PMTH339 Assignment 8

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Question 1

$$(1-x^2)y'' - xy' + \alpha^2 y = 0 \tag{1}$$

Divide through by $(1-x^2)$ to get

$$y'' - \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0 \tag{2}$$

Multiply by $I(x)$, where $I(x)$ is

$$I(x) = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1-x^2} \tag{3}$$

Therefore

$$\begin{aligned} I(x)(2) \implies 0 &= \sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{\alpha^2}{\sqrt{1-x^2}}y \\ &= (\sqrt{1-x^2}y')' + \frac{\alpha^2}{\sqrt{1-x^2}}y \end{aligned} \tag{4}$$

(4) is in the form $(p(x)y')' + q(x)y = 0$, with $p(x) = \sqrt{1-x^2}$ and $q(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$ defined and with continuous derivatives on the interval $(-1, 1)$.

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ hold certain properties that are useful here. Firstly, $T_n(x)$ and $U_n(x)$ solve (1) and thus (4) for $\alpha = n$. Secondly, the derivatives of $T_n(x)$ can be defined in terms of $U_{n-1}(x)$ as $T_n'(x) = nU_{n-1}(x)$. Finally, they hold the following values at $x = -1, 1$:

$$T_n(-1) = (-1)^n \qquad U_n(-1) = (n+1)(-1)^n \qquad T_n(1) = 1 \qquad U_n(1) = n+1 \tag{5}$$

From this, we can see that the Chebyshev polynomials T_n satisfy (4) and the following boundary conditions

$$T_n(-1) - \frac{1}{n^2}T_n'(-1) = 0 \qquad\qquad T_n(1) + \frac{1}{n^2}T_n'(1) = 0 \tag{6}$$

Therefore, the Chebyshev polynomials T_n are eigenfunctions of the linear operator $L[y] = (-py')'$ corresponding to non-negative integer n eigenvalues.

Let n and m be distinct non-negative integers. Then, as T_n and T_m are eigenfunctions corresponding to distinct eigenvalues, Theorem 19.1 implies that they are r-orthogonal for any function r. That is, the inner product $\langle rT_n, T_m\rangle = 0$. In particular, if $r(x) = \frac{1}{\sqrt{1-x^2}}$ we get that

$$\langle rT_n, T_m\rangle = \int_{-1}^{1}(1-x^2)^{-1}T_n(x)T_m(x)dx = 0$$

as required.

Question 2

$$u'' + \lambda u = 0 \tag{7}$$
$$u'(0) = u'(1) = 0 \tag{8}$$

We consider three cases for λ.

<u>$\boldsymbol{\lambda > 0}$</u>:
If $\lambda > 0$ then (7) has solution $u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. The boundary condition $u'(0) = 0$ implies that $-B\cos\sqrt{\lambda}0 = -B = 0$, so B must be 0. The second boundary condition gives $A\sin\sqrt{\lambda} = 0$. This has a non-trivial solution when $\sqrt{\lambda} = n\pi$, $n \in \mathbb{Z}^+$. Therefore the system (7), (8) has eigenvalues $\lambda_n = n^2\pi^2$ with corresponding eigenfunctions $\phi_n(x) = \cos n\pi x$.

<u>$\boldsymbol{\lambda = 0}$</u>:
If $\lambda = 0$ then (7) has solution $u(x) = Ax + B$. The first boundary condition requires $A = 0$, and $u(x) = B$ satisfies the second. Therefore the system has an eigenvalue $\lambda_0 = 0$ with corresponding eigenfunction $\phi_0(x) = 1$.

<u>$\boldsymbol{\lambda < 0}$</u>:
If $\lambda < 0$, then the differential equation has solution $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. The first boundary condition gives $0 = A\sqrt{-\lambda}e^0 - B\sqrt{-\lambda}e^0$ which holds as long as $A = B$. The second condition, gives $0 = A\sqrt{\lambda}(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}})$. However, this is only true for $\lambda = 0$, and so the system has no negative eigenvalues.

Therefore, (7), (8) has eigenvalues and eigenfunctions given by

$$\lambda_n = n^2\pi^2,\ \phi_n(x) = \cos n\pi x$$

for non-negative integer n.

Question 3

$$u'' + ku = F(x) \tag{9}$$
$$u'(0) = u'(1) = 0 \tag{10}$$

Two solutions to the homogenous differential equation are $u_1(x) = \cos\sqrt{k}x$ and $u_2(x) = \sin\sqrt{k}x$, which have Wronskian $W = \sqrt{k}$. The general solution of (9) is therefore

$$\begin{aligned}
u(x) &= \sin\sqrt{k}x\int_0^x \frac{\cos\sqrt{k}t}{\sqrt{k}}F(t)dt - \cos\sqrt{k}x\int_0^x \frac{\sin\sqrt{k}t}{\sqrt{k}}F(t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x \\
&= \frac{1}{\sqrt{k}}\int_0^x F(t)(\sin\sqrt{k}x\cos\sqrt{k}t - \cos\sqrt{k}x\sin\sqrt{k}t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x \\
&= \frac{1}{\sqrt{k}}\int_0^x \sin(\sqrt{k}(x-t))F(t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x
\end{aligned} \tag{11}$$

Taking the first derivative, we get

$$u'(x) = \int_0^x F(t)\cos(\sqrt{k}(x-t))dt - A\sqrt{k}\sin\sqrt{k}x + B\sqrt{k}\cos\sqrt{k}x \tag{12}$$

The first boundary requires $B = 0$. To satisfy the second boundary condition, we need to choose A so that

$$\int_0^1 F(t)\cos(\sqrt{k}(1-t))dt - \sqrt{k}A\sin\sqrt{k} = 0 \tag{13}$$

By Theorem 19.2, if k is not an eigenvalue of the homogeneous (7), (8), then (9), (10) has a unique solution. If k is an eigenvalue, then (13) only holds if

$$\int_0^1 F(t)\cos(n\pi(1-t))dt = 0 \tag{14}$$

in which case there are infinitely many solutions of the form $y(x)+k\phi_n(x)$, where $y(x)$ is a particular solution and $\phi_n(x)$ is the eigenfunction corresponding to eigenvalue $k_n = n^2\pi^2$.

Consider the case $F(x) = x$. If k is not an eigenvalue then we require A so that (13) holds. Evaluating the integral,

$$\begin{aligned}
\int_0^1 t\cos(\sqrt{k}(1-t))dt &= \frac{t\sin(\sqrt{k}(1-t))}{\sqrt{k}}\bigg|_0^1 - \frac{1}{\sqrt{k}}\int_0^1 \sin(\sqrt{k}(1-t))dt \\
&= \frac{\cos(\sqrt{k}(1-t))}{k}\bigg|_0^1 \\
&= \frac{1-\cos\sqrt{k}}{k}
\end{aligned}$$

Combining this with (13), we can find A as

$$\begin{aligned}
\frac{1-\cos\sqrt{k}}{k} &= \sqrt{k}A\sin\sqrt{k} \\
\implies A &= \frac{(1-\cos\sqrt{k})\csc\sqrt{k}}{k\sqrt{k}}
\end{aligned}$$

Therefore, if k is not an eigenvalue, the unique solution is

$$\begin{aligned}
u(x) &= \frac{1}{\sqrt{k}}\int_0^x t\sin(\sqrt{k}(x-t))dt + \cos\sqrt{k}x\frac{(1-\cos\sqrt{k})\csc\sqrt{k}}{k\sqrt{k}} \\
&= -\frac{t\cos\sqrt{k}(x-t)}{k}\bigg|_0^x + \cos\sqrt{k}x\frac{(1-\cos\sqrt{k})\csc\sqrt{k}}{k\sqrt{k}} \\
&= -\frac{x}{k} + \cos\sqrt{k}x\frac{(1-\cos\sqrt{k})\csc\sqrt{k}}{k\sqrt{k}}
\end{aligned}$$

If $k = k_n$ is an eigenvalue, then the infinite solutions are

$$u_n(x) = x + A\cos n\pi x$$

where A is an arbitrary constant.