PMTH339 Assignment 4

Jayden Turner (SN 220188234)

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Question 1

$$u'' + x^2 u = 0 \tag{1}$$

Consider power series solutions to (1), that is $u(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $u' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ and $u'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$. Substituting into (1),

$$\begin{aligned} 0 &= u'' + x^2 u \\ &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=-2}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} x^n ((n+1)(n+2)a_{n+2} + a_{n-2}) \end{aligned} \tag{2}$$

For (2) to hold, we require $a_2 = a_3 = 0$ and $(n+1)(n+2)a_{n+2} + a_{n-2} = 0, \forall n \in \mathbb{N} \cup \{0\}$. Using the initial conditions $u(0) = 1, u'(0) = 0$ gives $a_0 = 1, a_1 = 0$. These conditions give a recursive definition for the coefficients of the power series:

$$\{a_n\}_{n=0}^{\infty} = \begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_n = -\frac{a_{n-4}}{n(n-1)} \end{cases} \tag{3}$$

Note that $a_k = 0$ whenever k is not a multiple of 4, and that the terms are negative for odd multiples of 4 and positive for even multiples of 4. Therefore we can construct another sequence of coefficients $\{b_n\}_{n=0}^{\infty}$ as

$$b_n = (-1)^n \frac{1}{\prod_{k=1}^{n} (4k)(4k-1)} \tag{4}$$

such that $u(x) = \sum_{n=0}^{\infty} b_n x^{4n}$. Therefore, a power series solution to (1) is

$$u(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{\prod_{k=1}^{n} (4k)(4k-1)} x^{4n} \tag{5}$$

To determine for what values of x (5) converges, use the ratio test:

$$
\begin{aligned}
L &= \left| \frac{(-1)^{n+1} \frac{1}{\prod\limits_{k=1}^{n+1}(4k)(4k-1)} x^{4(n+1)}}{(-1)^{n} \frac{1}{\prod\limits_{k=1}^{n}(4k)(4k-1)} x^{4n}} \right| \\
&= \left| \frac{\prod\limits_{k=1}^{n}(4k)(4k-1)x^{4n+4}}{\prod\limits_{k=1}^{n+1}(4k)(4k-1)x^{4n}} \right| \\
&= x^4 \left| \frac{1}{4(n+1)(4(n+1)-1)} \right| \\
&= x^4 \left| \frac{1}{(4n+4)(4n+3)} \right|
\end{aligned}
$$

$\lim\limits_{n\to\infty} L = 0 < 1$, therefore the series (5) converges absolutely $\forall x \in \mathbb{R}$.

Question 3

The Chebyshev polynomials can be found with the following recursive definitions

$$\{T_n(x)\}_{n=0}^{\infty} = \begin{cases} T_0(x) = 1 \\ T_{n+1}(x) = xT_n(x) - (1-x^2)U_n(x) \end{cases} \qquad \{U_n(x)\}_{n=0}^{\infty} = \begin{cases} U_0(x) = 0 \\ U_{n+1}(x) = T_n(x) + xU_n(x) \end{cases}$$

The following table summarises the process of finding $T_5(x)$:

n	$T_n(x)$	$U_n(x)$
0	1	0
1	$xT_0(x) - (1-x^2)U_0(x)$ $= x$	$T_0(x) + xU_0(x)$ $= 1$
2	$xT_1(x) - (1-x^2)U_1(x)$ $= 2x^2 - 1$	$T_1(x) + xU_1(x)$ $= 2x$
3	$xT_2(x) - (1-x^2)U_2(x)$ $= x(2x^2-1) - (1-x^2)(2x)$ $= 2x^3 - x - 3x + 2x^3$ $= 4x^3 - 3x$	$T_2(x) + xU_2(x)$ $= 2x^2 - 1 + x(2x)$ $= 4x^2 - 1$
4	$xT_3(x) - (1-x^2)U_3(x)$ $= x(4x^3-3x) - (1-x^2)(4x^2-1)$ $= 4x^4 - 3x^2 - 4x^2 + 1 + 4x^4 - x^2$ $= 8x^4 - 8x^2 + 1$	$T_3(x) + xU_3(x)$ $= 4x^3 - 3x + x(4x^2-1)$ $= 4x^3 - 3x + 4x^3 - x$ $= 8x^3 - 4x$
5	$xT_4(x) - (1-x^2)U_4(x)$ $= x(8x^4 - 8x^2 + 1) - (1-x^2)(8x^3 - 4x)$ $= 8x^5 - 8x^3 + x - 8x^3 + 4x + 8x^5 - 4x^3$ $= 16x^5 - 20x^3 + 5x$	

Therefore the 5th Chebyshev polynomial is

$$T_5(x) = 16x^5 - 20x^3 + 5x \tag{6}$$

Question 4

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \tag{7}$$

a) To find a power series solution to (7), assume y is of the form $y = \sum\limits_{n=0}^{\infty} a_n x^n$. Then (7) requires

$$\begin{aligned}
0 &= (1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=-2}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} x^n\left((n+1)(n+2)a_{n+2} + a_n(n(n-1) + 2n - \alpha(\alpha+1))\right)
\end{aligned}$$

For this to hold, all coefficients in the series must be zero, thus the following recursion relation on a_n must hold:

$$\begin{aligned}
a_{n+2} &= a_n \frac{n(n-1) + 2n - \alpha(\alpha+1)}{(n+1)(n+2)} \\
&= a_n \frac{n^2 - n + 2n - \alpha^2 - \alpha}{(n+1)(n+2)} \\
&= a_n \frac{(n-\alpha)(n+\alpha) + n - \alpha}{n+1)(n+2)} \\
&= a_n \frac{(n-\alpha)(n+\alpha+1)}{(n+1)(n+2)}
\end{aligned} \tag{8}$$

As (8) relates a_{n+2} to a_n, two sub sequences of coefficients arise; one for even n and one for odd n:

$$a_{2n+2} = a_{2n}\frac{(2n-\alpha)(2n+\alpha+1)}{(2n+1)(2n+2)} \qquad a_{2n+3} = a_{2n+1}\frac{(2n+1-\alpha)(2n+\alpha+2)}{(2n+1)(2n+3)}$$

Now consider different values of positive integer α. If α is even (i.e. $\alpha = 2k$ for some $k \in \mathbb{Z}$), then the even sequence terminates when $n = k = \frac{\alpha}{2}$, while the odd sequence is infinite. Likewise, for odd α (i.e. $\alpha = 2k+1$ for some $k \in \mathbb{Z}$), then the odd sequence terminates when $n = k = \frac{\alpha-1}{2}$, while the even sequence is infinite.

Consider $\alpha = 0$. Then the even sequence terminates after $n = 0$. The condition $P_0(1) = 1$ then requires

$$1 = a_0 + \sum_{n=0}^{\infty} a_{2n+1}$$

Setting $a_1 = 0$ gives $a_{2n+1} = 0, \forall n \in \mathbb{Z}$. Hence $P_0(x) = a_0 = 1$. Now consider $\alpha = 1$. The odd sequence terminates after $n = 0$, and the condition $P_1(1) = 1$ gives

$$1 = a_1 + \sum_{n=0}^{\infty} a_{2n}$$

Setting $a_0 = 0$ gives $a_{2n} = 0, \forall n \in \mathbb{Z}$. Hence $a_1 = 1 \implies P_1(x) = x$. For $\alpha = 2$, the even sequence terminates after $n = 1$, and the condition $P_2(1) = 1$ gives

$$
\begin{aligned}
1 &= a_0 + a_2 + \sum_{n=0}^{\infty} a_{2n+1} \\
&= a_0 + a_0 \left(\frac{-2 \cdot 3}{1 \cdot 2} \right) + \sum_{n=0}^{\infty} a_{2n+1} \\
&= a_0(1-3) + \sum_{n=0}^{\infty} a_{2n+1} \\
&= -2a_0 + \sum_{n=0}^{\infty} a_{2n+1}
\end{aligned}
$$

Setting $a_1 = 1$ makes the terms of the infinite sum vanish, so $a_0 = -\frac{1}{2} \implies a_1 = \frac{3}{2}$. Therefore $P_2(x) = \frac{1}{2}(3x^2 - 1)$. For $\alpha = 3$, the odd sequence terminates after $n = 1$ and the condition $P_3(1) = 1$ gives

$$
\begin{aligned}
1 &= a_1 + a_3 + \sum_{n=0}^{\infty} a_{2n} \\
&= a_1 + a_1 \left(\frac{-2 \cdot 5}{2 \cdot 3} \right) + \sum_{n=0}^{\infty} a_{2n} \\
&= a_1 \left(1 - \frac{5}{3} \right) + \sum_{n=0}^{\infty} a_{2n} \\
&= -\frac{2}{3} a_1 + \sum_{n=0}^{\infty} a_{2n}
\end{aligned}
$$

Setting $a_0 = 0$ makes the terms of the infinite sum vanish, leaving $a_1 = -\frac{3}{2} \implies a_3 = \frac{5}{2}$. Therefore $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

b) The general solution to (7) with $\alpha = 1$ is $y = Ay_1 + By_2$, where $y_1 = P_1 = x$, and y_2 is another linearly independent solution. Dividing (7) by $1 - x^2$ gives

$$
y'' - \frac{2x}{1-x^2} y' + \frac{2}{1-x^2} y = 0 \tag{9}
$$

Therefore, the Wronskian of y_1 and y_2 can be calculated as

$$
\begin{aligned}
W &= e^{-\int \frac{-2}{1-x^2} dx} \\
&= e^{2 \int \frac{x}{1-x^2} dx}
\end{aligned}
$$

Substituting $u = 1 - x^2$

$$
\begin{aligned}
W &= e^{-2 \int \frac{x}{u} \frac{du}{-2x} dx} \\
&= e^{-\int \frac{1}{u} du} \\
&= e^{-\ln |u|} \\
&= \frac{1}{|u|} \\
&= \frac{1}{|1-x^2|} \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

for $|x| < 1$. Therefore y_2 is

$$\begin{aligned} y_2 &= y_1 \int \frac{W}{y_1^2} dx \\ &= x \int \frac{1}{x^2(1-x^2)} dx \end{aligned}$$

Setting $\frac{1}{x^2(1-x^2)} = \frac{A}{x} + \frac{B}{x}^2 + \frac{C}{1-x} + \frac{D}{1+x}$ gives $1 = Ax(1-x^2) + B(1-x^2) + Cx^2(1+x) + Dx^2(1-x)$. To find values for A, B, C, D, substitute various values of x:

$$\begin{aligned} x = 0 &\implies 1 = B \\ x = 1 &\implies 1 = 2C \implies C = \frac{1}{2} \\ x = -1 &\implies 1 = 2D \implies D = \frac{1}{2} \\ x = 2 &\implies 1 = -6A - 3 + 6 - 2 \implies A = 0 \end{aligned}$$

Therefore

$$\begin{aligned} y_2 &= x \int \frac{1}{x^2} + \frac{1}{2(1-x}) + \frac{1}{2(1+x)} dx \\ &= x \left(-\frac{1}{x} - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| \right) \\ &= \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1 \end{aligned}$$

Therefore, the general solution to (7) with $\alpha = 1$ and $|x| < 1$ is

$$y = Ax + \frac{Bx}{2} \ln\left(\frac{1+x}{1-x}\right) - B$$

where $A, B \in \mathbb{R}$ are constants.

Question 5

$$(1-x)y' - 2y = 0 \tag{10}$$

Suppose $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution to (10). Then

$$\begin{aligned} 0 &= (1-x) \sum_{n=0}^{\infty} n a_n x^{n-1} - 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=-1}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} x^n ((n+1) a_{n+1} - n a_n - 2 a_n) \end{aligned}$$

For this to hold we require $a_0 = \frac{1}{2} a_1$ and $(n+1)a_{n+1} = (n+2)a_n$. Further, for this sum to converge, we require

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}|x^n|\left|\frac{n+2}{n+1}\right| &< 1 \\
\implies \lim_{n\to\infty}|x|\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right| &< 1 \\
|x| &< 1
\end{aligned}
$$

Therefore any power series solution to (10) is only convergent on the interval $(-1,1)$. To find a solution, note that (10) is a seperable differential equation, so the solution y satisfies

$$
\begin{aligned}
\int \frac{1}{y}dy &= 2\int \frac{1}{1-x}dx \\
\ln y &= -2\ln|1-x| + c \\
y &= \frac{A}{(1-x)^2}
\end{aligned}
$$

The initial condition $y(0)=1$ gives that $A=1$. Therefore the solution is $y=\frac{1}{(1-x)^2}$. Observe that for $|x|<1$

$$
\begin{aligned}
y(x) &= \frac{1}{(1-x)^2} \\
&= \frac{d}{dx}\frac{1}{1-x} \\
&= \frac{d}{dx}\sum_{n=0}^{\infty} x^n \\
&= \sum_{n=0}^{\infty} nx^{n-1}
\end{aligned}
$$

which is a power series solution to (10).