PMTH332 Assignment 2

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Question 1

Let $\phi := \ln : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \ln x$. For $x, y \in H$, $\ln(xy) = \ln x + \ln y$, so ϕ is a homomorphism. Now let $\psi := \exp : \mathbb{R} \to \mathbb{R}^+, x \mapsto e^x$. Then $e^{x+y} = e^x e^y$, so ψ is a homomorphism. Further, $\psi = \phi^{-1}$, so ϕ is an isomorphism. Therefore, $G \cong H$.

Question 2

Let $\phi : G \to H$ and $\psi : H \to K$ be isomorphisms. Then there exists inverses $\phi^{-1} : H \to G$ and $\psi^{-1} : K \to H$ of ϕ and ψ. By Proposition 3.6, $\psi \circ \phi$ is a homomorphism. Observe that

$$\begin{aligned}
(\phi^{-1} \circ \psi^{-1}) \circ (\psi \circ \phi) &= \phi^{-1} \circ (\psi^{-1} \circ (\psi \circ \phi)) \\
&= \phi^{-1} \circ ((\psi^{-1} \circ \psi) \circ \phi) \\
&= \phi^{-1} \circ (id_H \circ \phi) \\
&= \phi^{-1} \circ \phi \\
&= id_G
\end{aligned}$$

$$\begin{aligned}
(\psi \circ \phi) \circ (\phi^{-1} \circ \psi^{-1}) &= (\psi \circ (\phi \circ (\phi^{-1} \circ \psi^{-1})) \\
&= \psi \circ ((\phi \circ \phi^{-1}) \circ \psi^{-1}) \\
&= \psi \circ (id_G \circ \psi^{-1} \\
&= \psi \circ \psi^{-1} \\
&= id_K
\end{aligned}$$

Therefore $\psi \circ \phi : G \to K$ has inverse $\phi^{-1} \circ \psi^{-1} : K \to G$ and is an isomorphism.

Question 3

Let $H \subseteq G$ and define $x \sim_H y :\leftrightarrow xy^{-1} \in H$.

Suppose $H \leq G$. Then if $x \in H$, $e = xx^{-1} \in H$. Therefore, $x \sim_H x$.

As H is a subgroup, for $x \in y$, $xy^{-1} \in H$. By the existence of inverses in groups, we thus have $(xy^{-1})^{-1} = yx^{-1} \in H$. Therefore, if $x \sim_H y$ then $y \sim_H x$.

If $x, y, z \in H$ we have $xy^{-1} \in H$ and $yz^{-1} \in H$. By closure of multiplication in groups, we then have $(xy^{-1})(yz^{-1}) = xz^{-1} \in H$. Therefore $\sim_H$ is transitive and thus satisfies all criteria of an equivalence relation

Conversely, suppose that $H \subseteq G$ and $x \sim_H y :\leftrightarrow xy^{-1} \in H$ defines an equivalence relation. Consider for $x \in H$ the equivalence class $[x] = \{y \in H | xy^{-1} \in H\}$. By reflexivity of $\sim_H$, we have $x \in [x] \implies xx^{-1} = e \in H$. Let $h \in H$ and suppose that $h \notin [x]$. Then $xh^{-1} \notin H$. However, as equivalence relations partition the set they act upon, there is some $y \in H$ such that $h \in [y] \implies yh^{-1} \in H$. Therefore, given $x, y \in H$ we must have that $xy^{-1} \in H$ and thus H is a subgroup of G.

Question 4

a) Let $\phi, \psi \in \mathrm{Aut}(G)$. Then both $\phi^{-1} : G \to G$ and $\psi^{-1} : G \to G$ exist and are in $\mathrm{Aut}(G)$. Further, $\upsilon := \phi \circ \psi^{-1} : G \to G$ is an isomorphism by the result of Question 2, and is therefore in $\mathrm{Aut}(G)$. Therefore, by Proposition 4.5, $\mathrm{Aut}(G) \leq S(G)$.

b) Let $g \in G$ and define $\phi_g : G \to G, x \mapsto gxg^{-1}$. As G is a group, it is closed under multiplication, so the function is well defined. Consider $x, y \in G$. Then,

$$\phi_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \phi_g(y)\phi_g(y)$$

so ϕ_g is a homomorphism. Now let $x, x' \in G$ such that $\phi_g(x) = \phi_g(x')$. Then,

$$\begin{aligned} gxg^{-1} &= gx'g^{-1} \\ gxg^{-1}g &= gx'g^{-1}g \\ g^{-1}xe &= g^{-1}gx'e \\ ex &= ex' \\ \implies x &= x' \end{aligned}$$

So ϕ_g is injective. Now let $y \in G$. Define $x = g^{-1}yg$. Then

$$\phi_g(x) = \phi_g(g^{-1}yg) = gg^{-1}yg^{-1}g = eye = y$$

So ϕ_g is surjective and thus bijective. Therefore, ϕ_g is an isomorphism from G to G and is thus an automorphism.

c) Let $x, y \in G$. Then $\mathrm{Inn}(xy) = \phi_{xy}$. Let $g \in G$. Then,

$$\phi_{xy}(g) = xyg(xy)^{-1} = xygy^{-1}x^{-1} = x(ygy^{-1})x^{-1} = \phi_x(\phi_y(g)) = (\phi_x \circ \phi_y)(g)$$

Therefore, $\mathrm{Inn}(xy) = \phi_{xy} = \phi_x \circ \phi_y$, so Inn is a homomorphism.