Mindscapes *Invitations to Further Thought*

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **Shake 'em up.** What did Georg Cantor do that "shook the foundations of infinity"?

2. **Detecting digits.** Here's a list of three numbers between 0 and 1:

   0.12345

   0.24242

   0.98765

   What's the first digit of the first number? What's the second digit of the second number? What's the third digit of the third number?

3. **Delving into digits.** Consider the real number

   0.12345678910111213141516. . . .

   Describe in words how this number is constructed. What's its 14th digit? What's the 25th digit? What's the 31st digit?

4. **Undercover friend (ExH).** Your friend gives you a list of three, five-digit numbers, but she only reveals one digit in each:

   3????

   ?8???

   ??2??

   Can you describe a five-digit number you know for certain will not be on her list? If so, give one; if not, explain why not.

5. **Underhanded friend.** Now your friend shows you a new list of three, five-digit numbers, again with only a few digits revealed:

   6????

   ?5???

   ?????

   Can you describe a five-digit number you know for certain will not be on her list? If so, give one; if not, explain why not.

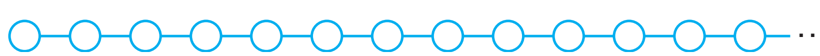
II. Solidifying Ideas

6. **Dodgeball.** Revisit the game of Dodgeball from Chapter 1, "Fun and Games." Play it several times with several people. Get the strategy down, and then explain to your opponents the underlying principle. Record the results of the games.

7. **Don't dodge the connection (S).** Explain the connection between the Dodgeball game and Cantor's proof that the cardinality of the reals is greater than the cardinality of the natural numbers.
8. **Cantor with 3's and 7's.** Rework Cantor's proof from the beginning. This time, however, if the digit under consideration is 3, then make the corresponding digit of M a 7; and if the digit is not 3, make the associated digit of M a 3.
9. **Cantor with 4's and 8's.** Rework Cantor's proof from the beginning. This time, however, if the digit under consideration is 4, then make the corresponding digit of M an 8; and if the digit is not 4, make the associated digit of M a 4.
10. **Think positive.** Prove that the cardinality of the positive real numbers is the same as the cardinality of the negative real numbers. (*Caution:* You need to describe a one-to-one correspondence; however, remember that you cannot list the elements in a table.)
11. **Diagonalization.** Cantor's proof is often referred to as "Cantor's diagonalization argument." Explain why this is a reasonable name.
12. **Digging through diagonals.** First, consider the following infinite collection of real numbers. Describe in your own words how these numbers are constructed (that is, describe the procedure for generating this list of numbers). Then, using Cantor's diagonalization argument, find a number not on the list. Justify your answer.

$$\begin{gathered}0.12345678910111213141516171 8\ldots\\0.2468101214161820222426283032\ldots\\0.3691215182124273033363942 45\ldots\\0.4812162024283236404448525660\ldots\\0.5101520253035404550556065 70\ldots\\\vdots\end{gathered}$$

13. **Coloring revisited (ExH).** In Mindscape III.35 of the previous section we considered the following infinite collection of circles and all the different ways of coloring the circles with either red or blue markers. Show that the set of all possible circle colorings has a greater cardinality than the set of all natural numbers.



14. **A penny for their thoughts.** Suppose you had infinitely many people, each one wearing a uniquely numbered button: 1, 2, 3, 4, 5, . . . (you can use all the people in the Hotel Cardinality if you don't know enough people yourself). You also have lots of pennies (infinitely many, so you're *really* rich; but don't try to carry them all around

at once). Now you give each person a penny; then ask everyone to flip his or her penny at the same time. Then ask them to shout out in order what they flipped (H for heads and T for tails). So you might hear: HHTHHHTTTHTTHTHTHTHHHHTH. . . or you might hear THTTTHTHHTTHTHTTTTTHHHTHTHTH. . . , and so forth. Consider the set of all possible outcomes of their flipping (all possible sequences of H's and T's). Does the set of possible outcomes have the same cardinality as the natural numbers? Justify your answer.

15. **The first digit (H).** Suppose that, in constructing the number *M* in the Cantor diagonalization argument, we declare that the first digit to the right of the decimal point of *M* will be 7, and then the other digits are selected as before (if the second digit of the second real number has a 2, we make the second digit of *M* a 4; otherwise, we make the second digit a 2, and so on). Show by example that the number *M* may, in fact, be a real number on our list.

III. Creating New Ideas

16. **Ones and twos (H).** Show that the set of all real numbers between 0 and 1 just having 1's and 2's after the decimal point in their decimal expansions has a greater cardinality than the set of natural numbers. (So, the number 0.112111122212122211112. . . is a number in this set, but 0.1161221212122122. . . is not, because it contains digits other than just 1's and 2's.)
17. **Pairs (S).** In Cantor's argument, is it possible to consider pairs of digits rather than single digits? That is, suppose we look at the first two digits of the first real number on our list; and, if they are not 22, then we make the first two digits of *M* be 22. If the first two digits are 22, then we make the first two digits of *M* be 44. Similarly, let the next two digits of the next real number on our list determine the next two digits of *M*, and so on. If this procedure would still produce a number *M* not on our list, then provide the details for such a method. If this procedure does not work, explain or illustrate why it does not.
18. **Three missing.** Given a list of real numbers, as in the Cantor argument, explain how to construct three different real numbers that are not on the list.
19. **No Vacancy (H).** Recall the Hotel Cardinality, described in Mindscapes II.16, II.17, and II.18 of the previous section. Create a collection of people so that it would be impossible for the night manager to give each person a room. Thus, for a really big group of people, a No Vacancy sign (or actually a Not Enough Room sign) might actually be necessary. Explain why it is not possible to give each person from your group a room.

20. **Just guess.** This is just a "guessing question." Do you think there are sets whose cardinality is actually larger than that of the set of real numbers? Or, do you think the infinity of reals is the largest infinity? Just make a guess and informally explain it.

IV. Further Challenges

21. **Nines.** Would Cantor's argument work if we used 2 and 9 instead of 2 and 4 as the digits? That is, for each digit we ask whether the digit is a 2. If it is 2, we make the analogous digit of M a 9, and otherwise we make the digit a 2. Using this method, we are not guaranteed that the number M we construct is not on our list. Provide a scenario in which the constructed number is on the list! (*Hint:* Remember from Chapter 2 that $0.1999999\ldots = 0.2$. The number 9 is key here.)
22. **Missing irrational.** Could you modify the diagonalization procedure so that the missing real you produce is a rational number? How could you modify the diagonalization argument so that the missing real number you produce is an irrational number? (*Hint*: Using the construction in this section, each digit of M is a 2 or a 4. Modify the construction so the 10th place, the 100th place, the 1000th place, and so on, are either a 3 or a 5, whereas the rest are 2's and 4's. Why will such a number be irrational?)

V. In Your Own Words

23. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.
24. **With a group of folks.** In a small group, discuss and work through Cantor's argument showing that the set of real numbers has a greater cardinality than the set of natural numbers. After your discussion, write a brief narrative describing the argument in your own words.