it exists—we first just do the best we can and then splice in a loop-d-loop circuit to augment our first attempt at tracing the given graph, and if we are still not done, we splice in another loop-d-loop circuit and so on until we have gone over every edge of the graph.

A Look BACK

We analyzed the question of when we can walk over a collection of bridges that connect various landmasses without going over the same bridge twice. This question was prompted by the famous Königsberg Bridge Puzzle from the year 1735. Landmasses connected by bridges were abstracted into the mathematical idea of a graph consisting of vertices and edges that connect vertices in pairs.

We found that a graph has an Euler circuit (that is, it can be traced starting anywhere and ending where we start without lifting our pencil or going over the same edge twice) if and only if it is connected and every vertex appears an even number of times in the list of edges. The study of graph theory goes far beyond Königsberg and leads to important insights into computer science, sociology, neuroscience, epidemiology, social networks, and many other applications.

Our strategy for developing this topic was to build from the concrete. We analyzed a specific question. Our method was to ignore the irrelevant and isolate essential ingredients from the situation. Our method of solution included looking for patterns and building insights from failed attempts. Then we took insights that we came to and extended them to apply to related situations. The strategies of investigation that we used here can generate insights and resolve issues far beyond mathematical ones.

Life Lessons

Analyze specific questions. Then generalize.

•

Find the essential ingredients.

•

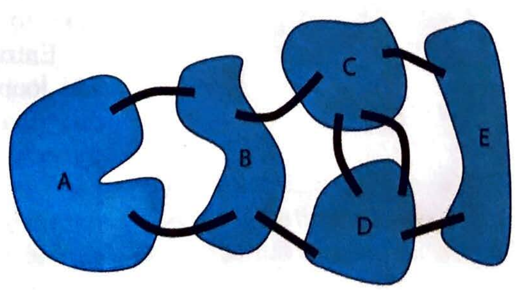
Take a solution to one question and see where else it can be applied.

MINDSCAPES Invitations to Further Thought

*In this section, Mindscapes marked (**H**) have hints for solutions at the back of the book. Mindscapes marked (**ExH**) have expanded hints at the back of the book. Mindscapes marked (**S**) have solutions.*

Developing Ideas

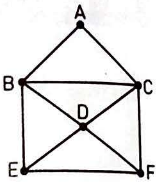
1. **Map maker, map maker make me a graph.** Represent this map using a graph, with a vertex (dot) for each landmass and an edge (line or arc) for each bridge.



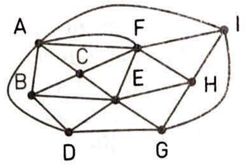
2. **Unabridged list.** Represent each landmass from Mindscape 1 as a vertex and give each vertex a letter. Represent each bridge from Mindscape 1 as an edge of your graph using a pair of letters as in the text. Make a list of the edges. What are the degrees of each vertex of your graph?

3. **Will the walk work?** Does your graph from Mindscapes 1 and 2 have an Euler circuit? That is, can you take a walk around the town shown in the map in Mindscape 1, cross each bridge exactly once, and return to where you started? If yes, describe such a walk by listing the edges. If not, explain why not.

4. **Walk around the house.** Is it possible to traverse this graph with a path that uses each edge exactly once and returns to the vertex at which you started? In other words, does the graph have an Euler circuit? If so, find one. If not, explain why not.



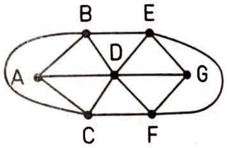
5. **Where's Waldoskova?** Do some research to find the location of Königsberg on a modern map. Does it still have the same name?



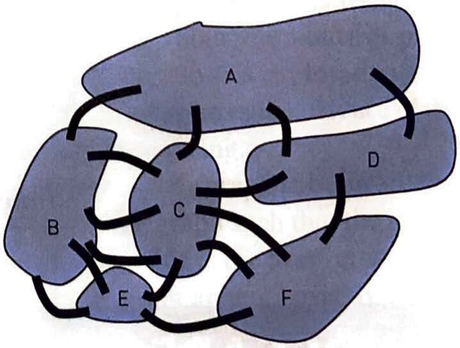
Solidifying Ideas

6. **Walk the line.** Does this graph above have an Euler circuit? If so, find one. If not, explain why not.

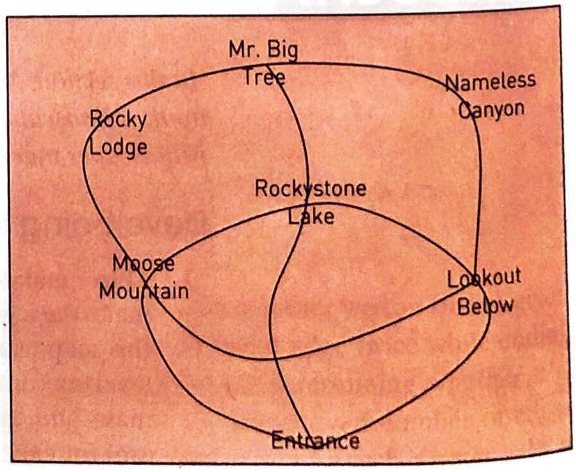
7. **Walkabout.** Does this graph have an Euler circuit? If so, find one. If not, explain why not.



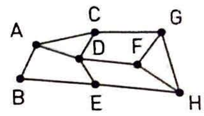
8. **Linking the loops.** In this map, the following walks can be taken from various starting points: CAADDFFC, FCCBB CCEEF, DCCBBEEBBAAD. Can these walks be spliced together to create one walk that starts on landmass C, crosses each bridge exactly once, and then returns to C? If so, find such a walk. If not, explain why not.



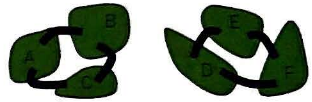
9. **Scenic drive. (S)** Here is a map of Rockystone National Park. One scenic drive is Entrance to Moose Mountain to Rockystone Lake to Lookout Below to Entrance. Can you add loop-d-loops to this drive to obtain a trip that traverses each road in the park exactly once and returns to the entrance? If so, find such a trip. If not, explain why not.



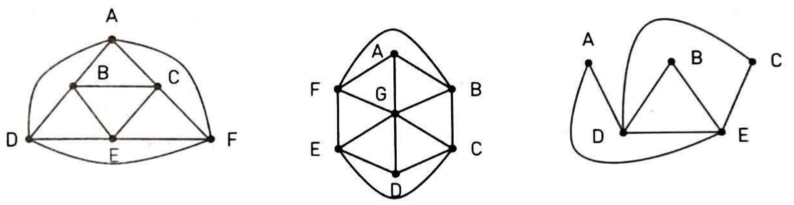
10. **Under-edged. (H)** Does this graph have an Euler circuit? If so, find one. If not, add the fewest number of new edges until such a circuit is possible. (Remember that edges can be curved.)



11. **No man is an island.** The country of Pelago consists of six islands. Create a graph to model the islands and bridges of Pelago. What is the degree of each vertex of your graph? Does your graph have an Euler circuit? Why or why not?



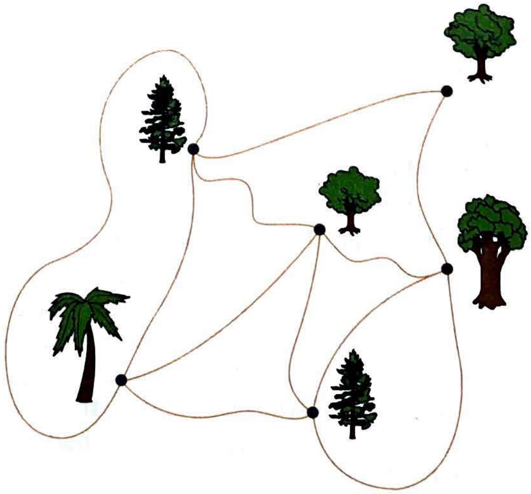
12. **Path-o-rama.** For each graph below, determine if the graph has an Euler circuit. If such a path is possible, present it. If not, explain why.



(a) (b) (c)

13. **Walk around the block.** Create a graph of the buildings on your campus that you use frequently, with the buildings (dorm, classroom, library, gym, etc.) as vertices, and edges joining vertices when they have a direct walkway or road connecting them. Does this graph have an Euler circuit? If so, find one. If not, explain why not.

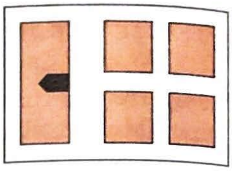
14. **Walking the dogs.** Your dogs, Abbey and Bear, love to walk in the town park, modeled with the graph below. Based on the drawing, what do the vertices in this graph represent? What do the edges represent? Abbey and Bear recently learned about Euler circuits and try to traverse each edge exactly once during their walk (as they visit many trees more than once). Will they be successful?



15. **Delivery query.** The next time you see a postal worker delivering mail, ask her how she plans her route. In particular, ask if she is fortunate enough not to have to retrace her steps down streets where she has already delivered the mail.

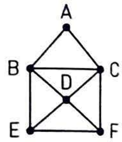
Creating New Ideas

16. **Snow job. (ExH)** Shown here is a map of the tiny town of Eulerville. The streets are white; the blocks are orange. After a winter storm, the village snowplow can clear a street with just one pass. Is it possible for the plow to start and end at the Town Hall (shown in black), clearing all the streets without traversing any street more than once? If so, find such a route. If not, explain why not.

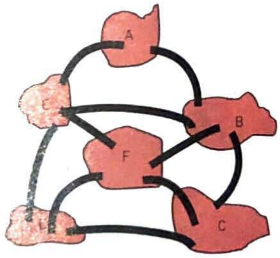


17. **Special delivery. (ExH)** Julia is the letter carrier for the town of Eulerville shown in the previous Mindscape. The streets are white; the blocks are orange. Assume that any street with a block on each side will have homes or business on both sides. Therefore Julia must walk along both sides of some streets to deliver the mail. The Post Office is located in the Town Hall (shown in black). Is there a route for Julia to use so she can start at the Post Office, deliver the mail along each street without having to retrace her steps, and return to the Post Office? If so, find such a route. If not, explain why not.

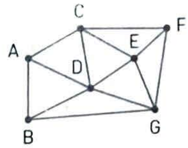
18. **Draw this old house.** Suppose you wanted to trace out the edges in this graph below without lifting your pen from the paper so that you draw each edge exactly once but with a new twist: You don't care if you return to the point where you started. Is it possible? If so, do it. If not, explain why you can't.



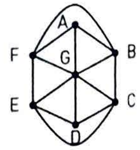
19. **Path of no return.** Consider this map showing a town with six landmasses and ten bridges. If you started on land mass C and walked as far as possible without going over the same bridge twice, where could you possibly get stuck? Why? What if you started on landmass D? What is special about C and D?



20. **Without a trace.** Is it possible to trace out either of these graphs without crossing an edge more than once? (Don't worry about ending up where you start.) If yes, describe the path. If no, explain.



(a)



(b)

21. **New Euler.** In the three previous Mindscapes, you were presented with graphs that had no Euler circuit because they had vertices with odd degree (an odd number of incident edges). But in three of the four graphs, you could find a path that traversed each edge exactly once. Such a path is called an *Euler path.* Each of your Euler paths started and ended at a vertex of odd degree. Did this have to happen for these graphs? If you had more than two vertices of odd degree, could an Euler path exist?

22. **New edge—new circuit.** Look at the graph for Mindscape 18. Draw your own copy of this graph and add a second edge from E to F. Does the new graph have an Euler circuit? If you wanted to, could you start and end that circuit at vertex E, with the new edge being the last edge you used?

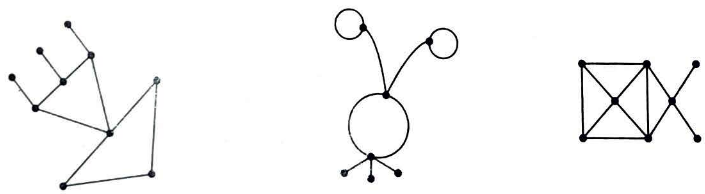
23. **New edge—new path.** Review your work for Mindscape 22. If you start your Euler circuit starting at E and end without using the new edge, what have you done? How does this relate to the question in Mindscape 18?

24. **Path to proof.** Suppose you have a connected graph in which every vertex has even degree except for two vertices with odd degree. If you add an edge between the two odd-degree vertices, what can you say about the resulting graph? Apply the reasoning from Mindscapes 22 and 23 to deduce that the original graph must have an Euler path that starts at one odd-degree vertex and ends at the other. Test your reasoning on graph (b) for Mindscape 20.

25. **No Euler no how.** Look at graph (a) for Mindscape 20. If you add one edge between two of the odd-degree vertices, does the resulting graph have an Euler circuit? Does this lead you to a conjecture about how many odd-degree vertices a graph can have and still have an Euler path?

Further Challenges

26. **Degree day. (S)** For each graph below, determine the degree of each vertex. (An edge that begins and ends at the same vertex is called a *loop*. Such an edge is counted twice when determining the degree of a vertex.) Then for each graph, compute the sum of all the degrees of the vertices in that graph. Count the number of edges in each graph. What do you notice?



27. **100 degrees of proof.** Review your work for Mindscape 26. Can you make a conjecture that relates the number of edges in any graph to the degrees of all the vertices in that graph? (You might discover a result known as the Handshake Theorem. Can you explain why it's called that?)

28. **Degrees in sequence.** Can you draw a graph that has six vertices with degrees 5, 4, 3, 2, 2, 2? (This list is called the *degree sequence* of the graph.) Is there more than one way to draw such a graph? What does the sum of the numbers in the degree sequence equal? Can you draw a graph with degree sequence 3, 3, 2, 1, 1? What about 4, 3, 2, 1, 1?

29. **Even Steven.** Review your work in Mindscape 28 to make a conjecture about the number of vertices of odd degree that a graph can have. Prove your claim.

30. **Little League lesson. (H)** You are in charge of scheduling the baseball games for your town's Little League. There are 11 teams in your league. Usually you play ten games in a season, but some of the coaches want to extend the season to 13 games. So every team would play 13 games (thus playing more than one game against some of the teams). How will you explain to the coaches why this plan can never work?

In Your Own Words

31. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.
32. **With a group of folks.** In a small group, discuss and actively work through the reasoning involved in proving that a connected graph with all vertices of even degree has an Euler circuit. After your discussion, write a brief narrative describing the methods in your own words.
33. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.
34. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—for which some of the strategies of thought presented in this section would provide effective methods for approaching and resolving them.

For the Algebra Lover

Here we celebrate the power of algebra as a powerful way of finding unknown quantities by naming them, of expressing infinitely many relationships and connections clearly and succinctly, and of uncovering pattern and structure.

35. **Finding V.** Suppose a graph has n vertices and $(1/2)n(n-1) = 15$ edges. How many vertices does the graph actually have?
36. **Still looking. (H)** Suppose a graph has n vertices and $(1/2)n(n-1) = 45$ edges. How many vertices does the graph actually have?
37. **Pigeon count.** Mary runs every morning. She follows an Euler circuit through the park near her house. Along the first edge she sees one pigeon, along the second edge she sees three, along the third edge she sees five, and so on. If there are ten edges total in Mary's run, how many pigeons does she see? Find an expression that gives the total number of pigeons she would see if there were n edges in her run.
38. **Edges vs. vertices.** Someone chalked a graph outside your dorm. You observe that the number of edges is twice the number of vertices. You also see that the number of edges times the number of vertices is equal to 30 plus four times the number of vertices. Let V be the number of vertices in the graph and E the number of edges. Write an equation showing the relationship between V and E and another equation involving EV. Use these two equations to find the value of V.
39. **Family values.** Your teacher claims to have a family of graphs where the graph with n vertices has $(1/2)n(n-1)$ edges, and n can be any natural number. How many edges does the graph with 100 vertices have? What about the graph with two vertices? With one vertex? Draw examples of these last two graphs.