PMTH399 Assignment 5

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Question 1

$$\alpha^2\left(\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}\right)=\frac{\delta u}{\delta t} \tag{1}$$

Consider solutions of the form $u(x, y, t) = F(x)G(y)H(T)$. Substituting into (1) gives

$$\alpha^2(F''GH + FG''H) = FGH'$$

Divide by $u = FGH$,

$$\alpha^2\left(\frac{F''}{F}+\frac{G''}{G}\right)=\frac{H'}{H} \tag{2}$$

As the left hand side of (2) is a function of x and y, and the right hand side is a function of t, both sides must be equal to a constant k for the equality to hold. Therefore, the following must hold

$$H' - kH = 0 \tag{3}$$

$$\frac{F''}{F}-\frac{k}{\alpha^2}=-\frac{G''}{G} \tag{4}$$

Similiarly, the left hand side of (4) is dependent on x, while the right hand side is dependent on y. Hence, both must be equal to a consant λ, so

$$G'' + \lambda G = 0 \tag{5}$$

$$F''-\left(\lambda+\frac{k}{\alpha^2}\right)=0 \tag{6}$$

Thus (3), (5) and (6) are ordinary differential equations that must be satisified by F, G and H if $u(x, y, t) = F(x)G(y)H(t)$ is a solution to (1).

Question 2

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\lambda u=0 \tag{7}$$

Consider solutions of the form $u(x, y) = F(x)G(y)$ such that $u(x, y) = 0$ on the boundaries of the unit square, and $u(x, y)$ is not uniformly zero inside the unit square. That is, $u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0$ for $0 \le x \le 1$ and $0 \le y \le 1$. Substituting the desired form of u into (7) gives

$$F''G + FG' + \lambda FG = 0$$

Divide by $u = FG$,

$$\frac{F''}{F} + \frac{G''}{G} + \lambda = 0$$
$$\implies \frac{F''}{F} + \lambda = -\frac{G''}{G} \tag{8}$$

As the left hand side of (8) is a function of x, and the right hand side is a function of y, both must be equal to a constant k. Hence,

$$G'' + kG = 0 \tag{9}$$
$$F'' + (\lambda - k)F = 0 \tag{10}$$

If $k < 0$, then $G(y) = C_1 e^{\sqrt{k}y} + C_2 e^{-\sqrt{k}y}$. However this does not statisfy the boundary conditions, as $G(0) = 0 = C_1 + C_2 \implies C_2 = -C_1$ and $G(1) = 0 = C_1(e^{\sqrt{k}} - e^{-\sqrt{k}})$, which does not hold for $k > 0$.

If $k = 0$, then $G(y) = C_1 y + C_2$. Respecting the boundary conditions, $G(0) = 0 = C_2$ and $G(1) = 0 = C_1$, so $G(y) = 0$. However, we are looking for solutions u that are not uniformly zero, so we disregard this case.

If $k > 0$, then $G(y) = C_1 \cos \sqrt{k}y + C_2 \sin \sqrt{k}y$. The boundary conditions require $G(0) = 0 = C_1$ and $G(1) = 0 = C_2 \sin \sqrt{k}$. For solutions $u \neq 0$, $C_2 \neq 0$, thus it must hold that $\sqrt{k} = m\pi \implies k = m^2\pi^2$ for $m \in \mathbb{Z}$.

Equation (10) then becomes

$$F'' + (\lambda - m^2\pi^2)F = 0 \tag{11}$$

Using the same reasoning for G, it must hold that $\lambda - m^2\pi^2 > 0$. In this case, F must be of the form $F(x) = A_1 \cos(\sqrt{\lambda - m^2\pi^2}x) + A_2 \sin(\sqrt{\lambda - m^2\pi^2}x)$. The boundary conditions require $F(0) = 0 = A_1$ and $F(1) = 0 = A_2 \sin(\sqrt{\lambda - m^2\pi^2})$. Therefore $\sqrt{\lambda - m^2\pi^2} = n\pi \implies \lambda = (n^2 + m^2)\pi^2$ for $n \in \mathbb{Z}$. Further, as $\lambda - m^2\pi^2 > 0$ we require that $n > 0$.

Therefore, when λ is of the form $\lambda = (n^2 + m^2)\pi^2$ where $n, m \in \mathbb{Z}$ such that $n > m > 0$, (7) has solutions of the form

$$u(x, y) = A \sin(n\pi x) \sin(m\pi y) \tag{12}$$

where A is a constant.

Question 3

$$\frac{\delta^2 u}{\delta x^2} + \lambda u = 0 \tag{13}$$

If $\lambda < 0$ then $u(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$. Respecting the boundary conditions, $u(0) = 0 = C_1 + C_2 \implies C_2 = -C_1$ and $u(1) - u'(1) = 0 = C_1(e^{\sqrt{\lambda}}(1 - \sqrt{\lambda}) - e^{-\sqrt{\lambda}}(1 + \sqrt{\lambda}))$. However this implies $C_1 = 0$ and as we are only interested in non-trivial solutions to (13), so we ignore this case.

If $\lambda = 0$ then $u(x) = C_1 x + C_2$. The boundary conditions are $u(0) = 0 = C_2$ and $u(1) - u'(1) = 0 = C_1 - C_1$. Thus for $\lambda = 0$, $u(x) = c$ for constant c is a solution.

If $\lambda > 0$, then $u(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x$. The boundary conditions require $u(0) = 0 = C_1$ and $u(1) - u'(1) = 0 = C_2 \sin \sqrt{\lambda} - C_2\sqrt{\lambda} \cos \sqrt{\lambda} \implies \sqrt{\lambda} = \tan \sqrt{\lambda}$.

Note that for $\lambda \geq 0$, $\sqrt{\lambda} = \tan \sqrt{\lambda}$ has infinite solutions. Therefore there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$

such that for $\lambda = \lambda_n$, (13) has a non-trivial solution. Note that $\lambda = 0$ solves $\sqrt{\lambda} = \tan\sqrt{\lambda}$, so this case is included in the sequence. The non-trivial solutions are

$$u(x) = \begin{cases} A & , \lambda_n = 0 \\ B \sin\sqrt{\lambda} x & , \text{otherwise} \end{cases}$$

for constants A and B.