# PMTH339 Assignment 8

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## Question 1

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0 \tag{1}$$

Divide through by $(1 - x^2)$ to get

$$y'' - \frac{x}{1 - x^2}y' + \frac{\alpha^2}{1 - x^2}y = 0 \tag{2}$$

Multiply by $I(x)$, where $I(x)$ is

$$I(x) = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln|1-x^2|} = \sqrt{1 - x^2} \tag{3}$$

Therefore

$$\begin{aligned} I(x)(2) \implies 0 &= \sqrt{1 - x^2}y'' - \frac{x}{\sqrt{1 - x^2}}y' + \frac{\alpha^2}{\sqrt{1 - x^2}}y \\ &= (\sqrt{1 - x^2}y')' + \frac{\alpha^2}{\sqrt{1 - x^2}}y \end{aligned} \tag{4}$$

(4) is in the form $(p(x)y')' + q(x)y = 0$, with $p(x) = \sqrt{1 - x^2}$ and $q(x) = \frac{\alpha^2}{\sqrt{1-x^2}}$ defined and with continuous derivatives on the interval $(-1, 1)$.

The Chebyshev polynomials $T_n(x)$ and $U_n(x)$ hold certain properties that are useful here. Firstly, $T_n(x)$ and $U_n(x)$ solve (1) and thus (4) for $\alpha = n$. Secondly, the derivatives of $T_n(x)$ can be defined in terms of $U_{n-1}(x)$ as $T_n'(x) = nU_{n-1}(x)$. Finally, they hold the following values at $x = -1, 1$:

$$T_n(-1) = (-1)^n \qquad U_n(-1) = (n + 1)(-1)^n \qquad T_n(1) = 1 \qquad U_n(1) = n + 1 \tag{5}$$

From this, we can see that the Chebyshev polynomials $T_n$ satisfy (4) and the following boundary conditions

$$T_n(-1) - \frac{1}{n^2}T_n'(-1) = 0 \qquad\qquad T_n(1) + \frac{1}{n^2}T_n'(1) = 0 \tag{6}$$

Therefore, the Chebyshev polynomials $T_n$ are eigenfunctions of the linear operator $L[y] = (-py')'$ corresponding to non-negative integer $n$ eigenvalues.

Let $n$ and $m$ be distinct non-negative integers. Then, as $T_n$ and $T_m$ are eigenfunctions corresponding to distinct eigenvalues, Theorem 19.1 implies that they are $r$-orthogonal for any function $r$. That is, the inner product $\langle rT_n, T_m\rangle = 0$. In particular, if $r(x) = \frac{1}{\sqrt{1-x^2}}$ we get that

$$\langle rT_n, T_m\rangle = \int_{-1}^{1} (1 - x^2)^{-1}T_n(x)T_m(x)dx = 0$$

as required.

## Question 2

$$u'' + \lambda u = 0 \tag{7}$$
$$u'(0) = u'(1) = 0 \tag{8}$$

We consider three cases for $\lambda$.

<u>$\boldsymbol{\lambda > 0}$</u>:
If $\lambda > 0$ then (7) has solution $u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. The boundary condition $u'(0) = 0$ implies that $-B\cos\sqrt{\lambda}0 = -B = 0$, so $B$ must be 0. The second boundary condition gives $A\sin\sqrt{\lambda} = 0$. This has a non-trivial solution when $\sqrt{\lambda} = n\pi$, $n \in \mathbb{Z}^+$. Therefore the system (7), (8) has eigenvalues $\lambda_n = n^2\pi^2$ with corresponding eigenfunctions $\phi_n(x) = \cos n\pi x$.

<u>$\boldsymbol{\lambda = 0}$</u>:
If $\lambda = 0$ then (7) has solution $u(x) = Ax + B$. The first boundary condition requires $A = 0$, and $u(x) = B$ satisfies the second. Therefore the system has an eigenvalue $\lambda_0 = 0$ with corresponding eigenfunction $\phi_0(x) = 1$.

<u>$\boldsymbol{\lambda < 0}$</u>:
If $\lambda < 0$, then the differential equation has solution $u(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$. The first boundary condition gives $0 = A\sqrt{-\lambda}e^0 - B\sqrt{-\lambda}e^0$ which holds as long as $A = B$. The second condition, gives $0 = A\sqrt{\lambda}(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}})$. However, this is only true for $\lambda = 0$, and so the system has no negative eigenvalues.

Therefore, (7), (8) has eigenvalues and eigenfunctions given by

$$\lambda_n = n^2\pi^2,\ \phi_n(x) = \cos n\pi x$$

for non-negative integer $n$.

## Question 3

$$u'' + ku = F(x) \tag{9}$$
$$u'(0) = u'(1) = 0 \tag{10}$$

Two solutions to the homogenous differential equation are $u_1(x) = \cos\sqrt{k}x$ and $u_2(x) = \sin\sqrt{k}x$, which have Wronskian $W = \sqrt{k}$. The general solution of (9) is therefore

$$\begin{aligned}
u(x) &= \sin\sqrt{k}x\int_0^x \frac{\cos\sqrt{k}t}{\sqrt{k}}F(t)dt - \cos\sqrt{k}x\int_0^x \frac{\sin\sqrt{k}t}{\sqrt{k}}F(t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x \\
&= \frac{1}{\sqrt{k}}\int_0^x F(t)(\sin\sqrt{k}x\cos\sqrt{k}t - \cos\sqrt{k}x\sin\sqrt{k}t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x \\
&= \frac{1}{\sqrt{k}}\int_0^x \sin(\sqrt{k}(x-t))F(t)dt + A\cos\sqrt{k}x + B\sin\sqrt{k}x \qquad (11)
\end{aligned}$$

Taking the first derivative, we get

$$u'(x) = \frac{1}{\sqrt{k}}\int_0^x F(t)\cos(\sqrt{k}(x-t))dt - A\sqrt{k}\sin\sqrt{k}x + B\sqrt{k}\cos\sqrt{k}x \tag{12}$$

The first boundary requires $B = 0$. To satisfy the second boundary condition, we need to choose $A$ so that

$$\int_0^1 F(t)\cos(\sqrt{k}(x-t))dt - kA\sin\sqrt{k} = 0 \tag{13}$$

## Question 4