Linear Algebra

Tutorial

Inner Product, Outer Product and Norms

(1) *Angle between two nonnegative vectors.* Let x and y be two nonzero n-vectors with nonnegative entries, *i.e.*, each $x_i \geq 0$ and each $y_i \geq 0$. Show that the angle between x and y lies between 0 and 90°. Draw a picture for the case when $n = 2$, and give a short geometric explanation. When are x and y orthogonal?

Coding:

- Initialize two vectors(2-D) with nonnegative entries and plot them.(use matplotlib)
- Modify the entries of vectors and observe the angle between the vectors.

(2) *Distance between Boolean vectors.* Suppose that x and y are Boolean n-vectors, which means that each of their entries is either 0 or 1. What is their distance $\|x - y\|$?

Coding:

- Initialize two vectors with boolean entries(0s and 1s)
- Calculate the distance between them
- Calculate hamming distance (Use scipy)

(3) *Reverse triangle inequality.* Suppose a and b are vectors of the same size. The triangle inequality states that $\|a+b\| \leq \|a\| + \|b\|$. Show that we also have $\|a+b\| \geq \|a\| - \|b\|$. *Hints.* Draw a picture to get the idea. To show the inequality, apply the triangle inequality to $(a+b)+(-b)$.

Coding:

- Initialize two random vectors
- Verify triangle inequality and reverse triangular inequality

(4) *When is the outer product symmetric?* Let a and b be n-vectors. The inner product is symmetric, *i.e.*, we have $a^T b = b^T a$. The outer product of the two vectors is generally *not* symmetric; that is, we generally have $ab^T \neq ba^T$. What are the conditions on a and b under which $ab = ba^T$? You can assume that all the entries of a and b are nonzero. (The conclusion you come to will hold even when some entries of a or b are zero.) *Hint.* Show that $ab^T = ba^T$ implies that a_i/b_i is a constant (*i.e.*, independent of i).

Matrices

(5) *Block matrix.* Assuming the matrix

$$K = \begin{bmatrix} I & A^T \\ A & 0 \end{bmatrix}$$

makes sense, which of the following statements must be true? ('Must be true' means that it follows with no additional assumptions.)

(a) K is square.

(b) A is square or wide.

(c) K is symmetric, *i.e.*, $K^T = K$.

(d) The identity and zero submatrices in K have the same dimensions.

(e) The zero submatrix is square.

(6) *Matrix sizes.* Suppose A, B, and C are matrices that satisfy $A + BB^T = C$. Determine which of the following statements are necessarily true. (There may be more than one true statement.)

(a) A is square.

(b) A and B have the same dimensions.

(c) A, B, and C have the same number of rows.

(d) B is a tall matrix.

(7) *Multiplication by a diagonal matrix.* Suppose that A is an $m \times n$ matrix, D is a diagonal matrix, and $B = DA$. Describe B in terms of A and the entries of D. You can refer to the rows or columns or entries of A.

Coding:

- Create a matrix A and diagonal matrix D
- Calculate B = DA
- Observe the entries of B

(8) **Trace of a square matrix.** The *trace* of a square matrix $A \in \mathbb{R}^{n \times n}$, $A = (a_{ij})$, is defined to be the sum of its diagonal elements:

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}\,.$$

It's obvious that trace is linear,

$$\mathrm{trace}(A + B) = \mathrm{trace}(A) + \mathrm{trace}(B) \qquad \mathrm{trace}(\alpha A) = \alpha\ \mathrm{trace}(A)\,,$$

and that

$$\mathrm{trace}(A^T) = \mathrm{trace}(A)\,.$$

Less obvious is the following fact.

a) For $A, B \in \mathbb{R}^{n \times n}$, show that

$$\mathrm{trace}(AB) = \mathrm{trace}(BA)\,.$$

Coding:

- Create a matrix A and B
- Calculate trace of A,B,A+B,AB and BA (use numpy.trace)
- Observe the values

b) The properties of the trace allow us to define an inner product of two square matrices A and B by

$$\langle A\,,\,B\rangle = \mathrm{trace}(AB^T)\,.$$

This inner product then defines a norm of A as

$$\|A\| = \{\mathrm{trace}(AA^T)\}^{1/2}\,.$$

What is $\|A\|$ in terms of the entries of A?

c) We can define a vectorize function, $\mathrm{vec}(A) \in \mathbb{R}^{n^2}$, which, given the matrix A, stacks up the columns of A into a vector of length n^2 (first column followed by the second column, etc.) Show that $\langle A\,,\,B\rangle = \mathrm{trace}(AB^T)\ = \mathrm{vec}(A)^T\,\mathrm{vec}(B)$.

Coding:

- Calculate values of <A,B> and <A,A>
- Flatten the matrices A,B and calculate their dot product
- Calculate frobenius norm of matrix A and compare it with ||A||s

Eigenvalues and Eigenvectors

(9) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

Coding:

- Initialize A
- Calculate eigenvalues and eigenvectors
- Calculate sum and product of eigenvalues and compare them to trace and determinant of A

(10) **If we shift to $A - 7I$, what are the eigenvalues and eigenvectors and how are they related to those of A?**

$$B = A - 7I = \begin{bmatrix} -6 & -1 \\ 2 & -3 \end{bmatrix}.$$

Coding:

- Calculate B
- Calculate eigenvalues and eigenvectors of B
- Compare the above to eigenvalues and eigenvectors of A

(11) Suppose that λ is an eigenvalue of A, and x is its eigenvector: $Ax = \lambda x$.

(a) Show that this same x is an eigenvector of $B = A - 7I$, and find the eigenvalue.

(b) Assuming $\lambda \neq 0$, show that x is also an eigenvector of A^{-1}—and find the eigenvalue.

Coding:

- Calculate inverse of A
- Calculate eigenvalues and eigenvectors for inverse of A
- Compare the above to eigenvalues and eigenvectors of A

Least Squares

(12)**Find the least squares solution to the problem with data,**

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Coding:

- Find pseudo inverse of A by your own code and using inbuilt function
- Find the solution by multiplying pseudo inverse of A and B

Additional Questions

(14) *Norm of matrix-vector product.* Suppose A is an $m \times n$ matrix and x is an n-vector. A famous inequality relates $\|x\|$, $\|A\|$, and $\|Ax\|$:

$$\|Ax\| \leq \|A\|\|x\|.$$

The left-hand side is the (vector) norm of the matrix-vector product; the right-hand side is the (scalar) product of the matrix and vector norms. Show this inequality. *Hints.* Let a_i^T be the ith row of A. Use the Cauchy–Schwarz inequality to get $(a_i^T x)^2 \leq \|a_i\|^2\|x\|^2$. Then add the resulting m inequalities.

(15) **Express the following statements in matrix language.** You can assume that all matrices mentioned have appropriate dimensions. Here is an example: "Every column of C is a linear combination of the columns of B" can be expressed as "$C = BF$ for some matrix F".

There can be several answers; one is good enough for us.

a) Suppose Z has n columns. For each i, row i of Z is a linear combination of rows $i, \ldots, n$ of Y.

b) W is obtained from V by permuting adjacent odd and even columns (*i.e.*, 1 *and* 2, 3 *and* 4, $\ldots$).

c) *Each column of P makes an acute angle with each column of Q.*

d) *Each column of P makes an acute angle with the corresponding column of Q.*

e) *The first k columns of A are orthogonal to the remaining columns of A.*