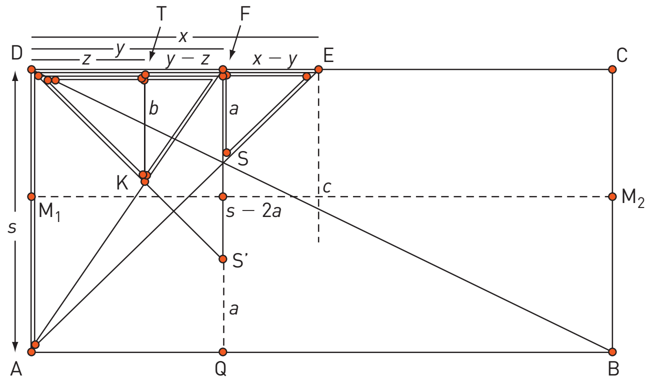


The eighth-grade construction

A Look Back

THE PYTHAGOREAN THEOREM can be proved by physically building with puzzle pieces a square whose sides are the hypotenuse of a right triangle and showing how those pieces can be reassembled into a thickened L shape with area equal to the sum of the squares of the lengths of the other two sides.

The Pythagorean Theorem is an idea that may at first seem abstract, but when we hold and manipulate physical triangles, that abstract idea can become concrete and real. Using several ways of learning about an idea helps us to understand it.

Experience ideas in as many ways as possible.

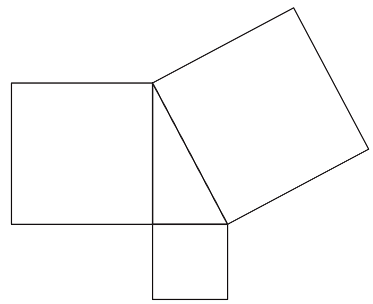
Mindscapes — Invitations to Further Thought

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

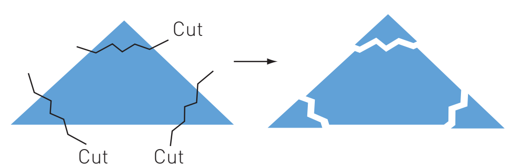
1. **The main event.** State the Pythagorean Theorem.
2. **Two out of three.** If a right triangle has legs of length 1 and 2, what is the length of the hypotenuse? If it has one leg of length 1 and a hypotenuse of length 3, what is the length of the other leg?

3. **Hypotenuse hype.** If a right triangle has legs of length 1 and x, what is the length of the hypotenuse?
4. **Assessing area.** Suppose you know the base of a rectangle has a length of 4 inches and a diagonal has a length of 5 inches. Find the area of the rectangle.
5. **Squares all around.** How does the figure below relate to the Pythagorean Theorem?



II. Solidifying Ideas

6. **Operating on the triangle.** Using a straightedge, draw a random triangle. Now carefully cut it out. Next amputate the angles by snipping through adjacent sides. Now move the angles together so the vertices all touch and the edges meet. What do you conclude about the sum of the angles of a triangle? Try this procedure with triangles having different dimensions.



7. **Excite your friends about right triangles.** Describe the proof of the Pythagorean Theorem to someone who has never seen it before. Try to get him or her inspired and intrigued by math! Record the event and the various reactions.
8. **Easy as 1, 2, 3?** Can there be a right triangle with sides of length 1, 2, and 3? Why or why not? Can you find a right triangle whose side lengths are consecutive natural numbers?
9. **Sky high (S).** On a sunny, warm day, a student decides to fly a kite on the college green just to relax. His kite takes off and soars. He lets all

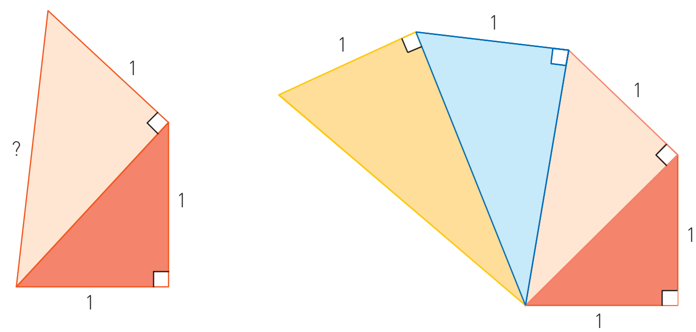
150 feet of the string out and attracts a crowd of onlookers. There is a slight breeze, and a spectator 90 feet away from the student notices that the kite is directly above her. Unlike a real kite, this math-question kite has the string going in a straight line from the student to the kite. How high is the kite from the ground?

10. **Sand masting (H).** The sailboat named Sand Bug has a tall mast. The backstay (the heavy steel cable that attaches the top of the mast to the back, or stern, of the sailboat) is made of 130 feet of cable. The base of the mast is located 50 feet from the stern of the boat. How tall is the mast?

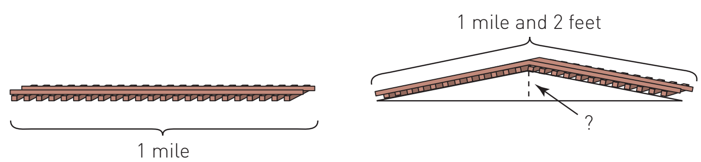
11. **Getting a pole on a bus.** For his 13th birthday, Adam was allowed to travel down to Sarah's Sporting Goods store to purchase a brand new fishing pole. With great excitement and anticipation, Adam boarded the bus on his own and arrived at Sarah's store. Although the collection of fishing poles was tremendous, there was only one pole for Adam and he bought it: a five-foot, one-piece fiberglass "Trout Troller 570" fishing pole. When Adam's bus arrived, the driver reported that Adam could not board the bus with the fishing pole. Objects longer than four feet were not allowed on the bus. In tears, Adam remained at the bus stop holding his beautiful five-foot Trout Troller. Sarah, seeing the whole ordeal, rushed out and said, "Don't cry, Adam! We'll get your fishing pole on the bus!" Sure enough, when the same bus and the same driver returned, Adam boarded the bus with his fishing pole and the driver welcomed him aboard with a smile. How was Sarah able to have Adam board the bus with his five-foot fishing pole without breaking the bus line rules and without cutting or bending the pole?

12. **The scarecrow (ExH).** In the 1939 movie *The Wizard of Oz*, when the brainless scarecrow is given the confidence to think by the Wizard (by merely handing him a diploma, by the way), the first words the scarecrow utters are, "The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side." An isosceles right triangle is just a right triangle having both legs the same length. Suppose that an isosceles right triangle has legs each of length 3. What is the length of the hypotenuse? Is the scarecrow's assertion valid? This question illustrates the true value of a diploma without studying.

13. **Rooting through a spiral.** Start with a right triangle with both legs having length 1. What is the length of the hypotenuse? Suppose we draw a line of length 1 perpendicular to the hypotenuse and then make a new triangle as illustrated. What is the length of this new hypotenuse? Suppose we continue in this manner. Describe a formula for the lengths of all the hypotenuses.



14. **Is it right? (H)** Suppose someone tells you that she has a triangle with sides having lengths 2.6, 8.1, and 8.6. Is this a right triangle? Why or why not? Is there an angle in the triangle larger than 90°? Justify your answer.
15. **Train trouble (H).** Train tracks are made of metal. Consequently, they expand when it's warm and shrink when it's cold. When riding in a train, you hear the clickety-clack of the wheels going over small gaps left in the tracks to allow for this expansion. Suppose you were a beginner at laying railroad tracks and forgot to put in the gaps. Instead, you made a track 1 mile long that was firmly fixed at each end. On a hot day, suppose the track expanded by 2 feet and therefore buckled up in the middle, creating a triangle.



Roughly how high would the midpoint be? Now you may appreciate the click-clack of the railroad track.

III. Creating New Ideas

16. **Does everyone have what it takes to be a triangle?** Suppose a friend comes up to you and says, "Hey, I just made a triangle with sides of length 2431; 5642; and 3210." How would you respond to him? What basic fact about triangles do you conclude from this dialogue with your friend?
17. **Getting squared away.** In our proof of the Pythagorean Theorem, we stated that the second figure is actually two perfect squares touching

along an edge. Prove that they are indeed both perfect squares. It will be useful for you to use the puzzle pieces from your kit to build the first big square again and carefully notice how the pieces all fit together.

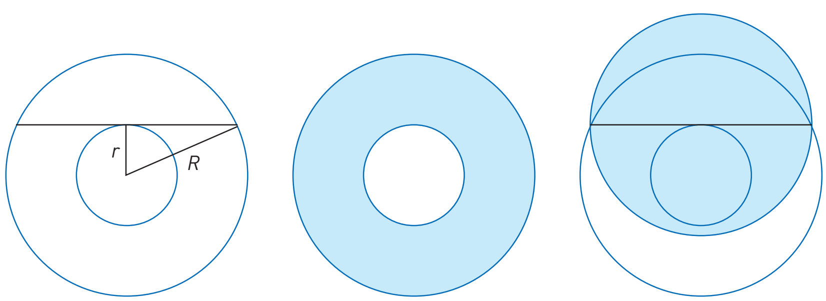
18. The practical side of Pythagoras. Suppose you are building a patio and you want to make certain that the sides of your patio meet at right angles. Using the converse of the Pythagorean Theorem that appears at the end of Mindscape IV.22, give a practical and easy method to check that the angle between two adjacent sides is 90°.

19. Pythagorean pizzas (H). You have a choice at the local pizza place: For the same price you can get either one large pizza or both a small and a medium. How can you determine which way you get more by using just the Pythagorean Theorem and knowing the diameters of the different sizes of pizza? Describe an easy procedure to figure out which deal to choose.

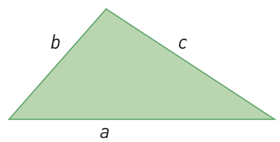
20. Natural right (S). Suppose r and s are any two natural numbers where r is bigger than s. Using the converse of the Pythagorean Theorem that appears at the end of Mindscape IV.22, show that the triangle having side lengths equal to $2rs$, $r^2 - s^2$, and $r^2 + s^2$ is actually a right triangle. Are there infinitely many different right triangles having all sides of integer lengths?

IV. Further Challenges

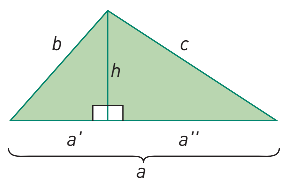
21. Well-rounded shapes. Suppose we have two circles having the same center. The small one has radius r, and the large one has radius R. Let's now consider two shapes. The first is the doughnut-like region between the two circles. The second is a disk whose diameter is the length of the line segment whose endpoints are on the large circle and whose center point touches the small circle at its north pole. What are the areas of these two shaded regions? How do their sizes compare with each other? (We note that the formula for the area of a circle of radius r equals πr^2.)



22. A Pythagorean Theorem for triangles other than right triangles. Suppose we have a triangle that is not a right triangle. For example, consider:



Now, if we drop a perpendicular line from the top vertex down to the side of length a and we cut that side into two pieces of lengths, a' and a'', we would have:



It turns out that $a^2 + b^2 = c^2 + 2aa'$. Use the Pythagorean Theorem with the two right triangles in the preceding picture to produce two equations. Subtract one equation from the other and notice that the h^2 terms drop out. Now, deduce the preceding formula (remember that $a = a' + a''$, so you can solve for a''). Once you have proved this formula, show what happens if the angle between sides a and b is 90°. Notice that you have actually proved the "converse" of the Pythagorean Theorem: If a triangle has sides with lengths a, b, and c satisfying $a^2 + b^2 = c^2$, then it is a right triangle.

V. In Your Own Words

23. With a group of folks. In a small group, discuss and actively work through the proof of the Pythagorean Theorem. After your discussion, write a brief narrative describing the proof in your own words.