Assignment 5

1. The heat equation in two space dimensions is

$$\alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}.$$

   Assuming separation solution of the form $u(x,\, y,\, t) = F(x)\, G(y)\, H(t)$, find ordinary differential equations satisfies by F, G, and H.

2. The partial differential equation (*Helmholtz*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0,$$

   together with the boundary condition $u(x,\, y) = 0$ on the boundary of the unit square $0 \leq x \leq 1, \quad 0 \leq y \leq 1$, always has the trivial solution $u(x, y) \equiv 0$, for all (x, y) inside the square.

   However there are some values of λ for which the boundary value problem has a non-trivial solution. Use separation of variables to find some of these values of λ and corresponding solutions.

3. The one-dimensional boundary value problem

$$\frac{d^2 u}{dx^2} + \lambda u = 0$$

   with

$$u(0) = 0 \qquad , \qquad u(1) - u'(1) = 0$$

   also has the solution $u \equiv 0$. Show that there exists an infinite sequence $\{\lambda_n\}$ of values of λ such that the problem has a non-trivial solution.