# PMTH339 Assignment 8

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## Question 1

$$
\begin{aligned}
(1-x^2)y'' &- xy' + \alpha^2 y = 0 \\
y'' &- \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0 \qquad (1)
\end{aligned}
$$

Define $I(x)$ by

$$
I(x) = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2} \qquad (2)
$$

Multiply (1) by (2) to get

$$
\begin{aligned}
\sqrt{1-x^2}y'' &- \frac{x}{\sqrt{1-x^2}}y' + \frac{\alpha^2}{\sqrt{1-x^2}} = 0 \\
(-\sqrt{1-x^2}y')' &+ \frac{\alpha^2}{\sqrt{1-x^2}}y = 0 \qquad (3)
\end{aligned}
$$

This is in the form $L[y] = \lambda r y$, where $r(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$. The Chebyshev polynomials $T_n$ solve (3) and satisfy the initial conditions

$$
a_1 T_n(-1) + a_2 T_n'(-1) = 0 \qquad b_1 T_n(1) + b_2 T_n'(1) = 0 \qquad (4)
$$

To see this, note that $T_n' = nU_{n-1}$. Therefore, given that $T_n(-1) = (-1)^n$, $U_n(-1) = (n+1)(-1)^n$, $T_n(1) = 1$ and $U_n(1) = n+1$, we can set $a_1 = 1, a_2 = \frac{1}{n^2}$ and $b_1 = 1, b_2 = \frac{1}{n^2}$ to get
Therefore, the Chebyshev polynomials are eigenfunctions of the system (3), (4), and by Theorem 9.1, are $r$-orthogonal, where $r(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$. That is,

$$
\int_{-1}^{1} (1-x^2)^{-1/2} T_m(x) T_n(x) dx = 0
$$

for $m \neq n$.

## Question 2

$$
\begin{aligned}
u'' + \lambda u &= 0 \qquad (5) \\
u'(0) = u'(1) &= 0 \qquad (6)
\end{aligned}
$$

If $\lambda > 0$, then $u$ is of the form

$$
u = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x
$$

Therefore $u' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$. The first condition gives $B = 0$. Therefore $u' = -A\sqrt{\lambda}\sin\sqrt{\lambda}x$. The second condition requires $-A\sqrt{\lambda}\sin\sqrt{\lambda} = 0$, which only holds for $\lambda = n^2\pi^2$, where $n$ is an integer.

If $\lambda = 0$, then $u = A$. For $\lambda < 0$, $u$ is of the form

$$u = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Taking the first derivative and applying the first condition, we get $A = B$. Making this substitution and applying the second condition, we get that

$$0 = \sqrt{-\lambda}A(e^{\sqrt{-\lambda}} - e^{-\sqrt{\lambda}})$$

Which only has solution when $\lambda = 0$, hence there are no solutions for $\lambda < 0$.

Therefore, the eigenvalues of the system (5), (6) are $\lambda_n = n^2\pi^2$ for integer $n \geq 0$, which correspond to eigenfunctions $\phi_n = \cos\sqrt{\lambda_n}x$. Note that this includes the case for $\lambda = 0$, which corresponds to eigenfunction $\phi_0 = \cos\sqrt{0}x = 1$.

# Question 3

# Question 4