To speak algebraically, Mr. M. is execrable, but Mr. C. is (x + 1)-ecrable.

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Understand simple things deeply.

Mindscapes *Invitations to Further Thought*

In this section, Mindscapes marked **(H)** *have hints for solutions at the back of the book. Mindscapes marked* **(ExH)** *have expanded hints at the back of the book. Mindscapes marked* **(S)** *have solutions.*

I. Developing Ideas

1. **Muchos mangos.** You inherit a large crate of mangos. The top layer has 18 mangos. Peering through the cracks in the side of the crate, you estimate there are five layers of mangos inside. About how many mangos did you inherit?
2. **Packing balls.** Your best friend is about to turn 21 and you want to send him a box full of Ping-Pong balls. You have a square box measuring 12 inches on each side and you wonder how many Ping-Pong balls would fit inside. Suppose you have just enough balls to cover the bottom of the box in a single layer. How could you estimate the number that would fill the box?
3. **Alternative rock.** You have an empty CD rack consisting of five shelves and you just bought five totally kickin' CDs. Can each CD go on a different shelf? What if you had six new CDs?
4. **The Byrds.** You have 16 new CDs to put on your empty five-shelf CD rack. Can you place the CDs so that each shelf contains three or fewer CDs? Can you arrange them so that each shelf contains exactly three?
5. **For the birds.** Explain the Pigeonhole principle.

II. Solidifying Ideas

6. **Treasure chest (ExH).** Someone offers to give you a million dollars ($1,000,000) in one-dollar ($1) bills. To receive the money, you must lie down; the million one-dollar bills will be placed on your stomach.

If you keep them on your stomach for 10 minutes, the money is yours! Do you accept the offer? Carefully explain your answer using quantitative reasoning.

7. **Order please.** Order the following numbers from smallest to largest: number of telephones on the planet; number of honest members of Congress; number of people; number of grains of sand; number of states in the United States; number of cars.

8. **Penny for your thoughts (H).** Two thousand years ago, a noble Arabian king wished to reward his minister of science. Although the modest minister resisted any reward from the king, the king finally forced him to state a desired reward. Impishly the minister said that he would be content with the following token: "Let us take a checkerboard. On the first square I would be most grateful if you would place one piece of gold. Then on the next square twice as much as before, thus placing two pieces, and on each subsequent square, placing twice as many pieces of gold as in the previous square. I would be most content with all the gold that is on the board once your majesty has finished." This sounded extremely reasonable, and the king agreed. Given that there are 64 squares on a checkerboard, roughly how many pieces of gold did the king have to give to our "modest" minister of science? Why did the king have him executed?

9. **Twenty-nine is fine.** Find the most interesting property you can, unrelated to size, that the number 29 has and that the number 27 does not have.

10. **Perfect numbers.** The only natural numbers that divide evenly into 6, other than 6 itself, are 1, 2, and 3. Notice that the sum of all those numbers equals the original number 6 ($1 + 2 + 3 = 6$). What is the next number that has the property of equaling the sum of all the natural numbers other than itself that divide evenly into it? Such numbers are called *perfect numbers*. No one knows whether or not there are infinitely many perfect numbers. In fact, no one knows whether there are *any* odd perfect numbers. These two unsolved mysteries are examples of long-standing open questions in the theory of numbers.

11. **Many fold (S).** Suppose you were able to take a large piece of paper of ordinary thickness and fold it in half 50 times. What would the height of the folded paper be? Would it be less than a foot? About one yard? As long as a street block? As tall as the Empire State Building? Taller than Mount Everest?

12. **Only one cake.** Suppose we had a room filled with 370 people. Will there be at least two people who have the same birthday?

13. **For the birds.** Years ago, before overnight delivery services and e-mail, people would send messages by carrier pigeon and would keep an ample supply of pigeons in pigeonholes on their rooftops.

Suppose you have a certain number of pigeons, let's say P of them, but you have only $P - 1$ pigeonholes. If every pigeon must be kept in a hole, what can you conclude? How does the principle we discussed in this section relate to this question?

14. **Sock hop (ExH).** You have 10 pairs of socks, five black and five blue, but they are not paired up. Instead, they are all mixed up in a drawer. It's early in the morning, and you don't want to turn on the lights in your dark room. How many socks must you pull out to guarantee that you have a pair of one color? How many must you pull out to have two good pairs (each pair is the same color)? How many must you pull out to be certain you have a pair of black socks?
15. **The last one.** Here is a game to be played with natural numbers. You start with any number. If the number is even, you divide it by 2. If the number is odd, you triple it (multiply it by 3), and then add 1. Now you repeat the process with this new number. Keep going. You win (and stop) if you get to 1. For example, if we start with 17, we would have:

$$17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 \rightarrow \text{we see a 1, so we win!}$$

Play four rounds of this game starting with the numbers 19, 11, 22, and 30. Do you think you will always win no matter what number you start with? No one knows the answer!

III. Creating New Ideas

16. **See the three.** What proportion of the first 1000 natural numbers have a 3 somewhere in them? For example, 135, 403, and 339 all contain a 3, whereas 402, 677, and 8 do not.
17. **See the three II (H).** What proportion of the first 10,000 natural numbers contain a 3?
18. **See the three III.** Explain why almost all million-digit numbers contain a 3.
19. **Commuting.** One hundred people in your neighborhood always drive to work between 7:30 and 8:00 a.m. and arrive 30 minutes later. Why must two people always arrive at work at the same time, within a minute?
20. **RIP (S).** The Earth has more than 6.8 billion people and almost no one lives 100 years. Suppose this longevity fact remains true. How do you know that some year soon, more than 50 million people will die?

IV. Further Challenges

21. Say the sequence. The following are the first few terms in a sequence. Can you figure out the next few terms and describe how to find all the terms in the sequence?

1
11
21
1211
111221
312211
. . .

22. Lemonade. You want to buy a new car, and you know the model you want. The model has three options, each one of which you can either take or not take, and you have a choice of four colors. So far 100,000 cars of this model have been sold. What is the largest number of cars that you can guarantee to have the same color and the same options as each other?

V. In Your Own Words

23. With a group of folks. In a small group, discuss and work through the reasoning for why there are two people on Earth having the same number of hairs on their bodies. After your discussion, write a brief narrative describing your analysis and conclusion in your own words.