Method of Lagrangian Multipliers.

Inequality constraints.

Prob: Minimize $f(x)$

subject to $g_j(x) \leq 0$; $j = 1, 2, \ldots, m$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \leftarrow$ decision vector.

$x_1, \ldots, x_n$: decision variables.

After introducing 'm' – positive slack variables

we get, $g_j(x) + s_j^2 = 0$; $j = 1, 2, \ldots, m$

Consider the Lagrangian fn.,

$L(x_1, \ldots, x_n, s_1, \ldots, s_m, \lambda_1, \ldots, \lambda_m) =$

$f(x) + \sum_{j=1}^{m} \lambda_j \left(g_j(x) + s_j^2\right)$ where

$\lambda_1, \lambda_2, \ldots, \lambda_m$ are Lagrange multipliers.

Necessary Conditions for its local/relative minima

Karush-Kuhn-Tucker or KKT-Conditions

$\dfrac{\partial L}{\partial x_i} = 0 \Rightarrow \dfrac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \dfrac{\partial g_j(x)}{\partial x_i} = 0$; $i = 1, 2, \ldots, n$ (Optimality)

$\dfrac{\partial L}{\partial \lambda_j} = 0 \Rightarrow g_j(x) + s_j^2 = 0$; $j = 1, 2, \ldots, m$

$\Rightarrow g_j(x) \leq 0$; $j = 1, \ldots, m$ (Feasibility)

$\dfrac{\partial L}{\partial s_j} = 0 \Rightarrow 2\lambda_j s_j = 0 \Rightarrow \lambda_j s_j = 0$; $j = 1, 2, \ldots, m$ (Complementary slackness)

$\lambda_j \geq 0$ (Non-negativity)

At a pt. x^* (Local minima)

Complementary slackness condition

$$\Rightarrow \lambda_j s_j = 0 \begin{cases} \lambda_j = 0, s_j \neq 0 & \text{(Case-I)} \\ \lambda_j \neq 0, s_j = 0 & \text{(Case-II)} \end{cases}$$

(Case I)

$\lambda_j = 0, s_j \neq 0$ & feasibility condition;

$$g_j(x^*) + s_j^2 = 0 \Rightarrow \underbrace{g_j(x^*) \leq 0}_{\text{inactive constraints}}$$

(Case II)

$\lambda_j \neq 0, s_j = 0$ & feasibility condition;

$$g_j(x^*) + s_j^2 = 0 \Rightarrow \underbrace{g_j(x^*) = 0}_{\text{active constraints.}}$$

For active constraints $\lambda_j > 0$ & for inactive constraints $\lambda_j = 0$. (For a minimization problem)

Opposite will be for a maximization problem.

~~Exercise~~

So KKT conditions are the necessary conditions.

Sufficient conditions (Minimization)

The KKT conditions are sufficient conditions if $f(x)$ is convex, and the feasible space is convex

In general,

$$L = f(x) + \sum_{j=1}^{m} \lambda_j (g_j(x) + s_j^2)$$

$\lambda_j \geq 0$ & x^* is local minimum

if $\nabla^2 L|_{x^*}$ is positive definite.

Exercise:

1). Minimize $f(x) = x_1^2 - 4x_1 + x_2^2 - 6x_2$

subject to $x_1 + x_2 \leq 3$

$-2x_1 + x_2 \leq 2$.

2). Maximize $f(x) = 3x + 4y$

subject to $x^2 + y^2 \leq 4$

$x \geq 1$.

3). Minimize $f(x) = x_1^2 + x_2$

subject to $x_1^2 + x_2^2 \leq 9$

$x_1 + x_2 \leq 1$.

show that the pt. (1, 0) doesn't satisfy the KKT conditions.

4). Minimize $f(x)$

subject to

$x_1 + x_2 = 90$

$f(x) = 6x_1^2 + 12x_2^2$.

- Use Newton's method to minimize the Powell f^n :

$$f(x_1,x_2,x_3,x_4) = (x_1+10x_2)^2+5(x_3-x_4)^2+(x_2-2x_3)^4+10(x_1-x_4)^4$$

Use the starting point $x^{(0)} = (3,-1,0,1)^T$. Compute for three iterations. You can take help of softwares for computing matrix inverses & matrix multiplications.

- Find the maximum value of

$$f(x) = 2\sin x - \frac{x^2}{10}$$ with an initial guess of $x^{(0)} = 2.5$. (Use Newton's method)

- How many basic sol^{n}s are there in the following linearly independent set of eq^{n}s? Find all of them.

$$2x_1 - x_2 + 3x_3 + x_4 = 6 \quad ; \quad x_1, x_2, x_3, x_4 \geq 0.$$
$$4x_1 - 2x_2 - x_3 + 2x_4 = 10$$

- Solve graphically the LPP

Min $Z = -2x_1 + x_2$

subject to $x_1 + x_2 \geq 6,\ 3x_1 + 2x_2 \geq 16,\ x_2 \leq 9\ ;\ x_1, x_2 \geq 0.$

- Solve graphically the LPP

Max $Z = 4x_1 + 7x_2$

subject to
$$12x_1 + 7x_2 \leq 42$$
$$5x_1 + 4x_2 \leq 20$$
$$2x_1 + 3x_2 \geq 6 \quad ; \quad x_1, x_2 \geq 0.$$