Linear Programming

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Linear Programming

A *linear programming problem* may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints*. The constraints maybe equalities or inequalities.

Linear programs are problems that can be expressed in standard matrix form as

$$\begin{aligned} \text{Manimize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ \text{and} \quad & x \geq 0 \end{aligned}$$

Here we assume that the matrix A has a full row rank.

Convexity

Definition (Convex Sets)

Let S be a vector space over an ordered field and $C \subseteq S$. We say C is **convex** if for all x and y in C, the line segment connecting x and y is included in C. This further means $t \cdot x + (1-t) \cdot y \in S$ for $x, y \in S$ when $0 \leq t \leq 1$.

Definition (Hyperplanes)

A **hyperplane** is a subset $X \subseteq \mathbb{R}^n$ defined by

$$X = \{(x_1, x_2, \ldots, x_n) : c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = b\}$$

for $c_1, c_2, \ldots, c_n, b \in \mathbb{R}$.

Convexity

Definition (Half Spaces)

A **half-space** is a subset $X \subseteq \mathbb{R}^n$ defined by

$$X = \{(x_1, x_2, \ldots, x_n) : c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \leq b\}$$

for $c_1, c_2, \ldots, c_n, b \in \mathbb{R}$.

Definition (Extreme Point)

A point $\mathbf{x}$ in a convex set C is said to be an **extreme point** of C if there are no two distinct points $\mathbf{x_1}$ and $\mathbf{x_2}$ in C such that $\mathbf{x} = \alpha \mathbf{x_1} + (1 - \alpha)\mathbf{x_2}$ for some $0 < \alpha < 1$.

Polytope, Feasible Region

Definition (Polyhedron, Polytope)

A **polyhedron** is the intersection of finitely many halfspaces $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. A **polytope** is a bounded polyhedron.

Definition (Feasible Solution)

A **feasible solution** to a linear program is a solution that satisfies all constraints.

Definition (Feasible Region)

The **feasible region** in a linear program is the set of all possible feasible solutions.

Slack Form

This is a more convenient form for describing the Simplex Algorithm for solving linear program. We are dealing with linear equality instead of inequality.

We change inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

to an equality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+1} = b_i$$

by a **slack variable** x_{n+1} where $x_{n+1} \geq 0$.

Polytope $\Leftrightarrow$ Feasible Region

Theorem (Polytope $\Leftrightarrow$ Feasible Region)

Any polytope $P \subseteq \mathbb{R}^{n-m}$ *corresponds to the feasible region of a linear program* $Ax = b, x \geq 0$ *(denoted as* $\{x : Ax = b, x \geq 0\}$*), and vice versa.*

Simplex Algorithm

This is an iteration process: we move from one vertex to another via an edge and stop when we get optimalization.

- Why does it suffice to consider vertices of the polytope only?
- How to obtain a vertex?
- How to implement "moving to another vertex via an edge"?
- When should we stop?

Why does it suffice to consider vertices of the polytope only?

Definition (Optimal Solution)

An optimal solution to a linear program is the feasible solution with the optimized objective function value.

Theorem (Optimal Solution can be Reached at a Vertex)

There exists a vertex in P that takes the optimal value (if the optimal objective value is finite).

How to obtain a vertex?

Theorem (Vertex $\Leftrightarrow$ Basic Feasible Solution)

A vertex of P corresponds to a basis of matrix A.

For a vertex x of the polytope, a basis B can be derived from extracting the column vectors corresponding to the non-zero x_i. The non-basis column vectors are denoted as N.

$$\min \quad c_B^T x_B + c_N^T x_T$$

$$\text{s.t.} \quad \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b$$

Vertex $\Leftrightarrow$ Basic Feasible Solution

$$\begin{aligned} \min \quad & c_B{}^T x_B + c_N{}^T x_T \\ \text{s.t.} \quad & \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \end{aligned}$$

Here, x is decomposed as $x = [x_B \quad x_N]^T$. Then, we have $x_N = 0$ and $x_B = B^{-1}b$. Thus, the corresponding objective value is

$$c^T x = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b.$$

How to implement "moving to another vertex via an edge"?

Theorem (Edge $\Leftrightarrow$ Non-basis Column Vector of A)

Let $x = [x_1, x_2, \ldots, x_n]^T$ be a vertex corresponding to basis $B = \{a_1, a_2, \ldots, a_m\}$. Consider a non-basis vector $a_e \notin B$. Suppose a_e can be decomposed as $a_e = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$. Let $\theta = \min_{a_i \in B, \lambda_i > 0} \frac{x_i}{\lambda_i} = \frac{x_l}{\lambda_l}$. Then, $x' = x - \theta\lambda$ is also a vertex corresponding to basis $B' = B - \{a_l\} \cup \{a_e\}$. Here $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m, 0, \ldots, -1, \ldots, 0]^T$.

$x' = x - \theta\lambda$ is a Feasible Solution

Recall that $x_1 a_1 + x_2 a_2 + \cdots x_m a_m = b$.

Also, $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m - a_e = 0$.

Thus,

$$(x_1 - \theta\lambda_1)a_1 + \cdots + (x_m - \theta\lambda_m)a_m + \cdots + \theta a_e = b.$$

Now, it suffices to prove $x' \geq 0$.

When should we stop?

Theorem

Consider a linear program in slack form

$$\begin{aligned} \textit{Manimize} \quad & c^T x \\ \textit{s.t.} \quad & Ax = b \\ \textit{and} \quad & x \geq 0 \end{aligned}$$

Let x be a vertex corresponding to the basis B. If $c^T - c_B^T B^{-1} A \geq 0$, then x is an optimal solution.

Time Complexity

Unfortunately, Simplex is not a polynomial-time algorithm.