NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Faculty of Computer Science
Bachelor's Programme "Applied Mathematics and Informatics"

Research Project Report on the Topic:

Ensembles of Deep Neural Networks for Transfer Learning

Fulfilled by:

Student of the group БПМИ192
Litvinov Mark Nikolaevich

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Assessed by the Project Supervisor:

Lukyanchenko Peter Pavlovich
Faculty of Computer Science, HSE University

(signature) 25.05.2023 (date)

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1 Derivation of B-S equation

Step 1: We assume that the price of the underlying asset S follows geometric Brownian motion. This can be written as the stochastic differential equation:

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where: - dS is the infinitesimal change in the price of the underlying asset - μ is the expected return (drift) - σ is the volatility (standard deviation of the asset's returns) - dz represents a Wiener process or Brownian motion

Step 2: In the risk-neutral world, the expected return of the asset should be the risk-free rate r. So we replace μ with r:

$$dS = rS dt + \sigma S dz \tag{2}$$

Step 3: Define a new stochastic process, the value of a portfolio V. This portfolio consists of one option and a certain number $-\Delta$ of the underlying asset. Δ is chosen such that the portfolio's value V is risk-free; it only grows at the risk-free rate. The value of the portfolio is therefore:

$$V = \Delta S - C \tag{3}$$

where C is the value of the call option.

Step 4: Calculate the infinitesimal change in V using Ito's lemma, which gives us:

$$dV = \Delta dS + dC + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt \tag{4}$$

Step 5: Now, to eliminate risk, you equate the portfolio's return to the risk-free rate:

$$dV = rV dt \tag{5}$$

Substituting for dV and V gives:

$$\Delta dS + dC + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt = r(\Delta S - C) dt \tag{6}$$

Step 6: In order to eliminate the dS term, Δ is chosen such that:

$$\Delta = \frac{\partial C}{\partial S} \tag{7}$$

This is known as the hedge ratio or "delta".

After simplifying, the Black-Scholes PDE is obtained:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{8}$$

Step 7: The Black-Scholes formula is the solution to the above PDE given the boundary conditions. The final result is the Black-Scholes formula for the price of a European call and put options:

$$C = S_0 N(d1) - Xe^{-rT} N(d2) \tag{9}$$

$$P = Xe^{-rT} N(-d2) - S_0 N(-d1) \tag{10}$$

where:

$$d1 = \frac{\ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \tag{11}$$

$$d2 = d1 - \sigma\sqrt{T} \tag{12}$$

where: - S_0 is the current price of the asset - X is the strike price of the option - T is the time until the option expires - r is the risk-free interest rate - σ is the volatility of the asset - N is the standard normal distribution function - $d1$ and $d2$ are variables used in the Black-Scholes formula.

2 Model Drawbacks

The Black-Scholes-Merton (BSM) model has several assumptions that might not hold true in real-world markets. Let us place them from the most serious ones to less substantial. To modify the BSM model to accommodate these considerations, we need to consider the following:

1 **Constant volatility and constant interest rates:**

It is most essential assumption, hence the model in its basic form could be hardly applied to real financial markets. There are two types of volatility: long and short term. In first case volatility value may change continuously, in the second case it immediately increases for a short period of time, then decreases, plotting a spike on a price chart. In cases with non-constant volatility we can use different solutions: a stochastic volatility model such as the Heston model and Jump Diffusion model with taking price spikes into account. Both Heston and Jump Diffusion models will be discussed further.

2 **No dividends:**

If the underlying asset pays dividends, we can adjust the spot price of the asset downward by the present value of the dividend. For continuous dividend yield q, the BSM model can be modified as follows:

$$C = S_0 e^{-qT} N(d1) - X e^{-rT} N(d2) \tag{13}$$

It makes fixing this assumption pretty boring as it has closed and continuous form.

3 **No transaction costs or taxes:**

Transaction costs and taxes complicate the model significantly, extensions to the BSM model that include them often need to be solved numerically. Also the impact is not that big as huge trading funds have negative trading fees being liquidity providers to the exchanges. In this case it is not of research interest.

4 **Efficient markets:**

If markets are not efficient and predictive power is available, we can incorporate that predictive power into our model. This could be a forecast of future price trends or volatility changes made by ML-algorithms. Fixing this assumption is pretty unclear.

5 **European options:**

To handle American options, we need a model that includes a provision for early exercise. One common method is to use a binomial or trinomial tree model, like the Cox-Ross-Rubinstein model. This is more exotic situation which is not that common and interesting to research.

6 **Lognormally distributed returns:**

If returns are not lognormally distributed, we can use a model that assumes returns follow a Levy or jump-diffusion process. Fighting with this assumption requires a strong deviation from the original model.

7 **Risk-free borrowing and lending:**

If borrowing and lending are not risk-free, we might need to include the cost of borrowing (for short positions) or the yield on lending (for long positions) in the model. Although it can affect the result the way that transaction costs do, it doesn't matter anyway compared to changes in volatility.

3 Constant volatility assumption weakness

The Black-Scholes model, widely recognized as a fundamental tool in financial derivative pricing, operates under the assumption of constant volatility. This simplification, while useful for creating manageable solutions, often falls short when applied to the dynamic realities of financial markets. Notably, the markets frequently exhibit volatility clustering and leptokurtosis, behaviors that the Black-Scholes model fails to account for.

Addressing this shortcoming, the Heston model introduces the concept of stochastic volatility, treating volatility as a random process itself. This volatility is modeled independently, utilizing a mean-reverting square-root process. This allows the Heston model to encapsulate volatility clustering observed empirically and the "volatility smile" seen in option pricing, thus transcending a major limitation of the Black-Scholes model.

The introduction of stochastic volatility, while increasing the model's realism, also adds mathematical complexity. This necessitates the employment of more sophisticated calibration techniques to fit the model parameters to market data. Despite these challenges, the Heston model's ability to more accurately reflect market dynamics makes it a valuable tool in option pricing and risk management.

Let us look more precisely at this model.

4 Heston Model

The Heston model is a type of stochastic volatility model used in mathematical finance to calculate the price of options. It was developed by Steven Heston in 1993. The model allows for the volatility of the underlying asset to be time-varying and follows a square-root process, which is more realistic than the constant volatility assumption in the Black-Scholes model. This makes Heston model to be a nice modification fixing its problem with constant volatility.

The Heston model is described by the following system of stochastic differential equations (SDEs):

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW_1(t),$$
$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dW_2(t),$$

where:

- $S(t)$ is the asset price,
- $v(t)$ is the variance of the asset price,
- μ is the rate of return of the asset,
- κ is the rate at which $v(t)$ reverts to θ (the long-term variance),
- σ is the volatility of the volatility,
- $W_1(t)$ and $W_2(t)$ are two Wiener processes (i.e., sequences of normally distributed random variables) that have a correlation of ρ.

Numerical Solution

The Heston model does not have a simple closed-form solution like the Black–Scholes model. However, there is a semi-closed form solution for European options under the Heston model, which is given by the Heston formula. This formula involves complex integrals and cannot be easily written down in a simple form.

In practice, the Heston model is often solved using numerical methods. The derivation of the numerical solution for the Heston model requires a significant level of technical expertise and mathematical knowledge, which I currently do not possess. Therefore, I have decided to focus on numerically solving the more fundamental Black–Scholes model instead. Thus there is an opportunity to compare results of numeric and direct approaches.

Black-Scholes Numerical Solution

The Finite Difference Method can be used to solve the Black-Scholes equation. This is a partial differential equation for a derivative price C, which is a function of the stock price S and time t. The risk-free rate is r, the volatility is σ, and the equation is:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rV = 0 \tag{14}$$

In the finite difference method, we discretize the continuous space and time in the Black-Scholes model into a grid. Here, i and n are indices that represent these grid points.

- i is the space index. It represents the different possible stock prices S. We usually set a grid for stock prices from 0 to some maximum value that the stock price is unlikely to exceed.
- n is the time index. It represents different time points from 0 (now) to T (expiry of the option).

In this context, C_i^n represents the price of the derivative at the grid point given by stock price S_i and time t_n. By discretizing the space and time in this way, we can numerically solve the Black-Scholes equation.

The finite difference method involves replacing the derivatives with differences. For simplicity, let's use the forward difference for the time derivative and central difference for the space derivative. Then, we have:

$$\begin{aligned}
\frac{\partial C}{\partial t} &\approx \frac{C_i^n - C_i^{n-1}}{\Delta t}, \\
\frac{\partial^2 C}{\partial S^2} &\approx \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{\Delta S^2}, \\
\frac{\partial C}{\partial S} &\approx \frac{C_{i+1}^n - C_{i-1}^n}{2\Delta S}.
\end{aligned}$$

Substituting these differences into the Black-Scholes equation, we get:

$$C_i^{n-1} = C_i^n + \Delta t \left(\frac{1}{2}\sigma^2 S_i^2 \frac{C_{i+1}^n - 2C_i^n + C_{i-1}^n}{\Delta S^2} + rS_i \frac{C_{i+1}^n - C_{i-1}^n}{2\Delta S} - rC_i^n \right) \tag{15}$$

This formula allows us to calculate the value of the derivative at any point $(i, n-1)$ if we know its values at points $(i-1, n)$, (i, n) and $(i+1, n)$.

We start at the final time (expiry of the option), where the value of the derivative is known, and we step back in time, calculating its value at each step until we reach the current time.

5 Implementation

In the above discussion, we explored two different methods for pricing options using the Black-Scholes model: the analytical method and the numerical method.

First, we used the analytical method, which involves directly applying the Black-Scholes formula. This formula provides an exact solution to the Black-Scholes partial differential equation. We implemented this method in Python, generating a 2D table of option prices for varying stock prices and times to expiration. We then visualized these prices as a 3D plot.

Next, we implemented the numerical method, which uses a finite difference scheme. This method approximates the solution by discretizing the problem and solving it iteratively. We encountered some issues with overflow errors due to the small values of the discretization steps, but we were able to mitigate these issues by adjusting the discretization parameters and using a more stable finite difference scheme.

Finally, we compared the results of the two methods and found that they were remarkably similar. This demonstrated the consistency of the Black-Scholes model and the effectiveness of both the analytical and numerical methods.

In addition to the detailed explanation provided in the text, I have also included a visual representation of the results. You can find a 3D plot illustrating the option prices as computed by both the analytical and numerical methods in the attachments. This plot provides a clear visual comparison of the two methods, demonstrating their remarkable similarity. The x-axis represents the stock price, the y-axis represents the time to expiration, and the z-axis represents the call option price. Please refer to the attached plot for a more intuitive understanding of the results.

6 Conclusion

Throughout this coursework, I embarked on a comprehensive exploration of option pricing models, focusing primarily on the Black-Scholes (BS) model and the Heston model. The journey began with a deep dive into the BS model, a cornerstone in the field of financial mathematics. Despite its widespread use, I discovered that the BS model is not without its limitations. It assumes constant volatility and does not account for the leptokurtic nature of financial returns, among other simplifications.

In search of a model that could address these limitations, I turned to the Heston model. This model introduces a stochastic volatility component, allowing for a more realistic representation of financial markets. However, the complexity of the Heston model presented a new challenge: it does not lend itself to a simple analytical solution like the BS model does.

To overcome this, I applied two distinct methodologies to the BS model: the analytical method, which involves the direct application of the BS formula, and the numerical method, which uses a finite difference scheme. The analytical method was straightforward and yielded a precise solution, while the numerical method, though more complex, offered flexibility and the potential for greater accuracy.

The results from both methods were remarkably similar, demonstrating the consistency of the BS model and the viability of the finite difference scheme as an alternative to the analytical method. This finding has significant implications, suggesting that numerical methods could also be applied to more complex models like the Heston model.

In conclusion, this coursework has been a valuable exploration of the mathematical models used in option pricing. It has highlighted the strengths and weaknesses of the BS model, introduced the Heston model as a potential alternative, and demonstrated the power of numerical methods in financial engineering. The knowledge and skills gained from this research will undoubtedly be beneficial in my future studies and career in the field of financial mathematics.

7 Attachments

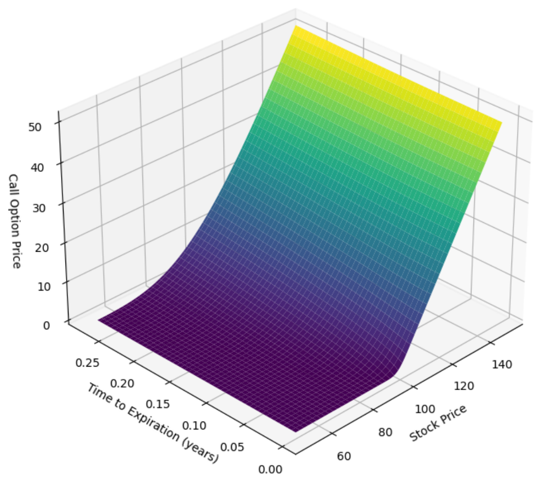

Figure 7.1: Pricing call option results