Linear Programming

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December 2, 2020

A *linear programming problem* may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints.* The constraints maybe equalities or inequalities.

Linear programs are problems that can be expressed in standard matrix form as

$$\begin{aligned} \text{Manimize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ \text{and} \quad & x \geq 0 \end{aligned}$$

Here we assume that the matrix A has a full row rank.

Definition 1 (Hyperplane)**.** $\{x : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\}$ (linear equality constraint).

Definition 2 (Halfspace)**.** $\{x : a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b\}$ (linear inequality constraint).

Definition 3 (Polyhedron)**.** The intersection of several half spaces.

Definition 4 (Polytope)**.** A bounded, nonempty polyhedron.

Slack form is a more convenient form for describing the Simplex Algorithm for solving linear program. We are dealing with linear equality instead of inequality.

We change inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

to an equality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+1} = b_i$$

by a **slack variable** x_{n+1} where $x_{n+1} \geq 0$.

Theorem 5 (Polytope $\Leftrightarrow$ Feasible Region)**.** *Any polytope $P \subseteq \mathbb{R}^{n-m}$ corresponds to the feasible region of a linear program $Ax = b, x \geq 0$, and vice versa.*

Proof. (Polytope $\Rightarrow$ Feasible Region) In the standard form, we write

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}
$$

Now, we write in slack form (let $s_1, s_2, \ldots, s_m$ be slack variables)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 &\leq b_2 \\
\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m &\leq b_m \\
x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m &\geq 0
\end{aligned}
$$

Thus, $(x_1, x_2, \ldots, x_n) \in P \Rightarrow (x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) \geq 0$.

(Feasible Region $\Rightarrow$ Polytope) Row reduction gives

$$
\begin{aligned}
x_1 + a'_{1,m+1}x_{m+1} + \cdots + a'_{1,n}x_n &= b'_1 \\
x_2 + a'_{2,m+1}x_{m+1} + \cdots + a'_{2,n}x_n &= b'_2 \\
\\
x_m + a'_{m,m+1}x_{m+1} + \cdots + a'_{m,n}x_n &= b'_m \\
x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n &\geq 0
\end{aligned}
$$

By removing positive variables $x_1, x_2, \ldots, x_m$ gives

$$
\begin{aligned}
a'_{1,m+1}x_{m+1} + \cdots + a'_{1,n}x_n &= b'_1 \\
a'_{2,m+1}x_{m+1} + \cdots + a'_{2,n}x_n &= b'_2 \\
\\
a'_{m,m+1}x_{m+1} + \cdots + a'_{m,n}x_n &= b'_m \\
x_{m+1}, \ldots, x_n &\geq 0
\end{aligned}
$$

Now, we define a polytope $P \subseteq \mathbb{R}^{n-m}$ as the intersection of m half-spaces. Thus, any feasible solution $x = (x_1, x_2, \ldots, x_n)$ correspond to $x_N = (x_{m+1}, \ldots, x_n)$.

$\square$

Theorem 6 (Optimal Solution can be Reached at a Vertex)**.** *There exists a vertex in P that takes the optimal value (if the optimal objective value is finite).*

Proof. Since P is closed and bounded, so $c^T x$ can reach its optimum in P. Define a point x_0 be the optimal solution. We want to show that there is a vertex at least as good as x_0.

x_0 can be represented as the convex combination of vertices of P, that is $x_0 = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k$, where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. Thus, $c^T x_0 = \lambda_1 c^T x_1 + \lambda_2 c^T x_2 + \cdots + \lambda_k c^T x_k$. Let x_i be the vertex with the minimal objective value $c^T x_i$, then

$$c^T x_0 = \lambda_1 c^T x_1 + \lambda_2 c^T x_2 + \cdots + \lambda_k c^T x_k \geq c^T x_i.$$

Thus, vertex x_i is also an optimal solution since $c^T x_i \leq c^T x_0$.

$\square$

Example 7 (A vertex of P corresponds to a basis of matrix A)**.**

$$\begin{aligned}
\text{Minimize} \quad & -x_1 - 14x_2 - 6x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \leq 4 \\
& x_1 \leq 2 \\
& x_3 \leq 3 \\
& 3x_2 + x_3 \leq 6 \\
\text{and} \quad & x_1, x_2, x_3 \geq 0
\end{aligned}$$

Turning into the slack form, we have

$$\begin{aligned}
\text{Minimize} \quad & -x_1 - 14x_2 - 6x_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 4 \\
& x_1 + x_5 = 2 \\
& x_3 + x_6 = 3 \\
& 3x_2 + x_3 + x_7 = 6 \\
\text{and} \quad & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned}$$

so

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's take the vertex $(x_1, x_2, x_3) = (0, 2, 0)$. The full solution is

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, 2, 0, 2, 2, 3, 0).$$

The column vectors for non-zero x_i are linearly independent and thus form a basis. Then, we can construct two other points $x' = x + \theta\lambda \in P$ and $x'' = x - \theta\lambda \in P$.

The basis is

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

Decomposing x above gives $x_B = [2 \quad 2 \quad 2 \quad 3]^T$ and $x_N = [0 \quad 0 \quad 0]^T$. Then,

$$b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

The basic feasible solution x respect to basis B is

$$x = [0 \quad 2 \quad 0 \quad 2 \quad 2 \quad 3 \quad 0]^T .$$

Thus, $(x_1, x_2, x_3) = (0, 2, 0)$ is a vertex of the polytope P.

Unfortunately, simplex is not a polynomial-time algorithm.

Consider

$$\begin{aligned} \text{Maximize} \quad & x_n \\ \text{s.t.} \quad & \delta \leq x_i \leq 1 \text{ for } i = 1, \ldots, n \\ & \delta x_{i-1} \leq x_i \leq 1 - \delta x_{i-1} \text{ for } i = 2, \ldots, n \\ \text{and} \quad & x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \end{aligned}$$

The Klee-Minty cube takes place when $n = 3, \delta = \dfrac{1}{4}$. The paper *How Good Is the Simplex Algorithm? In Inequalities* is given by V. Klee and G. L. Minty (1972).