Assignment 8.

1. Write the Chebyshev equation
$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$
in the form
$$(p(x)y')' + q(x)y = 0.$$
Prove that Chebyshev polynomials T_m and T_n have the orthogonality property
$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_m(x) T_n(x) dx = 0 \qquad \text{when } m \neq n.$$

2. Find the eigenvalues and corresponding eigenfunctions for the homogeneous two–point boundary value problem
$$\begin{aligned} u'' + \lambda u &= 0 \\ u'(0) = u'(1) &= 0. \end{aligned}$$

3. Apply the method of Lecture 19 to the non–homogeneous problem
$$\begin{aligned} u'' + ku &= F(x), \qquad k \text{ constant} \\ u'(0) = u'(1) &= 0. \end{aligned}$$
Find a series representation for u when $F(x) = x$.

   Also solve this problem by the more direct variation of parameters method.

4. * For the equation (4) in the proof of Theorem 17.1 prove that $\theta(1, \lambda)$ is an increasing function of λ. [Hint: It is just as easy to prove that $\theta(x, \lambda)$ is an increasing function of λ for *every* fixed x in $0 < x \leq 1$.]