PMTH339 Assignment 8

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Question 1

$$(1-x^2)y'' - xy' + \alpha^2 y = 0 \tag{1}$$

Divide through by $(1-x^2)$ to get

$$y'' - \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0 \tag{2}$$

Multiply by $I(x)$, where $I(x)$ is

$$I(x) = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1-x^2} \tag{3}$$

Therefore

$$\begin{aligned} I(x)(2) \implies 0 &= \sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{\alpha^2}{\sqrt{1-x^2}}y \\ &= (\sqrt{1-x^2}y')' + \frac{\alpha^2}{\sqrt{1-x^2}}y \end{aligned} \tag{4}$$

(4) is in the form $(p(x)y')' + q(x)y = 0$, with $p(x) = \sqrt{1-x^2}$ and $q(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$ defined and with continuous derivatives on the interval $(-1, 1)$.

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ hold certain properties that are useful here. Firstly, $T_n(x)$ and $U_n(x)$ solve (1) and thus (4) for $\alpha = n$. Secondly, the derivatives of $T_n(x)$ can be defined in terms of $U_{n-1}(x)$ as $T_n'(x) = nU_{n-1}(x)$. Finally, they hold the following values at $x = -1, 1$:

$$T_n(-1) = (-1)^n \qquad U_n(-1) = (n+1)(-1)^n \qquad T_n(1) = 1 \qquad U_n(1) = n+1 \tag{5}$$

From this, we can see that the Chebyshev polynomials T_n satisfy (4) and the following boundary conditions

$$T_n(-1) - \frac{1}{n^2}T_n'(-1) = 0 \qquad T_n(1) + \frac{1}{n^2}T_n'(1) = 0 \tag{6}$$

Therefore, the Chebyshev polynomials T_n are eigenfunctions of the linear operator $L[y] = (-py')'$ corresponding to non-negative integer n eigenvalues.

Let n and m be distinct non-negative integers. Then, as T_n and T_m are eigenfunctions corresponding to distinct eigenvalues, Theorem 19.1 implies that they are r-orthogonal for any function r. That is, the inner product $\langle rT_n, T_m\rangle = 0$. In particular, if $r(x) = \frac{1}{\sqrt{1-x^2}}$ we get that

$$\langle rT_n, T_m\rangle = \int_{-1}^{1}(1-x^2)^{-1}T_n(x)T_m(x)dx = 0$$

as required.

Question 2

$$u'' + \lambda u = 0 \tag{7}$$
$$u'(0) = u'(1) = 0 \tag{8}$$

If $\lambda < 0$, then (7) has solution $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ for some constants A and B. The boundary conditions (8) dictate that

$$u'(0) = 0 \implies A - B = 0 \implies A = B$$
$$u'(1) = 0 \implies e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}} = 0$$

This only holds for $\lambda = 0$, so there are no solutions for negative λ. If $\lambda = 0$, then (7) has solution $u(x) = Ax + B$. Using the boundary conditions, we get that $A = B = 0$. Therefore the trivial solution $u(x) = 0$ corresponds to eigenvalue $\lambda = 0$.

If $\lambda > 0$, then (7) has solution $u(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$ for some constants A and B. The boundary conditions (8) give

$$u'(0) = 0 \implies A\sqrt{\lambda}\cos 0 - B\sqrt{\lambda}\sin 0 = A\sqrt{\lambda} = 0 \implies A = 0$$
$$u'(1) = 0 \implies -B\sqrt{\lambda}\sin\sqrt{\lambda} = 0 \implies \sqrt{\lambda} = n\pi,\ n \in \mathbb{Z}$$

Therefore the system (7), (8) has positive eigenvalues $\lambda_n = n^2\pi^2$ for positive integer n. The corresponding eigenfunctions are $\phi_n(x) = \sin n\pi x$. Note that if we set $\lambda_0 = 0$, then $phi_0(x) = \sin 0 = 0$ is consistent with the conclusion for $\lambda = 0$.

Question 3

$$u'' + ku = F(x) \tag{9}$$
$$u'(0) = u'(1) = 0 \tag{10}$$

Question 4