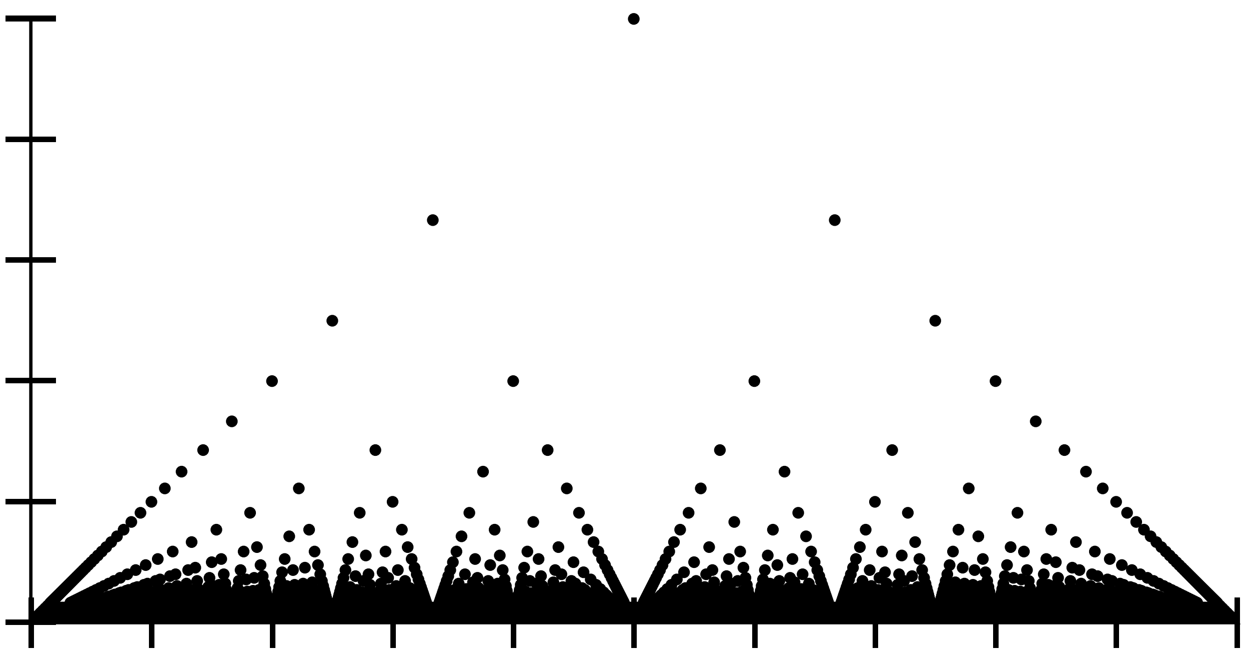
# Math 140A Foundations of Real Analysis I

Jack Yang

Last Updated: December 30, 2020



$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q}, \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

We will take for granted (and without proof) the basic properties of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the set of **natural numbers**), $\mathbb{Z}$ (the set of **integers**), and $\mathbb{Q}$ (the set of **rational numbers**).

We will explore in detail the properties of $\mathbb{R}$ (the set of **real numbers**). With great care and precision we will define what the real numbers are. All of our prior knowledge and beliefs about $\mathbb{R}$ will be held in suspicion until we can find proofs of those properties based on our formal definition of $\mathbb{R}$.

The longterm goal is to provide the logical and theoretical justification for calculus (140B) and go beyond (140C).

We will often focus on specific features of $\mathbb{R}$ and study those features in more abstract settings.

**Abstract**

Real number system. Basic point-set topology. Metric spaces. Numerical sequences and series. Functions of a real variable. Continuity.

# Contents

# 1 The Real Number System

We start the journey to analysis from the basic set theory.

We define a *set* $A$ to be any unordered collection of objects, for example, $\{0, 1, 2\}$ is a set, which means the set containing three elements 0, 1, and 2. If $x$ is an object, we say that $x$ *is an element of* $A$ or $x \in A$ if $x$ is in the collection; otherwise, we say $x \notin A$. For example, $1 \in S$, but $7 \notin S$. In general, if $x$ is an object and $A$ is a set, then either $x \in A$ is true or $x \in A$ is false. If $A$ is not a set, we leave the statement $x \in A$ undefined.

**Definition 1.1** (Empty Set)**.** There exists a set $\varnothing$, known as the **empty set**, which contains no elements, that is, for every object $x$, we have $x \notin \varnothing$.

Clearly, some sets seem to be larger than others. Also, it is possible that two sets are equal. One way to formalize this concept is through a *subset*.

**Definition 1.2** (Subset)**.** A set $A$ is a **subset** of a set $B$ if $x \in A$ implies $x \in B$, and we write $A \subseteq B$. That is, all members of $A$ are also members of $B$.

(a) Two sets $A$ and $B$ are **equal** if $A \subseteq B$ and $B \subseteq A$. We write $A = B$. That is, $A$ and $B$ contain exactly the same elements. If it is not true that $A$ and $B$ are equal, then we write $A \neq B$.

(b) A set $A$ is a **proper subset** of $B$ if $A \subset B$ and $A \neq B$. We write $A \subsetneq B$.

We have $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$, because every element in $\{1, 2, 3\}$ is also an element of $\{1, 2, 3, 4, 5\}$. In fact, $\{1, 2, 3\} \subsetneq \{1, 2, 3, 4, 5\}$.

**Proposition 1.3.** *Given any set $A$, $A \subseteq A$ and $\varnothing \subseteq A$.*

**Proposition 1.4** (Sets are Partially Ordered by Set Inclusion)**.** *Let $A$, $B$, and $C$ be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.*

**Example 1.5.** The following are sets including the standard notations.

(a) The set of **natural numbers**, $\mathbb{N} = \{0, 1, 2, \ldots\}$.

(b) The set of **integers**, $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$.

(c) The set of **rational numbers**, $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\right\}$.

(d) The set of **real numbers**, $\mathbb{R}$.

Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

There are many operations we want to do with sets.

**Definition 1.6** (Union)**.** The **union** $A \cup B$ of two sets is defined to be the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

**Definition 1.7** (Intersection)**.** The **intersection** $A \cap B$ of two sets is defined to be the set

$$A \cap B = \{x \in A : x \in B\}.$$

In other words, $A \cap B$ consists of all the elements which belong to both $A$ and $B$. Thus, for all objects $x$,

$$x \in A \cap B \iff x \in A \text{ and } x \in B.$$

We also need to intersect or union several sets at once. If there are only finitely many, then we simply apply the union or intersection operation several times. However, suppose we have an infinite collection of sets (a set of sets) $\{A_1, A_2, A_3, \ldots\}$. We define

$$\bigcup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\},$$
$$\bigcap_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

Two sets $A$ and $B$ are said to be *disjoint* if $A \cap B = \varnothing$. Note that this is not same as being *distinct*, $A \neq B$. We are also interested in the difference of two sets.

**Definition 1.8** (Difference Sets)**.** Given two sets $A$ and $B$, we define the set $A - B$ or $A \setminus B$ to be the set $A$ with any elements of $B$ removed

$$A \setminus B = \{x \in A : x \notin B\}.$$

We now give some properties of unions, intersections, and difference sets.

**Proposition 1.9** (Boolean Algebra)**.** *Let $A$, $B$, and $C$ be sets. Let $X$ be a set containing $A$, $B$, and $C$ as subsets.*

*(a) (Minimal Element) We have $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$.*

*(b) (Maximal Element) We have $A \cup X = X$ and $A \cap X = A$.*

*(c) (Identity) We have $A \cap A = A$ and $A \cup A = A$.*

*(d) (Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

*(e) (Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*

*(f) (Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

*(g) (partition) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \varnothing$.*

*(h) (De Morgan Laws) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.*

Lecture 1
Friday
October 2

## 1.1 Ordered Sets

A real number is a value that represents a quantity along a continuous number line. Real numbers have several specific features. One nice feature is that real numbers can be ordered.

**Definition 1.10** (Ordered Set)**.** Let $S$ be a set. An **order** on $S$, denoted $<$, is a relation satisfying

(a) (Trichotomy Law) for all $x, y \in S$, exactly one of the statements

$$x < y, \quad x = y, \quad x > y$$

is true.

(b) (Transitivity) for all $x, y, z, \in S$, if $x < y$ and $y < z$, then $x < z$.

An **ordered set** is a set of which an order is defined. For example, the integer set $\mathbb{Z}$ is ordered.

For convenience, we write

(a) $y > x$ to mean $x < y$.

(b) $x \leq y$ to mean $x < y$ or $x = y$.

**Definition 1.11** (Bounds)**.** Let $S$ be an ordered set and $E \subseteq S$.

(a) If there is $b \in S$ with $x \leq b$ for all $x \in E$, then we say $E$ is **bounded above** and call $b$ an **upper bound** to $E$.

(b) If there is $b \in S$ with $x \geq b$ for all $x \in E$, then we say $E$ is **bounded below** and call $b$ an **lower bound** to $E$.

In general, sets may not have upper or lower bounds. Consider the set of integers $\mathbb{Z}$ and the set of rational numbers $\mathbb{Q}$. The set of natural numbers has a lower bound 0 but does not have an upper bound.

**Definition 1.12** (Supremum and Infimum)**.** Let $S$ be an ordered set and $E \subseteq S$. We call $\alpha \in S$ the **least upper bound** of $E$ or the **supremum** of $E$, denoted $\alpha = \sup E$, if

(a) $\alpha$ is an upper bound to $E$.

(b) whenever $x \in S$ and $x < \alpha$, $x$ is not an upper bound to E.

The **greatest lower bound** or **infimum** of $E$ is defined similarly and denoted $\inf E$. The statement $\beta = \inf E$ means that $\beta$ is a lower bound of $E$ and that no $\gamma$ with $\gamma > \beta$ is a lower bound of $E$.

**Example 1.13.** Consider the set $E = \left\{ \frac{1}{n} : n \in \mathbb{Z}_+ \right\} \subseteq \mathbb{Q}$, we have $\sup E = 1$ and $\inf E = 0$ (this is not so obvious, for now we get by intuition).

Note that $\sup E \in E$ and $\inf E \notin E$. In general, $\sup E$ and $\inf E$ may or may not be element of $E$.

To distinguish maximum (or minimum) and supremum (or infimum), in this case we have $\inf E = 0$ but $\min E$ does not exist and $\sup E = \max E = 1$.

If the maximum exists, then it is the supremum. For an infinite set, it can happen that the maximum does not exist but the supremum does exist. Similar idea holds for minimum and infimum. In general, supremum and infimum are used in more general and abstract setting.

**Definition 1.14** (Least Upper Bound Property)**.** An ordered set $S$ has the **least upper bound property** if whenever $E \subseteq S$ is nonempty and bounded above, $\sup E$ exists.

**Example 1.15.** The set $\mathbb{Q}$ of rational numbers does not have the least-upper-bound property. Recall that $\sqrt{2} \notin \mathbb{Q}$. Let $A = \{p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 \leq 2\}$ and $B = \{p \in \mathbb{Q} : p > 0 \text{ and } p^2 \geq 2\}$. Then, $\mathbb{Q} = A \cup B$, $B$ is the set of upper bounds to $A$ and $A$ is the set of lower bounds to $B$.

But $A$ has no largest element and $B$ has no smallest element, so $\sup A$ and $\inf B$ do not exist (when using $\mathbb{Q}$). This implies that $\mathbb{Q}$ does not have the least-upper-bound property.

Lecture 2 Monday October 5

For $p \in \mathbb{Q}_{>0}$, set

$$q = p - \frac{p^2 - 2}{p + 2} \in \mathbb{Q}.$$

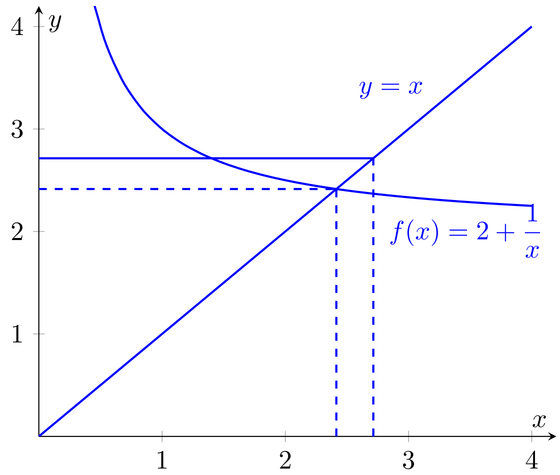
Then, $q = \frac{2p+2}{p+2}$, so

$$q^2 = 2 + \frac{2(p^2 - 2)}{(p+2)^2}.$$

Suppose $p \in A$, then $p^2 - 2 \leq 0$. If $p \leq 0$, then $p < 1$ and $1 \in A$. If $p > 0$, then $q > p$ since $q - p = -\frac{p^2 - 2}{p + 2} > 0$, and $q \in A$ since $q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2} \leq 0$. Thus, $p$ is not the largest element in $A$.

Suppose $p \in B$, then $p^2 - 2 > 0$. So, we get $0 < q < p$ since $q - p = -\frac{p^2 - 2}{p + 2} < 0$, and $q \in B$ since $q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2} > 0$. Thus, $q$ is not the smallest element of $B$.

To see what motivates the $q$ above, we first note that $x = 1 + \sqrt{2}$ is a solution to the equation $x = 2 + \frac{1}{x}$ (more generally, $x = 1 + \sqrt{k}$ is a solution to $x = \frac{k-1}{x} + 2$). Consider the function $f(x) = 2 + \frac{1}{x}$. From the graph, we note that for sufficiently small $\varepsilon$, $f(\sqrt{2} + 1 + \varepsilon)$ is closer to $\sqrt{2} + 1$ than $\sqrt{2} + 1 + \varepsilon$ is. Similarly, $f(\sqrt{2} + 1 - \varepsilon)$ is closer to $\sqrt{2} + 1$ than $\sqrt{2} + 1 - \varepsilon$ is.




Note that if $p \approx \sqrt{2}$, then $f(p+1) - 1$ will give us an approximation of $\sqrt{2}$, and so $\sqrt{2} \approx f(p+1) - 1 = \frac{1}{p+1} + 2 = \frac{p+2}{p+1}$.

Also, we note that if $p > \sqrt{2}$, then $f(p+1) < \sqrt{2} + 1$. For $p = \sqrt{2}$, we have $\sqrt{2}(f(p+1) - 1) = 2$, hence

$$\sqrt{2} = \frac{2}{f(p+1) - 1} = \frac{2p+2}{p+2}.$$

This gives us that if $p < \sqrt{2}$, then $f(p+1) - 1 > \sqrt{2}$.

Note that $p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2}$ is what we want to know.

**Theorem 1.16.** *If $S$ has the least-upper-bound property, then it has the greatest-lower-bound property, that is, if $E \subseteq S$ is nonempty and bounded below then $\inf E$ exists.*

*Proof.* Let $E \subseteq S$ be nonempty and bounded below. Let $A$ be the set of all lower bounds to $E$. Then $A$ is nonempty and bounded above, that is, every $e \in E$ is an upper bound.

So, $\alpha = \sup A$ exists by the least-upper-bound property (Definition 1.14). We will check $\alpha = \inf E$.

(Check $\alpha$ is a lower bound to $E$) Consider any $e \in E$. By definition $A$, we have $\forall A \in A$, $a \leq e$. Thus $e$ is an upper bound to $A$. Since $\alpha$ is the least upper bound to $A$, we have $\alpha \leq e$. Thus $\alpha$ is a lower bound.

(Check anything greater than $\alpha$ is not a lower bound) Suppose $x$ is a lower bound to $E$, then by definition, $x \in A$. Then, $x \leq \sup A = \alpha$.

We conclude that $\inf E = \alpha$ exists.

□

## 1.2 Fields

**Definition 1.17** (Field)**.** A **field** is a set $F$ with two binary operations $+$ and $\cdot$, called **addition** and **multiplication**, with the following properties

(a) (Commutativity) for all $a, b \in F$, $a + b = b + a$, $a \cdot b = b \cdot a$.

(b) (Associativity) for all $a, b, c \in F$, $(a+b)+c = a+(b+c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(c) (Identity) there are $0, 1 \in F$, with for all $a \in F$, $0 + a = a$ and $1 \cdot a = a$.

(d) (Inverse) for all $a \in F$, there is an element $-a \in F$ with $a + (-a) = 0$; also, for all $a \in F$ with $a \neq 0$, there is $\frac{1}{a} \in F$ with $a \cdot \frac{1}{a} = 1$.

(e) (Distributivity) for all $a, b, c \in F$, $a \cdot (b+c) = (a \cdot b) + (a \cdot c) = a \cdot b + a \cdot c$.

**Example 1.18.** These are fields

(a) $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

(b) $\mathbb{Q}(t) = \left\{ \frac{p(t)}{q(t)} : p, q \text{ are polynomials in } t \text{ with coefficients in } \mathbb{Q} \right\}$.

(c) Set of conjugacy classes $\bmod p$ for $p$ a prime.

**Example 1.19.** The set of natural numbers $\mathbb{N}$ is not a field. It violates (d) in Definition 1.17. Also, the set of integers $\mathbb{Z}$ is also not a field. It violates (d) as well.

Consider $a = 2$, then the multiplicative inverse is $\frac{1}{2}$, which is not a natural number nor an integer.

We write (in any field) $x - y$ in place of $x + (-y)$, $\frac{x}{y}$ in place of $x \cdot \left(\frac{1}{y}\right)$, 2 in place of $1 + 1$, similarly $2x$ in place of $x + x$, and $x^2$ in place of $x \cdot x$.

**Proposition 1.20.** *For a field $F$ and $x, y, z \in F$,*

*(a) if $x + y = x + z$, then $y = z$.*

*(b) if $x + y = 0$, then $y = -x$ (further it proves that additive inverse is unique).*

*(c) if $x + y = x$, then $y = 0$ (further it proves that additive identity is unique).*

*(d) $-(-x) = x$.*

*Proof.*

(a) If $x + y = x + z$, then $y = 0 + y = -x + (x + y) = -x + (x + z) = z$. That's it.

(b) Apply (a) with $z = -x$, then we're done.

(c) Apply (a) with $z = 0$, then we're done.

(d) Since $-x + x = x + (-x) = 0$, (b) implies $x = -(-x)$.

□

**Proposition 1.21.** *Let $F$ be a field. Then for all $x, y, z \in F$, with $x \neq 0$,*

(a) *if* $x \cdot y = x \cdot z$*, then* $y = z$*.*

(b) *if* $x \cdot y = 1$*, then* $y = \frac{1}{x}$ *(further it proves that multiplicative inverse is unique).*

(c) *if* $x \cdot y = x$*, then* $y = 1$ *(further it proves that multiplicative identity is unique).*

(d) $\frac{1}{1/x} = x$*.*

The proof is so similar to that of Proposition 1.20.

Lecture 3 Wednesday October 7

**Proposition 1.22.** *For any field* $F$ *and* $x, y \in F$*,*

(a) $0 \cdot x = 0$*. Note that* $0$ *is not necessary* $0 \in \mathbb{Z}$*, just a notation.*

(b) *if* $x \neq 0$ *and* $y \neq 0$*, then* $x \cdot y \neq 0$*.*

(c) $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ *(specifically,* $-x = (-1) \cdot x$*).*

(d) $(-x) \cdot (-y) = x \cdot y$*.*

*Proof.*

(a) We have $0 \cdot x + 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x$, so $0 \cdot x = 0$ by Proposition 1.20.

(b) Assume $x \neq 0$ and $y \neq 0$, but $xy = 0$. Then, (a) gives

$$1 = (x \cdot y) \cdot \left( \frac{1}{x} \cdot \frac{1}{y} \right) = 0 \cdot \left( \frac{1}{x} \cdot \frac{1}{y} \right) = 0,$$

that is a contradiction. Thus, we must have $x \cdot y \neq 0$.

(c) Since $(-x) \cdot y + x \cdot y = (-x + x) \cdot y = 0 \cdot y = 0$, so $(-x) \cdot y = -(x \cdot y)$ by Proposition 1.20.

Similarly, we have $x \cdot (-y) = -(x \cdot y)$.

(d) By Proposition 1.20, we have

$$(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = x \cdot y.$$

$\square$

**Definition 1.23** (Ordered Field)**.** An **ordered field** is a field $F$ with an ordering such that

(a) for all $x, y, z \in F$, if $y < z$, then $x + y < x + z$.

(b) for all $x, y \in F$, if $x > 0$ and $y > 0$, then $x \cdot y > 0$.

We call $x \in F$ **positive** if $x > 0$ and **negative** if $x < 0$.

TODO Explain

**Example 1.24.** These are ordered fields: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Q}(t)$ (defined in Example 1.18).

Here is a proof for the last field.

*Proof.* Recall the basic algebra fact that a polynomial in $t$ has value 0 for only finitely many values of $t \in \mathbb{R}$ (in fact, a polynomial of degree $n$ has at most $n$ roots), unless the polynomial is simply the constant 0. Since polynomials are continuous, this means that for any polynomial $p$, either $p = 0$ or else there is $t_0 \in \mathbb{R}$ such that $p(t)$ is always positive for $t > t_0$ or is always negative for $t > t_0$.

Suppose that $f, g, p, q$ are polynomials in $t$ with coefficients in $\mathbb{Q}$ and that $g \neq 0$ and $q \neq 0$. Note that $\frac{f}{g} - \frac{p}{q} = \frac{f \cdot g - p \cdot q}{g \cdot q}$ and therefore $\frac{f}{g} = \frac{p}{q}$ if and only if the numerator above, $f \cdot q - p \cdot g$, is 0. Thus, either $\frac{f}{g} = \frac{p}{q}$ or there is $t_0 \in \mathbb{R}$ such that $\frac{f(t)}{g(t)} - \frac{p(t)}{q(t)}$ does not change sign for $t > t_0$.

We define an ordering by declaring $\frac{p}{q} < \frac{f}{g}$ if and only if there exists a $t_0 \in \mathbb{R}$ such that $\frac{f(t)}{g(t)} - \frac{p(t)}{q(t)} > 0$ for all $t > t_0$. In other words, $\frac{p}{q} < \frac{f}{g}$ if and only if for all sufficiently large $t$, we have $\frac{p(t)}{q(t)} < \frac{f(t)}{g(t)}$.

By ordering of $\mathbb{Q}$, it is not difficult to check that $<$ satisfies the Trichotomy Law and transitivity, and that it satisfies the two properties needed to make $\mathbb{Q}(t)$ into an ordered field.

This order can be described in a more direct algebraic way. Recall that $\frac{f}{g} - \frac{p}{q} = \frac{f \cdot q - p \cdot g}{g \cdot q}$, it is true that $\frac{p}{q} < \frac{f}{g}$ if and only if both $\frac{f}{g} \neq \frac{p}{q}$ and the ratio of the leading coefficient of $f \cdot q - p \cdot g$ to the leading coefficient of $g \cdot q$ is positive. (Recall that the leading coefficient of a polynomial in $t$ is the coefficient of the largest power of $t$.)

$\square$

**Proposition 1.25.** *For an ordered field $F$ and $x, y, z \in F$,*

*(a) if $x > 0$, then $-x < 0$, and vice versa.*

*(b) if $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.*

*(c) if $x < 0$ and $y < z$, then $x \cdot y > x \cdot z$.*

*(d) if $x \neq 0$, then $x^2 > 0$ (this statement is reversible).*

*(e) if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.*

*Proof.*

(a) If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$. If $x < 0$, then $0 = x + (-x) < 0 + (-x)$, so $-x > 0$.

(b) Since $y < z$, we have $0 = y - y < z - y$, so $x \cdot (z - y) > 0$, and therefore

$$x \cdot z = x \cdot (z - y) + x \cdot y > 0 + x \cdot y = x \cdot y.$$

(c) By (a) and (b) together, we have $(-x) \cdot y < (-x) \cdot z$, so

$$x \cdot z = (-x) \cdot y + x \cdot y + x \cdot z < (-x) \cdot z + x \cdot y + x \cdot z = x \cdot y.$$

(d) If $x > 0$, then $x^2 > 0$ by Definition 1.23. If $x < 0$, then $-x > 0$, so $(-x)^2 > 0$, but $(-x)^2 = x^2$ by Proposition 1.21. (This is an application of Trichotomy Law.)

(e) Assume $0 < x < y$. If $z \leq 0$, then $yz \leq 0$ by (b). Since $y \cdot \frac{1}{y} = 1 > 0$, we must have $\frac{1}{y} > 0$. Similarly, we have $\frac{1}{x} > 0$ as well. Finally, multiply $x < y$ by positive element $\frac{1}{x} \cdot \frac{1}{y}$, so

$$\frac{1}{y} = x \cdot \frac{1}{x} \cdot \frac{1}{y} < y \cdot \frac{1}{x} \cdot \frac{1}{y} = \frac{1}{x}.$$

□

## 1.3 The Real Field

We now start contructing $\mathbb{R}$ only in terms of rationals. We will construct it in such a way that it satisfies the *existence theorem.*

**Theorem 1.26.** *There exists a unique ordered field having the least-upper-bound property. Moreover, this field contains $\mathbb{Q}$. We denote this field $\mathbb{R}$ and call its elements **real numbers**.*

A set $S$ has the *least upper bound* property if every nonempty subset of $S$ has an upper bound also has a least upper bound, so a set that satisfies this property will always have a least upper bound if bounded.

### Construction of Real Numbers by Dedekind Cut

We have seen the problem with $\mathbb{Q}$. Some subsets of $\mathbb{Q}$ have no least upper boundes. For example, the set $A = \{p : p^2 < 2 \text{ and } p > 0\}$ has no least upper bound, which means you can always find a number $r \in A$ such that $r > p$. Therefore, we cannot express $\sqrt{2}$ that we know of using only rational field since there are "gaps" in $\mathbb{Q}$. Therefore, we want to contruct a more *complete* field, which we call the **real field**.

How do we express $\sqrt{2}$ using only $\mathbb{Q}$? There are many elements in $\mathbb{Q}$ that are approaching $\sqrt{2}$. For example,

$$\frac{7}{5}, \frac{41}{29}, \frac{239}{169}, \frac{1393}{985}, \frac{3363}{2378}, \frac{577}{408}, \frac{99}{70}, \frac{17}{12}, \frac{3}{2}$$

We can "cut" the axis into two pieces.

**Definition 1.27** (Dedekind Cut)**.** A **Dedekind Cut** is any set $\alpha \subseteq \mathbb{Q}$ with the following three properties

(a) $\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$;

(b) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$;

(c) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

(a) basically says *Dedekind Cut* cannot be empty or the whole rational field $\mathbb{Q}$. (b) says that it is closed downward, which means it should contain everything to the left of the "cutting edge." (c) basically says it does not contain a biggest number.

**Lemma 1.28** (Dedekind Cut at $\sqrt{2}$)**.** *Let $L = \{x \in \mathbb{Q} : x \leq 0$ or $x^2 < 2\}$. Then, $L$ is a Dedekind Cut.*

*Proof.* (a) holds because $0 \in L$ (so $L \neq \varnothing$) and $3 \notin L$ because $3 > 0$ and $3^2 = 9 > 2$.

To check (b) holds, assume that $a < b$ and $b \in L$. We want to show $a \in L$. If $a \leq 0$, then $a \in L$ by the definition of $L$. So, suppose that $a > 0$. Then, $b > 0$ as well and also $b^2 < 2$. Since $0 < a < b$, we find that $a^2 < b^2$. Therefore, $a^2 < 2$ and so $a \in L$ as well. Thus, (b) holds.

To check (c) holds, we want to show that each element $a \in L$ is not a maximum element. In other words, we want to show that there exists $a' \in L$ with $a' \in a$. Consider $a \in L$. If $a \leq 1$, then we may take $a' = \frac{5}{4}$. Therefore, we assume that $a > 1$. Since $\left(\frac{3}{2}\right)^2 > 2$, we get $a < \frac{3}{2}$. We want to show that there exists a rational number $y > 0$ such that $(a+y)^2 < 2$. Assume that $y = \frac{1}{n}$ for some $n \in \mathbb{Z}_{>0}$. Thus, we need

$$a^2 + 2ay + y^2 < 2.$$

Since $a \geq 1$ and $n \geq 1$, we have

$$\frac{1}{n^2} \leq \frac{1}{n} \leq \frac{a}{n}.$$

Therefore,

$$a^2 + 2ay + y^2 = a^2 + 2 \cdot \frac{a}{n} + \frac{1}{n^2} < a^2 + 2 \cdot \frac{a}{n} + \frac{a}{n} = a^2 + 3 \cdot \frac{a}{n}$$

so that it suffices to find an $n$ such that $a^2 + 3 \cdot \frac{a}{n} < 2$. We can immediately get $3 \cdot \frac{a}{n} < 2 - a^2$, that is, $n > 3 \cdot \frac{a}{2 - a^2}$.

We conclude that $L$ is a *Dedekind Cut* since all three conditions satisfy.

□

Here is a useful result about *Dedekind Cut*.

**Lemma 1.29.** *Let $L$ be a Dedekind Cut and $a \notin L$. Then, $a$ is an upper bound for $L$, that is, every $b \in L$ satisfies $b < a$.*

*Proof.* Let $b \in L$. Then, $b \neq a$ because $a \notin L$ and $b \in L$. If $b > a$, then $a \in L$ by (b) in Definition 1.27, which is impossible. Hence, $b < a$.

□

Now, we have defined $\mathbb{R}$, we want to define an order on $\mathbb{R}$.

**Definition 1.30** (Orders on $\mathbb{R}$)**.** Let two sets $A$ and $B$ determine *Dedekind Cut*, and hence $\alpha, \beta \in \mathbb{R}$ respectively. We define $\alpha < \beta$ to mean $A$ is a proper subset of $B$.

Now, we define an order on $\mathbb{R}$. We need to define arithmetic operations on $\mathbb{R}$. We first define the addition.

**Proposition 1.31** (Addition on $\mathbb{R}$)**.** *Given Dedekind Cuts $L$ and $M$ define the subset $L + M$ of $\mathbb{Q}$ by*

$$L + M = \{a + b : a \in L, b \in M\}.$$

*Then $L + M$ is a Dedekind Cut.*

*Proof.* We will check $L+M$ satisfies all the conditions in Definition 1.27.

To check (a) holds, want to show that $L+M$ is a proper subset, that is to show $L+M \neq \varnothing$ and $L+M \neq \mathbb{Q}$. There are $a_0 \in L$ and $b_0 \in M$, which means $a_0 + b_0 \in L+M$, so $L+M \neq \varnothing$. Since $L$ and $M$ are valid *Dedekind Cuts*, so there are elements $u$ and $v$ such that $u \notin L$ and $v \notin M$, we want to show that $u+v \notin L+M$. Towards a contradiction, assume $u+v \in L+M$, then there are $a \in L$ and $b \in M$ such that $u+v = a+b$. By Lemma 1.29, since $a \in L$ and $u \notin L$, so $a < u$. Similarly, we have $b < v$. Therefore, $a+b < u+v$, a contradiction to our assumption. Thus, (a) holds for $L+M$.

To check (b) holds, assume $x < y$ and $y \in L+M$. Then, we can write $y = a+b$ for $a \in L$ and $b \in M$. Then,

$$x = a + b - (y - x) = a - (y - x) + b = a' + b$$

where $a' = a - (y - x) < a$. Since $a \in L$, so $a' \in L$ since $L$ is a valid *Dedekind Cut*. Hence, we can write $x = a' + b$ as $a' \in L$ and $b \in M$. Therefore, $x \in L+M$ and (b) holds.

To check (c) holds, let $a+b \in L+M$ with $a \in L$ and $b \in M$. Since $L$ has no maximum element, so there is $x \in L$ such that $a < x$. Then, $x+b \in L+M$ and $a+b < x+b$. Therefore, $a+b$ is not maximal in $L+M$. Since this holds for all $a+b \in L+M$, then $L+M$ has no maximal element.

We conclude that $L+M$ is a valid *Dedekind Cut*.

□

Further, consider $L, M \in \mathbb{R}$, the addition on $\mathbb{R}$ is defined to be

$$L + M = \{r + s : r \in L, s \in M\}.$$

By Proposition 1.31, we conclude that the addition we defined is binary since $L \in \mathbb{R}$ and $M \in \mathbb{R}$ imply $L + M \in \mathbb{R}$.

(a) (Commutativity) Since addition itself is commutative, it follows that $L+M = M+L$.

(b) (Associativity) This follows from the associative property in the rational field $\mathbb{Q}$.

(c) (Identity) We define our zero element $L_0$ to be the set of all negative rational numbers. Then, if $r \in L$ and $s \in L_0$, we have $r+s < r$. Therefore, $r+s \in L$. So, $L + L_0 \subseteq L$.

   Pick $a, b \in L$ and $b > a$. Then, $a - b \in L_0$ and $a = b + (a-b) \in L + L_0$. Thus, $L + L_0 \subseteq L$. We conclude that $L + L_0 = L$.

(d) (Inverse) Define a *Dedekind Cut*

$$-L = \{x \in \mathbb{Q} : \text{ there exists } y \notin L \text{ such that } y < -x\}.$$

   We want to show that the *Dedekind Cut* $-L$ has the property that if $r \in L$, then $-r \in -L$.

Towards a contradiction, assume the *Dedekind Cut* does not have the property as we state. Then, for all $y \notin L$, we have $y \geq r$. Hence, $L \subseteq \{x \in \mathbb{Q} : x \leq r\}$. But $r \in L$ is then a maximum for $L$, contradicting Definition 1.27 (c) for the *Dedekind Cut* $L$.

Now we want to show $L + (-L) = L_0$.

If $r \in L$ and $x \in -L$, then there exists $y \notin L$ such that $-y > x$, so $r + x < r - y$ and $r < y$ since $y \notin L$ by Definition 1.27 (b). Hence, $r + x < 0$ and $r + x \in L_0$.

If $s \in L_0$, then $s < 0 = r + (-r)$ for any $r \in L$. Then, we have $-r \in -L$, which implies that $s \in L + (-L)$ by Definition 1.27 (b). Hence, we conclude that $L + (-L) = L_0$.

**Proposition 1.32** (Multiplication on $\mathbb{R}_{>0}$). *Given Dedekind Cuts $L, M \in \mathbb{Q}_{>0}$ define the subset $L + M$ of $\mathbb{Q}_{>0}$ by*

$$L \cdot M = \{a \cdot b : a \in L, b \in M\}.$$

*Then, $L \cdot M$ is a Dedekind Cut.*

*Proof.* We will check $L \cdot M$ satisfies all the conditions in Definition 1.27.

(a) is immediate and similar to Proposition 1.31.

To check (b) holds, assume if $p \in L \cdot M$, pick $q \in \mathbb{Q}$ and $q < p$. Then, $p \leq a \cdot b$ for some positive $a \in L$ and $b \in M$. Then, $q < p \leq a \cdot b$. It follows immediately that $q \in L \cdot M$.

To check (c) holds, pick $t \in L$ and $t > a$. Let $k = t \cdot b$. Then, $p \leq a \cdot b < t \cdot b = k$, but $k = t \cdot b \in L \cdot M$.

We conclude that $L \cdot M$ is a *Dedekind Cut*.

□

Further, consider $L, M \in \mathbb{R}_{>0}$, the multiplication on $\mathbb{R}$ is defined to be

$$L \cdot M = \{p : p \leq a \cdot b \text{ for some } a \in L, b \in S, a > 0, b > 0\}.$$

By Proposition 1.32, we conclude that the multiplication on positive real field we defined is binary since $L \in \mathbb{R}$ and $M \in \mathbb{R}$ imply $L \cdot M \in \mathbb{R}_{>0}$.

(a) (Commutativity) This follows immediately from the commutative property of multiplication on $\mathbb{Q}$.

(b) (Associativity) This follows immediately from the associative property of multiplication on $\mathbb{Q}$.

(c) (Identity) We define our identity element $L_1$ to be

$$L_1 = \{q : q < 1\}.$$

It is immediate that $L_1$ is a *Dedekind Cut*, so $L_1 \in \mathbb{R}$. We want to prove that $L_1 \cdot L = L$.

Pick $x \in L_1 \cdot L$. Then, $x \leq q \cdot p$ for some $q \in L_1$ and $p \in L$, but $x \leq q \cdot p < p$ and $p \in L$, thus $x \in L$. Then, $L_1 \cdot L \subseteq L$.

Pick $p \in L$, there exists $q \in L$ and $q > p$. Then, $\frac{p}{q} < 1$. So, $\frac{p}{q} \in L_1$ and $p \leq q \cdot \frac{p}{q}$. So, $p \in L_1 \cdot L$, which implies $L \subseteq L_1 \cdot L$. We conclude that $L_1 \cdot L = L$.

(d) (Inverse) We define the multiplicative inverse of $L$ to be

$$M = \left\{ p : \text{ there exists } r > 0 \text{ such that } \frac{1}{p} - r \notin L \right\}.$$

We first need to show that $M$ is a *Dedekind Cut*.

To check (a) holds, it is clear that $M \neq \varnothing$. If we pick $p$ such that $\frac{1}{p} \in L$, then there exists $r > 0$ such that $\frac{1}{p} - r \in L$, so $p \notin M$. Therefore, $M \neq \mathbb{Q}$. To check (b) holds, if $p \in M$, then there exists $r > 0$ such that $\frac{1}{p} = r \notin L$. Pick $q < p$, then $\frac{1}{q} > \frac{1}{p}$. Then, $\frac{1}{q} - r > \frac{1}{p} - r \notin L$. So, $\frac{1}{q} - r \notin L$. Hence, $q \in M$. To check (c) holds, if $p \in M$, there exists $r > 0$ such that $\frac{1}{p} - r \notin L$. Pick $t = \frac{2 \cdot p}{2 - r \cdot p}$. Then, $\frac{1}{t} = \frac{1}{p} - \frac{1}{2}$, so $t > p$. Now, $\frac{1}{t} - \frac{r}{2} = \frac{1}{p} - r \notin L$, so $t \in M$. We conclude that $M$ is a valid *Dedekind Cut*.

Now, we want to prove that $L \cdot M = L_1$. If $p \in L \cdot M$, then $p \leq t \cdot s$ for some positive $t \in L$ and $s \in M$. Since $s \in M$, there exists $r > 0$ such that $\frac{1}{s} - r \notin L$. Since $t \in L$ and $\frac{1}{s} - r \notin L$, it follows that $t < \frac{1}{s} - r$. So, $t < \frac{1}{s}$. Thus, $t \cdot s < \frac{1}{s} \cdot s = 1$. So, $t \cdot s \in L_1$. Since $L_1$ is a valid *Dedekind Cut* and $t \cdot s \in L_1$ and $p \leq t \cdot s$, so $L \cdot M \subseteq L_1$.

Pick $0 < t < 1$, so $t \in L_1$. Pick positive $p \in L$. Since $0 < t < 1$, so $(1 - t) \cdot p < p$, which means $(1 - t) \cdot p \in L$. Assume we know the Archimedean Property (on $\mathbb{Q}$) which we will prove later (on $\mathbb{R}$), there exists a positive integer $n$ such that $p_1 = n \cdot (1 - t) \cdot p \in L$ and $p_2 = (n + 1) \cdot (1 - t) \cdot p \notin L$. Since $p_2 \notin L$ adn $p \in L$, so $p_2 > p$ and $p_2 \cdot (1 - t) > p \cdot (1 - t) = p_2 - p_1$. Therefore, $p_2 < \frac{p_1}{t}$. Since $p_2 \notin L$ and $p_2 < \frac{p_1}{t}$, it follows that $\frac{p_1}{t} \notin L$. Thus, $\frac{p_1}{t} - = \left(\frac{p_1}{t} - p_2\right) \notin L$, which implies $\frac{t}{p_2} \in M$. Finally, $t \leq p_1 \cdot \frac{t}{p_1} \in L \cdot M$, so $L_1 \subseteq L \cdot M$. We conclude that $L \cdot M = L_1$.

(e) (Distributivity) This is obvious.

Now, we can complete the definition of multiplication on whole real field $\mathbb{R}$ by setting $L \cdot L_0 = L_0 \cdot L = L_0$ and

$$L \cdot M = \begin{cases} (-L) \cdot (-M) & L < L_0, M < L_0 \\ (-L) \cdot M & L < L_0, M > L_0 \\ -(L \cdot (-M)) & L > L_0, M < L_0 \end{cases}.$$

Recall $\gamma = -(-\gamma)$, it is obvious to prove what we define for the multiplication on $\mathbb{R}$.

**Consequences of "Completeness"**

We worked hard to construct the real field $\mathbb{R}$ and proved that $\mathbb{R}$ is **an ordered field with least upper bound property**. Can the real number field defined as the set of all *Dedekind Cuts*, that is to express something that $\mathbb{Q}$ fails to express? Indeed, there are gaps in the rational numbers. There does not exist any rational number $x$ for which $x^2 = 2$. The set $\mathbb{Q}$ is an extension of $\mathbb{N}$, and $\mathbb{R}$ in turn is an extension of $\mathbb{Q}$. The next result indicates how $\mathbb{N}$ and $\mathbb{Q}$ sit inside of $\mathbb{R}$.

**Theorem 1.33.**

*(a) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $n \cdot x > y$.*

*(b) If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ with $x < p < y$.*

Lecture 4
Friday
October 9

*Proof.*

(a) Towards a contradiction, suppose for all $n \in \mathbb{N}$, we have $n \cdot x \leq y$.

Let $A = \{n \cdot x : n \in \mathbb{N}\}$. By our assumption, $A$ is bounded above by $y$, so by the least-upper-bound property, $\alpha = \sup A$ exists.

Since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound of $A$, which means there is $n \in \mathbb{N}$ with $n \cdot x > \alpha - x$. Then, $(n+1)x > \alpha$, contradicting $\alpha = \sup A$, and $(n+1)x \in A$.

(b) Since $x < y$, we have $y - x > 0$, so by (a), there is $n \geq 1$ such that $n \cdot (y - x) > 1$.

Apply (a) twice more, we get $m_1, m_2 \geq 1$ with $m_1, m_2 \in \mathbb{Z}$ and $m_1 > n \cdot x$ and $m_2 > -n \cdot x$, so $-m_2 < n \cdot x < m_1$.

So, the finite set $\{-m_2, -m_2 + 1, \ldots, m_1\}$ must contain the least $m$ with $n \cdot x < m$. Since $m$ is the least, so $m - 1 \leq n \cdot x < m$.

Therefore, $n \cdot x < m \leq n \cdot x + 1 < n \cdot y$ and $x < \dfrac{m}{n} < y$.

$\square$

Note that Theorem 1.33 (a) is known as the **Archimedean Property**, and Theorem 1.33 (b) says that $\mathbb{Q}$ is **dense** in $\mathbb{R}$ (we will define dense soon).

Archimedean Property is equivalent to

(a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.

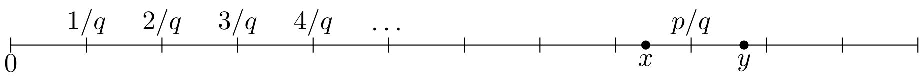
(b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\dfrac{1}{n} < y$.

There is a friendlier proof of density of $\mathbb{Q}$ in $\mathbb{R}$.

*Proof.* A rational number is a quotient of integers, so we want to find $p \in \mathbb{Z}$ and $q \in \mathbb{N} - \{0\}$ such that

$$x < \frac{p}{q} < y.$$

The first step is to choose the denominator $n$ large enough so that consecutive increments of size $\dfrac{1}{q}$ are too close together to "step over" the interval $(x, y)$.




By Archimedean Property, we may pick an $q \in \mathbb{N}$ sufficiently large enough so that

$$\frac{1}{q} < y - x.$$

Now, we want to show that $qx < p < qy$. With $q$ already chosen, the idea now is to choose $p$ to be the smallest integer greater than $qx$. In other words, pick $p \in \mathbb{Z}$ so that

$$p - 1 \leq qx < p.$$

We get $x < \frac{p}{q}$. Recall that $\frac{1}{q} < y - x$, then

$$\begin{aligned} p &\leq qx + 1 \\ &< q\left(y - \frac{1}{q}\right) + 1 \\ &= qy. \end{aligned}$$

Since $p < qy$ implies $\frac{p}{q} < y$, we have $x < \frac{p}{q} < y$, as desired.

□

**Corollary 1.34** (Density of Irrational Numbers)**.** *Given any two real numbers $x < y$, there exists an irrational number $t$ satisfying $x < t < y$.*

The rational numbers are everythere. They are among us. So are irrational numbers. What we are saying is that between any two real numbers there is a rational number. A problem we have encountered is that we are not guaranteed the existence of square roots in $\mathbb{Q}_{\geq 0}$. Fortunately, this has been remedied by contructing the real field $\mathbb{R}$.

**Theorem 1.35.** *If $x \in \mathbb{R}$ is positive and $n \in \mathbb{Z}_+$, then there is a unique real $y > 0$ with $y^n = x$. This number $y$ is denoted $\sqrt[n]{x}$ or $x^{1/n}$.*

*Proof.* We first prove the uniqueness. Equivalently, we want to show if $0 < y_1 < y_2$, then $y_1^n < y_2^n$.

Since $\frac{y_2}{y_1} > 1$, we have

$$\frac{y_2^n}{y_1^n} = \left(\frac{y_2}{y_1}\right)^n > 1$$

hence, $y_1^n < y_2^n$. So, if $y$ exists, then it must be **unique**.

We still need to prove the **existence** (otherwise, the proof of **uniqueness** is not meaningful).

Now, let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$.

(Check $E \neq \varnothing$) If $t = \frac{x}{x+1}$, then $t < 1$ and

$$t - x = \frac{x}{x+1} - x = -\frac{x^2}{x+1} < 0$$

so $t^n < t < x$ and hence $t \in E$.

(Check $1+x$ is upper bound to $E$) If $t > 1+x$, then $t^n > t > x$, so $t \notin E$. By the Theorem 1.26, $y = \sup E$ exists.

(Check $y^n = x$) Recall the identity

$$b^n - a^n = (b-a)(b^{n-1} + a \cdot b^{n-2} + \cdots + a^{n-2} \cdot b + a^{n-1}),$$

it follows that

$$b^n - a^n < (b-a)nb^{n-1}$$

when $0 < a < b$.

Towards a contradiction, suppose $y^n < x$. Pick $h$ with

$$0 < h < \min\left(1, \frac{x - y^n}{n \cdot (y+1)^{n-1}}\right)$$

that is, $h < 1$ (this is to make sure the second inequality sign in the next inequality holds) and $h < \dfrac{x - y^n}{n \cdot (y+1)^{n-1}}$ (this is our main construction for the next inequality, for the last inequality sign), then

$$(y+h)^n - y^n < h \cdot n \cdot (y+h)^{n-1} \le h \cdot n \cdot (y+1)^{n-1} < x - y^n.$$

So, $y + h \in E$ and $y + h > y$, contradicting $y$ being an upper bound to $E$.

Towards a contradiction, suppose $y^n > x$. Let $k = \dfrac{y^n - x}{n \cdot y^{n-1}}$, then $0 < k < y$. If $t \ge y - k$, then

$$y^n - t^n \le y^n - (y-k)^n < k \cdot n \cdot y^{n-1} = y^n - x$$

so $r^n > x$ and $\notin E$. Thus, $y - k$ is an upper bound to $E$ and $y - k < y$, contradicting to $y = \sup E$.

Finally, by Trichotomy Law, so $y^n = x$.

$\square$

*Remark* 1.36. It is possible to define decimal representations of real numbers.

Let $x \in \mathbb{R}_{>0}$. Let $n_0$ be the largest integer such that $n_0 \le x$. (Note that the existence of $n_0$ depends on the archimedean property of $\mathbb{R}$.) Having chosen $n_0, n_1, \ldots, n_{k-1}$, let $n_k$ be the largest integer such that

$$n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \le x.$$

Let $E = \left\{ n_0 + \dfrac{n_1}{10} + \cdots + \dfrac{n_k}{10^k} : k = 0, 1, 2, \ldots \right\}$. Then, $x = \sup E$. The decimal expansion of $x$ is $n_0.n_1n_2n_3\cdots$.

**Definition 1.37** (Extended Real Number System)**.** The **extended real number system** is the set $\mathbb{R} \cup \{-\infty, \infty\{$ where for all $x \in \mathbb{R}$,

(a) $-\infty < x < +\infty$.

(b) $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{+\infty} = 0$, $\dfrac{x}{-\infty} = 0$.

(c) If $x > 0$, then $x \cdot (+\infty) = \infty$, $x \cdot (-\infty) = -\infty$. If $x < 0$, then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = \infty$.

All other operations are left **undefined**.

*Remark* 1.38.

(a) The extended real number system is not a field since the operation $\infty - \infty$ is undefined (that is a contradiction to Definition 1.17). Also, $\pm\infty$ do not have multiplicative inverses.

(b) It is clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound.

(c) To distinguish $x \in \mathbb{R}$ from $-\infty$ and $+\infty$, we call $x$ **finite**.

## 1.4 The Complex Field

**Definition 1.39** (Complex Number)**.** The set of **complex numbers** is

$$\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$$

Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

For $x, y \in \mathbb{C}$, say $x = (a, b)$ and $y = (c, d)$, we define

$$x + y = (a + c, b + d),$$
$$x \cdot y = (ac - bd, ad + bc).$$

**Theorem 1.40.** *$\mathbb{C}$ is a field with $(0, 0)$ and $(1, 0)$ playing the rolds of additive identity $0$ and multiplicative identity $1$.*

*Proof.* The field axioms in Definition 1.17 are easy to verify as we use the field structure of $\mathbb{R}$. Here, we only prove for additive and multiplicative identities.

Let $x = (a, b) \in \mathbb{C}$. Write $-x = (-a, -b)$. Then,

$$x + (-x) = (a, b) + (-a, -b) = (0, 0).$$

Now, assume $x \neq 0$. Then, $(a, b) \neq (0, 0)$, so $a \neq 0$ or $b \neq 0$. Thus, $a^2 + b^2 > 0$. Write

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right).$$

Then,

$$x \cdot \frac{1}{x} = (a, b) \cdot \left(\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}\right) = (1, 0) = 1.$$

$\square$

Lecture 5
Monday
October 12

**Theorem 1.41.** *For all $a, b \in \mathbb{R}$,*

$$(a, 0) + (b, 0) = (a + b, 0) \qquad (a, 0) \cdot (b, 0) = (a \cdot b, 0).$$

This means we can identify $a \in \mathbb{R}$ with $(a, 0)$ and this identification preserves addition and multiplication, so we can view $\mathbb{R}$ as a subfield of $\mathbb{C}$.

**Definition 1.42.** $i = (0, 1)$.

**Theorem 1.43.** $i^2 = -1$.

*Proof.* $i^2 = (0,1)^2 = (0,1) \cdot (0,1) = (-1, 0)$. $\square$

**Theorem 1.44.** *If $a, b \in \mathbb{R}$, then $(a, b) = a + bi$.*

*Proof.* $a + bi = (a, 0) + (b, 0) \cdot (0, 1) = (a, 0) + (0, b) = (a, b)$. $\square$

**Definition 1.45.** For $z = a + bi \in \mathbb{C}$, we call $a$ the **real part** of $z$ and $b$ the **imaginary part** of $z$ and write $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$. We call $\bar{z} = a - bi$ the **complex conjugate** of $z$.

**Theorem 1.46.** *If $z, w \in \mathbb{C}$, then*

*(a)* $\overline{z + w} = \bar{z} + \bar{w}$.

*(b)* $\overline{zw} = \bar{z} \cdot \bar{w}$.

*(c)* $z + \bar{z} = 2\operatorname{Re}(z)$ *and* $z - \bar{z} = 2\operatorname{Im}(z)$.

*(d)* $z\bar{z} \in \mathbb{R}$ *and* $z\bar{z} > 0$ *when* $z \neq 0$.

*Proof.* (a), (b), and (c) are easy to check by calculation. (d) holds since if $z = a + bi$, then $z \cdot \bar{z} = a^2 + b^2$. $\square$

**Definition 1.47** (Absolute Value)**.** The **absolute value** of $z \in \mathbb{C}$ is defined $|z| = (z \cdot \bar{z})^{1/2}$.

If $x \in \mathbb{R}$, then $\bar{x} = x$ so $|x| = \sqrt{x^2}$ meaning $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

**Theorem 1.48.** *If $z, w \in \mathbb{C}$, then*

*(a)* $|z| > 0$ *unless* $z = 0$.

*(b)* $|\bar{z}| = |z|$.

*(c)* $|z \cdot w| = |z| \cdot |w|$.

*(d)* $|\operatorname{Re}(z)| \leq |z|$.

*(e)* $|z + w| \leq |z| + |w|$.

*Proof.* (a) and (b) are easy to check by calculation.

(c)
$$\begin{aligned}|z \cdot w| &= (zw \cdot \overline{zw})^{1/2} = (zw\bar{z}\overline{w})^{1/2} \\ &= (z\bar{z}w\overline{w})^{1/2} = (z\bar{z})^{1/2}(w\overline{w})^{1/2} \\ &= |z| \cdot |w|.\end{aligned}$$

(d) Say $z = a + bi$, then $a^2 \le a^2 + b^2$, so
$$|\operatorname{Re}(z)| = |a| = \sqrt{a^2} \le \sqrt{a^2 + b^2} = \sqrt{z \cdot \bar{z}} = |z|.$$

(e) Note $\overline{\bar{z} \cdot w} = z \cdot \overline{w}$, so
$$\bar{z} \cdot w + z \cdot \overline{w} = 2\operatorname{Re}(z \cdot w).$$

Therefore,
$$\begin{aligned}|z + w|^2 &= (z + w)(\bar{z} + \overline{w}) \\ &= z \cdot \bar{z} + w \cdot \bar{z} + \overline{w} \cdot z + w \cdot \overline{w} \\ &= |z|^2 + 2\operatorname{Re}(\overline{w} \cdot z) + |w|^2 \\ &\le |z|^2 + 2|\overline{w} \cdot z| + |w|^2 \\ &= |z|^2 + 2|w| \cdot |z| + |w|^2 \\ &= (|z| + |w|)^2.\end{aligned}$$

□

We conclude our discussion of complex numbers with an important and famous inequality, *Cauchy-Schwarz inequality.*

**Theorem 1.49** (Cauchy-Schwarz Inequality)**.** *If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{C}$, then*
$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 \le \sum_{k=1}^{n} |a_k|^2 \cdot \sum_{k=1}^{n} |b_k|^2.$$

If $(a_1, a_2, \dots, a_n) = \vec{a} \in \mathbb{R}^n$ and $(b_1, b_2, \dots, b_n) = \vec{b} \in \mathbb{R}^2$, this says
$$|\vec{a} \cdot \vec{b}|^2 \le |\vec{a}|^2 \cdot |\vec{b}|^2.$$

From the geometric intuition, we expect equality to hold precisely
$$B \cdot \vec{a} = C \cdot \vec{b}$$
where $B = \displaystyle\sum_{k=1}^{n} |b_k|^2$ and $C = \displaystyle\sum_{k=1}^{n} a_k \cdot \overline{b_k}$. This means that $\vec{a}$ and $\vec{b}$ are collinear.

*Proof.* Define $B$ and $C$ as above, let $A = \displaystyle\sum_{k=1}^{n} |a_k|^2$.

If $B = 0$, then $b_1 = b_2 = \cdots = b_n = 0$ and the conclusion is trivial.

So assume $B > 0$, we have

$$\begin{aligned}
0 &\leq \sum_{k=1}^{n} |B \cdot a_k - C \cdot b_k|^2 \\
&= \sum_{k=1}^{n} (B \cdot a_k - C \cdot b_k)(B \cdot \overline{a_k} - \overline{C} \cdot \overline{b_k}) \\
&= B^2 \sum_{k=1}^{n} |a_k|^2 - B \cdot \overline{C} \sum_{k=1}^{n} a_k \cdot \overline{b_k} - B \cdot C \sum_{k=1}^{n} \overline{a_k} \cdot b_k + |C|^2 \sum_{k=1}^{n} |b_k|^2 \\
&= B^2 \cdot A - B \cdot |C|^2 - B \cdot |C|^2 + |B| \cdot |C|^2 \\
&= B^2 \cdot A - B \cdot |C|^2 = B(B \cdot A - |C|^2).
\end{aligned}$$

Since $B > 0$, we get $B \cdot A - |C|^2 \geq 0$. This is our desired result.

$\square$

## 1.5 The Euclidean Space

Now, we know the Cauchy-Schwarz inequality, we say a little bit more about Euclidean space.

**Definition 1.50** (Vector)**.** For $k \in \mathbb{Z}_+$, we let $\mathbb{R}^k$ be the set of all $k$-tuples

$$\vec{x} = (x_1, x_2, \ldots, x_k), \quad x_i \in \mathbb{R}.$$

We call $\vec{x}$ a **point** or a **vector**. We call $\vec{0} = (0, 0, \ldots, 0)$ the **origin**.

*Remark* 1.51. $\mathbb{R}^k$ is an example of a **vector space**, with operations

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k) \text{ for } \vec{x}, \vec{y} \in \mathbb{R}^k$$
$$\alpha \cdot \vec{x} = (\alpha \cdot x_1, \alpha \cdot x_2, \ldots, \alpha \cdot x_k) \text{ for } x \in \mathbb{R}^k, \alpha \in \mathbb{R}.$$

**Definition 1.52** (Inner Product, Norm)**.** The **inner product** (or **dot product**) is

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i \cdot y_i.$$

The **norm** of $\vec{x} \in \mathbb{R}^k$ is

$$|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \left( \sum_{i=1}^{k} x_i^2 \right)^{1/2}.$$

The structure now defined (the vector space $\mathbb{R}^k$ with the above inner product and norm) is called **$k$-dimensional Euclidean Space**.

*Remark* 1.53.

(a) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$.

(b) $\vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$.

Lecture 6
Wednesday
October 14

**Theorem 1.54.** *If $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, then*

*(a)* $|\vec{x}| \geq 0$ *and* $|\vec{x}| = 0$ *if and only if* $\vec{x} = \vec{0}$.

*(b)* $|\alpha \cdot \vec{x}| = |\alpha| \cdot |\vec{x}|$.

*(c)* $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| \cdot |\vec{y}|$.

*(d)* $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$. *This is the Triangle inequality.*

*(e)* $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$.

*Proof.* (a) and (b) are obvious.

(c) This is an immediate consequence of the Cauchy-Schwarz Inequality.

(d)

$$\begin{aligned} |\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\ &= |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \\ &\leq |\vec{x}|^2 + 2|\vec{x}| \cdot |\vec{y}| + |\vec{y}|^2 \\ &= (|\vec{x}| + |\vec{y}|)^2. \end{aligned}$$

(e) By Triangle inequality,

$$\begin{aligned} |\vec{x} - \vec{z}| &= |(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})| \\ &\leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|. \end{aligned}$$

□

# 2 Introduction to Metric Space

Informally, a *set-theoretic function* $f$ taking a set $X$ to a set $B$ is a mapping that to each $x \in A$ assigns a unique $y \in Y$. We write $f : X \to Y$.

**Definition 2.1** (Cartesian Product)**.** Let $X$ and $Y$ be sets. The **Cartesian product** is the set of tuples defined as

$$A \times B = \{(x, y) : x \in X, y \in Y\}.$$

**Definition 2.2** (Function)**.** A **function** $f : X \to Y$ is a subset $f$ of $X \times Y$ such that for each $x \in X$, there is a unique $(x, y) \in f$. We then write $f(x) = y$.

The set $X$ is called the **domain** of $f$. The set

$$\{y \in Y : \text{there exists an } x \text{ such that } f(x) = y\}$$

is called the **range** of $f$.

It is possible that the range is a proper subset of $Y$, while the domain of $f$ is always equal to $X$.

**Definition 2.3** (Image, Preimage)**.** Let $f : X \to Y$. The **image** of $A \subseteq X$ is

$$f(A) = \{f(a) : a \in A\}.$$

The **preimage** of $B \subseteq Y$ is

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

For $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$.

**Definition 2.4** (Injective, Surjective, Bijective)**.** Let $X$ and $Y$ be sets and $f : X \to Y$ a function from $X$ to $Y$. We make the following definitions.

(a) $f$ is **injective** (or $f$ is an **injection**) if, for all $x, x' \in X$, we have $f(x) = f(x')$ implies that $x = x'$. In other words, different elements in $X$ get sent to different values in $Y$.

(b) $f$ is **surjective** (or $f$ is an **surjection**) if, for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. In other words, all possible values in $Y$ are achieved.

(c) $f$ is **bijective** (or $f$ is a **bijection**) if it is both injective and surjective.

## 2.1 Finite, Countable, and Uncountable Sets

The term **cardinality** is used in mathematics to refer to the size of a set. The cardinalities of finite sets can be compared simply by attaching a natural number to each set.

**Definition 2.5** (Finite, Countablity)**.** Two sets $X, Y$ **have equal cardinality**, denoted $|X| = |Y|$, if there exists a bijection $f : X \to Y$.

(a) $X$ is **finite** if $X = \varnothing$ or there exists $n \in \mathbb{Z}_+$, $|X| = |\{1, 2, \ldots, n\}|$, otherwise $X$ is **infinite**.

(b) $X$ is **countable** if it is finite of $|X| = |\mathbb{N}|$, othersie $X$ is **uncountable**.

**Example 2.6.**

(a) Let $E = \{2, 4, 6, \ldots\}$ be the set of even natural numbers, then $E$ is countable. Consider the following arrangement of sets $E$ and $\mathbb{N}$

$$\begin{array}{cccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ E: & 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

It is certainly true that $E$ is a proper subset of $\mathbb{N}$, and for this reason it may seem logical to say that $E$ is a "smaller" set than $\mathbb{N}$. We can give an explicit formula for a function $f : \mathbb{N} \to E$ given by $f(n) = 2n$, which sets up a one-to-one correspondence. The definition of cardinality is quite specific, and from this point of view, $E$ and $\mathbb{N}$ are equivalent and countable.

(b) Although $\mathbb{N}$ is contained in $\mathbb{Z}$ as a proper subset, we can show they are equivalent. This time, let $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ -\dfrac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

**Definition 2.7** (Sequence)**.** A **sequence** is a function $f$ with domain $\mathbb{N}$ or $\mathbb{Z}_+$. When $f(n) = x_n$ for each $n$, we write $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n\in\mathbb{Z}_+}$.

**Theorem 2.8.** *If $X$ is countable and $A \subseteq X$, then $A$ is countable.*

*Proof.* This is obvious if $A$ is finite. So we assume $A$ is infinite. Then, $X$ is infinite so $|X| = |\mathbb{N}|$. So, we can list elements of $X$ as

$$\{x_0, x_1, x_2, \ldots\}.$$

Let $n_0 \in \mathbb{N}$ be least with $x_{n_0} \in A$. Inductively, after choosing $n_, \ldots, n_{k-1}$, pick $n_k > n_{k-1}$ to be least with $x_{n_k} \in A$.

Now, define $f : \mathbb{N} \to A$ by $f(k) = x_{n_k}$. Then, $f$ is a bijection.

∎

**Theorem 2.9.** *If $(X_n)_{n\in\mathbb{N}}$ is a sequence of countable sets, then $\displaystyle\bigcup_{n\in\mathbb{N}} X_n$ is countable.*

*Proof.* For $n \in \mathbb{N}$, let $(X_{n,k})_{k\in\mathbb{N}}$ be a sequence in $X_n$, that uses every element of $X_n$ at least once.

$$\begin{array}{ccccc} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & \cdots \\ x_{2,1} & x_{2,2} & x_{2,3} & & \cdots \\ x_{3,1} & x_{3,2} & & & \cdots \\ x_{4,1} & & & & \cdots \\ \vdots & & & & \end{array}$$

Let $f$ be the sequence $x_{0,0}, x_{1,0}, x_{0,1}, x_{2,0}, x_{1,1}, x_{0,2}, \ldots$ Then, $f$ is onto.

Let $A = \{n \in \mathbb{N} : \forall k < n, f(k) \neq f(n)\}$. Then, $A$ is countable by Theorem 2.8 and $f : A \to \bigcup_{n\in\mathbb{N}} X_n$ is a bijection.

□

**Theorem 2.10.** *If $X$ is countable, then $X^n = X \times X \times \cdots \times X$ is countable.*

*Proof.* We use induction. For the base case $n = 1$, $X^1 = X$ is countable.

Assumen $X^{n-1}$ is countable, then each set $\{x\} \times X^{n-1}$ is countable, so $X^n = \bigcup_{x\in X} \{x\} \times X^{n-1}$ is countable by Theorem 2.9.

□

Lecture 7 Friday October 16

**Corollary 2.11** (Countability of $\mathbb{Q}$)**.** *The set of all rational numbers is countable.*

*Proof.* By Theorem 2.10, with $n = 2$, every subset of $\mathbb{Z}^2$ is countable.

Define $f : \mathbb{Q} \to \mathbb{Z}^2$ by setting $f(q) = (a, b)$ where $a, b \in \mathbb{Z}$ satisfy $b > 0$, $\dfrac{a}{b} = q$, and $a, b$ are coprime. Then $f$ is a bijection with its image, which is countable, so $\mathbb{Q}$ is countable.

□

**Theorem 2.12.** *The set $\{0, 1\}^{\mathbb{N}}$ of all functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable.*

*Proof.* Let $F \subseteq \{0, 1\}^{\mathbb{N}}$ be any countably infinite set. Say $F = \{f_0, f_1, f_2, \ldots\}$, each $f_i : \mathbb{N} \to \{0, 1\}$.

Define $g : \mathbb{N} \to \{0, 1\}$ by $g(n) = 1 - f_n(n)$. Then, for any $n \in \mathbb{N}$, $g \neq f_n$ since $g_n \neq f_n(n)$. So, $g \in \{0, 1\}^{\mathbb{N}} \setminus F$. Thus, $F \neq \{0, 1\}^{\mathbb{N}}$.

□

## 2.2 Definition of Metric Spaces

Recall that real numbers can be ordered, which is a great feature. Now, here is another nice feature. We can define a rule for how far apart two points are.

**Definition 2.13** (Metric Space)**.** A **metric space** is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to \mathbb{R}$ satisfies

(a) $\forall p, q \in X$, $d(p, p) = 0$; if $p \neq q$, then $d(p, q) > 0$.

(b) $\forall p, q \in X$, $d(p, q) = d(q, p)$.

(c) (Triangle inequality) $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

The function $d$ is called a **metric**.

**Example 2.14** (Real Field)**.** The set of real numbers $\mathbb{R}$ is a metric space with the metric $d(x, y) = |x - y|$.

(a) and (b) of the Definition 2.13 are obvious. The Triangle inequality (c) follows immediately from the standard Triangle inequality for real numbers

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

The metric is the *standard metric on* $\mathbb{R}$.

**Example 2.15** ($k$-dimensional Euclidean Space)**.** The $k$-dimensional Euclidean space $\mathbb{R}^k$ is a metric space with the metric $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$, which is furtuer dictated by the Pythagorean theorem.

**Example 2.16** ($L^p$ Space)**.** Another metric for the $k$-dimensional Euclidean space $\mathbb{R}^k$ is the $L^p$ metric

$$d_p(\vec{x}, \vec{y}) = \left( \sum_{i=1}^{k} (x_i - y_i)^p \right)^{1/p} \qquad (p > 1).$$

Also, we can define the $L^p$ infinity metric for $k$-dimensional Euclidean Space

$$d_\infty(\vec{x}, \vec{y}) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

**Example 2.17** (Complex Field)**.** The set of complex numbers $\mathbb{C}$ is a metric space with the metric $d(z, w) = |z - w|$.

Let $z = x + iy$. Recall the *complex modulus* $|z| = \sqrt{x^2 + y^2}$. Then for any two complex numbers $z = x_1 + iy_1$ and $w = x_2 + iy_2$, the distance is

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|.$$

**Example 2.18** (Set of Functions)**.** The set of $[0, 1]^{[0,1]}$ (that is, set of functions $f : [0, 1] \to [0, 1]$) is a metric space with the metric

$$d(f, g) = \sup \{|f(x) - g(x)| : x \in [0, 1]\}.$$

## 2.3 Open and Closed Sets

Think of a set or subset as your property, surrounded by a fence. The set is *open* if the entire fence belongs to your neighbor. As long as you stay on your property, you can get closer and closer to the fence, but you can never reach it. No matter how close you are to your neighbor's property, there is a buffer zone of your property between you and it.

The set is *closed* if you own the fence. Now, if you sit on your fence, there is nothing betwen you and your neighbor's property. What is some of the fence belongs to you and some belongs to your neighbor? Then the set is *neither open nor closed*.

To state this in proper mathematical language, we first need to define an *open ball*. Imagine a balloon of radius $r$, centered around a point $p$. The open ball of radius $r$ around $p$ consists of all points $q$ inside the balloon, but not the skin of the balloon itself.

**Definition 2.19** (Ball)**.** Let $(X, d)$ be a metric space. For $p \in X$, $r > 0$, the **ball of radius around** $x$ is

$$B_r(p) = \{q \in X : d(p, q) < r\}.$$

**Open Sets**

An open set includes none of the fence; however close a point in the open set is to the fence, you can always surround it with a ball of other points in the open set.

**Definition 2.20** (Interior Point, Open Set)**.** Let $(X, d)$ be a metric space and $E \subseteq X$. A point $p \in X$ is an **interior point** of $E$ if there exists $r > 0$ such that $B_r(p) \subseteq E$. We denote the set of interior points of $E$ as $E^{\circ}$.

$E \subseteq X$ is an **open set** if every point of $E$ is an interior point of $E$.

$E \subseteq X$ is a **neighborhood** of $p$ if $E$ is open and $p \in E$.

Intuitively, an open set $E$ is a set that does not include its "boundary." Wherever we are in $E$, we are allowed to "wiggle" a little bit and stay in $E$.

**Example 2.21** (Open Sets)**.**

(a) If $a < b$, then the interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is open. Indeed, if $x \in (a, b)$, set $r = \min\{x - a, b - x\}$. Both these numbers are strictly positive, since $a < x < b$, and so is their minimum. Then, the ball $\{y : |y - x| < r\}$ is a subset of $(a, b)$.

(b) The infinite intervals $(a, +\infty)$, $(-\infty, b)$ are also open, but the intervals $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ and $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.

*Remark* 2.22. Let $(X, d)$ be a metric space and $p, q \in X$. Define $r = \frac{1}{2}d(p, q)$. Then, $p \in B_r(p)$, $q \in B_r(q)$, and $B_r(p) \cap B_r(q) = \varnothing$ is the empty set.

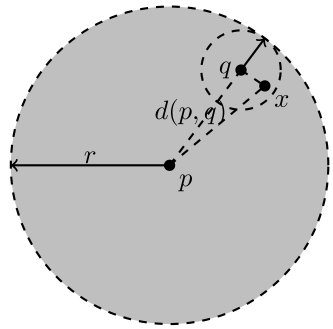
**Theorem 2.23.** *Let $(X, d)$ be a metric space. For $p \in X$, $r > 0$, the ball $B_r(p)$ is always open.*

*Proof.* Let a point $q \in B_r(p)$ and $x \in B_{r-d(p,q)}(q)$, then

$$\begin{aligned} d(p, x) &\leq d(p, q) + d(q, x) \\ &< d(p, q) + (r - d(p, q)) = r \end{aligned}$$

so $x \in B_r(p)$. Thus, $B_{r-d(p,q)} \subseteq B_r(p)$, so $q$ is an interior point and $B_r(p)$ is open.

□




Why we specify open set? Assume we know how to compute derivatives, if a function $f$ is defined on a set that is not open, and thus contains at least one point $x$ that is part of the fence, then talking of the derivative of $f$ at $x$ is meaningless. To compute $f'(x)$ we need to compute

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \left( f(x+h) - f(x) \right),$$

but $f(x+h)$ will not necessarily exist even for $h$ arbitrarily small, since $x+h$ may be outside the fence and thus not in the domain of $f$. This situation gets much worse in Euclidean space.

### Closed Sets

Unlike open sets, a closed set includes its fence, intuitively.

**Definition 2.24** (Limit Point, Closed Set)**.** Let $(X, d)$ be a metric space and $E \subseteq X$. A point $p \in X$ is a **limit point** of $E$ if for all $r > 0$,

$$(B_r(p) \setminus \{p\}) \cap E \neq \varnothing.$$

We denote the set of limit points of $E$ is $E'$.

$E \subseteq X$ is a **closed set** if every limit point of $E$ is a point of $E$, in short, $E' \subseteq E$.

Also, $E \subseteq X$ is **perfect** if $E' = E$, and $E \subseteq X$ is **dense** in $X$ if $E \cup E' = X$.

Intuitively, a set $E$ is closed if everything not in $E$ is some distance away from $E$. However, not every set is either open or closed. Generally, most subsets are neither.

For instance, the set of rationals $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed. Every neighborhood of a rational number contains irrational numbers, and every neighborhood of an irrational number contains rational numbers.

In general, "closed" does *not* mean "not open!"

**Theorem 2.25.** *Let $(X, d)$ be a metric space and $E \subseteq X$. If a point $p \in E'$, then for all $r > 0$,*

$$(B_r(p) \setminus \{p\}) \cap E$$

*is infinite.*

*Proof.* Let $r > 0$. Towards a contradiction, suppose not, that is, there exists $r > 0$, such that $(B_r(p) \setminus \{p\}) \cap E$ is finite.

Then,

$$t = \min \{d(p, q) : q \in (B_r(p) - \{q\}) \cap E\}$$

is positive. We have

$$\left(B_{t/2}(p) \setminus \{p\}\right) \cap E = \varnothing,$$

so $p \notin E'$.

□

**Corollary 2.26.** *Any finite set cannot have limit points.*

*Proof.* Let $S$ be a finite set and $S = \{s_1, s_2, \ldots, s_n\}$.

Towards a contradiction, suppose $S$ has a limit point $x_0$.

If $x_0 \in S$, then by Definition 2.24, for all $r > 0$, there exists $x \in S$ with $x \neq x_0$ such that $|x - x_0| < r$. Let $r = \min \{|s_i - s_j| : i \neq j\}$, then there is no $x \in S$ such that $|x - x_0| < r$ holds, a contradiction.

If $x_0 \notin S$, let $r = \min \{|x_0 - s| : s \in S\}$. Since the distance $|x_0 - s| > 0$, and there are only finitely many of them, so $r > 0$. Still, it is obvious that there is no $x \in S$ such that $|x - x_0| < r$, a contradiction.

Hence, any finite set cannot have limit points.

□

After discussing open and closed sets separately, we are interested in the relationship between them. The next theorem and corollary says the duality of open and closed sets.

**Theorem 2.27** (Duality of Open and Closed Sets)**.** *A set $E \subseteq X$ is open if and only if $E^{\mathsf{C}}$ is closed.*

*Proof.* Assume that $E^{\mathsf{C}}$ is closed. Pick an $x \in E$. Then, $x \notin E^{\mathsf{C}}$ and $x$ is not a limit point of $E^{\mathsf{C}}$ since a closed set contains all its limit points. Hence, there exists $r > 0$ such that $B_r(x) \cap E^{\mathsf{C}} = \varnothing$. Then, $B_r(x) \subseteq E$. Thus, $x$ is an interior point of $E$. So, $E$ is open.

Conversely, assume that $E$ is open. Let $x$ be a limit point of $E^{\mathsf{C}}$. Then, for all $r > 0$, we have $B_r(x) \cap E^{\mathsf{C}} \neq \varnothing$. So, $x$ is not an interior point of $E$, we have $x \notin E$, which means $x \in E^{\mathsf{C}}$.

□

**Corollary 2.28.** *A set $E \subseteq X$ is closed if and only if ites complement is open.*

Here is another concept we will be dealing with in the metric space. The definition of bounded set is intuitive. In the real field, given $E \subseteq \mathbb{R}$, boundedness means that for all $x \in E$, we have $|x| \leq M$ for an $M \in \mathbb{R}$. Now, we convert this idea to more general metric space.

**Definition 2.29** (Bounded Set)**.** Let $(X, d)$ be a metric space. $E \subseteq X$ is a **bounded set** if there exists $M \in \mathbb{R}$ and a point $p \in X$ such that $E \subseteq B_M(p)$.

Lecture 8
Monday
October 19

The following theorem paves the road for the behavior of intersections and unions of open sets and closed sets.

**Theorem 2.30** (De Morgan's Law)**.** *Let $\{E_\alpha\}$ be a finite (or infinite) collection of sets $E_\alpha$. Then,*

$$\left(\bigcup_\alpha E_\alpha\right)^{\complement} = \bigcap_\alpha E_\alpha^{\complement}.$$

*Proof.* We use element tracing to prove this theorem.

Let $x \in \left(\bigcup_\alpha E_\alpha\right)^{\complement}$, then $x \notin \bigcup_\alpha E_\alpha$, hence $x \notin E_\alpha$ for any $\alpha$, then $x \in E_\alpha^{\complement}$ for any $\alpha$, so we have $x \in \bigcap_\alpha E_\alpha^{\complement}$. Thus, $\left(\bigcup_\alpha E_\alpha\right)^{\complement} \subseteq \bigcap_\alpha E_\alpha^{\complement}$

Let $x \in \bigcap_\alpha E_\alpha^{\complement}$, then $x \in E_\alpha^{\complement}$ for every $\alpha$, hence $x \notin E_\alpha$ for any $\alpha$, so $x \notin \bigcup_\alpha E_\alpha$ and $x \in \left(\bigcup_\alpha E_\alpha\right)^{\complement}$. Thus, $\bigcap_\alpha E_\alpha^{\complement} \subseteq \left(\bigcup_\alpha E_\alpha\right)^{\complement}$.

Hence $\left(\bigcup_\alpha E_\alpha\right)^{\complement} = \bigcap_\alpha E_\alpha^{\complement}$.

□

A key fact about open sets is that a finite intersection of open sets is again open, as is an *arbitrary* union of open sets.

**Theorem 2.31.** *Let $(X, d)$ be a metric space. Let $A$ be any set (possibly uncountable).*

*(a) For any $\alpha \in A$, $U_\alpha \subseteq X$ is open, then $\bigcup_{\alpha \in A} U_\alpha$ is open.*

*(b) If $U_1, \ldots, U_n \subseteq X$ are open, then $\bigcap_{i=1}^{n} U_i$ is open.*

*Proof.* (a) If $x \in \bigcup_{\alpha \in A} U_\alpha$, then there exists a $\beta \in A$ with $x \in U_\beta$. Since $U_\beta$ is open, there exists an $r > 0$ such that $B_r(x) \subseteq U_\beta \subseteq \bigcup_{\alpha \in A} U_\alpha$, so $x$ is an interior point of $\bigcup_{\alpha \in A} U_\alpha$ so $\bigcup_{\alpha \in A} U_\alpha$ is open.

(b) Let $x \in \bigcap_{i=1}^{n} U_i$. Each $U_i$ is open, so we can pick $r_i > 0$ with $B_{r_i}(x_i) \subseteq U_i$.

Let $r = \min\{r_i : 1 \leq i \leq n\}$, then $r > 0$ and $B_r(x) \subseteq \bigcap_{i=1}^{n} U_i$. Thus, $\bigcap_{i=1}^{n} U_i$ is open.

□

Using De Morgan's Law (in short, the complement of an intersection is the union of the complements), we immediately get the following result for closed set, an opposite conclusion (between unions and intersections) from the open sets.

**Theorem 2.32.** *Let $(X, d)$ be a metric space. Let $A$ be any set (possibly uncountable).*

*(a) For any $\alpha \in A$, $F_\alpha \subseteq X$ is closed, then $\bigcap_{\alpha \in A} F_\alpha$ is closed.*

*(b) If $F_1, \ldots, F_n \subseteq X$ are closed, then $\bigcup_{i=1}^{n} F_i$ is closed.*

*Proof.*

(a) By Theorem 2.30, $\left(\bigcup_{\alpha \in A} F_\alpha\right)^{\mathsf{C}} = \bigcup_{\alpha \in A} F_\alpha^{\mathsf{C}}$ is open by Theorem 2.31 (a), so $\bigcap_{\alpha \in A} F_\alpha$ is closed.

(b) By taking complements, this is a result from Theorem 2.31 (b).

$\square$

Finiteness assumption is necessary in Theorem 2.31 (b) and Theorem 2.32 (b). Consider $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq \mathbb{R}$ is open but $\bigcap_{n \in \mathbb{Z}_+} U_n = \{0\}$ is not open. Similarly, consider $V_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$, which is closed, but $\bigcup_{n \in \mathbb{Z}_+} V_n = (-1, 1)$, which is open.

The reason why argument in Theorem 2.31 (b) where we take the minimum of a series of radii fails in the infinite case is that the sequence of infinitely many radii doesn't necessarily have a minimum.

Previously, in a metric space $(X, d)$ with $E \subseteq X$, we define an *interior* as a set of points in $E$ such that all "nearby" points of $X$ are also in its subset $E$. Now, we are interested in the set of points on the "edge" of $E$.

**Definition 2.33** (Closure)**.** Let $(X, d)$ be a metric space and $E \subseteq X$. The **closure** of $E$ is defined by the set

$$\overline{E} = E \cup E'.$$

Recall that $E'$ denote the set of all limit points of $E$ in $X$.

**Theorem 2.34.** *Let $(X, d)$ be a metric space and $E \subseteq X$. Then,*

*(a) $\overline{E}$ is closed.*

*(b) $E = \overline{E}$ if and only if $E$ is closed.*

*(c) if $F \subseteq X$ is closed and $E \subseteq F$, we have $\overline{E} \subseteq F$.*

*Proof.*

(a) Let $x \in (\overline{E})'$. If $x \in E$, then we are done.

Since $x \in \overline{E}$, so we assume that $x \notin E$. Let $r > 0$. Since $x \in (\overline{E})'$, we have $(B_r(x) \setminus \{x\}) \cap \overline{E} \neq \varnothing$. Pick $y \in (B_r(x) \setminus \{x\}) \cap \overline{E}$. Let $s = \min\{r - d(x, y), d(x, y)\} > 0$. We note that $B_s(y) \subseteq B_r(x) \setminus \{x\}$. Since

$y \in \overline{E}$, we must have $B_s(y) \cap E \neq \varnothing$, so there is a point $p \in B_s(y) \cap E$, and we have $p \in (B_r(x) \setminus \{x\}) \cap E$. Thus $(B_r(x) \setminus \{x\}) \cap E \neq \varnothing$ and hence $x \in E' \subseteq \overline{E}$.

(b) Suppose that $E = \overline{E}$. Since $\overline{E}$ is closed by Part (a), we conclude that $E$ is closed.

Conversely, suppose that $E$ is closed. Then, by Definition 2.24, $E' \subseteq E$. Hence, $E = E \cup E' = \overline{E}$.

(c) We first show a quick result that if $E \subseteq F$, then $E' \subseteq F'$.

Let $p \in E'$, then $p$ is a limit point of $E$. Then, for all $r > 0$, we have $(B_r(p) - \{p\}) \cap E \neq \varnothing$, which means there exists a $q \in (B_r(p) - \{p\}) \cap E$ such that $q \in E$, so $q \in F$. Then, $q \in (B_r(p) - \{p\}) \cap F$, so $q$ is a limit point of $F$. Thus, $E' \subseteq F'$.

So, if $F$ is closed and $E \subseteq F$, then $\overline{E} = E \cup E' \subseteq E \cup F' \subseteq F$.

$\square$

Let $(X, d)$ be a metric space and $E \subseteq X$. By Theorem 2.34 (a) and (c), $\overline{E}$ is the *smallest* closed subset of $X$ that contains $E$.

Let $X = \mathbb{R}$ we are familiar with, we will get a nice property of the supremum of a set on the real line.

**Theorem 2.35.** *If $E \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup E \in \overline{E}$.*

*Proof.* Let $y = \sup E$. If $y \in E$, then $y \in \overline{E}$ and we are done.

So, assume $y \notin E$ and let $y > 0$. Since $y = \sup E$ and $y - r < y$, there must be $x \in E$ with $x > y - r$. Since $y = \sup E$ and $y \notin E$, we have $x < y$. So, $x \in (B_r(y) \setminus \{y\}) \cap E$. We conclude $y \in E' \subseteq \overline{E}$.

$\square$

This result will be fundamental to our rigorous development of the differential calculus.

### Relative Topology

Let $(X, d)$ be a metric space. If $Y \subseteq X$ and let $(Y, d_Y)$ is a metric space where $d_Y$ is the restriction of $d$ to $Y \times Y$ for $y_1, y_2 \in Y$, we have $d_Y(y_1, y_2) = d(y_1, y_2)$.

**Definition 2.36** (Relatively Open Sets)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. If $E \subseteq Y \subseteq X$, we say $E$ is **open relative to** $Y$ if $E$ is open in $(Y, d_Y)$. Equivalently, $E$ is open relative to $Y$ if for any point $p \in E$, there exists $r > 0$ such that $B_r(p) \cap Y \subset E$.

**Theorem 2.37.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. If $E \subseteq Y \subseteq X$, then $E$ is open relative to $Y$ if and only if there is an open set $U \subseteq X$ with $E = U \cap Y$.*

*Proof.* Assume $E$ is open relative to $Y$. For each $p \in E$, we pick $r_p > 0$ with $B_{r_p}(p) \cap Y \subseteq E$. Let $U = \bigcup_{p \in E} B_{r_p}(p)$. Then, $U$ is open and $U \cap Y \subseteq E$. For every $p \in E$, we have $p \in B_{r_p}(p) \subseteq U \cap Y$, so $E \subseteq U \cap Y$. Thus, $E = U \cap Y$.

Assume there is an open set $U \subseteq X$ with $E = U \cap Y$. Let $p \in E$. Then, $p \in U$ and $U$ is open, so there is $r > 0$ with $B_r(p) \subseteq U$. Hence, $B_r(p) \cap Y \subseteq U \cap Y = E$. So, $E$ is open relative to $Y$.

$\square$

Lecture 9
Wednesday
October 21

## 2.4 Compact Sets

Recall that finite sets have a lot of great properties, they are bounded and closed sets. Also, finite sets have no limit points. In the real field, they contain their supremum and infimum.

**Definition 2.38** (Open Cover)**.** Let $(X, d)$ be a metric space. An **open cover** of $E \subseteq X$ is a collection $\{U_\alpha : \alpha \in A\}$ of open subsets $U_\alpha \subseteq X$ with $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.

**Definition 2.39** (Compact Sets)**.** Let $(X, d)$ be a metric space. A subset $K \subseteq X$ is **compact** if every open cover $\{U_\alpha : \alpha \in A\}$ of $K$ contains a **finite subcover**, meaning that there are $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. In other words, $K$ is compact if the following statement is true

For every collection $\{U_\alpha : \alpha \in A\}$ with each $U_\alpha$ open,

$$K \subseteq \bigcup_{\alpha \in A} U_\alpha \Rightarrow \exists n \, \exists \alpha_1, \alpha_2, \ldots, \alpha_n \in A, K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

By the definition above, finite sets are always compact. Say, let $F$ be finite, so we can write $F = \{f_1, f_2, \ldots, f_n\}$. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of $F$, that is, $F \subseteq \bigcup_{\alpha \in A} U_\alpha$. For each $1 \leq i \leq n$, we have $f_i \in U_{\alpha_i}$ for some $\alpha_i \in A$. Then, the finite set $F$ is contained in the finite subcover $U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$. Thus, finite set is compact.

**Example 2.40.** The set $\left[\frac{1}{2}, 1\right) \subseteq \mathbb{R}$ has cover $\bigcup_{n=3}^{\infty} V_n$, where $V_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$. Another cover may be $\{(0, 2)\}$, but it is a boring cover. However, $\left[\frac{1}{2}, 1\right)$ is not compact since there is a open cover with no finite subcover. The cover $V_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ has no finite subcover. Since $n \to \infty$ is required for $\{V_n\}$ to cover all the points in the interval.

**Example 2.41.** $\mathbb{Z} \subseteq \mathbb{R}$ is not compact. To prove this, *all we need to do is find a open cover that has no finite subcover.* For the open cover $\{B_r(x) : x \in \mathbb{Z}, r > 0\}$, the balls covering individual integers, we cannot even remove a single one of them, otherwise $\mathbb{Z}$ cannot be covered.

Now, we begin to explore the properties of compact sets. The first theorem says that compactness is an intrinsic property, which means a set is compact no matter which metric space we are interested, unlike open and closed sets.

**Theorem 2.42.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Assume $K \subseteq Y \subseteq X$. Then, $K$ is compact relative to $Y$ if and only if $K$ is compact relative to $X$.*

*Proof.* Assume that $K$ is compact relative to $Y$. Suppose $K \subseteq \bigcup_{\alpha \in A} U_\alpha$, each $U_\alpha$ open in $X$. Then, $U_\alpha \cap Y$ is open relative to $Y$ by Theorem 2.37 and since $K \subseteq Y$, we have $K \subseteq Y \cap \left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A}(Y \cap U_\alpha)$, so there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$ by Definition 2.39, so $K$ is compact relative to $X$.

Assume that $K$ is compact relative to $X$. Suppose $K \subseteq \bigcup_{\alpha \in A} V_\alpha$ each $V_\alpha$ open relative to $Y$. By Theorem 2.37, there are open sets in $X$, $U_\alpha \subseteq X$ with $V_\alpha = U_\alpha \cap Y$. Then, $K \subseteq \bigcup_{\alpha \in A} U_\alpha$, so there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Since $K \subseteq Y$, so $K \subseteq Y \cap \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) = \bigcup_{i=1}^{n}(U_{\alpha_i} \cap Y) = \bigcup_{i=1}^{n} V_{\alpha_i}$. So $K$ is compact relative to $Y$.

□

So, we may sensibly speak of $K$ as being compact without reference to the ambient metric space; in particular, this is always equivalent to $K$ being compact as a metric space in its own right.

The following theorems pave a road for our journey in compactness. In short, compact sets are closed and bounded.

**Theorem 2.43.** *Comapct sets are always bounded.*

*Proof.* Let $(X, d)$ be a metric space. Let $K \subseteq X$ be compact. Consider the open cover $K \subseteq \bigcup_{p \in K} B_1(p)$. Since $K$ is comapct, so there exists $p_1, p_2, \ldots, p_n$ such that $K \subseteq \bigcup_{i=1}^{n} B_1(p_i)$.

Then, given any two points $q, r \in K$, we have $q \in B_1(p_i)$ and $r \in B_1(p_j)$ for some $1 \le i \le n$ and $1 \le j \le n$. By the triangle inequality,

$$d(r, q) \le d(q, p_i) + d(p_i, p_j) + d(p_j, r) \le 2 + d(p_i, p_j).$$

It follows that the distance between any two points in $K$ is at most $\max\{d(p_i, p_j)\} + 2$, so $K$ is bounded.

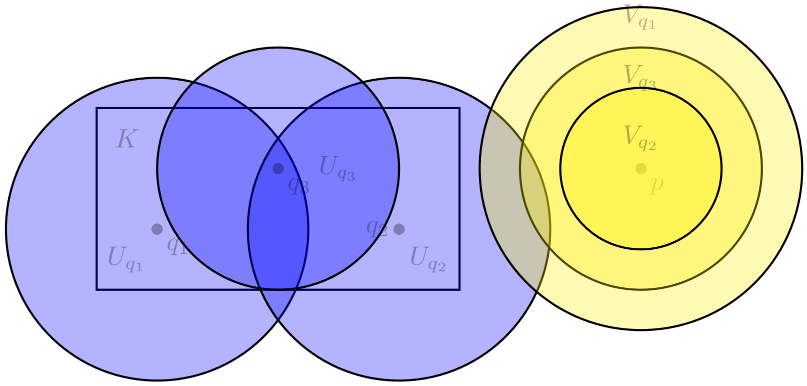
□

**Theorem 2.44.** *Compact sets are always closed.*

*Proof.* Let $(X, d)$ be a metric space. Let $K \subseteq X$ be compact, we will show $X \setminus K$ is open, so pick $p \in X \setminus K$.

For each $q \in K$, let $r = \frac{1}{3} d(p, q)$, set

$$U_q = B_r(q) \quad \text{and} \quad V_q = B_r(p).$$




We have $K \subseteq \bigcup_{q \in K} U_q$, so by compactness, there are $q_1, q_2, \ldots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$.

Then, let $r_0 = \frac{1}{3}\min\{d(p, q_1), d(p, q_2), \ldots, d(p, q_n)\}$, we have $\bigcap_{i=1}^{n} V_{q_i} = B_{r_0}(p)$ is disjoint with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$ hence a ball around $p$ is contained $X \setminus K$, that is $B_{r_0}(p) \subseteq X \setminus K$. Thus $K$ is closed.

□

So no matter how you expand the universe that $K$ lives in, you will never construct points which are limit points of $K$.

**Theorem 2.45.** *Let $(X, d)$ be a metric space and $K \subseteq X$. If $K$ is compact and $F \subseteq K$ is closed, then $F$ is compact as well.*

*Proof.* Say $\{U_\alpha : \alpha \in A\}$ is an open cover of $F$. $F^{\complement}$ is open so $\{F^{\complement}\} \cup \{U_\alpha : \alpha \in A\}$ is an open cover of $K$. So there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ with $F \subseteq K \subseteq F^{\complement} \cup \bigcup_{i=1}^{n} U_{\alpha_i}$, so $F \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

□

**Corollary 2.46.** *If $K$ is compact and $F$ is closed, then $K \cap F$ is compact.*

The next theorem connects finite with infinite.

**Theorem 2.47** (Finite Intersection Property)**.** *If $K_\alpha \subseteq X$ is compact for every $\alpha \in A$ and if the intersection of every finite collection from $\{K_\alpha : \alpha \in A\}$ is nonempty, then $\bigcap_{\alpha \in A} K_\alpha \neq \varnothing$.*

*Proof.* Towards a contradiction, assume $\bigcap_{\alpha \in A} K_\alpha = \varnothing$. Fix any $K \in \{K_\alpha : \alpha \in A\}$. Then,

$$K \subseteq X = X \setminus \varnothing = X \setminus \bigcap_{\alpha \in A} K_\alpha = \bigcup_{\alpha \in A} (X \setminus K_\alpha)$$

and each $X \setminus K_\alpha$ is open, so there are $\alpha_1, \ldots, \alpha_n \in A$ with $K \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i})$. Then,

$$K \cap \bigcap_{i=1}^{n} K_{\alpha_i} \subseteq \bigcup_{i=1}^{n} (X \setminus K_{\alpha_i}) \cap \bigcap_{i=1}^{n} K_{\alpha_i} = \varnothing$$

so contradiction here.

□

**Corollary 2.48.** *If $K_n \neq \varnothing$ is compact and $K_{n+1} \subseteq K_n$ for all $n$, then* $\bigcap_{n\in\mathbb{N}} K_n \neq \varnothing$.

**Theorem 2.49.** *Let $(X,d)$ be a metric space and $K \subseteq X$. If $K$ is compact and $E \subseteq K$ is infinite, then $E' \cap K \neq \varnothing$.*

Lecture 10 Friday October 23

*Proof.* Towards a contradiction, assume $E' \cap K = \varnothing$, which means $E$ has no limit points in $K$. Further, this means for each $q \in K$ there is $r_q > 0$ with $(B_{r_q}(q) \setminus \{q\}) \cap E = \varnothing$, meaning that $U_q = B_{r_q}(q)$ satisfies $U_q \cap E \subseteq \{q\}$.

Then $K \subseteq \bigcup_{q\in K} U_q$ so by compactness, there are $q_1, \ldots, q_n \in K$ with $K \subseteq \bigcup_{i=1}^{n} U_{q_i}$. Then,

$$|E \cap K| \leq \left| E \cap \bigcup_{i=1}^{n} U_{q_i} \right| = \left| \bigcup_{i=1}^{n} (E \cap U_{q_i}) \right| = |\{q_1, \ldots, q_n\}| = n.$$

Since $E \subseteq K$, so $|E| = |E \cap K| \leq n$, which means $E$ is finite, that's a contradiction.

□

Recall that $K$ is compact imples that $K$ is closed, thus $E' \subseteq K' \subseteq K$.

**Theorem 2.50** (Nested Interval Property)**.** *Let $I_n = [a_n, b_n]$ with $a_n \leq b_n$ be a closed set on $\mathbb{R}$, and $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, then* $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$.

*Proof.* For all $n, m \in \mathbb{N}$, we have $a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$, so $b_m$ is an upperbound to $\{a_n : n \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. So, $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ exists and $a_m \leq \alpha \leq b_m$ for all $m$. Thus, $\alpha \in \bigcap_{n\in\mathbb{N}} I_n$.

□

This theorem is not true if we consider open intervals. For example, we have $\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right) = \varnothing$.

Further, we can use Nested Interval Property to prove that $\mathbb{R}$ is uncountable. The proof is indeed elegant and nice.

**Theorem 2.51** (Uncountability of Real Numbers)**.** *The set of real numbers is uncountable.*

*Proof.* Towards a contradiction, suppose $\mathbb{R}$ is countable, so we can list the elements of $\mathbb{R}$ as $\{x_1, x_2, x_3, \ldots\}$.

Pick a closed interval $I_1$ such that $x_1 \notin I_1$. Then, pick $I_2$ such that $I_2 \subseteq I_1$ and $x_2 \notin I_2$. Pick $I_3$ such that $I_3 \subseteq I_2$ and $x_3 \notin I_3$. Repeat this process. From Theorem 2.50, we know that $\bigcap_{n\in\mathbb{N}} I_n$ is nonempty, so there is some element in our union that is not on the list $\{x_1, x_2, x_3, \ldots\}$. Hence $\mathbb{R}$ is uncountable.

□

Now, we shift our focus to the Euclidean space.

**Theorem 2.52.** *Suppose $C_n = [a_{n_1}, b_{n_1}] \times [a_{n_2}, b_{n_2}] \times \cdots \times [a_{n_k}, b_{n_k}] \subseteq \mathbb{R}^k$ and $C_n \neq \varnothing$ and $C_{n+1} \subseteq C_n$ for all $n$. Then* $\bigcap_{n\in\mathbb{N}} C_n \neq \varnothing$.

*Proof.* This is obvious since

$$\bigcap_{n\in\mathbb{N}} C_n = \left(\bigcap_{n\in\mathbb{N}} [a_{n_1}, b_{n_1}]\right) \times \cdots \times \left(\bigcap_{n\in\mathbb{N}} [a_{n_k}, b_{n_k}]\right) \neq \varnothing.$$

□

**Theorem 2.53** (Compactness of $k$-cell)**.** *$C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k]$ is compact.*

*Proof.* Define $\delta = \sqrt{\sum_{i=1}^{k} |b_i - a_i|^2}$, the length of the longest diagonal. Then, we have $|x - y| < \delta$ for all $\vec{x}, \vec{y} \in C$.

Towards a contradiction, suppose $\{U_\alpha : \alpha \in A\}$ is an open cover of $C$ having no finite subcover of $C$.

We cut at the midpoint of each side of $C$ to divide $C$ into $2^k$ many rectangles. By our assumption, at least one of them does not admit a finite subcover, call it $C_1$. Proceeding inductively by repeating this process, we build a sequence $(C_n)_{n\in\mathbb{Z}_+}$ such that

(a) $C \supseteq C_1 \supseteq C_2 \supseteq \cdots$

(b) $C_n$ does not admit a finite subcover from $\{U_\alpha : \alpha \in A\}$

(c) For all $\vec{x}, \vec{y} \in C_n$, we have $|\vec{x} - \vec{y}| \leq 2^{-n} \cdot \delta$.

By Theorem 2.52, $\bigcap_{n\in\mathbb{Z}_+} C_n \neq \varnothing$. We pick $\vec{z} \in \bigcap_{n\in\mathbb{Z}_+} C_n$. We pick an $\alpha \in A$ such that $\vec{z} \in U_\alpha$. Since $U_\alpha$ is open, there exists $r > 0$ such taht $B_r(\vec{z}) \subseteq U_\alpha$. We pick an $n$ with $2^{-n} \cdot \delta < r$, then (c) implies $C_n \subseteq U_\alpha$ since $\vec{z} \in C_n$. Thus, $C_n$ admits a finite subcover, which contradicts (b).

Hence, we conclude that $C$ is compact.

□

Recall that the compact sets are always closed and bounded. In general, it is very rare for the converse to be true, meaing that closed and bounded sets are compact. One very important and special case is $\mathbb{R}^k$ under the Euclidean space. Sets in the Euclidean space are compact if and only if they are closed and bounded.

**Theorem 2.54.** *Let $E \subseteq \mathbb{R}^k$, the following are equivalent*

*(a) $E$ is closed and bounded.*

*(b) $E$ is compact.*

*(c) Every infinite subset of $E$ has a limit point in $E$.*

*Proof.*

(a) $\Rightarrow$ (b). By assumption, $E$ is bounded, which means $E \subseteq C$ for some $C = [a_1, b_1] \times \cdots \times [a_k, b_k]$. Since $C$ is compact and $E \subseteq C$ is closed, so $E$ is compact.

(b) $\Rightarrow$ (c). This is Theorem 2.49.

(c) $\Rightarrow$ (a). Assume that every infinite subset of $E$ has a limit point in $E$. We want to show that $E$ is closed and bounded.

Towards a contradiction, assume that $E$ is not bounded. Then, there exists a point $x_1 \in E$ such that $|x_1| > 1$, for otherwise that $E$ would be contained in a unit closed ball around the origin of $\mathbb{R}^k$. We proceed this process inductively, by finding a point $x_2 \in E$ such that $|x_2| > 1 + \max\{2, |x_1|\}$. After picking the point $x_{n-1}$ (for $n \geq 3$), we can pick a point $x_n \in E$ such that $|x_n| > 1 + \max\{n, |x_1|, \ldots, |x_{n-1}\}$. Otherwise, $E$ would be contained in a closed ball of radius $1 + \max\{n, |x_1|, \ldots, |x_{n-1}|\}$ and centered at the origin.

We have inductively chosen a sequence $(x_n)_{n\in\mathbb{Z}_+}$ of distinct points of $E$ such that for every $n \in \mathbb{Z}_+$, we have $|x_n| > n$ and $|x_n| > |x_i|$ for all $i \in \{1, 2, \ldots, n-1\}$.

Now, define $S = \{x_n : n \in \mathbb{Z}_+\}$. This is an infinite subset of $E$. We want to show that $S$ has no limit points in $\mathbb{R}^k$ and hence no limit points in $E$. For any $m, n \in \mathbb{Z}_+$ such that $n > m$, we have

$$|x_n - x_m| \geq |x_n| - |x_m| \geq 1 + \max\{n, |x_1|, \ldots, |x_{n-1}|\} - |x_m| > 1.$$

So if some point $x \in \mathbb{R}^k$ is a limit point of $S$, then there would be infinitely many $n \in \mathbb{Z}_+$ such that $|x_n - x| < \frac{1}{4}$, and for any two distinct points $x_m$ and $x_n$ of $S$, we have

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

which contradicts what we did above about the distance between any two distinct points of $S$. So, $S$ fails to admit a limit point in $\mathbb{R}^k$ and hence in $E$. We conclude that $E$ must be bounded.

Again, towards a contradiction, assume that $E$ is not closed. Then, $E$ has a limit point $x_0 \in \mathbb{R}^k \setminus E$. Since $x_0$ is a limit point of $E$, for all $r > 0$, $B_r(x_0)$ contains a point of $E$ distinct from the point $x_0$ itself (in fact infinitely many points of $E$).

Thus, there exists a point $x_1 \in E$ such that $0 < |x_1 - x_0| < \frac{1}{2}$. Again, there exists a point $x_2 \in E$ such that $0 < |x_2 - x_0| < \min\left\{|x_1 - x_0|, \frac{1}{3}\right\}$. Assuming that the point $x_{n-1}$ has been chosen, we can pick a point $x_n \in E$ such taht $0 < |x_n - x_0| < \min\left\{|x_1 - x_0|, \ldots, |x_{n-1} - x_0|, \frac{1}{n+1}\right\}$. Thus, we inductively define a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $E$ such that $x_n \neq x_m$ for all $n \neq m$ and also $0 < |x_n - x_0| < \frac{1}{n}$ for all $n > 0$.

Now, we define $S = \{x_n : n \in \mathbb{N}\}$. This is also an infinite subset of $E$. We want to show that $x_0$ is the only limit point of $S$, that is, to show that $x_0$ is a limit point of $S$ but no other point $y \in \mathbb{R}^k$ can be a limit point of $S$.

Let $\delta > 0$. Then, by the Archimedean Property, we can find $n_\delta \in \mathbb{N}$ such that $n_\delta > \frac{1}{\delta}$, and so for all $n \in \mathbb{N}$ such that $n \geq n_\delta$, we have $0 < |x_n - x_0| < \frac{1}{n+1} < \frac{1}{n_\delta} < \delta$, which implies that $x_0$ is indeed a limit point of $S$. Now, if $y \in \mathbb{R}^k$ and $y \neq x_0$, then $|y - x_0| > 0$, so we can find an integer $N > 0$ such that $N > \frac{2}{|y - x_0|}$. So, for every $n \in \mathbb{N}$ such that $n \geq N$, we have

$$0 < |x_n - x_0| < \frac{1}{n} \leq \frac{1}{N} < \frac{|y - x_0|}{2}$$

and hence

$$|x_n - y| \geq |y - x_0| - |x_n - x_0| \geq |y - x_0| - \frac{|y - x_0|}{2} > \frac{|y - x_0|}{3}.$$

So for $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{2}\min\left\{|x_1 - y|, \ldots, |x_N - y|, \frac{|y - x_0|}{3}\right\}$, then there is no point of $S$ that lies in $B_\varepsilon(y)$, other than the point $y$ itself if $y \in S$, that is, $S \cap (B_\varepsilon(y) \setminus \{y\}) = \varnothing$, which implies that the point $y$ cannot be a limit point of $S$.

However, we assumed that $y \in \mathbb{R}^k\{x_0\}$. Then, $x_0$ is the only limit point of $S$, but it contradicts to our assumption that $x_0 \notin E$. We conclude that $E$ must be closed.

So, if every infinite subset of $E \subseteq \mathbb{R}^k$ has a limit point in $E$, then $E$ must be closed and bounded.

□

Equivalence of (a) and (b) is known as the *Heine-Borel Theorem*, for Euclidean space. Equivalence of (b) and (c) applies for all metric spaces.

**Theorem 2.55** (Bolzano-Weierstrass). *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

*Proof.* Take the closure and apply (a) $\Rightarrow$ (c) in Theorem 2.54.

□

Lecture 11

Monday

October 26

## 2.5 Perfect Sets

Recall that in a metric space $(X, d)$ and $E \subseteq X$, if $p$ is a limit point, then for all $r > 0$, the set $(B_r(p)\setminus\{p\})\cap E$ is infinite. So, if $U$ is open and $U \cap E' \neq \varnothing$, then $U \cap E$ is infinite.

**Corollary 2.56.** *For any metric space $(X, d)$,*

$$\overline{B_r(p)} \subseteq \{q \in X : d(p, q) \leq r\}.$$

*Proof.* We note that $B_r(p) \subseteq \{q \in X : d(p, q) \leq r\}$, so it is suffice to check $B_r(p)' \subseteq \{q \in X : d(p, q) \leq r\}$.

Let $x \in B_r(p)'$. Towards a contradiction, suppose that $d(p, x) > r$. Set $R = d(p, x) - r > 0$. Since $x \in B_r(p)'$, there exists a point $y \in (B_R(x) \setminus \{x\}) \cap B_r(p)$. Then,

$$d(p, x) \leq d(p, y) + d(y, x) < r + R = d(p, x)$$

so $d(p, x) < d(p, x)$, a contradiction.

Then, $d(p, x) \leq r$ and $x \in \{q \in X : d(p, q) \leq r\}$.

□

**Definition 2.57** (Perfect Sets). Let $(X, d)$ be a metric space and $E \subseteq X$. $E$ is **perfect** if $E$ is closed and every point of $E$ is a limit point of $E$.

The empty set, all closed intervals, and the real line itself are perfect sets. Well, another example is the Cantor Set we will introduce later.

**Theorem 2.58.** *Let $P$ be a nonempty perfect set in $\mathbb{R}^k$. Then $P$ is uncountable.*

*Proof.* Since $P$ is perfect, that is $P' = P = \varnothing$, we must have that $P$ is infinite. Towards a contradiction, suppose $P$ is countable, that is

$$P = \{\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots\}.$$

We will inductively build sets $V_n$ with $n \in \mathbb{N}$ satisfying following

(a) $V_n$ is open

(b) $V_n \cap P \neq \varnothing$

(c) When $n \geq 1$, $\overline{V_n} \subseteq V_{n-1}$

(d) When $n \geq 1$, $\vec{x}_{n-1} \notin V_n$

To begin, set $V_0 = B_1(\vec{x_0})$. Now, inductively assume that $V_0, \ldots, V_n$ have been defined. (a) and (b) imply that $V_n \cap P$ is infinite, so we can pick $\vec{y} \in V_n \cap P$ with $\vec{y} \neq \vec{x_n}$.

Let $0 < r < d(\vec{y}, \vec{x_n})$ be small enough so that $B_r(\vec{y}) \subseteq V_n$. Now, set $V_{n+1} = B_{r/2}(\vec{y})$. Set $K_n = \overline{V_n} \cap P$. Then, $K_n$ is compact and nonempty and $K_{n+1} \subseteq K_n$, so by Finite Intersection Property, there exists $\vec{z} \in \bigcap_{n\in\mathbb{N}} K_n$. Sicne each $K_n \subseteq P$, so $\vec{z} \in P$. Hence there exists an $n \in \mathbb{N}$ such that $\vec{z} = \vec{x_n} \in P$, but (d) assumes that $\vec{z} = \vec{x_n} \notin K_{n+1}$, which is a contradiction. We conclude that every nonempty perfect set in $\mathbb{R}^k$ is uncountable.

□

When $k = 1$, we get that the real number field is uncountable, which is the following corollary.

**Corollary 2.59.** *For all $a, b \in \mathbb{R}$, $[a, b]$ is perfect and hence uncountable. In particular, $\mathbb{R}$ is uncountable.*

## 2.6 The Cantor Set

**Definition 2.60** (Cantor Set)**.** The **Cantor Ternary Set** is constructed by iteratively remove the middle third from a set of line segments. One starts by deleting the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ from the interval $[0, 1]$, leaving two line segments

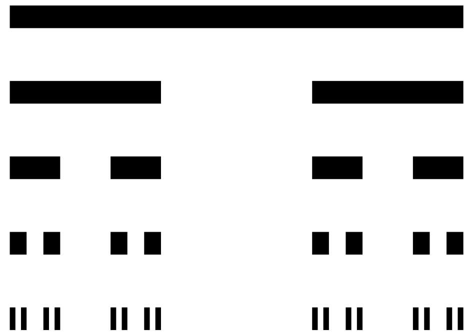
$$P_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Next, the open middle third of each of those remaining segment is deleted, leaving four line segments

$$P_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

The process continues infinitely. We define the **Cantor Ternary Set** to be

$$P = \bigcap_{n=1}^{\infty} P_n.$$



Here are two quick observations (properties) of the Cantor Set $P$.

(a) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$.

(b) $E_n$ is the union of $2^n$ intervals, each with length $3^{-n}$.

**Theorem 2.61** (Compactness)**.** *The Cantor Set $P$ is compact.*

This is obvious since $P$ is clearly a closed subset of the compact set $[0,1]$.

**Theorem 2.62.** *The Cantor Set P has a length of zero, which means that it contains no segments (intervals).*

*Proof.* We will prove that the complement of Cantor Set $P$ relative to $[0,1]$ has the length 1.

From the construction of the Cantor Set $P$, we note that at the $n$th step, we remove $2^{n-1}$ intervals, all of which are of length $\frac{1}{3^n}$. Therefore, the sum of the length of all intervals removed is

$$\sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{1}{3^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-2/3} = 1.$$

□

This theorem is actually saying that the Lebesgue measure of the Cantor Set $P$ is 0.

**Theorem 2.63.** *The Cantor Set $P$ is perfect.*

*Proof.* We need to prove that any point$p \in P$ is a limit point of $P$, that is $P \subseteq P'$.

Let $p \in P$. Let $S$ be any segment containing $p$ (this serves as a neighborhood). Let $I_n$ be that interval of $E_n$ which contains $p$. Pick a sufficiently large $n$ such that $I_n \subseteq S$. Let $p_n$ be an end point of $I_n$ such that $p_n \neq p$. It follows from the construction of the Cantor Set $P$ that $p_n \in P$. Since $p_n \in S$, we conclude that $p$ is a limit point of $P$. Therefore, $P$ is perfect.

□

Before we diving into more fun facts on the Cantor Set, we first have a look at base 3 expansion of a number.

We consider base 3 expansions of numbers $x \in [0,1]$. We will always take $a_j$ is equal to 0, 1, or 2 in what follows.

Suppose we have a sequence $(a_n)_{n\geq 1}$ in $\{0,1,2\}^{\mathbb{N}}$. We associate to this sequence as a number $\sum_{k=1}^{\infty}\frac{a_k}{3^k}$. The sequence $b_n = \sum_{k=1}^{n} a_k \cdot 3^{-k}$ is increasing in $n$, and bounded above by $2\sum_{k=1}^{\infty}\frac{1}{3^k} = \frac{2}{3}\cdot\frac{1}{1-1/3} = 1$ so the infinite sum makes sense as the least upper bound of $b_n$.

We next see that this map $\{0,1,2\}^{\mathbb{N}} \mapsto [0,1]$ is onto (but not one to one).

Suppose $x \in [0,1)$. Let $a_1$ be the largest integer such that $\frac{a_1}{3} \leq x$. Then, since $0 \leq x < 1$, we have $0 \leq a_1 \leq 2$ and

$$0 \leq x - \frac{a_1}{3} < \frac{1}{3}.$$

Now, let $a_2$ be the largest integer such that $\frac{a_2}{3^2} \leq x - \frac{a_1}{3}$. Then, we must have $0 \leq a_2 \leq 2$ as well, and

$$0 \leq x - \frac{a_1}{3} - \frac{a_2}{3^2} < \frac{1}{3^2}.$$

In general, we will take $a_j$ inductively such that

$$0 \leq x - \sum_{k=1}^{n}\frac{a_k}{3^k} < \frac{1}{3^n}.$$

In short, $a_j$'s are the largest integers from 0, 1, or 2, so that $\sum_{k=1}^{n}\frac{a_k}{3^k} \leq x$. Further, we have

$$x = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{a_k}{3^k} = \sum_{k=1}^{\infty}\frac{a_k}{3^k}.$$

This shows that the map is onto.

For $x = 1$, we take $a_j = 2$ for all $j$, and note that

$$\sum_{k=1}^{n}\frac{2}{3^k} = \frac{2}{3}\sum_{k=0}^{n-1}\frac{1}{3^k} = \frac{2}{3}\cdot\frac{1-(1/3)^n}{1-1/3} = 1 - \frac{1}{3^n}.$$

Thus, $\sum_{k=1}^{\infty}\frac{2}{3^k} = 1$.

We now determine how the map $\{0,1,2\}^{\mathbb{N}} \mapsto [0,1]$ can fail to be one to one. Assume we have two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$\sum_{k=1}^{\infty}\frac{a_k}{3^k} = \sum_{k=1}^{\infty}\frac{b_k}{3^k}.$$

Suppose that $n$ is the first position where $a_n \neq b_n$, and assume $a_n > b_n$. Then,

$$\frac{a_n - b_n}{3^n} + \sum_{k=n+1}^{\infty}\frac{a_k}{3^k} = \sum_{k=n+1}^{\infty}\frac{b_k}{3^k}.$$

Since

$$\frac{a_n - b_n}{3^n} \geq \frac{1}{3^n} \text{ and } \sum_{k=n+1}^{\infty}\frac{b_k}{3^k} \leq \sum_{k=n+1}^{\infty}\frac{2}{3^k} = \frac{1}{3^n},$$

so the only way we can have equality is if $a_n - b_n = 1$ and $a_k = 0$ and $b_k = 2$ for $k \geq n+1$. That is, the sequence $a_k$ has terminal 0's and $b_k$ has terminal 2's. For example,

$$0.1202000000\cdots = 0.1201222222\cdots .$$

Other than this kind of case, the ternary expansion of $x \in [0,1]$ is unique.

**Proposition 2.64** (More Fun Facts on the Cantor Set)**.** *Let $P$ denote the Cantor Set.*

*(a) Let $x = 0.\overline{a_1a_2a_3\ldots}$ be the base 3 expansion of a number $x \in [0,1]$. Then, $x \in P$ if and only if $a_n \in \{0,2\}$ for all $n \geq 1$.*

*(b) $P$ is uncountable.*

*(c) $\frac{1}{4} \in P$, but $\frac{1}{4}$ is not an endpoint of any of the intervals in any of the sets $E_k$ for $k \in \mathbb{N}$, where $E_k$ is defined before.*

*Proof.*

(a) It is equivalent to show that the map $\{0,2\}^{\mathbb{N}} \mapsto P$ is a bijection.

Assume $a_n \in \{0,2\}$ for all $n \geq 1$, which means $x$ can be represented precisely in the form $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. Then,

$$\frac{1}{3}x = \sum_{n=1}^{\infty} \frac{b_n}{3^n} \text{ where } b_1 = 0 \text{ and } b_n = a_{n-1} \text{ if } n \geq 2$$

$$\frac{1}{3}x + \frac{2}{3} = \sum_{n=1}^{\infty} \frac{b_n}{3^n} \text{ where } b_1 = 2 \text{ and } b_n = a_{n-1} \text{ if } n \geq 2.$$

Thus, $x \in P_1$ if and only if it is a ternary expansion where either $a_1 = 0$ or $a_1 = 2$, since it is of the form $\frac{1}{3}y$ or $\frac{1}{3}y + \frac{2}{3}$ for some $y \in [0,1]$. By induction, we note that $x \in P_k$ if and only if it is the ternary expansion where $a_n$ is either 0 or 2 for $1 \leq n \leq k$. It follows that the ternary expansions where every digit is either 0 or 2 belong to $P_n$ for every $n$, hence give an element of $P$.

Assume $x \in P$. If the ternary expansion of $x$ is unique, then every $a_n$ is either 0 or 2. If $x$ has two ternary expansions, let $k$ be the first digit they differ. Then, $a_n$ is either 0 or 2 for $1 \leq n < k$, and for $n > k$, $a_n$ is either identically 0 or identically 2. One of the two expansions has $a_k = 1$, the other must have $a_k = 0$ or 2. So, $x$ has one expansion where $a_n \neq 1$ for every $n \geq 1$.

We conclude that $x \in P$ if and only if $a_n \in \{0,2\}$ for all $n \geq 1$.

(b) It is equivalent to construct a function $f : P \to [0,1]$ that is surjective. By (a), for any $x = 0.\overline{a_1a_2a_3\ldots}$, $x \in P$ if and only if $a_n \in \{0,2\}$ for all $n \geq 1$.

Then, we construct a function $f : P \to [0,1]$ which replaces all the 2's by 1's, and intertprets the sequence as a binary representation of a real number, that is,

$$f\left(\sum_{n=1}^{\infty} a_n \cdot 3^{-n}\right) = \sum_{n=1} \frac{a_n \cdot 2^{-n}}{2}.$$

For any $y \in [0,1]$, its binary representation can be translated into a ternary representation of a number $x \in P$ by replacing all the 1's by 2's, so the range of $f$ is $[0,1]$. Thus, $|P| \geq |[0,1]|$, which means $P$ is uncountable.

(c) We note that $\frac{1}{4} = \frac{2}{9-1} = \frac{2}{9} \cdot \frac{1}{1-1/9}$, so the ternary expansion of $\frac{1}{4}$ is $0.\overline{02}$. Thus, by (b), $\frac{1}{4} \in P$.

Further, note that $x \in P_n$ is an endpoint if $x = 0$, $x = 1$, or $x = 3^{-n}$ for some $n \in \mathbb{N}$. Clearly, $\frac{1}{4} \neq 0, 1$, and for all $n \in \mathbb{N}$, $\frac{1}{4} \neq 3^{-n}$. We conclude that $\frac{1}{4}$ is not an endpoint as well.

□

Lecture 12
Friday
October 30

## 2.7 Connected Sets

**Definition 2.65** (Separated Sets)**.** Let $(X,d)$ be a metric space and $A, B \subseteq X$. We say $A$ and $B$ are **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, that is, if no point of $A$ lies in the closure of $B$ adn no point of $B$ lies in the closure of $A$.

"Separated" is a stronger definition than "disjoint."

(a) $(0,1)$ and $(1,2)$ are separated (and disjoint).

(b) $(0,1]$ and $(1,2)$ are not separated but disjoint.

**Definition 2.66** (Connected Set)**.** Let $(X,d)$ be a metric space and $E \subseteq X$. We say $E$ is **connected** if $E$ is not a union of two nonempty separated sets.

As with compactness, this is an intrinsic property of $E$, irrespective of the alrger metric space in which it lives. More explicitly, $E$ is connected in $X$ if and only if $E$ is connected in $E$.

Note that if $X = A \cup B$ with $A$ and $B$ separated, then $B = A^{\complement}$. And $\overline{A} \cap B = \varnothing$, so $\overline{A} = A$, which means $A$ is closed. Similarly, $B$ is closed. So, $A$ and $B$ are both closed and open since their complements are closed.

**Theorem 2.67.** *$E \subseteq \mathbb{R}$ is connected if and only if for all $x \leq y \in E$, $[x,y] \subseteq E$.*

*Proof.* We prove both sides by contrapositive.

Assume there are $x \leq y \in E$ with $[x,y] \nsubseteq E$. Pick $x < z < y$ with $z \notin E$. Set $A = E \cap (-\infty, z)$ and $B = E \cap (z, +\infty)$. Then, $A$ and $B$ are nonempty since $x \in A$ and $y \in B$. And since $\overline{A} \subseteq (-\infty, z]$ and $\overline{B} \subseteq [z, +\infty)$, so $A$ and $B$ are separated. Also, with $A \cup B = E$, thus $E$ is not connected.

Now, assume that $E$ is not connected. Let $A$ and $B$ be nonempty, separated, and $E = A \cup B$. Pick $x \in A$ and $y \in B$. Without loss of generality, we can assume $x < y$. Set $z = \sup(A \cap [x,y])$. Then, $z \in \overline{A \cap [x,y]} \subseteq \overline{A}$, hence $z \notin B$ by the assumption that $E$ is not connected. If $z \notin A$, then $z \in E$, hence

$[x,y] \nsubseteq E$ as $z \in [x,y] \setminus E$. If $z \in A$, then $z \notin \overline{B}$. Since $z \in \overline{(z,y]}$, we must have $(z,y] \nsubseteq B$, so there exists $z' \in (z,y] \setminus B$. Also, $z' \notin A$ since $z' > z$. Thus, $z' \in [x,y] \setminus (A \cup B) = [x,y] \setminus E$. So, $[x,y] \nsubseteq E$.

□

Recall the construction of Cantor Set $P$, it is not so hard to conclude that $P$ is disconnected.

**Proposition 2.68** (Cantor Set)**.** *The Cantor Set is totally disconnected.*

*Proof.* We note that the subsets $P_n$ are not connected, for each $n \geq 1$, the set $P_n$ is the union of disjoint closed intervals. The intervals that make up $P_n$ each have length $\frac{1}{3^{n-1}}$. For $x, y \in P$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{3^{N-1}} < |x-y|$. This means that $x$ and $y$ must be in different closed intervals inside $P_N$. Therefore, the only connected sets are singletons, sets with one element. We conclude that the Cantor Set is totally disconnected.

□

# 3 Sequences and Series

So far, we have introduced sets as well as the number systems. Now, we will study sequences of numbers. Sequences are, basically, countably many numbers arranged in an order that may or may not exhibit certain patterns.

**Definition 3.1** (Sequence)**.** A sequence $(p_n)_{n\in\mathbb{N}}$ in $X$ is a function $f : \mathbb{N} \to X$ that maps $n$ to a point $p_n \in X$.

We often think of a sequence $f : \mathbb{N} \to X$ as, and denote it by, the list of numbers,

$$f(0), f(1), f(2), \dots$$

For example, we can define a sequence in $\mathbb{R}$ inductively. Given $x_0 = 2$, we define $x_{n+1} = \dfrac{x_n + 1}{2}$. In this case, it might be difficult to derive a formula for the function $f$, but that does not matter very much for our purpose.

I tend to refer $\mathbb{N}$ as all nonnegative integers, so I will clearly define the domain of each sequence is natural numbers $\mathbb{N}$ or positive integers $\mathbb{Z}_+$. Sometimes we can define the domain by the function $f$. If $f(n) = \dfrac{1}{n}$, then the sequence is definitely defined on $\mathbb{Z}_+$.

## 3.1 Convergent Sequences

A limit describes how a sequence $(p_n)$ behaves "eventually" as $n$ gets larger, in a sense that we make explicit below.

**Definition 3.2** (Convergence)**.** A sequence $(p_n)_{n\in\mathbb{N}}$ in a metric space $(X, d)$ **converges** if there is a point $p \in X$ with the following property: for every $\varepsilon > 0$, there exists a positive integer $N$ such that for all $n \geq N$, then $d(p_n, p) < \varepsilon$.

In this case, we say $(p_n)_{n\in\mathbb{N}}$ **converges to** $p$ or has a **limit** $p$ and write $p_n \to p$ or $\lim\limits_{n\to\infty} p_n = p$.

If $(p_n)_{n\in\mathbb{N}}$ does not converge, we say it **diverges**.

The definition is a bit intimidating at first, but what it says is that for any open ball bround the limit, the elements of a sequence eventually stay in the ball.

An equivalent definition for convergence is that for every $\varepsilon > 0$, we have $d(p_n, p) < \varepsilon$ for all but finitely many $n$.

Convergence and divergence can depend on the metric space $(X, d)$ as well as $(p_n)$. For example, the sequence $p_n = \dfrac{1}{2^n}$ converges as a sequence in $\mathbb{R}$ but not as a sequence in $(0, 1)$. We say "$(p_n)$ converges in $X$" if the ambient space may otherwise be ambiguous.

**Definition 3.3.** The **range** of $(p_n)_{n\in\mathbb{N}}$ is $\{p_n : n \in \mathbb{N}\}$. Further, $(p_n)_{n\in\mathbb{N}}$ is **bounded** if the range is bounded.

Since sequences allow repetitions, the range of a sequence can be finite. For instance, consider the range of $p_n = (-1)^n \in \mathbb{R}$.

**Example 3.4.** The sequence $p_n = \dfrac{n+1}{n}$ converges and $p_n \to 1$.

We want to show that given an $\varepsilon > 0$, there is a positive integer $N$ that works. Note that

$$\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon$$

holds if $n > \dfrac{1}{\varepsilon}$ for positive $n$ $\varepsilon$. Therefore, we can consider any $N > \dfrac{1}{\varepsilon}$.

*Proof.* Given $\varepsilon > 0$, pick $N = \lceil \frac{1}{\varepsilon} \rceil + 1$. Then, for all $n \geq N$, we have $n > \dfrac{1}{\varepsilon}$. Hence, $\dfrac{1}{n} < \varepsilon$. So,

$$|p_n - p| = \left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| < \varepsilon.$$

□

It should be clear that $p_n \to p$ fi and only if the sequence $(d(p_n, p))$ converges to 0 in $\mathbb{R}$. It is easy to see that a sequence $((x_n, y_n))$ in a product metric space $X \times Y$ converges to $(x, y)$ if and only if $x_n \to x$ and $y_n \to y$.

Convergence of sequences is also related to several convepts in the metric space, as follows.

**Theorem 3.5.** *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in a metric space $(X, d)$.*

*(a) $(p_n)$ converges to $p \in X$ if and only if for every $\varepsilon > 0$, there exists a positive integer $N$ such that for all $n \geq N$, then $p_n \in B_\varepsilon(p)$.*

*(b) If $(p_n)$ converges to $p \in X$ and $p' \in X$, then $p = p'$. (This is basically saying the limit of a sequence is unique.)*

*(c) If $(p_n)$ converges, then $(p_n)$ is bounded.*

*(d) If $E \subseteq X$ and $p$ is a limit point of $E$, then there exists a sequence $(p_n)$ in $E$ such that $p_n \to p$.*

*(e) If $p_n \to p$ and $p_n \in E$ for all $n$, then $p \in \overline{E}$.*

*Proof.*

(a) This follows from the fact

$$d(p_n, p) < \varepsilon \iff p_n \in B_\varepsilon(p).$$

(b) Towards a contradiction, assume $p_n \to p$ and $p_n \to p'$ with $p \neq p'$. Let $\varepsilon = d(p, p') > 0$. Pick $N_1$ such that for any $n \geq N_1$, $d(p_n, p) < \dfrac{\varepsilon}{2}$ and $N_2$ such that for any $n \geq N_2$, $d(p_n, p') < \dfrac{\varepsilon}{2}$. Let $N = \max\{N_1, N_2\}$. Then, $n \geq N$ implies

$$\varepsilon = d(p, p') \leq d(p, p_n) + d(p_n, p') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So this says $\varepsilon < \varepsilon$, a contradiction. Thus, we are done.

(c) Suppose $p_n \to p$, then there exists a positive integer $N$ such that for all $n \geq N$, we have $d(p_n, p) < 1$, that is choosing a ball with radius 1. Set

$$r = \max\{1, d(p_1, p), \ldots, d(p_N, p)\}$$

then $d(p_n, p) \leq r$ for all $n$. Thus, $\{p_n : n \in \mathbb{N}\}$ is bounded.

(d) For each $n \in \mathbb{Z}_+$, there exists a point $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. In short, pick $p_n \in B_{1/n}(p) \cap E$. Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $N \cdot \varepsilon > 1$ by Archimedean property. For $n \geq N$, we have

$$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $p_n \to p$.

(e) If $p \in E$, then we are done.

So, assume $p \notin E$. Then, for every $r > 0$, there exists an $n \in \mathbb{N}$ such that

$$p_n \in B_r(p) \cap E = (B_r(p) \setminus \{p\}) \cap E.$$

Thus, $(B_r(p) \setminus \{p\}) \cap E \neq \varnothing$ so $p \in E'$.

Hence, $p \in \overline{E}$.

$\square$

When we are dealing with multiple sequences, it makes sense to define the arithmetic operations on them. We deduce the following for sequences of real or complex numbers.

**Theorem 3.6** (Limit Rules). *Suppose $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$. Then,*

*(a)* $\lim_{n\to\infty}(s_n + t_n) = s + t$.

*(b) For all $c \in \mathbb{C}$, $\lim_{n\to\infty}(s_n + c) = s + c$ and $\lim_{n\to\infty}(c \cdot s_n) = c \cdot s$.*

*(c)* $\lim_{n\to\infty} s_n \cdot t_n = s \cdot t$.

*(d)* $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ *if $s \neq 0$ and $s_n \neq 0$ for all $n \in \mathbb{N}$.*

*(e) If $s_n \leq t_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.*

*Proof.*

(a) For $\varepsilon > 0$, pick $N_1$ such that for any $n \geq N_1$, $|s_n - s| < \frac{\varepsilon}{2}$ and $N_2$ such that for any $n \geq N_2$, $|t_n - t| < \frac{\varepsilon}{2}$. Then, for $n \geq \max\{N_1, N_2\}$, we have

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $s_n + t_n \to s + t$.

(b) This is a trivial result, left as an exercise.

(c) By arithmetic manipulation, we use a trick

$$s_n t_n = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) + st.$$

For $\varepsilon > 0$, pick $N_1$ such that for any $n \geq N_1$, $|s - s_n| < \sqrt{\varepsilon}$, and $N_2$ such that for any $n \geq N_2$, $|t - t_n| < \sqrt{\varepsilon}$. Then, for $n \geq \max\{N_1, N_2\}$, we have $|(s_n - s)(t_n - t)| < \varepsilon$, so that

$$\lim_{n\to\infty}(s_n - s)(t_n - t) = 0.$$

Now, apply (a) and (b), we get

$$\lim_{n\to\infty} s_n \cdot t_n = s \cdot t.$$

(d) By arithmetic manipulation, we consider the trick

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|ss_n|}.$$

Pick $m \in \mathbb{N}$ such that for all $n \geq m$, $|s_n - s| < \frac{1}{2}|s|$. Then, for $n \geq m$, $|s_n| > \frac{1}{2}|s|$ by triangle inequality. Now, let $\varepsilon > 0$ and pick $N \geq m$ such that for all $n \geq N$,

$$|s_n - s| < \frac{1}{2}|s|^2\varepsilon.$$

Then, for $n \geq N$, we have

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s - s_n|}{|ss_n|} < \frac{1}{\frac{1}{2}|s|^2} \cdot |s - s_n| < \varepsilon.$$

Thus $\frac{1}{s_n} \to \frac{1}{s}$.

(e) For every $n$, we have $t_n - s_n$ lies in the closed set $[0, +\infty) \subseteq \mathbb{R} \subseteq \mathbb{C}$. Thus,

$$t - s = \lim_{n\to\infty} t_n - \lim_{n\to\infty} s_n = \lim_{n\to\infty}(t_n - s_n) \in [0, +\infty)$$

so $s \leq t$.

$\square$

Lecture 13

November 2

Monday

Now, we discuss the limit operations in the Euclidean space $\mathbb{R}^k$.

**Theorem 3.7.** *Suppose $\vec{x_n} \in \mathbb{R}^k$ for $n \in \mathbb{N}$ and $\vec{x_n} = (\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{k,n})$. Then, $\vec{x_n}$ converges to $\vec{x} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ if and only if*

$$\lim_{n\to\infty} \alpha_{i,n} = \alpha_i \quad 1 \leq i \leq k.$$

*Proof.* Assume that $\vec{x_n} \to \vec{x}$. Then, by the definition of the norm (distance in $\mathbb{R}^k$) we have

$$\begin{aligned} |\vec{x_n} - \vec{x}| &= \left( \sum_{i=1}^{k} (\alpha_{i,n} - \alpha_i)^2 \right)^{1/2} \\ &\geq \left[ (\alpha_{i,n} - \alpha_i)^2 \right]^{1/2} \\ &\geq |\alpha_{i,n} - \alpha_i| \end{aligned}$$

for all $1 \leq i \leq k$. Hence, $|\alpha_{i,n} - \alpha_i| \leq |\vec{x_n} - \vec{x}|$ immediately shows that $\lim\limits_{n\to\infty} \alpha_{i,n} = \alpha_i$.

Assume that $\lim\limits_{n\to\infty} \alpha_{i,n} = \alpha_i$ for all $1 \leq i \leq k$ holds, then to each $\varepsilon > 0$ there exists an integer $N$ such that for all $n \geq N$,

$$|\alpha_{i,n} - \alpha_i| < \frac{\varepsilon}{\sqrt{k}} \quad 1 \leq i \leq k.$$

Hence, $|\vec{x_n} - \vec{x}| = \left( \sum_{i=1}^{k} |\alpha_{i,n} - \alpha_i|^2 \right)^{1/2} < \varepsilon.$

□

**Theorem 3.8.** *Let $(\vec{x_n})$ and $(\vec{y_n})$ be sequences in $\mathbb{R}^k$ with $\vec{x_n} \to \vec{x}$ and $\vec{y_n} \to \vec{y}$. Let $(\beta_n)$ be a sequence in $\mathbb{R}$ with $\beta_n \to \beta$. Then,*

*(a)* $\lim\limits_{n\to\infty} \vec{x_n} + \vec{y_n} = \vec{x} + \vec{y}$.

*(b)* $\lim\limits_{n\to\infty} \vec{x_n} \cdot \vec{y_n} = \vec{x} \cdot \vec{y}$.

*(c)* $\beta_n \cdot \vec{x_n} = \beta \cdot \vec{x}$.

*Proof.* This follows from Theorem 3.6 and Theorem 3.7.

□

## 3.2 Subsequence

It is useful to sometimes consider only some terms of a sequence. Simply, a subsequence of $(p_n)$ is a sequence that contains only some of the numbers from $(p_n)$ in the same order.

**Definition 3.9** (Subsequence)**.** Given a sequence $(p_n)$, consider a sequence $(n_k)$ of positive integers such that $n_1 < n_2 < n_3 < \cdots$. Then, the sequence $(p_{n_k})$ is called a **subsequence** of $(p_n)$. If $(p_{n_k})$ converges, its limit is called a **subsequential limit** of $(p_n)$.

For an arbitrary subsequence, we have the following proposition about convergence.

**Theorem 3.10.** *A sequence $(p_n)$ converges to $p$ if and only if every subsequence $(p_n)$ converges to $p$.*

If we remove "every" in this theorem, it will be wrong. Below is a counterexample.

**Example 3.11.** Consider the following sequence

$$1, \pi, \frac{1}{2}, \pi, \frac{1}{3}, \pi, \frac{1}{4}, \pi, \cdots$$

This sequence is divergent, but it has a convergent subsequence, namely

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$$

Thus, a divergent sequence may have a convergent subsequence.

This example basically says that existence of a convergent subsequence does not imply convergence of the sequence itself.

**Example 3.12.** The set of natural numbers $\mathbb{N}$ does not contain any convergent subsequence. Thus, not every sequence contains a convergent subsequence.

**Example 3.13.** Consider we are using the metric space $(X, d)$ with $X = \mathbb{Q}$. Consider the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \cdots$$

does not converge in $\mathbb{Q}$, and its subsequence does not converge in $\mathbb{Q}$. While we switch our $\mathbb{Q}$ to $\mathbb{R}$, it will converge to $\pi$. Thus, not every bounded sequence contains a convergent subsequence. For general metric spaces, this is not true.

Here are two useful results for the subsequence. The following theorem connects infinite with subsequential limit.

**Theorem 3.14.** *A point $q$ in a metric space $(X, d)$ is a subsequential limit of $(p_n)_{n\in\mathbb{N}}$ if and only if for all $r > 0$, the set $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite.*

*Proof.* Assume $(p_{n_i})$ is a subsequence with $p_{n_i} \to q$. Let $r > 0$. Pick an $N \in \mathbb{N}$ such that for all $i \geq N$, we have $d(p_{n_i}, q) < r$. Then,

$$\{n_N, n_{N+1}, n_{N+2}, \cdots\} \subseteq \{n \in \mathbb{N} : p_n \in B_r(q)\}.$$

Since $\{n_N, n_{N+1}, n_{N+2}, \cdots\}$ is infinite, so $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite.

Now, assume that for all $r > 0$, $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite. Then, pick any $n_1$ with $p_{n_1} \in B_1(q)$. Once $n_1 < \cdots n_{i-1}$ have been defined, pick $n_i > n_{i-1}$ with $p_{n_i} \in B_{1/i}(q)$. This defines a subsequence $(p_{n_i})_{i\in\mathbb{N}}$. Fix an $\varepsilon > 0$. We pick an $N \in \mathbb{Z}_+$ with $\dfrac{1}{N} < \varepsilon$. Then, for all $i \geq N$,

$$d(p_{n_i}, q) < \frac{1}{i} \leq \frac{1}{N} < \varepsilon$$

since $p_{n_i} \in B_{1/i}(q)$. Thus, $p_{n_i} \to q$.

$\square$

**Corollary 3.15.** *If $q$ is a limit point of the set $\{p_n : n \in \mathbb{N}\}$, then $q$ is a subsequential limit of $(p_n)$.*

*Proof.* Fix $r > 0$. Define a set $I = \{n \in \mathbb{N} : p_n \in B_r(q)\}$. Then,

$$\{p_i : i \in I\} = \{p_n : n \in \mathbb{N}\} \cap B_r(q)$$

and the set on the right is infinite since $q \in \{p_n : n \in \mathbb{N}\}'$. Thus, $\{p_i : i \in I\}$ is infinite, so $I$ must be infinite as well.

□

**Theorem 3.16.**

*(a) If $(p_n)$ is a sequence in a compact metric space $(X, d)$, then $(p_n)$ has a subsequential limit.*

*(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

*Proof.* (a) Let $E = \{p_n : n \in \mathbb{N}\}$.

If $E$ is finite, then there must be some $p \in E$ and $n_1 < n_2 < n_3 < \cdots$ with $p_{n_i} = p$ for all $i$, so $p_{n_i} \to p$.

If $E$ is infinite, then $E' \neq \varnothing$ by Theorem 2.49. Thus, $(p_n)$ has a subsequential limit by Corollary 3.15.

(b) This follows from (a) and Heine-Borel Theorem.

□

**Theorem 3.17.** *The set of all subsequential limits of $(p_n)$ is a closed set.*

*Proof.* Let $E^*$ be the set of all subsequential limits of $(p_n)$. Let $q \in (E^*)'$. Fix $r > 0$. Since $q \in (E^*)'$, so $(B_{r/2}(q) \setminus \{q\}) \cap E^* \neq \varnothing$. We can pick a point $x \in (B_{r/2}(q) \setminus \{q\}) \cap E^*$. Let another point $w \in B_{r/2}(x)$, then $d(q, w) \leq d(q, x) + d(x, w) < \dfrac{r}{2} + \dfrac{r}{2} = r$ and hence $w \in B_r(q)$. Thus, $B_{r/2}(x) \subseteq B_r(q)$.

Since $x \in E^*$, so $\left\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\right\}$ is infinite by Theorem 3.14. Therefore, $\{n \in \mathbb{N} : p_n \in B_r(q)\}$ is infinite since $\left\{n \in \mathbb{N} : p_n \in B_{r/2}(x)\right\} \subseteq \{n \in \mathbb{N} : p_n \in B_r(q)\}$. By Theorem 3.14, $q$ is a subsequential limit, which means $q \in (E^*)'$. We can conclude that $E^*$ is closed.

□

## 3.3 Cauchy Sequence

Previously, we talk about the convergence of a sequence. However, the definition of convergence implies that we know the limit of a sequence and then we can proceed a proof with $\varepsilon$. If we don't know whether the sequence has a limit, can we say whether it is convergent or not?

Also, we note that the convergence of a sequence is not an intrinsic property. Consider $p_n = \dfrac{1}{2^n}$. The sequence converges as a sequence in $\mathbb{R}$ but not in $(0, 1)$ since $\{0\}$ is not in the open set $(0, 1)$. This motivates the *Cauchy sequence*. The basic idea is that if a sequence is convergent, then the successive points in the sequence must be closer and closer.

**Definition 3.18** (Cauchy Sequence)**.** A sequence $(p_n)$ in a metric space $(X, d)$ is called a **Cauchy sequence** if for every $\varepsilon > 0$, there is a positive integer $N$ such that for all $n, m \geq N$, $d(p_n, p_m) < \varepsilon$.

This property is intrinsic because it does not require a limit point $p$ whose existence may depend on the choice of the metric space. We will see that any convergent sequence is *Cauchy* later.

**Definition 3.19** (Diameter)**.** Let $(X, d)$ be a metric space. The **diameter** of nonempty $E \subseteq X$ is

$$\operatorname{diam} E = \sup\{d(p, q) : p, q \in E\}$$

if the supremum exists, otherwise $\operatorname{diam} E = +\infty$.

A sequence $(p_n)$ is Cauchy if and only if $\lim_{n\to\infty} \operatorname{diam}\{p_n, p_{n+1}, p_{n+2}, \cdots\} = 0$. Lecture 14

**Theorem 3.20.** *Let $(X, d)$ be a metric space.* Wednesday

November 4

*(a) If $E$ is a nonempty subset of $X$ and $\operatorname{diam} E$ exists, then*

$$\operatorname{diam} \overline{E} = \operatorname{diam} E.$$

*(b) If $K_n$ is compact and nonempty, and $K_{n+1} \subseteq K_n$, and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, then $\bigcap_{n\in\mathbb{N}} K_n$ is a singleton, the set with only one element.*

*Proof.*

(a) It is obvious that $\operatorname{diam} \overline{E} \geq \operatorname{diam} E$ since $E \subseteq \overline{E}$.

Let $p, q \in \overline{E}$. Let $\varepsilon > 0$ and pick $p', q' \in E$ with $d(p, p') < \varepsilon$ and $d(q, q') < \varepsilon$. Then,

$$\begin{aligned} d(p, q) &\leq d(p, p') + d(p', q') + d(q', q) \\ &< \varepsilon + d(p', q') + \varepsilon = 2\varepsilon + \operatorname{diam} E. \end{aligned}$$

This holds for all $\varepsilon > 0$, so $d(p, q) \leq \operatorname{diam} E$.

This holds for all $p, q \in \overline{E}$, so $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$.

(b) Set $K = \bigcap_{n\in\mathbb{N}} K_n$. Then, $K \neq \varnothing$ by Finite Intersection Property (Theorem 2.47) and $\operatorname{diam} K \leq \operatorname{diam} K_n$ for all $n \in \mathbb{N}$ since $K \subseteq K_n$. Thus, $\operatorname{diam} K = 0$ and hence $K$ consists of a single point.

□

**Theorem 3.21.** *For any metric space $(X, d)$, if sequence $(p_n)$ converges, then $(p_n)$ is Cauchy.*

*Proof.* Let $p_n \to p$. Let $\varepsilon > 0$. Pick an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(p_n, p) < \frac{\varepsilon}{2}$. Then, for $n, m \geq N$ (do not use $n + 1$ as $m$, this is not from the definition of Cauchy sequence), we have

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(p_n)$ is Cauchy.

□

This theorem basically says convergence implies *Cauchy* in any metric space, but the converse does not always hold. We need to guarantee we are in a comacpt metric space.

**Theorem 3.22.** *If the metric space $(X, d)$ is compact and $(p_n)$ is Cauchy, then $(p_n)$ converges.*

*Proof.* Set $E_n = \{p_n, p_{n+1}, p_{n+2}, \cdots\}$. Since $(p_n)$ is Cauchy, $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$ by Theorem 3.20 (a). $X$ is compact, so $\overline{E_n}$ is compact and $\overline{E_n} \supseteq \overline{E_{n+1}}$. So, by Theorem 3.20 (b), $\bigcap\limits_{n\in\mathbb{N}} \overline{E_n} = \{p\}$ for some $p \in X$. Now, let $\varepsilon > 0$ and pick $N$ such that $\operatorname{diam} \overline{E_N} < \varepsilon$. Then, for $n \geq N$, we have $p_n \in E_n \subseteq E_N$ and $p \in \overline{E_N}$, so $d(p_n, p) \leq \operatorname{diam} \overline{E_N} < \varepsilon$. Thus, $p_n \to p$.

□

In the metric space $(\mathbb{Q}, d)$ with $d(x, y) = |x - y|$, consider the sequence $p_n = \left(1 + \dfrac{1}{n}\right)^n$. The sequence $(p_n)$ is *Cauchy* (assume we know how to prove it by Mean Value Theorem). Later, we will know that $p_n \to e$, but $e$ is not rational. Again, $\mathbb{Q}$ is not compact, so the compactness is essential in this result.

**Theorem 3.23.** *In $k$-dimensional Euclidean space $\mathbb{R}^k$, every Cauchy sequence converges.*

*Proof.* Assume that $(\vec{x_n})$ is a Cauchy sequence in $\mathbb{R}^k$. Pick an $N \in \mathbb{N}$ such that $\operatorname{diam}\{\vec{x_N}, \vec{x_{N+1}}, \cdots\} < 1$. Then, for $n \geq N$,

$$|\vec{x_n}| \leq |\vec{x_N}| + |\vec{x_n} - \vec{x_N}| < |\vec{x_N}| + 1$$

so $\{\vec{x_0}, \vec{x_1}, \cdots\} \subseteq B_r(\vec{0})$ where $r = 1 + \min\{|\vec{x_0}|, \cdots, |\vec{x_N}|\}$. By Heine-Borel Theorem, $(\vec{x_N})$ is a sequence in the compact set $\overline{B_r(\vec{0})}$, hence it converges by Theorem 3.22.

□

For $k = 1$, we can conclude that in the metric space $(\mathbb{R}, d)$, every Cauchy sequence converges. This motivates the following definition.

**Definition 3.24** (Complete)**.** A metric space $(X, d)$ is **complete** if every Cauchy sequence converges.

**Example 3.25.** Compact spaces, Euclidean spaces, and closed subsets of *these* are complete.

However, the set of rational numbers $\mathbb{Q}$ is NOT complete.

$\mathbb{R}$ is the smallest complete metric space containing $\mathbb{Q}$ (Cauchy construction).

**Definition 3.26** (Monotonicity)**.** A sequence $(s_n)$ in $\mathbb{R}$ is

(a) **monotone increasing** if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$.

(b) **monotone decreasing** if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

(c) **monotone** if either of the above.

**Theorem 3.27.** *Suppose $(s_n)$ is monotone. Then $(s_n)$ converges if and only if $(s_n)$ is bounded.*

*Proof.* Without loss of generality, let's say $(s_n)$ is monotone increasing (the other case is similar).

Assume $(s_n)$ converges. Then the result follows from Theorem 3.5 (c).

Assume $(s_n)$ is bounded, so $s = \sup\{s_n : n \in \mathbb{N}\}$ exists. Let $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound to $\{s_n : n \in \mathbb{N}\}$. So, there is an $N \in \mathbb{N}$ such that $s_N > s - \varepsilon$. Then, for $n \geq N$, $s - \varepsilon < s_N \leq s_n \leq s$, hence $|s - s_n| < \varepsilon$. We conclude that $s_n \to s$.

□

**Cauchy Construction of Real Numbers**

Again, recall the definition of a real field, that is an ordered field with the least upper bound property.

**Theorem 3.28.** *There exists an ordered field $\mathbb{R}$ with the least upper bound property and containing $\mathbb{Q}$ as an ordered subfield.*

Before starting the proof, we remark that we do not need to know what $\mathbb{R}$ is to talk about distances in $\mathbb{Q}$ and *Cauchy sequences*. Indeed, the number $|p - q| \in \mathbb{Q}$ for any $p, q \in \mathbb{Q}$ and satisfies the definition of *metric*.

**Proposition 3.29.** *There are Cauchy sequences in $\mathbb{Q}$ that do not converge to any element in $\mathbb{Q}$.*

*Proof.* It is suffice to prove that $\mathbb{Q}$ neither satisfies the least upper bound nor the greatest lower bound property. We modify Example 1.15 to the following recursively defined sequence

$$p_{n+1} = p_n - \frac{p_n^2 - 2}{p_n + 2} = 2 - \frac{2}{p_n + 2}$$

which is increasing if $p_1^2 < 2$ and decreasing if $p_1^2 > 2$ by induction. The sequence is a Cauchy sequence. (We omit the proof here.) If the limit exists, say $L = \lim_{n\to\infty} p_n$, then taking limits gives that $L^2 = 2$, whence the limit cannot exist in $\mathbb{Q}$.

□

There are other recursive relations yielding Cauchy sequences whose limits $L$, if they exist, must satisfy $L^2 = 2$, for example $p_{n+1} = \dfrac{p_n}{2} + \dfrac{1}{p_n}$.

Before constructing the real numbers, we first discuss the *relation*.

**Definition 3.30** (Relation)**.** Let $X$ be a set. A **relation** $R$ on $X$ is a subset of the Cartesian product $X \times X$, that is, $R \subseteq X \times X$.

We often denotes $xRy$ or $x \sim y$ (this is what I prefer) to mean $(x, y) \in R$.

**Definition 3.31** (Equivalence Relation)**.** An **equivalence relation** on a set $X$ is a relation satisfying the following properties

(a) **Reflexivity**: $x \sim x$ for all $x \in X$.

(b) **Symmetry**: if $x \sim y$, then $y \sim x$, for all $x, y \in X$.

(c) **Transitivity**: if $x \sim y$ and $y \sim z$, then $x \sim z$, for all $x, y, z \in X$.

The relation on the set of all sets defined by

$$A \sim B \iff \text{there exists a bijection } f : A \to B$$

is an equivalence relation. *Reflexivity* holds because the identity mapping $\mathrm{id} : A \to A$ mapping $x$ to $x$ is a bijection. *Symmetry* holds because a bijection $f : A \to B$ has an inverse mapping $f^{-1} : B \to A$ that is still a bijection. *Transitivity* holds because if $f : A \to B$ and $g : B \to C$ are bijectison, then the *composed map*

$$g \circ f : A \to C \text{ defined by } (g \circ f)(x) = g(f(x))$$

is a bijection.

**Definition 3.32** (Equivalence Class)**.** Given an equivalence relation $\sim$ on a set $X$ and $x \in X$, we define the **equivalence clas** $[x]$ of $x$ to be the set

$$[x] = \{y \in X : y \sim x\}.$$

Any element in the equivalence class is called a **representative** of the equivalence class.

We often denote the set of equivalence classes by $X/\sim$.

The idea of Cauchy construction of the real field is the following. Since $\mathbb{Q}$ lacks limits of Cauchy sequences in $\mathbb{Q}$, we add those "limits" representing them simply by the Cauchy sequences themselves. The elements of $\mathbb{Q}$ can be viewed as constant Cauchy sequences in $\mathbb{Q}$. Thus, the real number $\sqrt{2}$ would be represented by the Cauchy sequence $(p_n)$ in the proof of Proposition 3.29. However, we need to take into consideration that different Cauchy sequences may yield the same limits as we provided previously. It is to avoid this ambiguity that equivalences enter into the blueprint.

**Definition 3.33.** Two Cauchy sequences $(p_n)$ and $(q_n)$ in $\mathbb{Q}$ are said to be **equivalent** if the sequence $(|p_n - q_n|)$ in $\mathbb{Q}$ converges to 0.

We denote $(p_n) \sim (q_n)$ if $(p_n)$ and $(q_n)$ are equivalent.

We denote by $[p_n]$ teh equivalence class of $(p_n)$ adn define $\mathbb{R}$ to be the *set of all equivalence classes of Cauchy sequences on* $\mathbb{Q}$, that is,

$$\mathbb{R} = \{\text{Cauchy sequences in } \mathbb{Q}\}/\sim .$$

Now, we know what real numbers are. Except ... do we? We have a formal construction, which produces a set of objects, but the real numbers we know have lots of structure to them, as spelled out by the field axioms, order axioms, and least upper bound property.

We note that $\mathbb{Q}$ can be identified with a subset of $\mathbb{R}$ by identifying $q \in \mathbb{Q}$ with the equivalence class of the *constant Cauchy sequence* $p_n = p$. It is clear that $(p_n) \sim (q_n)$ if and only if $p = q$, so the function $\mathbb{Q} \to \mathbb{R}$ mapping $q$ to $[q_n]$ (the equivalence class of $(q_n)$) is injective. Hence, $\mathbb{Q}$ can be identified with the image of this function, which is a subset of the newly defined set $\mathbb{R}$.

**Definition 3.34.** Let $(p_n)$ and $(q_n)$ be two *Cauchy sequences* in $\mathbb{Q}$. We define two operations $+$ and $\cdot$ on $\mathbb{R}$ that are compatible with the ones on the subset $\mathbb{Q}$.

(a) We define $[(p_n)] + [(q_n)]$ to be the equivalence class of the sequence $(p_n + q_n)$.

(b) We define $[(p_n)] \cdot [(q_n)]$ to be the equivalence class of the sequence $(p_n \cdot q_n)$.

We have to prove that these operations are well-defined, that is, that they are independent of the choice of representative for the equivalence classes of Cauchy sequences. This is taken care of by the following.

**Lemma 3.35.** *If $(p'_n) \sim (p_n)$ and $(q'_n) \sim (q_n)$, then $(p'_n + q'_n) \sim (p_n + q_n)$ and $(p'_n \cdot q'_n) \sim (p_n \cdot q_n)$.*

*Proof.* By definition, $(p'_n) \sim (p_n)$ and $(q'_n) \sim (q_n)$ means that $\lim_{n\to\infty} |p'_n - p_n| = 0$ and $\lim_{n\to\infty} |q'_n - q_n| = 0$. By the triangle inequality,

$$|(p'_n + q'_n) - (p_n + q_n)| \leq |p'_n - p_n| + |q'_n - q_n|$$

whence $\lim_{n\to\infty} |(p'_n + q'_n) - (p_n + q_n)| = 0$. It follows that $(p'_n + q'_n) \sim (p_n + q_n)$, as desired.

To prove the remaining part, we use the trick

$$|p'_n q'_n - p_n q_n| = |q'_n(p'_n - p_n) + p_n(q'_n - q_n)| \leq |q'_n||p'_n - p_n| + |p_n||q'_n - q_n|,$$

and the fact that any Cauchy sequence is bounded (Theorem 3.36) to conclude that $\lim_{n\to\infty} |p'_n q'_n - p_n q_n| = 0$. □

**Theorem 3.36.** *If $(p_n)$ is a Cauchy sequence, then it is bounded, that is, there exists some sufficiently large number $M$ such that $|p_n| \leq M$ for all $n$.*

*Proof.* Since $(p_n)$ is Cauchy, so let $\varepsilon = 1$, there exists an $N \in \mathbb{N}$ such that $|p_m - p_n| < 1$ for all $m, n > N$. Thus, $|p_{N+1} - p_n| < 1$ for $n \geq N$. We can rewrite this as $p_{N+1} - 1 < p_n < p_{N+1} + 1$. This means that $|p_n| < \max\{|p_{N+1} - 1|, |p_{N+1} + 1|\}$. So, set

$$M = \max\{|p_0|, |p_1|, \ldots, |p_N|, |p_{N+1} - 1|, |p_{N+1} + 1|\}.$$

Then, for any term $p_n$, if $n < N$, then $|p_n|$ appears in the list above and so $|p_n| \leq M$; if $n \geq N$, then $|p_n|$ is less than at least one of the last two entries in that list, so $|p_n| \leq M$. Hence, $M$ is a bound for the sequence. We conclude that Cauchy sequence is bounded.

□

So, now that we have well-defined operations, it behooves us to prove that the real field $\mathbb{R}$, equipped with them, is a field. It is tedious to work through the proof of commutativity and associativity of these binary operations.

It is not so hard to verify that the *identity element under addition* is the sequence $\{0, 0, 0, \ldots\}$ and the *identity element under multiplication* is the sequence $\{1, 1, 1, \ldots\}$. Also, it is not hard to show that the *inverse element under addition* of $[p_n]$ is $[-p_n]$, but the *inverse element under multiplication* is a bit tricky, which we will prove below.

**Theorem 3.37** (Multiplicative Inverse)**.** *Given any real number $p \neq 0$, there exists a real number $q$ such that $p \cdot q = 1$.*

*Proof.* The premise is that $s$ is nonzero, which means that $p$ is not in the equivalence class of $\{0, 0, 0, \ldots\}$. In other words, $p = [(p_n)]$ where $p_n - 0$ does *not* converges to 0. From this, we are to deduce the existence of a real number $q = [(q_n)]$ such that $p \cdot q = [(p_n \cdot q_n)]$ is the same equivalence class as $[\{1, 1, 1, \ldots\}]$. Doing so is actually a consequence of the fact that nonzero rational numbers have multiplicative inverses, but there is a subtle difficulty. Just because $p$ is nonzero, that is, $(p_n)$ does not tend to 0, there is no reason that any number of the terms in $(p_n)$ cannot equal to 0. However, it turns out that *eventually*, $p_n \neq 0$, so there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $p_n \neq 0$.

Now, define a sequence $(q_n)$ of rational numbers as follows. For $n < N$, $q_n = 0$, and for $n \geq N$, $q_n = \dfrac{1}{p_n}$. This makes sense since for a$n \geq N$, $p_n$ is a nonzero rational number, so $\dfrac{1}{p_n}$ exists. Then, $p_n \cdot q_n$ is equal to $p_n \cdot 0 = 0$ for $n < N$, and equals $p_n \cdot q_n = p_n \cdot \dfrac{1}{p_n} = 1$ for $n \geq N$. Then, if we look at the sequence $\{1, 1, 1, \ldots\}$, we have $\{1, 1, 1, \ldots\} - \{a_n \cdot b_n\}$ is the sequence which is $1 - 0 = 1$ for $n < N$ and equals $1 - 1 = 0$ for $n \geq N$. Since this sequence is *eventually* equal to 0, it converges to 0, so $[(p_n \cdot q_n)] = \{1, 1, 1, \ldots\} = 1 \in \mathbb{R}$. This shows that $q = [(q_n)]$ is a multiplicative inverse to $p = [(p_n)]$.

□

Our choice of $(q_n)$ to start with 0's was arbitrary. We could have chosen each term randomly from $\mathbb{Q}$ until the $N$th stage. The only important choice is that $q_n = \dfrac{1}{p_n}$ for $n \geq N$.

Now, we want $\mathbb{R}$ to be an *ordered field*, which means there exists an order relation $<$ on $\mathbb{R}$ which respects the field operations.

**Definition 3.38** (Order)**.** For two Cauchy sequences $(p_n)$ and $(q_n)$ in $\mathbb{Q}$, $(p_n) < (q_n)$ if and only if there exists an $N \in \mathbb{N}$ and $r \in \mathbb{Q}_{>0}$ such that $p_n + r < q_n$ for all $n \geq N$.

**Lemma 3.39.** *Assume that $(p_n) < (q_n)$. If $(p'_n) \sim (p_n)$ and $(q'_n) \sim (q_n)$, then $(p'_n) < (q'_n)$.*

*Proof.* By assumption $(p_n) < (q_n)$, then there exists an $N_1 \in \mathbb{N}$ and $r \in \mathbb{Q}_{>0}$ such that $q_n > p_n + r$ for all $n \geq N_1$. Since $(p'_n) \sim (p_n)$ and $(q'_n) \sim (q_n)$, there are $N_2 \in \mathbb{Z}$ and $N_3 \in \mathbb{Z}$ such that $-\dfrac{r}{3} \leq p_n - p'_n \leq \dfrac{r}{3}$ for all $n \geq N_2$ and $-\dfrac{r}{3} \leq q_n - q'_n \leq \dfrac{r}{3}$ for all $n \geq N_3$.

Let $N = \max\{N_1, N_2, N_3\}$. For $n \geq N$, we have

$$q'_n \geq q_n - \frac{r}{3} > (p_n + r) - \frac{r}{3} = p_n + \frac{2}{3}r \geq (p'_n - \frac{r}{3}) + \frac{2}{3}r = p'_n + \frac{r}{3}.$$

Thus, we conclude that $(q'_n) > (p'_n)$ by definition.

□

By Theorem 3.39, we may define for any $[(p_n)], [(q_n)] \in \mathbb{R}$, $[(p_n)] < [(q_n)]$ if and only if $(p_n) < (q_n)$ for any representatives $(p_n)$ and $(q_n)$.

**Theorem 3.40.** *The relation $<$ is an order relation on $\mathbb{R}$ satisfying $p > q$ implies $p + r > q + r$ for $r \in \mathbb{R}$.*

*Proof.* Let $p = [(p_n)]$, $q = [(q_n)]$, and $r = [(r_n)]$. Since $p > q$, that is $p - q > 0$, we know that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $p_n - q_n > 0$. So, $p_n > q_n$ for $n \geq N$. Now, adding $r_n$ to both sides of this inequality, we have $p_n + r_n > q_n + r_n$ for $n \geq N$. Note that $(p_n + r_n) - (q_n + r_n) = p_n - q_n$ does not converge to 0, by the assumption that $p - q > 0$. Thus, by Definition 3.38, this means that $p + r = [(p_n + r_n)] > [(q_n + r_n)] = q + r$.

□

The proofs of the other order axioms are similar.

So, we have constructed an *ordered field* $\mathbb{R}$. The thing that really distinguished $\mathbb{R}$ from $\mathbb{Q}$ is the *least upper bound property.* To complete the construction, we need to show that this "crazy" set $\mathbb{R}$ of equivalence classes of Cauchy sequences really satisfies the *least upper bound property*!

Before getting directly to this proof, let us pause to prove that $\mathbb{R}$ has the *Archimedean Property.* Of course, once we have proven that $\mathbb{R}$ has the *least upper bound property*, we will know it is *Archimedean*, but we can actually use the *Archimedean Property* to help prove the *least upper bound property* in this case.

**Theorem 3.41.** *The real field $\mathbb{R}$ has the Archimedean Property, that is, if $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that $n \cdot x > y$.*

*Proof.* Let $p, q \in \mathbb{R}_{>0}$. We want to find an $m \in \mathbb{N}$ such that $m \cdot p > q$. Recall that, by $m$ in this context, we mean $[\{m, m, m, \ldots\}]$ as an equivalence class of a constant sequence. So, letting $p = [(p_n)]$ and q $= [(q_n)]$, what we need to show is that there exists an $m$ such that

$$[\{m, m, m, \ldots\}] \cdot [(p_n)] = [(m \cdot p_n)] > [(q_n)].$$

Now, we want to show that $[(m \cdot p_n)] > [(q_n)]$, that is to show that there exists an $N \in \mathbb{N}$ such that $m \cdot p_n - q_n > 0$ for all $n \geq N$, while $m \cdot p_n - q_n \nrightarrow 0$.

Towards a contradiction, assume that for every $m \in \mathbb{N}$ and $N \in \mathbb{N}$, there exists an $n \geq N$ such that $m \cdot p_n \leq q_n$. Since $(q_n)$ is a Cauchy sequence, then $(q_n)$ is bounded, which means there exists an $M \in \mathbb{Q}$ such that $q_n \leq M$ for all $n \in \mathbb{N}$. Now, by the *Archimedean Property for the rational numbers*, given any small rational number $\varepsilon > 0$, there exists an $m$ such that $\dfrac{M}{m} < \dfrac{\varepsilon}{2}$. Fix this $m$. Then if $m \cdot p_n \leq q_n$, we have $p_n \leq \dfrac{q_n}{m} \leq \dfrac{M}{m} < \dfrac{\varepsilon}{2}$.

Now, since $(p_n)$ is Cauchy, so there exists an $N \in \mathbb{N}$ such that for $n, k \geq N$, $|p_n - p_k| < \dfrac{\varepsilon}{2}$. By our assumption, we have an $n \geq N$ such taht $m \cdot p_n \leq q_n$, which means that $p_n < \dfrac{\varepsilon}{2}$. Then, for every $k \geq N$, we have $p_k - p_n < \dfrac{\varepsilon}{2}$, so $p_k < p_n + \dfrac{\varepsilon}{2} < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$. Hence, $p_k < \varepsilon$ for all $k \geq N$. This proves that $p_k \to 0$, which contradicts the fact that $[(p_n)] = p > 0$.

Thus, there exists indeed some $m \in \mathbb{N}$ such that $mp_n - q_n > 0$ for all sufficiently large $n$. Last, we want to show that $mp_n - q_n \nrightarrow 0$. Actually, it is possible that $mp_n - q_n \to 0$. For example, $(p_n) = \{1, 1, 1, \ldots\}$ and

$(q_n) = \{m, m, m, \ldots\}$. If this is the case, then we pick a larger $m$, which means we can take instead $m+1$. Since $p = [(p_n)] > 0$, we have $p_n > 0$ for all sufficiently large $n$, so $(m+1)p_n - q_n = mp_n - q_n + p_n > p_n > 0$ for all sufficiently large $n$, so $m+1$ works fine as $m$ did in this regard. Since $mp_n - q_n \to 0$, we have $(m+1)p_n - q_n = (mp_n - q_n) + p_n \nrightarrow 0$ since $p = [(p_n)] > 0$ (so $p_n \nrightarrow 0$). If $mp_n - q_n \nrightarrow 0$, then the proof is automatically complete.

We conclude that $\mathbb{R}$ has the *Archimedean Property.*

$\square$

**Theorem 3.42.** *Given any real number $r$, and any small (rational) number $\varepsilon > 0$, there exists a rational number $q$ such that $|r - q| < \varepsilon$.*

*Proof.* The real number $r$ is represented by a Cauchys equence $(r_n)$. So, given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_n - a_m| < \varepsilon$. We pick a fixed $k \geq N$, we can take the rational number $q$ given by $q = [\{p_k, p_k, p_k, \ldots\}]$. Then, we have $r - q = [(p_n - p_k)_{n\in\mathbb{N}}]$, and $q - r = [(p_k - p_n)_{n\in\mathbb{N}}]$. Now, since $k \geq N$, we note that for $n \geq N$, $p_n - p_k < \varepsilon$ and $p_k - p_n < \varepsilon$, which means that $|r - q| < \varepsilon$.

$\square$

And now, we want to prove that $\mathbb{R}$ has the *least upper bound property.* Let $S \subseteq \mathbb{R}$ be a nonempty subset, and let $M$ be an upper bound for $S$. We are going to construct two sequences of real numbers, $(p_n)$ and $(q_n)$. Since $S$ is nonempty, there exists some element $s_0 \in S$. Now, we go through inductively to produce the numbers $p_0, p_1, p_2, \ldots$ and $q_0, q_1, q_2, \ldots$.

First, set $p_0 = M$ and $q_0 = s_0$.

Second, suppose that we have already defined $p_n$ adn $q_n$, consider the number $r_n = \dfrac{p_n + q_n}{2}$, the average between $p_n$ and $q_n$. If $r_n$ is an upper bound for $S$, then define $p_{n+1} = r_n$ adn $q_{n+1} = q_n$. If $r_n$ is not an upper bound for $S$, then define $p_{n+1} = p_n$ and $q_{n+1} = r_n$.

Since $s_0 < M$, it is not hard to prove by induction that $(p_n)$ is a monotone decreasing sequence since $p_{n+1} \leq p_n$ and $(q_n)$ is a monotone increasing sequence since $q_{n+1} \geq q_n$. This gives us the following.

**Lemma 3.43.** *$(p_n)$ and $(q_n)$ are Cauchy sequences of real numbers.*

*Proof.* Note that each $q_n \leq M$ for all $n$. Since $(q_n)$ is monotone increasing, so $(q_n)$ is Cauchy, by Theorem 3.21 and Theorem 3.27. For $(p_n)$, we have $p_n \geq s_0$ for all $n$, so $-p_n \leq -s_0$. Since $(p_n)$ is monotone increasing, similarly, $(-p_n)$ is Cauchy. We conclude that $(p_n)$ and $(q_n)$ are both Cauchy.

$\square$

**Lemma 3.44.** *There exists a real number $p$ such that $p_n \to p$ and $q_n \to p$.*

*Proof.* Fix a term $p_n$ in the sequence $(p_n)$. By Theorem 3.42, there exists $s_n \in \mathbb{Q}$ such that $|p_n - s_n| < \dfrac{1}{n}$. Consider the sequence $(s_n)$ in $\mathbb{Q}$. We want to show that this sequence is Cauchy first.

Fix $\varepsilon > 0$. By the *Archimedean Property*, we pick an $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{3}$. Since $(p_n)$ is Cauchy, so there exists an $M \in \mathbb{N}$ such that $|p_n - p_m| < \dfrac{\varepsilon}{3}$ for $n, m \geq M$. Then, so long as $n, m \geq \max\{N, M\}$, we have

$$\begin{aligned}
|s_n - s_m| &= |(s_n - p_n) + (p_n - p_m) + (p_m - s_m)| \\
&\leq |s_n - p_n| + |p_n - p_m| + |p_m - s_m| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus, $(s_n)$ is a Cauchy sequence in $\mathbb{Q}$, so it represents a real number $s = [(s_n)]$. We want to show that $p_n - p \to 0$. Let $\tilde{s_n} = [\{s_n, s_n, s_n, \ldots\}]$, it is immediate that $\tilde{s_n} - p \to 0$ (this is precisely equivalent to that $(s_n)$ is Cauchy). We have constructed that $p_n - \tilde{s_n} < \dfrac{1}{n}$, so if $\tilde{s_n} \to p$ and $p_n - \tilde{s_n} \to 0$, then $p_n \to p$.

So, $(p_n)$, a monotone decreasing sequence of upper bounds for $S$, tends to $p \in \mathbb{R}$. Our final destination is to prove that $p$ is the *least upper bound* of $S$. We still need to prove one thing, that is $q_n \to p$.

If $r_n$ is an upper bound for $S$, we have

$$p_{n+1} - q_{n+1} = r_n - q_n = \frac{p_n + q_n}{2} - q_n = \frac{p_n - q_n}{2}.$$

If $r_n$ is not an upper bound for $S$, we have

$$p_{n+1} - q_{n+1} = p_n - r_n = p_n = \frac{p_n + q_n}{2} = \frac{p_n - q_n}{2}.$$

Now, this means that $p_1 - q_1 = \dfrac{1}{2}(M - s)$, and so $p_2 - q_2 = \dfrac{1}{2}(p_1 - q_1) = \left(\dfrac{1}{2}\right)^2 (M - s)$, and by induction, $p_n - q_n = 2^{-n}(M - s)$. Since $M > s$, so $M - s > 0$, and since $2^{-n} < \dfrac{1}{n}$, by the *Archimedean Property*, for any $\varepsilon > 0$, $2^{-n}(M - s) < \varepsilon$ for all sufficiently large $n$. Thus, $p_n - q_n < 2^{-n}(M - s) < \varepsilon$ as well, and so $p_n - q_n \to 0$. Since $p_n \to p$, so $q_n \to p$ as well.

□

Finally, we arrive our destination.

**Theorem 3.45.** *$\mathbb{R}$ has the least upper bound property.*

*Proof.* We show that $p$ is an upper bound first. Towards a contradiction, suppose it is not, so there exists some $s \in S$ such that $p < s$. Then, let $\varepsilon = s - p > 0$. Since $p_n \to p$ and $(p_n)$ is monotone decreasing, there exists an $n \in \mathbb{N}$ such that $p_n - p < \varepsilon$, which means $p_n < p + \varepsilon = p + (s - p) = s$. Since $p_n$ is an upper bound for $S$, this is a contradiction. We conclude that $p$ is an upper bound for $S$.

Now, we know that for each $n \in \mathbb{N}$, $q_n$ is not an upper bound, which means that for each $n \in \mathbb{N}$, there exists an $s_n \in S$ such that $q_n \leq s_n$. By Lemma 3.44, $q_n \to p$, and since $(q_n)$ is monotone increasing, this means that for $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $q_n > p - \varepsilon$ for $n \geq N$. Hence, $s_n \geq q_n > p - \varepsilon$ for all $n \geq N$ as well. In particular, for $\varepsilon > 0$, there exists an $s \in S$ such that $s > p - \varepsilon$, which means that no element less than $p$ can be an upper bound for $S$. We conclude that $p$ is the *least upper bound* for $S$, so $\sup S$ exists.

□

This concludes our construction of these real numbers. Hurrah! Remember that they are only real (in the vernacular sense) inasmuch as they are well-approximated by rational numbers. So, there does exist an ordered field with the *least upper bound property*.

## 3.4 Upper and Lower Limits

**Definition 3.46.** For a sequence $(s_n)$ in $\mathbb{R}$, we write

(a) $\lim\limits_{n\to\infty} = +\infty$ or $s_n \to +\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n \geq M$.

(b) $\lim\limits_{n\to\infty} = -\infty$ or $s_n \to -\infty$ if for all $M \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n \leq M$.

When either of the above holds, $(s_n)$ still diverges. $\mathbb{R} \cup \{-\infty, +\infty\}$ is not a metric space since we do not allow "infinite distance" and $d(p, q) < \infty$ for $p, q \in X$.

Given $(s_n)$ (as always, in $\mathbb{R}$), let $E$ consist of $x \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that there exists a subsequence $s_{n_k} \to x$. Note that $E$ is never empty, because if $s_n$ is bounded, it must contain a subsequential limit, by compactness. If $s_n$ is not bounded, then it has either $+\infty$ or $-\infty$ as a subsequential limit.

**Definition 3.47** (Upper Limit, Lower Limit)**.** Let $(s_n)$ be a sequence in $\mathbb{R}$. Let $E$ be the set of all subsequential limits of $(s_n)$ (including $+\infty, -\infty$ if appropriate).

(a) The **upper limit** or **limit supremum** of $(s_n)$, denoted $\limsup\limits_{n\to\infty} s_n$ is $\sup E \in \mathbb{R} \cup \{-\infty, +\infty\}$.

(b) The **lower limit** or **limit infimum** of $(s_n)$, denoted $\liminf\limits_{n\to\infty} s_n$ is $\inf E \in \mathbb{R} \cup \{-\infty, +\infty\}$.

**Theorem 3.48.** *Let $(s_n)$ be a sequence in $\mathbb{R}$. Let $E$ be the set of all subsequential limits of $(s_n)$ (including $+\infty, -\infty$ if appropriate).*

*(a)* $\limsup\limits_{n\to\infty} s_n \in E$.

*(b) If $x > \limsup\limits_{n\to\infty} s_n$, then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < x$.*

*Moreover,* $\limsup\limits_{n\to\infty}$ *is the unique extended real number with these properties.*

Lecture 15

*Proof.* Friday

November 6

(a) If $\limsup\limits_{n\to\infty} s_n \in \mathbb{R}$, then $\limsup\limits_{n\to\infty} s_n = \sup E \in \overline{E} = E$ by Theorem 2.35 and Theorem 3.17.

If $\limsup\limits_{n\to\infty} s_n = +\infty$, then $E$ is not bounded above by anything in $\mathbb{R}$, hence $\{s_n : n \in \mathbb{N}\}$ is not bounded above in $\mathbb{R}$, so there is a subsequence $(s_{n_k})$ with $s_{n_k} \to \infty$. Thus, $\limsup\limits_{n\to\infty} s_n = +\infty \in E$.

If $\limsup\limits_{n\to\infty} s_n = -\infty$, then $E = \{-\infty\}$ hence $\limsup\limits_{n\to\infty} s_n \in E$.

(b) Towards a contradiction, suppose $s_n \geq x$ for infinitely $n$. Then $(s_n)$ has a subsequence in $[x, +\infty)$, hence has a subsequential limit $y \in [x, +\infty]$. Thus, $\limsup\limits_{n\to\infty} s_n = \sup E \geq y \geq x$ since $y \in E$, contradicting $x > \limsup\limits_{n\to\infty} s_n$.

Lastly, suppose $p < q$ both satisfy (a) and (b). Choose $x$ such that $p < x < q$. Applying (b) to $p$ and $x$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n < x$. It follows that every subsequential limit of $(s_n)$ is in $[-\infty, x]$. So, $E \subseteq [-\infty, x]$. Thus, $q$ cannot be a subsequential limit and therefore cannot satisfy (a).

□

**Corollary 3.49.** *If* $\liminf_{n\to\infty} s_n > x$, *there exists an* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $s_n > x$.

This is the lim inf version of Theorem 3.48 (b). The proof is similar.

**Example 3.50.** Consider the sequence $s_n = (-1)^n \left(1 + \frac{1}{2^n}\right)$. Then,

$$\limsup_{n\to\infty} s_n = 1 \qquad \liminf_{n\to\infty} s_n = -1.$$

Note that these values don't bound the sequence but are eventual bounds, up to substraction or addition by $\varepsilon > 0$.

*Remark* 3.51. $\lim_{n\to\infty} s_n$ exists and equals to $s$ if and only if

$$\limsup_{n\to\infty} s_n = s = \liminf_{n\to\infty} s_n.$$

Also, the sequence can have uncountably many subsequences with distinct limits. Consider $(s_n)$ be a sequence that enumerates the set of rational numbers $\mathbb{Q}$. Then, every real number is a subsequential limit.

**Theorem 3.52.** *If* $s_n \geq t_n$ *for* $n \geq N$, *where* $N$ *is fixed, then*

$$\limsup_{n\to\infty} s_n \geq \limsup_{n\to\infty} t_n \qquad \liminf_{n\to\infty} s_n \geq \liminf_{n\to\infty} t_n.$$

*Proof.* Let $t^* = \limsup_{n\to\infty} t_n$ and $s^* = \limsup_{n\to\infty} s_n$. Towards a contradiction, assume that $s^* > t^*$ and set $\varepsilon = \frac{s^* - t^*}{2}$ and $x = s^* - \varepsilon$. Note that $x > t^*$.

By Theorem 3.48 (b), there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, we have $t_n < x$. On the other hand, by Theorem 3.48 (a), $s^* \in E$ and so there is a subsequence $(s_{n_k})$ that converges to $s^*$. This implies that there exists $n_k \geq \max\{N_0, N\}$ such that $s_{n_k} \in (s^* - \varepsilon, s^* + \varepsilon)$, and we have $s_{n_k} > s^* - \varepsilon = x > t_{n_k}$, contradicting that $s_n \leq t_n$ for $n \geq N$.

It is similar to prove that $\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$ by Theorem 3.48.

□

## 3.5 More on Sequences

Before introducing some sequences, we first provide a useful result, **Binomial Theorem**.

**Definition 3.53** (Binomial Coefficient)**.** For $n, k \in \mathbb{N}$ and $0 \leq k \leq n$, we define

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}.$$

This is pronounced "$n$ **chooses** $k$."

**Lemma 3.54** (Pascal's Identity)**.** *For $n, k \in \mathbb{N}$ and $0 \le k \le n$, we have*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This is easier to check by calculation or by combinatorial property.

**Theorem 3.55** (Binomial Theorem)**.** *For $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

*Proof.* We prove by induction.

For $n = 1$, the formula becomes

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y$$

and this is forever true.

Now, assume $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ holds, then we want to show that it also holds for $n+1$. Then,

$$\begin{aligned}
(x+y)^{n+1} &= (x+y) \cdot (x+y)^n \\
&= (x+y) \left( \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) \\
&= x \cdot \left( \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) + y \cdot \left( \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \right) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1} \\
&= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k} y^{k+1} + \binom{n}{n} y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k} y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n} \left[ \binom{n}{k} + \binom{n}{k-1} \right] x^{n+1-k} y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + y^{n+1}.
\end{aligned}$$

Note that in the last step of calculation, we apply the Pascal's Identity given above. Since $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$, we can rewrite this last equation and obtain

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k.$$

$\square$

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \le x_n \le s_n$ for $n \ge N$ where $N$ is some fixed number, and if $s_n \to 0$, then $x_n \to 0$.

**Theorem 3.56.**

*(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*

*(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.*

*(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.*

*(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.*

*(e) If $z \in \mathbb{C}$ and $|z| < 1$, then $\lim_{n\to\infty} z^n = 0$.*

*Proof.*

(a) Let $\varepsilon > 0$. By Archimedean Property, there exists an $N \in \mathbb{N}$ such that $N > \left(\frac{1}{\varepsilon}\right)^{1/p}$. Then, if $n \ge N$, we have

$$\frac{1}{n} \le \frac{1}{N} < \varepsilon^{1/p}.$$

So,

$$\left|\frac{1}{n^p} - 0\right| = \left(\frac{1}{n}\right)^p \le \left(\frac{1}{N}\right)^p < \varepsilon$$

for $n \ge N$. We are done.

(b) It is obvious if $p = 1$.

Assume $p > 1$. Let $x_n = \sqrt[n]{p} - 1$. Then, $x_n > 0$. By Binomial Theorem,

$$1 + n \cdot x_n \le (1 + x_n)^n = p$$

so $0 < x_n \le \frac{p-1}{n}$ and thus $x_n \to 0$.

If $0 < p < 1$, then by Theorem 3.6,

$$1 = \frac{1}{1} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{p}}} = \lim_{n\to\infty} \sqrt[n]{p}.$$

(c) Let $x_n = \sqrt[n]{n} - 1$. Then, $x_n > 0$ and

$$\binom{n}{2} \cdot x_n^2 = \frac{n \cdot (n-1)}{2} \cdot x_n^2 \le (1 + x_n)^n = n$$

so $0 < x_n \le \sqrt{\frac{2}{n-1}}$ thus $x_n \to 0$ (whenever $\frac{2}{n-1} < \varepsilon^2$ we have $\sqrt{\frac{2}{n-1}} < \varepsilon$).

(d) Fix $k \in \mathbb{N}$ with $k > \alpha$. When $n > 2k$,

$$(1+p)^n > \binom{n}{k} \cdot p^k = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \cdot p^k > \left(\frac{n}{2}\right)^k \cdot \frac{p^k}{k!} = \frac{n^k \cdot p^k}{2^k \cdot k!}.$$

Thus,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}}.$$

Since $k - \alpha > 0$, so

$$\frac{2^k \cdot k!}{p^k} \cdot \frac{1}{n^{k-\alpha}} \to 0$$

by (a) and Theorem 3.6.

(e) Apply (d) with $\alpha = 0$ and $p = \frac{1}{|z|} - 1$, we find $|z|^n \to 0$. Since $|z^n| = |z|^n$, we obtain $z^n \to 0$.

□

## 3.6 Series

The idea of a series is to make sense of **summing** an infnite sequence of numbers.

**Definition 3.57** (Partial Sum, Series)**.** Let $(a_n)$ be a sequence in $\mathbb{C}$. For each $n \in \mathbb{N}$, we can sum the first $n$ terms of this sequence to get

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_n.$$

Then $(s_n)$ is also a sequence. $s_n$ is called the $n$**th partial sum**.

The expressions $a_0 + a_1 + a_2 + \cdots$ and $\sum_{n \in \mathbb{N}} a_n$ are called **(infinite) series** and denote the value $\lim_{n\to\infty} s_n$ when it exists.

**Definition 3.58** (Convergence of Series)**.** Let $(a_n)$ and $(s_n)$ be defined as the definition above. If $(s_n)$ converges to a number $s$, then we say that the infinite series $\sum_{n=0}^{\infty} a_n$ **converges** to $s$ and we write $\sum_{n=0}^{\infty} a_n = s$. If $(s_n)$ diverges, then we say that the series $\sum_{n=0}^{\infty} a_n$ diverges.

Recall Theorem 3.21, a sequence of real numbers converges if and only if it is a Cauchy sequence. Given a sequence $(a_n)$, we apply this theorem to the corresponding sequence of partial sums $(s_n)$ to get the following. The series $\sum_{n=0}^{\infty} a_n$ converges if and only if $(s_n)$ is a Cauchy sequence, that is, if and only if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $m \geq n \geq N$, then $|s_m - s_{n-1}| < \varepsilon$ (I use $n-1$ instead of $n$ to make the following display looks nicer). Now,

$$s_m - s_{n-1} = \sum_{k=0}^{m} a_k - \sum_{k=0}^{n-1} a_k = \sum_{k=n}^{m} a_k.$$

So, we have the following result.

**Theorem 3.59** (Cauchy Criterion)**.** $\sum_{n=0}^{\infty} a_n$ *converges if and only if for every* $\varepsilon > 0$*, there exists an* $N \in \mathbb{N}$ *such that for* $m \geq n \geq N$*, then*

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon.$$

*Proof.* This follows from the Cauchy criterion for sequence convergence (Theorem 3.21) and

$$|s_m - s_n| = \left| \sum_{k=n}^{m} a_k \right|.$$

□

**Theorem 3.60.** *If* $\sum_{n=0}^{\infty} a_n$ *converges, then* $a_n \to 0$*.*

*Proof.* This follows from the prior theorem by taking $m = n$.

□

Lecture 16 Monday November 9

*Remark* 3.61 (Harmonic Series)*.* The converse of Theorem 3.60 is false. Consider $a_n = \frac{1}{n}$. The **harmonic series** is the series $\sum_{n=1}^{\infty} a_n$. We define the partial sum

$$s_n = \sum_{k=1}^{n} a_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

In fact, $s_{2^k} \geq \frac{k}{2} + 1$. This helps completing the rest of the proof for $\sum_{n=1}^{\infty} a_n$ diverges but $a_n \to 0$.

**Theorem 3.62.** *If* $a_n \geq 0$*, then* $\sum_{n=0}^{\infty} a_n$ *converges if and only if its partial sums are bounded.*

*Proof.* Since $a_n \geq 0$ for all $n \in \mathbb{N}$, then its partial sums are monotonically increasing, in other words, $s_{n+1} \geq s_n$ for all $n$. Then, Theorem 3.27 says monotonic sequence converges if it is bounded. That's it.

□

We note from the harmonic series that for a series to converge, the terms not only have to tend to zero, but they have to go to zero "fast enough." The harmonic series does not go to zero fast enough at end. If we know about convergence of a certain series, we can use the following *comparison test* to see if the terms of another series go to zero "fast enough."

**Theorem 3.63** (Comparison Test)**.**

*(a) If* $|a_n| \leq c_n$ *for* $n \geq N_0$ *and* $\sum_{n=0}^{\infty} c_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges as well.*

*(b) If* $a_n \geq d_n \geq 0$ *and* $\sum_{n=0}^{\infty} d_n$ *diverges, then* $\sum_{n=0}^{\infty} a_n$ *diverges.*

*Proof.*

(a) Given $\varepsilon > 0$, there exists an $N \geq N_0$ such that for all $m \geq n \geq N$, $\sum_{k=n}^{m} c_k \leq \varepsilon$. Thus,

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_n| \leq \sum_{k=n}^{m} c_k \leq \varepsilon$$

so $\sum_{n=0}^{\infty} a_n$ converges by Theorem 3.59.

(b) This is the contrapositive of (a).

□

To apply *comparison test*, we need to have a standard about "fast." This motivates us to define *geometric series.*

**Definition 3.64** (Geometric Series)**.** For $x \in \mathbb{C}$, $\sum_{n=0}^{\infty} x^n$ is called a **geometric series**.

**Theorem 3.65** (Convergence of Geometric Series)**.** *If $x \in \mathbb{C}$ and $|x| < 1$, then*

$$\sum_{n=0}^{\infty} x_n = \frac{1}{1-x}.$$

*If $|x| \geq 1$, then the series diverges.*

*Proof.* Note that

$$(1-x) \cdot \sum_{k=0}^{n} x^k = 1 - x^{n+1}$$

so

$$s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

thus $\lim_{n\to\infty} s_n = \frac{1}{1-x}$ when $|x| < 1$.

When $|x| \geq 1$, we have $x^n \nrightarrow 0$ hence $\sum_{n=0}^{\infty} x^n$ diverges by contrapositive version of Theorem 3.60.

□

We now introduce the *Cauchy Condensation Test* to test whether a series of non-negative but decreasing terms is convergent.

**Theorem 3.66** (Cauchy Condensation Test)**.** *Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

*converges.*

*Proof.* For both series, they converge if and only if their partial sums are bounded by Theorem 3.59.

Let

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n \\ t_k &= a_1 + 2a_2 + \cdots + 2^k a_{2^k} \end{aligned}.$$

Assume that $(t_k)$ is convergent, then for $n < 2^k$, we have

$$\begin{aligned} s_n &\le a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\le a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k. \end{aligned}$$

Hence, we have $s_n \le t_k$. For sufficiently large $k$, if $t_k \to t$, we have $s_n \le t_k \le t$. But when $k$ is arbitrarily large, then $2^k$ is also arbitrarily large. Hence, $n < 2^k$ implies that $n$ can also be arbitrarily large. Hence, for sufficiently large $n$, $(s_n)$ is monotone increasing and bounded. Hence, $(s_n)$ converges.

Conversely, assume that $(s_n)$ is convergent. Then, for $n > 2^k$, we have

$$\begin{aligned} s_n &\ge a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\ge \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\ &= \frac{1}{2} t_k. \end{aligned}$$

Hence, we have $2s_n \ge t_k$. Then, for sufficiently large $n$, we have $2s \ge 2s_n \ge t_k$. Similarly, for sufficiently large $k$, $(t_k)$ is monotone increasing and bounded. Thus, $(t_k)$ converges.

□

**Theorem 3.67** ($p$-series)**.** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$.

*Proof.* If $p \le 0$, the series is clearly divergent since $\frac{1}{n^p} \nrightarrow 0$.

Assume $p > 0$. By Theorem 3.66, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k$$

converges. This is a geometric series, so it converges if and only if $2^{1-p} < 1$ if and only if $p > 1$.

□

Here is an interesting fact. The quantity $\sum_{n=1}^{\infty} \frac{1}{n^q}$, whn it converges, is called $\zeta(q)$, the *Riemann-zeta function of* $q$. This function is very important in number theory, and in particular in the distribution of the primes; there is a very famous unsolved problem regarding this function, the *Riemann hypothesis*.

We have not learned about log function yet, but for the sake of an example, let's pretend we know what it is. log means the logarithmic function with base 10.

**Theorem 3.68.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *converges if* $p > 1$ *and diverges if* $p \le 1$.

*Proof.* If $p \le 0$, series diverges since $\dfrac{1}{n} \le \dfrac{1}{n(\log n)^p}$ for $n \ge 11$ by Theorem 3.63.

Assume $p > 0$, the terms are monotone decreasing and positive. By Theorem 3.66, the series converges if and only if

$$\sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k \cdot (\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p \cdot (\log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges. By Theorem 3.67, this happens if and only if $p > 1$.

□

**The Number $e$**

**Definition 3.69** (The Number $e$)**.** $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.

Note that $\dfrac{1}{n!} \le \dfrac{1}{2^{n-1}}$, so $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}$.

**Theorem 3.70.** $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Lecture 17

Friday

November 13

*Proof.* Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \qquad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the Binomial Theorem, we have

$$\begin{aligned} t_n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{1}{n^k} + \cdots + \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \\ &\quad + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \\ &\le s_n \le e \end{aligned}$$

so $\displaystyle\limsup_{n\to\infty} t_n \le e$. Here we take the upper limit simply because we don't know whether $t_n$ has a limit. But we do know that it must have an upper limit.

Now, fix $m \in \mathbb{N}$. If $n \ge m$, then

$$t_n \ge 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

Holding $m$ fixed and taking $\liminf$ over $n$ on both sides, we get

$$\liminf_{n\to\infty} t_n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m$$

by Theorem 3.52. Taking limit as $m \to \infty$, we obtain

$$\liminf_{n\to\infty} t_n \ge \lim_{m\to\infty} s_m = e.$$

Since $e \le \liminf_{n\to\infty} t_n \le \limsup_{n\to\infty} t_n \le e$, so we conclude that

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = e$$

and $(t_n)$ converges to $e$.

□

We can also estimate how fast $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ is converging. Let $s_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$, we have

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} = \frac{1}{n! \cdot n}. \end{aligned}$$

Hence, $0 < e - s_n < \dfrac{1}{n! \cdot n}$. This means that this series is converging very quickly to $e$. This sets up the fact that $e$ is irrational.

**Theorem 3.71** (Irrationality of $e$)**.** *The number $e$ is irrational.*

*Proof.* Towards a contradiction, assume $e$ is rational and we can write $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$. Let $s_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$. Then, since $0 < e - s_q < \dfrac{1}{q! \cdot q}$, we have

$$0 < q! \cdot (e - s_q) < \frac{1}{q}.$$

By assumption, $q! \cdot e$ is an integer. Also,

$$q! \cdot s_q = q!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$$

is an integer, so $q! \cdot (e - s_q)$ is an integer strictly between 0 and $\dfrac{1}{q}$. Since $q \ge 1$, so there must be an integer between 0 and 1. This is a contradiction.

□

Further, the number $e$ is not **algebraic**, which means $e$ is not a root of a polynomial with integer coefficients, in other words,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where all $a_n \in \mathbb{Z}$. Actually, there are only countably many algebraic numbers. The proof is left as an exercise.

### The Root and Ratio Tests

**Theorem 3.72** (Root Test)**.** *Consider a series* $\sum_{n=1}^{\infty} a_n$ *and set* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

*(a) If* $\alpha < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

*(b) If* $\alpha > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

*(c) If* $\alpha = 1$*, then the test is inconclusive.*

*Proof.*

(a) Suppose $\alpha < 1$. Let $\beta = \frac{1+\alpha}{2}$. Then, $\alpha < \beta < 1$. There exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\sqrt[n]{|a_n|} < \beta$ by Theorem 3.48. Thus, $|a_n| < \beta^n$ for $n \geq N$. Since $0 < \beta < 1$, so $\sum_{n=N}^{\infty} \beta^n$ converges. By the comparison test, we conclude that $\sum_{n=1}^{\infty}$ converges.

(b) Suppose $\alpha > 1$. Then, by Theorem 3.48, we have $\alpha$ is a subsequential limit of $(\sqrt[n]{|a_n|})$, which means there exists a subsequence $(\sqrt[n_k]{|a_{n_k}|})$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Since $\alpha > 1$, there exists an $K \in \mathbb{N}$ such that $|a_{n_k}| > 1$ for all $k \geq K$. Thus, $\sum_{n=1}^{\infty} a_n$ diverges since $a_n \not\to 0$.

(c) For $\alpha = 1$, consider $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The first series diverges, but the second converges. Thus, the test is inconclusive.

$\square$

**Theorem 3.73** (Ratio Test)**.** *Let* $(a_n)$ *be a sequence such that for any* $n \in \mathbb{N}$, $a_n \neq 0$.

*(a)* $\sum_{n=1}^{\infty} a_n$ *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

*(b)* $\sum_{n=1}^{\infty} a_n$ *diverges if there exists an* $N \in \mathbb{N}$ *such that* $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq N$*. That is,* $(|a_n|)$ *is nondecreasing.*

*Proof.*

(a) Let $\beta = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Let $\gamma = \frac{\beta+1}{2}$. Since $\beta < 1$, we have $\beta < \gamma < 1$. Then, there exists an $N \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < \beta$ for all $n \geq N$. That is, $|a_{n+1}| < \gamma |a_n|$ for $n \geq N$. By induction, we can prove that

$$|a_{N+k}| < \gamma^k |a_N|$$

for $k \geq 1$. In other words, $|a_n| < \gamma^{n-N}|a_N|$ for all $n \geq N$. Now, the series $\sum_{n=N}^{\infty} \gamma^{n-N}|a_N| = \sum_{n=N}^{\infty} \frac{a_N}{\gamma^N} \cdot \gamma^n$ converges since it is a geometric series with $N$ fixed and also $0 < \gamma < 1$. Therefore, by the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

(b) This is immediate since $a_n \nrightarrow 0$. Here are more details. Since $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq N$, we have $|a_{n+1}| \geq |a_n|$ for all $n \geq N$. By induction, we can prove that $|a_n| \geq |a_N|$ for $n \geq N$. Since $|a_N| > 0$ by assumption $a_n \neq 0$, we have $a_n \nrightarrow 0$. Thus, by the contrapositive version of Theorem 3.60, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges.

□

Actually, Root Test is always more accurate than the Ratio Test, but sometimes the Root Test is harder to evaluate. Further, the Ratio Test cannot be improved (more precisely, the obvious ways that it could be improved do not work).

If we change the condition for divergence to $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1$, this will fail. Consider the following example.

**Example 3.74.** Consider the following series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

or more directly, $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n^2}$. A fun fact is that the series converges to $\frac{\pi^2}{6}$ (just ignore it for now).

The upper and lower limits using Root Test are

$$\liminf_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} \cdot \frac{1}{\sqrt[n]{n}} = 1.$$

Note that if the limit exists, then its limit superior and limit inferior are equal.

The upper and lower limits using Ratio Test are

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1.$$

By the hypothesis, the series diverges, but it doesn't.

Both of Root Test and Ratio Test give no information. To prove the series diverges, we use the Comparison Test. We note that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \\
&\leq 1 + \sum_{n=2}^{\infty} \frac{1}{n^2 - n} \\
&= 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \to 2.
\end{aligned}$$

However, this is not the most accurate bound, at least we prove the convergence.

If we change the condition for divergence to $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \geq 1$, this will fail. Consider the following example.

**Example 3.75.** Consider the following series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

or more specifically, $\sum_{n=1}^{\infty} a_n$ where

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{(n+1)/2} & n \text{ is odd} \\ \left(\frac{1}{3}\right)^{n/2} & n \text{ is even} \end{cases}.$$

The series converges to $\frac{3}{2}$ (the sum of two geometric series).

The upper and lower limits using Toot test are

$$\begin{aligned} \liminf_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}} \\ \limsup_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

The Root Test indicates convergence.

However, if we use Ratio Test, we get

$$\lim_{n\to\infty}\frac{a_{2n+1}}{a_{2n}} = \lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{3}{2}\right)^n = +\infty,$$

so $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$. The Ratio Test gives no information, so in the possible improvement stated before this example, the hypothesis is true but the conclusion is false.

With these two examples, we notice that both tests will malfunction sometimes, but both of improved versions of the Ratio Test do not give the correct result. In general, the Root Test is powerful than the Ratio Test, but sometimes the Root Test is harder to evaluate. The reason is given in the following theorem.

**Theorem 3.76.** *For any sequence $(c_n)$ such that $c_n > 0$ for all $n \in \mathbb{N}$,*

$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n} \underbrace{\leq}_{(a)} \liminf_{n\to\infty}\sqrt[n]{c_n} \underbrace{\leq}_{(b)} \limsup_{n\to\infty}\sqrt[n]{c_n} \underbrace{\leq}_{(c)} \limsup_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

*Proof.* (b) is immediate. We will prove (c), while (a) is similar.

Let $\alpha = \limsup_{n\to\infty}\frac{c_{n+1}}{c_n}$. If $\alpha = +\infty$, then we are done.

So, assume $\alpha < +\infty$, that means $\alpha$ is finite. Pick a $\beta > \alpha$. By Theorem 3.48, there exists an $N \in \mathbb{N}$ such that $\frac{c_{n+1}}{c_n} < \beta$ for all $n \geq N$. Similar to the proof of the Ratio Test, we can get a recurrence relation

$c_{N+p} < \beta^p \cdot c_N$. Let $n = N + p$, then $p = n - N$, so we have $c_n < \beta^{n-N} \cdot c_N$. Taking the $n$th root on both sides, we get $\sqrt[n]{c_n} < \beta^{1-N/n} \cdot \sqrt[n]{c_N} = \beta \cdot \sqrt[n]{\beta^{-N} \cdot c_N}$, so $\limsup_{n\to\infty} \sqrt[n]{a_n} \le \beta$. This is true for all $\beta > \alpha$. In other words, let $\beta = \alpha + \varepsilon$ for all $\varepsilon > 0$, we get

$$\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha + \varepsilon.$$

This is true for all $\varepsilon > 0$, so this implies that $\limsup_{n\to\infty} \sqrt[n]{c_n} \le \alpha$ (by contradiction). We conclude that $\limsup_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$.

$\square$

Lecture 18 Monday November 16

**Power Series**

**Definition 3.77** (Power Series)**.** Given a sequence $(c_n)$ of complex numbers and $z \in \mathbb{C}$, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a **power series**.

This is a fairly natural generalization of the polynomial, but whether it actually makes sense as a quantity depends on the convergence of the series. Convergence depends on the value of $z$.

**Theorem 3.78** (Convergence for Power Series)**.** *For a power series $\sum_{n=0}^{\infty} c_n z^n$, set $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R = \frac{1}{\alpha}$ (if $\alpha = 0$, set $R = +\infty$; if $\alpha = +\infty$, set $R = 0$). Then, $\sum_{n=0}^{\infty} c_n z^n$ converges when $|z| < R$ and diverges when $|z| > R$. Further, $R$ is called the* ***radius of convergence****.*

*Proof.* Apply the Root Test, we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z| \cdot \alpha = \frac{|z|}{R}.$$

$\square$

Note that we haven't said anything about what happens when $|z| = R$. In that case, it's difficult to say anything without considering the particular power series at hand.

**Example 3.79.**

(a) For series $\sum_{n=0}^{\infty} z_n$, we have $R = 1$. It diverges when $|z| = 1$ since $z^n \not\to 0$. On the other hand, for series $\sum_{n=0}^{\infty} n^n z^n$, we have $R = 0$.

(b) For series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, we have $R = +\infty$. In this case, the Ratio Test is easier to apply than the Root Test.

(c) For series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$, we have $R = 1$. It converges for all $z$ with $|z| = 1$, by the Comparison Test, since $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$.

(d) For series $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n}$, we have $R = 1$. It diverges when $z = 1$. It converges for $|z| = 1$ but $z \neq 1$. (The last statement will be proved later.)

**Summation by Parts**

The following theorem is interesting since it provides a discrete "version" (analogue) of integration by parts,

$$\int_a^b f\,dg = [f \cdot g]_a^b - \int_a^b g\,df.$$

**Theorem 3.80** (Summation by Parts). *For sequences $(a_n)$ and $(b_n)$, define $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$. Then, for $0 \leq p \leq q$,*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

*Proof.* This is a direct algebraic proof. We rewrite the left-hand side

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \qquad \text{Note the shift of index} \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.\end{aligned}$$

□

Indeed, in this discrete case, $A_n$ is like the function $dg$, the original function of $a_n$, which is like $g\,dx$. We note that $b_n$ is like $f$, so that $b_{n+1} - b_n$ is somewhat like $df$.

The motivation of summation by parts is the observation that convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ does not imply $\sum_{n=1}^{\infty} a_n b_n$ converges. Here I index from $n = 1$, since the sequence defined in the counterexample is usually undefined as $n = 0$. I will provide a counterexample later.

**Theorem 3.81** (Dirichlet's Test). *If the partial sums of $\sum_{n=0}^{\infty} a_n$ are bounded and $b_0 \geq b_1 \geq b_2 \geq \cdots \geq 0$ with $\lim_{n\to\infty} b_m = 0$, then $\sum_{n=0}^{\infty} a_n b_n$ converges.*

*Proof.* Define $A_{-1} = 0$ and $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$ as in Theorem 3.80. We pick an $M \in \mathbb{R}$ such that for all

$n \in \mathbb{N}$, we have $|A_n| \le M$. Fix an $\varepsilon > 0$ and pick an $N \in \mathbb{N}$ with $b_N < \dfrac{\varepsilon}{2M}$. For $q \ge p \ge N$,

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\le \sum_{n=p}^{q-1} |A_n| \cdot (b_n - b_{n+1}) + |A_q| \cdot b_q + |A_{p-1}| \cdot b_p \\ &\le M \cdot \left[\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right] \\ &= M \cdot (b_p - b_q + b_q + b_p) \\ &= 2M \cdot b_p \le 2M \cdot b_N < \varepsilon.\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} a_n b_n$ converges by Cauchy criterion.

□

Monotone increasing $(b_n)$ also works in the *Dirichlet Test*.

**Theorem 3.82** (Alternating Series Test)**.** *Suppose* $|c_1| \ge |c_2| \ge \cdots$, $c_{2m-1} \ge 0$, *and* $c_{2m} \le 0$ *for* $m \ge 1$ *(opposite is also fine) and* $\lim_{n\to\infty} c_n = 0$. *Then,* $\sum_{n=1}^{\infty} c_n$ *converges.*

*Proof.* Apply Theorem 3.81 with $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

□

Consider $a_n = b_n = (-1)^n \dfrac{1}{\sqrt{n}}$. By Theorem 3.82, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. However, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \dfrac{1}{n}$ is the harmonic series, which diverges.

**Theorem 3.83.** *Suppose* $\sum_{n=1}^{\infty} c_n z^n$ *has radius of convergence* $1$, $c_0 \ge c_1 \ge \cdots$ *and* $\lim_{n\to\infty} c_n = 0$. *Then,* $\sum_{n=1}^{\infty} c_n z^n$ *converges for all* $z$ *with* $|z| = 1$ *except possibly* $z = 1$.

*Proof.* Apply Theorem 3.81 with $a_n = z^n$ and $b_n = c_n$. We note that if $|z| = 1$ and $z \ne 1$, then

$$\left|\sum_{k=0}^{n} a_k\right| = \left|\sum_{k=0}^{n} z^k\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \le \frac{2}{|1 - z|}.$$

□

The Theorem 3.82 is pretty remarkable, because it's a rare case in which convergence is not dependent upon the rate at which the terms of the series converge to 0.

### Absolute Convergence

**Definition 3.84** (Absolute Convergence). The series $\sum_{n=0}^{\infty} a_n$ **converges absolutely** if $\sum_{n=0}^{\infty} |a_n|$ converges. If $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ dieverges, we say $\sum_{n=0}^{\infty} a_n$ **converges non-absolutely**.

**Proposition 3.85** (Absolute Convergence Implies Convergence). *If $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it converges.*

*Proof.* By Cauchy criterion,

$$\left|\sum_{n=p}^{q} a_n\right| \leq \sum_{n=p}^{q} |a_n|$$

and the right-hand side gets arbitrarily small for sufficiently large $n$ because $\sum_{n=0}^{\infty} |a_n|$ converges.

□

The converse to this theorem is not true. There exist series which are convergent non-absolutely. Recall the harmonic series, we modify to an alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is convergent (but it is not convergent absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by *Cauchy condensation*).

*Remark* 3.86. For series of positive terms, absolute convergence is the same as convergence.

The Comparison Test, Root Test, and Ratio Test demonstrate absolute convergence, and therefore cannot give any information about non-absolutely convergent series.

Lecture 19
Wednesday
November 18

### Addition and Multiplication of Series

Now, we get to some operations on series which are seemingly safe but in fact require the condition of absolute convergence in order to be safe.

**Theorem 3.87.** *If $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, then $\sum_{n=0}^{\infty} (a_n + b_n) = A + B$ and $\sum_{n=0}^{\infty} c \cdot a_n = c \cdot A$ for any fixed $c \in \mathbb{C}$.*

*Proof.* Define $A_n = \sum_{k=0}^{n} a_k$ and $B_n = \sum_{k=0}^{n} b_k$. Then,

$$A_n + B_n = \sum_{k=0}^{n} (a_k + b_k).$$

So,

$$A + B = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = \lim_{n\to\infty} (A_n + B_n) = \sum_{n=0}^{\infty} (a_n + b_n)$$

and

$$c \cdot A = c \cdot \lim_{n\to\infty} A_n = \lim_{n\to\infty} c \cdot A_n = \sum_{n\to\infty} c \cdot a_n.$$

□

Thus, defining addition and scalar multiplication of series are not so hard, and is as well-behaved as we desire to get. However, defining series multiplication is much less obvious. Consider the product

$$(a_0 + a_1 + a_2 + \cdots) \cdot (b_0 + b_1 + b_2 + \cdots).$$

We need to make sure all the terms hit each other and group them by sum of their indices. The result is

$$a_0b_0 + (a_1b_0 + a_0b_1) + (a_2b_0 + a_1b_1 + a_0b_2) + \cdots$$

This motivates the following definition.

**Definition 3.88** (Cauchy Product)**.** The **Cauchy Product** of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

for $n \geq 0$.

**Example 3.89.** Consider $a_n = b_n = (-1)^n \cdot \dfrac{1}{\sqrt{n+1}}$. By Theorem 3.82, both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge but not absolutely. Then,

$$\begin{aligned} |c_n| &= \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \\ &= \left| \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \right| \\ &\geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2n+2}{n+2} \end{aligned}$$

so $c_n \not\to 0$ and $\sum_{n=1}^{\infty} c_n$ diverges.

The last inequality comes from

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

Fortunately, things are nicer with the assumption of absolute convergence.

**Theorem 3.90** (Mertens)**.** *Suppose* $\sum_{n=0}^{\infty} a_n = A$ *and* $\sum_{n=0}^{\infty} b_n = B$ *with* $\sum_{n=0}^{\infty} a_n$ *converging absolutely. Let* $\sum_{n=0}^{\infty} c_n$ *be the Cauchy product. Then,* $\sum_{n=0}^{\infty} c_n = A \cdot B$.

*Proof.* Let

$$A_n = \sum_{k=0}^{n} a_k \qquad B_n = \sum_{k=0}^{n} b_k \qquad C_n = \sum_{k=0}^{n} c_k \qquad \beta_n = B_n - B$$

($\beta_n$ is defined similarly as the "error term"). Then,

$$\begin{aligned} C_n &= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + \cdots + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_nB + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0. \end{aligned}$$

Let

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0.$$

We want to show that $C_n \to AB$ as $n \to \infty$. Since $A_nB \to AB$ as $n \to \infty$ by Theorem 3.87, it suffices to show that $\gamma_n \to 0$ as $n \to \infty$. Let $\alpha = \sum_{n=0}^{\infty} |a_n|$. Fix an $\varepsilon > 0$. Since $\beta_n \to 0$ by the assumption $\sum_{n=0}^{\infty} b_n = B$, we can pick an $N \in \mathbb{N}$ such that $|\beta_n| < \varepsilon$ for all $n \geq N$. Then, for all $n \geq N$,

$$\begin{aligned}
|\gamma_n| &\leq |a_0||\beta_n| + |a_1||\beta_{n-1}| + \cdots + |a_{n-N}||\beta_N| + |a_{n-N+1}||\beta_{N-1} + \cdots + |a_n||\beta_0| \\
&< \varepsilon \cdot (|a_0| + |a_1| + \cdots + |a_{n-N}|) + |a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0| \\
&\leq \varepsilon \cdot \alpha + |a_{n-N+1}||\beta_{N-1} + \cdots + |a_n||\beta_0|.
\end{aligned}$$

For sufficiently large $n$, $|a_{n-N+1}||\beta_{N-1}| + \cdots + |a_n||\beta_0|$ will be less than $\varepsilon$ since it converges to 0. Thus, for sufficiently large $n$, $|\gamma_n| < \varepsilon \cdot (\alpha + 1)$. Hence, $\gamma_n \to 0$ as $n \to \infty$.

□

**Theorem 3.91** (Abel's Test)**.** *If* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, *and* $\sum_{n=0}^{\infty} c_n$ *converge to* $A$, $B$, *and* $C$, *where* $\sum_{n=0}^{\infty} c_n$ *is the Cauchy Product, then* $\sum_{n=0}^{\infty} c_n$ *converges and* $C = A \cdot B$.

Here no assumption is made concerning absolute convergence. We will give a simple proof, which depends on the continuity of power series, in the future 140B.

### Rearrangements

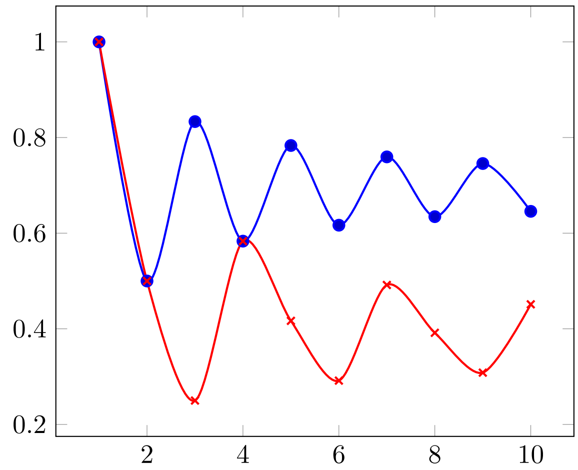
One feature of finite sums is that no matter how we rearrange the terms in a sequence, the total sum is the same. However, it is complex for infinite series. If all the terms are non-negative, then rearrangement does not alter the summation. For alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots,$$

the rearrangement gives

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots \\
=&\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \cdots \\
=&\frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right).
\end{aligned}$$

Thus, it is not always same for the alternating series.




This is the visualization of first 10 terms of the summation in the original order (blue curve) and in the order we rearrange (red curve). This enlightens us to explore the *rearrangement*. We first rigorously define what is a *rearrangement*.

**Definition 3.92** (Rearrangement)**.** If $(k_n)$ is a sequence in $\mathbb{N}$ using each positive inter precisely once, and if $\sum_{n=1}^{\infty} a_n$ is a series and we set $a'_n = a_{k_n}$, then $\sum_{n=1}^{\infty} a'_n$ is called a **rearrangement** of $\sum_{n=1}^{\infty} a_n$.

"Using each positive integer precisely once" means that $k_n : \mathbb{N} \to \mathbb{N}$ is a one-to-one and onto map. We simply sum the series in a different order. Does the summation in a different order matter a lot? Sometimes.

Now, we explore the behavior of rearrangement for convergent series. We split convergence into cases: absolute convergence and non-absolute convergence.

**Theorem 3.93.** *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then every rearrangement converges to the same value.*

*Proof.* Let $(k_n)$ be a sequence in $\mathbb{N}$ using each natural number precisely once. Let $\sum_{n=1}^{\infty} a'_n = \sum_{n=1}^{\infty} a_{k_n}$ be a rearrangement and let $\sum_{n=1}^{\infty} a_n = s$, $s_n = \sum_{k=1}^{n} a_k$, and $s'_n = \sum_{k=1}^{n} a'_k$. Fix an $\varepsilon > 0$, since $\sum_{n=1}^{\infty}$ converges absolutely, so there exists an $N \in \mathbb{N}$ such that $\sum_{k=n}^{m} |a_k| < \frac{\varepsilon}{2}$ for $m \geq n \geq N$ and $|s_n - s| < \frac{\varepsilon}{2}$ for $n \geq N$.

Now, we pick a sufficiently large $p$ such that $\{1, 2, 3, \dots, N\} \subseteq \{k_1, k_2, k_3, \dots, k_p\}$. We note that $p \geq N$. Then, if $n \geq p$, $s_n - s'_n$ contains no terms $a_k$ where $k < N$. Thus,

$$|s_n - s'_n| \leq \sum_{k=l}^{m} |a_k| < \frac{\varepsilon}{2} \text{ for } m \leq l \leq p.$$

Hence,

$$|s'_n - s| \leq |s'_n - s_n| + |s_n - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq p$. We conclude that every rearrangement of an absolutely convergent series converges to the same value.

□

Lecture 20
Friday
November 20

In fact, Riemann proved the following general statement for non-absolute convergence.

**Theorem 3.94** (Riemann's Rearrangement)**.** *Suppose* $\sum_{n=1}^{\infty} a_n$ *converges non-absolutely and* $-\infty \leq \alpha \leq \beta \leq +\infty$*. Then, there is a rearrangement* $\sum_{n=1}^{\infty} a'_n$ *with partial sums* $s'_n$ *satisfying*

$$\liminf_{n\to\infty} s'_n = \alpha \qquad \limsup_{n\to\infty} s'_n = \beta.$$

*Proof.* Set

$$p_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad q_n = \begin{cases} -a_n & \text{if } a_n \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

We note that $p_n \geq 0$ and $q_n \geq 0$ and $a_n = p_n - q_n$. We first need to prove that $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverges.

Towards a contradiction, assume $\sum_{n=1}^{\infty} p_n$ diverges. Then, $\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty}(p_n - a_n)$ will converge and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty}(p_n + q_n)$ will converge, which contradicts to the definition of non-absolute convergence. We conclude that $\sum_{n=1}^{\infty} p_n$ diverges. Similarly, $\sum_{n=1}^{\infty} q_n$ diverges.

Now, let $P_1, P_2, P_3, \ldots$ be the nonnegative terms from $a_1, a_2, a_3, \ldots$, in the order in which they order. Also, let $Q_1, Q_2, Q_3, \ldots$ be the absolute value of the strictly negative terms from $a_1, a_2, a_3, \ldots$, also in their original order.

Then, the series $\sum_{n=1}^{\infty} P_n$ adn $\sum_{n=1}^{\infty} Q_n$ only differ from $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ by zero terms. Thus, $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} Q_n$ diverge.

We pick real-valued sequences $(\alpha_n)$ and $(\beta_n)$ such that $\beta_1 > 0$, $\alpha_n < \beta_n$, $\alpha_n \to \alpha$, and $\beta_n \to \beta$. For example, $\beta_1 = |\beta| + \frac{1}{2}$, $\beta_n = |\beta| + \frac{1}{2^n}$, and $\alpha_n = \alpha - \frac{1}{2^n}$ when $\alpha, \beta \in \mathbb{R}$.

Let $m_1, k_1 \in \mathbb{Z}_+$ be *least* with

$$P_1 + P_2 + \cdots + P_{m_1} > \beta_1$$
$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} < \alpha_1.$$

Let $m_2, k_2 \in \mathbb{Z}_+$ be *least* with

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2$$
$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots Q_{k_2} < \alpha_2.$$

We proceed inductively, let $m_n, k_n \in \mathbb{Z}_+$ be *least* with

$$\begin{aligned} x_n &= P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} \\ &\quad + P_{m_1+1} + \cdots - Q_{k_{n-1}} + P_{m_{n-1}+1} + \cdots + P_{m_n} > \beta_n \\ y_n &= P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} \\ &\quad + P_{m_1+1} + \cdots + P_{m_n} - Q_{k_{n-1}+1} - \cdots - Q_{k_n} < \alpha_n. \end{aligned}$$

Then, $|x_n - \beta_n| < P_{m_n}$ adn $|y_n - \alpha_n| < Q_{k_n}$.

Since $\beta_n \to \beta$, $P_n \to 0$, we have $x_n \to \beta$. Since $\alpha_n \to \alpha$, $Q_n \to 0$, we have $y_n \to \alpha$. We still need to prove that there is no element less than $\alpha$ or greater tham $\beta$ can be a subsequential limit of the partial sums.

We first note that partial sums are increasing from $y_{n-1}$ to $x)n$ and decreasing from $x_n$ to $y_n$. Let $\beta' > \beta$. Then, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\beta_n + |P_n| < \beta'$ since $\beta_n \to \beta$ adn $P_n \to 0$. So, if $s'_n$ is a partial sum between $y_{n-1}$ and $x_n$ with $n \geq N$, then for $m_n \geq n \geq N$,

$$s'_n \leq x_n < \beta_n + P_{m_n} < \beta'.$$

So, eventually all partial sums are strictly less than $\beta'$. So, $\beta = \limsup_{n\to\infty} s_n$ by Theorem 3.48. Similar arguments work for $\liminf$.

We conclude that there exists a rearrangement $\sum_{n=1}^{\infty}$ with partial sums $s'_n$ such that $\liminf_{n\to\infty} s'_n = \alpha$ and $\limsup_{n\to\infty} s'_n = \beta$ for $-\infty \leq \alpha \leq \beta \leq +\infty$.

□

To summarize, rearranging series is safe when the series is *absolutely convergent*, but is somewhat dangerous otherwise. A series which is *non-absolutely convergent* can in fact be rearranged to *any* value (or rearranged to diverge, note that we use the *extended real number system* in Theorem 3.94).

# 4 Limits and Continuity

Consider the function $f : A \to \mathbb{R}$. Recall that a limit point $c$ of $A$ is a point with the property that $B_r(c)$ with $r > 0$ intersects $A$ in some point other than $c$. Equivalently, $c$ is a limit point of $A$ if and only if $c = \lim_{n\to\infty} x_n$ for some sequence $(x_n)$ in $A$ with $x_n \neq c$. Limit points of $A$ do not necessarily belong to the set $A$ unless $A$ is closed.

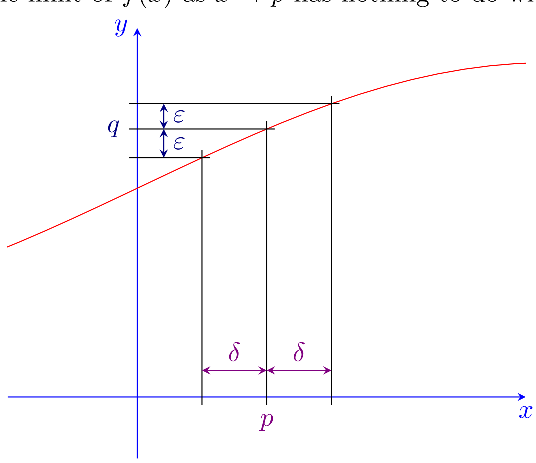
Now, we will get into a more abstract setting. We will define the limit and continuity in more general metric spaces.

## 4.1 Limits of Functions

**Definition 4.1** (Limit)**.** Let $(X, d_X)$ and $(Y, d_Y)$ are metric spaces. Suppose $E \subseteq X$, $f : E \to Y$, and $p$ is a limit point of $E$. For a point $q \in Y$, we say the **limit of** $f$ **at** $p$ **is** $q$ and write "$f(x) \to q$ as $x \to p$" or $\lim_{x\to p} f(x) = q$ if

$$\text{for all } \varepsilon > 0, \text{ there exists a } \delta > 0, \text{ such that } 0 < d_X(x,p) < \delta \Longrightarrow d_Y(f(x), q) < \varepsilon.$$

It may be that $p \in X$ but $p \notin E$, so $f(p)$ is not defined. Even if $p \in E$, it can happen that $f(p) \neq \lim_{x\to p} f(x)$. In our definition, we say $0 < d_X(x,p) < \delta$. Since the distance is greater than zero by default, this implies that $x \neq p$. In other words, the limit of $f(x)$ as $x \to p$ has nothing to do with the value of $f$ at $x = p$.




This is a visualization of *limit* in the argument with $\delta$ and $\varepsilon$.

The following theorem connects the sequence and limit of a function together.

**Theorem 4.2** (Limit in a Sequential Perspective)**.** *Let $(X, d_X)$ and $(Y, d_Y)$ are metric spaces. Suppose $E \subseteq X$, $f : E \to Y$, and $p$ is a limit point of $E$. Then, $\lim_{x\to p} f(x) = q$ if and only if for all sequences $(p_n)$ in $E$ such that if $p_n \neq p$ and $p_n \to p$ for all $n \in \mathbb{N}$, then $f(p_n) \to q$.*

*Proof.* Assume $\lim_{x\to p} f(x) = q$. Let $(p_n)$ be a sequence in $E$ such that $p_n \neq p$ and $p_n \to p$ for all $n \in \mathbb{N}$. Fix an $\varepsilon > 0$ and pick a $\delta > 0$ such that for all $x \in E$, $0 < d_X(x,p) < \delta$ implies that $d_Y(f(x), q) < \varepsilon$. Since $p_n \to p$, there exists an $N \in \mathbb{N}$ such that $0 < d_X(p, p_n) < \delta$ for all $n \geq N$. Then, for all $n \geq N$, we have $d_Y(f(p_n), q) < \varepsilon$. Thus, $f(p_n) \to q$.

Conversely, it is weird and hard to prove directly since the condition "for all sequences $(p_n)$ in $E$" is hard to use, so we prove by contrapositive. Assume $f(x) \nrightarrow q$ as $x \to p$. Then, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists a point $x \in E$ such that $0 < d_X(x,p) < \delta$ and $d_Y(f(x),q) > \varepsilon$. Define $\delta_n = \dfrac{1}{n}$ for each $n \geq 1$, we can obtain $p_n \in E$ such that $0 < d_X(p_n,p) < \dfrac{1}{n}$ and $d_Y(f(p_n),p) \geq \varepsilon$. Then, $p_n \to p$ but $f(p_n)$ does not converge to $f(p)$.

□

**Corollary 4.3** (Uniqueness of Limit)**.** *If $f$ has a limit at $p$, then the limit is unique.*

*Proof.* Since the limit of a sequence is unique, then so is the limit of function by Theorem 4.2.

□

We need to define some operations on multiple functions. The definitions of addition, multiplication, and division are what we expected.

**Definition 4.4.** Let $E \subseteq X$. Let $f, g : E \to \mathbb{C}$. We define new functions

(a) $(f+g)(x) = f(x) + g(x)$.

(b) $(f \cdot g)(x) = f(x) \cdot g(x)$.

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ when $g(x) \neq 0$.

If $f, g : E \to \mathbb{R}$, we write $f \leq g$ if for any point $x \in E$, we have $f(x) \leq g(x)$.

Similarly, if $f, g : E \to \mathbb{R}^k$, we define

(a) $(\vec{f} + \vec{g})(x) = \vec{f}(x) + \vec{g}(x)$.

(b) $(\vec{f} \cdot \vec{g})(x) = \vec{f}(x) \cdot \vec{g}(x)$.

(c) for all $\lambda \in \mathbb{R}$, $(\lambda \vec{f})(x) = \lambda \cdot \vec{f}(x)$.

**Theorem 4.5** (Limit Rules)**.** *Let $(X,d)$ be a metric space, $E \subseteq X$, and $p$ is a limit point of $E$. Let $f, g : E \to \mathbb{C}$ and $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$. Then*

*(a)* $\lim_{x \to p} (f+g)(x) = A + B$.

*(b)* $\lim_{x \to p} (f \cdot g)(x) = A \cdot B$.

*(c)* $\lim_{x \to p} \left(\dfrac{f}{g}\right)(x) = \dfrac{A}{B}$ *if* $B \neq 0$.

*Similarly, if $\vec{f}, \vec{g} : E \to \mathbb{R}^k$ and $\lim_{x \to p} \vec{f}(x) = \vec{A}$ and $\lim_{x \to p} \vec{g}(x) = \vec{B}$, then*

*(a)* $\lim_{x \to p} (\vec{f} + \vec{g})(x) = \vec{A} + \vec{B}$.

*(b)* $\lim_{x\to p}(\vec{f}\cdot\vec{g})(x) = \vec{A}\cdot\vec{B}$.

*Proof.* This follows from Theorem 3.6, Theorem 3.8, and Theorem 4.2.

$\square$

Lecture 22
Monday
November 30

## 4.2 Continuous Functions

**Definition 4.6** (Continuous Functions)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $E \subseteq X$. Let $f : E \to Y$. We say $f$ is **continuous** at $p \in E$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in E$,

$$d_X(x, p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon.$$

Furthermore, if $f$ is continuous at every $p \in E$, then we say $f$ is **continuous on** $E$ or **continuous** in short.

Intuitively, this means that we can restrict our output by restricting input.

Compare this with the definition of *limit* of a function, we see that $f$ has to be defined at point $p$ in order to be continuous at $p$.

**Theorem 4.7.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $E \subseteq X$. If $p \in E \setminus E'$ (this means $p$ is an isolated point), then every function $f : E \to Y$ is continuous at $p$. If $p \in E \cap E'$, then $f : E \to Y$ is continuous at $p$ if and only if $\lim_{x\to p} f(x) = f(p)$.*

*Proof.* If $p \in E \setminus E'$, then there is a $\delta > 0$ with $(B_\delta(p) \setminus \{p\}) \cap E = \varnothing$, so for all $x \in E$, $d_X(x, p) < \delta$ implies $x = p$, then $f(x) = f(p)$ means $d_Y(f(x), f(p)) = 0 < \varepsilon$ for all $\varepsilon > 0$. Thus, this $\delta$ works for all $\varepsilon > 0$.

The second statement is immediate from Definition 4.6.

$\square$

We now turn into composition of functions. Continuous functions have a nice property that a continuous function of a continuous function is continuous.

**Theorem 4.8.** *Suppose $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ are metric spaces and $E_X \subseteq X$ and $E_Y \subseteq Y$. Let $f : E_X \to E_Y$ and $g : E_Y \to Z$. Define $h : E \to Z$ by $h(p) = g(f(p))$. If $f$ is continuous at $p$ and $g$ is continuous at $f(p)$, then $h$ is continuous at $p$.*

*Proof.* Let $\varepsilon > 0$. Since $g$ is continuous at $f(p)$, so there exists a $r > 0$ such that for all $y \in E_Y$,

$$d_Y(y, f(p)) < r \Longrightarrow d_Z(g(y), g(f(p))) < \varepsilon.$$

Since $f$ is continuous at $p$, there is a $\delta > 0$ such that for all $x \in E_X$,

$$d_X(x, p) < \delta \Longrightarrow d_Y(y, f(p)) < r.$$

If follows that for all $x \in E_X$,

$$d_X(x,p) < \delta \Longrightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon.$$

We conclude that $h$ is continuous at $x = p$.

$\square$

Before diving into more nice interpretations and properties of continuity, we first introduce the inverse mapping.

**Definition 4.9** (Image)**.** Given a mapping $f : X \to Y$, consider the set $E \subseteq X$. The **image** of $E$ is defined as

$$f(E) = \{f(x) : x \in E\}.$$

**Definition 4.10** (Inverse Image)**.** Given a mapping $f : X \to Y$, an inverse mapping $f^{-1} : Y \to X$. Consider a set $C \in Y$. Then the **inverse image** of $C$ is defined as

$$f^{-1}(C) = \{x \in X : f(x) \in C\}.$$

Using these two definitions, we can discover some interesting properties for the inverse mapping.

**Proposition 4.11.** *Let $f : X \to Y$, $A, B \subseteq X$, and $C, D \subseteq Y$.*

*(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.*

*(b) $A \subseteq f^{-1}(f(A))$. (This is an important and interesting property. You can construct a function to see what happens.)*

*(c) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.*

*(d) $f(f^{-1}(C)) \subseteq C$. (This is also interesting.)*

This is true for any function $f$. We can use these propositions to explore the interaction between continuity and topology.

**Theorem 4.12.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. The mapping $f : X \to Y$ is on $X$ if and only if for all open sets $V \subseteq Y$, $f^{-1}(V) \subseteq X$ is open.*

*Proof.* Assume $f$ is continuous and let $V \subseteq Y$ be open. Let $p \in f^{-1}(V)$. Then, $f(p) \in V$. Since $V$ is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(p)) \subseteq V$, which means that for all $y \in Y$, $d_Y(y, f(p)) < \varepsilon \Longrightarrow y \in V$. Since $f$ is continuous, there exists a $\delta > 0$ such that for all $x \in X$, $d_X(x,p) < \delta \Longrightarrow d_Y(f(x), f(p)) < \varepsilon$. It follows that $f(B_\varepsilon(p)) \subseteq V$, which means that $B_\varepsilon(p) \subseteq f^{-1}(p)$. Thus, $f^{-1}(V)$ is open.

Assume $f^{-1}(V)$ is open for all open sets $V \subseteq Y$. Fix a point $p \in X$ and let $\varepsilon > 0$. Set $V = B_\varepsilon(f(p))$. Then, $V$ is open so $f^{-1}(V)$ is open. Since $p \in f^{-1}(V)$, there exists a $\delta > 0$ with $B_\delta(p) \subseteq f^{-1}(V)$. So, if

$x \in X$ satisfies $d_X(x,p) < \delta$, then $x \in B_\delta(p) \subseteq f^{-1}(V)$, so $f(x) \in V = B_\varepsilon(f(p))$ and thus $d_Y(f(x), f(p)) < \varepsilon$. We conclude that $f$ is continuous.

□

A more general result is the following.

**Proposition 4.13.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $E \subseteq X$. The mapping $f : E \to Y$ is on $X$ if and only if for all open sets $V \subseteq Y$, $f^{-1}(V)$ is relatively open in $E$.*

If we switch the inverse map to forward map, then the statement becomes: if $f : X \to Y$ is continuous and $V \subseteq X$ is open, then $f(V)$ is open in $Y$. This is false. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. We take $V = (-1, 1)$, then $f(V) = [0, 1)$, which is not open.

*Proof.* Consider $E$ as a metric space in its own right, and apply Theorem 4.12.

□

**Corollary 4.14.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. The mapping $f : X \to Y$ is on $X$ if and only if for all closed sets $C \subseteq Y$, $f^{-1}(C)$ is closed.*

*Proof.* This follows from Theorem 4.12 together with duality between open and closed sets and the fact that for all sets $D \subseteq Y$, $f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)$.

□

Also, the closedness does not preserve as we switch from the inverse image. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \dfrac{1}{1+x^2}$. Then $C = [0, +\infty)$ is closed in $\mathbb{R}$, but $f(C) = (0, 1]$ is not closed in $\mathbb{R}$.

We can now use the fact that the composite of continuous functions is continuous as a powerful tool to generate new continuous functions from previously known ones.

**Theorem 4.15** (Continuous Complex-Valued Functions)**.** *Let $f, g : X \to \mathbb{C}$ be continuous functions. Then, so are $f + g$, $f \cdot g$, and $\dfrac{f}{g}$ (if for all $x \in X$, $g(x) \neq 0$).*

*Proof.* At isolated points, there is nothing to prove.

At limit points, this follows from Theorem 4.5 and Theorem 4.7.

□

**Theorem 4.16** (Continuous Vector-Valued Functions)**.**

*(a) Let $f_1, f_2, \dots, f_k : X \to \mathbb{R}$ and define $\vec{f} : X \to \mathbb{R}^k$ by*

$$\vec{f}(x) = (f_1(x), f_2(x), \dots, f_k(x)) .$$

*Then $\vec{f}$ is continuous if and only if $f_i$ is continous for $1 \leq i \leq k$.*

*(b) If $\vec{f}, \vec{g} : X \to \mathbb{R}^k$ are continuous, then so are $\vec{f} + \vec{g}$ and $\vec{f} \cdot \vec{g}$.*

*Proof.*

(a) This follows from Theorem 3.8, Theorem 4.2, and Theorem 4.7

(b) This follows from Theorem 4.5 and Theorem 4.7.

□

Lecture 23 Wednesday December 2

For $1 \leq i \leq k$, the mapping from $\mathbb{R}^k$ to $\mathbb{R}$ given by $\vec{x} = (x_1, x_2, \ldots, x_k) \mapsto x_i$ is continuous on $\mathbb{R}^k$. Then, for $n_1, n_2, \ldots, n_k \in \mathbb{N}$, $(x_1, x_2, \ldots, x_k) \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is continuous on $\mathbb{R}^k$. So polynomials given by $p(\vec{x}) = \sum c_{n_1,n_2,\ldots,n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ (where $c_{n_1,n_2,\ldots,n_k} \in \mathbb{C}$ are fixed and all but finitely many are 0) are continuous. Additionally, rational functions $\dfrac{P(\vec{x})}{Q(\vec{x})}$ ($P$ and $Q$ are polynomials) are continuous on their domain.

From the triangle inequality, we can show that $||\vec{x}| - |\vec{y}|| \leq |\vec{x} - \vec{y}|$. Hence, the mapping $\vec{x} \mapsto |\vec{x}|$ is continuous.

## 4.3 Continuity and Topology

We have touched with limit perspective, sequence perspective, and a bit topological perspective (in open and closed sets) of the continuity. Now, we continue with continuity and compactness.

**Definition 4.17** (Boundedness)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is **bounded** if there is a $q \in Y$ and $M > 0$ with $f(X) \subseteq B_M(q)$.

### Continuity and Compactness

**Theorem 4.18.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. If $f : X \to Y$ is continuous and $X$ is compact, then $f(X)$ is compact.*

*Proof.* Let $\{V_\alpha : \alpha \in A\}$ be an open cover of $f(X)$. Since $f$ is continuous, by Theorem 4.12, each of the sets $f^{-1}(V_\alpha)$ is open. Since $X$ is compact, so there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$. Then, we have

$$f(X) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})\right) = \bigcup_{i=1}^{n} f\left(f^{-1}(V_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}.$$

We conclude $f(X)$ is compact.

□

This theorem says that if $f$ is a continuous mapping, while it is possible to map the open interval $(a, b)$ into $\mathbb{R}$, it is impossible to map a closed interval $[a, b]$ onto the entire $\mathbb{R}$ since the mapping of a compact set is also compact because of the continuity of function $f$. Compact sets are "small" in some sense. Hence, if $f$ is continuous, then if it takes in a "small" input (closed interval), then the output must also be relatively small.

In the proof for Theorem 4.18, we have used the relation $f\left(f^{-1}(E)\right) \subseteq E$, valid for $E \subseteq Y$. If $E \subseteq X$, then $f^{-1}(f(E)) \supseteq E$. Equality need not hold in either case.

Unlike open and closed sets, compactness is preserved in the "forward" direction of a function, instead of the inverse one. It is still not true that if $f : X \to Y$ is continuosu and $K \subseteq Y$ is compact then $f^{-1}(K)$ is compact. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \dfrac{1}{1+x^2}$. Then $K = [0, 1]$ is compact, but $f^{-1}(K) = \mathbb{R}$ since for all $x$, we have $0 < f(x) \leq 1$, so $f(x) \in K$. Clearly $f^{-1}(K) = \mathbb{R}$ is not compact.

**Theorem 4.19.** *If $f : X \to \mathbb{R}^k$ is continuous and $X$ is compact, then $f(X)$ is closed and bounded.*

*Proof.* This follows from Theorem 4.18 and the Heine-Borel Theorem.

□

**Theorem 4.20.** *Let $(X, d_X)$ be a compact metric space. Let $f : X \to \mathbb{R}$. Set $M = \sup_{x\in X} f(x)$ and $m = \inf_{x\in X} f(x)$. Then, there are $p, q \in X$ with $f(p) = M$ and $f(q) = m$.*

*Proof.* Since $f(X)$ is compact in $\mathbb{R}$, so it is closed and bounded, which means $M = \sup_{x\in X} f(x)$ and $m = \inf_{x\in X} f(x)$ exist. Since $f(X)$ is closed, then $\sup_{x\in X} f(x) \in f(X)$. Similarly, $\inf_{x\in X} f(x) \in f(X)$.

□

This theorem says that there exist $p, q \in X$ such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$, that is, $f$ attains its maximum at $p$ and $f$ attains its minimum at $q$.

**Theorem 4.21.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f : X \to Y$. If $X$ is compact and $f$ is a continuous bijection, then $f^{-1} : Y \to X$ is continuous.*

*Proof.* Since $\left(f^{-1}\right)^{-1} = f$, then Corollary 4.14 tells us that $f^{-1}$ is continuous if and only if $f(C)$ is closed for all closed sets $C \subseteq X$. Let $C \subseteq X$ be closed. Then, $C$ is compact, so by Theorem 4.18, $f(C)$ is compact, hence $f(C)$ is closed. Thus, we conclude that $f^{-1}$ is continuous.

□

### Uniform Continuity

By Definition 4.6, we can easily prove that the functions $f(x) = 3x + 1$ and $g(x) = x^2$ are everywhere continuous. Are there differences in these two functions?

**Definition 4.22** (Uniform Continuity)**.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$. We say $f$ is **uniformly continuous** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(p), f(q)) < \varepsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Recall that to say "$f$ is continuous at $X$" means that $f$ is continuous at each individual point $x \in X$. Uniform continuity is a strictly stronger property. The key distinction between asserting that $f$ is "uniformly continuous on $X$" versus simply "continuous on $X$" is that, given an $\varepsilon > 0$, a single $\delta > 0$ can be chosen that works simultaneously for all points $x \in X$. To say that a function is *not* uniformly continuous on a set $X$, then, does not necessarily mean it is not continuous at some point. Rather, it means that there is some $\varepsilon' > 0$ for which no single $\delta > 0$ is a suitable response for all $x \in X$.

**Example 4.23.** The function $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

*Proof.* Let $\varepsilon = 2$. We pick an arbitrary $\delta > 0$. Let $n_\delta \in \mathbb{N}$ such that $\frac{1}{n_\delta} < \delta$. Further, let $x_\delta = n_\delta + \frac{1}{n_\delta}$ and $y_\delta = n_\delta$. Then, $|x_\delta - y_\delta| = \frac{1}{n_\delta} < \delta$ while

$$f(x_\delta) - f(y_\delta) = \left(n_\delta + \frac{1}{n_\delta}\right)^2 - n_\delta^2 = 2 + \frac{1}{n_\delta^2} > \varepsilon.$$

We conclude that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

□

That's the difference between the continuity of $f9x) = 3x + 1$ and $f(x) = x^2$. We can easily check that linear function is *uniformly continuous.*

If we don't consider continuity on whole $\mathbb{R}$, can $f(x) = x^2$ be uniformly continuous sometimes?

Indeed, the function $f(x) = x^2$ is uniformly continuous on any bounded interval $[a, b]$.

For any $x, y \in [a, b]$, we get

$$\begin{aligned} |x^2 - y^2| &= |x + y||x - y| \\ &\leq (|x| + |y|)|x - y| \\ &\leq 2\max\{|a|, |b|\}|x - y|. \end{aligned}$$

Evidently, every uniformly continuous function is continuous. Is it possible that continuity and uniform continuity are same in context? Two concepts are equivalent on the compact sets.

**Theorem 4.24.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$. If $f$ is continuous on $X$ and $X$ is compact, then $f$ is uniformly continuous on $X$.*

*Proof.* Let $\varepsilon > 0$. Since $f$ is continuous, for each $p \in X$, we can pick $\delta_p > 0$ such that for all $q \in X$, $d_X(p, q) < \delta_p \implies d_Y(f(p), f(q)) < \varepsilon$. Set $V_p = B_{\delta_p/2}(p)$.

**Claim 4.24.1.** *If $q \in V_p$, $x \in X$, and $d_X(x, q) < \frac{1}{2}\delta_p$, then $d_Y(f(x), f(q)) < \varepsilon$.*

*Proof of Claim 4.24.1.* Since $q \in V_p$, then $d_X(p, q) < \frac{1}{2}\delta_p$, and $d_X(x, q) < \frac{1}{2}\delta_p$, by triangle inequality, we have

$$d_X(p, x) \leq d_X(p, q) + d_X(q, x) < \frac{\delta_p}{2} + \frac{\delta_p}{2} = \delta_p$$

and $d_X(p, q) < \frac{1}{2}\delta_p < \delta_p$ so

$$d_Y(f(p), f(q)) < \frac{\varepsilon}{2} \qquad d_Y(f(x), f(p)) < \frac{\varepsilon}{2}$$

and by triangle inequality again,

$$d_Y(f(x), f(q)) \leq d_Y(f(x), f(p)) + d_Y(f(p), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Then, let $\{V_p : p \in X\}$ be an open cover of $X$. Since $X$ is compact, so there exists $p_1, p_2, \dots, p_n$ with $X \subseteq \bigcup_{i=1}^{n} V_{p_i}$. Set $\delta = \frac{1}{2}\{\delta_{p_1}, \delta_{p_2}, \dots, \delta_{p_n}\}$. Consider $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$. Since $X \subseteq \bigcup_{i=1}^{n} V_{p_i}$, there is $1 \leq i \leq n$ such that $x_1 \in V_{p_i}$. Now, Claim 4.24.1 implies $d_Y(f(x_1), f(x_2)) < \varepsilon$.

□

Lecture 24 Friday December 4

We now proceed to show that compactness is essential in the hypotheses of Theorem 4.18, Theorem 4.19, Theorem 4.20, and Theorem 4.24.

**Theorem 4.25.** *Let $E \subseteq \mathbb{R}$ be a non-compact set. Then,*

*(a) there exists a continuous function $f : E \to \mathbb{R}$ but it is not bounded;*

*(b) there exists a continuous and bounded function $f : E \to \mathbb{R}$ but it has no maximum.*

*(c) if in addition $E$ is bounded, then there exists a continuous function $f : E \to \mathbb{R}$ but it is not uniformly continuous.*

*Proof.* Assume that $E$ is bounded. By Heine-Borel Theorem, $E$ is not closed, so there exists a point $x_0 \in E' \setminus E$.

For (a) and (c), consider $f(x) = \dfrac{1}{x - x_0}$ for $x \in E$.

**Claim 4.25.1.** *$f$ is not bounded.*

*Proof of Claim 4.25.1.* Let $M > 0$. Since $x_0 \in E'$, we can fine an $x \in E$ such that $|x - x_0| < \dfrac{1}{M}$. For this $x$, we have

$$|f(x)| = \frac{1}{|x - x_0|} > M.$$

THus, $f$ is not bounded.

□

**Claim 4.25.2.** *$f$ is not uniformly continuous.*

*Proof of Claim 4.25.2.* Let $\varepsilon > 0$ and $\delta > 0$. First pick any $p \in E$ such that $|p - x_0| < \dfrac{\delta}{2}$. Since $f$ is not bounded on $\left(x_0 - \dfrac{\delta}{2}, x_0 + \dfrac{\delta}{2}\right) \cap E$, we can find $q \in E$ with $|q - x_0| < \dfrac{\delta}{2}$ and $|f(q)| > |f(p)| + \varepsilon$. Then,

$$|p - q| \leq |p - x_0| + |x_0 - q| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

but

$$|f(q) - f(p)| \geq |f(q)| - |f(p)| > \varepsilon$$

by construction. Thus, $f$ is not uniformly continuous.

$\square$

For (b), consider $g(x) = \dfrac{1}{1+(x-x_0)^2}$ for $x \in E$.

**Claim 4.25.3.** *$g$ is bounded and for all $x \in E$, $g(x) < 1$ but $\sup_{x\in E} g(x) = 1$.*

*Proof of Claim 4.25.3.* Clearly for all $x \in E$, we have $0 < g(x) < 1$ and $g$ is bounded. Let $\varepsilon > 0$. Pick an $x \in E$ with $|x - x_0| < \sqrt{\dfrac{1}{1-\varepsilon} - 1}$. For this $x$,

$$g(x) = \frac{1}{1+(x-x_0)^2} > \frac{1}{1+\dfrac{1}{1-\varepsilon}-1} = 1-\varepsilon,$$

which means $1-\varepsilon$ is not an upper bound of $g(x)$. Thus, $\sup_{x\in E} g(x) = 1$.

$\square$

Now, assume $E$ is not bounded.

For (a), set $h(x) = x$ for all $x \in E$.

For (b), set $s(x) = \dfrac{x^2}{1+x^2}$ for $x \in E$.

**Claim 4.25.4.** *$s$ is bounded and for all $x \in E$, $s(x) < 1$ and $\sup_{x\in E} s(x) = 1$.*

*Proof of Claim 4.25.4.* It is clear that for all $x \in E$, $0 \leq s(x) < 1$ and $s$ is bounded. Let $\varepsilon > 0$. Pick $x \in E$ such that $|x| > \sqrt{\dfrac{1}{\dfrac{1}{1-\varepsilon}-1}}$. For this $x$,

$$s(x) = \frac{x^2}{1+x^2} = \left(\frac{1}{x^2}+1\right)^{-1} > \left(\frac{1}{1-\varepsilon}-1+1\right)^{-1} = 1-\varepsilon,$$

which means $1-\varepsilon$ is not an upper bound of $s(x)$. Thus, $\sup_{x\in E} s(x) = 1$.

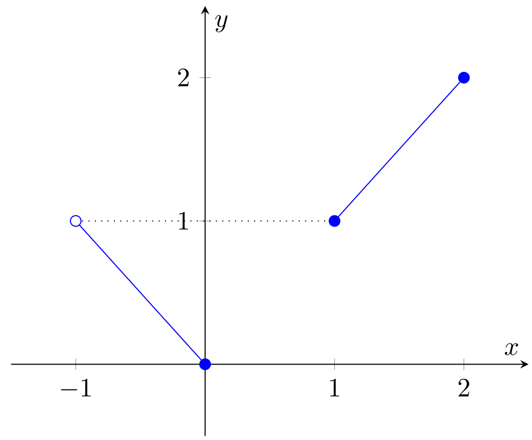
$\square$

$\square$

Theorem 4.25 (c) is not true if boundedness is not assumed. $\mathbb{Z}$ is not compact, but every function $f : \mathbb{Z} \to \mathbb{R}$ is uniformly continuous. To prove this, we take $\delta < 1$ in Definition 4.22.

Also, compactness is also essential in Theorem 4.21.

**Example 4.26.** Define $f : (-1, 0] \cup [1, 2] \to \mathbb{R}$ by $f(x) = |x|$. Then, $f$ is a continuous bijection but $f^{-1}$ is not continuous.




**Continuity and Connectedness**

Up to now, we have explored some nice properties in continuity and compactness, with a stronger version of continuity, uniform continuity. Now, we start to explore continuity in connectedness.

**Theorem 4.27.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$. If $E \subseteq X$ is connected and $f$ is continuous, then $f(E)$ is connected.*

*Proof.* We prove by contrapositive. Suppose $f(E)$ is not connected. Say, $A, B \subseteq Y$ are nonempty and separated and $A \cup B = f(E)$. Set $G = f^{-1}(A) \cap E$ and $H = f^{-1}(B) \cap E$. Then, $E = G \cup H$ and $G, H$ are nonempty.

Since $A \subseteq \bar{A}$, we have $G \subseteq f^{-1}(\bar{A})$. Since $f$ is continuous, $f^{-1}(\bar{A})$ is closed so $\bar{G} \subseteq f^{-1}(\bar{A})$. Therefore,

$$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(\bar{B}) \subseteq f^{-1}(\bar{A} \cap B) = f^{-1}(\varnothing) = \varnothing.$$

Similarly, $G \cap \bar{H} = \varnothing$. Thus, $G$ and $H$ are separated and $E$ is not connected. We conclude that $f(E)$ is connected if $E \subseteq X$ is connected and $f$ is continuous.

□

This theorem says that a continuous mapping preserves connectedness, which leads to Intermediate Value Theorem. Recall that $E \subseteq \mathbb{R}$ is connected if and only if for all $x, y \in E$ and $x < z < y$, then $z \in E$.

For the inverse mapping, connectedness is not preserved. Consider $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Then, $E = \{1\}$ is connected, but $f^{-1}(E) = \{-1, 1\}$ is not connected.

**Intermediate Value Theorem**

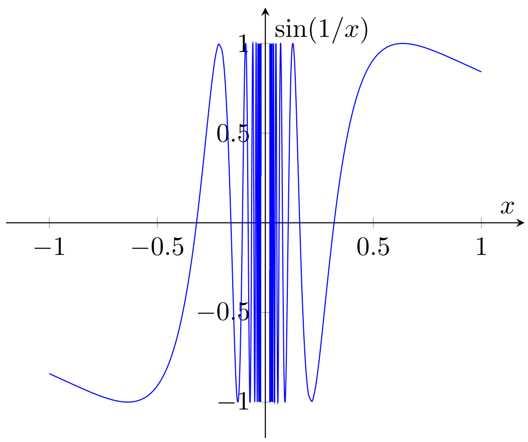
**Theorem 4.28** (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $c \in \mathbb{R}$ satisfies $f(a) < c < f(b)$, there there exists an $x \in (a, b)$ such that $f(x) = c$. A similar result holds, of course, if $f(a) > f(b)$.*

*Proof.* Since $[a, b]$ is connected, so by Theorem 4.27, $f([a, b])$ is also connected. Since $f(a), f(b) \in f([a, b])$ and $f(a) < f(c) < f(b)$, then we conclude by Theorem 2.67 that $c \in f([a, b])$. That is, there exists $x \in [a, b]$ such that $f(x) = c$. Since $c \neq f(a)$, so $x \neq a$. Similarly, $x \neq b$. We are done.

□

*Remark* 4.29. The converse of Intermediate Value Theorem is false. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$




$f$ is not continuous at $x = 0$. However, $f$ has the intermediate value property, that is, for any $a < b$ and any $y$ such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, there exists a $c \in [a, b]$ such that $f(c) = y$.

Lecture 25
Monday
December 7

## 4.4 Discontinuity

**Definition 4.30** (Discontinuity)**.** If $f$ is not continuous at $x$ and $x$ is in the domain of $f$, we say that $f$ is **discontinuous at** $x$.

Let $f : X \to \mathbb{R}$. We now introduce the notion of left and right limits, which can be thought of as two seperate "halves" of the limit $\lim_{x \to t : x \in X} f(x)$.

**Definition 4.31** (One-sided Limits)**.** Suppose $f$ is a real-valued function defined on $(a, b)$.

(a) For $a \leq x < b$, we write $f(x+) = q$ or $\lim_{t \to x^+} f(t) = q$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - q| < \varepsilon$ for all $t \in (x, x + \delta) \subseteq (a, b)$.

(b) For $a < x \leq b$, we write $f(x-) = q$ or $\lim_{t \to x^-} f(t) = q$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - q| < \varepsilon$ for all $t \in (x - \delta, x) \subseteq (a, b)$.

We can reformulate these definitions in terms of limits of sequences. Let $f : (a, b) \to \mathbb{R}$. For $a \leq x < b$, we say that $f(x+) = q$ if $f(t_n) \to q$ for all sequences $t_n \to x$ with $x < t_n < b$. Similarly, for $a < x \leq b$, we say that $f(x-) = q$ if $f(t_n) \to q$ for all sequences $t_n \to x$ with $a < t_n < x$.

**Lemma 4.32.** *Let $f : (a, b) \to \mathbb{R}$. $\lim\limits_{t\to x} f(t)$ exists if and only if $f(x+) = f(x-)$ and when this occurs, $\lim\limits_{t\to x} f(x)$ is equal to $f(x+) = f(x-)$.*

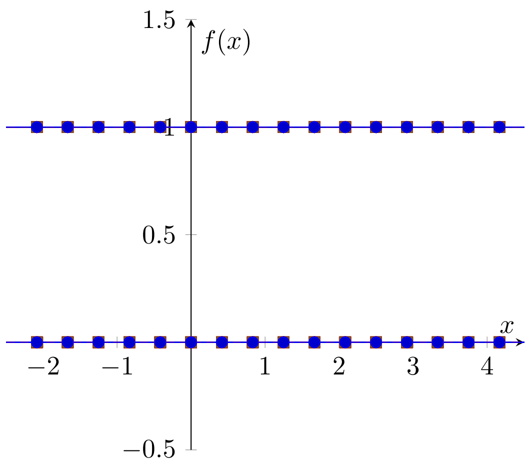
**Definition 4.33** (Types of Discontinuity)**.** If $f$ is discontinuous at $x$ and both $f(x+)$ and $f(x-)$ exist then we say $f$ has a **discontinuity of the first kind at** $x$ or **simple discontinuity at** $x$. Otherwise, the discontinuity is said to be of the **second kind**.

Simple discontinuity happens if both $f(x+)$ and $f(x-)$ exist and either $f(x+) \neq f(x-)$ or $f(x+) = f(x-)$ but $f(x) \neq f(x+) = f(x-)$.

**Example 4.34** (Dirichlet Function)**.** Consider the *Dirichlet Function* defined as

$$\mathscr{D}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

The *Dirichlet Function* is not continuous everywhere.



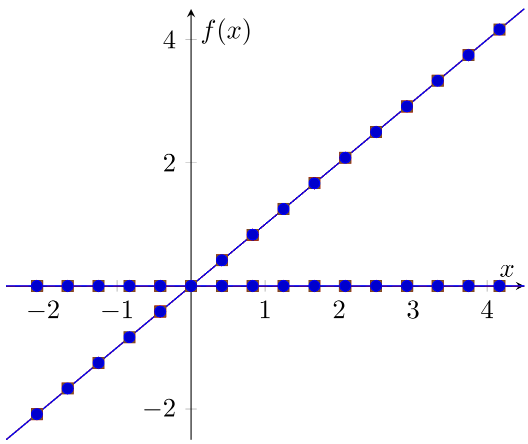

*Proof.* For any $c \in \mathbb{R}$, we can find a sequence $(x_n)$ in $\mathbb{Q}$ and sequence $(y_n)$ in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to c$ and $y_n \to c$. But $\mathscr{D}(x_n) = 1$ and $\mathscr{D}(y_n) = 0$ for all $n$. Hence $\lim\limits_{n\to\infty} \mathscr{D}(x_n) \neq \lim\limits_{n\to\infty} \mathscr{D}(y_n)$. This suggests that $\mathscr{D}$ is not continuous at $c$. Thus, *Dirichlet Function* is not continuous everywhere since $c$ is arbitrary. □

Specifically, the *Dirichlet Function* has the discontinuity of the second kind, since for all $p \in \mathbb{R}$, $(p - \delta, p)$ contains both rationals and irrationals. Hence the image $\{\mathscr{D}(x) : x \in (p - \delta, p)\}$ will oscillate between 0 and 1. Hence, the condition $d(f(p), f(x)) < \varepsilon$ will not work as we make $\varepsilon < 1$.

**Corollary 4.35** (Modified *Dirichlet Function*)**.** *We modify the Dirichlet Function $\mathscr{D}(x)$ a little bit to*

$$f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

*Then, $f(x)$ is continuous at $x = 0$ and discontinuous at $x \neq 0$ and has discontinuity of second kind at all $x \neq 0$.*




**Example 4.36** (More Proper Functions)**.**

(a) Consider the piecewise function

$$f(x) = \begin{cases} x+2 & -3 \leq x < -2 \\ -x-2 & -2 \leq x < 0 \\ x+2 & 0 \leq x \leq 1 \end{cases}.$$

$f(x)$ is continuous on $[-3, 1] \setminus \{0\}$ and has simple continuity at $x = 0$.

(b) Assume we know what sin function is and its properties. Consider the function

$$g(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

$g(x)$ is continuous on $\mathbb{R} \setminus \{0\}$ and has discontinuity of second kind at $x = 0$.

## 4.5 Monotone Functions

Classifying a set of discontinuities for an arbitrary function $f$ is somewhat complex, so it is interesting that describing the set of discontinuities is fairly straightforward for the class of monotone functions.

**Definition 4.37** (Monotone Function)**.** A function $f : (a, b) \to \mathbb{R}$ is **monotone increasing** if whenever $a < x < y < b$, we have $f(x) \leq f(y)$. Similarly, $f$ is **monotone decreasing** if whenever $a < x < y < b$, we have $f(x) \geq f(y)$. We say that $f$ is **monotone** if it is monotone increasing or monotone decreasing.

Continuous functions are not necessarily monotone, consider the function $f(x) = x^2$ on $\mathbb{R}$. Also, monotone functions are not necessarily continuous, consider the piecewise functions.

We further investigate the monotone functions through the limit "from the left" and "from the right."

**Theorem 4.38.** *Let $f : (a,b) \to \mathbb{R}$ be monotone increasing. Then, for every $x \in (a,b)$, $f(x-)$ and $f(x+)$ exist and*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t).$$

*Furthermore, if $a < x < y < b$, then $f(x+) \leq f(y-)$. Similar property holds when $f$ is monotone decreasing.*

*Proof.* The set $\{f(t) : a < t < x\}$ is bounded above by $f(x)$, so $\alpha = \sup\limits_{a<t<x} f(t)$ exists by least upper bound property and $\alpha \leq f(x)$.

Fix $\varepsilon > 0$. Since $\alpha - \varepsilon$ is not an upper bound to $\{f(t) : a < t < x\}$, there exists a $\delta > 0$ with $\alpha - \varepsilon < f(x-\delta) \leq \alpha$. So, for any $t \in (x-\delta, x)$, $\alpha - \varepsilon < f(x-\delta) \leq f(t) \leq \alpha$, so $|f(t) - \alpha| < \varepsilon$. Thus, $f(x-) = \alpha$. A similar argument shows that $f(x+) = \inf\limits_{x<t<b} f(t)$ and $f(x+) \geq f(x)$.

Now, suppose $a < x < y < b$. Pick any $c$ such that $x < c < y$. Then,

$$f(x+) = \inf_{x<t<b} f(t) \leq f(c) \leq \sup_{a<t<y} f(t) = f(y-).$$

□

**Corollary 4.39.** *Monotone functions have no discontinuities of the second kind.*

**Theorem 4.40.** *If $f$ is monotone on $(a,b)$, then it only has countably many discontinuities on $(a,b)$.*

*Proof.* Assume that $f$ is monotone increasing. Let $E$ be the set of discontinuities in $(a,b)$. For each $x \in E$, pick $r(x) \in \mathbb{Q}$ satisfy $f(x-) < r(x) < f(x+)$. Then, $r : E \to \mathbb{Q}$ is an injection since if $x_1 < x_2$, then by Theorem 4.38, $r(x_1) < f(x_1+) < f(x_2-) < r(x_2)$. Since $\mathbb{Q}$ is countable, it follows that $E$ is countable.

□

Lecture 26 Wednesday December 9

**Example 4.41.** Given any countable set $E \subseteq (a,b)$, there is a monotone increasing function $f : (a,b) \to \mathbb{R}$ such that $E$ is the set of discontinuities of $f$.

*Proof.* Let $E = \{e_1, e_2, e_3, \ldots\}$. Fix a sequence $(c_n)$ of positive real numbers such that $\sum\limits_{n=1}^{\infty} c_n$ converges. Define, for $x \in (a,b)$,

$$I_x = \{n : e_n < x\} \quad I_x^+ = \{n : e_n \leq x\}.$$

Define $f(x) = \sum\limits_{n \in I_x} c_n$ (this converges because $\sum\limits_{n=1}^{\infty} c_n$ converges absolutely). Then,

(a) $f$ is monotone increasing;

(b) $f(e_n+) - f(e_n-) = c_n > 0$;

(c) $f$ is not continuous on $(a,b) \setminus E$.

(a) holds since $x < t \Longrightarrow I_x \subseteq I_t \Longrightarrow f(x) \leq f(t)$.

For (b) and (c), it suffices to show that for all $x \in (a, b)$,

$$f(x-) = f(x) \text{ and } f(x+) = \sum_{n \in I_x^+} c_n.$$

Since

$$f(e_k+) - f(e_k-) = \sum_{n \in I_{e_n}^+ \setminus I_{e_n}} c_n = c_k$$

and for $x \in (a, b) \setminus E$, we have $I_x = I_x^+$ and thus $f(x-) = f(x+)$, so $f$ is continuous at $x$.

We want to show that for $x \in (a, b)$, $f(x-) = f(x+)$ and $f(x+) = \sum_{n \in I_x^+} c_n$. Note that when $t < x$, $[t, x) \cap \{e_1, e_2, \ldots, e_N\} = \varnothing$ implies that for $1 \leq i \leq N$,

$$(e_i < t \Leftrightarrow e_i < x) \Longrightarrow I_x \setminus I_t \subseteq \{e_{N+1}, \ldots\} \Longrightarrow 0 \leq f(x) - f(t) \leq \sum_{n > N} c_n.$$

When $x < t$,

$$\begin{aligned}(x, t) \cap \{e_1, e_2, \ldots, e_N\} = \varnothing &\Longrightarrow \text{for all } 1 \leq i \leq N, e_i \leq x \Leftrightarrow e_i < t \\ &\Longrightarrow I_t \setminus I_x^+ \subseteq \{e_{N+1}, e_{N+2}, \ldots\} \\ &\Longrightarrow 0 \leq f(t) - \sum_{n \in I_x^+} c_n \leq \sum_{n > N} c_n.\end{aligned}$$

So, given $\varepsilon > 0$ and $x \in (a, b)$, pick $N$ with $\sum_{n > N} c_n < \varepsilon$ and choose $\delta > 0$ small enough so that $(x - \delta, x)$ and $(x, x + \delta)$ are disjoint with $\{e_1, e_2, \ldots, e_N\}$. Then,

$$\begin{aligned} t \in (x - \delta, x) &\Longrightarrow |f(t) - f(x)| < \varepsilon \\ t \in (x, x + \delta) &\Longrightarrow \left| f(t) - \sum_{n \in I_x^+} c_n \right| < \varepsilon. \end{aligned}$$

Thus, $f(x-) = f(x)$ and $f(x+) = \sum_{n \in I_x^+} c_n$.

□

## 4.6 Infinite Limits and Limits at Infinity

Previously, toward a linear function $f(x)$, we may conclude that "as $x$ goes to infinity, $f(x)$ goes to infinity." Currently, we don't have the tools to formalize this idea because infinity cannot live in a metric space. Recall that a set $U \subseteq \mathbb{R}$ is a **neighborhood** of $x \in \mathbb{R}$ if $U$ is open and $x \in U$. By further developing limits of functions and defining neighborhoods in the extended real number system, we can handle these cases.

**Definition 4.42** (Neighborhood at Infinity)**.** A **neighborhood of** $+\infty$ is a set of the form $(M, +\infty)$ for $M \in \mathbb{R}$. A **neighborhood of** $-\infty$ is a set of the form $(-\infty, -M)$ for $M \in \mathbb{R}$.

We have seen that $\lim_{x\to p} f(x) = q$ if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - q| < \varepsilon$. Equivalently, for every sequence $(x_n)$ converging to $p$ with $x_n \neq p$, $f(x_n) \to q$. This is also equivalent to say that for every neighborhood $U$ of $q$, there exists a neighborhood $V$ of $p$ with $x \in V$ implies $f(x) \in U$.

**Definition 4.43** (Limits at Infinity)**.** Let $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$. For $x, y \in \mathbb{R} \cup \{-\infty, +\infty\}$, we write $\lim_{t\to x} f(t) = y$ or $f(t) = y$ as $t \to x$ if

(a) *either* $x \in E'$ *or* $E$ is not bounded above and $x = +\infty$ *or* $E$ is not bounded below and $x = -\infty$

(b) *and* for every neighborhood $V$ of $y$ there exists a neighborhood $U$ of $x$ such that $f(t) \in V$ for all $t \in E$, $x \neq t \in U$.

Now, we will generalize some limit rules in the extended real number system.

**Theorem 4.44.** *Let $E \subseteq \mathbb{R}$. Let $f, g : E \to \mathbb{R}$. Suppose $x, a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\lim_{t\to x} f(t) = a$, and $\lim_{t\to x} g(t) = b$. Then,*

*(a) if $\lim_{t\to x} f(t) = a'$, then $a = a'$*

*(b) $\lim_{t\to x}(f + g)(t) = a + b$*

*(c) $\lim_{t\to x}(f \cdot g)(t) = a \cdot b$*

*(d) $\lim_{t\to x} \left(\frac{f}{g}\right)(t) = \frac{a}{b}$*

*provided that right-hand side is defined. $+\infty + (-\infty)$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{a}{0}$ are not defined.*

*Proof.*

(a) Towards a contradiction, assume $a \neq a'$. Say $a < a'$ ($a > a'$ is similar). Then, there is $r \in \mathbb{R}$ such that $a < r < a'$. Then, $V = (-\infty, r)$ and $V' = (r, +\infty)$ are neighborhoods of $a$ and $a'$ respectively. So there are neighborhoods $U$ and $U'$ of $x$ such that for all $t \in E$,

$$x \neq t \in U \Longrightarrow f(t) \in V$$
$$x \neq t \in U' \Longrightarrow f(t) \in V'.$$

Then, $U \cap U'$ is a neighborhood of $x$ so we can find a $t \in E$ with $x \neq t \in U \cap U'$. Then,

$$f(t) \in V \cap V' = (-\infty, r) \cap (r, +\infty) = \varnothing,$$

a contradiction. Hence, $a = a'$.

□

That's the end of Math 140A.