmatrix}^{-1} \times \begin{vmatrix} -0.4 \\ -0.4 \end{vmatrix} = \begin{vmatrix} 0.2 \\ 0.6 \end{vmatrix}$$

$\pi_1 = 0.2$, $\pi_2 = 0.6$, $\pi_3 = (1-(0.2+0.6)) = 0.2$

---

d. Given that the taxi driver is currently in town 2 and is waiting to pick up his first customer for the day, what is the probability that the first time he picks up a passenger to town 2 is when he picks up his third passenger for the day?

$f_{22}(3) \longrightarrow$

2 — 1 — 1 — 2
2 — 1 — 3 — 2
2 — 3 — 1 — 2
2 — 3 — 3 — 2

$$f_{22}(3) = (0.1 \times 0.3 \times 0.2) + (0.1 \times 0.5 \times 0.4)$$
$$(0.1 \times 0.4 \times 0.2) + (0.1 \times 0.2 \times 0.4)$$

$$\boxed{f_{22}(3) = 0.042}$$

e. Given that he is currently in town 2, what is the probability that his third passenger from now will be going to town 1?

$$P_{21}(3)$$

$$P = \begin{vmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{vmatrix}^{1}$$

$$P = \begin{vmatrix} 0.31 & 0.42 & 0.27 \\ 0.15 & 0.7 & 0.15 \\ 0.24 & 0.48 & 0.28 \end{vmatrix}^{2}$$

$$P = \begin{vmatrix} 0.243 & 0.506 & 0.251 \\ \boxed{0.175} & 0.65 & 0.175 \\ 0.232 & 0.544 & 0.224 \end{vmatrix}^{3}$$

$$\boxed{P_{21}(3) = 0.175}$$

1) There are three brands of coffee A, B and C. A study is conducted and the results indicate brand-switching model obeys the following scheme:

$A \longrightarrow A$ with probability 0.70, $A \longrightarrow B$ with probability 0.20, $A \longrightarrow C$ with probability 0.10,

$B \longrightarrow A$ with probability 0.40, $B \longrightarrow B$ with probability 0.25, $B \longrightarrow C$ with probability 0.35,

$C \longrightarrow A$ with probability 0.35, $C \longrightarrow B$ with probability 0.45, $C \longrightarrow C$ with probability 0.20.

a) Determine the matrix of one –step transition probabilities.

$$P = \begin{vmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.25 & 0.35 \\ 0.35 & 0.45 & 0.20 \end{vmatrix}$$

---

b) Obtain the probability distribution of states after 4 transitions, given that the initial of states is given by the vector $u^{(0)} = [0.5 \;\; 0.2 \;\; 0.3]$.

$$P(4) = u^{(0)} \cdot P^{(4)} = [0.5 \;\; 0.2 \;\; 0.3] \cdot \begin{vmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.25 & 0.35 \\ 0.35 & 0.45 & 0.20 \end{vmatrix}^4$$

$$= [0.55760937 \quad 0.25894812 \quad 0.1834425]$$

---

2) Consider a 3-state Markov chain (M. C.) with one-step transition matrix

$$P = \begin{bmatrix} 0.45 & 0.15 & 0.40 \\ 0.35 & 0.40 & 0.25 \\ 0.20 & 0.45 & 0.35 \end{bmatrix}$$

and initial distribution of states $u^{(0)} = [0.30 \;\; 0.25 \;\; 0.45]$.

a) Determine the distribution of states after 3 transitions.

$$P(3) = u^{(0)} \cdot P^3 = \begin{bmatrix} 0.3 & 0.25 & 0.45 \end{bmatrix} \cdot \begin{bmatrix} 0.45 & 0.15 & 0.40 \\ 0.35 & 0.40 & 0.25 \\ 0.20 & 0.45 & 0.35 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0.3333937 & 0.3339812 & 0.332625 \end{bmatrix}$$

---

b) Determine the steady - distribution of states.

$$0.45\pi_1 + 0.35\pi_2 + 0.2\pi_3 = 0 \quad ①$$

$$0.15\pi_1 + 0.4\pi_2 + 0.45\pi_3 = 0 \quad ②$$

$$0.4\pi_1 + 0.25\pi_2 + 0.35\pi_3 = 0 \quad ③$$

$$\boxed{\begin{aligned} -0.55\pi_1 + 0.35\pi_2 &= -0.2 \\ 0.15\pi_1 - 0.6\pi_2 &= -0.45 \end{aligned}} \longrightarrow \begin{aligned} &-0.75\pi_1 + 0.15\pi_2 \\ &-0.3\pi_1 - 1.05\pi_2 \end{aligned}$$

$$\begin{vmatrix} -0.75 & 0.15 \\ -0.3 & -1.05 \end{vmatrix}^{-1} \cdot \begin{vmatrix} -0.2 \\ -0.45 \end{vmatrix} = \begin{vmatrix} 1/3 \\ 1/3 \end{vmatrix}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = \frac{1}{3}, \quad \pi_2 = \frac{1}{3}, \quad \pi_3 = \frac{1}{3}$$

3) Markov chain has 4 states S_1, S_2, S_3, S_4 with one-step transition matrix

$$Q = \begin{bmatrix} 0.27 & 0.18 & 0.15 & 0.40 \\ 0.33 & 0.25 & 0.12 & 0.30 \\ 0.17 & 0.30 & 0.25 & 0.28 \\ 0.23 & 0.27 & 0.20 & 0.30 \end{bmatrix}$$

and the initial distribution of states $u^{(0)} = [0.20 \quad 0.40 \quad 0.10 \quad 0.30]$. Determine the steady distribution $\pi = [\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4]$.

$$0.27\pi_1 + 0.33\pi_2 + 0.17\pi_3 + 0.23\pi_4$$
$$0.18\pi_1 + 0.25\pi_2 + 0.3\pi_3 + 0.27\pi_4$$
$$0.15\pi_1 + 0.12\pi_2 + 0.25\pi_3 + 0.2\pi_4$$
$$0.4\pi_1 + 0.3\pi_2 + 0.28\pi_3 + 0.3\pi_4$$

$$\left.\begin{matrix} -0.73\pi_1 + 0.33\pi_2 + 0.17\pi_3 + 0.23\pi_4 \\ 0.18\pi_1 - 0.75\pi_2 + 0.3\pi_3 + 0.27\pi_4 \\ 0.15\pi_1 + 0.12\pi_2 - 0.75\pi_3 + 0.2\pi_4 \\ 0.4\pi_1 + 0.3\pi_2 + 0.28\pi_3 - 0.7\pi_4 \end{matrix}\right\} \rightarrow \text{Using Calculator}$$

$\pi_1 = 0.25434 \quad \pi_2 = 0.2474 \quad \pi_3 = 0.1763 \quad \pi_4 = 0.3219$

Determine the distribution of states after 2 ansitions.

$u^{(0)} = [0.20 \quad 0.40 \quad 0.10 \quad 0.30]$.

$$P(2) = u^{(0)} \cdot Q^2$$

$$= [0.2 \ 0.4 \ 0.1 \ 0.3] \cdot \begin{bmatrix} 0.27 & 0.18 & 0.15 & 0.40 \\ 0.33 & 0.25 & 0.12 & 0.30 \\ 0.17 & 0.30 & 0.25 & 0.28 \\ 0.23 & 0.27 & 0.20 & 0.30 \end{bmatrix}$$

$$= [0.2558 \quad 0.24547 \quad 0.17479 \quad 0.32394]$$

Suppose the process {X(t) , t >= 0 } be a poisson process having rate x = 2

(1) P [X(2) = 4 , X(5) = 12 , X(9) = 16]

$$P[X(2)=4 \;,\; X(5)-X(2)=8 \;,\; X(9)-X(5)=4]$$

$$P[X(2)=4] \cdot P[X(5)-X(2)=8] \cdot P[X(9)-X(5)=4]$$

$\lambda t=4$, $\lambda t=6$, $\lambda t=8$

$n=4$ $\quad$ $n=8$ $\quad$ $n=4$

$$\frac{4^4 \times e^{-4}}{4!} \times \frac{6^8 \cdot e^{-6}}{8!} \times \frac{8^4 \cdot e^{-8}}{4!} = 0.001154958$$

---

(2) P [X(1.5) = 10 , X(3.5) = 18 , X(5) = 20]

$$P[X(1.5)=10, X(3.5)-X(1.5)=8, X(5)-X(3.5)=2]$$

$$P[X(1.5)=10] \cdot P[X(3.5)-X(1.5)=8] \cdot P[X(5)-X(3.5)=2]$$

$\lambda t=3$, $\lambda t=4$, $\lambda t=3$

$n=10$ $\quad$ $n=8$ $\quad$ $n=2$

$$\frac{3^{10} \times e^{-3}}{10!} \times \frac{4^8 \times e^{-4}}{8!} \times \frac{3^2 \times e^{-3}}{2!} = 0.000005403$$

Suppose the process {X(t) , t >= 0 } be a poisson process having rate x = 10

(1) P [X(6) = 15]

$$\frac{\lambda t^{n} \cdot e^{-\lambda t}}{n!} = \frac{60^{15} \cdot e^{-60}}{15!} = \boxed{3.14847 \times 10^{-12}}$$

---

(2) P[X(6) = 15 , X(20) = 30]

$$P[X(6)=15 \, , X(20)-X(6)=15]$$

$$P[X(6)=15] \times P[X(20)-X(6)=15] \quad \cdots\cdots\rightarrow \quad \begin{matrix} \lambda t = 140 \\ n = 15 \end{matrix}$$

$$= \frac{60^{15} \cdot e^{-60}}{15!} \times \frac{140^{15} \times e^{-140}}{15!} = \boxed{5.9196 \times 10^{-53}}$$

---

(3) P [X(20) = 30 / X(6) = 15]

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{5.9196 \times 10^{-53}}{3.14847 \times 10^{-12}} = \boxed{1.88015 \times 10^{-41}}$$