PMTH332 Assignment 3

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Question 1

Consider $g \in \ker(\text{Inn})$. That is, $g \in G$ such that $\phi_g = \text{id}_G$. Then

$$\begin{aligned} \phi_g(x) = gxg^{-1} = x, \forall x \in G \\ \iff gx = xg, \forall x \in G \\ \iff g = xgx^{-1}, \forall x \in G \\ \iff g \in C(G) \end{aligned}$$

Therefore, $\ker(\text{Inn}) \subseteq C(G)$. Now let $x \in C(G)$. Then x commutes with all elements of G, i.e.

$$\begin{aligned} gxg^{-1} = x, \forall x \in G \\ \iff gx = xg, \forall x \in G \\ \iff xgx^{-1} = g, \forall x \in G \end{aligned} \tag{1}$$

By definition, $\text{Inn}(x) = \phi_x : G \to G$ is defined as $\phi_x(y) = xyx^{-1}$. By (1), $\phi_x(y) = xyx^{-1} = y, \forall y \in G$, so $\phi_x = \text{id}_G$ i.e. $\phi_x \in \ker(\text{Inn})$. Hence $C(G) \subseteq \ker(\text{Inn}) \implies \ker(\text{Inn}) = C(G)$.

Question 2

Let H be a subgroup of G of index two. That is, H has two cosets in G. Take $g \in G$. As the cosets of H are the equivalence classes of the equivalence relation $\sim_H$, these cosets partition G into two subsets. Therefore, for the left cosets of H there are two possibilities:

$$\begin{aligned} g \in H \implies gH = H \\ g \notin H \implies gH = G \backslash H \end{aligned}$$

Likewise, for the right cosets of H,

$$\begin{aligned} g \in H \implies Hg = H \\ g \notin H \implies Hg = G \backslash H \end{aligned}$$

Thus $gH = Hg, \forall g \in G$, which by Lemma 6.8, implies H is normal in G.

Question 3

Let $f_g : G \to G$, $x \mapsto gx$ be defined for all $g \in G$. Then f_g^{-1} exists and is given by $f_{g^{-1}} = g^{-1}x$. Thus, each f_g is a bijection. Consider the set $H := \{f_g | g \in G\}$ with the binary operation of composition of functions. Then, as each element of H has an inverse such that $f_g \circ f_{g^{-1}} = \text{id}_G$, where $\text{id}_G = e_H$ is the neutral element of H, H is a group. Specifically, it is a group of bijections on $|G| = n$ elements i.e. $H \subseteq S_n$.

Define $\phi : G \to K, g \mapsto f_g$. To show that this is a homomorphism, observe that

$$\begin{aligned}
\phi(xy)(g) &= f_{xy}(g) \\
&= xyg \\
&= x(yg) \\
&= f_x(f_y(g)) \\
&= (f_x \circ f_y)(g)
\end{aligned}$$

By definition, ϕ is surjective. Let $g \in G$ such that $\phi(g) = \mathrm{id}_G$. That is, $\forall x \in G, f_g(x) = gx = x \implies g = e$ by cancellation. Therefore the kernal of ϕ is trivial and so ϕ is injective. Thus, ϕ is a bijective homomorphism i.e. an isomorphism, and

$$G \cong H \leq S_n$$

Question 4

i) As H and N are subgroups of G, $e \in H, N \implies e \in HN$, so HN is nonempty. Take $x, y \in HN$ such that $x = h_1 n_1$ and $y = h_2 n_2$. Then

$$\begin{aligned}
xy^{-1} &= h_1 n_1 (h_2 n_2)^{-1} \\
&= h_1 n_1 n_2^{-1} h_2^{-1} \\
&= h_1 h_2^{-1} (h_2 n_1 n_2^{-1} h_2^{-1})
\end{aligned}$$

As H is a group, $h_1 h_2^{-1} \in H$. As N is a normal subgroup, $h_2 n_1 n_2^{-1} h_2^{-1} \in N$. Therefore, $xy^{-1} \in HN$ given $x, y \in HN$, hence $HN \leq G$.

ii) As H and N are subgroups of G, given $h \in H$ and $n \in N$, we have $h = he \in HN$ and $n = en \in HN$. Therefore HN contains both H and N.

Let K be another subgroup of G containing H and N. Then K is closed as a group, so K contains all elements that are products of other elements of K. Therefore K contains all elements of the form $hn, h \in H, n \in N$ i.e. K contains HN. Therefore as HN is contained in any other subgroup of G containing H and N, HN is the smallest group to do so.

iii) Suppose H is normal in G. Then it holds that $gH = Hg$ and $gN = Ng, \forall g \in G$. Therefore,

$$\begin{aligned}
gHN &= \{ghn | h \in H, n \in N\} \\
&= \{hgn | h \in H, n \in N\} \\
&= \{hng | h \in H, n \in N\} \\
&= HNg
\end{aligned}$$

thus HN is normal in G.