PMTH339 Assignment 1

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Question 1

$$y' + \frac{3}{x}y = \frac{2}{x^2} \tag{1}$$

Set $I(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$, and multiply both sides of (1) to get

$$x^3y' + 3x^2y = 2x \tag{2}$$

By the chain rule, the LHS of (2) is equal to $\frac{d}{dx}(x^3y)$. Integrating both sides with respect to x, we solve for the general solution $y(x)$ as follows:

$$\begin{aligned} \int \frac{d}{dx}(x^3y)dx &= \int 2xdx + c \\ x^3y &= x^2 + c \\ \implies y(x) &= \frac{1}{x} + \frac{c}{x^2} \end{aligned} \tag{3}$$

for $c \in \mathbb{R}$ and $x \neq 0$. Given the condition $y(2) = -1$, we can solve for the constant c.

$$\begin{aligned} -1 &= \frac{1}{2} + \frac{c}{4} \\ -\frac{3}{2} &= \frac{c}{4} \\ \implies c &= -6 \end{aligned}$$

Therefore, the particular solution to the differential equation (1) with the initial condition $y(2) = -1$ is

$$y(x) = \frac{1}{x} - \frac{6}{x^2}$$

Question 2

$$y' = \frac{1+y^2}{1+x^2} \tag{4}$$

Equation (4) is of the form $y' = p(x)q(y)$ and $q(y) = 1+y^2 \neq 0$, so is therefore a seperable first order differential equation. As such, it can be solved by evaluating

$$\int \frac{1}{1+y^2}dy = \int \frac{1}{1+x^2}dx + c \tag{5}$$

for $c \in \mathbb{R}$, and solving for y. Both integrands are recognised as the derivative of arctan. Therefore the general solution $y(x)$ can be found:

$$
\begin{aligned}
(5) \implies \arctan(y) &= \arctan(x) + c \\
\implies y(x) &= \tan(\arctan(x) + c) \qquad (6)
\end{aligned}
$$

Using the intial condition $y(2) = 1$ gives

$$
\begin{aligned}
1 &= \tan(\arctan(2) + c) \\
\arctan(1) &= \arctan(2) + c \\
\implies c &= \frac{\pi}{4} - \arctan(2) \qquad (7)
\end{aligned}
$$

Therefore, the particular solution of the differential equation (4) corresponding to initial condition $y(2) = 1$ is

$$y(x) = \tan(\arctan(x) + \frac{\pi}{4} - \arctan(2)) \qquad (8)$$

To find where this solution is valid, we note that $\tan x$ is defined for $|x| < \frac{\pi}{2}$. We therefore find bounds on x as

$$
\begin{aligned}
-\frac{\pi}{2} &< \arctan x + \frac{\pi}{4} - \arctan(2) < \frac{\pi}{2} \\
\arctan(2) - \frac{3\pi}{4} &< \arctan x < \arctan(2) + \frac{\pi}{4} \\
\tan\left(\arctan(2) - \frac{3\pi}{4}\right) &< x < \tan\left(\arctan(2) + \frac{\pi}{4}\right)
\end{aligned}
$$

Question 3

$$y' = \frac{y + x}{y - x} \qquad (9)$$

Let $y = ux$, where $u \in \mathbb{R}$. This is a solution of (9) if and only if

$$
\begin{aligned}
u &= \frac{ux + x}{ux - x} \\
&= \frac{u + 1}{u - 1} \\
\implies u^2 - 2u - 1 &= 0 \qquad (10)
\end{aligned}
$$

Equation (10) has roots $u = 1 \pm \sqrt{2}$, so therefore $y = (1 \pm \sqrt{2})x$ are solutions to the differential equation.

Question 4

$$y'' + 4y' + 5y = 0 \qquad (11)$$

Equation (11) is a second order homogenous differential equation. Therefore $y = e^{\lambda x}$ is a solution if and only if $\lambda^2 + 4\lambda + 5 = 0$. The solutions to this polynomial are $\lambda = -2 \pm i$. Therefore, the general solution and its first derivative are

$$
\begin{aligned}
y &= 2^{-2x}(C_1 \cos x + C_2 \sin x) \qquad (12) \\
y' &= e^{-2x}((C_2 - 2C_1)\cos x - (C_1 - 2C_2)\sin x) \qquad (13)
\end{aligned}
$$

for $C_1, C_2 \in \mathbb{C}$. Given the initial condition $y(0) = 1, y'(0) = 0$, we get

$$\begin{aligned}
(12) \implies 1 &= C_1 \cdot 1 + C_2 \cdot 0 \\
\therefore C_1 &= 1 \\
(13) \implies 0 &= C_2 - 2C_1 \\
\therefore C_2 &= 2
\end{aligned}$$

Therefore the particular solution according to this initial condition is $y(x) = e^{-2x}(\cos x + 2\sin x)$. Given the initial condition $y(0) = 12, y'(0) = -5$, we get

$$\begin{aligned}
(11) \implies C_1 &= 12 \\
(13) \implies -5 &= C_2 - 2C_1 \\
\therefore C_2 &= 19
\end{aligned}$$

Therefore the particular solution according to this initial condition is $y(x) = e^{-2x}(12\cos x + 19\sin x)$.