PMTH332 Assignment 5

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Question 1

Consider the two non-zero matrices $A, B \in M(2; R)$ defined by

$$A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$$

then $AB = 0$, so $M(2; R)$ has zero divisors. Consider matrix C given by

$$C = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

then $AC = \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix}$ and $CA = 0$, so $M(2; R)$ is not commutative.

Question 2

Let F be a finite integral domain and take non-zero $a \in F$. Define F^* to be the set of non-zero elements of F, and define the map $\phi : F^* \to F^*$ by $\phi(x) = ax$.

Suppose that for $x, y \in F^*$, $\phi(x) = \phi(y)$. Then

$$ax = ay \iff ax - ay = 0 \iff a(x - y) = 0$$

As F is an integral domain, it has no zero-divisors. Therefore the above implies that either $a = 0$ or $x - y = 0$. As a is non-zero by choice, it must hold that $x = y$. Therefore ϕ is injective. Further, as $F*$ is finite, ϕ must be surjective. Therefore, as $1 \in F^*$, there exists $x \in F^*$ so that $\phi(x) = ax = 1$. Hence, every non-zero element of F has multiplicative inverse, so F is a field.

Question 3

Let R be a non-trivial integral domain with 1. The characteristic of R is the integer n such that $n\mathbb{Z} = \ker \epsilon$, where $\epsilon : \mathbb{Z} \to R$ is the unique homomorphism of unital rings, given by $\epsilon(n) = n \cdot 1_R$.

Given that $n\mathbb{Z} = \{x \in \mathbb{Z} | x = nm, m \in \mathbb{Z}\}$ and $\ker \epsilon = \{x \in \mathbb{Z} | x \cdot 1_R = 0_R\}$, we can deduce that if n is the characteristic of R, then $nm \cdot 1_R = (n \cdot 1_R)(m \cdot 1_R) = 0_R$ for all $m \in mathbbZ$. $n = 0$ satisfies this, in which case $\mathbb{Z}$ is a subring of R.

Suppose $n \neq 0$ and n is not prime. Then $n = kp$ for some prime $p < n$ and natural number k. n being the characteristic of R, we must have that for non-zero $a \in R$, $na \cdot 1_R = (k \cdot 1_R)(p \cdot 1_R)a = 0_R$. As R is an integral domain, R has no zero-divisors. Hence it must hold that either $k = 0$ or $p = 0$. This is a contradiction of the assumption that n is not prime, hence n must be so.

Therefore, the characteristic of an integral domain with 1 must be either 0 or prime.

Question 4

Given an integral domain D, consider $\widetilde{D} = \{(b,a)|a,b \in D, a \neq 0\}$, and the equivalence relation $(b,a)\ (d,c) \iff bc = ad$. Let F be the set of equivalence classes of this equivalence relation.