PMTH339 Assignment 3

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Question 1

$$y'' = 2 \tag{1}$$

By inspection of the homogeneous DE $y'' = 0$, $y_1 = 1$ and $y_2 = x$ are a fundamental pair of solutions with Wronskian $W = y_1 y_2' - y_1' y_2 = 1$. By Theorem 6.1, the unique solution y to (1) satisfying $y(0) = y'(0) = 0$ is given by

$$\begin{aligned}
y(x) &= y_2(x) \int_0^x \frac{y_1(t)}{W(t)} g(t) dt - y_1(x) \int_0^x \frac{y_2(t)}{W(t)} g(t) dt \\
&= x \int_0^x 2dt - \int_0^x 2tdt \\
&= 2x^2 - x^2 \\
&= x^2
\end{aligned}$$

Question 2

$$y'' + n^2 y = \cos mx \tag{2}$$

The homogeneous DE $y'' + n^2 y = 0$ has characteristic equation $\lambda^2 + n^2 = 0$, with solutions $\lambda = \pm in$. Thus, the homogeneous DE has solutions $y_1 = A \cos nx$ and $y_2 = B \sin nx$.

Assume solution to (2) is of the form $y = C_1 \cos mx + C_2 \sin mx$. Differentiating, we get $y' = -C_1 \sin mx + C_2 \cos mx$ and $y'' = -C_1 \cos mx - C_2 \sin mx$. For this to be a solution, we require

$$\begin{aligned}
\cos mx &= y'' + n^2 y \\
&= -C_1 \cos mx - C_2 \sin mx + n^2 (C_1 \cos mx + C_2 \sin mx) \\
&= C_1 (n^2 - 1) \cos mx + C_2 (n^2 - 1) \sin mx
\end{aligned}$$

Equating coefficients, we thus require

$$\begin{aligned}
C_1 (n^2 - 1) = 1 &\implies C_1 = \frac{1}{n^2 - 1} \\
C_2 (n^2 - 1) = 0 &\implies C_2 = 0
\end{aligned}$$

where $n \neq \pm 1$. Therefore the general solution to (2) is

$$y(x) = A \cos nx + B \sin nx + \frac{1}{n^2 - 1} \cos mx \tag{3}$$

A and B can be found according to the initial conditions $y(0) = 1$ and $y'(0) = 0$:

$$
\begin{aligned}
y(0) = 1 &\implies 1 = A + \frac{1}{n^2-1} \implies A = \frac{n^2-2}{n^2-1} \\
y'(0) = 0 &\implies 0 = B
\end{aligned}
$$

Therefore the particular solution according to the above initial conditions is

$$y(x) = \frac{1}{n^2-1}(\cos mx + (n^2-2)\cos nx)$$

Question 3

$$(2x+1)y'' + (4x-2)y' - 8y = 0 \tag{4}$$

$y_1 = e^{-2x} \implies y_1' = -2e^{-2x} \implies y_1'' = 4e^{-2x}$. Substituting these into (4),

$$
\begin{aligned}
(2x+1)y_1'' + (4x-2)y_1' - 8y_1 &= (2x+1)(4e^{-2x}) + (4x-2)(-2e^{-2x}) - 8e^{-2x} \\
&= 8xe^{-2x} + 4e^{-2x} - 8xe^{-2x} + 4e^{-2x} - 8e^{-2x} \\
&= 0
\end{aligned}
$$

so y_1 is indeed a solution of (4). Dividing by the coefficient of y'', (4) becomes

$$y'' + \frac{4x-2}{2x+1}y' - \frac{8}{2x+1}y = 0 \tag{5}$$

as long as $x \neq -\frac{1}{2}$. Given y_1, Theorem 5.1 guarantees a linearly independent second solution y_2 given by

$$y_2(x) = y_1(x)\int \frac{W(x)}{y_1(x)^2}dx \tag{6}$$

where $W(x)$ is the Wronskian, calculated as follows:

$$
\begin{aligned}
W(x) &= e^{-\int p(x)dx} \\
&= e^{-\int \frac{4x-2}{2x+1}dx}
\end{aligned}
$$

Let $u = 2x+1$. Then $x = \frac{u-1}{2}$ and $\frac{du}{dx} = 2$. Then,

$$
\begin{aligned}
W(x) &= e^{-\int \frac{u-2}{u}du} \\
&= e^{2\int \frac{1}{u}du}e^{-\int 1du} \\
&= e^{2\ln u}e^{-u} \\
&= u^2e^{-u} \\
&= (2x+1)^2e^{-2x-1}
\end{aligned}
$$

Using (6), a second solution to (4) is

$$
\begin{aligned}
y_2(x) &= y_1(x)\int \frac{W(x)}{y_1(x)^2}dx \\
&= e^{-2x}\int \frac{(2x+1)^2e^{-2x-1}}{e^{-4x}}dx \\
&= e^{-2x}\int (2x+1)^2e^{2x-1}dx \\
&= e^{-2x-2}\int (2x+1)^2e^{2x+1}dx
\end{aligned}
$$

Let $u = 2x + 1$, then the integral becomes

$$y_2(x) = \frac{e^{-2x-2}}{2} \int u^2 e^u du$$

Set $v = u^2, w' = e^u$ and integrate by parts to get $\int u^2 e^u du = u^2 e^u - 2 \int ue^u du$. Integrating by parts again, setting $v = u, w' = e^u$, this becomes $\int u^2 e^u = u^2 e^u - 2ue^u + 2e^u$. Therefore,

$$\begin{aligned} y_2(x) &= e^{-2x-2}(u^2 e^u - 2ue^u + 2e^u) \\ &= e^{-2x-2}(e^{2x+1})((2x+1)^2 - 2(2x+1) + 2) \\ &= \frac{4x^2+1}{2e} \end{aligned}$$

Question 4

$$(4x^2 - x)y'' + 2(2x-1)y' - 4y = 0 \tag{7}$$

$$(4x^2 - x)y'' + 2(2x-1)y' - 4y = 12x^2 - 6x \tag{8}$$

$y_1 = \frac{1}{x} \implies y_1' = -\frac{1}{x^2} \implies y_1'' = \frac{2}{x^3}$. Therefore, substituting y_1 into (7),

$$\begin{aligned} (4x^2 - x)y_1'' + 2(2x-1)y_1' - 4y_1 &= (4x^2 - x)\frac{2}{x^3} + 2(2x-1)\left(-\frac{1}{x^2}\right) - \frac{4}{x} \\ &= \frac{8}{x} - \frac{2}{x^2} - \frac{4}{x} + \frac{2}{x^2} - \frac{4}{x} \\ &= 0 \end{aligned}$$

so y_1 indeed solves (7). Rearrange (7) and (8)to get

$$y'' + \frac{2(2x-1)}{x(4x-1)}y' - \frac{4}{x(4x-1)} = 0 \tag{9}$$

$$y'' + \frac{2(2x-1)}{x(4x-1)}y' - \frac{4}{x(4x-1)} = \frac{12x-6}{4x-1} \tag{10}$$

where $x \neq 0, \frac{1}{4}$. Similiarly to Question 3, Theorem 5.1 guarantees a second solution by (6) using the Wronskian

$$\begin{aligned} W &= e^{-2\int \frac{2x-1}{x(4x-1)} dx} \\ &= e^{-2\left(\int \frac{1}{x} - \frac{2}{4x-1} dx\right)} \\ &= e^{-2\int \frac{1}{x} dx + 5\int \frac{1}{4x-1} dx} \\ &= e^{-2\ln x} e^{\ln(4x-1)} \\ &= \frac{4x-1}{x^2} \end{aligned}$$

which exists and is non-zero when $x \neq 0, \frac{1}{4}$. This second solution y_2 is then

$$
\begin{aligned}
y_2 &= y_1 \int \frac{W(x)}{y_1(x)^2} dx \\
&= \frac{1}{x} \int \frac{4x-1}{x^2} x^2 dx \\
&= \frac{1}{x} \int 4x - 1 dx \\
&= \frac{1}{x}(2x^2 - x) \\
&= 2x - 1
\end{aligned}
$$

Now, given linearly independent y_1 and y_2, Theorem 6.1 says that the unique solution y to (10) that satisfies initial conditions $y(x_0) = y'(x_0) = 0$ is given by

$$
y(x) = y_2(x) \int_{x_0}^{x} \frac{y_1(t)}{W(t)} g(t) dt - y_1(x) \int_{x_0}^{x} \frac{y_2(t)}{W(t)} g(t) dt \tag{11}
$$

x_0 is an arbitrary point in the domain. In this case, choose $x_0 = 1$. Then (11) evalutates as

$$
\begin{aligned}
y(x) &= (2x-1) \int_1^x \frac{1}{t} \frac{t^2}{4t-1} 6(2t-1) 4t - 1 dt - \frac{1}{x} \int_1^x (2t-1) \frac{t^2}{4t-1} \frac{6(2t-1)}{4t-1} dt \\
&= 6(2x-1) \int_1^x \frac{t(2t-1)}{(4t-1)^2} dt - \frac{6}{x} \int_1^x \frac{t^2(2t-1)^2}{(4t-1)^2} dt \\
&= 6(2x-1) \int_1^x \frac{1}{8} - \frac{1}{8(4t-1)^2} dt - \frac{6}{x} \int_1^x \frac{1}{4} t^2 - \frac{1}{8} t - \frac{1}{64} + \frac{1}{64(4t-1)^2} dt \\
&= \frac{3(2x-1)}{4} \left[t + \frac{1}{4(4t-1} \right]_1^x - \frac{6}{x} \left[\frac{1}{12} t^3 - \frac{1}{16} t^2 - \frac{1}{64} t - \frac{1}{256(4t-1)} \right]_1^x \\
&= \frac{6x-3}{4} \left(x + \frac{1}{4(4x-1)} - \frac{1}{12} \right) - \frac{6}{x} \left(\frac{1}{12} x^3 - \frac{1}{16} x^2 - \frac{1}{64} x - \frac{1}{256(4x-1)} + \frac{1}{768} \right) \\
&= \frac{3}{2} x^2 + \frac{3x}{8(4x-1)} - \frac{3}{24} x - \frac{3}{4} x - \frac{3}{16(4x-1)} + \frac{3}{48} - \frac{1}{2} x^2 + \frac{3}{8} x + \frac{3}{32} + \frac{3}{128x(4x-1)} - \frac{1}{128x} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{6x-3}{16(4x-1)} + \frac{3}{128x(4x-1)} - \frac{1}{128x} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{8x(6x-3) + 3 - (4x-1)}{128x(4x-1)} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{48x^2 - 24x + 3 - 4x + 1}{128x(4x-1)} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{4(12x^2 - 7x + 1)}{128x(4x-1)} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{(3x-1)(4x-1)}{32x(4x-1)} \\
&= x^2 - \frac{1}{2} x + \frac{1}{8} + \frac{3x-1}{32x} \\
&= x^2 - \frac{1}{2} x + \frac{7}{32} - \frac{1}{32x}
\end{aligned}
$$

Therefore, the general solution to (8) is

$$
y(x) = \frac{A}{x} + B(2x-1) + x^2 - \frac{1}{2} x + \frac{7}{32} - \frac{1}{32x}
$$

where A, B are constants.

Question 5

$$y'' = \frac{y'}{x} + \frac{x^2}{y'} \tag{12}$$

Multiply both sides by y' to get

$$y'y'' = \frac{(y')^2}{x} + x^2 \tag{13}$$

Let $\phi = (y')^2$. Then $\phi' = 2y'y''$. Therefore, (13) is transformed into the first order DE

$$\phi' - \frac{2}{x}\phi = 2x^2 \tag{14}$$

Let $I(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$. Multiply (14) by I to get

$$\frac{1}{x^2}\phi' - \frac{2}{x^3}\phi = 2 \tag{15}$$

By the chain rule, the left hand side is equal to $\frac{d}{dx}\left(\frac{1}{x^2}\phi\right)$. Integrate both sides with respect to x to get

$$\begin{aligned}
\frac{1}{x^2}\phi &= 2x + C \\
\phi &= 2x^3 + Cx^2 \\
(y')^2 &= 2x^3 + Cx^2 \\
y' &= \sqrt{2x^3 + Cx^2} \qquad (16)
\end{aligned}$$

Using the intial condition $y'(2) = 4$ yields $C = 0$. Substituting this value and integrating both sides with respect to x again,

$$\begin{aligned}
y' &= \sqrt{2x^3} \\
&= \sqrt{2}x^{\frac{3}{2}} \\
y(x) &= \frac{2\sqrt{2}}{5}x^{\frac{5}{2}} + D \qquad (17)
\end{aligned}$$

Using the initial condition $y'(2) = 0$ yields $D = -\frac{16}{5}$. Therefore, the solution to (12) satisfying the given initial conditions is

$$y(x) = \frac{2\sqrt{2}}{5}x^{\frac{5}{2}} - \frac{16}{5}$$