Terse Notes on Riemannian Geometry

Tom Fletcher

January 26, 2010

These notes cover the basics of Riemannian geometry, Lie groups, and symmetric spaces. This is just a listing of the basic definitions and theorems with no in-depth discussion or proofs. Some exercises are included at the end of each section to give you something to think about. See the references cited within for more complete coverage of these topics.

Many geometric entities are representable as Lie groups or symmetric spaces. Transformations of Euclidean spaces such as translations, rotations, scalings, and affine transformations all arise as elements of Lie groups. Geometric primitives such as unit vectors, oriented planes, and symmetric, positive-definite matrices can be seen as points in symmetric spaces. This chapter is a review of the basic mathematical theory of Lie groups and symmetric spaces.

The various spaces that are described throughout these notes are all generalizations, in one way or the other, of Euclidean space, $\mathbb{R}^n$. Euclidean space is a topological space, a Riemannian manifold, a Lie group, and a symmetric space. Therefore, each section will use $\mathbb{R}^n$ as a motivating example. Also, since the study of geometric transformations is stressed, the reader is encouraged to keep in mind that $\mathbb{R}^n$ can also be thought of as a transformation space, that is, as the set of translations on $\mathbb{R}^n$ itself.

1 Topology

The study of a topological spaces arose from the desire to generalize the notion of continuity on Euclidean spaces to more general spaces. Topology is a fundamental building block for the theory of manifolds and function spaces. This section is a review of the basic concepts needed for the study of differentiable manifolds. For a more thorough introduction see [14]. For several examples of topological spaces, along with a concise reference for definitions, see [21].

1.1 Basics

Remember that continuity of a function on the real line is phrased in terms of open intervals, i.e., the usual ϵ-δ definition. A topology defines which subsets of a set X are "open", much in the same way an interval is open. As will be seen at the end of this subsection, open sets in $\mathbb{R}^n$ are made up of unions of open balls of the form $B(x, r) = \{y \in \mathbb{R}^n : \|x-y\| < r\}$. For a general set X this concept of open sets can be formalized by the following set of axioms.

Definition 1.1. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that
(1) $\varnothing$ and X are in $\mathcal{T}$.
(2) The union of an arbitrary collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of a finite collection of elements of $\mathcal{T}$ is in $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a **topological space**. However, it is a standard abuse of notation to leave out the topology $\mathcal{T}$ and simply refer to the topological space X. Elements of $\mathcal{T}$ are called **open sets**. A set $C \subset X$ is a **closed set** if it's complement, $X - C$, is open. Unlike doors, a set can be both open and closed, and there can be sets that are neither open nor closed. Notice that the sets $\varnothing$ and X are both open and closed.

Example 1.1. Any set X can be given a topology consisting of only $\varnothing$ and X being open sets. This topology is called the **trivial topology** on X. Another simple topology is the **discrete topology** on X, where any subset of X is an open set.

Definition 1.2. A **basis** for a topology on a set X is a collection $\mathcal{B}$ of subsets of X such that
(1) For each $x \in X$ there exists a $B \in \mathcal{B}$ containing x.
(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \subset B_1 \cap B_2$ such that $x \in B_3$.

The basis $\mathcal{B}$ generates a topology $\mathcal{T}$ by defining a set $U \subset X$ to be open if for each $x \in U$ there exists a basis element $B \in \mathcal{B}$ with $x \in B \subset U$. The reader can check that this does indeed define a topology. Also, the reader should check that the generated topology $\mathcal{T}$ consists of all unions of elements of $\mathcal{B}$.

Example 1.2. The motivating example of a topological space is Euclidean space $\mathbb{R}^n$. It is typically given the standard topological structure generated by the basis of open balls $B(x,r) = \{y \in \mathbb{R}^n : \|x-y\| < r\}$ for all $x \in \mathbb{R}^n, r \in \mathbb{R}$. Therefore, a set in $\mathbb{R}^n$ is open if and only if it is the union of a collection of open balls. Examples of closed sets in $\mathbb{R}^n$ include sets of discrete points, vector subspaces, and closed balls, i.e., sets of the form $\bar{B}(x,r) = \{y \in \mathbb{R}^n : \|x-y\| \leq r\}$.

1.2 Subspace and Product Topologies

Here are two simple methods for constructing new topologies from existing ones. These constructions arise often in the study of manifolds. It is left as an exercise to check that these two definitions lead to valid topologies.

Definition 1.3. Let X be a set with topology $\mathcal{T}$ and $Y \subset X$. Then Y can be given the **subspace topology** $\mathcal{T}'$, in which the open sets are given by $U \cap Y \in \mathcal{T}'$ for all $U \in \mathcal{T}$.

Definition 1.4. Let X and Y be topological spaces. The **product topology** on the product set $X \times Y$ is generated by the basis elements $U \times V$, for all open sets $U \in X$ and $V \in Y$.

1.3 Metric spaces

Notice that the topology on $\mathbb{R}^n$ is defined entirely by the Euclidean distance between points. This method for defining a topology can be generalized to any space where a distance is defined.

Definition 1.5. A **metric space** is a set X with a function $d : X \times X \to \mathbb{R}$ that satisfies
(1) $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) + d(y,z) \geq d(x,z)$.

The function d above is called a **metric** or **distance function**. Using the distance function of a metric space, a basis for a topology on X can be defined as the collection of open balls $B(x,r) = \{y \in X : d(x,y) < r\}$ for all

$x \in X, r \in \mathbb{R}$. From now on when a metric space is discussed, it is assumed that it is given this topology.

One special property of metric spaces will be important in the review of manifold theory.

Definition 1.6. A metric d on a set X is called **complete** if every Cauchy sequence converges in X. A Cauchy sequence is a sequence $x_1, x_2, \ldots \in X$ such that for any $\epsilon > 0$ there exists an integer N such that $d(x_i, x_j) < \epsilon$ for all $i, j > N$.

1.4 Continuity

As was mentioned at the beginning of this section, topology developed from the desire to generalize the notion of continuity of mappings of Euclidean spaces. That generalization is phrased as follows:

Definition 1.7. Let X and Y be topological spaces. A mapping $f : X \to Y$ is **continuous** if for each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X.

It is easy to check that for a function $f : \mathbb{R} \to \mathbb{R}$ the above definition is equivalent to the standard ϵ-δ definition.

Definition 1.8. Again let X and Y be topological spaces. A mapping $f : X \to Y$ is a **homeomorphism** if it is bijective and both f and f^{-1} are continuous. In this case X and Y are said to be **homeomorphic**.

When X and Y are homeomorphic, there is a bijective correspondence between both the points and the open sets of X and Y. Therefore, as topological spaces, X and Y are indistinguishable. This means that any property or theorem that holds for the space X that is based only on the topology of X also holds for Y.

1.5 Various Topological Properties

This section is a discussion of some special properties that a topological space may possess. The particular properties that are of interest are the ones that are important for the study of manifolds.

Definition 1.9. A topological space X is said to be **Hausdorff** if for any two distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$.

Notice that any metric space is a Hausdorff space. Given any two distinct points x, y in a metric space X, we have $d(x,y) > 0$. Then the two open balls $B(x,r)$ and $B(y,r)$, where $r = \frac{1}{2}d(x,y)$, are disjoint open sets containing x and y, respectively. However, not all topological spaces are Hausdorff. For example, take any set X with more than one point and give it the trivial topology, i.e., $\varnothing$ and X as the only open sets.

Definition 1.10. Let X be a topological space. A collection $\mathcal{O}$ of open subsets of X is said to be an **open cover** if $X = \bigcup_{U \in \mathcal{O}} U$. A topological space X is said to be **compact** if for any open cover $\mathcal{O}$ of X there exists a finite subcollection of sets from $\mathcal{O}$ that covers X.

The Heine-Borel theorem (see [15], Theorem 2.41) gives intuitive criteria for a subset of $\mathbb{R}^n$ to be compact. It states that the compact subsets of $\mathbb{R}^n$ are exactly the closed and bounded subsets. Thus, for example, a closed ball $\bar{B}(x,r)$ is compact, as is the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The sphere, like Euclidean space, will be an important example throughout these notes.

Definition 1.11. A **separation** of a topological space X is a pair of disjoint open sets U, V such that $X = U \cup V$. If no separation of X exists, it is said to be **connected**.

Exercises

1. Give an example of a topology on the three point set $X = \{a, b, c\}$ where there exists a set that is neither open nor closed. Give an example of a topology in which a set other than $\emptyset$ or X is both open and closed.

2. Let Y be a subspace of a topological space X, with basis $\mathcal{B}$. Show that the sets $\{B \cap Y : B \in \mathcal{B}\}$ form a basis for the subspace topology of Y.

3. Let X and Y be topological spaces. Show that an equivalent definition for continuity of a mapping $f : X \to Y$ is that for any closed set $C \subset Y$, $f^{-1}(C)$ is closed in X.

4. Let X and Y be topological spaces and $f : X \to Y$ be a continuous mapping. Show that if X is compact, then its image, $f(X)$, is also compact.

2 Differentiable Manifolds

Differentiable manifolds are spaces that locally behave like Euclidean space. Much in the same way that topological spaces are natural for talking about continuity, differentiable manifolds are a natural setting for calculus. Notions such as differentiation, integration, vector fields, and differential equations make sense on differentiable manifolds. This section gives a review of the basic construction and properties of differentiable manifolds. A good introduction to the subject may be found in [2]. For a comprehensive overview of differential geometry see [16–20]. Other good references include [1, 6, 13].

2.1 Topological Manifolds

A manifold is a topological space that is locally equivalent to Euclidean space. More precisely,

Definition 2.1. A **manifold** is a Hausdorff space M with a countable basis such that for each point $p \in M$ there is a neighborhood U of p that is homeomorphic to $\mathbb{R}^n$ for some integer n.

At each point $p \in M$ the dimension n of the $\mathbb{R}^n$ in Definition 2.1 turns out to be unique (Exercise 1). If the integer n is the same for every point in M, then M is called a ***n*-dimensional** manifold. The simplest example of a manifold is $\mathbb{R}^n$, since it is trivially homeomorphic to itself. Likewise, any open set of $\mathbb{R}^n$ is also a manifold.

2.2 Differentiable Structures on Manifolds

The next step in the development of the theory of manifolds is to define a notion of differentiation of manifold mappings. Differentiation of mappings in Euclidean space is defined as a local property. Although a manifold is locally homeomorphic to Euclidean space, more structure is required to make differentiation possible. First, recall that a function on Euclidean space $f : \mathbb{R}^n \to \mathbb{R}$ is **smooth** or $\boldsymbol{C^\infty}$ if all of its partial derivatives exist. A mapping of Euclidean spaces $f : \mathbb{R}^m \to \mathbb{R}^n$ can be thought of as a n-tuple of real-valued functions on $\mathbb{R}^m$, $f = (f^1, \ldots, f^n)$, and f is smooth if each f^i is smooth.

Given two neighborhoods U, V in a manifold M, two homeomorphisms $x : U \to \mathbb{R}^n$ and $y : V \to \mathbb{R}^n$ are said to be $\boldsymbol{C^\infty}$**-related** if the mappings $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ are C^∞. The

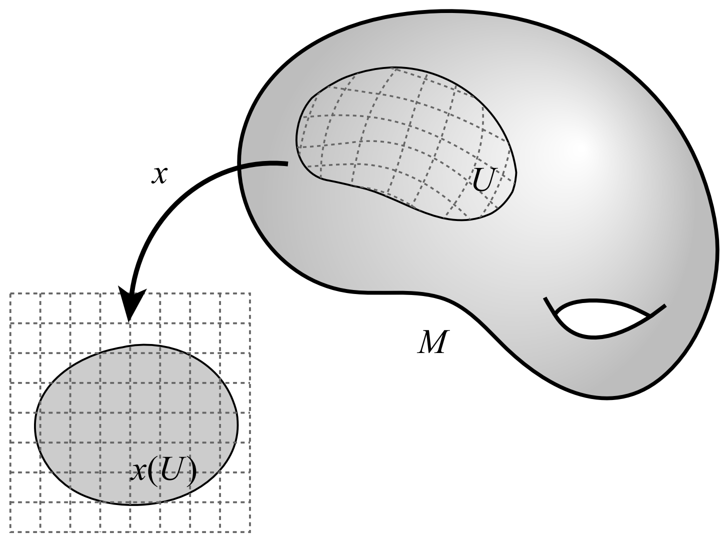

Figure 1: A local coordinate system (x, U) on a manifold M.

pair (x, U) is called a **chart** or **coordinate system**, and can be thought of as assigning a set of coordinates to points in the neighborhood U (see Figure 1). That is, any point $p \in U$ is assigned the coordinates $x^1(p), \ldots, x^n(p)$. As will become apparent later, coordinate charts are important for writing local expressions for derivatives, tangent vectors, and Riemannian metrics on a manifold. A collection of charts whose domains cover M is called an **atlas**. Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are said to be **compatible** if any pair of charts $(x, U) \in \mathcal{A}$ and $(y, V) \in \mathcal{A}'$ are C^∞-related.

Definition 2.2. An atlas $\mathcal{A}$ on a manifold M is said to be **maximal** if for any compatible atlas $\mathcal{A}'$ on M any coordinate chart $(x, U) \in \mathcal{A}'$ is also a member of $\mathcal{A}$.

Definition 2.3. A **smooth structure** on a manifold M is a maximal atlas $\mathcal{A}$ on M. The manifold M along with such an atlas is termed a **smooth manifold**.

The next theorem demonstrates that it is not necessary to define every coordinate chart in a maximal atlas, but rather, one can define enough compatible coordinate charts to cover the manifold.

Theorem 2.1. Given a manifold M with an atlas $\mathcal{A}$, there is a unique maximal atlas $\mathcal{A}'$ such that $\mathcal{A} \subset \mathcal{A}'$.

Proof. It is easy to check that the unique maximal atlas $\mathcal{A}'$ is given by the set of all charts that are C^∞-related to all charts in $\mathcal{A}$. □

Example 2.1. The easiest example of a differentiable manifold is Euclidean space, in which the differentiable structure can be defined by the global chart given by the identity map on $\mathbb{R}^n$.

Example 2.2. Another simple example of a smooth manifold can be constructed as the graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$. Recall that the graph of f is the set $M = \{(x, f(x)) : x \in \mathbb{R}^n\}$, which is a subset of $\mathbb{R}^{n+1}$. Now we can see M is a smooth n-dimensional manifold by considering the global chart given by the projection mapping $\pi : M \to \mathbb{R}^n$, defined as $\pi(x, f(x)) = x$.

Example 2.3. Consider the sphere S^2 as a subset of $\mathbb{R}^3$. The upper hemisphere $U = \{(x, y, z) \in S^2 : z > 0\}$ is an open neighborhood in S^2. Now consider the homeomorphism $\phi : S^2 \to \mathbb{R}^2$ given by

$$\phi : (x, y, z) \mapsto (x, y).$$

This gives a coordinate chart (ϕ, U). Similar charts can be produced for the lower hemisphere, and for hemispheres in the x and y dimensions. The reader may check that these charts are C^∞-related and cover S^2. Therefore, these charts make up an atlas on S^2 and by Theorem 2.1 there is a unique maximal atlas containing these charts that makes S^2 a smooth manifold. A similar argument can be used to show that the n-dimensional sphere, S^n, for any $n \geq 1$ is also a smooth manifold.

2.3 Smooth Functions and Mappings

Now consider a function $f : M \to \mathbb{R}$ on the smooth manifold M. This function is said to be a **smooth function** if for every coordinate chart (x, U) on M the function $f \circ x^{-1} : U \to \mathbb{R}$ is smooth. More generally, a mapping $f : M \to N$ of smooth manifolds is said to be a **smooth mapping** if for each coordinate chart (x, U) on M and each coordinate chart (y, V) on N the mapping $y \circ f \circ x^{-1} : x(U) \to y(V)$ is a smooth mapping. Notice that the mapping of manifolds was converted locally to a mapping of Euclidean spaces, where differentiability is easily defined. We'll denote the space of all smooth functions on a smooth manifold M as $C^\infty(M)$. This space forms an algebra under pointwise addition and multiplication of functions and multiplication by real constants.

As in the case of topological spaces, there is a desire to know when two smooth manifolds are equivalent. This should mean that they are homeomorphic as topological spaces and also that they have equivalent smooth structures. This notion of equivalence is given by

Definition 2.4. Given two smooth manifolds M, N, a bijective mapping $f : M \to N$ is called a **diffeomorphism** if both f and f^{-1} are smooth mappings.

Example 2.4. Two manifolds may be diffeomorphic even though they have two unique differentiable structures, i.e., atlases that are not compatible with each other. For example, consider the manifold $\hat{\mathbb{R}}$, topologically equivalent to the real line, but with differentiable structure given by the global chart $\phi : \hat{\mathbb{R}} \to \mathbb{R}$, defined as $\phi(x) = x^3$. This chart is a homeomorphism and smooth, but it's inverse is not smooth at $x = 0$. Therefore, ϕ is not C^∞-related to the identity map, and the resulting atlas is not compatible with the standard differentiable structure on $\mathbb{R}$. However, $\hat{\mathbb{R}}$ is diffeomorphic to $\mathbb{R}$ by the mapping ϕ itself. So, $\hat{\mathbb{R}}$ is in this sense equivalent to $\mathbb{R}$ with its usual manifold structure, and we say that $\hat{\mathbb{R}}$ and $\mathbb{R}$ have the same differentiable structure up to diffeomorphism.

Interestingly, $\mathbb{R}$ has only one differentiable structure up to diffeomorphism. However, $\mathbb{R}^4$ has several unique differentiable structures that are not diffeomorphic to each other (in fact, an entire continuum of differentiable structures!). This is the only example of a Euclidean space with "non-standard" differentiable structure; in all other dimensions there is only the familar differentiable structure on $\mathbb{R}^n$.

2.4 Tangent Spaces

Given a manifold $M \subset \mathbb{R}^d$, it is possible to associate a linear subspace of $\mathbb{R}^d$ to each point $p \in M$ called the **tangent space** at p. This space is denoted T_pM and is intuitively thought of as the linear subspace that best approximates M in a neighborhood of the point p. Vectors in this space are called **tangent vectors** at p.

Tangent vectors can be thought of as directional derivatives. Consider a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. Then given any smooth function[1] $f : M \to \mathbb{R}$, the composition $f \circ \gamma$ is a smooth function, and the following derivative exists:

$$\frac{d}{dt}(f \circ \gamma)(0).$$

This leads to an equivalence relation $\sim$ between smooth curves passing through p. Namely, if γ_1 and γ_2 are two smooth curves passing through the point p at $t = 0$, then $\gamma_1 \sim \gamma_2$ if

$$\frac{d}{dt}(f \circ \gamma_1)(0) = \frac{d}{dt}(f \circ \gamma_2)(0),$$

for any smooth function $f : M \to \mathbb{R}$. A tangent vector is now defined as one of these equivalence classes of curves. It can be shown (see [1]) that these equivalence classes form a vector space, i.e., the tangent space T_pM, which has the same dimension as M. Given a local coordinate system (x, U) containing p, a basis for the tangent space T_pM is given by the partial derivative operators $\partial/\partial x^i|_p$, which are the tangent vectors associated with the coordinate curves of x. We can write an arbitrary vector $v \in T_pM$ using these standard coordinate vectors as a basis:

$$v = \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x^i}\right|_p, \quad \text{where } v_i \in \mathbb{R}.$$

Example 2.5. Again, consider the sphere S^2 as a subset of $\mathbb{R}^3$. The tangent space at a point $p \in S^2$ is the set of all vectors in $\mathbb{R}^3$ perpendicular to p, i.e., $T_pS^2 = \{v \in \mathbb{R}^3 : \langle v, p\rangle = 0\}$. This is of course a two-dimensional vector space, and it is the space of all tangent vectors at the point p for smooth curves lying on the sphere and passing through the point p.

A **vector field** on a manifold M is a function that smoothly assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. This mapping is smooth in the sense that the components of the vectors may be written as smooth functions in any local coordinate system. A vector field may be seen as an operator $X : C^\infty(M) \to C^\infty(M)$ that maps a smooth function $f \in C^\infty(M)$ to the smooth function $Xf : p \mapsto X_pf$. In other words, the directional derivative is applied at each point on M. Given a coordinate system (x, U),

[1]Strictly speaking, the tangent vectors at p are defined as directional derivatives of smooth **germs** of functions at p, which are equivalence classes of functions that agree in some neighborhood of p.

the partial derivatives $\partial/\partial x^i$ are a vector field, and an arbitrary vector field X can be written

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x^i}, \quad \text{where } X_i \in C^\infty(M).$$

For two manifolds M and N a smooth mapping $\phi : M \to N$ induces a linear mapping of the tangent spaces $\phi_* : T_pM \to T_{\phi(p)}N$ called the **differential** of ϕ. It is given by $\phi_*(X_p)f = X_p(f \circ \phi)$ for any vector $X_p \in T_pM$ and any smooth function $f \in C^\infty(M)$. A smooth mapping of manifolds does not always induce a mapping of vector fields (for instance, when the mapping is not onto). However, a related concept is given in the following definition.

Definition 2.5. Given a mapping of smooth manifolds $\phi : M \to N$, a vector field X on M and a vector field Y on N are said to be **ϕ-related** if $\phi_*(X(p)) = Y(q)$ holds for each $q \in N$ and each $p \in \phi^{-1}(q)$.

Exercises

1. Prove that in Definition 2.1 the n for a fixed $x \in M$ must be unique.
2. Show that the charts in the atlas $\mathcal{A}'$ in Theorem 2.1 are C^∞-related.
3. Prove that a differentiable manifold can always be specified with a countable number of charts. Give an example of a manifold that cannot be specified with only a finite number of charts.
4. The **general linear group** on $\mathbb{R}^n$ is the space of all nonsingular $n \times n$ matrices, denoted $GL(n) = \{A \in \mathbb{R}^{n \times n} \,|\, \det(A) \neq 0\}$. Prove that $GL(n)$ is a differentiable manifold. (Hint: Use the fact that it is a subspace of Euclidean space and that det is a continuous function.)
5. Given two smooth mappings $\phi : M \to N$ and $\psi : N \to P$, with M, N, P all smooth manifolds, show that the composition $\psi \circ \phi : M \to P$ is a smooth mapping.

3 Riemannian Geometry

As mentioned at the beginning of this chapter, the idea of distances on a manifold will be important in the definition of manifold statistics. The notion of distances on a manifold falls into the realm of Riemannian geometry.

This section briefly reviews the concepts needed. A good crash course in Riemannian geometry can be found in [12]. Also, see the books [2, 9, 16, 17].

Recall the definition of length for a smooth curve in Euclidean space. Let $\gamma : [a, b] \to \mathbb{R}^d$ be a smooth curve segment. Then at any point $t_0 \in [a, b]$ the derivative of the curve $\gamma'(t_0)$ gives the velocity of the curve at time t_0. The length of the curve segment γ is given by integrating the speed of the curve, i.e.,

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

The definition of the length functional thus requires the ability to take the norm of tangent vectors. On manifolds this is handled by the definition of a Riemannian metric.

3.1 Riemannian Metrics

Definition 3.1. A **Riemannian metric** on a manifold M is a function that smoothly assigns to each point $p \in M$ an inner product $\langle \cdot, \cdot \rangle$ on the tangent space T_pM. A **Riemannian manifold** is a smooth manifold equipped with such a Riemannian metric.

Now the norm of a tangent vector $v \in T_pM$ is defined as $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$. Given local coordinates $x^1, \ldots, x^n$ in a neighborhood of p, the coordinate vectors $v^i = \partial/\partial x^i$ at p form a basis for the tangent space T_pM. The Riemannian metric may be expressed in this basis as an $n \times n$ matrix g, called the metric tensor, with entries given by

$$g_{ij} = \langle v^i, v^j \rangle.$$

The g_{ij} are smooth functions of the coordinates $x^1, \ldots, x^n$.

Given a smooth curve segment $\gamma : [a, b] \to M$, the length of γ can be defined just as in the Euclidean case as

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt, \tag{1}$$

where now the tangent vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}M$, and the norm is given by the Riemannian metric at $\gamma(t)$.

Given a manifolds M and a manifold N with Riemannian metric $\langle \cdot, \cdot \rangle$, a mapping $\phi : M \to N$ induces a metric $\phi^*\langle \cdot, \cdot \rangle$ on M defined as

$$\phi^*\langle X_p, Y_p \rangle = \langle \phi_*(X_p), \phi_*(Y_p) \rangle.$$

This metric is called the **pull-back** metric induced by ϕ, as it maps the metric in the opposite direction of the mapping ϕ.

3.2 Geodesics

In Euclidean space the shortest path between two points is a straight line, and the distance between the points is measured as the length of that straight line segment. This notion of shortest paths can be extended to Riemannian manifolds by considering the problem of finding the shortest smooth curve segment between two points on the manifold. If $\gamma : [a, b] \to M$ is a smooth curve on a Riemannian manifold M with endpoints $\gamma(a) = x$ and $\gamma(b) = y$, a **variation of** γ **keeping endpoints fixed** is a family α of smooth curves:

$$\alpha : (-\epsilon, \epsilon) \times [a, b] \to M,$$

such that

1. $\alpha(0, t) = \gamma(t)$,
2. $\tilde{\alpha}(s_0) : t \mapsto \alpha(s_0, t)$ is a smooth curve segment for fixed $s_0 \in (-\epsilon, \epsilon)$,
3. $\alpha(s, a) = x$, and $\alpha(s, b) = y$ for all $s \in (-\epsilon, \epsilon)$.

Now the shortest smooth path between the points $x, y \in M$ can be seen as finding a critical point for the length functional (1), where the length of $\tilde{\alpha}$ is considered as a function of s. The path $\gamma = \tilde{\alpha}(0)$ is a critical path for L if

$$\frac{dL(\tilde{\alpha}(s))}{ds}\bigg|_{s=0} = 0.$$

It turns out to be easier to work with the critical paths of the **energy functional**, which is given by

$$E(\gamma) = \int_a^b \|\gamma'(t)\|^2 dt.$$

It can be shown (see [16]) that a critical path for E is also a critical path for L. Conversely, a critical path for L, once reparameterized proportional to arclength, is a critical path for E. Thus, assuming curves are parameterized proportional to arclength, there is no distinction between curves with minimal length and those with minimal energy. A critical path of the functional E is called a **geodesic**.

Given a chart (x, U) a geodesic curve $\gamma \subset U$ can be written in local coordinates as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Using any such coordinate system, γ satisfies the following differential equation (see [16] for details):

$$\frac{d^2\gamma^k}{dt^2} = -\sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t)) \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt}. \tag{2}$$

The symbols Γ_{ij}^k are called the **Christoffel symbols** and are defined as

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right),$$

where g^{ij} denotes the entries of the inverse matrix g^{-1} of the Riemannian metric.

Example 3.1. In Euclidean space $\mathbb{R}^n$ the Riemannian metric is given by the identity matrix at each point $p \in \mathbb{R}^n$. Since the metric is constant, the Christoffel symbols are zero. Therefore, the geodesic equation (2) reduces to

$$\frac{d^2\gamma^k}{dt^2} = 0.$$

The only solutions to this equation are straight lines, so geodesics in $\mathbb{R}^n$ must be straight lines.

Given two points on a Riemannian manifold, there is no guarantee that a geodesic exists between them. There may also be multiple geodesics connecting the two points, i.e., geodesics are not guaranteed to be unique. Moreover, a geodesic does not have to be a *global* minimum of the length functional, i.e., there may exist geodesics of different lengths between the same two points. The next two examples demonstrate these issues.

Example 3.2. Consider the plane with the origin removed, $\mathbb{R}^2 - \{0\}$, with the same metric as $\mathbb{R}^2$. Geodesics are still given by straight lines. There does not exist a geodesic between the two points $(1, 0)$ and $(-1, 0)$.

Example 3.3. Geodesics on the sphere S^2 are given by great circles, i.e., circles on the sphere with maximal diameter. This fact will be shown later in the section on symmetric spaces. There are an infinite number of equal-length geodesics between the north and south poles, i.e., the meridians. Also, given any two points on S^2 that are not antipodal, there is a unique great circle between them. This great circle is separated into two geodesic segments between the two points. One geodesic segment is longer than the other.

The idea of a global minimum of length leads to a definition of a distance metric $d : M \times M \to \mathbb{R}$ (not to be confused with the Riemannian metric). It is defined as

$$d(p, q) = \inf\{L(\gamma) : \gamma \text{ a smooth curve between } p \text{ and } q\}.$$

If there is a geodesic γ between the points p and q that realizes this distance, i.e., if $L(\gamma) = d(p, q)$, then γ is called a **minimal geodesic**. Minimal geodesics are guaranteed to exist under certain conditions, as described by the following definition and the Hopf-Rinow Theorem below.

Definition 3.2. A Riemannian manifold M is said to be **complete** if every geodesic segment $\gamma : [a, b] \to M$ can be extended to a geodesic from all of $\mathbb{R}$ to M.

The reason such manifolds are called "complete" is revealed in the next theorem.

Theorem 1 (Hopf-Rinow)**.** *If M is a complete, connected Riemannian manifold, then the distance metric $d(\cdot, \cdot)$ induced on M is complete. Furthermore, between any two points on M there exists a minimal geodesic.*

Example 3.4. Both Euclidean space $\mathbb{R}^n$ and the sphere S^2 are complete. A straight line in $\mathbb{R}^n$ can extend in both directions indefinitely. Also, a great circle in S^2 extends indefinitely in both directions (even though it wraps around itself). As guaranteed by the Hopf-Rinow Theorem, there is a minimal geodesic between any two points in $\mathbb{R}^n$, i.e., the unique straight line segment between the points. Also, between any two points on the sphere there is a minimal geodesic, i.e., the shorter of the two great circle segments between the two points. Of course, for antipodal points on S^2 the minimal geodesic is not unique.

Given initial conditions $\gamma(0) = p$ and $\gamma'(0) = v$, the theory of second-order partial differential equations guarantees the existence of a unique solution to the defining equation for γ (2) at least locally. Thus, there is a unique geodesic γ with $\gamma(0) = p$ and $\gamma'(0) = v$ defined in some interval $(-\epsilon, \epsilon)$. When the geodesic γ exists in the interval $[0, 1]$, the **Riemannian exponential map** at the point p (see Figure 2), denoted $\mathrm{Exp}_p : T_pM \to M$, is defined as

$$\mathrm{Exp}_p(v) = \gamma(1).$$

If M is a complete manifold, the exponential map is defined for all vectors $v \in T_pM$.

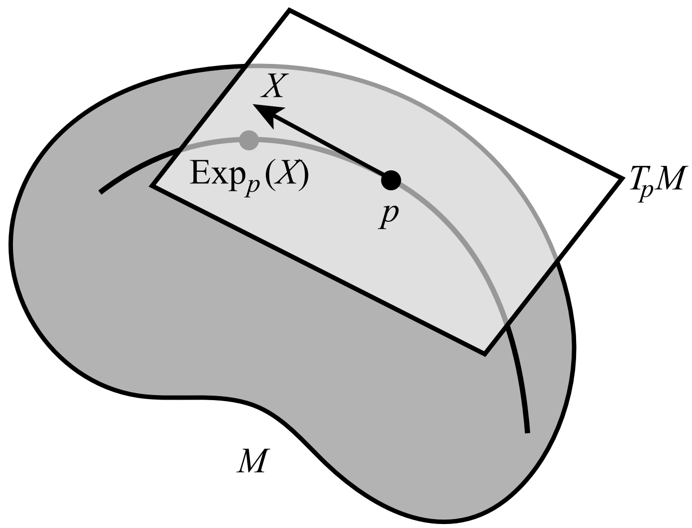

Figure 2: The Riemannian exponential map.

Theorem 2. *Given a Riemannian manifold M and a point $p \in M$, the mapping* Exp_p *is a diffeomorphism in some neighborhood $U \subset T_pM$ containing* 0*.*

This theorem implies that the Exp_p has an inverse defined at least in the neighborhood $\mathrm{Exp}_p(U)$ of p, where U is the same as in Theorem 2. Not surprisingly, this inverse is called the **Riemannian log map** and denoted by $\mathrm{Log}_p : \mathrm{Exp}_p(U) \to T_pM$.

Definition 3.3. An **isometry** is a diffeomorphism $\phi : M \to N$ of Riemannian manifolds that preserves the Riemannian metric. That is, if $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$ are the metrics for M and N, respectively, then $\phi^* \langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_M$.

It follows from the definitions that an isometry preserves the length of curves. That is, if c is a smooth curve on M, then the curve $\phi \circ c$ is a curve of the same length on N. Also, the image of a geodesic under an isometry is again a geodesic.

4 Lie Groups

The set of all possible translations of Euclidean space $\mathbb{R}^n$ is again the space $\mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is transformed by the vector $v \in \mathbb{R}^n$ by vector addition, $p + v$. This transformation has a unique inverse transformation, namely, translation by the negated vector, $-v$. The operation of translation is a smooth mapping of the space $\mathbb{R}^n$. Composing two translations (i.e., addition in $\mathbb{R}^n$) and inverting a translation (i.e., negation in $\mathbb{R}^n$) are also smooth mappings. A set of transformations with these properties, i.e., a smooth manifold with smooth group operations, is known as a Lie group. Many other interesting transformations of Euclidean space are Lie groups, including rotations, reflections, and magnifications. However, Lie groups also arise more generally as smooth transformations of manifolds. This section is a brief introduction to Lie groups. More detailed treatments may be found in [2, 4–6, 8, 16].

It is assumed that the reader knows the basics of group theory (see [7] for an introduction), but the definition of a group is listed here for reference.

Definition 4.1. A **group** is a set G with a binary operation, denoted here by concatenation, such that

1. $(xy)z = x(yz)$, for all $x, y, z \in G$,
2. there is an **identity**, $e \in G$, satisfying $xe = ex = x$, for all $x \in G$,
3. each $x \in G$ has an **inverse**, $x^{-1} \in G$, satisfying $xx^{-1} = x^{-1}x = e$.

As stated at the beginning of this section, a Lie group adds a smooth manifold structure to a group.

Definition 4.2. A **Lie group** G is a smooth manifold that also forms a group, where the two group operations,

$$\begin{array}{llll} (x, y) \mapsto xy & : & G \times G \to G & \textit{Multiplication} \\ x \mapsto x^{-1} & : & G \to G & \textit{Inverse} \end{array}$$

are smooth mappings of manifolds.

Example 4.1. The space of all $n \times n$ non-singular matrices forms a Lie group called the **general linear group**, denoted $GL(n)$. The group operation is matrix multiplication, and $GL(n)$ can be given a smooth manifold structure as an open subset of $\mathbb{R}^{n^2}$. The equations for matrix multiplication and inverse are smooth operations in the entries of the matrices. Thus, $GL(n)$ satisfies the requirements of a Lie group in Definition 4.2. A **matrix group** is any closed subgroup of $GL(n)$. Matrix groups inherit the smooth structure of $GL(n)$ as a subset of $\mathbb{R}^{n^2}$ and are thus also Lie groups. The books [3,5] focus on the theory of matrix groups.

Example 4.2. The $n \times n$ rotation matrices are a closed matrix subgroup of $GL(n)$ and thus form a Lie group. This group is called the **special orthogonal group** and is defined as $SO(n) = \{R \in GL(n) : R^T R = I \text{ and } \det(R) = 1\}$. This space is a closed and bounded subset of R^{n^2}, so it is compact by the Heine-Borel theorem.

Given a point y in a Lie group G, it is possible to define the following two diffeomorphisms:

$$
\begin{aligned}
L_y &: x \mapsto yx && \textit{(Left multiplication)} \\
R_y &: x \mapsto xy && \textit{(Right multiplication)}
\end{aligned}
$$

A vector field X on a Lie group G is called **left-invariant** if it is invariant under left multiplication, i.e., $L_{y*}X = X$ for every $y \in G$. **Right-invariant** vector fields are defined similarly. A left-invariant (or right-invariant) vector field is uniquely defined by its value on the tangent space at the identity, $T_e G$.

Recall that vector fields on G can be seen as operators on the space of smooth functions, $C^\infty(G)$. Thus two vector fields X and Y can be composed to form another operator XY on $C^\infty(G)$. However, the operator XY is not necessarily vector field. Surprisingly, however, the operator $XY - YX$ is a vector field on G. This leads to a definition of the **Lie bracket** of vector fields X, Y on G, defined as

$$[X, Y] = XY - YX. \tag{3}$$

Definition 4.3. A **Lie algebra** is a vector space V equipped with a bilinear product $[\cdot, \cdot] : V \times V \to V$, called a **Lie bracket**, that satisfies

(1) $[X, Y] = -[Y, X]$,

(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$,

for all $X, Y, Z \in V$.

The tangent space of a Lie group G, typically denoted $\mathfrak{g}$ (a German Fraktur font), forms a Lie algebra. The Lie bracket on $\mathfrak{g}$ is induced by the Lie bracket on the corresponding left-invariant vector fields. If X, Y are two vectors in $\mathfrak{g}$, then let $\tilde{X}, \tilde{Y}$ be the corresponding unique left-invariant vector fields on G. Then the Lie bracket on $\mathfrak{g}$ is given by

$$[X, Y] = [\tilde{X}, \tilde{Y}](e).$$

The Lie bracket provides a test for whether the Lie group G is commutative. A Lie group G is commutative if and only if the Lie bracket on the corresponding Lie algebra $\mathfrak{g}$ is zero, i.e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example 4.3. The Lie algebra for Euclidean space $\mathbb{R}^n$ is again $\mathbb{R}^n$. The Lie bracket is zero, i.e., $[X, Y] = 0$ for all $X, Y \in \mathbb{R}^n$. In fact, the Lie bracket for the Lie algebra of any commutative Lie group is always zero.

Example 4.4. The Lie algebra for $GL(n)$ is $\mathfrak{gl}(n)$, the space of all real $n \times n$ matrices. The Lie bracket operation for $X, Y \in \mathfrak{gl}(n)$ is given by

$$[X, Y] = XY - YX.$$

Here the product XY denotes actual matrix multiplication, which turns out to be the same as composition of the vector field operators (compare to (3)). All Lie algebras corresponding to matrix groups are subalgebras of $\mathfrak{gl}(n)$.

Example 4.5. The Lie algebra for the rotation group $SO(n)$ is $\mathfrak{so}(n)$, the space of skew-symmetric matrices. A matrix A is skew-symmetric if $A = -A^T$.

The following theorem will be important later.

Theorem 3. *A direct product $G_1 \times \cdots \times G_n$ of Lie groups is also a Lie group.*

4.1 Lie Group Exponential and Log Maps

Definition 4.4. A mapping of Lie groups $\phi : G_1 \to G_2$ is called a **Lie group homomorphism** if it is a smooth mapping and a homomorphism of groups, i.e., $\phi(e_1) = e_2$, where e_1, e_2 are the respective identity elements of G_1, G_2, and $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G_1$.

The image of a Lie group homomorphism $h : \mathbb{R} \to G$ is called a **one-parameter subgroup**. A one-parameter subgroup is both a smooth curve and a subgroup of G. This does not mean, however, that any one-parameter subgroup is a Lie subgroup of G (it can fail to be an imbedded submanifold of G, which is required to be a Lie subgroup of G). As the next theorem shows, there is a bijective correspondence between the Lie algebra and the one-parameter subgroups.

Theorem 4. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Given any vector $X \in \mathfrak{g}$ there is a unique Lie group homomorphism $h_X : \mathbb{R} \to G$ such that $h'_X(0) = X$.*

The **Lie group exponential map**, $\exp : \mathfrak{g} \to G$, not to be confused with the Riemannian exponential map, is defined by

$$\exp(X) = h_X(1).$$

Example 4.6. For the Lie group $\mathbb{R}^n$ the unique Lie group homomorphism $h_X : \mathbb{R} \to \mathbb{R}^n$ in Theorem 4 is given by $h_X(t) = tX$. Therefore, one-parameter subgroups are given by straight lines at the origin. The Lie group exponential map is the identity. In this case the Lie group exponential map is the same as the Riemannian exponential map at the origin. This is not always the case, however, as will be shown later.

For matrix groups the Lie group exponential map of a matrix $X \in \mathfrak{gl}(n)$ is computed by the formula

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{4}$$

This series converges absolutely for all $X \in \mathfrak{gl}(n)$.

Example 4.7. For the Lie group of 3D rotations, $SO(3)$, the matrix exponential map takes a simpler form. For a matrix $X \in \mathfrak{so}(3)$ the following identity holds:

$$X^3 = -\theta X, \quad \text{where } \theta = \sqrt{\frac{1}{2}\operatorname{tr}(X^T X)}.$$

Substituting this identity into the infinite series (4), the exponential map for $\mathfrak{so}(3)$ can now be reduced to

$$\exp(X) = \begin{cases} I, & \theta = 0, \\ I + \dfrac{\sin\theta}{\theta} X + \dfrac{1 - \cos\theta}{\theta^2} X^2, & \theta \in (0, \pi). \end{cases}$$

The Lie group log map for a rotation matrix $R \in SO(3)$ is given by

$$\log(R) = \begin{cases} I, & \theta = 0, \\ \dfrac{\theta}{2\sin\theta}(R - R^T), & |\theta| \in (0, \pi), \end{cases}$$

where $\operatorname{tr}(R) = 2\cos\theta + 1$.

The exponential map for 3D rotations has an intuitive meaning. Any vector $X \in \mathfrak{so}(3)$, i.e., a skew-symmetric matrix, may be written in the form

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

If $v = (x, y, z) \in \mathbb{R}^3$, then the rotation matrix given by the exponential map $\exp(X)$ is a 3D rotation by angle $\theta = \|v\|$ about the unit axis $v/\|v\|$.

References

[1] L. Auslander and R. E. MacKenzie. *Introduction to Differentiable Manifolds.* Dover, 1977.

[2] W. M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* Academic Press, 2nd edition, 1986.

[3] M. L. Curtis. *Matrix Groups.* Springer-Verlag, 1984.

[4] J. J. Duistermaat and J. A. C. Kolk. *Lie Groups.* Springer, 2000.

[5] B. C. Hall. *Lie groups, Lie algebras, and representations: an elementary introduction.* Springer-Verlag, 2003.

[6] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, 1978.

[7] I. N. Herstein. *Topics in Algebra.* John Wiley and Sons, 2nd edition, 1975.

[8] K. Kawakubo. *The Theory of Transformation Groups.* Oxford University Press, 1991.

[9] J. M. Lee. *Riemannian Manifolds: An Introduction to Curvature.* Springer, 1997.

[10] J. M. Lee. *Introduction to Topological Manifolds.* Springer, 2000.

[11] J. M. Lee. *Introduction to Smooth Manifolds.* Springer, 2002.

[12] J. W. Milnor. *Morse Theory.* Princeton University Press, 1963.

[13] J. W. Milnor. *Topology from the Differentiable Viewpoint.* Princeton University Press, 1997.

[14] J. R. Munkres. *Topology: A First Course.* Prentice-Hall, 1975.

[15] W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 1976.

[16] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 1. Publish or Perish, 3rd edition, 1999.

[17] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 2. Publish or Perish, 3rd edition, 1999.

[18] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 3. Publish or Perish, 3rd edition, 1999.

[19] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 4. Publish or Perish, 3rd edition, 1999.

[20] M. Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 5. Publish or Perish, 3rd edition, 1999.

[21] L. A. Steen and J. A. Seebach. *Counterexamples in Topology*. Dover, 1995.