Mindscapes Invitations to Further Thought

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **At one with the universe.** Below is a sketch of a 1-dimensional space. Identify the points $x = 3$, $y = -5/2$, and $z = 2.25$.

   How does the fact that you need only *one* number to identify a particular point relate to the dimension of the space?

2. **Are we there yet?** Why does the information "$x = 4$" *not* specify a unique point in the plane? What does this say about the dimension of the plane?
3. **Plain places.** Plot the following points in the plane: (2, 1), (−1/2, 0), (π, −3). Is there a point in the plane for which you need exactly three numbers to specify its location?
4. **Big stack.** If you take a huge number of sheets of paper and stack them up, what do you get? What is the dimension of the structure you built?
5. **A bigger stack.** If you take a huge number of different 17-dimensional spaces and stack them up, what do you get? Guess its dimension.

II. Solidifying Ideas

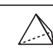
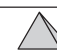
6. **On the level in two dimensions.** Pictured in the chart below are level slices of objects in two dimensions. That is, we took a 2-dimensional object and made several parallel slices with a line at different levels. What are the objects?

	Level 1	Level 2	Level 3	Level 4	Level 5
Object 1	•	• •	• •	• •	•
Object 2	———	• •	• •	• •	•
Object 3	———	• •	• •	• •	———
Object 4	•	•	•	•	•

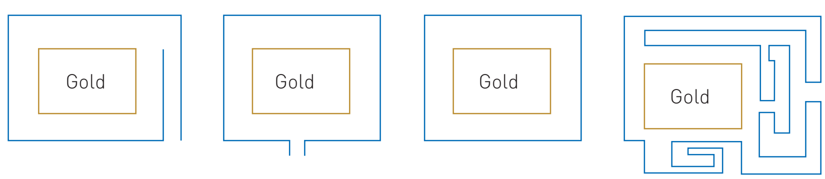
7. **On the level in three dimensions (S).** Pictured in the chart below are level slices of objects in three dimensions. That is, we made several parallel slices with a plane at different levels. What are the objects?

	Level 1	Level 2	Level 3	Level 4	Level 5
Object 1	•	○	◯	○	•
Object 2	•	△	△	△	▲
Object 3	●	○	◯	○	●
Object 4	•	⬭	○ ○	⬭	•

8. **On the level in four dimensions.** Pictured in the chart below are level slices of objects in four dimensions. That is, we made several parallel slices with 3-dimensional space at different levels. What are the objects?

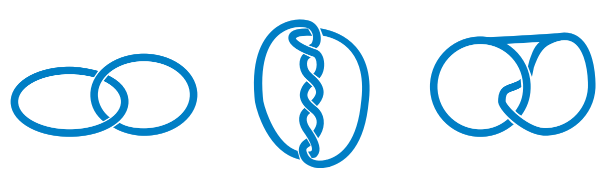
	Level 1	Level 2	Level 3	Level 4	Level 5
Object 1	•	⊖	⊖	⊖	∘
Object 2					
Object 3	•	•	•	•	•
Object 4	•				

9. **Tearible 2's.** In the pictures below, describe how you would remove the gold bar from the various barriers just by bending and moving the walls in the 2-dimensional plane. Indicate which pictures require that you tear the barrier to remove the bar.

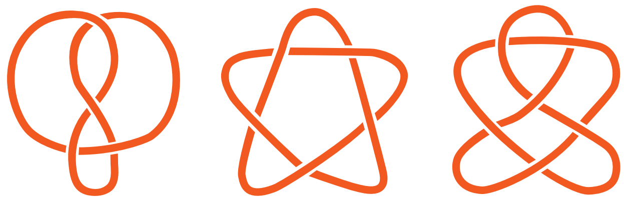


10. **Dare not to tear?** For the figures in the Tearible 2's that required tearing to remove the gold, describe how you could remove the gold without tearing by using the third dimension.

11. **Unlinking (H).** Using the fourth dimension, describe how you would unlink the pictured pairs of objects.



12. **Unknotting.** Describe how you would unknot the following knots using the fourth dimension.



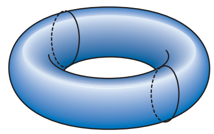
13. **Latitude.** Some 4-dimensional person has an object. She shows you 3-dimensional cross-sectional slices of it. The slices look like a circle in one level of 4-space with increasingly smaller circles at each level above and below until they end at a point at the top and a point at the bottom. What is the object?
14. **Edgy hypercubes (H).** Produce drawings of the regular cube, the 4-dimensional cube, the 5-dimensional cube, and the 6-dimensional cube.
15. **Hypercube computers (ExH).** Parallel processing computers use 4, 5, and higher dimensions by locating a processor at each vertex of the cube. One processor sends information to processors that are attached to it by an edge. The distance between two vertices is defined to be the minimum number of edges required to create a path from one of the vertices to the other. What is the longest distance between vertices on a 3-dimensional cube? 4-dimensional cube? 5-dimensional cube? In general, an n-dimensional cube?

III. Creating New Ideas

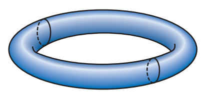
16. ***N*-dimensional triangles (ExH).** We saw how to build cubes in all dimensions; how about triangles? A 0-dimensional triangle is just a point. A 1-dimensional triangle is a line segment; you know what a 2-dimensional triangle looks like; a 3-dimensional triangle is a tetrahedron. What is the pattern? We take the triangle we just created and then add a new point in the next dimension "above" the triangle. If we draw new edges from the vertices of the triangles to our new point, then we have a next higher dimensional triangle. Sketch a 4-dimensional triangle and then a 5-dimensional triangle. Fill in the following table.

Triangle's Dimension	Number of Vertices	Number of Edges	Number of 2-Dimensional Faces	Number of 3-Dimensional Faces
1				
2				
3				
4				
5				
n (in general)				

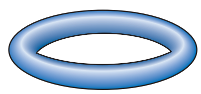
17. **Doughnuts in dimensions.** Suppose we have a mysterious object in four dimensions. If we slice the object in half with a 3-dimensional slice, we'd see the surface of a hollow doughnut.



If we take a 3-dimensional slice just above or below that first slice, we'd see the surface of another doughnut—this one thinner than the first.



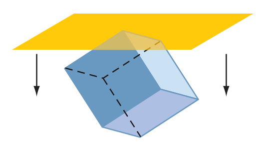
As we slice higher (and lower) we see doughnuts whose waistlines get thinner and thinner, until finally we just see a circle.



What is the 4-dimensional object if we know the level 3-dimensional slices? Why is the answer "a circle of spheres"?

18. **Assembly required (S).** As promised in the preceding section, here is your chance to glue together a 4-cube. Draw a picture of the unfolded 4-dimensional cube as eight 3-dimensional cubes. Indicate in your drawing which faces get glued together to reassemble the 4-cube in four dimensions.

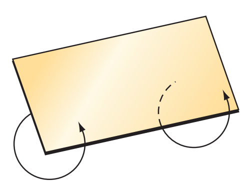
19. **Slicing the cube.** Take a 3-dimensional cube balancing on a vertex and imagine slicing it with many parallel planes starting with this one. Sketch the various types of level curves, that is, cross-sectional slices, we'd see. For example, the first few would look like triangles that are increasing in size. Continue sketching the slices and make sure to include all the shapes we'd see.



Move plane down and slice cube.

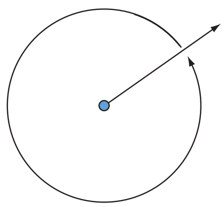
20. **4D swinger.** The plane is just a half line swung around a point. Three-dimensional space is a half plane swung around a line.

    Suppose we make a circle with a dot inside it on the half plane. When we swing the half plane around to make 3-dimensional space, the circle and the point produce objects in 3-dimensional space.

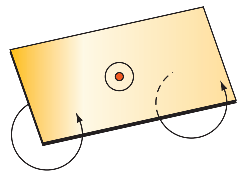


Swing a half plane around a line and make a 3D space.

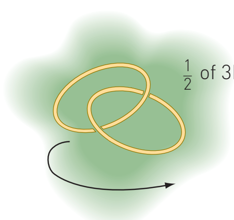
Describe the objects. What pair of objects do we produce if we swing only the point around but keep the circle fixed?



Swing around and sweep out a plane.



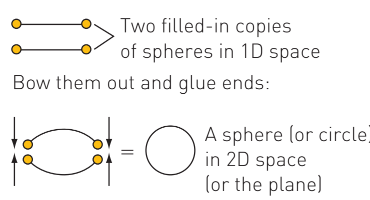
Now what is 4-dimensional space? It is half 3-space swung around a plane. Hard to see? Yes—but try. What do we get if we take a pair of linking circles in the half 3-space and swing it around to make 4-dimensional space? What if we take a sphere in the half 3-space with a point inside? Swing the point around, but leave the sphere still. Do we get a sphere linked with a circle? Explain your answer as best as you can.



IV. Further Challenges

21. Spheres without tears. A sphere is the set of points at a fixed distance from a given point. A sphere in 1-dimensional space is just two points. To make a sphere in 2-dimensional space, we take two copies of the sphere in 1-dimensional space, fill each one in (color in all the points between the points), and then glue the outer edge of one of the filled-in spheres in 1-dimensional space to the outer edge of the other (this requires us to bend each sphere out a bit). This process produces a circle (a sphere in 2-dimensional space). Generalize this procedure to produce a sphere in 3-dimensional space (include pictures), and then use this method to describe how to construct a sphere in 4-dimensional space.

A sphere in 1D space!



22. **Linking.** Start with two linked circles in 3-dimensional space. Make one of the circles the equator of a sphere in 4-dimensional space by constructing increasingly smaller circles as we rise and descend in levels of 4-dimensional space. Can we pull the sphere and circle apart in 4-dimensional space, or are they linked? Explain your answer.

V. In Your Own Words

23. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.
24. **With a group of folks.** In a small group, discuss and actively work through how to untie a knot in 4-dimensional space. After your discussion, write a brief narrative describing the process in your own words.
25. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.
26. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.