Homework 0

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1.

(1) $\{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$

(2) $P(X=(0,0,0))=(1-0.6)^3=0.064$

Probability of two 1's and one 0's $= 0.6\times 0.6\times 0.4\times\binom{3}{1}=0.432$

2.

(1)

A denotes the event that the alarm is triggered

B denotes the event that there was a burglar

$$P(B\mid A)=\frac{P(A\mid B)P(B)}{P(A)}=\frac{0.97\times 0.005}{0.005\times 0.97+0.995\times 0.002}=\frac{0.00485}{0.00485+0.00199}\approx 0.709 \tag{1}$$

(2)

Let E denote the event that there was an earthquake

$$\begin{aligned}
P(B\mid E\cap A)&=\frac{P(B\cap E\cap A)}{P(E\cap A)}\\
&=\frac{P(A\mid B\cap E)\times P(B\cap E)}{P(A\mid E)\times P(E)}\\
&=\frac{P(A\mid B\cap E)\times P(B\cap E)}{(P(A\mid E\cap B)+P(A\mid E\cap\neg B))\times P(E)}\\
&=\frac{(1-0.03\times 0.99)\times 0.005\times 0.001}{((1-0.03\times 0.99)+0.02)\times 0.001}\\
&\approx 0.0049
\end{aligned}$$

3.

We can rewrite the likelihood function as $P(X \mid \Theta) = \prod_{i=1}^{K} P(i)^{n_i}$, where $n_i = \sum_{j=1}^{N} \delta(i, x_j)$

Our goal is to find a Θ that maximizes $P(X \mid \Theta)$, where $\Theta_k = P(X = k \mid \Theta)$

$$\arg\max_{\Theta} P(X \mid \Theta) = \arg\max_{\Theta} \log P(X \mid \Theta) = \arg\max_{\Theta} \frac{\log P(X \mid \Theta)}{N} \tag{2}$$

Substituing the rewritten $P(X \mid \Theta)$, we get:

$$\arg\max_{\Theta} \sum_{i=1}^{K} \frac{n_i}{N} \log \Theta_i \tag{3}$$

Let's define $q_i = \frac{n_i}{N}$

$$\sum_{i=1}^{K} q_i \log \Theta_i = \sum_{i=1}^{K} q_i \log \frac{\Theta_i}{q_i} + \sum_{i=1}^{K} q_i \log q_i = -KL(q||\Theta) - H(q) \tag{4}$$

$$\arg\max_{\Theta}(-KL(q||\Theta) - H(q)) = \arg\max_{\Theta}(-KL(q||\Theta)) = \arg\min_{\Theta} KL(q||\Theta) \tag{5}$$

Since $KL(q||\Theta) \geq 0$ with equality if and only if $q = \Theta$ as shown in the answer to the next problem, we have $q = \Theta$. So $p_k = \Theta_k = q_k = \frac{\sum_{n=1}^{N} \delta(x^{(n)}, k)}{N}$

4.

(1) $1 \times 0.7 + 0 \times 0.3 = 0.7$

(2) $-0.7 \log_2 0.7 - (1 - 0.7) \log_2 (1 - 0.7) \approx 0.881$

(3) $\sum_{x \in X} P(X) \log \frac{P(x)}{Q(x)} = 0.7 \times \log \frac{0.7}{0.4} + 0.3 \times \log \frac{0.3}{0.6} \approx 0.08$

(4)

Since $\log a \leq a - 1$ for all $a > 0$, with equality if and only if $a = 1$, we have

$$KL(P||Q) = -\sum_{x \in X} P(x) \log \frac{Q(x)}{P(x)} \geq -\sum_{x \in X} P(x)(\frac{Q(x)}{P(x)} - 1) = -\sum_{x \in X} Q(x) + \sum_{x \in X} P(x) = 0 \tag{6}$$