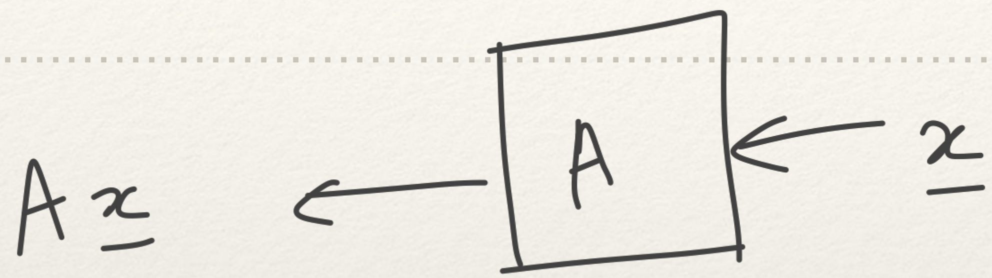
Professional program in AI

Eigenvalues and eigenvectors

May 30, 2019

System interpretation



Matrix defined by how it "operates on" input vectors

Examples: scalar matrix, diagonal matrix, rotation matrix

Eigenvalues and Eigenvectors

$A \in \mathbb{R}^{n \times n}$, λ is an eigen value of A

if $A\underline{x} = \lambda \underline{x}$ for some $\underline{x} \neq \underline{0}$.

$\underline{x}$ is an eigen vector corresponding to λ.

Examples:

Symmetric matrices

A real symmetric $n \times n$ matrix has n orthogonal eigen vectors:

$$A\underline{v}_1 = \lambda_1 \underline{v}_1, \quad A\underline{v}_2 = \lambda_2 \underline{v}_2, \quad ---, \quad A\underline{v}_n = \lambda_n \underline{v}_n$$

where $v_i^T v_j = 0$ for $i \neq j$ and $\|v_i\|_2 = 1$.

Can be written as $AV = VD$ or $A = VDV^T$

Quadratic forms

Given a symmetric matrix A,

$$x^T A x = \sum_{i,j=1}^{n} x_i x_j A_{ij}$$

Examples

Optima of quadratic forms

For any symmetric matrix A with largest eigen value λ_1 and smallest eigen value λ_n,

$$\lambda_n \underline{x}^T \underline{x} \leq \underline{x}^T A \underline{x} \leq \lambda_1 \underline{x}^T \underline{x}$$

For any vector $\underline{x}$.

Positive semi definite matrices

A (symmetric) matrix S is positive semi definite

if $\underline{x}^T S \underline{x} \geq 0$ for all $\underline{x}$.

S is positive semi-definite if and only if

all eigen values of S are non-negative

Examples: Covariance matrix, Gram matrix

Matrix norms

$$A\underline{x} \longleftarrow \boxed{A} \longleftarrow \underline{x}$$

$$\|A\| = \max \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} = \lambda_{max}$$

Rank of a matrix

Rank of a (symmetric) matrix is the number of non-zero eigen values.

If rank is n, the matrix has an inverse, and $A^{-1} = V D^{-1} V^T$

Low rank approximation