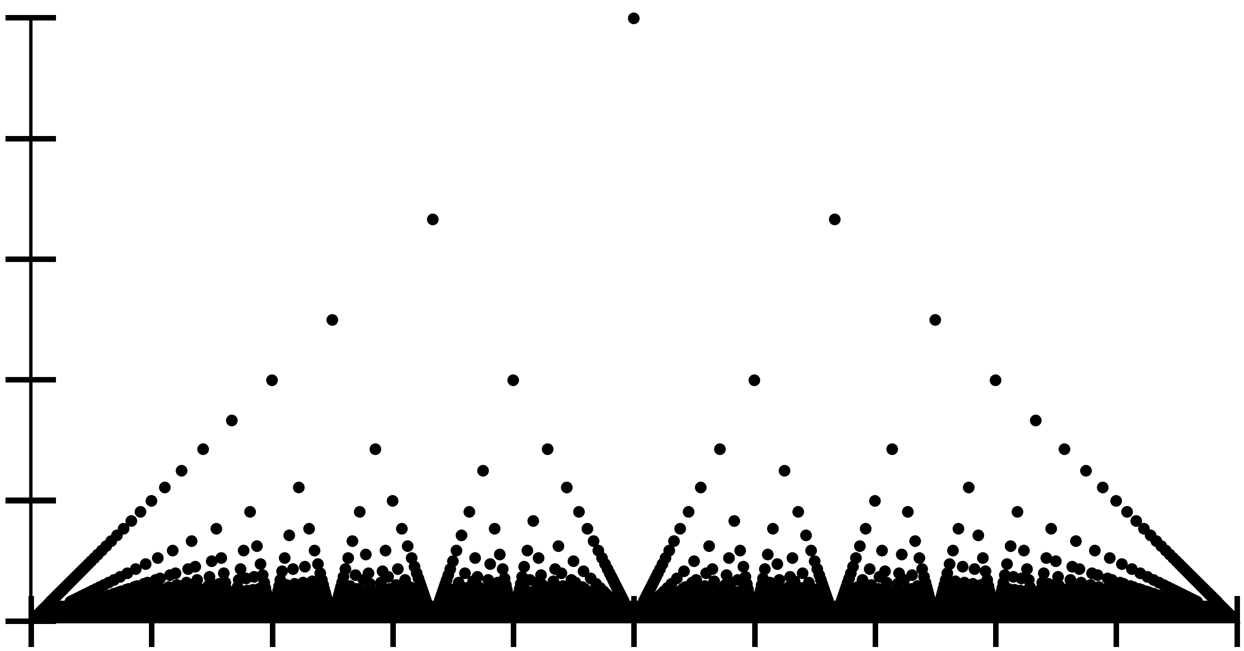
Math 140A Foundations of Real Analysis I

Jack Yang

Last Updated: 2020/12/16 at 14:53:08



$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q}, \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

We will take for granted (and without proof) the basic properties of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the set of **natural numbers**), $\mathbb{Z}$ (the set of **integers**), and $\mathbb{Q}$ (the set of **rational numbers**).

We will explore in detail the properties of $\mathbb{R}$ (the set of **real numbers**). With great care and precision we will define what the real numbers are. All of our prior knowledge and beliefs about $\mathbb{R}$ will be held in suspicion until we can find proofs of those properties based on our formal definition of $\mathbb{R}$.

The longterm goal is to provide the logical and theoretical justification for calculus (140B) and go beyond (140C).

We will often focus on specific features of $\mathbb{R}$ and study those features in more abstract settings.

Abstract

Real number system. Basic point-set topology. Metric spaces. Numerical sequences and series. Functions of a real variable. Continuity.

Contents

1 The Real Number System

A **set** is a collection of objects called elements. A set with no objects is called the **empty set** and is denoted by $\varnothing$. We will use the language of sets to talk about analysis.

Most of the time we will consider sets of numbers. For example, if $S = \{0, 1, 2\}$, that is, the set containing the three elements 0, 1, and 2. We write $1 \in S$ to denote that number 1 belongs to the set S. Similarly, we write $7 \notin S$ to denote that the number 7 is not in S.

Definition 1.1 (Subset)**.** A set A is a **subset** of a set B if $x \in A$ implies $x \in B$, and we write $A \subseteq B$. That is, all members of A are also members of B.

(a) Two sets A and B are **equal** if $A \subseteq B$ and $B \subseteq A$. We write $A = B$. That is, A and B contain exactly the same elements. If it is not true that A and B are equal, then we write $A \neq B$.

(b) A set A is a **proper subset** of B if $A \subset B$ and $A \neq B$. We write $A \subsetneq B$.

Example 1.2. The following are sets including the standard notations.

(a) The set of **natural numbers**, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

(b) The set of **integers**, $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$.

(c) The set of **rational numbers**, $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

(d) The set of **real numbers**, $\mathbb{R}$.

Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

There are many operations we want to do with sets.

Definition 1.3.

(a) A **union** of two sets A and B is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

(b) An **intersection** of two sets A and B is defined as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

(c) A **complement of** B **relative to** A (or **set-theoretic difference** of A and B) is defined as

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

We say **complement of** B and write B^{C} instead of $A \setminus B$ if the set A is either the entire universe or is the obvious set containing B, and is understood from context.

(d) We say sets A and B are **disjoint** if $A \cap B = \varnothing$.

Theorem 1.4 (De Morgan's Laws)**.** *Let A, B, C be sets. Then*

$$(B \cup C)^{\complement} = B^{\complement} \cap C^{\complement},$$
$$(B \cap C)^{\complement} = B^{\complement} \cup C^{\complement},$$

or, more generally,

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C),$$
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

We also need to intersect or union several sets at once. If there are only finitely many, then we simply apply the union or intersection operation several times. However, suppose we have an infinite collection of sets (a set of sets) $\{A_1, A_2, A_3, \ldots\}$. We define

$$\bigcup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\},$$
$$\bigcap_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Lecture 1
Friday
October 2

1.1 Ordered Sets

A real number is a value that represents a quantity along a continuous number line. Real numbers have several specific features. One nice feature is that real numbers can be ordered. The symbol for the set of real numbers is $\mathbb{R}$.

Definition 1.5 (Ordered Set)**.** Let S be a set. An **order** on S, denoted $<$, is a relation satisfying

(a) (Trichotomy Law) for all $x, y \in S$, exactly one of the statements

$$x < y, \quad x = y, \quad x > y$$

is true.

(b) (Transitivity) for all $x, y, z, \in S$, if $x < y$ and $y < z$, then $x < z$.

An **ordered set** is a set of which an order is defined. For example, the integer set $\mathbb{Z}$ is ordered.

For convenience, we write

(a) $y > x$ to mean $x < y$.

(b) $x \leq y$ to mean $x < y$ or $x = y$.

Definition 1.6 (Bounds)**.** Let S be an ordered set and $E \subseteq S$.

(a) If there is $b \in S$ with $x \leq b$ for all $x \in E$, then we say E is **bounded above** and call b an **upper bound** to E.

(b) If there is $b \in S$ with $x \geq b$ for all $x \in E$, then we say E is **bounded below** and call b an **lower bound** to E.

Definition 1.7 (Supremum and Infimum)**.** Let S be an ordered set and $E \subseteq S$. We call $\alpha \in S$ the **least upper bound** of E or the **supremum** of E, denoted $\alpha = \sup E$, if

(a) α is an upper bound to E.

(b) whenever $x \in S$ and $x < \alpha$, x is not an upper bound to E.

The **greatest lower bound** or **infimum** of E is defined similarly and denoted $\inf E$. The statement $\beta = \inf E$ means that β is a lower bound of E and that no γ with $\gamma > \beta$ is a lower bound of E.

Example 1.8. Consider the set $E = \left\{\frac{1}{n} : n \in \mathbb{Z}_+\right\} \subseteq \mathbb{Q}$, we have $\sup E = 1$ and $\inf E = 0$ (this is not so obvious, for now we get by intuition).

Note that $\sup E \in E$ and $\inf E \notin E$. In general, $\sup E$ and $\inf E$ may or may not be element of E.

To distinguish maximum (or minimum) and supremum (or infimum), in this case we have $\inf E = 0$ but $\min E$ does not exist and $\sup E = \max E = 1$.

If the maximum exists, then it is the supremum. For an infinite set, it can happen that the maximum does not exist but the supremum does exist. Similar idea holds for minimum and infimum. In general, supremum and infimum are used in more general and abstract setting.

Definition 1.9 (Least Upper Bound Property)**.** An ordered set S has the **least upper bound property** if whenever $E \subseteq S$ is nonempty and bounded above, $\sup E$ exists.

Example 1.10. The set $\mathbb{Q}$ of rational numbers does not have the least-upper-bound property. Recall that $\sqrt{2} \notin \mathbb{Q}$. Let $A = \{p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 \leq 2\}$ and $B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 \geq 2\}$. Then, $\mathbb{Q} = A \cup B$, B is the set of upper bounds to A and A is the set of lower bounds to B.

But A has no largest element and B has no smallest element, so $\sup A$ and $\inf B$ do not exist (when using $\mathbb{Q}$). This implies that $\mathbb{Q}$ does not have the least-upper-bound property.

Lecture 2
Monday
October 5

For $p \in \mathbb{Q}_{>0}$, set

$$q = p - \frac{p^2 - 2}{p + 2} \in \mathbb{Q}.$$

Then, $q = \dfrac{2p + 2}{p + 2}$, so

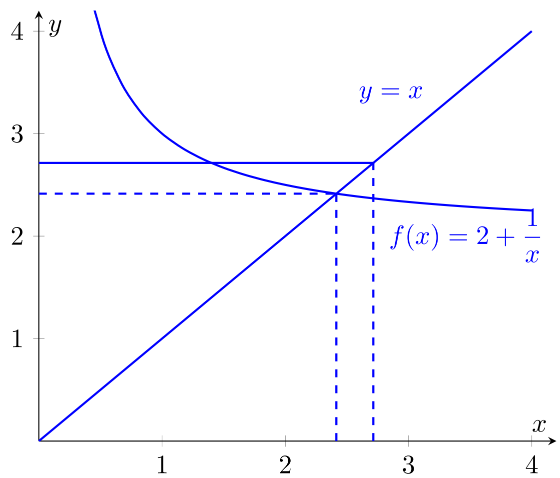
$$q^2 = 2 + \frac{2(p^2 - 2)}{(p + 2)^2}.$$

Suppose $p \in A$, then $p^2 - 2 \leq 0$. If $p \leq 0$, then $p < 1$ and $1 \in A$. If $p > 0$, then $q > p$ since $q - p = -\dfrac{p^2 - 2}{p + 2} > 0$, and $q \in A$ since $q^2 - 2 = \dfrac{2(p^2 - 2)}{(p + 2)^2} \leq 0$. Thus, p is not the largest element in A.

Suppose $p \in B$, then $p^2 - 2 > 0$. So, we get $0 < q < p$ since $q - p = -\dfrac{p^2-2}{p+2} < 0$, and $q \in B$ since $q^2 - 2 = \dfrac{2(p^2-2)}{(p+2)^2} > 0$. Thus, q is not the smallest element of B.

TODO Motivation

Remark 1.11. To see what motivates the q above, we first note that $x = 1+\sqrt{2}$ is a solution to the equation $x = 2 + \dfrac{1}{x}$ (more generally, $x = 1+\sqrt{k}$ is a solution to $x = \dfrac{k-1}{x} + 2$). Consider the function $f(x) = 2 + \dfrac{1}{x}$. From the graph, we note that for sufficiently small ε, $f(\sqrt{2}+1+\varepsilon)$ is closer to $\sqrt{2}+1$ than $\sqrt{2}+1+\varepsilon$ is. Similarly, $f(\sqrt{2}+1-\varepsilon)$ is closer to $\sqrt{2}+1$ than $\sqrt{2}+1-\varepsilon$ is.



Note that if $p \approx \sqrt{2}$, then $f(p+1) - 1$ will give us an approximation of $\sqrt{2}$, and so $\sqrt{2} \approx f(p+1) - 1 = \dfrac{1}{p+1} + 2 = \dfrac{p+2}{p+1}$.

Also, we note that if $p > \sqrt{2}$, then $f(p+1) < \sqrt{2}+1$. For $p = \sqrt{2}$, we have $\sqrt{2}(f(p+1)-1) = 2$, hence

$$\sqrt{2} = \frac{2}{f(p+1)-1} = \frac{2p+2}{p+2}.$$

This gives us that if $p < \sqrt{2}$, then $f(p+1) - 1 > \sqrt{2}$.

Note that $p - \dfrac{p^2-2}{p+2} = \dfrac{2p+2}{p+2}$ is what we want to know.

Theorem 1.12. *If S has the least-upper-bound property, then it has the greatest-lower-bound property, that is, if $E \subseteq S$ is nonempty and bounded below then* $\inf E$ *exists.*

Proof. Let $E \subseteq S$ be nonempty and bounded below. Let A be the set of all lower bounds to E. Then A is nonempty and bounded above, that is, every $e \in E$ is an upper bound.

So, $\alpha = \sup A$ exists by the least-upper-bound property (Definition 1.9). We will check $\alpha = \inf E$.

(Check α is a lower bound to E) Consider any $e \in E$. By definition A, we have $\forall A \in A$, $a \leq e$. Thus e is an upper bound to A. Since α is the least upper bound to A, we have $\alpha \leq e$. Thus α is a lower bound.

(Check anything greater than α is not a lower bound) Suppose x is a lower bound to E, then by definition, $x \in A$. Then, $x \leq \sup A = \alpha$.

We conclude that $\inf E = \alpha$ exists.

□

1.2 Fields

Definition 1.13 (Field)**.** A **field** is a set F with two binary operations $+$ and $\cdot$, called **addition** and **multiplication**, with the following properties

(a) (Commutativity) for all $a, b \in F$, $a + b = b + a$, $a \cdot b = b \cdot a$.

(b) (Associativity) for all $a, b, c \in F$, $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(c) (Identity) there are $0, 1 \in F$, with for all $a \in F$, $0 + a = a$ and $1 \cdot a = a$.

(d) (Inverse) for all $a \in F$, there is an element $-a \in F$ with $a + (-a) = 0$; also, for all $a \in F$ with $a \neq 0$, there is $\frac{1}{a} \in F$ with $a \cdot \frac{1}{a} = 1$.

(e) (Distributivity) for all $a, b, c \in F$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c) = a \cdot b + a \cdot c$.

Example 1.14. These are fields

(a) $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

(b) $\mathbb{Q}(t) = \left\{ \frac{p(t)}{q(t)} : p, q \text{ are polynomials in } t \text{ with coefficients in } \mathbb{Q} \right\}$.

(c) Set of conjugacy classes $\bmod p$ for p a prime.

Remark 1.15. We write (in any field) $x - y$ in place of $x + (-y)$, $\frac{x}{y}$ in place of $x \cdot \left(\frac{1}{y}\right)$, 2 in place of $1 + 1$, similarly $2x$ in place of $x + x$, and x^2 in place of $x \cdot x$.

Proposition 1.16. *For a field F and $x, y, z \in F$,*

(a) if $x + y = x + z$, then $y = z$.

(b) if $x + y = 0$, then $y = -x$ (further it proves that additive inverse is unique).

(c) if $x + y = x$, then y = 0 (further it proves that additive identity is unique).

(d) $-(-x) = x$.

Proof.

(a) If $x + y = x + z$, then

$$y = 0 + y = -x + (x + y) = -x + (x + z) = z.$$

That's it.

(b) Apply (a) with $z = -x$, then we're done.

(c) Apply (a) with $z = 0$, then we're done.

(d) Since $-x + x = x + (-x) = 0$, (b) implies $x = -(-x)$.

$\square$

Proposition 1.17. *Let F be a field. Then for all $x, y, z \in F$, with $x \neq 0$,*

(a) if $x \cdot y = x \cdot z$, then $y = z$.

(b) if $x \cdot y = 1$, then $y = \frac{1}{x}$ (further it proves that multiplicative inverse is unique).

(c) if $x \cdot y = x$, then $y = 1$ (further it proves that multiplicative identity is unique).

(d) $\frac{1}{1/x} = x$.

The proof is so similar to that of Proposition 1.16. Lecture 3 Wednesday October 7

Proposition 1.18. *For any field F and $x, y \in F$,*

(a) $0 \cdot x = 0$. Note that 0 is not necessary $0 \in \mathbb{Z}$, just a notation.

(b) if $x \neq 0$ and $y \neq 0$, then $x \cdot y \neq 0$.

(c) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ (specifically, $-x = (-1) \cdot x$).

(d) $(-x) \cdot (-y) = x \cdot y$.

Proof.

(a) We have $0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x$, so $0 \cdot x = 0$ by Proposition 1.16.

(b) Assume $x \neq 0$ and $y \neq 0$, but $xy = 0$. Then, (a) gives

$$1 = (x \cdot y) \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0 \cdot \left(\frac{1}{x} \cdot \frac{1}{y}\right) = 0,$$

that is a contradiction. Thus, we must have $x \cdot y \neq 0$.

(c) Since $(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0$, so $(-x) \cdot y = -(x \cdot y)$ by Proposition 1.16.

Similarly, we have $x \cdot (-y) = -(x \cdot y)$.

(d) By Proposition 1.16, we have

$$(-x) \cdot (-y) = -\left(x \cdot (-y)\right) = -\left(-(x \cdot y)\right) = x \cdot y.$$

$\square$

Definition 1.19 (Ordered Field)**.** An **ordered field** is a field F with an ordering such that

(a) for all $x, y, z \in F$, if $y < z$, then $x + y < x + z$.

(b) for all $x, y \in F$, if $x > 0$ and $y > 0$, then $x \cdot y > 0$.

We call $x \in F$ **positive** if $x > 0$ and **negative** if $x < 0$.

Example 1.20. These are ordered fields: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Q}(t)$ (defined in Example 1.14). TODO Explain

Proposition 1.21. *For an ordered field F and $x, y, z \in F$,*

(a) if $x > 0$, then $-x < 0$, and vice versa.

(b) if $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.

(c) if $x < 0$ and $y < z$, then $x \cdot y > x \cdot z$.

(d) if $x \neq 0$, then $x^2 > 0$ (this statement is reversible).

(e) if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof.

(a) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$. If $x < 0$, then $0 = x + (-x) < 0 + (-x)$, so $-x > 0$.

(b) Since $y < z$, we have $0 = y - y < z - y$, so $x \cdot (z - y) > 0$, and therefore

$$x \cdot z = x \cdot (z - y) + x \cdot y > 0 + x \cdot y = x \cdot y.$$

(c) By (a) and (b) together, we have $(-x) \cdot y < (-x) \cdot z$, so

$$x \cdot z = (-x) \cdot y + x \cdot y + x \cdot z < (-x) \cdot z + x \cdot y + x \cdot z = x \cdot y.$$

(d) If $x > 0$, then $x^2 > 0$ by Definition 1.19. If $x < 0$, then $-x > 0$, so $(-x)^2 > 0$, but $(-x)^2 = x^2$ by Proposition 1.17. (This is an application of Trichotomy Law.)

(e) Assume $0 < x < y$. If $z \leq 0$, then $yz \leq 0$ by (b). Since $y \cdot \frac{1}{y} = 1 > 0$, we must have $\frac{1}{y} > 0$. Similarly, we have $\frac{1}{x} > 0$ as well. Finally, multiply $x < y$ by positive element $\frac{1}{x} \cdot \frac{1}{y}$, so

$$\frac{1}{y} = x \cdot \frac{1}{x} \cdot \frac{1}{y} < y \cdot \frac{1}{x} \cdot \frac{1}{y} = \frac{1}{x}.$$

$\square$

1.3 The Real Field

Theorem 1.22. *There exists a unique ordered field having the least-upper-bound property. Moreover, this field contains $\mathbb{Q}$. We denote this field $\mathbb{R}$ and call its elements **real numbers**.* TODO Proof here

The set $\mathbb{Q}$ is an extension of $\mathbb{N}$, and $\mathbb{R}$ in turn is an extension of $\mathbb{Q}$. The next result indicates how $\mathbb{N}$ and $\mathbb{Q}$ sit inside of $\mathbb{R}$.

Theorem 1.23.

(a) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $n \cdot x > y$.

(b) If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ with $x < p < y$.

Lecture 4

Friday

October 9

Proof.

(a) Towards a contradiction, suppose for all $n \in \mathbb{N}$, we have $n \cdot x \leq y$.

Let $A = \{n \cdot x : n \in \mathbb{N}\}$. By our assumption, A is bounded above by y, so by the least-upper-bound property, $\alpha = \sup A$ exists.

Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound of A, which means there is $n \in \mathbb{N}$ with $n \cdot x > \alpha - x$. Then, $(n+1)x > \alpha$, contradicting $\alpha = \sup A$, and $(n+1)x \in A$.

(b) Since $x < y$, we have $y - x > 0$, so by (a), there is $n \geq 1$ such that $n \cdot (y - x) > 1$.

Apply (a) twice more, we get $m_1, m_2 \geq 1$ with $m_1, m_2 \in \mathbb{Z}$ and $m_1 > n \cdot x$ and $m_2 > -n \cdot x$, so $-m_2 < n \cdot x < m_1$.

So, the finite set $\{-m_2, -m_2 + 1, \ldots, m_1\}$ must contain the least m with $n \cdot x < m$. Since m is the least, so $m - 1 \leq n \cdot x < m$.

Therefore, $n \cdot x < m \leq n \cdot x + 1 < n \cdot y$ and $x < \dfrac{m}{n} < y$.

$\square$

Remark 1.24. Note that Theorem 1.23 (a) is known as the **Archimedean Property**, and Theorem 1.23 (b) says that $\mathbb{Q}$ is **dense** in $\mathbb{R}$ (we will define dense soon).

Archimedean Property is equivalent to

(a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

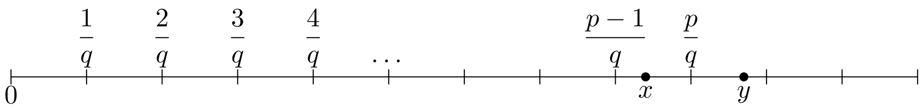
(b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\dfrac{1}{n} < y$.

There is a friendlier proof of density of $\mathbb{Q}$ in $\mathbb{R}$.

A rational number is a quotient of integers, so we want to find $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$ such that

$$x < \frac{p}{q} < y.$$

The first step is to choose the denominator n large enough so that consecutive increments of size $\dfrac{1}{q}$ are too close together to "step over" the interval (x, y).



By Archimedean Property, we may pick an $q \in \mathbb{N}$ sufficiently large enough so that

$$\frac{1}{q} < y - x.$$

Now, we want to show that $qx < p < qy$. With q already chosen, the idea now is to choose p to be the smallest integer greater than qx. In other words, pick $p \in \mathbb{Z}$ so that

$$p - 1 \leq qx < p.$$

We get $x < \dfrac{p}{q}$. Recall that $\dfrac{1}{q} < y - x$, then

$$\begin{aligned} p &\leq qx + 1 \\ &< q\left(y - \frac{1}{q}\right) + 1 \\ &= qy. \end{aligned}$$

Since $p < qy$ implies $\dfrac{p}{q} < y$, we have $x < \dfrac{p}{q} < y$, as desired.

Corollary 1.25 (Density of Irrational Numbers)**.** *Given any two real numbers $x < y$, there exists an irrational number t satisfying $x < t < y$.*

Theorem 1.26. *If $x \in \mathbb{R}$ is positive and $n \in \mathbb{Z}_+$, then there is a unique real $y > 0$ with $y^n = x$. This number y is denoted $\sqrt[n]{x}$ or $x^{1/n}$.*

Proof. We first prove the uniqueness. Equivalently, we want to show if $0 < y_1 < y_2$, then $y_1^n < y_2^n$.

Since $\dfrac{y_2}{y_1} > 1$, we have

$$\frac{y_2^n}{y_1^n} = \left(\frac{y_2}{y_1}\right)^n > 1$$

hence, $y_1^n < y_2^n$. So, if y exists, then it must be **unique**.

We still need to prove the **existence** (otherwise, the proof of **uniqueness** is not meaningful).

Now, let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$.

(Check $E \neq \varnothing$) If $t = \dfrac{x}{x+1}$, then $t < 1$ and

$$t - x = \frac{x}{x+1} - x = -\frac{x^2}{x+1} < 0$$

so $t^n < t < x$ and hence $t \in E$.

(Check $1 + x$ is upper bound to E) If $t > 1 + x$, then $t^n > t > x$, so $t \notin E$. By the Theorem 1.22, $y = \sup E$ exists.

(Check $y^n = x$) Recall the identity

$$b^n - a^n = (b-a)(b^{n-1} + a \cdot b^{n-2} + \cdots + a^{n-2} \cdot b + a^{n-1}),$$

it follows that

$$b^n - a^n < (b-a)nb^{n-1}$$

when $0 < a < b$.

Towards a contradiction, suppose $y^n < x$. Pick h with

$$0 < h < \min\left(1, \frac{x - y^n}{n \cdot (y+1)^{n-1}}\right)$$

that is, $h < 1$ (this is to make sure the second inequality sign in the next inequality holds) and $h < \dfrac{x - y^n}{n \cdot (y+1)^{n-1}}$ (this is our main construction for the next inequality, for the last inequality sign), then

$$(y+h)^n - y^n < h \cdot n \cdot (y+h)^{n-1} \leq h \cdot n \cdot (y+1)^{n-1} < x - y^n.$$

So, $y + h \in E$ and $y + h > y$, contradicting y being an upper bound to E.

Towards a contradiction, suppose $y^n > x$. Let $k = \dfrac{y^n - x}{n \cdot y^{n-1}}$, then $0 < k < y$. If $t \geq y - k$, then

$$y^n - t^n \leq y^n - (y-k)^n < k \cdot n \cdot y^{n-1} = y^n - x$$

so $r^n > x$ and $\notin E$. Thus, $y - k$ is an upper bound to E and $y - k < y$, contradicting to $y = \sup E$.

Finally, by Trichotomy Law, so $y^n = x$.

□

Remark 1.27. It is possible to define decimal representations of real numbers.

Let $x \in \mathbb{R}_{>0}$. Let n_0 be the largest integer such that $n_0 \leq x$. (Note that the existence of n_0 depends on the archimedean property of $\mathbb{R}$.) Having chosen $n_0, n_1, \ldots, n_{k-1}$, let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x.$$

Let $E = \left\{n_0 + \dfrac{n_1}{10} + \cdots + \dfrac{n_k}{10^k} : k = 0, 1, 2, \ldots\right\}$. Then, $x = \sup E$. The decimal expansion of x is $n_0.n_1n_2n_3\cdots$.

Definition 1.28 (Extended Real Number System)**.** The **extended real number system** is the set $\mathbb{R} \cup \{-\infty, \infty\{$ where for all $x \in \mathbb{R}$,

(a) $-\infty < x < +\infty$.

(b) $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{+\infty} = 0$, $\dfrac{x}{-\infty} = 0$.

(c) If $x > 0$, then $x \cdot (+\infty) = \infty$, $x \cdot (-\infty) = -\infty$. If $x < 0$, then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = \infty$.

All other operations are left **undefined**.

Remark 1.29.

(a) The extended real number system is not a field since the operation $\infty - \infty$ is undefined (that is a contradiction to Definition 1.13).

(b) It is clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound.

(c) To distinguish $x \in \mathbb{R}$ from $-\infty$ and $+\infty$, we call x **finite**.

1.4 The Complex Field

Definition 1.30 (Complex Number)**.** The set of **complex numbers** is

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

For $x, y \in \mathbb{C}$, say $x = (a, b)$ and $y = (c, d)$, we define

$$x + y = (a + c, b + d),$$
$$x \cdot y = (ac - bd, ad + bc).$$

Theorem 1.31. *$\mathbb{C}$ is a field with $(0, 0)$ and $(1, 0)$ playing the rolds of additive identity 0 and multiplicative identity 1.*

Proof. The field axioms in Definition 1.13 are easy to verify as we use the field structure of $\mathbb{R}$. Here, we only prove for additive and multiplicative identities.

Let $x = (a, b) \in \mathbb{C}$. Write $-x = (-a, -b)$. Then,

$$x + (-x) = (a, b) + (-a, -b) = (0, 0).$$

Now, assume $x \neq 0$. Then, $(a, b) \neq (0, 0)$, so $a \neq 0$ or $b \neq 0$. Thus, $a^2 + b^2 > 0$. Write

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} \right).$$

Then,

$$x \cdot \frac{1}{x} = (a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2} \right) = (1, 0) = 1.$$

□

Lecture 5
Monday
October 12

Theorem 1.32. *For all $a, b \in \mathbb{R}$,*

$$(a, 0) + (b, 0) = (a + b, 0) \qquad (a, 0) \cdot (b, 0) = (a \cdot b, 0).$$

This means we can identify $a \in \mathbb{R}$ with $(a, 0)$ and this identification preserves addition and multiplication, so we can view $\mathbb{R}$ as a subfield of $\mathbb{C}$.

Definition 1.33. $i = (0, 1)$.

Theorem 1.34. $i^2 = -1$.

Proof. $i^2 = (0,1)^2 = (0,1)\cdot(0,1) = (-1,0)$.

$\square$

Theorem 1.35. *If $a, b \in \mathbb{R}$, then $(a, b) = a + bi$.*

Proof. $a + bi = (a,0) + (b,0)\cdot(0,1) = (a,0) + (0,b) = (a,b)$.

$\square$

Definition 1.36. For $z = a + bi \in \mathbb{C}$, we call a the **real part** of z and b the **imaginary part** of z and write $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$. We call $\bar{z} = a - bi$ the **complex conjugate** of z.

Theorem 1.37. *If $z, w \in \mathbb{C}$, then*

(a) $\overline{z + w} = \bar{z} + \bar{w}$.

(b) $\overline{zw} = \bar{z} \cdot \bar{w}$.

(c) $z + \bar{z} = 2\operatorname{Re}(z)$ *and* $z - \bar{z} = 2\operatorname{Im}(z)$.

(d) $z\bar{z} \in \mathbb{R}$ *and* $z\bar{z} > 0$ *when* $z \neq 0$.

Proof. (a), (b), and (c) are easy to check by calculation. (d) holds since if $z = a + bi$, then $z \cdot \bar{z} = a^2 + b^2$.

$\square$

Definition 1.38 (Absolute Value)**.** The **absolute value** of $z \in \mathbb{C}$ is defined $|z| = (z \cdot \bar{z})^{1/2}$.

Remark 1.39. If $x \in \mathbb{R}$, then $\bar{x} = x$ so $|x| = \sqrt{x^2}$ meaning $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Theorem 1.40. *If $z, w \in \mathbb{C}$, then*

(a) $|z| > 0$ *unless* $z = 0$.

(b) $|\bar{z}| = |z|$.

(c) $|z \cdot w| = |z| \cdot |w|$.

(d) $|\operatorname{Re}(z)| \leq |z|$.

(e) $|z + w| \leq |z| + |w|$.

Proof. (a) and (b) are easy to check by calculation.

(c)

$$\begin{aligned}|z \cdot w| &= (zw \cdot \overline{zw})^{1/2} = (zw\bar{z}\bar{w})^{1/2} \\ &= (z\bar{z}w\bar{w})^{1/2} = (z\bar{z})^{1/2}(w\bar{w})^{1/2} \\ &= |z| \cdot |w|.\end{aligned}$$

(d) Say $z = a + bi$, then $a^2 \le a^2 + b^2$, so

$$|\operatorname{Re}(z)| = |a| = \sqrt{a^2} \le \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}} = |z|.$$

(e) Note $\overline{\bar{z} \cdot w} = z \cdot \bar{w}$, so

$$\bar{z} \cdot w + z \cdot \bar{w} = 2\operatorname{Re}(z \cdot w).$$

Therefore,

$$\begin{aligned}|z + w|^2 &= (z + w)(\bar{z} + \bar{w}) \\ &= z \cdot \bar{z} + w \cdot \bar{z} + \bar{w} \cdot z + w \cdot \bar{w} \\ &= |z|^2 + 2\operatorname{Re}(\bar{w} \cdot z) + |w|^2 \\ &\le |z|^2 + 2|\bar{w} \cdot z| + |w|^2 \\ &= |z|^2 + 2|w| \cdot |z| + |w|^2 \\ &= (|z| + |w|)^2.\end{aligned}$$

□

Theorem 1.41 (Cauchy-Schwarz Inequality)**.** *If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{C}$, then*

$$\left|\sum_{k=1}^{n} a_k \bar{b_k}\right|^2 \le \sum_{k=1}^{n} |a_k|^2 \cdot \sum_{k=1}^{n} |b_k|^2.$$

If $(a_1, a_2, \ldots, a_n) = \vec{a} \in \mathbb{R}^n$ and $(b_1, b_2, \ldots, b_n) = \vec{b} \in \mathbb{R}^2$, this says

$$|\vec{a} \cdot \vec{b}|^2 \le |\vec{a}|^2 \cdot |\vec{b}|^2.$$

From the geometric intuition, we expect equality to hold precisely

$$B \cdot \vec{a} = C \cdot \vec{b}$$

where $B = \displaystyle\sum_{k=1}^{n} |b_k|^2$ and $C = \displaystyle\sum_{k=1}^{n} a_k \cdot \bar{b_k}$. This means that $\vec{a}$ and $\vec{b}$ are collinear.

Proof. Define B and C as above, let $A = \displaystyle\sum_{k=1}^{n} |a_k|^2$.

If $B = 0$, then $b_1 = b_2 = \cdots = b_n = 0$ and the conclusion is trivial.

So assume $B > 0$, we have

$$\begin{aligned} 0 &\le \sum_{k=1}^{n} |B \cdot a_k - C \cdot b_k|^2 \\ &= \sum_{k=1}^{n} (B \cdot a_k - C \cdot b_k)(B \cdot \bar{a_k} - \bar{C} \cdot \bar{b_k}) \\ &= B^2 \sum_{k=1}^{n} |a_k|^2 - B \cdot \bar{C} \sum_{k=1}^{n} a_k \cdot \bar{b_k} - B \cdot C \sum_{k=1}^{n} \bar{a_k} \cdot b_k + |C|^2 \sum_{k=1}^{n} |b_k|^2 \\ &= B^2 \cdot A - B \cdot |C|^2 - B \cdot |C|^2 + |B| \cdot |C|^2 \\ &= B^2 \cdot A - B \cdot |C|^2 = B(B \cdot A - |C|^2). \end{aligned}$$

Since $B > 0$, we get $B \cdot A - |C|^2 \ge 0$. This is our desired result.

□

1.5 The Euclidean Space

Definition 1.42 (Vector)**.** For $k \in \mathbb{Z}_+$, we let $\mathbb{R}^k$ be the set of all k-tuples

$$\vec{x} = (x_1, x_2, \ldots, x_k), \quad x_i \in \mathbb{R}.$$

We call $\vec{x}$ a **point** or a **vector**. We call $\vec{0} = (0, 0, \ldots, 0)$ the **origin**.

Remark 1.43. $\mathbb{R}^k$ is an example of a **vector space**, with operations

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k) \text{ for } \vec{x}, \vec{y} \in \mathbb{R}^k$$
$$\alpha \cdot \vec{x} = (\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_k) \text{ for } x \in \mathbb{R}^k, \alpha \in \mathbb{R}.$$

Definition 1.44 (Inner Product, Norm)**.** The **inner product** (or **dot product**) is

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i \cdot y_i.$$

The **norm** of $\vec{x} \in \mathbb{R}^k$ is

$$|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}.$$

The structure now defined (the vector space $\mathbb{R}^k$ with the above inner product and norm) is called **k-dimensional Euclidean Space**.

Remark 1.45.

(a) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

(b) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$.

Lecture 6
Wednesday
October 14

Theorem 1.46. *If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then* *(a)* $|\vec{x}| \geq 0$ *and* $|\vec{x}| = 0$ *if and only if* $\vec{x} = \vec{0}$.

(b) $|\alpha \cdot \vec{x}| = |\alpha| \cdot |\vec{x}|$.

(c) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$.

(d) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$. *This is the Triangle inequality.*

(e) $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$.

Proof. (a) and (b) are obvious.

(c) This is an immediate consequence of the Cauchy-Schwarz Inequality.

(d)

$$\begin{aligned} |\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \\ &\leq |\vec{x}|^2 + 2|\vec{x}| \cdot |\vec{y}| + |\vec{y}|^2 \\ &= (|\vec{x}| + |\vec{y}|)^2. \end{aligned}$$

(e) By Triangle inequality,

$$\begin{aligned} |\vec{x} - \vec{z}| &= |(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})| \\ &\leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|. \end{aligned}$$

□

2 Introduction to Metric Space

Informally, a *set-theoretic function* f taking a set X to a set B is a mapping that to each $x \in A$ assigns a unique $y \in Y$. We write $f : X \to Y$.

Definition 2.1 (Cartesian Product)**.** Let X and Y be sets. The **Cartesian product** is the set of tuples defined as

$$A \times B = \{(x, y) : x \in X, y \in Y\}.$$

Definition 2.2 (Function)**.** A **function** $f : X \to Y$ is a subset f of $X \times Y$ such that for each $x \in X$, there is a unique $(x, y) \in f$. We then write $f(x) = y$.

The set X is called the **domain** of f. The set

$$\{y \in Y : \text{there exists an } x \text{ such that } f(x) = y\}$$

is called the **range** of f.

It is possible that the range is a proper subset of Y, while the domain of f is always equal to X.

Definition 2.3 (Image, Preimage)**.** Let $f : X \to Y$. The **image** of $A \subseteq X$ is

$$f(A) = \{f(a) : a \in A\}.$$

The **preimage** of $B \subseteq Y$ is

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

For $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$.

Definition 2.4 (Injective, Surjective, Bijective)**.** Let X and Y be sets and $f : X \to Y$ a function from X to Y. We make the following definitions.

(a) f is **injective** (or f is an **injection**) if, for all $x, x' \in X$, we have $f(x) = f(x')$ implies that $x = x'$. In other words, different elements in X get sent to different values in Y.

(b) f is **surjective** (or f is an **surjection**) if, for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. In other words, all possible values in Y are achieved.

(c) f is **bijective** (or f is a **bijection**) if it is both injective and surjective.

2.1 Finite, Countable, and Uncountable Sets

The term **cardinality** is used in mathematics to refer to the size of a set. The cardinalities of finite sets can be compared simply by attaching a natural number to each set.

Definition 2.5 (Finite, Countablity)**.** Two sets X, Y **have equal cardinality**, denoted $|X| = |Y|$, if there exists a bijection $f : X \to Y$.

(a) X is **finite** if $X = \varnothing$ or there exists $n \in \mathbb{Z}_+$, $|X| = |\{1, 2, \ldots, n\}|$, otherwise X is **infinite**.

(b) X is **countable** if it is finite of $|X| = |\mathbb{N}|$, othersie X is **uncountable**.

Example 2.6.

(a) Let $E = \{2, 4, 6, \ldots\}$ be the set of even natural numbers, then E is countable. Consider the following arrangement of sets E and $\mathbb{N}$

$$\begin{array}{lccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ E: & 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

It is certainly true that E is a proper subset of $\mathbb{N}$, and for this reason it may seem logical to say that E is a "smaller" set than $\mathbb{N}$. We can give an explicit formula for a function $f : \mathbb{N} \to E$ given by $f(n) = 2n$, which sets up a one-to-one correspondence. The definition of cardinality is quite specific, and from this point of view, E and $\mathbb{N}$ are equivalent and countable.

(b) Although $\mathbb{N}$ is contained in $\mathbb{Z}$ as a proper subset, we can show they are equivalent. This time, let $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ -\dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Definition 2.7 (Sequence)**.** A **sequence** is a function f with domain $\mathbb{N}$ or $\mathbb{Z}_+$. When $f(n) = x_n$ for each n, we write $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n\in\mathbb{Z}_+}$.

Theorem 2.8. *If X is countable and $A \subseteq X$, then A is countable.*

Proof. This is obvious if A is finite. So we assume A is infinite. Then, X is infinite so $|X| = |\mathbb{N}|$. So, we can list elements of X as

$$\{x_0, x_1, x_2, \ldots\}.$$

Let $n_0 \in \mathbb{N}$ be least with $x_{n_0} \in A$. Inductively, after choosing $n_, \ldots, n_{k-1}$, pick $n_k > n_{k-1}$ to be least with $x_{n_k} \in A$.

Now, define $f : \mathbb{N} \to A$ by $f(k) = x_{n_k}$. Then, f is a bijection.

$\square$

Theorem 2.9. *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of countable sets, then $\bigcup_{n\in\mathbb{N}} X_n$ is countable.*

Proof. For $n \in \mathbb{N}$, let $(X_{n,k})_{k\in\mathbb{N}}$ be a sequence in X_n, that uses every element of X_n at least once.

$$\begin{array}{ccccc} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & & \cdots \\ x_{3,1} & x_{3,2} & & & \cdots \\ x_{4,1} & & & & \cdots \\ \vdots & & & & \end{array}$$

Let f be the sequence $x_{0,0}, x_{1,0}, x_{0,1}, x_{2,0}, x_{1,1}, x_{0,2}, \ldots$ Then, f is onto.

Let $A = \{n \in \mathbb{N} : \forall k < n, f(k) \neq f(n)\}$. Then, A is countable by Theorem 2.8 and $f : A \to \bigcup_{n\in\mathbb{N}} X_n$ is a bijection.

□

Theorem 2.10. *If X is countable, then $X^n = X \times X \times \cdots \times X$ is countable.*

Proof. We use induction. For the base case $n = 1$, $X^1 = X$ is countable.

Assumen X^{n-1} is countable, then each set $\{x\} \times X^{n-1}$ is countable, so $X^n = \bigcup_{x\in X} \{x\} \times X^{n-1}$ is countable by Theorem 2.9.

□

Lecture 7

Friday

October 16

Corollary 2.11 (Countability of $\mathbb{Q}$)**.** *The set of all rational numbers is countable.*

Proof. By Theorem 2.10, with $n = 2$, every subset of $\mathbb{Z}^2$ is countable.

Define $f : \mathbb{Q} \to \mathbb{Z}^2$ by setting $f(q) = (a, b)$ where $a, b \in \mathbb{Z}$ satisfy $b > 0$, $\frac{a}{b} = q$, and a, b are coprime. Then f is a bijection with its image, which is countable, so $\mathbb{Q}$ is countable.

□

Theorem 2.12. *The set $\{0, 1\}^{\mathbb{N}}$ of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable.*

Proof. Let $F \subseteq \{0, 1\}^{\mathbb{N}}$ be any countably infinite set. Say $F = \{f_0, f_1, f_2, \ldots\}$, each $f_i : \mathbb{N} \to \{0, 1\}$.

Define $g : \mathbb{N} \to \{0, 1\}$ by $g(n) = 1 - f_n(n)$. Then, for any $n \in \mathbb{N}$, $g \neq f_n$ since $g_n \neq f_n(n)$. So, $g \in \{0, 1\}^{\mathbb{N}} \setminus F$. Thus, $F \neq \{0, 1\}^{\mathbb{N}}$.

□

2.2 Definition of Metric Spaces

Recall that real numbers can be ordered, which is a great feature. Now, here is another nice feature. We can define a rule for how far apart two points are.

Definition 2.13 (Metric Space)**.** A **metric space** is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ satisfies

(a) $\forall p, q \in X$, $d(p, p) = 0$; if $p \neq q$, then $d(p, q) > 0$.

(b) $\forall p, q \in X$, $d(p, q) = d(q, p)$.

(c) (Triangle inequality) $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

The function d is called a **metric**.

Example 2.14 (Real Field)**.** The set of real numbers $\mathbb{R}$ is a metric space with the metric $d(x, y) = |x - y|$.

(a) and (b) of the Definition 2.13 are obvious. The Triangle inequality (c) follows immediately from the standard Triangle inequality for real numbers

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

The metric is the *standard metric on* $\mathbb{R}$.

Example 2.15 (k-dimensional Euclidean Space)**.** The k-dimensional Euclidean space $\mathbb{R}^k$ is a metric space with the metric $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$, which is furtuer dictated by the Pythagorean theorem.

Example 2.16 (L^p Space)**.** Another metric for the k-dimensional Euclidean space $\mathbb{R}^k$ is the L^p metric

$$d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{k} (x_i - y_i)^p \right)^{1/p} \qquad (p > 1).$$

Also, we can define the L^p infinity metric for k-dimensional Euclidean Space

$$d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

Example 2.17 (Complex Field)**.** The set of complex numbers $\mathbb{C}$ is a metric space with the metric $d(z, w) = |z - w|$.

Let $z = x + iy$. Recall the *complex modulus* $|z| = \sqrt{x^2 + y^2}$. Then for any two complex numbers $z = x_1 + iy_1$ and $w = x_2 + iy_2$, the distance is

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|.$$

Example 2.18 (Set of Functions)**.** The set of $[0, 1]^{[0,1]}$ (that is, set of functions $f : [0, 1] \to [0, 1]$) is a metric space with the metric

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\}.$$

2.3 Open and Closed Sets

Definition 2.19 (Ball)**.** Let (X, d) be a metric space. For $p \in X$, $r > 0$, the **ball of radius around** x is

$$B_r(p) = \{q \in X : d(p, q) < r\}.$$

Definition 2.20 (Interior Point, Open Set)**.** Let (X, d) be a metric space and $E \subseteq X$. A point $p \in X$ is an **interior point** of E if there exists $r > 0$ such that $B_r(p) \subseteq E$. We denote the set of interior points of E as E°.

$E \subseteq X$ is an **open set** if every point of E is an interior point of E.

$E \subseteq X$ is a **neighborhood** of p if E is open and $p \in E$.

Intuitively, an open set E is a set that does not include its "boundary." Wherever we are in E, we are allowed to "wiggle" a little bit and stay in E.

Example 2.21 (Open Sets)**.**

(a) If $a < b$, then the interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is open. Indeed, if $x \in (a, b)$, set $r = \min\{x - a, b - x\}$. Both these numbers are strictly positive, since $a < x < b$, and so is their minimum. Then, the ball $\{y : |y - x| < r\}$ is a subset of (a, b).

(b) The infinite intervals $(a, +\infty)$, $(-\infty, b)$ are also open, but the intervals $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ and $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.

Remark 2.22. Let (X, d) be a metric space and $p, q \in X$. Define $r = \dfrac{1}{2}d(p, q)$. Then, $p \in B_r(p)$, $q \in B_r(q)$, and $B_r(p) \cap B_r(q) = \varnothing$ is the empty set.

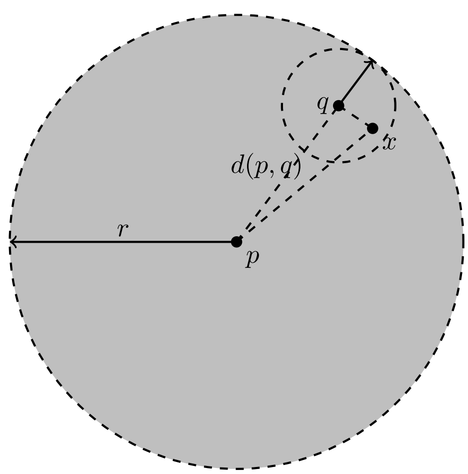
Theorem 2.23. *Let (X, d) be a metric space. For $p \in X$, $r > 0$, the ball $B_r(p)$ is always open.*

Proof. Let a point $q \in B_r(p)$ and $x \in B_{r-d(p,q)}(q)$, then

$$\begin{aligned} d(p, x) &\leq d(p, q) + d(q, x) \\ &< d(p, q) + (r - d(p, q)) = r \end{aligned}$$

so $x \in B_r(p)$. Thus, $B_{r-d(p,q)} \subseteq B_r(p)$, so q is an interior point and $B_r(p)$ is open.

□



Definition 2.24 (Limit Point, Closed Set)**.** Let (X, d) be a metric space and $E \subseteq X$. A point $p \in X$ is a **limit point** of E if for all $r > 0$,

$$(B_r(p) \setminus \{p\}) \cap E \neq \varnothing.$$

We denote the set of limit points of E is E'.

$E \subseteq X$ is a **closed set** if every limit point of E is a point of E, in short, $E' \subseteq E$.

Also, $E \subseteq X$ is **perfect** if $E' = E$, and $E \subseteq X$ is **dense** in X if $E \cup E' = X$.

Intuitively, a set E is closed if everything not in E is some distance away from E. However, not every set is either open or closed. Generally, most subsets are neither.

For instance, the set of rationals $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed. Every neighborhood of a rational number contains irrational numbers, and every neighborhood of an irrational number contains rational numbers.

Theorem 2.25. *Let (X, d) be a metric space and $E \subseteq X$. If a point $p \in E'$, then for all $r > 0$,*

$$(B_r(p) \setminus \{p\}) \cap E$$

is finite.

Proof. Let $r > 0$. Towards a contradiction, suppose not, that is, there exists $r > 0$, such that $(B_r(p) \setminus \{p\}) \cap E$ is finite.

Then,

$$t = \min\{d(p,q) : q \in (B_r(p) - \{q\}) \cap E\}$$

is positive. We have

$$\left(B_{t/2}(p) \setminus \{p\}\right) \cap E = \varnothing,$$

so $p \notin E'$.

□

Corollary 2.26. *Any finite set cannot have limit points.*

Proof. Let S be a finite set and $S = \{s_1, s_2, \ldots, s_n\}$.

Towards a contradiction, suppose S has a limit point x_0.

If $x_0 \in S$, then by Definition 2.24, for all $r > 0$, there exists $x \in S$ with $x \neq x_0$ such that $|x - x_0| < r$. Let $r = \min\{|s_i - s_j| : i \neq j\}$, then there is no $x \in S$ such that $|x - x_0| < r$ holds, a contradiction.

If $x_0 \notin S$, let $r = \min\{|x_0 - s| : s \in S\}$. Since the distance $|x_0 - s| > 0$, and there are only finitely many of them, so $r > 0$. Still, it is obvious that there is no $x \in S$ such that $|x - x_0| < r$, a contradiction.

Hence, any finite set cannot have limit points.

□

Theorem 2.27 (Duality of Open and Closed Sets)**.** *A set $E \subseteq X$ is open if and only if $E^{\complement}$ is closed.*

Proof. Assume that $E^{\complement}$ is closed. Pick an $x \in E$. Then, $x \notin E^{\complement}$ and x is not a limit point of $E^{\complement}$ since a closed set contains all its limit points. Hence, there exists $r > 0$ such that $B_r(x) \cap E^{\complement} = \varnothing$. Then, $B_r(x) \subseteq E$. Thus, x is an interior point of E. So, E is open.

Conversely, assume that E is open. Let x be a limit point of $E^{\complement}$. Then, for all $r > 0$, we have $B_r(x) \cap E^{\complement} \neq \varnothing$. So, x is not an interior point of E, we have $x \notin E$, which means $x \in E^{\complement}$.

□

Corollary 2.28. *A set $E \subseteq X$ is closed if and only if ites complement is open.*

Definition 2.29 (Bounded Set)**.** Let (X, d) be a metric space. $E \subseteq X$ is a **bounded set** if there exists $M \in \mathbb{R}$ and a point $p \in X$ such that $E \subseteq B_M(p)$.

Lecture 8

Monday

October 19

Theorem 2.30. *Let $\{E_\alpha\}$ be a finite (or infinite) collection of sets E_α. Then,*

$$\left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}} = \bigcap_\alpha E_\alpha^{\mathsf{c}}.$$

Proof. We use element tracing to prove this theorem.

Let $x \in \left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}}$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for any α, then $x \in E_\alpha^{\mathsf{c}}$ for any α, so we have $x \in \bigcap_\alpha E_\alpha^{\mathsf{c}}$. Thus, $\left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}} \subseteq \bigcap_\alpha E_\alpha^{\mathsf{c}}$

Let $x \in \bigcap_\alpha E_\alpha^{\mathsf{c}}$, then $x \in E_\alpha^{\mathsf{c}}$ for every α, hence $x \notin E_\alpha$ for any α, so $x \notin \bigcup_\alpha E_\alpha$ and $x \in \left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}}$. Thus, $\bigcap_\alpha E_\alpha^{\mathsf{c}} \subseteq \left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}}$.

Hence $\left(\bigcup_\alpha E_\alpha\right)^{\mathsf{c}} = \bigcap_\alpha E_\alpha^{\mathsf{c}}$.

□

Theorem 2.31. *Let (X, d) be a metric space. Let A be any set (possibly uncountable).*

(a) For any $\alpha \in A$, $U_\alpha \subseteq X$ is open, then $\bigcup_{\alpha \in A} U_\alpha$ is open.

(b) For any $\alpha \in A$, $F_\alpha \subseteq X$ is closed, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(c) If $U_1, \ldots, U_n \subseteq X$ are open, then $\bigcap_{i=1}^{n} U_i$ is open.

(d) If $F_1, \ldots, F_n \subseteq X$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.

Proof.

(a) If $x \in \bigcup_{\alpha \in A} U_\alpha$, then there exists a $\beta \in A$ with $x \in U_\beta$. Since U_β is open, there exists an $r > 0$ such that $B_r(x) \subseteq U_\beta \subseteq \bigcup_{\alpha \in A} U_\alpha$, so x is an interior point of $\bigcup_{\alpha \in A} U_\alpha$ so $\bigcup_{\alpha \in A} U_\alpha$ is open.

(b) By Theorem 2.30, $\left(\bigcup_{\alpha \in A} F_\alpha\right)^{\mathsf{c}} = \bigcup_{\alpha \in A} F_\alpha^{\mathsf{c}}$ is open by (a), so $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(c) Let $x \in \bigcap_{i=1}^{n} U_i$. Each U_i is open, so we can pick $r_i > 0$ with $B_{r_i}(x_i) \subseteq U_i$.

Let $r = \min\{r_i : 1 \le i \le n\}$, then $r > 0$ and $B_r(x) \subseteq \bigcap_{i=1}^{n} U_i$. Thus, $\bigcap_{i=1}^{n} U_i$ is open.

(d) By taking complements, this is a result from Part (c).

□

Remark 2.32. Finiteness assumption is nnecessary in (c) and (d) in Theorem 2.31. Consider $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq \mathbb{R}$ is open but $\bigcap_{n\in\mathbb{Z}_+} U_n = \{0\}$ is not open. Similarly, consider $V_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$, which is closed, but $\bigcup_{n\in\mathbb{Z}_+} V_n = (-1,1)$, which is open.

The reason why argument in (c) where we take the minimum of a series of radii fails in the infinite case is that the sequence of infinitely many radii doesn't necessarily have a minimum.

Definition 2.33 (Closure)**.** Let (X,d) be a metric space and $E \subseteq X$. The **closure** of E is defined by the set

$$\overline{E} = E \cup E'.$$

Recall that E' denote the set of all limit points of E in X.

Theorem 2.34. *Let (X,d) be a metric space and $E \subseteq X$. Then,*

(a) $\overline{E}$ is closed.

(b) $E = \overline{E}$ if and only if E is closed.

(c) if $F \subseteq X$ is closed and $E \subseteq F$, we have $\overline{E} \subseteq F$.

Proof.

(a) Let $x \in (\overline{E})'$. If $x \in E$, then we are done.

Since $x \in \overline{E}$, so we assume that $x \notin E$. Let $r > 0$. Since $x \in (\overline{E})'$, we have $(B_r(x)\setminus\{x\})\cap\overline{E} \neq \varnothing$. Pick $y \in (B_r(x)\setminus\{x\})\cap\overline{E}$. Let $s = \min\{r - d(x,y), d(x,y)\} > 0$. We note that $B_s(y) \subseteq B_r(x)\setminus\{x\}$. Since $y \in \overline{E}$, we must have $B_s(y)\cap E \neq \varnothing$, so there is a point $p \in B_s(y)\cap E$, and we have $p \in (B_r(x)\setminus\{x\})\cap E$. Thus $(B_r(x)\setminus\{x\})\cap E \neq \varnothing$ and hence $x \in E' \subseteq \overline{E}$.

(b) Suppose that $E = \overline{E}$. Since $\overline{E}$ is closed by Part (a), we conclude that E is closed.

Conversely, suppose that E is closed. Then, by Definition 2.24, $E' \subseteq E$. Hence, $E = E \cup E' = \overline{E}$.

(c) We first show a quick result that if $E \subseteq F$, then $E' \subseteq F'$.

Let $p \in E'$, then p is a limit point of E. Then, for all $r > 0$, we have $(B_r(p) - \{p\}) \cap E \neq \varnothing$, which means there exists a $q \in (B_r(p) - \{p\}) \cap E$ such that $q \in E$, so $q \in F$. Then, $q \in (B_r(p) - \{p\}) \cap F$, so q is a limit point of F. Thus, $E' \subseteq F'$.

So, if F is closed and $E \subseteq F$, then $\overline{E} = E \cup E' \subseteq E \cup F' \subseteq F$.

□

Remark 2.35. Let (X, d) be a metric space and $E \subseteq X$. By Theorem 2.34 (a) and (c), $\overline{E}$ is the *smallest* closed subset of X that contains E.

Theorem 2.36. *If $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E \in \overline{E}$.*

Proof. Let $y = \sup E$. If $y \in E$, then $y \in \overline{E}$ and we are done.

So, assume $y \notin E$ and let $y > 0$. Since $y = \sup E$ and $y - r < y$, there must be $x \in E$ with $x > y - r$. Since $y = \sup E$ and $y \notin E$, we have $x < y$. So, $x \in (B_r(y) \setminus \{y\}) \cap E$. We conclude $y \in E' \subseteq \overline{E}$.

□

Let (X, d) be a metric space. If $Y \subseteq X$ and let (Y, d_Y) is a metric space where d_Y is the restriction of d to $Y \times Y$ for $y_1, y_2 \in Y$, we have $d_Y(y_1, y_2) = d(y_1, y_2)$.

Definition 2.37 (Relatively Open Sets)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. If $E \subseteq Y \subseteq X$, we say E is **open relative to** Y if E is open in (Y, d_Y). Equivalently, E is open relative to Y if for any point $p \in E$, there exists $r > 0$ such that $B_r(p) \cap Y \subset E$.

Theorem 2.38. *Let (X, d_X) and (Y, d_Y) be metric spaces. If $E \subseteq Y \subseteq X$, then E is open relative to Y if and only if there is an open set $U \subseteq X$ with $E = U \cap Y$.*

Proof. Assume E is open relative to Y. For each $p \in E$, we pick $r_p > 0$ with $B_{r_p}(p) \cap Y \subseteq E$. Let $U = \bigcup_{p \in E} B_{r_p}(p)$. Then, U is open and $U \cap Y \subseteq E$. For every $p \in E$, we have $p \in B_{r_p}(p) \subseteq U \cap Y$, so $E \subseteq U \cap Y$. Thus, $E = U \cap Y$.

Assume there is an open set $U \subseteq X$ with $E = U \cap Y$. Let $p \in E$. Then, $p \in U$ and U is open, so there is $r > 0$ with $B_r(p) \subseteq U$. Hence, $B_r(p) \cap Y \subseteq U \cap Y = E$. So, E is open relative to Y.

□

Lecture 9
Wednesday
October 21

2.4 Compact Sets

Recall that finite sets have a lot of great properties, they are bounded and closed sets. Also, finite sets have no limit points. In the real field, they contain their supremum and infimum.

Definition 2.39 (Open Cover)**.** Let (X, d) be a metric space. An **open cover** of $E \subseteq X$ is a collection $\{U_\alpha : \alpha \in A\}$ of open subsets $U_\alpha \subseteq X$ with $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Definition 2.40 (Compact Sets)**.** Let (X, d) be a metric space. A subset $K \subseteq X$ is **compact** if every open cover $\{U_\alpha : \alpha \in A\}$ of K contains a **finite subcover**, meaning that there are $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. In other words, K is compact if the following statement is true

For every collection $\{U_\alpha : \alpha \in A\}$ with each U_α open,

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha \Rightarrow \exists n \, \exists \alpha_1, \alpha_2, \ldots, \alpha_n \in A, K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Remark 2.41. Finite sets are always compact.

Example 2.42. The set $\left[\frac{1}{2}, 1\right) \subseteq \mathbb{R}$ has cover $\bigcup_{n=3}^{\infty} V_n$, where $V_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Another cover may be $\{(0, 2)\}$, but it is a boring cover. However, $\left[\frac{1}{2}, 1\right)$ is not compact since there is a open cover with no finite subcover. The cover $V_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ has no finite subcover. Since $n \to \infty$ is required for $\{V_n\}$ to cover all the points in the interval.

Example 2.43. $\mathbb{Z} \subseteq \mathbb{R}$ is not compact. To prove this, *all we need to do is find a open cover that has no finite subcover.* For the open cover $\{B_r(x) : x \in \mathbb{Z}, r > 0\}$, the balls covering individual integers, we cannot even remove a single one of them, otherwise $\mathbb{Z}$ cannot be covered.

Theorem 2.44. *Let (X, d_X) and (Y, d_Y) be metric spaces. Assume $K \subseteq Y \subseteq X$. Then, K is compact relative to Y if and only if K is compact relative to X.*

Proof. Assume that K is compact relative to Y. Suppose $K \subseteq \bigcup_{\alpha \in A} U_\alpha$, each U_α open in X. Then, $U_\alpha \cap Y$ is open relative to Y by Theorem 2.38 and since $K \subseteq Y$, we have $K \subseteq Y \cap \left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} (Y \cap U_\alpha)$, so there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ by Definition 2.40, so K is compact relative to X.

Assume that K is compact relative to X. Suppose $K \subseteq \bigcup_{\alpha \in A} V_\alpha$ each V_α open relative to Y. By Theorem 2.38, there are open sets in X, $U_\alpha \subseteq X$ with $V_\alpha = U_\alpha \cap Y$. Then, $K \subseteq \bigcup_{\alpha \in A} U_\alpha$, so there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Since $K \subseteq Y$, so $K \subseteq Y \cap \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) = \bigcup_{i=1}^{n} (U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i}$. So K is compact relative to Y.

□

Theorem 2.45. *Compact sets are always closed.*

Proof. Let (X, d) be a metric space. Let $K \subseteq X$ be compact, we will show $X \setminus K$ is open, so pick $p \in X \setminus K$.

For each $q \in K$, let $r = \frac{1}{3} d(p, q)$, set

$$U_q = B_r(q) \quad \text{and} \quad V_q = B_r(p).$$

We have $K \subseteq \bigcup_{q \in K} U_q$, so by compactness, there are $q_1, q_2, \ldots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$.

Then, let $r_0 = \frac{1}{3} \min\{d(p, q_1), d(p, q_2), \ldots, d(p, q_n)\}$, we have $\bigcap_{i=1}^{n} V_{q_i} = B_{r_0}(p)$ is disjoint with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$ hence a ball around p is contained $X \setminus K$, that is $B_{r_0}(p) \subseteq X \setminus K$. Thus K is closed.

□

Theorem 2.46. *Let (X, d) be a metric space and $K \subseteq X$. If K is compact and $F \subseteq K$ is closed, then F is compact as well.*

Proof. Say $\{U_\alpha : \alpha \in A\}$ is an open cover of F. $F^{\complement}$ is open so $\{F^{\complement}\} \cup \{U_\alpha : \alpha \in A\}$ is an open cover of K. So there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $F \subseteq K \subseteq F^{\complement} \cup \bigcup_{i=1}^{n} U_{\alpha_i}$, so $F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

□

Corollary 2.47. *If K is compact and F is closed, then $K \cap F$ is compact.*

Theorem 2.48 (Finite Intersection Property)**.** *If $K_\alpha \subseteq X$ is compact for every $\alpha \in A$ and if the intersection of every finite collection from $\{K_\alpha : \alpha \in A\}$ is nonempty, then $\bigcap_{\alpha \in A} K_\alpha \neq \varnothing$.*

Proof. Towards a contradiction, assume $\bigcap_{\alpha \in A} K_\alpha = \varnothing$. Fix any $K \in \{K_\alpha : \alpha \in A\}$. Then,

$$K \subseteq X = X \setminus \varnothing = X \setminus \bigcap_{\alpha \in A} K_\alpha = \bigcup_{\alpha \in A} (X \setminus K_\alpha)$$

and each $X \setminus K_\alpha$ is open, so there are $\alpha_1, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i})$. Then,

$$K \cap \bigcap_{i=1}^{n} K_{\alpha_i} \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i}) \cap \bigcap_{i=1}^{n} K_{\alpha_i} = \varnothing$$

so contradiction here.

□

Corollary 2.49. *If $K_n \neq \varnothing$ is compact and $K_{n+1} \subseteq K_n$ for all n, then $\bigcap_{n \in \mathbb{N}} K_n \neq \varnothing$.*

Theorem 2.50. *Let (X, d) be a metric space and $K \subseteq X$. If K is compact and $E \subseteq K$ is infinite, then $E' \cap K \neq \varnothing$.*

Lecture 10
Friday
October 23

Proof. Towards a contradiction, assume $E' \cap K = \varnothing$, which means E has no limit points in K. Further, this means for each $q \in K$ there is $r_q > 0$ with $(B_{r_q}(q) \setminus \{q\}) \cap E = \varnothing$, meaning that $U_q = B_{r_q}(q)$ satisfies $U_q \cap E \subseteq \{q\}$.

Then $K \subseteq \bigcup_{q \in K} U_q$ so by compactness, there are $q_1, \ldots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$. Then,

$$|E \cap K| \leq \left| E \cap \bigcup_{i=1}^{n} U_{q_i} \right| = \left| \bigcup_{i=1}^{n} (E \cap U_{q_i}) \right| = |\{q_1, \ldots, q_n\}| = n.$$

Since $E \subseteq K$, so $|E| = |E \cap K| \leq n$, which means E is finite, that's a contradiction.

□

Remark 2.51. K is compact, so K is closed. Thus, $E' \subseteq K' \subseteq K$.

Theorem 2.52 (Nested Interval Property)**.** *Let* $I_n = [a_n, b_n]$ *with* $a_n \leq b_n$ *be a closed set on* $\mathbb{R}$*, and* $I_{n+1} \subseteq I_n$ *for all* $n \in \mathbb{N}$*, then* $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$.

Proof. For all $n, m \in \mathbb{N}$, we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$, so b_m is an upperbound to $\{a_n : n \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. So, $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ exists and $a_m \leq \alpha \leq b_m$ for all m. Thus, $\alpha \in \bigcap_{n\in\mathbb{N}} I_n$. □

This theorem is not true if we consider open intervals. For example, we have $\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Theorem 2.53 (Uncountability of Real Numbers)**.** *The set of real numbers is uncountable.*

Proof. Towards a contradiction, suppose $\mathbb{R}$ is countable, so we can list the elements of $\mathbb{R}$ as $\{x_1, x_2, x_3, \ldots\}$.

Pick a closed interval I_1 such that $x_1 \notin I_1$. Then, pick I_2 such that $I_2 \subseteq I_1$ and $x_2 \notin I_2$. Pick I_3 such that $I_3 \subseteq I_2$ and $x_3 \notin I_3$. Repeat this process. From Theorem 2.52, we know that $\bigcap_{n\in\mathbb{N}} I_n$ is nonempty, so there is some element in our union that is not on the list $\{x_1, x_2, x_3, \ldots\}$. Hence $\mathbb{R}$ is uncountable. □

Theorem 2.54. *Suppose* $C_n = [a_{n_1}, b_{n_1}] \times [a_{n_2}, b_{n_2}] \times \cdots \times [a_{n_k}, b_{n_k}] \subseteq \mathbb{R}^k$ *and* $C_n \neq \varnothing$ *and* $C_{n+1} \subseteq C_n$ *for all* n*. Then* $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$.

Proof. This is obvious since

$$\bigcap_{n\in\mathbb{N}} C_n = \left(\bigcap_{n\in\mathbb{N}} [a_{n_1}, b_{n_1}]\right) \times \cdots \times \left(\bigcap_{n\in\mathbb{N}} [a_{n_k}, b_{n_k}]\right) \neq \varnothing.$$

□

Theorem 2.55 (Compactness of k-cell)**.** $C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ *is compact.*

Proof. Define $\delta = \sqrt{\sum_{i=1}^{k} |b_i - a_i|^2}$, the length of the longest diagonal.

Towards a contradiction, suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of C having no finite subcover of C.

Cut at the midpoint of each side of C to divide C into 2^k many rectangles.

TODO

□

Theorem 2.56. *Let* $E \subseteq \mathbb{R}^k$*, the following are equivalent*

(a) *E is closed and bounded.*

(b) *E is compact.*

(c) *Every infinite subset of E has a limit point in E.*

Proof.

(a) ⇒ (b). By assumption, E is bounded, which means $E \subseteq C$ for some $C = [a_1, b_1] \times \cdots \times [a_k, b_k]$. Since C is compact and $E \subseteq C$ is closed, so E is compact.

(b) ⇒ (c). This is Theorem 2.50.

(c) ⇒ (a). Let $K \subseteq \mathbb{R}^k$ have the property that every infinite subset of K has a limit point in K.

(1) Towards a contradiction, assume that K is not bounded. Then, for each $n \in \mathbb{N}$, there exists $\vec{x_n} \in E$ with $|\vec{x_n}| > n$. Define $S = \{\vec{x_n} : n \in \mathbb{N}\}$. Let $\vec{p} \in \mathbb{R}^k$. Then,

□

Remark 2.57. Equivalence of (a) and (b) is known as the *Heine-Borel Theorem*, for Euclidena space. Equivalence of (b) and (c) applies for all metric spaces.

Theorem 2.58 (Bolzano-Weierstrass)**.** *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$*.*

Proof. Since the set is bounded, it lives in a compact k-cell. Infinite subsets of compact sets have limit points, so the set has a limit point.

□

Lecture 11
Monday
October 26

2.5 Perfect Sets

Recall that in a metric space (X, d) and $E \subseteq X$, if p is a limit point, then for all $r > 0$, the set $(B_r(p) \setminus \{p\}) \cap E$ is infinite. So, if U is open and $U \cap E' \neq \varnothing$, then $U \cap E$ is infinite.

Corollary 2.59. *For any metric space* (X, d)*,*

$$\overline{B_r(p)} \subseteq \{q \in X : d(p, q) \leq r\}.$$

Definition 2.60 (Perfect Sets)**.** Let (X, d) be a metric space and $E \subseteq X$. E is **perfect** if E is closed and every point of E is a limit point of E.

Theorem 2.61. *Let P be a nonempty perfect set in* $\mathbb{R}^k$*. Then P is uncountable.*

Proof. Since $P' = P \neq \varnothing$, we must have that P is infinite. Towards a contradiction, suppose

$$p = \{\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots\}.$$

We will inductively build sets V_n with $n \in \mathbb{N}$ satisfying following

(a) V_n is open

(b) $V_n \cap P \neq \varnothing$

(c) When $n \geq 1$, $\overline{V_n} \subseteq V_{n-1}$

(d) When $n \geq 1$, $\vec{x}_{n-1} \notin V_n$

To begin, set $V_0 = B_1(\vec{x_0})$. Now, inductively assume that $V_0, \ldots, V_n$ have been defined. (a) and (b) imply that $V_n \cap P$ is infinite, so we can pick $\vec{y} \in V_n \cap P$ with $\vec{y} \neq \vec{x_n}$.

TODO. (I am confused.)

□

Corollary 2.62. *For all $a, b \in \mathbb{R}$, $[a, b]$ is perfect and hence uncountable. In particular, $\mathbb{R}$ is uncountable.*

2.6 The Cantor Set

Definition 2.63 (Cantor Set)**.** The **Cantor Ternary Set** is constructed by iteratively remove the middle third from a set of line segments. One starts by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the interval $[0, 1]$, leaving two line segments

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Next, the open middle third of each of those remaining segment is deleted, leaving four line segments

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

The process continues infinitely. We define the **Cantor Ternary Set** to be

$$P = \bigcap_{n=1}^{\infty} E_n.$$

Remark 2.64. Here are two quick observations (properties) of the Cantor Set P

(a) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$.

(b) E_n is the union of 2^n intervals, each with length 3^{-n}.

Theorem 2.65 (Compactness)**.** *The Cantor Set P is compact.*

Theorem 2.66. *The Cantor Set P has a length of zero, which means that it contains no segments (intervals).*

Remark 2.67. This theorem is actually saying that the Lebesgue measure of the Cantor Set P is 0.

Theorem 2.68. *The Cantor Set P is perfect.*

Proposition 2.69 (More Fun Facts on Ternary Expansion)**.** *Let P denote the Cantor Set.*

(a) Let $x = \overline{0.a_1a_2a_3\ldots}$ be the base 3 expansion of a number $x \in [0, 1]$. Then, $x \in P$ if and only if $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$.

(b) P is uncountable.

(c) $\frac{1}{4} \in P$, but $\frac{1}{4}$ is not an endpoint of any of the intervals in any of the sets E_k for $k \in \mathbb{N}$, where E_k is defined before.

Lecture 12
Friday
October 30

2.7 Connected Sets

Definition 2.70 (Separated Sets)**.** Let (X, d) be a metric space and $A, B \subseteq X$. We say A and B are **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, that is, if no point of A lies in the closure of B adn no point of B lies in the closure of A.

Definition 2.71 (Connected Set)**.** Let (X, d) be a metric space and $E \subseteq X$. We say E is **connected** if E is not a union of two nonempty separated sets.

Remark 2.72. "Separated" is a stronger definition than "disjoint."

(a) $(0, 1)$ and $(1, 2)$ are separated (and disjoint).

(b) $(0, 1]$ and $(1, 2)$ are not separated but disjoint.

Theorem 2.73. *$E \subseteq \mathbb{R}$ is connected if and only if for all $x \leq y \in E$, $[x, y] \subseteq E$.*

Proof. We prove both sides by contrapositive.

Assume there are $x \leq y \in E$ with $[x, y] \nsubseteq E$. Pick $x < z < y$ with $z \notin E$. Set $A = E \cap (-\infty, z)$ and $B = E \cap (z, +\infty)$. Then, A and B are nonempty since $x \in A$ and $y \in B$. And since $\overline{A} \subseteq (-\infty, z]$ and $\overline{B} \subseteq [z, +\infty)$, so A and B are separated. Also, with $A \cup B = E$, thus E is not connected.

Now, assume that E is not connected. Let A and B be nonempty, separated, and $E = A \cup B$. Pick $x \in A$ and $y \in B$. Without loss of generality, we can assume $x < y$. Set $z = \sup(A \cap [x, y])$. Then, $z \in \overline{A \cap [x, y]} \subseteq \overline{A}$, hence $z \notin B$ by the assumption that E is not connected. If $z \notin A$, then $z \in E$, hence $[x, y] \nsubseteq E$ as $z \in [x, y] \setminus E$. If $z \in A$, then $z \notin \overline{B}$. Since $z \in \overline{(z, y]}$, we must have $(z, y] \nsubseteq B$, so there exists $z' \in (z, y] \setminus B$. Also, $z' \notin A$ since $z' > z$. Thus, $z' \in [x, y] \setminus (A \cup B) = [x, y] \setminus E$. So, $[x, y] \nsubseteq E$.

$\square$

3 Sequences and Series

So far, we have introduced sets as well as the number systems. Now, we will study sequences of numbers. Sequences are, basically, countably many numbers arranged in an order that may or may not exhibit certain patterns.

Definition 3.1 (Sequence)**.** A sequence $(p_n)_{n\in\mathbb{N}}$ in X is a function $f : \mathbb{N} \to X$ that maps n to a point $p_n \in X$.

3.1 Convergent Sequences

Definition 3.2 (Convergence)**.** A sequence $(p_n)_{n\in\mathbb{N}}$ in a metric space (X, d) **converges** if there is a point $p \in X$ with the following property: for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, then $d(p_n, p) < \varepsilon$.

In this case, we say $(p_n)_{n\in\mathbb{N}}$ **converges to** p or has a **limit** p and write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

If $(p_n)_{n\in\mathbb{N}}$ does not converge, we say it **diverges**.

The definition is a bit intimidating at first, but what it says is that for any open ball bround the limit, the elements of a sequence eventually stay in the ball.

Definition 3.3. The **range** of $(p_n)_{n\in\mathbb{N}}$ is $\{p_n : n \in \mathbb{N}\}$. Further, $(p_n)_{n\in\mathbb{N}}$ is **bounded** if the range is bounded.

Since sequences allow repetitions, the range of a sequence can be finite. For instance, consider the range of $p_n = (-1)^n \in \mathbb{R}$.

Example 3.4. The sequence $p_n = \dfrac{n+1}{n}$ converges and $p_n \to 1$.

We want to show that given an $\varepsilon > 0$, there is a positive integer N that works. Note that

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon$$

holds if $n > \dfrac{1}{\varepsilon}$ for positive n ε. Therefore, we can consider any $N > \dfrac{1}{\varepsilon}$.

Proof. Given $\varepsilon > 0$, pick $N = \lceil\frac{1}{\varepsilon}\rceil + 1$. Then, for all $n \geq N$, we have $n > \dfrac{1}{\varepsilon}$. Hence, $\dfrac{1}{n} < \varepsilon$. So,

$$|p_n - p| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon.$$

□

Theorem 3.5. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d).*

(a) (p_n) converges to $p \in X$ if and only if for every $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, then $p_n \in B_\varepsilon(p)$.

(b) If (p_n) converges to $p \in X$ and $p' \in X$, then $p = p'$. (This is basically saying the limit of a sequence is unique.)

(c) If (p_n) converges, then (p_n) is bounded.

(d) If $E \subseteq X$ and p is a limit point of E, then there exists a sequence (p_n) in E such that $p_n \to p$.

(e) If $p_n \to p$ and $p_n \in E$ for all n, then $p \in \overline{E}$.

Proof.

(a) This follows from the fact

$$d(p_n, p) < \varepsilon \iff p_n \in B_\varepsilon(p).$$

(b) Towards a contradiction, assume $p_n \to p$ and $p_n \to p'$ with $p \neq p'$. Let $\varepsilon = d(p, p') > 0$. Pick N_1 such that for any $n \geq N_1$, $d(p_n, p) < \dfrac{\varepsilon}{2}$ and N_2 such that for any $n \geq N_2$, $d(p_n, p') < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then, $n \geq N$ implies

$$\varepsilon = d(p, p') \leq d(p, p_n) + d(p_n, p') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So this says $\varepsilon < \varepsilon$, a contradiction. Thus, we are done.

(c) Suppose $p_n \to p$, then there exists a positive integer N such that for all $n \geq N$, we have $d(p_n, p) < 1$, that is choosing a ball with radius 1. Set

$$r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}$$

then $d(p_n, p) \leq r$ for all n. Thus, $\{p_n : n \in \mathbb{N}\}$ is bounded.

(d) For each $n \in \mathbb{Z}_+$, there exists a point $p_n \in E$ such that $d(p_n, p) < \dfrac{1}{n}$. In short, pick $p_n \in B_{1/n}(p) \cap E$. Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $N \cdot \varepsilon > 1$ by Archimedean property. For $n \geq N$, we have

$$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $p_n \to p$.

(e) If $p \in E$, then we are done.

So, assume $p \notin E$. Then, for every $r > 0$, there exists an $n \in \mathbb{N}$ such that

$$p_n \in B_r(p) \cap E = (B_r(p) \setminus \{p\}) \cap E.$$

Thus, $(B_r(p) \setminus \{p\}) \cap E \neq \varnothing$ so $p \in E'$.

Hence, $p \in \overline{E}$.

$\square$

Theorem 3.6 (Limit Rules)**.** *Suppose $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$. Then,*

(a) $\lim_{n\to\infty}(s_n + t_n) = s + t$.

(b) For all $c \in \mathbb{C}$, $\lim_{n\to\infty}(s_n + c) = s + c$ and $\lim_{n\to\infty}(c \cdot s_n) = c \cdot s$.

(c) $\lim_{n\to\infty} s_n \cdot t_n = s \cdot t$.

(d) $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ *if $s \neq 0$ and $s_n \neq 0$ for all $n \in \mathbb{N}$.*

(e) If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.

Proof.

(a) For $\varepsilon > 0$, pick N_1 such that for any $n \geq N_1$, $|s_n - s| < \frac{\varepsilon}{2}$ and N_2 such that for any $n \geq N_2$, $|t_n - t| < \frac{\varepsilon}{2}$. Then, for $n \geq \max\{N_1, N_2\}$, we have

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $s_n + t_n \to s + t$.

(b) This is a trivial result, left as an exercise.

(c) By arithmetic manipulation, we use a trick

$$s_n t_n = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) + st.$$

For $\varepsilon > 0$, pick N_1 such that for any $n \geq N_1$, $|s - s_n| < \sqrt{\varepsilon}$, and N_2 such that for any $n \geq N_2$, $|t - t_n| < \sqrt{\varepsilon}$. Then, for $n \geq \max\{N_1, N_2\}$, we have $|(s_n - s)(t_n - t)| < \varepsilon$, so that

$$\lim_{n\to\infty}(s_n - s)(t_n - t) = 0.$$

Now, apply (a) and (b), we get

$$\lim_{n\to\infty} s_n \cdot t_n = s \cdot t.$$

(d) By arithmetic manipulation, we consider the trick

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|s s_n|}.$$

Pick $m \in \mathbb{N}$ such that for all $n \geq m$, $|s_n - s| < \frac{1}{2}|s|$. Then, for $n \geq m$, $|s_n| > \frac{1}{2}|s|$ by triangle inequality. Now, let $\varepsilon > 0$ and pick $N \geq m$ such that for all $n \geq N$,

$$|s_n - s| < \frac{1}{2}|s|^2\varepsilon.$$

Then, for $n \geq N$, we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|ss_n|} < \frac{1}{\frac{1}{2}|s|^2} \cdot |s - s_n| < \varepsilon.$$

Thus $\dfrac{1}{s_n} \to \dfrac{1}{s}$.

(e) For every n, we have $t_n - s_n$ lies in the closed set $[0, +\infty) \subseteq \mathbb{R} \subseteq \mathbb{C}$. Thus,

$$t - s = \lim_{n\to\infty} t_n - \lim_{n\to\infty} s_n = \lim_{n\to\infty} (t_n - s_n) \in [0, +\infty)$$

so $s \leq t$.

□

Lecture 13 November 2 Monday

Theorem 3.7. *Suppose $\vec{x_n} \in \mathbb{R}^k$ for $n \in \mathbb{N}$ and $\vec{x_n} = (\alpha_{1,n}, \alpha_{2,n}, \dots, \alpha_{k,n})$. Then, $\vec{x_n}$ converges to $\vec{x} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ if and only if*

$$\lim_{n\to\infty} \alpha_{i,n} = \alpha_i \quad 1 \leq i \leq k.$$

Proof. Assume that $\vec{x_n} \to \vec{x}$. Then, by the definition of the norm (distance in $\mathbb{R}^k$) we have

$$\begin{aligned} |\vec{x_n} - \vec{x}| &= \left(\sum_{i=1}^{k} (\alpha_{i,n} - \alpha_i)^2\right)^{1/2} \\ &\geq \left[(\alpha_{i,n} - \alpha_i)^2\right]^{1/2} \\ &\geq |\alpha_{i,n} - \alpha_i| \end{aligned}$$

for all $1 \leq i \leq k$. Hence, $|\alpha_{i,n} - \alpha_i| \leq |\vec{x_n} - \vec{x}|$ immediately shows that $\lim_{n\to\infty} \alpha_{i,n} = \alpha_i$.

Assume that $\lim_{n\to\infty} \alpha_{i,n} = \alpha_i$ for all $1 \leq i \leq k$ holds, then to each $\varepsilon > 0$ there exists an integer N such that for all $n \geq N$,

$$|\alpha_{i,n} - \alpha_i| < \frac{\varepsilon}{\sqrt{k}} \quad 1 \leq i \leq k.$$

Hence, $|\vec{x_n} - \vec{x}| = \left(\sum_{i=1}^{k} |\alpha_{i,n} - \alpha_i|^2\right)^{1/2} < \varepsilon$.

□

Theorem 3.8. *Let $(\vec{x_n})$ and $(\vec{y_n})$ be sequences in $\mathbb{R}^k$ with $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$. Let (β_n) be a sequence in $\mathbb{R}$ with $\beta_n \to \beta$. Then,*

(a) $\lim_{n\to\infty} \vec{x_n} + \vec{y_n} = \vec{x} + \vec{y}$.

(b) $\lim_{n\to\infty} \vec{x_n} \cdot \vec{y_n} = \vec{x} \cdot \vec{y}$.

(c) $\beta_n \cdot \vec{x_n} = \beta \cdot \vec{x}$.

Proof. This follows from Theorem 3.6 and Theorem 3.7.

□

3.2 Subsequence

Definition 3.9 (Subsequence)**.** Given a sequence (p_n), consider a sequence (n_k) of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then, the sequence (p_{n_k}) is called a **subsequence** of (p_n). If (p_{n_k}) converges, its limit is called a **subsequential limit** of (p_n).

Theorem 3.10. *A sequence (p_n) converges to p if and only if every subsequence (p_n) converges to p.*

Example 3.11. Consider the following sequence

$$1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \frac{1}{4}, \pi, \cdots$$

This sequence is divergent, but it has a convergent subsequence, namely

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

Thus, a divergent sequence may have a convergent subsequence.

Example 3.12. The set of natural numbers $\mathbb{N}$ does not contain any convergent subsequence. Thus, not every sequence contains a convergent subsequence.

Example 3.13. Consider we are using the metric space (X, d) with $X = \mathbb{Q}$. Consider the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots$$

does not converge in $\mathbb{Q}$, and its subsequence does not converge in $\mathbb{Q}$. While we switch our $\mathbb{Q}$ to $\mathbb{R}$, it will converge to π. Thus, not every bounded sequence contains a convergent subsequence. For general metric spaces, this is not true.

Theorem 3.14. *A point q in a metric space (X, d) is a subsequential limit of $(p_n)_{n\in\mathbb{N}}$ if and only if for all $r > 0$, the set $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite.*

Proof. Assume (p_{n_i}) is a subsequence with $p_{n_i} \to q$. Let $r > 0$. Pick an $N \in \mathbb{N}$ such that for all $i \geq N$, we have $d(p_{n_i}, q) < r$. Then,

$$\{n_N, n_{N+1}, n_{N+2}, \cdots\} \subseteq \{n \in \mathbb{N} : p_n \in B_r(q)\}.$$

Since $\{n_N, n_{N+1}, n_{N+2}, \cdots\}$ is infinite, so $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite.

Now, assume that for all $r > 0$, $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite. Then, pick any n_1 with $p_{n_1} \in B_1(q)$. Once $n_1 < \cdots n_{i-1}$ have been defined, pick $n_i > n_{i-1}$ with $p_{n_i} \in B_{1/i}(q)$. This defines a subsequence $(p_{n_i})_{i\in\mathbb{N}}$. Fix an $\varepsilon > 0$. We pick an $N \in \mathbb{Z}_+$ with $\frac{1}{N} < \varepsilon$. Then, for all $i \geq N$,

$$d(p_{n_i}, q) < \frac{1}{i} \leq \frac{1}{N} < \varepsilon$$

since $p_{n_i} \in B_{1/i}(q)$. Thus, $p_{n_i} \to q$.

□

Corollary 3.15. *If q is a limit point of the set $\{p_n : n \in \mathbb{N}\}$, then q is a subsequential limit of (p_n).*

Proof. Fix $r > 0$. Define a set $I = \{n \in \mathbb{N} : p_n \in B_r(q)\}$. Then,

$$\{p_i : i \in I\} = \{p_n : n \in \mathbb{N}\} \cap B_r(q)$$

and the set on the right is infinite since $q \in \{p_n : n \in \mathbb{N}\}'$. Thus, $\{p_i : i \in I\}$ is infinite, so I must be infinite as well.

□

Theorem 3.16.

(a) If (p_n) is a sequence in a compact metric space (X, d), then (p_n) has a subsequential limit.

(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. (a) Let $E = \{p_n : n \in \mathbb{N}\}$.

If E is finite, then there must be some $p \in E$ and $n_1 < n_2 < n_3 < \cdots$ with $p_{n_i} = p$ for all i, so $p_{n_i} \to p$.

If E is infinite, then $E' \neq \varnothing$ by Theorem 2.50. Thus, (p_n) has a subsequential limit by Corollary 3.15.

(b) This follows from (a) and Heine-Borel Theorem.

□

Theorem 3.17. *The set of all subsequential limits of (p_n) is a closed set.*

Proof. Let E^* be the set of all subsequential limits of (p_n). Let $q \in (E^*)'$. Fix $r > 0$. Since $q \in (E^*)'$, so $(B_{r/2}(q) \setminus \{q\}) \cap E^* \neq \varnothing$. We can pick a point $x \in (B_{r/2}(q) \setminus \{q\}) \cap E^*$. Let another point $w \in B_{r/2}(x)$, then $d(q, w) \leq d(q, x) + d(x, w) < \dfrac{r}{2} + \dfrac{r}{2} = r$ and hence $w \in B_r(q)$. Thus, $B_{r/2}(x) \subseteq B_r(q)$.

Since $x \in E^*$, so $\left\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\right\}$ is infinite by Theorem 3.14. Therefore, $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite since $\left\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\right\} \subseteq \{n \in \mathbb{N} : p_n \in B_r(q)\}$. By Theorem 3.14, q is a subsequential limit, which means $q \in (E^*)'$. We can conclude that E^* is closed.

□

3.3 Cauchy Sequence

Previously, we talk about the convergence of a sequence. However, the definition of convergence implies that we know the limit of a sequence and then we can proceed a proof with ε. If we don't know whether the sequence has a limit, can we say whether it is convergent or not? This motivates the Cauchy sequence. The basic idea is that if a sequence is convergent, then the successive points in the sequence must be closer and closer.

Definition 3.18 (Cauchy Sequence)**.** A sequence (p_n) in a metric space (X, d) is called a **Cauchy sequence** if for every $\varepsilon > 0$, there is a positive integer N such that for all $n, m \geq N$, $d(p_n, p_m) < \varepsilon$.

Definition 3.19 (Diameter)**.** Let (X, d) be a metric space. The **diameter** of nonempty $E \subseteq X$ is

$$\operatorname{diam} E = \sup\{d(p, q) : p, q \in E\}$$

if the supremum exists, otherwise $\operatorname{diam} E = +\infty$.

Remark 3.20. A sequence (p_n) is Cauchy if and only if $\lim_{n\to\infty} \operatorname{diam}\{p_n, p_{n+1}, p_{n+2}, \cdots\} = 0$.

Lecture 14
Wednesday
November 4

Theorem 3.21. *Let (X, d) be a metric space.*

(a) If E is a nonempty subset of X and $\operatorname{diam} E$ *exists, then*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

(b) If K_n is compact and nonempty, and $K_{n+1} \subseteq K_n$, and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, then $\bigcap_{n\in\mathbb{N}} K_n$ is a singleton, the set with only one element.

Proof.

(a) It is obvious that $\operatorname{diam} \overline{E} \geq \operatorname{diam} E$ since $E \subseteq \overline{E}$.

Let $p, q \in \overline{E}$. Let $\varepsilon > 0$ and pick $p', q' \in E$ with $d(p, p') < \varepsilon$ and $d(q, q') < \varepsilon$. Then,

$$\begin{aligned} d(p, q) &\leq d(p, p') + d(p', q') + d(q', q) \\ &< \varepsilon + d(p', q') + \varepsilon = 2\varepsilon + \operatorname{diam} E. \end{aligned}$$

This holds for all $\varepsilon > 0$, so $d(p, q) \leq \operatorname{diam} E$.

This holds for all $p, q \in \overline{E}$, so $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$.

(b) Set $K = \bigcap_{n\in\mathbb{N}} K_n$. Then, $K \neq \varnothing$ by Finite Intersection Property (Theorem 2.48) and $\operatorname{diam} K \leq \operatorname{diam} K_n$ for all $n \in \mathbb{N}$ since $K \subseteq K_n$. Thus, $\operatorname{diam} K = 0$ and hence K consists of a single point.

□

Theorem 3.22. *For any metric space (X, d), if sequence (p_n) converges, then (p_n) is Cauchy.*

Proof. Let $p_n \to p$. Let $\varepsilon > 0$. Pick an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(p_n, p) < \frac{\varepsilon}{2}$. Then, for $n, m \geq N$ (do not use $n + 1$ as m, this is not from the definition of Cauchy sequence), we have

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, (p_n) is Cauchy.

□

Theorem 3.23. *If the metric space (X, d) is compact and (p_n) is Cauchy, then (p_n) converges.*

Proof. Set $E_n = \{p_n, p_{n+1}, p_{n+2}, \cdots\}$. Since (p_n) is Cauchy, $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$ by Theorem 3.21 (a). X is compact, so $\overline{E_n}$ is compact and $\overline{E_n} \supseteq \overline{E_{n+1}}$. So, by Theorem 3.21 (b), $\bigcap\limits_{n\in\mathbb{N}} \overline{E_n} = \{p\}$ for some $p \in X$. Now, let $\varepsilon > 0$ and pick N such that $\operatorname{diam} \overline{E_N} < \varepsilon$. Then, for $n \geq N$, we have $p_n \in E_n \subseteq E_N$ and $p \in \overline{E_N}$, so $d(p_n, p) \leq \operatorname{diam} \overline{E_N} < \varepsilon$. Thus, $p_n \to p$.

□

Theorem 3.24. *In k-dimensional Euclidean space $\mathbb{R}^k$, every Cauchy sequence converges.*

Proof. Assume that $(\vec{x_n})$ is a Cauchy sequence in $\mathbb{R}^k$. Pick an $N \in \mathbb{N}$ such that $\operatorname{diam}\{\vec{x_N}, \vec{x_{N+1}}, \cdots\} < 1$. Then, for $n \geq N$,

$$|\vec{x_n}| \leq |\vec{x_N}| + |\vec{x_n} - \vec{x_N}| < |\vec{x_N}| + 1$$

so $\{\vec{x_0}, \vec{x_1}, \cdots\} \subseteq B_r(\vec{0})$ where $r = 1 + \min\{|\vec{x_0}|, \cdots, |\vec{x_N}|\}$. By Heine-Borel Theorem, $(\vec{x_N})$ is a sequence in the compact set $\overline{B_r(\vec{0})}$, hence it converges by Theorem 3.23.

□

Definition 3.25 (Complete)**.** A metric space (X, d) is **complete** if every Cauchy sequence converges.

Example 3.26. Compact spaces, Euclidean spaces, and closed subsets of *these* are complete.

However, the set of rational numbers $\mathbb{Q}$ is NOT complete.

Remark 3.27. $\mathbb{R}$ is the smallest complete metric space containing $\mathbb{Q}$ (Cauchy construction).

Definition 3.28 (Monotonicity)**.** A sequence (s_n) in $\mathbb{R}$ is

(a) **monotone increasing** if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.

(b) **monotone decreasing** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

(c) **monotone** if either of the above.

Theorem 3.29. *Suppose (s_n) is monotone. Then (s_n) converges if and only if (s_n) is bounded.*

Proof. Without loss of generality, let's say (s_n) is monotone increasing (the other case is similar).

Assume (s_n) converges. Then the result follows from Theorem 3.5 (c).

Assume (s_n) is bounded, so $s = \sup\{s_n : n \in \mathbb{N}\}$ exists. Let $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound to $\{s_n : n \in \mathbb{N}\}$. So, there is an $N \in \mathbb{N}$ such that $s_N > s - \varepsilon$. Then, for $n \geq N$, $s - \varepsilon < s_N \leq s_n \leq s$, hence $|s - s_n| < \varepsilon$. We conclude that $s_n \to s$.

□

3.4 Upper and Lower Limits

Definition 3.30. For a sequence (s_n) in $\mathbb{R}$, we write

(a) $\lim\limits_{n\to\infty} = +\infty$ or $s_n \to +\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n \geq M$.

(b) $\lim\limits_{n\to\infty} = -\infty$ or $s_n \to -\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n \leq M$.

Remark 3.31. When either of the above holds, (s_n) still diverges. $\mathbb{R} \cup \{-\infty, +\infty\}$ is not a metric space since we do not allow "infinite distance" and $d(p, q) < \infty$ for $p, q \in X$.

Definition 3.32 (Upper Limit, Lower Limit)**.** Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of (s_n) (including $+\infty, -\infty$ if appropriate).

(a) The **upper limit** or **limit supremum** of (s_n), denoted $\limsup\limits_{n\to\infty} s_n$ is $\sup E \in \mathbb{R} \cup \{-\infty, +\infty\}$.

(b) The **lower limit** or **limit infimum** of (s_n), denoted $\liminf\limits_{n\to\infty} s_n$ is $\inf E \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Theorem 3.33. *Let (s_n) be a sequence in $\mathbb{R}$. Let E be the set of all subsequential limits of (s_n) (including $+\infty, -\infty$ if appropriate).*

(a) $\limsup\limits_{n\to\infty} s_n \in E$.

(b) If $x > \limsup\limits_{n\to\infty} s_n$, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < x$.

Moreover, $\limsup\limits_{n\to\infty}$ *is the unique extended real number with these properties.*

Lecture 15

Friday

November 6

Proof.

(a) If $\limsup\limits_{n\to\infty} s_n \in \mathbb{R}$, then $\limsup\limits_{n\to\infty} s_n = \sup E \in \overline{E} = E$ by Theorem 2.36 and Theorem 3.17.

If $\limsup\limits_{n\to\infty} s_n = +\infty$, then E is not bounded above by anything in $\mathbb{R}$, hence $\{s_n : n \in \mathbb{N}\}$ is not bounded above in $\mathbb{R}$, so there is a subsequence (s_{n_k}) with $s_{n_k} \to \infty$. Thus, $\limsup\limits_{n\to\infty} s_n = +\infty \in E$.

If $\limsup\limits_{n\to\infty} s_n = -\infty$, then $E = \{-\infty\}$ hence $\limsup\limits_{n\to\infty} s_n \in E$.

(b) Towards a contradiction, suppose $s_n \geq x$ for infinitely n. Then (s_n) has a subsequence in $[x, +\infty)$, hence has a subsequential limit $y \in [x, +\infty]$. Thus, $\limsup\limits_{n\to\infty} s_n = \sup E \geq y \geq x$ since $y \in E$, contradicting $x > \limsup\limits_{n\to\infty} s_n$.

Lastly, suppose $p < q$ both satisfy (a) and (b). Choose x such that $p < x < q$. Applying (b) to p and x, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < x$. It follows that every subsequential limit of (s_n) is in $[-\infty, x]$. So, $E \subseteq [-\infty, x]$. Thus, q cannot be a subsequential limit and therefore cannot satisfy (a).

$\square$

Corollary 3.34. *If $\liminf\limits_{n\to\infty} s_n > x$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n > x$.*

This is the lim inf version of Theorem 3.33 (b). The proof is similar.

Example 3.35. Consider the sequence $s_n = (-1)^n \left(1 + \frac{1}{2^n}\right)$. Then,

$$\limsup_{n\to\infty} s_n = 1 \qquad \liminf_{n\to\infty} s_n = -1.$$

Remark 3.36. $\lim_{n\to\infty} s_n$ exists and equals to s if and only if

$$\limsup_{n\to\infty} s_n = s = \liminf_{n\to\infty} s_n.$$

Theorem 3.37. *If $s_n \geq t_n$ for $n \geq N$, where N is fixed, then*

$$\limsup_{n\to\infty} s_n \geq \limsup_{n\to\infty} t_n \qquad \liminf_{n\to\infty} s_n \geq \liminf_{n\to\infty} t_n.$$

3.5 More on Sequences

Before introducing some sequences, we first provide a useful result, **Binomial Theorem**.

Definition 3.38 (Binomial Coefficient)**.** For $n, k \in \mathbb{N}$ and $0 \leq k \leq n$, we define

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}.$$

This is pronounced "n **chooses** k."

Lemma 3.39 (Pascal's Identity)**.** *For $n, k \in \mathbb{N}$ and $0 \leq k \leq n$, we have*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This is easier to check by calculation or by combinatorial property.

Theorem 3.40 (Binomial Theorem)**.** *For $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. We prove by induction.

For $n = 1$, the formula becomes

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y$$

and this is forever true.

Now, assume $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ holds, then we want to show that it also holds for $n+1$. Then,

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)\cdot(x+y)^n \\
&= (x+y)\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\
&= x\cdot\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) + y\cdot\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
&= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}y^{k+1} + \binom{n}{n}y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]x^{n+1-k}y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^k + y^{n+1}.
\end{aligned}$$

Note that in the last step of calculation, we apply the Pascal's Identity given above. Since $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$, we can rewrite this last equation and obtain

$$(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k.$$

□

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \le x_n \le s_n$ for $n \ge N$ where N is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

Theorem 3.41.

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) If $z \in \mathbb{C}$ and $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.

Proof.

(a) Let $\varepsilon > 0$. By Archimedean Property, there exists an $N \in \mathbb{N}$ such that $N > \left(\frac{1}{\varepsilon}\right)^{1/p}$. Then, if $n \geq N$, we have

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon^{1/p}.$$

So,

$$\left|\frac{1}{n^p} - 0\right| = \left(\frac{1}{n}\right)^p \leq \left(\frac{1}{N}\right)^p < \varepsilon$$

for $n \geq N$. We are done.

(b) It is obvious if $p = 1$.

Assume $p > 1$. Let $x_n = \sqrt[n]{p} - 1$. Then, $x_n > 0$. By Binomial Theorem,

$$1 + n \cdot x_n \leq (1 + x_n)^n = p$$

so $0 < x_n \leq \dfrac{p-1}{n}$ and thus $x_n \to 0$.

If $0 < p < 1$, then by Theorem 3.6,

$$1 = \frac{1}{1} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{1}{p}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\dfrac{1}{p}}} = \lim_{n\to\infty} \sqrt[n]{p}.$$

(c) Let $x_n = \sqrt[n]{n} - 1$. Then, $x_n > 0$ and

$$\binom{n}{2} \cdot x_n^2 = \frac{n \cdot (n-1)}{2} \cdot x_n^2 \leq (1 + x_n)^n = n$$

so $0 < x_n \leq \sqrt{\dfrac{2}{n-1}}$ thus $x_n \to 0$ (whenever $\dfrac{2}{n-1} < \varepsilon^2$ we have $\sqrt{\dfrac{2}{n-1}} < \varepsilon$).

(d) Fix $k \in \mathbb{N}$ with $k > \alpha$. When $n > 2k$,

$$(1+p)^n > \binom{n}{k} \cdot p^k = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \cdot p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!} = \frac{n^k \cdot p^k}{2^k \cdot k!}.$$

Thus,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}}.$$

Since $k - \alpha > 0$, so

$$\frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}} \to 0$$

by (a) and Theorem 3.6.

(e) Apply (d) with $\alpha = 0$ and $p = \dfrac{1}{|z|} - 1$, we find $|z|^n \to 0$. Since $|z^n| = |z|^n$, we obtain $z^n \to 0$.

$\square$

3.6 Series

The idea of a series is to make sense of **summing** an infnite sequence of numbers.

Definition 3.42 (Partial Sum, Series)**.** Let (a_n) be a sequence in $\mathbb{C}$. For each $n \in \mathbb{N}$, we can sum the first n terms of this sequence to get

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_n.$$

Then (s_n) is also a sequence. s_n is called the n**th partial sum**.

The expressions $a_0 + a_1 + a_2 + \cdots$ and $\sum_{n\in\mathbb{N}} a_n$ are called **(infinite) series** and denote the value $\lim_{n\to\infty} s_n$ when it exists.

Definition 3.43 (Convergence of Series)**.** Let (a_n) and (s_n) be defined as the definition above. If (s_n) converges to a number s, then we say that the infinite series $\sum_{n=0}^{\infty} a_n$ **converges** to s and we write $\sum_{n=0}^{\infty} a_n = s$. If (s_n) diverges, then we say that the series $\sum_{n=0}^{\infty} a_n$ diverges.

Recall Theorem 3.22, a sequence of real numbers converges if and only if it is a Cauchy sequence. Given a sequence (a_n), we apply this theorem to the corresponding sequence of partial sums (s_n) to get the following. The series $\sum_{n=0}^{\infty} a_n$ converges if and only if (s_n) is a Cauchy sequence, that is, if and only if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $m \geq n \geq N$, then $|s_m - s_{n-1}| < \varepsilon$ (I use $n-1$ instead of n to make the following display looks nicer). Now,

$$s_m - s_{n-1} = \sum_{k=0}^{m} a_k - \sum_{k=0}^{n-1} a_k = \sum_{k=n}^{m} a_k.$$

So, we have the following result.

Theorem 3.44 (Cauchy Criterion)**.** *$\sum_{n=0}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m \geq n \geq N$, then*

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon.$$

Proof. This follows from the Cauchy criterion for sequence convergence (Theorem 3.22) and

$$|s_m - s_n| = \left| \sum_{k=n}^{m} a_k \right|.$$

□

Theorem 3.45. *If $\sum_{n=0}^{\infty} a_n$ converges, then $a_n \to 0$.*

Proof. This follows from the prior theorem by taking $m = n$.

□

Lecture 16
Monday
November 9

Remark 3.46 (Harmonic Series). The converse of Theorem 3.45 is false. Consider $a_n = \frac{1}{n}$. The **harmonic series** is the series $\sum_{n=1}^{\infty} a_n$. We define the partial sum

$$s_n = \sum_{k=1}^{n} a_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

In fact, $s_{2^k} \geq \frac{k}{2} + 1$. This helps completing the rest of the proof for $\sum_{n=1}^{\infty} a_n$ diverges but $a_n \to 0$.

Theorem 3.47. *If $a_n \geq 0$, then $\sum_{n=0}^{\infty} a_n$ converges if and only if its partial sums are bounded.*

Proof. Since $a_n \geq 0$ for all $n \in \mathbb{N}$, then its partial sums are monotonically increasing, in other words, $s_{n+1} \geq s_n$ for all n. Then, Theorem 3.29 says monotonic sequence converges if it is bounded. That's it. □

Theorem 3.48 (Comparison Test)**.**

(a) If $|a_n| \leq c_n$ for $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges as well.

(b) If $a_n \geq d_n \geq 0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Proof.

(a) Given $\varepsilon > 0$, there exists an $N \geq N_0$ such that for all $m \geq n \geq N$, $\sum_{k=n}^{m} c_k \leq \varepsilon$. Thus,

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_n| \leq \sum_{k=n}^{m} c_k \leq \varepsilon$$

so $\sum_{n=0}^{\infty} a_n$ converges by Theorem 3.44.

(b) This is the contrapositive of (a).

□

Definition 3.49 (Geometric Series)**.** For $x \in \mathbb{C}$, $\sum_{n=0}^{\infty} x^n$ is called a **geometric series**.

Theorem 3.50 (Convergence of Geometric Series)**.** *If $x \in \mathbb{C}$ and $|x| < 1$, then*

$$\sum_{n=0}^{\infty} x_n = \frac{1}{1-x}.$$

If $|x| \geq 1$, then the series diverges.

Proof. Note that

$$(1-x)\cdot\sum_{k=0}^{n}x^k = 1-x^{n+1}$$

so

$$s_n=\sum_{k=0}^{n}x^k=\frac{1-x^{n+1}}{1-x}$$

thus $\lim_{n\to\infty} s_n = \dfrac{1}{1-x}$ when $|x|<1$.

When $|x|\geq 1$, we have $x^n \nrightarrow 0$ hence $\sum_{n=0}^{\infty}x^n$ diverges by contrapositive version of Theorem 3.45. ☐

Theorem 3.51 (Cauchy Condensation Test)**.** *Suppose $a_1\geq a_2\geq a_3\geq\cdots\geq 0$. Then, the series $\sum_{n=1}^{\infty}a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty}2^k a_{2^k}=a_1+2a_2+4a_4+8a_8+\cdots$$

converges.

Proof. For both series, they converge if and only if their partial sums are bounded by Theorem 3.44.

Let

$$\begin{aligned} s_n &= a_1+a_2+\cdots+a_n \\ t_k &= a_1+2a_2+\cdots+2^k a_{2^k} \end{aligned}.$$

Assume that (t_k) is convergent, then for $n<2^k$, we have

$$\begin{aligned} s_n &\leq a_1+(a_2+a_3)+\cdots+(a_{2^k}+\cdots+a_{2^{k+1}-1}) \\ &\leq a_1+2a_2+\cdots+2^k a_{2^k} \\ &= t_k. \end{aligned}$$

Hence, we have $s_n\leq t_k$. For sufficiently large k, if $t_k\to t$, we have $s_n\leq t_k\leq t$. But when k is arbitrarily large, then 2^k is also arbitrarily large. Hence, $n<2^k$ implies that n can also be arbitrarily large. Hence, for sufficiently large n, (s_n) is monotone increasing and bounded. Hence, (s_n) converges.

Conversely, assume that (s_n) is convergent. Then, for $n>2^k$, we have

$$\begin{aligned} s_n &\geq a_1+a_2+(a_3+a_4)+\cdots+(a_{2^{k-1}+1}+\cdots+a_{2^k}) \\ &\geq \frac{1}{2}a_1+a_2+2a_4+\cdots+2^{k-1}a_{2^k} \\ &= \frac{1}{2}t_k. \end{aligned}$$

Hence, we have $2s_n\geq t_k$. Then, for sufficiently large n, we have $2s\geq 2s_n\geq t_k$. Similarly, for sufficiently large k, (t_k) is monotone increasing and bounded. Thus, (t_k) converges. ☐

Theorem 3.52 (p-series)**.** $\sum_{n=1}^{\infty}\dfrac{1}{n^p}$ *converges if $p>1$ and diverges if $p\leq 1$.*

Proof. If $p \le 0$, the series is clearly divergent since $\frac{1}{n^p} \nrightarrow 0$.

Assume $p > 0$. By Theorem 3.51, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k$$

converges. This is a geometric series, so it converges if and only if $2^{1-p} < 1$ if and only if $p > 1$.

□

We have not learned about log function yet, but for the sake of an example, let's pretend we know what it is. log means the logarithmic function with base 10.

Theorem 3.53. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$.

Proof. If $p \le 0$, series diverges since $\frac{1}{n} \le \frac{1}{n(\log n)^p}$ for $n \ge 11$ by Theorem 3.48.

Assume $p > 0$, the terms are monotone decreasing and positive. By Theorem 3.51, the series converges if and only if

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot (\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p \cdot (\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges. By Theorem 3.52, this happens if and only if $p > 1$.

□

Definition 3.54 (The Number e)**.** $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Note that $\frac{1}{n!} \le \frac{1}{2^{n-1}}$, so $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by comparison with the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}$.

Theorem 3.55. $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Lecture 17

Friday

November 13

Proof. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \qquad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the Binomial Theorem, we have

$$\begin{aligned} t_n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k} + \cdots + \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ &\le s_n \le e \end{aligned}$$

so $\limsup_{n\to\infty} t_n \le e$. Here we take the upper limit simply because we don't know whether t_n has a limit. But we do know that it must have an upper limit.

Now, fix $m \in \mathbb{N}$. If $n \geq m$, then

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

Holding m fixed and taking $\liminf$ over n on both sides, we get

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m$$

by Theorem 3.37. Taking limit as $m \to \infty$, we obtain

$$\liminf_{n\to\infty} t_n \geq \lim_{m\to\infty} s_m = e.$$

Since $e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e$, so we conclude that

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = e$$

and (t_n) converges to e.

□

We can also estimate how fast $\sum_{n=0}^{\infty} \frac{1}{n!}$ is converging. Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, we have

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n! \cdot n}. \end{aligned}$$

Hence, $0 < e - s_n < \frac{1}{n! \cdot n}$. This means that this series is converging very quickly to e. This sets up the fact that e is irrational.

Theorem 3.56 (Irrationality of e)**.** *The number e is irrational.*

Proof. Towards a contradiction, assume e is rational and we can write $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then, since $0 < e - s_q < \frac{1}{q! \cdot q}$, we have

$$0 < q! \cdot (e - s_q) < \frac{1}{q}.$$

By assumption, $q! \cdot e$ is an integer. Also,

$$q! \cdot s_q = q!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$$

is an integer, so $q! \cdot (e - s_q)$ is an integer strictly between 0 and $\frac{1}{q}$. Since $q \geq 1$, so there must be an integer between 0 and 1. This is a contradiction.

□

Further, the number e is not **algebraic**, which means e is not a root of a polynomial with integer coefficients, in other words,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all $a_n \in \mathbb{Z}$. Actually, there are only countably many algebraic numbers. The proof is left as an exercise.

Theorem 3.57 (Root Test). *Consider a series* $\sum_{n=1}^{\infty} a_n$ *and set* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

(a) If $\alpha < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(b) If $\alpha > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(c) If $\alpha = 1$, *then the test is inconclusive.*

Proof. (a) Suppose $\alpha < 1$. Let $\beta = \frac{1+\alpha}{2}$. Then, $\alpha < \beta < 1$. There exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\sqrt[n]{|a_n|} < \beta$ by Theorem 3.33. Thus, $|a_n| < \beta^n$ for $n \geq N$. Since $0 < \beta < 1$, so $\sum_{n=N}^{\infty} \beta^n$ converges. By the comparison test, we conclude that $\sum_{n=1}^{\infty}$ converges.

(b) Suppose $\alpha > 1$. Then, by Theorem 3.33, we have α is a subsequential limit of $(\sqrt[n]{|a_n|})$, which means there exists a subsequence $(\sqrt[n_k]{|a_{n_k}|})$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Since $\alpha > 1$, there exists an $K \in \mathbb{N}$ such that $|a_{n_k}| > 1$ for all $k \geq K$. Thus, $\sum_{n=1}^{\infty} a_n$ diverges since $a_n \not\to 0$.

(c) For $\alpha = 1$, consider $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The first series diverges, but the second converges. Thus, the test is inconclusive.

□

Theorem 3.58 (Ratio Test). *Let* (a_n) *be a sequence such that for any* $n \in \mathbb{N}$, $a_n \neq 0$.

(a) $\sum_{n=1}^{\infty} a_n$ *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

(b) $\sum_{n=1}^{\infty} a_n$ *diverges if there exists an* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq N$. *That is,* $(|a_n|)$ *is nondecreasing.*

Proof.

(a) Let $\beta = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Let $\gamma = \frac{\beta+1}{2}$. Since $\beta < 1$, we have $\beta < \gamma < 1$. Then, there exists an $N \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for all $n \geq N$. That is, $|a_{n+1}| < \gamma|a_n|$ for $n \geq N$. By induction, we can prove that

$$|a_{N+k}| < \gamma^k |a_N|$$

for $k \geq 1$. In other words, $|a_n| < \gamma^{n-N}|a_N|$ for all $n \geq N$. Now, the series $\sum_{n=N}^{\infty} \gamma^{n-N}|a_N| = \sum_{n=N}^{\infty} \frac{a_N}{\gamma^N} \cdot \gamma^n$ converges since it is a geometric series with N fixed and also $0 < \gamma < 1$. Therefore, by the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

(b) This is immediate since $a_n \nrightarrow 0$. Here are more details. Since $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N$, we have $|a_{n+1}| \geq |a_n|$ for all $n \geq N$. By induction, we can prove that $|a_n| \geq |a_N|$ for $n \geq N$. Since $|a_N| > 0$ by assumption $a_n \neq 0$, we have $a_n \nrightarrow 0$. Thus, by the contrapositive version of Theorem 3.45, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

□

Actually, Root Test is always more accurate than the Ratio Test, but sometimes the Root Test is harder to evaluate. Further, the Ratio Test cannot be improved (more precisely, the obvious ways that it could be improved do not work).

If we change the condition for divergence to $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, this will fail. Consider the following example.

Example 3.59. Consider the following series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

or more directly, $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n^2}$. A fun fact is that the series converges to $\frac{\pi^2}{6}$ (just ignore it for now).

The upper and lower limits using Root Test are

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \frac{1}{\sqrt[n]{n}} = 1.$$

Note that if the limit exists, then its limit superior and limit inferior are equal.

The upper and lower limits using Ratio Test are

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1.$$

By the hypothesis, the series diverges, but it doesn't.

Both of Root Test and Ratio Test give no information. To prove the series diverges, we use the Comparison Test. We note that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&\leq 1 + \sum_{n=2}^{\infty} \frac{1}{n^2 - n} \\
&= 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \to 2.
\end{aligned}$$

However, this is not the most accurate bound, at least we prove the convergence.

If we change the condition for divergence to $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \geq 1$, this will fail. Consider the following example.

Example 3.60. Consider the following series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

or more specifically, $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{(n+1)/2} & n \text{ is odd} \\ \left(\frac{1}{3}\right)^{n/2} & n \text{ is even} \end{cases}.$$

The series converges to $\frac{3}{2}$ (the sum of two geometric series).

The upper and lower limits using Toot test are

$$\begin{aligned} \liminf_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}} \\ \limsup_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

The Root Test indicates convergence.

However, if we use Ratio Test, we get

$$\lim_{n\to\infty}\frac{a_{2n+1}}{a_{2n}} = \lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{3}{2}\right)^n = +\infty,$$

so $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$. The Ratio Test gives no information, so in the possible improvement stated before this example, the hypothesis is true but the conclusion is false.

With these two examples, we notice that both tests will malfunction sometimes, but both of improved versions of the Ratio Test do not give the correct result. In general, the Root Test is powerful than the Ratio Test. The reason is given in the following theorem.

Theorem 3.61. *For any sequence (c_n) such that $c_n > 0$ for all $n \in \mathbb{N}$,*

$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n} \underbrace{\leq}_{(a)} \liminf_{n\to\infty}\sqrt[n]{c_n} \underbrace{\leq}_{(b)} \limsup_{n\to\infty}\sqrt[n]{c_n} \underbrace{\leq}_{(c)} \limsup_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

Proof. (b) is immediate. We will prove (c), while (a) is similar.

Let $\alpha = \limsup_{n\to\infty}\frac{c_{n+1}}{c_n}$. If $\alpha = +\infty$, then we are done.

So, assume $\alpha < +\infty$, that means α is finite. Pick a $\beta > \alpha$. By Theorem 3.33, there exists an $N \in \mathbb{N}$ such that $\frac{c_{n+1}}{c_n} < \beta$ for all $n \geq N$. Similar to the proof of the Ratio Test, we can get a recurrence relation

$c_{N+p} < \beta^p \cdot c_N$. Let $n = N + p$, then $p = n - N$, so we have $c_n < \beta^{n-N} \cdot c_N$. Taking the nth root on both sides, we get $\sqrt[n]{c_n} < \beta^{1-N/n} \cdot \sqrt[n]{c_N} = \beta \cdot \sqrt[n]{\beta^{-N} \cdot c_N}$. Since $\beta > \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$ is arbitrary, so

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

□

Lecture 18

Monday

November 16

Definition 3.62 (Power Series)**.** Given a sequence (c_n) of complex numbers and $z \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a **power series**.

This is a fairly natural generalization of the polynomial, but whether it actually makes sense as a quantity depends on the convergence of the series. Convergence depends on the value of z.

Theorem 3.63 (Convergence for Power Series)**.** *For a power series $\sum_{n=0}^{\infty} c_n z^n$, set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$ (if $\alpha = 0$, set $R = +\infty$; if $\alpha = +\infty$, set $R = 0$). Then, $\sum_{n=0}^{\infty} c_n z^n$ converges when $|z| < R$ and diverges when $|z| > R$. Further, R is called the **radius of convergence**.*

Proof. Apply the Root Test, we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z| \cdot \alpha = \frac{|z|}{R}.$$

□

Note that we haven't said anything about what happens when $|z| = R$. In that case, it's difficult to say anything without considering the particular power series at hand.

Example 3.64.

(a) For series $\sum_{n=0}^{\infty} z_n$, we have $R = 1$. It diverges when $|z| = 1$ since $z^n \not\to 0$. On the other hand, for series $\sum_{n=0}^{\infty} n^n z^n$, we have $R = 0$.

(b) For series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, we have $R = +\infty$. In this case, the Ratio Test is easier to apply than the Root Test.

(c) For series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$, we have $R = 1$. It converges for all z with $|z| = 1$, by the Comparison Test, since $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$.

(d) For series $\sum_{n=0}^{\infty} \frac{z^n}{n}$, we have $R = 1$. It diverges when $z = 1$. It converges for $|z| = 1$ but $z \neq 1$. (The last statement will be proved later.)

The following theorem is interesting since it provides a discrete "version" (analogue) of integration by parts,

$$\int_a^b f \cdot g \, dx = -\int_a^b F \cdot g' \, dx + \left[F \cdot g\right]_a^b$$

where $F' = f$.

Theorem 3.65 (Summation by Parts)**.** *For sequences (a_n) and (b_n), define $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. Then, for $0 \leq p \leq q$,*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof. This is a direct algebraic proof. We rewrite the left-hand side

$$\begin{aligned}
\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\
&= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \qquad \text{Note the shift of index} \\
&= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.
\end{aligned}$$

□

The motivation of summation by parts is the observation that convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ does not imply $\sum_{n=1}^{\infty} a_n b_n$ converges. Here I index from $n = 1$, since the sequence defined in the counterexample is usually undefined as $n = 0$. I will provide a counterexample later.

Theorem 3.66 (Dirichlet's Test)**.** *If the partial sums of $\sum_{n=0}^{\infty} a_n$ are bounded and $b_0 \geq b_1 \geq b_2 \geq \cdots \geq 0$ with $\lim_{n\to\infty} b_m = 0$, then $\sum_{n=0}^{\infty} a_n b_n$ converges.*

Proof. Define $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$ as in Theorem 3.65. We pick an $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, we have $|A_n| \leq M$. Fix an $\varepsilon > 0$ and pick an $N \in \mathbb{N}$ with $b_N < \frac{\varepsilon}{2M}$. For $q \geq p \geq N$,

$$\begin{aligned}
\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\
&\leq \sum_{n=p}^{q-1} |A_n| \cdot (b_n - b_{n+1}) + |A_q| \cdot b_q + |A_{p-1}| \cdot b_p \\
&\leq M \cdot \left[\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p\right] \\
&= M \cdot (b_p - b_q + b_q + b_p) \\
&= 2M \cdot b_p \leq 2M \cdot b_N < \varepsilon.
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} a_n b_n$ converges by Cauchy criterion.

$\square$

Monotone increasing (b_n) also works.

Theorem 3.67 (Alternating Series Test)**.** *Suppose* $|c_1| \geq |c_2| \geq \cdots$*,* $c_{2m-1} \geq 0$*, and* $c_{2m} \leq 0$ *for* $m \geq 1$ *(opposite is also fine) and* $\lim_{n\to\infty} c_n = 0$*. Then,* $\sum_{n=1}^{\infty} c_n$ *converges.*

Proof. Apply Theorem 3.66 with $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

$\square$

Consider $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$. By Theorem 3.67, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. However, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which diverges.

Theorem 3.68. *Suppose* $\sum_{n=1}^{\infty} c_n z^n$ *has radius of convergence* 1*,* $c_0 \geq c_1 \geq \cdots$ *and* $\lim_{n\to\infty} c_n = 0$*. Then,* $\sum_{n=1}^{\infty} c_n z^n$ *converges for all* z *with* $|z| = 1$ *except possibly* $z = 1$*.*

Proof. Apply Theorem 3.66 with $a_n = z^n$ and $b_n = c_n$. We note that if $|z| = 1$ and $z \neq 1$, then

$$\left|\sum_{k=0}^{n} a_k\right| = \left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - z|}.$$

$\square$

The Theorem 3.67 is pretty remarkable, because it's a rare case in which convergence is not dependent upon the rate at which the terms of the series converge to 0.

Definition 3.69 (Absolute Convergence)**.** The series $\sum_{n=0}^{\infty} a_n$ **converges absolutely** if $\sum_{n=0}^{\infty} |a_n|$ converges. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ dieverges, we say $\sum_{n=0}^{\infty} a_n$ **converges non-absolutely**.

Proposition 3.70 (Absolute Convergence Implies Convergence)**.** *If* $\sum_{n=0}^{\infty} a_n$ *converges absolutely, then it converges.*

Proof. By Cauchy criterion,

$$\left|\sum_{n=p}^{q} a_n\right| \leq \sum_{n=p}^{q} |a_n|$$

and the right-hand side gets arbitrarily small for sufficiently large n because $\sum_{n=0}^{\infty} |a_n|$ converges.

$\square$

Remark 3.71. For series of positive terms, absolute convergence is the same as convergence.

Remark 3.72. The Comparison Test, Root Test, and Ratio Test demonstrate absolute convergence, and therefore cannot give any information about non-absolutely convergent series.

Lecture 19 Wednesday November 18

Now, we get to some operations on series which are seemingly safe but in fact require the condition of absolute convergence in order to be safe.

Theorem 3.73. *If* $\sum_{n=0}^{\infty} a_n = A$ *and* $\sum_{n=0}^{\infty} b_n = B$*, then* $\sum_{n=0}^{\infty}(a_n + b_n) = A + B$ *and* $\sum_{n=0}^{\infty} c \cdot a_n = c \cdot A$ *for any fixed* $c \in \mathbb{C}$*.*

Proof. Define $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$. Then,

$$A_n + B_n = \sum_{k=0}^{n}(a_k + b_k).$$

So,

$$A + B = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = \lim_{n\to\infty}(A_n + B_n) = \sum_{n=0}^{\infty}(a_n + b_n)$$

and

$$c \cdot A = c \cdot \lim_{n\to\infty} A_n = \lim_{n\to\infty} c \cdot A_n = \sum_{n\to\infty} c \cdot a_n.$$

□

Thus, defining addition and scalar multiplication of series are not so hard, and is as well-behaved as we desire to get. However, defining series multiplication is much less obvious. Consider the product

$$(a_0 + a_1 + a_2 + \cdots) \cdot (b_0 + b_1 + b_2 + \cdots).$$

We need to make sure all the terms hit each other and group them by sum of their indices. The result is

$$a_0b_0 + (a_1b_0 + a_0b_1) + (a_2b_0 + a_1b_1 + a_0b_2) + \cdots$$

This motivates the following definition.

Definition 3.74 (Cauchy Product)**.** The **Cauchy Product** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for $n \geq 0$.

Example 3.75. Consider $a_n = b_n = (-1)^n \cdot \dfrac{1}{\sqrt{n+1}}$. By Theorem 3.67, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge but not absolutely. Then,

$$\begin{aligned}
|c_n| &= \left|\sum_{k=0}^{n} a_k b_{n-k}\right| \\
&= \left|\sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}}\right| \\
&\geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2n+2}{n+2}
\end{aligned}$$

so $c_n \nrightarrow 0$ and $\sum_{n=1}^{\infty} c_n$ diverges.

The last inequality comes from

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

Fortunately, things are nicer with the assumption of absolute convergence.

Theorem 3.76 (Mertens). *Suppose* $\sum_{n=0}^{\infty} a_n = A$ *and* $\sum_{n=0}^{\infty} b_n = B$ *with* $\sum_{n=0}^{\infty} a_n$ *converging absolutely. Let* $\sum_{n=0}^{\infty} c_n$ *be the Cauchy product. Then,* $\sum_{n=0}^{\infty} c_n = A \cdot B$.

Proof. Let

$$A_n = \sum_{k=0}^{n} a_k \qquad B_n = \sum_{k=0}^{n} b_k \qquad C_n = \sum_{k=0}^{n} c_k \qquad \beta_n = B_n - B$$

(β_n is defined similarly as the "error term"). Then,

$$\begin{aligned} C_n &= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + \cdots + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0 \\ &= a_0(B+\beta_n) + a_1(B+\beta_{n-1}) + \cdots + a_n(B+\beta_0) \\ &= A_nB + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0. \end{aligned}$$

Let

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0.$$

We want to show that $C_n \to AB$ as $n \to \infty$. Since $A_nB \to AB$ as $n \to \infty$ by Theorem 3.73, it suffices to show that $\gamma_n \to 0$ as $n \to \infty$. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Fix an $\varepsilon > 0$. Since $\beta_n \to 0$ by the assumption $\sum_{n=0}^{\infty} b_n = B$, we can pick an $N \in \mathbb{N}$ such that $|\beta_n| < \varepsilon$ for all $n \geq N$. Then, for all $n \geq N$,

$$\begin{aligned} |\gamma_n| &\leq |a_0||\beta_n| + |a_1||\beta_{n-1}| + \cdots + |a_{n-N}||\beta_N| + |a_{n-N+1}||\beta_{N-1} + \cdots + |a_n||\beta_0| \\ &< \varepsilon \cdot (|a_0| + |a_1| + \cdots + |a_{n-N}|) + |a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0| \\ &\leq \varepsilon \cdot \alpha + |a_{n-N+1}||\beta_{N-1} + \cdots + |a_n||\beta_0|. \end{aligned}$$

For sufficiently large n, $|a_{n-N+1}||\beta_{N-1} + \cdots + |a_n||\beta_0|$ will be less than ε since it converges to 0. Thus, for sufficiently large n, $|\gamma_n| < \varepsilon \cdot (\alpha + 1)$. Hence, $\gamma_n \to 0$ as $n \to \infty$.

□

Theorem 3.77 (Abel's Test). *If* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, *and* $\sum_{n=0}^{\infty} c_n$ *converge to* A, B, *and* C, *where* $\sum_{n=0}^{\infty} c_n$ *is the Cauchy Product, then* $\sum_{n=0}^{\infty} c_n$ *converges and* $C = A \cdot B$.

One feature of finite sums is that no matter how we rearrange the terms in a sequence, the total sum is the same. However, it is complex for infinite series. If all the terms are non-negative, then rearrangement does not alter the summation. For alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots,$$

the rearrangement gives

$$\begin{aligned}&1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots\\ =&\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\cdots\\ =&\frac{1}{2}\cdot\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right).\end{aligned}$$

Thus, it is not always same for the alternating series.

Definition 3.78 (Rearrangement)**.** If (k_n) is a sequence in $\mathbb{N}$ using each natural number precisely once, and if $\sum_{n=0}^{\infty} a_n$ is a series and we set $a'_n = a_{k_n}$, then $\sum_{n=0}^{\infty} a'_n$ is called a **rearrangement** of $\sum_{n=0}^{\infty} a_n$.

Theorem 3.79. *If $\sum_{n=0}^{\infty} a_n$ converges absolutely, then every rearrangement converges to the same value.*

Proof. Let (k_n) be a sequence in $\mathbb{N}$ using each natural number precisely once. Fix an $\varepsilon > 0$ and pick an $N \in \mathbb{N}$ with $\sum_{i=n}^{m} |a_i| < \varepsilon$ for all $m \geq n \geq N$. Now, pick a p such that $\{0, 1, 2, \ldots, N\} \subseteq \{k_1, k_2, \ldots, k_p\}$. Then, for $n > \max\{p, N\}$, we have

$$\left|\sum_{i=0}^{n} a_{k_i} - \sum_{i=0}^{n} a_i\right| < \varepsilon.$$

□

Lecture 20 Friday November 20

Theorem 3.80 (Riemann's Rearrangement)**.** *Suppose $\sum_{n=0}^{\infty} a_n$ converges non-absolutely and $-\infty \leq \alpha \leq \beta \leq +\infty$. Then, there is a rearrangement $\sum_{n=0}^{\infty} a'_n$ with partial sums s'_n satisfying*

$$\liminf_{n\to\infty} s'_n = \alpha \qquad \limsup_{n\to\infty} s'_n = \beta.$$

Proof. □

4 Limits and Continuity

Consider the function $f : A \to \mathbb{R}$. Recall that a limit point c of A is a point with the property that $B_r(c)$ with $r > 0$ intersects A in some point other than c. Equivalently, c is a limit point of A if and only if $c = \lim_{n\to\infty} x_n$ for some sequence (x_n) in A with $x_n \neq c$. Limit points of A do not necessarily belong to the set A unless A is closed.

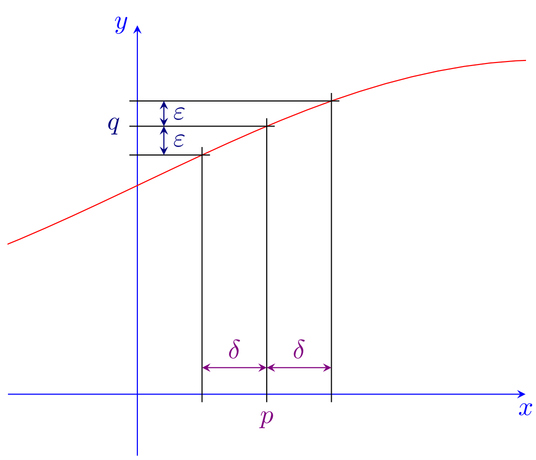
Now, we will get into a more abstract setting. We will define the limit and continuity in more general metric spaces.

4.1 Limits of Functions

Definition 4.1 (Limit)**.** Let (X, d_X) and (Y, d_Y) are metric spaces. Suppose $E \subseteq X$, $f : E \to Y$, and p is a limit point of E. For a point $q \in Y$, we say the **limit of** f **at** p **is** q and write "$f(x) \to q$ as $x \to p$" or $\lim_{x\to p} f(x) = q$ if

$$\text{for all } \varepsilon > 0, \text{ there exists a } \delta > 0, \text{ such that } 0 < d_X(x,p) < \delta \Longrightarrow d_Y(f(x), q) < \varepsilon.$$

Remark 4.2. It may be that $p \in X$ but $p \notin E$, so $f(p)$ is not defined. Even if $p \in E$, it can happen that $f(p) \neq \lim_{x\to p} f(x)$.



Theorem 4.3 (Limit in a Sequential Perspective)**.** *Let (X, d_X) and (Y, d_Y) are metric spaces. Suppose $E \subseteq X$, $f : E \to Y$, and p is a limit point of E. Then, $\lim_{x\to p} f(x) = q$ if and only if for all sequences (p_n) in E such that if $p_n \neq p$ and $p_n \to p$ for all $n \in \mathbb{N}$, then $f(p_n) \to q$.*

Proof. Assume $\lim_{x\to p} f(x) = q$. Let (p_n) be a sequence in E such that $p_n \neq p$ and $p_n \to p$ for all $n \in \mathbb{N}$. Fix an $\varepsilon > 0$ and pick a $\delta > 0$ such that for all $x \in E$, $0 < d_X(x,p) < \delta$ implies that $d_Y(f(x), q) < \varepsilon$. Since $p_n \to p$, there exists an $N \in \mathbb{N}$ such that $0 < d_X(p, p_n) < \delta$ for all $n \geq N$. Then, for all $n \geq N$, we have $d_Y(f(p_n), q) < \varepsilon$. Thus, $f(p_n) \to q$.

Conversely, it is weird and hard to prove directly since the condition "for all sequences (p_n) in E" is hard to use, so we prove by contrapositive. Assume $f(x) \not\to q$ as $x \to p$. Then, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists a point $x \in E$ such that $0 < d_X(x,p) < \delta$ and $d_Y(f(x),q) > \varepsilon$. Define $\delta_n = \dfrac{1}{n}$ for each $n \geq 1$, we can obtain $p_n \in E$ such that $0 < d_X(p_n,p) < \dfrac{1}{n}$ and $d_Y(f(p_n),p) \geq \varepsilon$. Then, $p_n \to p$ but $f(p_n)$ does not converge to $f(p)$.

□

Corollary 4.4 (Uniqueness of Limit)**.** *If f has a limit at p, then the limit is unique.*

Proof. Since the limit of a sequence is unique, then so is the limit of function by Theorem 4.3.

□

Definition 4.5. Let $E \subseteq X$. Let $f, g : E \to \mathbb{C}$. We define new functions

(a) $(f+g)(x) = f(x) + g(x)$.

(b) $(f \cdot g)(x) = f(x) \cdot g(x)$.

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ when $g(x) \neq 0$.

If $f, g : E \to \mathbb{R}$, we write $f \leq g$ if for any point $x \in E$, we have $f(x) \leq g(x)$.

Similarly, if $f, g : E \to \mathbb{R}^k$, we define

(a) $(\vec{f} + \vec{g})(x) = \vec{f}(x) + \vec{g}(x)$.

(b) $(\vec{f} \cdot \vec{g})(x) = \vec{f}(x) \cdot \vec{g}(x)$.

(c) for all $\lambda \in \mathbb{R}$, $(\lambda \vec{f})(x) = \lambda \cdot \vec{f}(x)$.

Theorem 4.6 (Limit Rules)**.** *Let (X,d) be a metric space, $E \subseteq X$, and p is a limit point of E. Let $f, g : E \to \mathbb{C}$ and $\lim_{x\to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$. Then*

(a) $\lim_{x\to p}(f+g)(x) = A + B$.

(b) $\lim_{x\to p}(f \cdot g)(x) = A \cdot B$.

(c) $\lim_{x\to p}\left(\dfrac{f}{g}\right)(x) = \dfrac{A}{B}$ *if* $B \neq 0$.

Similarly, if $\vec{f}, \vec{g} : E \to \mathbb{R}^k$ and $\lim_{x\to p} \vec{f}(x) = \vec{A}$ and $\lim_{x\to p} \vec{g}(x) = \vec{B}$, then

(a) $\lim_{x\to p}(\vec{f} + \vec{g})(x) = \vec{A} + \vec{B}$.

(b) $\lim_{x\to p}(\vec{f} \cdot \vec{g})(x) = \vec{A} \cdot \vec{B}$.

Proof. This follows from Theorem 3.6, Theorem 3.8, and Theorem 4.3.

□

Lecture 22
Monday
November 30

4.2 Continuous Functions

Definition 4.7 (Continuous Functions)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and $E \subseteq X$. Let $f : E \to Y$. We say f is **continuous** at $p \in E$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in E$,

$$d_X(x, p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon.$$

Furthermore, if f is continuous at every $p \in E$, then we say f is **continuous on** E or **continuous** in short.

Intuitively, this means that we can restrict our output by restricting input.

Theorem 4.8. *Let (X, d_X) and (Y, d_Y) be metric spaces and $E \subseteq X$. If $p \in E \setminus E'$ (this means p is an isolated point), then every function $f : E \to Y$ is continuous at p. If $p \in E \cap E'$, then $f : E \to Y$ is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.*

Proof. If $p \in E \setminus E'$, then there is a $\delta > 0$ with $(B_\delta(p) \setminus \{p\}) \cap E = \varnothing$, so for all $x \in E$, $d_X(x, p) < \delta$ implies $x = p$, then $f(x) = f(p)$ means $d_Y(f(x), f(p)) = 0 < \varepsilon$ for all $\varepsilon > 0$. Thus, this δ works for all $\varepsilon > 0$.

The second statement is immediate from Definition 4.7.

□

We now turn into composition of functions. Continuous functions have a nice property that a continuous function of a continuous function is continuous.

Theorem 4.9. *Suppose (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces and $E_X \subseteq X$ and $E_Y \subseteq Y$. Let $f : E_X \to E_Y$ and $g : E_Y \to Z$. Define $h : E \to Z$ by $h(p) = g(f(p))$. If f is continuous at p and g is continuous at $f(p)$, then h is continuous at p.*

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(p)$, so there exists a $r > 0$ such that for all $y \in E_Y$,

$$d_Y(y, f(p)) < r \Longrightarrow d_Z(g(y), g(f(p))) < \varepsilon.$$

Since f is continuous at p, there is a $\delta > 0$ such that for all $x \in E_X$,

$$d_X(x, p) < \delta \Longrightarrow d_Y(y, f(p)) < r.$$

If follows that for all $x \in E_X$,

$$d_X(x, p) < \delta \Longrightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon.$$

We conclude that h is continuous at $x = p$.

□

Before diving into more nice interpretations and properties of continuity, we first introduce the inverse mapping.

Definition 4.10 (Image)**.** Given a mapping $f : X \to Y$, consider the set $E \subseteq X$. The **image** of E is defined as

$$f(E) = \{f(x) : x \in E\} .$$

Definition 4.11 (Inverse Image)**.** Given a mapping $f : X \to Y$, an inverse mapping $f^{-1} : Y \to X$. Consider a set $C \in Y$. Then the **inverse image** of C is defined as

$$f^{-1}(C) = \{x \in X : f(x) \in C\} .$$

Theorem 4.12. *Let (X, d_X) and (Y, d_Y) be metric spaces. The mapping $f : X \to Y$ is on X if and only if for all open sets $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open.*

Proof. Assume f is continuous and let $V \subseteq Y$ be open. Let $p \in f^{-1}(V)$. Then, $f(p) \in V$. Since V is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(p)) \subseteq V$, which means that for all $y \in Y$, $d_Y(y, f(p)) < \varepsilon \implies y \in V$. Since f is continuous, there exists a $\delta > 0$ such that for all $x \in X$, $d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \varepsilon$. It follows that $f(B_\varepsilon(p)) \subseteq V$, which means that $B_\varepsilon(p) \subseteq f^{-1}(p)$. Thus, $f^{-1}(V)$ is open.

Assume $f^{-1}(V)$ is open for all open sets $V \subseteq Y$. Fix a point $p \in X$ and let $\varepsilon > 0$. Set $V = B_\varepsilon(f(p))$. Then, V is open so $f^{-1}(V)$ is open. Since $p \in f^{-1}(V)$, there exists a $\delta > 0$ with $B_\delta(p) \subseteq f^{-1}(V)$. So, if $x \in X$ satisfies $d_X(x, p) < \delta$, then $x \in B_\delta(p) \subseteq f^{-1}(V)$, so $f(x) \in V = B_\varepsilon(f(p))$ and thus $d_Y(f(x), f(p)) < \varepsilon$. We conclude that f is continuous.

□

Corollary 4.13. *Let (X, d_X) and (Y, d_Y) be metric spaces. The mapping $f : X \to Y$ is on X if and only if for all closed sets $C \subseteq Y$, $f^{-1}(C)$ is closed.*

Proof. This follows from Theorem 4.12 together with duality between open and closed sets and the fact that for all sets $D \subseteq Y$, $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$.

□

We now turn to complex-valued and vector-valued functions, and to functions defined on subsets of $\mathbb{R}^k$.

Theorem 4.14 (Continuous Complex-Valued Functions)**.** *Let $f, g : X \to \mathbb{C}$ be continuous functions. Then, so are $f + g$, $f \cdot g$, and $\dfrac{f}{g}$ (if for all $x \in X$, $g(x) \neq 0$).*

Proof. At isolated points, there is nothing to prove.

At limit points, this follows from Theorem 4.6 and Theorem 4.8.

□

Theorem 4.15 (Continuous Vector-Valued Functions)**.**

(a) Let $f_1, f_2, \ldots, f_k : X \to \mathbb{R}$ and define $\vec{f} : X \to \mathbb{R}^k$ by

$$\vec{f}(x) = (f_1(x), f_2(x), \ldots, f_k(x)) .$$

Then $\vec{f}$ is continuous if and only if f_i is continous for $1 \leq i \leq k$.

(b) If $\vec{f}, \vec{g} : X \to \mathbb{R}^k$ are continuous, then so are $\vec{f} + \vec{g}$ and $\vec{f} \cdot \vec{g}$.

Proof.

(a) This follows from Theorem 3.8, Theorem 4.3, and Theorem 4.8

(b) This follows from Theorem 4.6 and Theorem 4.8.

□

Lecture 23 Wednesday December 2

For $1 \leq i \leq k$, the mapping from $\mathbb{R}^k$ to $\mathbb{R}$ given by $\vec{x} = (x_1, x_2, \ldots, x_k) \mapsto x_i$ is continuous on $\mathbb{R}^k$. Then, for $n_1, n_2, \ldots, n_k \in \mathbb{N}$, $(x_1, x_2, \ldots, x_k) \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is continuous on $\mathbb{R}^k$. So polynomials given by $p(\vec{x}) = \sum c_{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ (where $c_{n_1, n_2, \ldots, n_k} \in \mathbb{C}$ are fixed and all but finitely many are 0) are continuous. Additionally, rational functions $\dfrac{P(\vec{x})}{Q(\vec{x})}$ (P and Q are polynomials) are continuous on their domain.

From the triangle inequality, we can show that $||\vec{x}| - |\vec{y}|| \leq |\vec{x} - \vec{y}|$. Hence, the mapping $\vec{x} \mapsto |\vec{x}|$ is continuous.

4.3 Continuity and Topology

We have touched with limit perspective, sequence perspective, and a bit topological perspective (in open and closed sets) of the continuity. Now, we continue with continuity and compactness.

Definition 4.16 (Boundedness)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **bounded** if there is a $q \in Y$ and $M > 0$ with $f(X) \subseteq B_M(q)$.

Theorem 4.17. *Let (X, d_X) and (Y, d_Y) be metric spaces. If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let $\{V_\alpha : \alpha \in A\}$ be an open cover of $f(X)$. Since f is continuous, by Theorem 4.12, each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, so there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$. Then, we have

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}.$$

We conclude $f(X)$ is compact.

□

This theorem says that if f is a continuous mapping, while it is possible to map the open interval (a, b) into $\mathbb{R}$, it is impossible to map a closed interval $[a, b]$ onto the entire $\mathbb{R}$ since the mapping of a compact set is also compact because of the continuity of function f. Compact sets are "small" in some sense. Hence, if f is continuous, then if it takes in a "small" input (closed interval), then the output must also be relatively small.

Remark 4.18. In the proof for Theorem 4.17, we have used the relation $f\left(f^{-1}(E)\right) \subseteq E$, valid for $E \subseteq Y$. If $E \subseteq X$, then $f^{-1}(f(E)) \supseteq E$. Equality need not hold in either case.

Theorem 4.19. *If $f : X \to \mathbb{R}^k$ is continuous and X is compact, then $f(X)$ is closed and bounded.*

Proof. This follows from Theorem 4.17 and the Heine-Borel Theorem. □

Theorem 4.20. *Let (X, d_X) be a compact metric space. Let $f : X \to \mathbb{R}$. Set $M = \sup_{x\in X} f(x)$ and $m = \inf_{x\in X} f(x)$. Then, there are $p, q \in X$ with $f(p) = M$ and $f(q) = m$.*

Proof. Since $f(X)$ is compact in $\mathbb{R}$, so it is closed and bounded, which means $M = \sup_{x\in X} f(x)$ and $m = \inf_{x\in X} f(x)$ exist. Since $f(X)$ is closed, then $\sup_{x\in X} f(x) \in f(X)$. Similarly, $\inf_{x\in X} f(x) \in f(X)$. □

This theorem says that there exist $p, q \in X$ such that $f(q) \le f(x) \le f(p)$ for all $x \in X$, that is, f attains its maximum at p and f attains its minimum at q.

Theorem 4.21. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. If X is compact and f is a continuous bijection, then $f^{-1} : Y \to X$ is continuous.*

Proof. Since $\left(f^{-1}\right)^{-1} = f$, then Corollary 4.13 tells us that f^{-1} is continuous if and only if $f(C)$ is closed for all closed sets $C \subseteq X$. Let $C \subseteq X$ be closed. Then, C is compact, so by Theorem 4.17, $f(C)$ is compact, hence $f(C)$ is closed. Thus, we conclude that f^{-1} is continuous. □

By Definition 4.7, we can easily prove that the functions $f(x) = 3x + 1$ and $g(x) = x^2$ are everywhere continuous. Are there differences in these two functions?

Definition 4.22 (Uniform Continuity)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$. We say f is **uniformly continuous** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Recall that to say "f is continuous at X" means that f is continuous at each individual point $x \in X$. Uniform continuity is a strictly stronger property. The key distinction between asserting that f is "uniformly continuous on X" versus simply "continuous on X" is that, given an $\varepsilon > 0$, a single $\delta > 0$ can be chosen that works simultaneously for all points $x \in X$. To say that a function is *not* uniformly continuous on a set X, then, does not necessarily mean it is not continuous at some point. Rather, it means that there is some $\varepsilon' > 0$ for which no single $\delta > 0$ is a suitable response for all $x \in X$.

Evidently, every uniformly continuous function is continuous. Two concepts are equivalent on the compact sets.

Theorem 4.23. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$. If f is continuous on X and X is compact, then f is uniformly continuous on X.*

Proof. Let $\varepsilon > 0$. Since f is continuous, for each $p \in X$, we can pick $\delta_p > 0$ such that for all $q \in X$, $d_X(p, q) < \delta_p \implies d_Y(f(p), f(q)) < \varepsilon$. Set $V_p = B_{\delta_p/2}(p)$.

Claim 4.23.1. *If $q \in V_p$, $x \in X$, and $d_X(x, q) < \frac{1}{2}\delta_p$, then $d_Y(f(x), f(q)) < \varepsilon$.*

Proof of Claim 4.23.1. Since $q \in V_p$, then $d_X(p, q) < \frac{1}{2}\delta_p$, and $d_X(x, q) < \frac{1}{2}\delta_p$, by triangle inequality, we have

$$d_X(p, x) \leq d_X(p, q) + d_X(q, x) < \frac{\delta_p}{2} + \frac{\delta_p}{2} = \delta_p$$

and $d_X(p, q) < \frac{1}{2}\delta_p < \delta_p$ so

$$d_Y(f(p), f(q)) < \frac{\varepsilon}{2} \qquad d_Y(f(x), f(p)) < \frac{\varepsilon}{2}$$

and by triangle inequality again,

$$d_Y(f(x), f(q)) \leq d_Y(f(x), f(p)) + d_Y(f(p), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Then, let $\{V_p : p \in X\}$ be an open cover of X. Since X is compact, so there exists $p_1, p_2, \ldots, p_n$ with $X \subseteq \bigcup_{i=1}^{n} V_{p_i}$. Set $\delta = \frac{1}{2}\{\delta_{p_1}, \delta_{p_2}, \ldots, \delta_{p_n}\}$. Consider $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$. Since $X \subseteq \bigcup_{i=1}^{n} V_{p_i}$, there is $1 \leq i \leq n$ such that $x_1 \in V_{p_i}$. Now, Claim 4.23.1 implies $d_Y(f(x_1), f(x_2)) < \varepsilon$.

□

Lecture 24 Friday December 4

We now proceed to show that compactness is essential in the hypotheses of Theorem 4.17, Theorem 4.19, Theorem 4.20, and Theorem 4.23.

Theorem 4.24. *Let $E \subseteq \mathbb{R}$ be a non-compact set. Then,*

(a) there exists a continuous function $f : E \to \mathbb{R}$ but it is not bounded;

(b) there exists a continuous and bounded function $f : E \to \mathbb{R}$ but it has no maximum.

(c) if in addition E is bounded, then there exists a continuous function $f : E \to \mathbb{R}$ but it is not uniformly continuous.

Proof. Assume that E is bounded. By Heine-Borel Theorem, E is not closed, so there exists a point $x_0 \in E' \setminus E$.

For (a) and (c), consider $f(x) = \dfrac{1}{x - x_0}$ for $x \in E$.

Claim 4.24.1. *f is not bounded.*

Proof of Claim 4.24.1. Let $M > 0$. Since $x_0 \in E'$, we can fine an $x \in E$ such that $|x - x_0| < \frac{1}{M}$. For this x, we have

$$|f(x)| = \frac{1}{|x - x_0|} > M.$$

THus, f is not bounded.

□

Claim 4.24.2. *f is not uniformly continuous.*

Proof of Claim 4.24.2. Let $\varepsilon > 0$ and $\delta > 0$. First pick any $p \in E$ such that $|p - x_0| < \frac{\delta}{2}$. Since f is not bounded on $\left(x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right) \cap E$, we can find $q \in E$ with $|q - x_0| < \frac{\delta}{2}$ and $|f(q)| > |f(p)| + \varepsilon$. Then,

$$|p - q| \leq |p - x_0| + |x_0 - q| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

but

$$|f(q) - f(p)| \geq |f(q)| - |f(p)| > \varepsilon$$

by construction. Thus, f is not uniformly continuous.

□

For (b), consider $g(x) = \frac{1}{1 + (x - x_0)^2}$ for $x \in E$.

Claim 4.24.3. *g is bounded and for all $x \in E$, $g(x) < 1$ but $\sup_{x \in E} g(x) = 1$.*

Proof of Claim 4.24.3. Clearly for all $x \in E$, we have $0 < g(x) < 1$ and g is bounded. Let $\varepsilon > 0$. Pick an $x \in E$ with $|x - x_0| < \sqrt{\frac{1}{1 - \varepsilon} - 1}$. For this x,

$$g(x) = \frac{1}{1 + (x - x_0)^2} > \frac{1}{1 + \frac{1}{1 - \varepsilon} - 1} = 1 - \varepsilon,$$

which means $1 - \varepsilon$ is not an upper bound of $g(x)$. Thus, $\sup_{x \in E} g(x) = 1$.

□

Now, assume E is not bounded.

For (a), set $h(x) = x$ for all $x \in E$.

For (b), set $s(x) = \frac{x^2}{1 + x^2}$ for $x \in E$.

Claim 4.24.4. *s is bounded and for all $x \in E$, $s(x) < 1$ and $\sup_{x \in E} s(x) = 1$.*

Proof of Claim 4.24.4. It is clear that for all $x \in E$, $0 \leq s(x) < 1$ and s is bounded. Let $\varepsilon > 0$. Pick $x \in E$ such that $|x| > \sqrt{\frac{1}{\frac{1}{1 - \varepsilon} - 1}}$. For this x,

$$s(x) = \frac{x^2}{1 + x^2} = \left(\frac{1}{x^2} + 1\right)^{-1} > \left(\frac{1}{1 - \varepsilon} - 1 + 1\right)^{-1} = 1 - \varepsilon,$$

which means $1-\varepsilon$ is not an upper bound of $s(x)$. Thus, $\sup_{x\in E} s(x) = 1$.

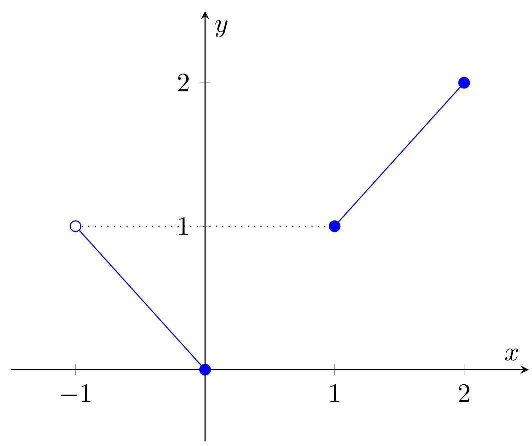
□

□

Remark 4.25. Theorem 4.24 (c) is not true if boundedness is not assumed. $\mathbb{Z}$ is not compact, but every function $f : \mathbb{Z} \to \mathbb{R}$ is uniformly continuous. To prove this, we take $\delta < 1$ in Definition 4.22.

Also, compactness is also essential in Theorem 4.21.

Example 4.26. Define $f : (-1, 0] \cup [1, 2] \to \mathbb{R}$ by $f(x) = |x|$. Then, f is a continuous bijection but f^{-1} is not continuous.



Up to now, we have explored some nice properties in continuity and compactness, with a stronger version of continuity, uniform continuity. Now, we start to explore continuity in connectedness.

Theorem 4.27. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$. If $E \subseteq X$ is connected and f is continuous, then $f(E)$ is connected.*

Proof. We prove by contrapositive. Suppose $f(E)$ is not connected. Say, $A, B \subseteq Y$ are nonempty and separated and $A \cup B = f(E)$. Set $G = f^{-1}(A) \cap E$ and $H = f^{-1}(B) \cap E$. Then, $E = G \cup H$ and G, H are nonempty.

Since $A \subseteq \bar{A}$, we have $G \subseteq f^{-1}(\bar{A})$. Since f is continuous, $f^{-1}(\bar{A})$ is closed so $\bar{G} \subseteq f^{-1}(\bar{A})$. Therefore,

$$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(\bar{B}) \subseteq f^{-1}(\bar{A} \cap B) = f^{-1}(\varnothing) = \varnothing.$$

Similarly, $G \cap \bar{H} = \varnothing$. Thus, G and H are separated and E is not connected. We conclude that $f(E)$ is connected if $E \subseteq X$ is connected and f is continuous.

□

This theorem says that a continuous mapping preserves connectedness, which leads to Intermediate Value Theorem. Recall that $E \subseteq \mathbb{R}$ is connected if and only if for all $x, y \in E$ and $x < z < y$, then $z \in E$.

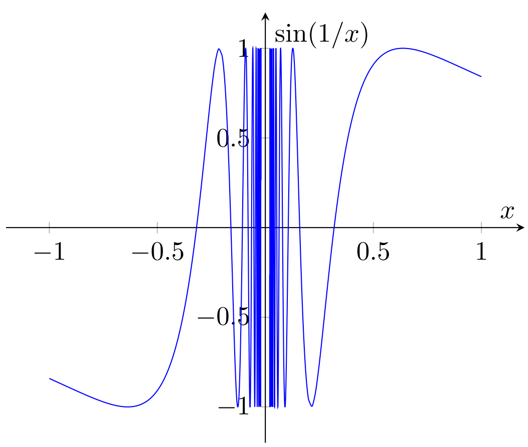
Theorem 4.28 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $c \in \mathbb{R}$ satisfies $f(a) < c < f(b)$, there there exists an $x \in (a, b)$ such that $f(x) = c$. A similar result holds, of course, if $f(a) > f(b)$.*

Proof. Since $[a, b]$ is connected, so by Theorem 4.27, $f([a, b])$ is also connected. Since $f(a), f(b) \in f([a, b])$ and $f(a) < f(c) < f(b)$, then we conclude by Theorem 2.73 that $c \in f([a, b])$. That is, there exists $x \in [a, b]$ such that $f(x) = c$. Since $c \neq f(a)$, so $x \neq a$. Similarly, $x \neq b$. We are done.

□

Remark 4.29. The converse of Intermediate Value Theorem is false. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$



f is not continuous at $x = 0$. However, f has the intermediate value property, that is, for any $a < b$ and any y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, there exists a $c \in [a, b]$ such that $f(c) = y$.

Lecture 25
Monday
December 7

4.4 Discontinuity

Definition 4.30 (Discontinuity)**.** If f is not continuous at x and x is in the domain of f, we say that f is **discontinuous at** x.

Let $f : X \to \mathbb{R}$. We now introduce the notion of left and right limits, which can be thought of as two seperate "halves" of the limit $\lim_{x \to t : x \in X} f(x)$.

Definition 4.31 (One-sided Limits)**.** Suppose f is a real-valued function defined on (a, b).

(a) For $a \leq x < b$, we write $f(x+) = q$ or $\lim_{t \to x^+} f(t) = q$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - q| < \varepsilon$ for all $t \in (x, x + \delta) \subseteq (a, b)$.

(b) For $a < x \leq b$, we write $f(x-) = q$ or $\lim_{t \to x^-} f(t) = q$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - q| < \varepsilon$ for all $t \in (x - \delta, x) \subseteq (a, b)$.

We can reformulate these definitions in terms of limits of sequences. Let $f : (a,b) \to \mathbb{R}$. For $a \le x < b$, we say that $f(x+) = q$ if $f(t_n) \to q$ for all sequences $t_n \to x$ with $x < t_n < b$. Similarly, for $a < x \le b$, we say that $f(x-) = q$ if $f(t_n) \to q$ for all sequences $t_n \to x$ with $a < t_n < x$.

Lemma 4.32. *Let $f : (a,b) \to \mathbb{R}$. $\lim\limits_{t\to x} f(t)$ exists if and only if $f(x+) = f(x-)$ and when this occurs, $\lim\limits_{t\to x} f(x)$ is equal to $f(x+) = f(x-)$.*

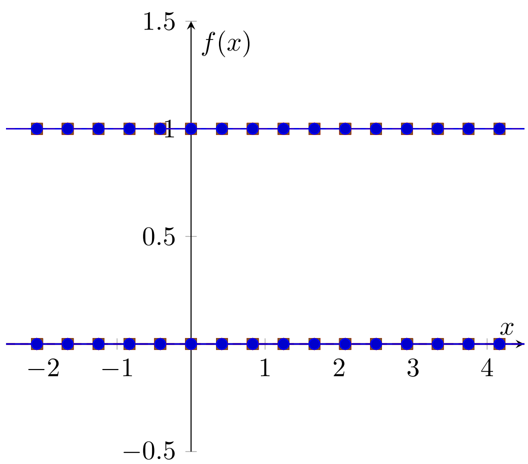
Definition 4.33 (Types of Discontinuity)**.** If f is discontinuous at x and both $f(x+)$ and $f(x-)$ exist then we say f has a **discontinuity of the first kind at** x or **simple discontinuity at** x. Otherwise, the discontinuity is said to be of the **second kind**.

Simple discontinuity happens if both $f(x+)$ and $f(x-)$ exist and either $f(x+) \neq f(x-)$ or $f(x+) = f(x-)$ but $f(x) \neq f(x+) = f(x-)$.

Example 4.34 (Dirichlet Function)**.** Consider the *Dirichlet Function* defined as

$$\mathscr{D}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

The *Dirichlet Function* is not continuous everywhere.



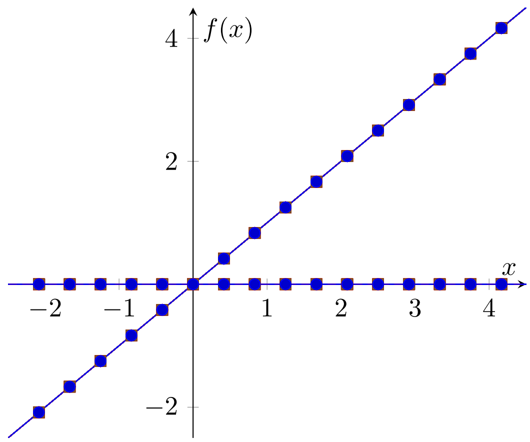
Proof. For any $c \in \mathbb{R}$, we can find a sequence (x_n) in $\mathbb{Q}$ and sequence (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to c$ and $y_n \to c$. But $\mathscr{D}(x_n) = 1$ and $\mathscr{D}(y_n) = 0$ for all n. Hence $\lim\limits_{n\to\infty} \mathscr{D}(x_n) \neq \lim\limits_{n\to\infty} \mathscr{D}(y_n)$. This suggests that $\mathscr{D}$ is not continuous at c. Thus, *Dirichlet Function* is not continuous everywhere since c is arbitrary. □

Specifically, the *Dirichlet Function* has the discontinuity of the second kind, since for all $p \in \mathbb{R}$, $(p-\delta, p)$ contains both rationals and irrationals. Hence the image $\{\mathscr{D}(x) : x \in (p-\delta, p)\}$ will oscillate between 0 and 1. Hence, the condition $d(f(p), f(x)) < \varepsilon$ will not work as we make $\varepsilon < 1$.

Corollary 4.35 (Modified *Dirichlet Function*)**.** *We modify the Dirichlet Function $\mathscr{D}(x)$ a little bit to*

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

Then, $f(x)$ is continuous at $x = 0$ and discontinuous at $x \neq 0$ and has discontinuity of second kind at all $x \neq 0$.



Example 4.36 (More Proper Functions)**.**

(a) Consider the piecewise function

$$f(x) = \begin{cases} x+2 & -3 \leq x < -2 \\ -x-2 & -2 \leq x < 0 \\ x+2 & 0 \leq x \leq 1 \end{cases}.$$

$f(x)$ is continuous on $[-3, 1] \setminus \{0\}$ and has simple continuity at $x = 0$.

(b) Assume we know what sin function is and its properties. Consider the function

$$g(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

$g(x)$ is continuous on $\mathbb{R} \setminus \{0\}$ and has discontinuity of second kind at $x = 0$.

4.5 Monotone Functions

Classifying a set of discontinuities for an arbitrary function f is somewhat complex, so it is interesting that describing the set of discontinuities is fairly straightforward for the class of monotone functions.

Definition 4.37 (Monotone Function)**.** A function $f : (a, b) \to \mathbb{R}$ is **monotone increasing** if whenever $a < x < y < b$, we have $f(x) \leq f(y)$. Similarly, f is **monotone decreasing** if whenever $a < x < y < b$, we have $f(x) \geq f(y)$. We say that f is **monotone** if it is monotone increasing or monotone decreasing.

Continuous functions are not necessarily monotone, consider the function $f(x) = x^2$ on $\mathbb{R}$. Also, monotone functions are not necessarily continuous, consider the piecewise functions.

We further investigate the monotone functions through the limit "from the left" and "from the right."

Theorem 4.38. *Let $f : (a, b) \to \mathbb{R}$ be monotone increasing. Then, for every $x \in (a, b)$, $f(x-)$ and $f(x+)$ exist and*

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$, then $f(x+) \le f(y-)$. Similar property holds when f is monotone decreasing.

Proof. The set $\{f(t) : a < t < x\}$ is bounded above by $f(x)$, so $\alpha = \sup\limits_{a<t<x} f(t)$ exists by least upper bound property and $\alpha \le f(x)$.

Fix $\varepsilon > 0$. Since $\alpha - \varepsilon$ is not an upper bound to $\{f(t) : a < t < x\}$, there exists a $\delta > 0$ with $\alpha - \varepsilon < f(x - \delta) \le \alpha$. So, for any $t \in (x - \delta, x)$, $\alpha - \varepsilon < f(x - \delta) \le f(t) \le \alpha$, so $|f(t) - \alpha| < \varepsilon$. Thus, $f(x-) = \alpha$. A similar argument shows that $f(x+) = \inf\limits_{x<t<b} f(t)$ and $f(x+) \ge f(x)$.

Now, suppose $a < x < y < b$. Pick any c such that $x < c < y$. Then,

$$f(x+) = \inf_{x<t<b} f(t) \le f(c) \le \sup_{a<t<y} f(t) = f(y-).$$

$\square$

Corollary 4.39. *Monotone functions have no discontinuities of the second kind.*

Theorem 4.40. *If f is monotone on (a, b), then it only has countably many discontinuities on (a, b).*

Proof. Assume that f is monotone increasing. Let E be the set of discontinuities in (a, b). For each $x \in E$, pick $r(x) \in \mathbb{Q}$ satisfy $f(x-) < r(x) < f(x+)$. Then, $r : E \to \mathbb{Q}$ is an injection since if $x_1 < x_2$, then by Theorem 4.38, $r(x_1) < f(x_1+) < f(x_2-) < r(x_2)$. Since $\mathbb{Q}$ is countable, it follows that E is countable.

$\square$

Lecture 26 Wednesday December 9

Example 4.41. Given any countable set $E \subseteq (a, b)$, there is a monotone increasing function $f : (a, b) \to \mathbb{R}$ such that E is the set of discontinuities of f.

Proof. Let $E = \{e_1, e_2, e_3, \ldots\}$. Fix a sequence (c_n) of positive real numbers such that $\sum\limits_{n=1}^{\infty} c_n$ converges. Define, for $x \in (a, b)$,

$$I_x = \{n : e_n < x\} \quad I_x^+ = \{n : e_n \le x\}.$$

Define $f(x) = \sum\limits_{n \in I_x} c_n$ (this converges because $\sum\limits_{n=1}^{\infty} c_n$ converges absolutely). Then,

(a) f is monotone increasing;

(b) $f(e_n+) - f(e_n-) = c_n > 0$;

(c) f is not continuous on $(a,b) \setminus E$.

(a) holds since $x < t \implies I_x \subseteq I_t \implies f(x) \le f(t)$.

For (b) and (c), it suffices to show that for all $x \in (a,b)$,

$$f(x-) = f(x) \text{ and } f(x+) = \sum_{n \in I_x^+} c_n.$$

Since

$$f(e_k+) - f(e_k-) = \sum_{n \in I_{e_n}^+ \setminus I_{e_n}} c_n = c_k$$

and for $x \in (a,b) \setminus E$, we have $I_x = I_x^+$ and thus $f(x-) = f(x+)$, so f is continuous at x.

We want to show that for $x \in (a,b)$, $f(x-) = f(x+)$ and $f(x+) = \sum_{n \in I_x^+} c_n$. Note that when $t < x$, $[t,x) \cap \{e_1, e_2, \ldots, e_N\} = \varnothing$ implies that for $1 \le i \le N$,

$$(e_i < t \Leftrightarrow e_i < x) \implies I_x \setminus I_t \subseteq \{e_{N+1}, \ldots\} \implies 0 \le f(x) - f(t) \le \sum_{n > N} c_n.$$

When $x < t$,

$$\begin{aligned}(x,t) \cap \{e_1, e_2, \ldots, e_N\} = \varnothing &\implies \text{for all } 1 \le i \le N, e_i \le x \Leftrightarrow e_i < t \\ &\implies I_t \setminus I_x^+ \subseteq \{e_{N+1}, e_{N+2}, \ldots\} \\ &\implies 0 \le f(t) - \sum_{n \in I_x^+} c_n \le \sum_{n > N} c_n.\end{aligned}$$

So, given $\varepsilon > 0$ and $x \in (a,b)$, pick N with $\sum_{n > N} c_n < \varepsilon$ and choose $\delta > 0$ small enough so that $(x - \delta, x)$ and $(x, x+\delta)$ are disjoint with $\{e_1, e_2, \ldots, e_N\}$. Then,

$$t \in (x - \delta, x) \implies |f(t) - f(x)| < \varepsilon$$

$$t \in (x, x + \delta) \implies \left| f(t) - \sum_{n \in I_x^+} c_n \right| < \varepsilon.$$

Thus, $f(x-) = f(x)$ and $f(x+) = \sum_{n \in I_x^+} c_n$.

□

4.6 Infinite Limits and Limits at Infinity

Previously, toward a linear function $f(x)$, we may conclude that "as x goes to infinity, $f(x)$ goes to infinity." Currently, we don't have the tools to formalize this idea because infinity cannot live in a metric space. Recall that a set $U \subseteq \mathbb{R}$ is a **neighborhood** of $x \in \mathbb{R}$ if U is open and $x \in U$. By further developing limits of functions and defining neighborhoods in the extended real number system, we can handle these cases.

Definition 4.42 (Neighborhood at Infinity)**.** A **neighborhood of** $+\infty$ is a set of the form $(M, +\infty)$ for $M \in \mathbb{R}$. A **neighborhood of** $-\infty$ is a set of the form $(-\infty, -M)$ for $M \in \mathbb{R}$.

We have seen that $\lim_{x\to p} f(x) = q$ if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-p| < \delta$ implies $|f(x) - q| < \varepsilon$. Equivalently, for every sequence (x_n) converging to p with $x_n \neq p$, $f(x_n) \to q$. This is also equivalent to say that for every neighborhood U of q, there exists a neighborhood V of p with $x \in V$ implies $f(x) \in U$.

Definition 4.43 (Limits at Infinity)**.** Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. For $x, y \in \mathbb{R} \cup \{-\infty, +\infty\}$, we write $\lim_{t\to x} f(t) = y$ or $f(t) = y$ as $t \to x$ if

(a) *either* $x \in E'$ *or* E is not bounded above and $x = +\infty$ *or* E is not bounded below and $x = -\infty$

(b) *and* for every neighborhood V of y there exists a neighborhood U of x such that $f(t) \in V$ for all $t \in E$, $x \neq t \in U$.

Now, we will generalize some limit rules in the extended real number system.

Theorem 4.44. *Let $E \subseteq \mathbb{R}$. Let $f, g : E \to \mathbb{R}$. Suppose $x, a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\lim_{t\to x} f(t) = a$, and $\lim_{t\to x} g(t) = b$. Then,*

(a) if $\lim_{t\to x} f(t) = a'$, then $a = a'$

(b) $\lim_{t\to x}(f+g)(t) = a+b$

(c) $\lim_{t\to x}(f \cdot g)(t) = a \cdot b$

(d) $\lim_{t\to x} \left(\frac{f}{g}\right)(t) = \frac{a}{b}$

provided that right-hand side is defined. $+\infty + (-\infty)$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{a}{0}$ are not defined.

Proof.

(a) Towards a contradiction, assume $a \neq a'$. Say $a < a'$ ($a > a'$ is similar). Then, there is $r \in \mathbb{R}$ such that $a < r < a'$. Then, $V = (-\infty, r)$ and $V' = (r, +\infty)$ are neighborhoods of a and a' respectively. So there are neighborhoods U and U' of x such that for all $t \in E$,

$$x \neq t \in U \Longrightarrow f(t) \in V$$
$$x \neq t \in U' \Longrightarrow f(t) \in V'.$$

Then, $U \cap U'$ is a neighborhood of x so we can find a $t \in E$ with $x \neq t \in U \cap U'$. Then,

$$f(t) \in V \cap V' = (-\infty, r) \cap (r, +\infty) = \varnothing,$$

a contradiction. Hence, $a = a'$.

$\square$

That's the end of Math 140A.