PMTH332 Assignment 4

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Question 1

Let M and N be normal subgroups of G such that $M \trianglelefteq N \trianglelefteq G$. Consider $N/M \trianglelefteq G/M$. If M is a maximal normal subgroup of G, then either

$$\begin{aligned} &N = G \\ \implies &N/M = G/M \end{aligned} \qquad \text{or} \qquad \begin{aligned} &N = M \\ \implies &N/M = M/M = \{M\} \end{aligned}$$

Therefore the only two normal subgroups of G/M are G/M and $\{M\}$, which is the identity element. Therefore G/M is simple.

Suppose $M \trianglelefteq G$ and G/M is simple. If $M \trianglelefteq N \trianglelefteq G$, then $N/M \trianglelefteq G/M$. As G/M is simple, either $N/M = G/M \implies N = G$ or $N/M = \{M\} \implies N = M$. Therefore M is a maximal normal subgroup of G.

Question 2

Let $m = \mathrm{ord}(a_1, ..., a_n)$. Then

$$(a_1, ..., a_n)^m = (a_1^m, ..., a_n^m) = (e_1, ..., e_n) \tag{1}$$

where e_i is the identity element of group G_i. Therefore, for each a_i, there exists an integer q_i such that $m = q_i \mathrm{ord}(a_i)$. As m is the smallest positive integer such that (1) holds, m must be the least common multiple of the orders of each a_i. That is,

$$\mathrm{ord}(a_1, ..., a_n) = \mathrm{lcm}(\mathrm{ord}(a_1), ..., \mathrm{ord}(a_n))$$

Question 3

Consider the group

$$(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) = \{(a, b) | a \in \mathbb{Z}/m\mathbb{Z}, b \in \mathbb{Z}/m\mathbb{Z}\} \tag{2}$$

This group has mn elements and is cyclic. By the result of Question 2, the order of (2) is $d = \mathrm{lcm}(m, n)$.

By the classification of cyclic groups, this group is isomorphic to $\mathbb{Z}/mn\mathbb{Z}$ if and only if the order $d = mn$. That is, if and only if $d = \mathrm{lcm}(m, n) = mn = \mathrm{lcm}(m, n)\mathrm{gcd}(m, n)$. Therefore, $(\mathbb{Z}/m\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/mn\mathbb{Z}$ if and only if $\mathrm{gcd}(m, n) = 1$.

Question 4

The prime factorisation of 324 is $324 = 2^2 \cdot 3^4$. By the first Sylow theorem, G has subgroups of orders $2, 2^2 = 4, 3, 3^2 = 9, 3^3 = 27$ and $3^4 = 81$. Let H be a subgroup of G of order 10. Then there is an element $g \in G$ of order 10. By Corollary 6.17, it must hold that $\text{ord}(g)||G|$. However, $10 \nmid 324$, so this is a contradiction, and G has no subgroups of order 10.