PMTH339 Assignment 2

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Question 1

$$x^2y'' + 3xy' + y = 0 \tag{1}$$

For $x < 0, y = (-x)^s$ is a solution to (1) iff

$$\begin{aligned} x^2 s(s-1)x^{s-2} - 3sxx^{s-1} + x^2 &= 0 \\ x^s(s^2 - 4s + 1) &= 0 \\ \implies s &= 2 \pm \sqrt{3} \end{aligned}$$

Therefore, $y_1 = A(-x)^{2+\sqrt{3}}$ and $y_2 = B(-x)^{2-\sqrt{3}}$ are solutions. To show that these form a fundamental pair, consider the Wronskian W:

$$\begin{aligned} W &= y_1y_2' - y_2y_1' \\ &= A(-x)^{2+\sqrt{3}}(-B)(2-\sqrt{3})(-x)^{1-\sqrt{3}} - B(-x)^{2-\sqrt{3}}(-A)(2+\sqrt{3})(-x)^{1+\sqrt{3}} \\ &= AB((\sqrt{3}-2)(-x)^3 + (\sqrt{3}+2)(-x)^3) \\ &= -2AB\sqrt{3}x^3 \end{aligned}$$

which is never zero for $x < 0$. Hence y_1 and y_2 form a fundamental pair of solutions for (1), and the general solution for $x < 0$ is

$$y = A(-x)^{2+\sqrt{3}} + B(-x)^{2-\sqrt{3}} \tag{2}$$

it's derivative is

$$y' = -A(2+\sqrt{3})(-x)^{1+\sqrt{3}} - B(2-\sqrt{3})(-x)^{1-\sqrt{3}} \tag{3}$$

Substituting the initial conditions $y(-1) = 3$ and $y'(-1) = 4$ into (2) and (3) gives the pair of equations for A and B

$$A + B = 3 \qquad\qquad -(2+\sqrt{3})A - (2-\sqrt{3})B = 4$$

Solving this system gives us $A = \frac{3}{2} - \frac{5\sqrt{3}}{3}$ and $B = \frac{3}{2} + \frac{5\sqrt{3}}{3}$. Therefore the particular solution according to the given initial conditions and for $x < 0$ is

$$y(x) = \left(\frac{3}{2} - \frac{5\sqrt{3}}{3}\right)(-x)^{2+\sqrt{3}} + \left(\frac{3}{2} + \frac{5\sqrt{3}}{3}\right)(-x)^{2-\sqrt{3}}$$

Question 2

$$y''' - 3y' + 2y = 0 \tag{4}$$

Let $y_1 = e^x$. Then $y_1' = y_1''' = e^x$. Therefore

$$y_1''' - 3y' + 2y = e^x - 3e^x + 2e^x = 0$$

So y_1 is a solution. Let $y_2 = xe^x$. Then $y_2' = xe^x + e^x$, $y_2' = xe^x + 2e^x$ and $y_2''' = xe^x + 3e^x$. Therefore

$$y_2''' - 3y_2' + 2y_2 = xe^x + 3e^x - 3xe^x - 3e^x + 2xe^x = 0$$

So y_2 is a solution. Let $y_3 = e^{-2x}$. Then $y_3' = -2e^{-2x}$, $y_3'' = 4e^{-2x}$ and $y_3''' = -8e^{-2x}$. Therefore,

$$y_3''' - 3y_3' + 2y_3 = -8e^{-2x} + 6e^{-2x} + 2e^{-2x} = 0$$

So y_3 is a solution. Let $y = Ay_1 + By_2 + Cy_3$ satisfy the initial conditions $y(0) = 1, y'(0) = 0, y''(0) = 0$. This gives rise to the system of equations for A, B and C:

$$\begin{aligned} A + C &= 1 \\ A + 2B - 2C &= 0 \\ A + 2B + 4C &= 0 \end{aligned}$$

which has solution $A = \frac{8}{3}$, $B = -2$ and $C = \frac{1}{3}$. Substitute y with these coefficients into (4) to check if y is a solution. Firstly, note that

$$\begin{aligned} y &= \frac{8}{3}e^x - 2xe^x + \frac{1}{3}e^{-2x} \\ y' &= \frac{2}{3}e^x - 2xe^x - \frac{2}{3}e^{-2x} \\ y'' &= -\frac{4}{3}e^x - 2xe^x + \frac{4}{3}e^{-2x} \\ y''' &= -\frac{10}{3}e^x - 2xe^x - \frac{8}{3}e^{-2x} \end{aligned}$$

Therefore,

$$\begin{aligned} y''' - 3y' + 2y &= \left(-\frac{10}{3} - 2 + \frac{16}{3}\right)e^x + (-2 + 6 - 4)xe^x + \left(-\frac{8}{3} + \frac{6}{3} + \frac{2}{3}\right)e^{-2x} \\ &= 0e^x + 0xe^x + 0e^{-2x} \\ &= 0 \end{aligned}$$

Therefore y as defined is a solution to third order linear differential equation (4) satisfying the given initial conditions.

Question 3

$$(1 - x^2)y'' - xy' + 4y = 0 \tag{5}$$

$$\implies y'' - \frac{x}{1 - x^2}y' + \frac{4}{1 - x^2}y = 0 \tag{6}$$

The existence-uniqueness theorem states that if $p(x) = -\frac{x}{1-x^2}$ and $q(x) = \frac{4}{1-x^2}$ are continuous on an interval I, then there exists a unique solution on I that satisfies given initial conditions. p and q are continous everywhere except at $1 - x^2 = 0 \implies x = \pm 1$. Therefore, solutions to (5) are guaranteed on the intervals $I_1 = (-\infty, -1)$, $I_2 = (-1, 1)$ and $I_3 = (1, \infty)$.

Let $y_1 = 1 - 2x^2$. Then $y_1' = -4x$ and $y_1'' = -4$. Substituting into (5),

$$(1 - x^2)(-4) - x(-4x) + 4(1 - 2x^2) = -4 + 4x^2 + 4x^2 + 4 - 8x^2 = 0$$

and so y_1 is a solution. To find a second solution y_2, we first calculate the Wronskian W as

$$W(x) = e^{-\int p(x)dx} = e^{\int \frac{x}{1-x^2}dx}$$

Let $u = 1 - x^2 \implies \frac{du}{dx} = -2x$. Therefore the expression becomes

$$W(x) = e^{-\frac{1}{2}\int \frac{1}{u}du} = e^{-\frac{1}{2}\ln u} = u^{-1/2} = \frac{1}{\sqrt{1-x^2}}$$

y_2 can then be calculated as

$$\begin{aligned} y_2 &= y_1 \int \frac{W}{y_1^2} dx \\ &= (1 - 2x^2) \int \frac{1}{\sqrt{1-x^2}(1-2x^2)^2} dx \end{aligned}$$

For the integrand, let $x = \sin u$. Then the integral becomes

$$\int \frac{\cos u}{\sqrt{1-\sin^2 u}(1-2\sin^2 u)^2} du = \int \frac{1}{\cos^2(2u)} du = \frac{1}{2}\tan(2u) + c$$

where $c \in \mathbb{R}$. Substituting $u = \sin^{-1} x$,

$$\begin{aligned} \frac{1}{2}\tan(2u) &= \frac{\sin 2u}{2\cos 2u} \\ &= \frac{\sin u \cos u}{\cos^2 u - \sin^2 u} \\ &= \frac{x\sqrt{1-\sin^2 u}}{1 - \sin^2 u - \sin^2 u} \\ &= \frac{x\sqrt{1-x^2}}{1-2x^2} \end{aligned}$$

Therefore we get a second solution

$$\begin{aligned} y_2 &= (1 - 2x^2)\left(\frac{x\sqrt{1-x^2}}{1-2x^2} + c\right) \\ &= x\sqrt{1-x^2} - 2cx^2 + c \end{aligned}$$

for $c \in \mathbb{R}$

Question 4

$$y' = x^2 + y^2 \tag{7}$$

Substitute for some function u, $y = -\frac{u'}{u}$. Thus $y' = -\left(\frac{u''u-(u')^2}{u^2}\right)$ and (7) becomes

$$\begin{aligned} \frac{(u')^2 - uu''}{u^2} &= x^2 + \frac{(u')^2}{u^2} \\ (u')^2 - uu'' &= x^2u^2 + (u')^2 \\ \implies u'' + x^2u &= 0 \end{aligned} \tag{8}$$

Question 5 of Assignment 1 asks about the behaviour of the solution to (7) with the initial condition $y(0) = 0$. With the above substitution, we require then that $y(0) = \frac{u'(0)}{u(0)} = 0 \implies u'(0) = 0 \implies u(0) = c$ for some constant $c \in \mathbb{R}\backslash\{0\}$. Therefore, an equivalent problem to Question 5 of Assignment 1 is to show that the solution $u(x)$ to the second order differential equation (8) satisfying the initial conditions $u(0) = 0, u'(0) = c$ has a vertical asymptote.