Mindscapes *Invitations to Further Thought*

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

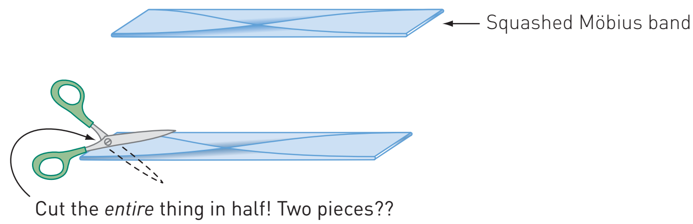
I. Developing Ideas

1. **One side to every story.** What is a Möbius band?
2. **Maybe Möbius.** How can you look at a loop of paper and determine if it's a Möbius band?
3. **Singin' the blues.** Take an ordinary strip of white paper. It has two sides. Color one side blue and leave the other side white. Now use the strip to make a Möbius band. What happens to the blue side and the white side?
4. **Who's blue now?** Take an ordinary strip of white paper. It has two sides. Color one side blue and leave the other side white. Now give one end of the strip two half twists (also know as a full twist). Tape the ends together. Do you get a Möbius band?
5. **Twisted sister.** Your sister holds a strip of paper. She gives one end a half twist, then she gives the other end a half twist in the same direction, then she tapes the ends together. Does she get a Möbius band?

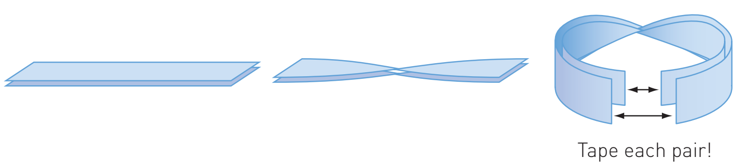
II. Solidifying Ideas

6. **Record reactions.** Explain to a friend how to make a Möbius band and how to cut it up in the various ways described in this section. Make your friend guess the outcomes before he or she does the cutting. Explain why they work as they do. Record your friend's reactions and thoughts.
7. **The unending proof.** Take a strip of paper and write on one side: "Möbius bands have only one side; in fact while." Next turn it over on its long edge and write "reading *The Heart of Mathematics*, I learned that." Now tape the strip to make a Möbius band. Read the band. This provides another proof that the Möbius band has only one side.
8. **Two twists.** Take a strip of paper, put two half twists in it, and glue the ends together. Cut it lengthwise along the center core line. What do you get? Can you explain why?
9. **Two twists again.** Take a strip of paper, put two half twists in it, and glue the ends together. Now cut it lengthwise while hugging the right edge. What do you get?

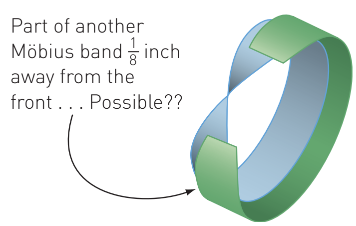
10. **Three twists (H).** Take a strip of paper, put three half twists in it, and glue the ends together. Cut it lengthwise along the center core line. What do you get? Find an interesting object hidden in all that tangle.
11. **Möbius length (S).** Use the method on page 347 for identifying edges of a Möbius band to find out how long a band we get when we cut the Möbius band lengthwise along the center line. Give the length in terms of the length of the original Möbius band.
12. **Möbius lengths.** Use the edge identification diagram of a Möbius band to find the lengths of the two bands we get when we cut the Möbius band by hugging the right edge. Give the lengths in terms of the length of the original Möbius band.
13. **Squash and cut.** Take a Möbius band and squash it flat on the table. Cut it like a buzz saw. What do you get?



14. **Two at once.** Take two strips of paper and put them on top of each other. Twist them together as though you were making a Möbius band, tape the tops together, then tape the bottoms. What do you have? What do you get if you cut it lengthwise down the center core if you keep the two bands together?



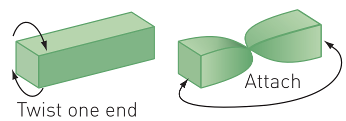
15. **Parallel Möbius.** Is it possible to have two Möbius bands of the same length situated parallel in space so that one hovers over the other exactly 1/8 inch away? Explain why or why not.



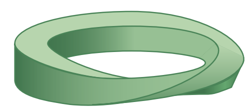
16. **Puzzling.** Suppose you have a collection of jigsaw pieces as shown on page 388. They can be put together to form a strip. Can they be assembled into a Möbius band? Can you explain why or why not?

17. **Möbius triangle.** Make a 1-inch-wide Möbius band, lay it like a circle on a table, and carefully flatten it (thus making three folds). What shape do you have? What is the shortest flattened Möbius band that you can make? Please build one.

18. **Thickened Möbius.** Imagine a Möbius band thickened so the edge is as thick as the side. We'll call this a *thickened Möbius band*. How many edges does it have?



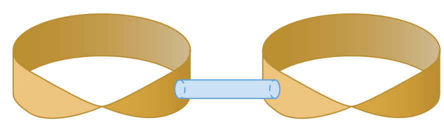
19. **Thickened faces.** How many faces (sides) does a thickened Möbius band have (see Mindscape II.18)?



20. **Thick then thin.** Suppose we take a Möbius band, thicken it to make a thickened Möbius band (see Mindscape II.18), and then shrink the original face to make it an edge. Is this new object a Möbius band?

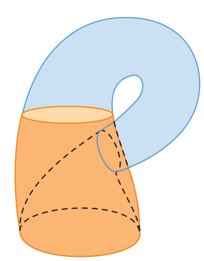
21. **Drawing the band (ExH).** Imagine you have a Möbius band made of cloth with a drawstring around the edge. Can you draw the drawstring completely together? Why or why not?

22. **Tubing (H).** Suppose we take two Möbius bands and make a small hole in each. We then glue to each hole a tube connecting the two bands. How many edges does this new object have? How many sides?



23. **Bug out (ExH).** Suppose you are a ladybug on the "outer" surface of the Klein bottle. Describe a path on the surface of the bottle that you can travel that would get you to the "inside" where a special gentleman bug is waiting.

24. **Open cider.** Consider the Klein bottle half filled with apple cider, as illustrated. Describe how you could pour out a glass of cider without opening the bottle.



25. **Rubber Klein (S).** Suppose you have a rectangular sheet of rubber. Carefully illustrate how you would associate and then glue the edges of the sheet together to build a Klein bottle. Draw a sequence of pictures illustrating the construction.

III. Creating New Ideas

26. One edge. Using the method on page 379 for identifying edges of a Möbius band, prove that the Möbius band has only one edge.

27. Twist of fate (S). Using the edge-identification diagram of the Möbius band, prove that, when you cut a Möbius band lengthwise down the center, you have two half twists.

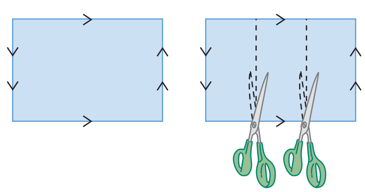
28. Linked together. Using the edge-identification diagram of the Möbius band, prove that, when you cut the Möbius band by hugging the right edge, the two pieces you get are interlocked.

29. Count twists. Using the edge-identification diagram of the Möbius band, determine the number of half twists each band has after you cut the Möbius band while hugging the right edge.

30. Don't cross. Can you draw a curve that does not intersect itself and that goes around a Möbius band three times without crossing over the edge? Experiment and explain your answer.

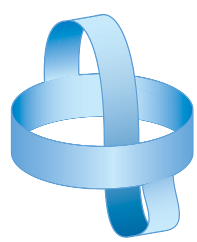
31. Twisted up (H). Suppose you are given a band of paper with a lot of twists in it. How can you tell without counting whether you have an even number of half twists or an odd number? Can you deduce a general fact about what you would have if there are an odd number of half twists and what you would have if there are an even number of half twists? Try. (*Hint:* Experiment by drawing on various physical models.)

32. Klein cut. Consider a rubber rectangular sheet with the edges associated as to create a Klein bottle. Suppose we make two cuts as indicated. How many pieces would we have? What would those pieces look like?



33. Find a band. Find a Möbius band on the surface of a Klein bottle. (The answer to the Rubber Klein story [Mindscape II.25] may help.)

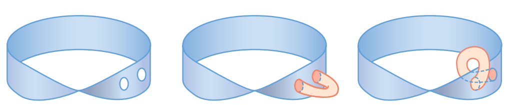
34. Holy Klein. Show that the figure on the right is equivalent to a Klein bottle with a hole.



35. Möbius Möbius. Show that the Klein bottle is two Möbius bands glued together on their edges.

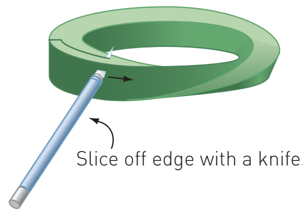
IV. Further Challenges

36. Attaching tubes. Consider a Möbius band with two small holes. Suppose we connect these holes with a tube. We could glue the tube to the holes in two different ways, as illustrated. Are these pictures equivalent? (*Hint:* Slide one end of the tube around the band.)



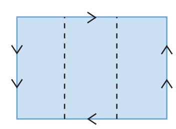
37. Möbius map (H). Using felt tip color pens that soak through both sides of the paper, draw a map on the Möbius band that has five countries, each of which touches the other four; that is, each and every country shares a border with the four other countries. Similarly, draw a map on the Möbius band that has six countries, each of which shares a border with the five other countries.

38. Thick slices. Thicken a Möbius band and then carefully cut (slice) the band along two adjacent sides straddling a common edge. How many pieces are you left with? If there is more than one piece, what is their relationship to one another?



39. Bagel slices. If we take a bagel and slice it in the usual way, we notice that the newly cut face is equivalent to an untwisted looped strip. Suppose we use a peculiar cutting method whereby, instead of cutting along a closed untwisted loop, we cut along a Möbius band. Into how many pieces will our bagel fall? Would it fit in a toaster?

40. Gluing and cutting. Consider a rectangular sheet of rubber with its edges identified, as shown here. Suppose we make two cuts as indicated. How many objects would we be left with? Describe the objects.



V. In Your Own Words

41. Personal perspectives. Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

42. With a group of folks. In a small group, discuss and work through the reasoning for how we get two interlocked strips when we cut the Möbius band 1/3 inch off the right edge. After your discussion, write a brief narrative describing the method in your own words.

43. Creative writing. Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. Power beyond the mathematics. Provide several real-life issues—ideally, from your own experience—that some of the strategies of thought presented in this section would effectively approach and resolve.