PMTH339 Assignment 6

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24 August 2018

Question 1

The Bessel equation of order $\alpha = 0$ is

$$x^2y'' + xy + x^2y = 0 \tag{1}$$

Let $y(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)d\phi$. Then the first and second derivatives are

$$\begin{aligned} y' &= \frac{1}{\pi}\int_0^\pi \frac{d}{dx}(\cos(x\sin\phi)d\phi) = -\frac{1}{\pi}\int_0^\pi \sin\phi\sin(x\sin\phi)d\phi \\ y'' &= -\frac{1}{\pi}\int_0^\pi \frac{d}{dx}(\sin\phi\sin(x\sin\phi))d\phi = -\frac{1}{\pi}\int_0^\pi \sin^2\phi\cos(x\sin\phi)d\phi \end{aligned}$$

Substituting into (1),

$$\begin{aligned} x^2y'' + xy' + x^2y &= \frac{1}{\pi}\int_0^\pi -x^2\sin^2\phi\cos(x\sin\phi) - x\sin\phi\sin(x\sin\phi) + x^2\cos(x\sin\phi)d\phi \\ &= \frac{1}{\pi}\int_0^\pi x^2\cos(x\sin\phi)(1-\sin^2\phi) - x\sin\phi\sin(x\sin\phi)d\phi \\ &= \frac{1}{\pi}\int_0^\pi x^2\cos(x\sin\phi)(1-\sin^2\phi) - x\sin\phi\sin(x\sin\phi)d\phi \\ &= \frac{1}{\pi}\int_0^\pi x^2\cos^2\phi\cos(x\sin\phi) - x\sin\phi\sin(x\sin\phi)d\phi \end{aligned}$$

Let $u = x\cos\phi$. Then $\frac{du}{d\phi} = -x\sin\phi$ and $u(0) = x, u(\pi) = -x$. Continuing the above,

$$\begin{aligned} &= \frac{1}{\pi}\int_x^{-x} (u^2\cos(\sqrt{x^2-u^2}) - x\sin\phi\sin(\sqrt{x^2-u^2}))\frac{du}{-x\sin\phi} \\ &= \frac{1}{\pi}\left(\int_{-x}^x \frac{u^2}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2})du - \int_{-x}^x \sin(\sqrt{x^2-u^2})du\right) \end{aligned}$$

Integrating the right integral by parts, set $v = \sin(\sqrt{x^2-u^2}) \implies v' = -\frac{u}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2})$ and $w' = 1 \implies w = u$. The integral then becomes

$$\begin{aligned} &= \frac{1}{\pi}\left(\int_{-x}^x \frac{u^2}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2})du - \left(u\sin(\sqrt{x^2-u^2}\Big|_{-x}^x + \int_{-x}^x \frac{u^2}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2})du\right)\right) \\ &= \frac{1}{\pi}\left(\int_{-x}^x \frac{u^2}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2}) - \frac{u^2}{\sqrt{x^2-u^2}}\cos(\sqrt{x^2-u^2})du\right) \\ &= 0 \end{aligned}$$

Hence y is a solution to (1). Further, $y(0) = \frac{1}{\pi}\int_0^\pi \cos(0)d\phi = \frac{1}{\pi}\int_0^\pi d\phi = 1$. Therefore, by the uniqueness of solutions to ordinary differential equations, it must hold that $y(x) = J_0(x)$.

Question 2

The Bessel equation of order $\alpha = \frac{1}{2}$ is

$$x^2y'' + xy + \left(x^2 - \frac{1}{4}\right)y = 0 \tag{2}$$

Let $y_1 = \sqrt{x}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$. Then $y_1' = \sum_{n=0}(n+\frac{1}{2})a_n x^{n-\frac{1}{2}}$ and $y_1'' = \sum_{n=0}(n+\frac{1}{2})(n-\frac{1}{2})a_n x^{n-\frac{3}{2}}$. For this to be a solution to (2) we require

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}\left(n^2 - \frac{1}{4} + n + \frac{1}{2} - \frac{1}{4}\right) + \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}} \\
&= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} n(n+1) + \sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{2}} \\
&= 2a_1 x^{3/2} + \sum_{n=2}^{\infty} x^{n+\frac{1}{2}}\left(n(n+1)a_n + a_{n-2}\right)
\end{aligned}$$

For this to hold we require $a_1 = 0$ and $a_n = -\frac{a_{n-2}}{n(n+1)}$. $a_1 = 0$ implies that $a_n = 0$ for all odd n. Choosing $a_0 = 1$, consider the claim that $a_{2n} = \frac{(-1)^n}{(2n+1)!}$. Clearly this holds for $n = 0$. Suppose it holds for $n = k$. Then,

$$\begin{aligned}
a_{2(k+1)} &= a_{2k+2} \\
&= -\frac{a_{2k}}{(2k+2)(2k+3)} \\
&= -\frac{(-1)^k}{(2k+1)!(2k+2)(2k+3)} \\
&= \frac{(-1)^{k+1}}{(2k+3)!} \\
&= \frac{(-1)^{k+1}}{(2(k+1)+1)!}
\end{aligned}$$

Therefore, by the principle of induction, it holds that $a_{2n} = \frac{(-1)^n}{(2n+1)!}$ for all integers $n \geq 0$. Thus, y_1 becomes

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} \\
&= x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}
\end{aligned} \tag{3}$$

Similiarly, set $y_2 = x^{-\frac{1}{2}}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}$ and so $y_2' = \sum_{n=0}^{\infty}(n-\frac{1}{2})a_n x^{n-\frac{3}{2}}$ and $y_2'' = \sum_{n=0}^{\infty}(n-\frac{1}{2})(n-\frac{3}{2})a_n x^{n-\frac{5}{2}}$. For this to be a solution to (2) we require

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{2}}\left(n^2 - 2n + \frac{3}{4} + n - \frac{1}{2} - \frac{1}{4}\right) + \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}} \\
&= \sum_{n=0}^{\infty} a_n x^n (n-1) + \sum_{n=2}^{\infty} a_{n-2} x^{n-\frac{1}{2}} \\
&= \sum_{n=2}^{\infty} x^{n-\frac{1}{2}}\left(a_n n(n-1) + a_{n-2}\right)
\end{aligned}$$

For this to hold we require $a_n = -\frac{a_{n-2}}{n(n-1)}$. Taking $a_0 = 1, a_1 = 0$ we have that $a_n = 0$ for all odd n. Consider the claim that $a_{2n} = \frac{(-1)^n}{(2n)!}$. This holds for $n = 0$. Suppose it holds for $n = k$. Then,

$$\begin{aligned}
a_{2(k+1)} &= a_{2k+2} \\
&= -\frac{a_{2k}}{(2k+2)(2k+1)} \\
&= -\frac{(-1)^k}{(2k+2)(2k+1)(2k)!} \\
&= \frac{(-1)^{k+1}}{(2k+2)!} \\
&= \frac{(-1)^{k+1}}{(2(k+1))!}
\end{aligned}$$

Therefore the formula for a_{2n} holds by induction, and y_2 becomes

$$y_2(x) = x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{4}$$

The power series in (3) and (4) are of $\sin x$ and $\cos x$, respectively. Therefore, the two derived solutions are

$$y_1(x) = \frac{\sin x}{\sqrt{x}} \qquad \text{and} \qquad y_2(x) = \frac{\cos x}{\sqrt{x}}$$

Question 3

By Theorem 12.2, (1) has a solution of the form

$$y_2(x) = J_0(x) \ln x + \sum_{n=0}^{\infty} g_k x^k \tag{5}$$

where the wronskian of J_0 and y_2 is $W = \frac{1}{x}$. Therefore

$$\begin{aligned}
y_2 &= J_0 \int \frac{W}{J_0^2} dx \\
&= J_0 \int \frac{1}{x J_0^2}
\end{aligned}$$

By Theorem 12.1,

$$\begin{aligned}
J_0(x) &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^2 m \\
&= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + ... \\
\implies J_0^2 &= 1 - \frac{x^2}{4} + \frac{3x^4}{32} - \frac{5x^6}{768} + ... \\
\implies (xJ_0^2)^{-1} &= x^{-1} \left(1 - \frac{x^2}{4} + \frac{3x^4}{32} - \frac{5x^6}{768} + ...\right)^{-1}
\end{aligned}$$

Applying a geometric series expansion for $(1-t)^{-1}$, where $t = \frac{x^2}{4} - \frac{3x^4}{32} + \frac{5x^6}{768} - ...$,

$$
\begin{aligned}
(xJ_0^2)^{-1} &= x^{-1}\left(1+\frac{x^2}{4}-\frac{3x^4}{32}+\frac{5x^6}{768}+\frac{x^4}{16}-\frac{3x^6}{64}+...\right)\\
&= \frac{1}{x}+\frac{x}{4}-\frac{x^3}{32}-\frac{31x^5}{768}+...
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_2(x) &= J_0\int \frac{1}{x}+\frac{x}{4}-\frac{x^3}{32}-\frac{31x^5}{768}+...\\
&= J_0\ln x + J_0\left(\frac{x^2}{8}-\frac{x^4}{128}-\frac{31x^6}{4608}+...\right)\\
&= J_0\ln x + \left(1-\frac{x^2}{4}+\frac{x^4}{64}-\frac{x^6}{2304}+...\right)\left(\frac{x^2}{8}-\frac{x^4}{128}-\frac{31x^6}{4608}+...\right)\\
&= J_0\ln x + \frac{x^2}{8}-\frac{5x^4}{128}-\frac{13x^6}{4608}+...
\end{aligned}
$$

which is of the form of (5). Hence $g_k = 0$ for odd k and the first few terms for even k are $g_2 = \frac{1}{8}$, $g_4 = -\frac{5}{128}$ and $g_6 = -\frac{13}{4608}$.

Question 4

Let

$$
y = 1+\sum_{n=1}^{\infty} a_n x^n \tag{6}
$$

be a solution to the hypergeometric equation

$$
x(1-x)y''+(c-(1+a+b)x)y'-aby=0 \tag{7}
$$

for constant a, b and c. The first and second derivatives of (6) are

$$
y' = \sum_{n=1}^{\infty} na_nx^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=1}^{\infty} n(n-1)a_nx^{n-2}
$$

Substituting these into (7),

$$
\begin{aligned}
0 &= \sum_{n=1}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=1}^{\infty} n(n-1)a_nx^n + \sum_{n=1}^{\infty} cna_nx^{n-1} - \sum_{n=1}^{\infty}(1+a+b)na_nx^n\\
&\qquad - ab - \sum_{n=1}^{\infty} aba_nx^n\\
&= \sum_{n=1}^{\infty} x^{n-1}a_n(n(n-1)+cn) - \sum_{n=1}^{\infty} x^na_n(n(n-1)+(1+a+b)n+ab) - ab\\
&= \sum_{n=0}^{\infty} x^na_{n+1}(n(n+1)+c(n+1)) - \sum_{n=1}^{\infty} x^na_n(n(n+a+b)+ab)) - ab\\
&= a_1c - ab + \sum_{n=1}^{\infty} x^n(a_{n+1}(n+1)(n+c) - a_n(n+a)(n+b))
\end{aligned}
$$

For this to hold we require $a_1 = \frac{ab}{c}$ and $a_{n+1} = a_n\frac{(n+a)(n+b)}{(n+1)(n+c)}$

Consider the claim that

$$a_n = \frac{1}{n!} \prod_{m=0}^{n-1} \frac{(m+a)(m+b)}{m+c} \tag{8}$$

for integer $n \geq 1$. $a_1 = \frac{ab}{c}$ satisfies the claim. Suppose the claim holds for $n = k$. Then

$$\begin{aligned}
a_{k+1} &= a_k \frac{(k+a)(k+b)}{(k+1)(k+c)} \\
&= \frac{1}{(k+1)k!} \frac{(k+a)(k+b)}{k+c} \prod_{m=0}^{k-1} \frac{(m+a)(m+b)}{m+c} \\
&= \frac{1}{(k+1)!} \prod_{m=0}^{k} \frac{(m+a)(m+b)}{m+c} \\
&= \frac{1}{(k+1)!} \prod_{m=0}^{(k+1)-1} \frac{(m+a)(m+b)}{m+c}
\end{aligned}$$

and so a_{k+1} satisfies (8) whenever a_k does. Therefore (8) holds for all integers $n \geq 1$. Therefore, (7) has solutions of the form

$$y = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \prod_{m=0}^{n-1} \frac{(m+a)(m+b)}{m+c}$$