PMTH339 Assignment 8

Jayden Turner (SN 220188234)

28 September 2018

Question 1

$$(1-x^2)y'' - xy' + \alpha^2 y = 0 \tag{1}$$

Divide through by $(1-x^2)$ to get

$$y'' - \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0 \tag{2}$$

Multiply by $I(x)$, where $I(x)$ is

$$I(x) = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1-x^2} \tag{3}$$

Therefore

$$\begin{aligned} I(x)(2) \implies 0 &= \sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{\alpha^2}{\sqrt{1-x^2}}y \\ &= (\sqrt{1-x^2}y')' + \frac{\alpha^2}{\sqrt{1-x^2}}y \end{aligned} \tag{4}$$

(4) is in the form $(p(x)y')' + q(x)y = 0$, with $p(x) = \sqrt{1-x^2}$ and $q(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$ defined and with continuous derivatives on the interval $(-1, 1)$.

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ hold certain properties that are useful here. Firstly, $T_n(x)$ and $U_n(x)$ solve (1) and thus (4) for $\alpha = n$. Secondly, the derivatives of $T_n(x)$ can be defined in terms of $U_{n-1}(x)$ as $T_n'(x) = nU_{n-1}(x)$. Finally, they hold the following values at $x = -1, 1$:

$$T_n(-1) = (-1)^n \qquad U_n(-1) = (n+1)(-1)^n \qquad T_n(1) = 1 \qquad U_n(1) = n+1 \tag{5}$$

Question 2

Question 3

Question 4