Math 184 Enumerative Combinatorics

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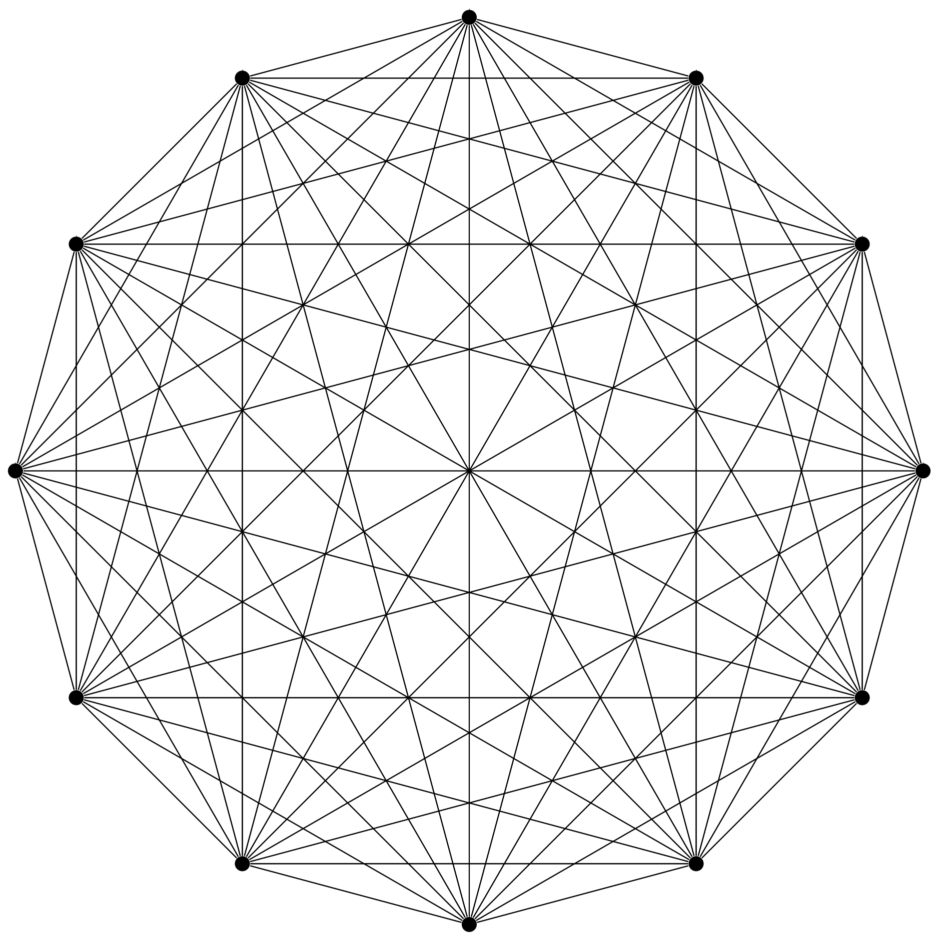
Last Updated: January 4, 2021



Abstract

Enumeration of combinatorial structures (permutations, integer partitions, set partitions). Bijections. Inclusion-exclusion. Ordinary generating functions. Exponential generating functions.

Contents

1 There are a Lot of Them: Permutations and Combinations **3**

1.1 Counting . . . 4

1.2 Permutations of Sets . . . 6

1.3 Combinations of Sets . . . 9

1.4 Permutations of Multisets . . . 11

1.5 Combinations of Multisets . . . 12

1.6 Finite Probability . . . 13

2 Seven is More Than Six: The Pigeonhole Principle **16**

2.1 Basic Version of the Pigeonhole Principle . . . 16

2.2 Generalized Version of the Pigeonhole Principle . . . 18

2.3 Introduction to Graph Theory . . . 18

2.4 Ramsey's Theorem . . . 19

3 No Matter How You Slice It: Binomial Theorem **25**

3.1 Binomial Theorem . . . 25

3.2 Unimodality of Binomial Coefficients . . . 28

3.3 Extremal Set Theory . . . 29

4 No Overcount: The Sieving Methods **32**

4.1 The Inclusion-Exclusion Principle . . . 32

4.2 Combinations with Multisets . . . 33

4.3 Derangements . . . 34

5 A Function Is Worth Many Numbers: Generating Functions **37**

5.1 Sequences . . . 37

5.2 Linear Recurrence Relation . . . 38

5.3 Ordinary Generating Functions . . . 42

5.4 Exponential Generating Functions . . . 45

5.5 Homogeneous Linear Recurrence Relation . . . 49

5.6 Nonhomogeneous Linear Recurrence Relations . . . 53

5.7 Catalan Numbers . . . 56

6 Additional Topics on Sequences **60**

6.1 Difference Sequences . . . 60

6.2 Stirling Numbers . . . 62

1 There are a Lot of Them: Permutations and Combinations

Given two sets X and Y, a map $f : X \to Y$ is a subset S of $X \times Y$ such that for each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in S$. We write the elements of S also as $(x, f(x))$.

Definition 1.1 (Injective, Surjective, and Bijective)**.** Let X and Y be sets and $f : X \to Y$ a function from X to Y. We make the following definitions.

(a) f is **injective** (or f is an **injection**) if, for all $x, x' \in X$, we have $f(x) = f(x')$ implies that $x = x'$. In other words, different elements in X get sent to different values in Y.

(b) f is **surjective** (or f is an **surjection**) if, for all $y \in Y$, there is some $x \in X$ such that $f(x) = y$. In other words, all possible values in Y are achieved.

(c) f is **bijective** (or f is a **bijection**) if it is both injective and surjective.

Definition 1.2 (Cardinality)**.** Let n be a positive integer. A finite non-empty set X has **cardinality** n if there is a bijection $X \to \{1, 2, \ldots, n\}$.

Theorem 1.3. *Let X and Y be finite sets.*

(a) If there is an injection $f : X \to Y$, then $|X| \leq |Y|$.

(b) If there is a surjection $f : X \to Y$, then $|X| \geq |Y|$.

(c) If there is a bijection $f : X \to Y$, then $|X| = |Y|$.

Proof. We write the elements of X as $X = \{x_1, x_2, \ldots, x_n\}$, so $|X| = n$.

(a) The elements $f(x_1), f(x_2), \ldots, f(x_n)$ are all distinct elements of Y since f is an injection, so Y contains a subset of size n, and hence $|Y| \geq n = |X|$.

(b) If there is a surjection $f : X \to Y$, then every element of Y is of the form $f(x_i)$ for some i. This means that Y has at most n elements (some of the values could coincide) which means that $|Y| \leq n = |X|$.

(c) By Parts (a) and (b), if there is a bijection $f : X \to Y$, then we would have $|X| \leq |Y|$ and $|X| \geq |Y|$ and hence $|X| = |Y|$.

□

Proposition 1.4. *Let X and Y be finite sets and $f : X \to Y$ a function. Then f is a bijection if any of the following 2 properties hold:*

(a) f is injective,

(b) f is surjective,

(c) $|X| = |Y|$.

Proof. We write the elements of X as $X = \{x_1, x_2, \ldots, x_n\}$, so $|X| = n$.

We check all possibilities. If (a) and (b) hold, then f is a bijection by definition.

Suppose that (a) and (c) hold. Since f is injective, the elements $f(x_1), f(x_2), \ldots, f(x_n)$ give n distinct elements of Y. But since $|X| = |Y|$, they must account for all of the elements of Y. This means that f is surjective since every element of Y is of the form $f(x_i)$ for some i. Hence f is bijective.

Suppose that (b) and (c) hold. Since f is surjective, every element of Y is of the form $f(x_i)$ for some i. Since $|Y| = |X| = n$, the n elements $f(x_1), f(x_2), \ldots, f(x_n)$ have to all be distinct (since they account for all of the elements of Y). Hence f is injective, and so f is bijective.

□

Given two functions $f : X \to Y$ and $g : Y \to X$, we say that they are inverses if $f \circ g$ is the identity function on Y, that is, $f(g(y)) = y$ for all $y \in Y$, and if $g \circ f$ is the identity function on X, that is, $g(f(x)) = x$ for all $x \in X$.

Proposition 1.5. *The function $f : X \to Y$ is a bijection if and only if there exists an inverse $g : Y \to X$.*

Lecture 1
Friday
October 2

1.1 Counting

Let's start with addition. It is one formulation of the principle that the whole is equal to the sum of its parts.

Definition 1.6 (Partition)**.** Let S be a set. A **partition** of S is a collection $S_1, S_2, \ldots, S_m$ of subsets of S such that each element of S is in exactly one of those subsets

$$S = S_1 \cup S_2 \cup \cdots \cup S_m,$$

$$S_i \cap S_j = \varnothing \text{ for } i \neq j.$$

Thus, the sets $S_1, S_2, \ldots, S_m$ are pairwise disjoint sets, and their union is S. The subsets $S_1, S_2, \ldots, S_m$ are called the parts of the partition.

Example 1.7. Let $S = \{a, b, c, d\}$. Then, consider two sets $S_1 = \{a, c\}$ and $S_2 = \{b, c\}$. Obviously, we have $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$, so S_1 and S_2 are parts of the partition.

Proposition 1.8 (Addition Principle)**.** *Suppose that a set S is partitioned into pairwise disjoint parts $S_1, S_2, \ldots, S_m$. The number of objects in S can be determined by finding the number of objects in each of the parts, and adding the numbers so obtained*

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

Recall that the Cartesian product of sets X and Y is defined to be the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

In other words, $X \times Y$ contains all the ordered pairs (x, y) with $x \in X$ and $y \in Y$. We are also interested in the cardinality of $X \times Y$. In fact, in general, for any sets X and Y, we have

$$|X \times Y| = |X| \cdot |Y|.$$

Proposition 1.9. *Let X and Y be finite sets. Then,*

$$|X \times Y| = |X| \cdot |Y|.$$

Proof. If either X or Y is empty, then $X \times Y = \varnothing$, so the claim holds.

Thus, assume that X and Y are both non-empty. There are positive integers n and m, and bijections

$$f : \{1, \ldots, n\} \to X$$
$$g : \{1, \ldots, m\} \to Y.$$

Let

$$\begin{aligned} h : \{1, \ldots, n \cdot m\} &\to X \times Y \\ : x &\mapsto \begin{cases} (f(x), g(1)) & \text{if } 1 \leq x \leq n \\ (f(x-n), g(2)) & \text{if } n+1 \leq x \leq 2n \\ \quad\vdots & \quad\vdots \\ (f(x-(m-1)n), g(m)) & \text{if } (m-1)n+1 \leq x \leq mn \end{cases} \end{aligned}$$

More compactly, $h : x \mapsto (f(x - in), g(i+1))$ if $i \cdot n + 1 \leq x \leq (i+1) \cdot n$ for $i = 0, 1, \ldots, m-1$.

This is a bijection.

□

Proposition 1.10 (Multiplication Principle)**.** *Let S be a set of ordered pairs (a, b) of objects, where the first object a comes from a set of cardinality p, and for each choice of object a there are q choices for object b. Then the cardinality of S is $p \cdot q$*

$$|S| = p \cdot q.$$

Corollary 1.11 (More Generalized Product Principle)**.** *Let $X_1, X_2, \ldots, X_n$ be finite sets. Then,*

$$|X_1 \times X_2 \times \ldots \times X_n| = |X_1| \cdot |X_2| \cdot \ldots \cdot |X_n|.$$

Example 1.12. How many 4-digit number are there whose digits are either 1, 2, 3, 4, or 5?

Let that 4-digit number be $\overline{pqrs}$, where p, q, r, and s are the digits in the respective positions. For each digit, we have 5 possibilities. Hence, by Corollary 1.11, we have 5^4 possibilities.

Plus we require that digits are distinct, for example, 1532 is ok but 1531 is not ok. Still, let that 4-digit number be $\overline{pqrs}$, where p, q, r, and s are the digits in the respective positions. For p, we have 5 possibilities; for q, we have $5-1=4$ possibilities; for r, we have $4-1=3$ possibilites; for s, we have $3-1=2$ possibilities. Hence, by Corollary 1.11, we have $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possibilities.

Forget the requirement for distinct digits, how many 4-digit numbers are there with digits 1, 2, 3, 4, or 5 that are even? In this case, for p, q, and r digits, we have 5 possibilities for each. Since our destined numbers are even, so the digit s should be even, that is, either 2 or 4. Hence, by Corollary 1.11, we have $5^3 \cdot 2 = 250$ possibilities.

Proposition 1.13 (Subtraction Principle)**.** *Let A be a finite set and let U be a larger finite set containing A. Let*

$$A^{\complement} = U - A = \{x \in U : x \notin A\}$$

be the ***complement of*** *A* ***in*** *U. Then the number $|A|$ of objects in A is given by the rule*

$$|A| = |U| - |A^{\complement}|.$$

Example 1.14. Passwords consist of 6 symbols.

- Digits: $0, 1, 2, \ldots, 9$ (10)
- Lowercase letters: $a, b, c, \ldots, z$ (26)

We want to know how many such passwords have a repeated symbol.

Let A be the collection of passwords with repeated symbols and $A^{\complement}$ be the collection of passwords without repeated symbols. We are given that there are $(10+26)^6 = 36^6$ possibilities of passwords, which means $|A| + |A^{\complement}| = 36^6$. Also, we have $|A^{\complement}| = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31$. Therefore, we get $|A|$ via Proposition 1.13.

Proposition 1.15 (Division Principle)**.** *Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by*

$$k = \frac{|S|}{\textit{number of objects in a part}}.$$

Lecture 2
Monday
October 5

1.2 Permutations of Sets

Definition 1.16 (Permutation)**.** The arrangement of different objects into a linear order using each object exactly once is called a **permutation** of these objects. Let S be a set of n elements. An r**-permutation** of S is an ordered arrangement of r elements in S.

Example 1.17. Let $S = \{a, b, c\}$, then there are three 1-permutations of S

$$\{a\}, \qquad \{b\}, \qquad \{c\}.$$

There are six 2-permutations of S

$$\{ab\}, \quad \{ac\}, \quad \{ba\}, \quad \{bc\}, \quad \{ca\}, \quad \{cb\}.$$

There are six 3-permutations of S

$$\{abc\}, \quad \{acb\}, \quad \{bac\}, \quad \{bca\}, \quad \{cab\}, \quad \{cba\}.$$

There are no 4-permutations of S since S has fewer than four elements.

We denote by $P(n,r)$ the number of r-permutations of a set with n elements. If $r > n$, then $P(n,r) = 0$. Clearly $P(n,1) = n$ for each positive integer n.

Theorem 1.18. *For n and r positive integers with $r \leq n$,*

$$P(n,r) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1).$$

Proof. In constructing an r-permutation of a set with n elements, we can choose the first item in n ways, the second item in $n-1$ ways, whatever the choice of the first item, so on and so forth, and the rth item in $n-(r-1)$ ways, whatever the choice of the first $r-1$ items. By the Corollary 1.11, the r items can be chosen in $n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$ ways.

□

To count permutations in general, we define the **factorial** as follows: $0! = 1$ and if n is a positive integer, then $n! = n \cdot (n-1)!$. Now, we may write

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Here are the first few values

$$0! = 1, \quad 1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad 5! = 120.$$

In the Example 1.17, we had 6 permutations of 3 elements, and $6 = 3!$. This will hold more generally.

Proposition 1.19. *If S has n elements and $n > 0$, then there are $n!$ different permutations of S.*

This is a direct result of Theorem 1.18.

Example 1.20 (Poker)**.** Poker is a deck of cards that consists of 52 cards (without 2 Jokers), divided into 4 suits of 13 ranks (values) each. Thirteen ranks are $\mathrm{A}, 2, 3, \ldots, 10, \mathrm{J}, \mathrm{Q}, \mathrm{K}$. Four suits are clubs, diamonds, hearts, and spades. The suits are all of equal value, that is, no suit is higher than any other suit. So, given a deck of cards, the total number of orderings of a deck of cards is $52!$.

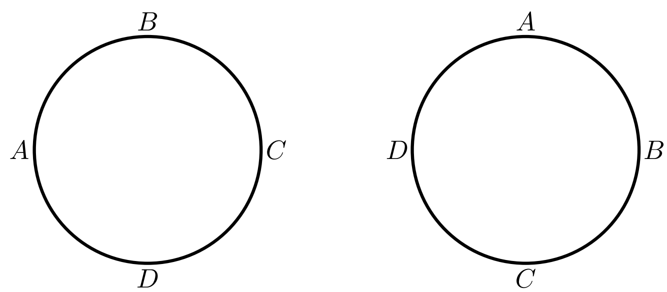
(a) Now, how many orderings are there if all the suits are together? The ordering for each suit is $13!$. There are $4!$ ways to order the four suits and with order of the suits, there are $13! \cdot 13! \cdot 13! \cdot 13!$ ways to order the cards. Thus the total number of ways ordering a deck of cards with each suits together is $4! \cdot (13!)^4$.

(b) How many ways can a poker hands be dealt? A poker hand is a set of 5 cards. Here order matters and we are drawing without replacement. So, the first card has 52 possibilities, the second card has 51 possibilities, ..., the fifth card has 48 possibilites. Therefore, the total number of ways of poker hands is $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$.

(c) How many poker hands are there? In this case, order does not matter. We will leave this question to next section.

Example 1.21 (Circular Permutations)**.** Suppose we have 4 people Alex, Benny, Chris, and Derek. How many ways can these four people sit at a round table?

What matters here is who is sitting next to who on the right and left sides. That is, we care the relative position! For example, the following two seating tables are equivalent, or same.



Since we are free to rotate the people, any circular permutation can be rotated so that A is in a fixed position. Now that A is fixed, the permutations of A, B, C, and D can be identified with the linear permutations of B, C, and D. There are 3! linear permutations of B, C, and D. Hence, there are 3! ways that four people sit at a round table.

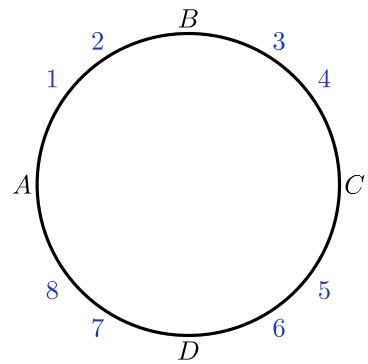
Theorem 1.22. *Let S be a set of n elements. The number of circular permutation of S is $(n-1)!$. More generally, the number of circular r-permutations of an n-element set S is $\binom{n}{r} \cdot r!$.*

Proof. We have $\binom{n}{r}$ ways to select r elements at the table and $(r-1)!$ ways to place the r elements at the circular table.

□

Example 1.23. How many ways 4 men and 8 women sit at a round table it there are 2 women between consecutive men?

This is an application of circular permutation, so we only consider the relative position. Below is one way we can seat these people (denote four men as A to D and eight women as 1 to 8).



By Theorem 1.22, we have 3! ways to seat 4 men. Now, we have 8 seats for 8 women, and there are 8! ways to seat 8 women. Finally, we have $3! \times 8!$ ways.

1.3 Combinations of Sets

Definition 1.24 (Combination)**.** Let S be a set of n elements. A **combination** of a set S denotes an unordered selection of the elements of S. The result of such a selection is a subset A of the elements of S, $A \subseteq S$.

We write $\binom{n}{r}$ the number of r-subsets of a set with n elements and we read "n chooses r" for this notation.

So, in Example 1.20, there are $\binom{52}{5}$ poker hands.

Remark 1.25 (Some Special Cases). We define some special cases and some intuitions.

(a) If $r > n$, we define $\binom{n}{r} = 0$.

(b) If $n = 0$, $r > 0$, we define $\binom{0}{r} = 0$.

(c) If $r = 0$, we define $\binom{n}{0} = 1$.

(d) By intuition, $\binom{n}{1} = n$ and $\binom{n}{n} = 1$. In particular, $\binom{0}{0} = 1$.

Theorem 1.26. *For* $0 \leq r \leq n$*,*

$$P(n, r) = r!\binom{n}{r}.$$

Hence,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. Let S be a set with n elements. Each r-permutation of S starts from exactly one way as a result of carrying out the two tasks: choosing r elements from S and arranging the chosen r elements in some order.

The number of ways to carry out choosing r elements from S is, by definition, the number $\binom{n}{r}$. The number of ways to carry out arranging the chosen r elements in some order is $P(r, r) = r!$. By Multiplication

Principle (Proposition 1.10), we have $P(n,r) = r! \cdot \binom{n}{r}$. With $P(n,r) = \frac{n!}{(n-r)!}$, we obtain

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

□

Lecture 3 Wednesday October 7

Theorem 1.27. *For $0 \leq r \leq n$,*

$$\binom{n}{r} = \binom{n}{n-r}.$$

Proof. Let $S = \{1, 2, \ldots, n\}$. Note that picking r elements of S is equivalent to not picking $n - r$ elements from S.

□

Theorem 1.28 (Pascal's Identity)**.** *For any $k \geq 0$, we have*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. The right hand side is the number of subsets of $\{1, 2, \ldots, n+1\}$ of size k. There are 2 types of such subsets: those that contain $n+1$ and those that do not. Note that the subsets do contain $n+1$ are naturally in bijection with the subsets of $\{1, 2, \ldots, n\}$ of size $k-1$: to get such subset, delete $n+1$. Those that do not contain $n+1$ are naturally already in bijection with the subsets of $\{1, 2, \ldots, n\}$ of size k. The two sets don't overlap and their sizes are $\binom{n}{k-1}$ and $\binom{n}{k}$, respectively.

□

Lemma 1.29 (Number of Subsets)**.** *There are 2^n subsets of a set of size n.*

Example 1.30. We use **words** (finite ordered sequence whose entries are drawn from some set A, which we call the **alphabet**), to show Lemma 1.29.

Given a subset $S \subseteq \{1, 2, \ldots, n\}$, we define a word w_S of length n in the alphabet $\{0, 1\}$ as follows. If $i \in S$, then the ith entry of w_S is 1, and otherwise the entry is 0. This defines a function

$$f : \{\text{subsets of } \{1, 2, \ldots, n\}\} \to \{\text{words of length } n \text{ on } \{0, 1\}\}.$$

We can also define an inverse function: given such a word w, we send it to the subset of positions where there is a 1 in w. We omit the check that these two functions are inverse to one another. So f is a bijection, and Proposition 1.11 tells us that there are 2^n words of length n on $\{0, 1\}$.

Theorem 1.31. *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. The left hand side counts the number of subsets of $\{1, 2, \ldots, n\}$ of some size k where k ranges from 0 to n. But all subsets of $\{1, 2, \ldots, n\}$ are accounted for and by Lemma 1.29, 2^n is the number of all subsets of $\{1, 2, \ldots, n\}$.

□

Example 1.32. There are 100 students at a school and three dormitories, A, B, and C, with capacities 25, 35, and 40, respectively.

(a) How many ways are there to fill the dormitories?

We are drawing without replacement and order does not matter. To assign students into dormitory A, we randomly choose 25 students from 100 total students, so there are $\binom{100}{25}$ possibilities. Then, to assign students into dormitory B, we randomly choose 35 students from the remaining $100 - 25 = 75$ students, so there are $\binom{75}{35}$ possibilities. Finally, there are 40 rest students automatically assigned to dormitory C, so there are $\binom{40}{40} = 1$ possibility. Thus, there are

$$\#A = \binom{100}{25} \cdot \binom{75}{35}$$

ways to fill the dormitories.

(b) Suppose that, of the 100 students, 50 are men and 50 are women and that A is an all-men's dorm, B is an all-women's dorm, and C is co-ed. How many ways are there to fill the dormitories?

We are still drawing without replacement and order does not matter. To assign 50 men to dormitory A with capacity 25, there are $\binom{50}{25}$ possibilities. To assign 50 women to dormitory B with capacity 35, there are $\binom{50}{35}$ possibilities. Then, the rest of 25 men and 15 women automatically go to dormitory C, so there are $\binom{40}{40} = 1$ possibility. Thus, there are

$$\#B = \binom{50}{25} \cdot \binom{50}{35}$$

ways to fill the dormitories.

1.4 Permutations of Multisets

An important variation of subset is the notion of a multiset. Given a set S, a **multiset** of S is like a subset, but we allow elements to be repeated. Said another way, a subset of S can be thought of as a way of assigning either a 0 or 1 to an element, based on wheteher it gets included. A multiset is then a way to assign some non-negative integer to each element, where numbers bigger than 1 mean we have picked them multiple times.

For example, if $S = \{1 \cdot a, 2 \cdot d\}$ (with one repeaded element d), sometimes switch betwen 2 letters does not make changes, since we have two d's.

Example 1.33. How many permutations are there of the letters of the word ADDRESSES?

We are dealing with the multiset $S = \{1 \cdot A, 2 \cdot D, 1 \cdot R, 2 \cdot E, 3 \cdot S\}$. At the beginning, we have $\binom{9}{1}$ ways of placing A. Then, we have $\binom{8}{2}$ ways of placing D, $\binom{6}{1}$ ways of placing R, $\binom{5}{2}$ ways of placing E, and finally $\binom{3}{3}$ ways of placing S. (Note that the order can change, but the number of ways for each letter's placement can change, too.) Then, we have

$$\begin{aligned}\#A &= \binom{9}{1}\binom{8}{2}\binom{6}{1}\binom{5}{2}\binom{3}{3} \\ &= \frac{9!}{1! \cdot \cancel{8!}} \cdot \frac{\cancel{8!}}{2! \cdot \cancel{6!}} \cdot \frac{\cancel{6!}}{1! \cdot \cancel{5!}} \cdot \frac{\cancel{5!}}{2! \cdot \cancel{3!}} \cdot \frac{\cancel{3!}}{3! \cdot \cancel{0!}} \\ &= \frac{9!}{1! \cdot 2! \cdot 1! \cdot 2! \cdot 3!}.\end{aligned}$$

Theorem 1.34. *Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let the size of S be $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals*

$$\frac{n!}{n_1! \cdot n_2! \cdots n_k!}.$$

Remark 1.35. Let S be a multiset with objects of k different types, where each object has an **infinite** repetition number. Then the number of r-permutation of S is k^r. Lecture 4 Friday October 9

In constructing an r-permutation of S, we can choose the first item to be an object of any one of the k types. Similarly, the second item can be an object of any one of the k types, and so on. Since all repetition numbers of S are infinite, the number of different choices for any item is always k and it does not depend on the choices of any previous items. By Proposition 1.11, the r items can be chosen in k^r ways.

Example 1.36 (r-permutations for Finite Repetitions)**.** Let $S = \{3 \cdot a, 4 \cdot b, 5 \cdot\}$ with $|S| = 3 + 4 + 5 = 12$. The number of 12-permutations is

$$\frac{12!}{3! \cdot 4! \cdot 5!}.$$

How many 11-permutations are there? The answer is still $\frac{12!}{3! \cdot 4! \cdot 5!}$.

(Faster explanation) Any 11-permutation will determine the last element in a 12-permutation.

(Slower calculation) Consider 3 cases: one a is removed (now $\{2 \cdot a, 4 \cdot b, 5 \cdot c\}$); one b is removed (now $\{3 \cdot a, 3 \cdot b, 5 \cdot c\}$); and one c is removed (now $\{3 \cdot a, 4 \cdot b, 4 \cdot c\}$). Since these cases do not intersect, by Proposition 1.8, we have

$$\frac{11!}{2! \cdot 4! \cdot 5!} + \frac{11!}{3! \cdot 3! \cdot 5!} + \frac{11!}{3! \cdot 4! \cdot 4!} = \frac{11!}{3! \cdot 4! \cdot 5!} \cdot (3 + 4 + 5) = \frac{12!}{3! \cdot 4! \cdot 5!}.$$

1.5 Combinations of Multisets

If S is a multiset, then an r-combination of S is an unordered selection of r of the objects of S.

Example 1.37. Let $S = \{1 \cdot a, 3 \cdot b\}$ with $|S| = 4$. Then, 2-combinations of S include $\{a, b\}$ and $\{b, b\}$.

Theorem 1.38 (Stars and Bars)**.** *Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S equals*

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

Example 1.39 (Combination of Doughnuts)**.** A bakery has 4 types of doughnuts, namely, Chocolate, Plain, Strawberry, and Jelly. If a box of doughnuts contains half dozen (6 doughnuts). How many different types of boxes are there?

(Observation) In this case, we have 6 stars (correspond to 6 doughnuts) and 3 bars (here we want to divide 6 stars to 4 parts, which means we just need 3 "dividers"). Therefore, a permutation of 6 stars and 3 bars will correspond to a box of donuts. For example, (if we arrance Chocolate, Plain, Strawberry, and Jelly accordingly)

$$* \quad * \quad \Big| \quad \Big| \quad * \quad * \quad * \quad * \quad \Big|$$

we get 2 Chocolate doughnuts, 0 Plain doughnuts, 4 Strawberry doughnuts, and 0 Jelly doughnuts.

Actually, what we do here is choosing 3 slots among 9 slots, that is, $\binom{9}{3} = \binom{9}{6}$ variations of boxes.

Example 1.40 (Integral Solutions of Equations)**.** Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 30.$$

We are interested in the number of integral solutions if $x_i \geq 0$ for all $i \in \{1, 2, 3, 4\}$. One solution could be $x_1 = 30, x_2 = x_3 = x_4 = 0$.

In this case, we can convert to we have 30 stars and we need 3 bars to divide (just place these bars randomly), so there are $\binom{33}{3}$ set of integral solutions.

What if $x_1 \geq 2, x_2 \geq 0, x_3 \geq -5, x_4 \geq 8$? Now, substitute the variables such that we are in the situation as above. Let $y_1 = x_1 - 2 \geq 0, y_2 = x_2 \geq 0, y_3 = x_3 + 5 \geq 0, y_4 = x_2 - 8 \geq 0$, then $y_1 + y_2 + y_3 + y_4 = 25$. By Theorem 1.38, we have $\binom{28}{3}$ set of integral solutions.

Example 1.41 (Bagel Combinations)**.** A bagel store sells 6 types of bagels, namely, 1 through 6. We will choose 15 bagels. How many choices are there? Lecture 5 Monday October 12

In this case, this is a straightforward application of Theorem 1.38 with 15 stars and 5 bars, so there are $\binom{20}{5}$ choices.

1.6 Finite Probability

There is an experiment (or a random process) which when carried out results in one of a finite set of outcomes. We assume that each outcome is *equally likely*. We say that the experiment is carried out *randomly*.

Definition 1.42 (Sample Space)**.** The **sample space** Ω is the set of all the possible outcomes of the experiment, with the assumption $|\Omega| < \infty$. Elements of Ω are called **sample points** and typically denoted by ω.

Definition 1.43 (Event)**.** An **event** is a subset E of the sample space Ω. The collection of events in Ω is denoted by $\mathcal{F}$.

Definition 1.44 (Probability)**.** The **probability measure** $\mathbb{P}$ is a function from $\mathcal{F}$ into the real numbers. If the sample space Ω has finitely many elements and each outcome is equally likely then for any event $E \subseteq \Omega$, we have

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

so $0 \le \mathbb{P}(E) \le 1$ and $\mathbb{P}(\varnothing) = 0$, $\mathbb{P}(\Omega) = 1$.

Example 1.45 (Continuation of Example 1.41)**.** If we choose 15 bagels at random, what is the probability that we have at least one bagel of each type?

Let $E = \{\text{At least one bagel from each type}\}$ and $\Omega = \{\text{All possible outcomes}\}$, with $|\Omega| = \binom{20}{5}$. We want to find $|E|$.

Let x_i be the number of bagels of type i, for $1 \le i \le 6$. Consider the linear system $\sum_{i=1}^{6} x_i = 15$ with $x_i \ge 1$ and we want to find the number of integral solutions of this system.

Now, grab 1 bagel from each type, then we have 9 bagels left to choose, that is, a direct application of Theorem 1.38 with 9 stars and 5 bars. So, $|E| = \binom{14}{5}$. Therefore,

$$\mathbb{P}(E) = \frac{\binom{14}{5}}{\binom{20}{5}}.$$

Remark 1.46. Here, we assume there are infinite bagels of each type. If we set up further constraints on the numbers of bagels of each type, for example, we only have 2 bagels of the first type and 9 bagels of the third type, etc, we need further knowledge in Inclusion-Exclusion to check all the cases.

Example 1.47 (Flush)**.** What is the probability that a poker hand contains a flush, that is, five cards of the same suit?

Let $E = \{\text{Flush}\}$. We have $|\Omega| = \binom{52}{5}$ poker hands. Also, we have 4 ways of fixing the suit and $\binom{13}{5}$ ways for picking 5 cards from the given suit, so

$$|E| = 4 \cdot \binom{13}{5}$$

and the probability of a flush is

$$\mathbb{P}(E) = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}.$$

Here are more examples on counting in general and I am lazy to put into a probability setting.

Example 1.48. We have 20 different books and 5 shelves. Each shelf can hold 20 books.

(a) How many ways can we place the books if we only care about the number of books on the shelves? That is, we don't care about the location of the books.

Again, this is a direct application of Theorem 1.38 with $20+4=24$ stars and 4 bars, so we have $\binom{24}{4}$ ways.

(b) How many ways can we place the books if we care about the location of each book, but we don't care about the ordering on the shelves?

For each book, we only care about which shelf it belongs to, so there are 5^20 ways.

(c) How many ways can we place these books if we care about the order on the shelves?

First we need to decide the number of books on each shelf, that is $\binom{24}{4}$ ways. Now, we want to fix the distribution of the books, which turns into a permutation case, so there are $20!$ ways of distributing books. Thus, we have $\binom{24}{4}\cdot 20!$ ways if order matters.

Example 1.49. There are $2n+1$ identical books. We have 3 shelves. We want to place the books in a way that each pair of shelves together contains more books than the other shelf.

Let x_i be the number of books on shelf i, for $1 \leq i \leq 3$. The linear system in this case is $x_1+x_2+x_3 = 2n+1$, with constraints $x_1+x_2 > x_3$, $x_1+x_3 > x_2$, and $x_2+x_3 > x_1$. Also, we need $x_i \geq 0$ for all $1 \leq i \leq 3$.

The number of ways to distribute $2n+1$ books on those three shelves is given by Theorem 1.38 with $2n+1$ stars (books) and 3 bars (shelves), so the result is $\binom{2n+3}{2}$.

We apply the substraction principle here, to substract the number of unwanted bad distributions. The bad situation is that either one shelf has too many books, that is at least $n+1$ books so that there is more books at this shelf than on other two together.

Assume that the first shelf has too many books, our constraints to the linear system $x_1+x_2+x_3 = 2n+1$ become to $x_1 \geq n+1$ and $x_2, x_3 \geq 0$. We want to find the number of integral solutions that satisfy the above condition. Let $y_1 = x_1-(n+1)$, $y_2 = x_2$, and $y_3 = x_3$, now we have that $y_i \geq 0$ for $1 \leq i \leq 3$. The new linear system is $y_1+y_2+y_3 = n$. Applying Theorem 1.38 with n stars and 3 bars, the number of integral solution to the system we discuss is $\binom{n+2}{2}$.

We do the same for assuming the second shelf or the third shelf has too many books. Then, our final answer is given by

$$\begin{aligned}
\binom{2n+3}{2} - 3\cdot\binom{n+2}{2} &= \frac{(2n+3)!}{2!\cdot(2n+3-2)!} - 3\cdot\frac{(n+2)!}{2!\cdot(n+2-2)!} \\
&= \frac{(2n+3)\cdot(2n+2)}{2} - 3\cdot\frac{(n+2)\cdot(n+1)}{2} \\
&= \frac{4n^2+10n+6-3\cdot(n^2+3n+2)}{2} \\
&= \frac{n^2+n}{2} = \frac{n\cdot(n+1)}{2} = \binom{n+1}{2}.
\end{aligned}$$

2 Seven is More Than Six: The Pigeonhole Principle

Lecture 6
Wednesday
October 14

2.1 Basic Version of the Pigeonhole Principle

The following is really obvious, but is a very important tool. The proof illustrates how to make "obvious" things rigorous.

Theorem 2.1 (Pigeonhole Principle)**.** *Let n, k be positive integers with $n > k$. If n objects are placed into k boxed, then there is a box that has at least 2 objects in it.*

Proof. We will do proof by contradiction. So suppose that the statement is false. Then each box has either 0 or 1 object in it. Let m be the number of boxes that have 1 object in it. Then, there are m objects total and hence $n = m$. However, $m \leq k$ since there are k boxes, but this contradicts our assumption that $n > k$. $\square$

Here is an illustration that $n+1$ objects (pigeons) are distributed into n boxes (holes).

$$
\begin{array}{ccc}
\text{Pigeons} & & \text{Holes} \\
1 & \longrightarrow & 1 \\
2 & \longrightarrow & 2 \\
\vdots & & \vdots \\
n & \longrightarrow & n \\
n+1 & &
\end{array}
$$

A simple example is that among 13 students, that there are 2 who have their birthday in the same month.

Example 2.2. If $n+1$ integers are chosen from the set $\{1, 2, 3, \ldots, 2n\}$, then there exists 2 chosen numbers that differ by 1.

$$
\begin{array}{ccc}
\text{Pigeons} & & \text{Holes} \\
a_1 & \longrightarrow & \{1, 2\} \\
a_2 & \longrightarrow & \{3, 4\} \\
a_3 & \longrightarrow & \{5, 6\} \\
\vdots & & \vdots \\
a_n & \longrightarrow & \{2n-1, 2n\} \\
a_{n+1} & &
\end{array}
$$

Divide the set $\{1, 2, \ldots, 2n\}$ into subsets consist of 2 consecutive elements as

$$\{1, 2\}, \{3, 4\}, \ldots, \{2n - 1, 2n\}.$$

We want to choose $n + 1$ numbers (our pigeons) from n subsets that consist of 2 consecutive elements (our holes). By the Pigeonhole Principle, we should have at least one of the subsets must have both elements chosen at the end.

Example 2.3. Given m integers $a_1, a_2, \ldots, a_m$. There exists $k, l \in \mathbb{Z}$ where $0 \leq k < l \leq m$ such that

$$a_{k+1} + a_{k+2} + \cdots + a_l$$

is divisible by m.

Consider the sums $a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \cdots + a_m$. If one of the sums above is divisible by m, then we are done.

So, we assume that all sums listed above are not divisible by m. Then, for each sum, there is a remainder r, we obtain after dividing by m,

$$r = \{1, 2, \ldots, m - 1\}.$$

By the Pigeonhole Principle, there exists k and l with $k < l$ such that

$$a_1 + a_2 + \cdots + a_k = bm + r$$

$$a_1 + a_2 + \cdots + a_l = cm + r$$

so we note that by substraction, $a_{k+1} + \cdots + a_l = (c - b)m$, thus the sum $a_{k+1} + \cdots + a_l$ is divisible by m.

Example 2.4. We have 10 people, each with age between 1 and 60. Then, there exists 2 disjoint subsets of people such that the sum of their ages are the same.

By Lemma 1.29, there are 2^{10} subsets of S, that is, a set of 10 people. We should consider the empty set, so we still have 1023 subsets. These are our pigeons.

The sums of their ages are our desired holes. If we only take 1 person with the age 1, then this is our minimum sum of ages. If we take 10 people with the age 60, then this our maximum sum of ages. Therefore, we have 600 holes.

By the Pigeonhole Principle, there must be two age groups S_i and S_j for $1 \leq i < j \leq 1023$ sharing the same sum. By removing the common elements in S_i and S_j if there is, we obtain 2 disjoint subsets of people, whose sum of ages is the same.

Theorem 2.5 (Erdős-Szekeres)**.** *Let $x_1, x_2, \ldots, x_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Then, there exists either an increasing subsequence of length $n + 1$, or a decreasing subsequence of length $n + 1$.*

Proof. Towards a contradiction, suppose there does not exist an increasing subsequence of length $n + 1$ and not exist a decreasing subsequence of length $n + 1$.

For each $x_i \in \mathbb{R}$, we give a label (s_i, t_i) where s_i is the length of the longest decreasing subsequence ending at x_i and t_i is the length of the longest increasing subsequence ending at x_i.

By assumption, we have $1 \leq s_i \leq n$ and $1 \leq t_i \leq n$, then the number of distinct labels is n^2. We have $n^2 + 1$ labels and n^2 distinct labels.

By the Pigeonhole Principle, there are 2 labels that are the same. Then, there exists $i \neq j$ such that $(s_i, t_i) = (s_j, t_j)$ with corresponding element x_i and x_j.

If $x_i < x_j$, then the longest increasing subsequence ending at x_i has length $t_i = t_j$. The longest increasing subsequence ending at x_j has length $\geq t_j + 1$, which is a contradiction.

If $x_i > x_j$, then the longest decreasing subsequence ending at x_i has length $s_i = s_j$. The longest decreasing subsequence ending at x_j has length $\geq s_j + 1$, which is a contradiction.

This completes the proof.

□

Lecture 7
Friday
October 16

2.2 Generalized Version of the Pigeonhole Principle

Theorem 2.6 (Strong Form of the Pigeonhole Principle)**.** *Let n, m, and r be positive integers and suppose that $n > rm$. If n objects are placed into m boxes, then there is a box that contains at least $r+1$ objects in it.*

If we set $r = 1$, then this is exactly the basic version of the Pigeonhole Principle.

Proof. We can again do this via proof by contradiction. Suppose the statement is false and label the boxes 1 up to m. Let b_i be the number of objects in box number i. Then, $b_i \leq r$ by our assumption. Further, we have

$$n = b_1 + b_2 + \cdots + b_m \leq r + r + \cdots + r = rm,$$

but this contradicts the assumption that $n > rm$.

□

Example 2.7. A bag contains 100 apples, 100 bananas, 100 oranges, and 100 pears. We pick a piece of fruit every minute. How long until it is guaranteed a dozen fruit of the same kind chosen?

Our worst case is that we pick 11 pieces of fruit for every kind, so at most we need to pick $4 \cdot 11 + 1 = 45$ pieces of fruit to get a dozen.

2.3 Introduction to Graph Theory

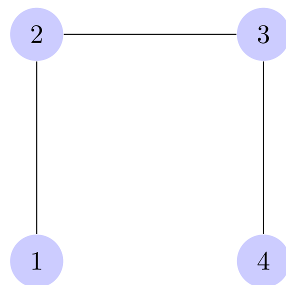
The origin of the graph theory is said to be Euler's solution of the bridges of Königsberg problem.

The problem was to find a path which starts and ends at the same point, and crosses each bridge exactly once.

Given a set V, let $\binom{V}{2}$ denote the set of 2-element subsets of V.

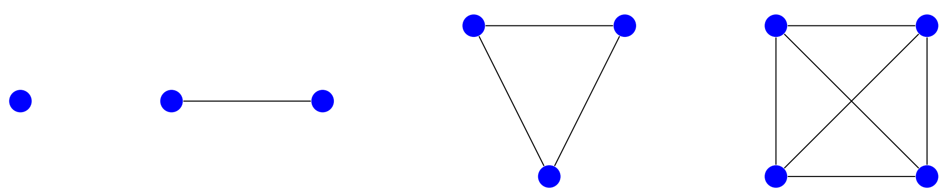
Definition 2.8 (Graph)**.** A **graph** G is a pair of sets (V, E) where V is the set of **vertices**, and E is a multiset from $V \cup \binom{V}{2}$, called the **edges**. The edges in V are called **loops**. Given an edge, the vertices that uses are called its **endpoints** (this could be the same in the case of a loop). If there are no loops and each pair of vertices has at most one edge between them, then the graph is called **simple**.

Example 2.9. The picture below represents the graph G with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$.



Definition 2.10 (Complete Graph)**.** A **complete graph** with n vertices is the graph with n vertices every two of which are adjacent. In other words, all edges are present. We denote K_n as the complete graph with n vertices.

Example 2.11 (Complete Graphs)**.** Here are some examples for K_n for $1 \leq n \leq 4$. Note that K_3 is a triangle and we often refer to K_3 as a triangle.



K_1 K_2 K_3 K_4

2.4 Ramsey's Theorem

Given K_n, suppose we color every edge red or blue. Can we guarantee any structure? Can we guarantee a monochromatic K_3, that is, a red K_3 or a blue K_3?

Theorem 2.12 ($K_6 \to K_3, K_3$)**.** *No matter how the edges of K_6 are colored with the colors red and blue, there is always a red K_3 (three of the original six points with the three line segments between them are all colored red) or a blue K_3 (three of the original six points with the three line segments between them are all colored blue), in short, a monochromous triangle.*

Proof. Fix one point p of K_6. It meets other five edges. By the Theorem 2.6, 3 edges emanating out the vertex p have the same color. Without the loss of generality, we can assume the color is blue.

Assuming the 3 points along the 3 edges emanating out the vertex p are denoted as p_1, p_2, p_3.

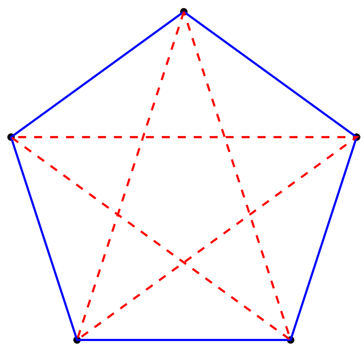
If there exists a blue edge among p_1, p_2, p_3, then we have a blue K_3.

If all edges among p_1, p_2, p_3 are red, then we have a red K_3.

□

What about a larger K_n? For larger complete graphs K_n with $n \geq 6$, we all guarantee a red K_3 or a blue K_3 since K_6 is contained in K_n with $n > 6$.

What about a smaller K_n? For $n = 5$, there is *some* way to color the edges of K_5 without creating a red K_3 or a blue K_3. Below shows a pentagon with the edges of the pentagon are blue and the edges of the inscribed pentagram are red.



Definition 2.13 (Ramsey Number)**.** Let $m, n \geq 2$ be integers. The **Ramsey Number** $r(m, n)$ is the smallest integer p such that no matter how you color the edges of K_p with red or blue, you are guaranteed either a red K_m or a blue K_n.

We show that $r(3, 3) = 6$. How about $r(4, 4)$?

Proposition 2.14 (Easy Properties of Ramsey Number)**.**

(a) $r(m, n) = r(n, m)$.

(b) $r(2, n) = n$, $r(m, 2) = m$.

Theorem 2.15. *If* $m \geq 2$ *and* $n \geq 2$*, then* $r(m, n) \leq r(m, n-1) + r(m-1, n)$.

Proof. Let $d = r(m, n-1) + r(m-1, n)$. Consider a red-blue coloring of the edges of K_d. Fix a vertex v_0. There are $d - 1$ edges emanating out of v_0. By the Pigeonhole Principle, among $d - 1$ edges, there are at least $r(m, n-1)$ edges that are blue, or there are at least $r(m-1, n)$ edges that are red.

Case 1. There are at least $r(m, n-1)$ edges emanating out of v_0 that are blue. Consider the graph on these $r(m, n-1)$ points. By Definition 2.13, this graph either has a red K_m (then we are done) or it has a blue K_{n-1}. In the second case, we just need to add in v_0 to get a blue K_n.

Case 2. There are at least $r(m-1, n)$ edges emanating out of v_0 that are red. Similar argument works as above case.

□

Corollary 2.16 (Upper Bound of Ramsey Number)**.** *For any* $m, n \geq 2$*,* $r(m, n)$ *does exist and*

$$r(m,n) \leq \binom{m+n-2}{m-1}.$$

Can we guarantee a red K_m or a blue K_n on smaller complete graphs?

Lecture 8

Monday

October 19

Proof. We prove by induction.

Base case 1. For $m = 2$, $n \geq 2$, we want to show that $r(2, n) \leq binomn1 = n$. If there exists a red edge, then there is a red K_2. Otherwise, if there are no red edges, then there is a blue K_n.

Base case 2. For $m \geq 2$, $n = 2$, we want to show that $r(m, 2) \leq \binom{m}{m-1} = m$. We will get either a blue K_2 or a red K_m by the similar argument as above.

Inductive step. Assume the statement holds for $m' < m$ or $n' < n$, we want to show that

$$r(m,n) \leq \binom{m+n-2}{m-1}.$$

Denote $p = \binom{m+n-2}{m-1}$. Fix a vertex, say v_0. By Pascal's Identity, we have

$$\begin{aligned} p - 1 &= \binom{m+n-2}{m-1} - 1 \\ &= \binom{(m-1)+n-2}{(m-1)-1} + \binom{m+(n-1)-2}{m-1} - 1. \end{aligned}$$

By the string form of Pigeonhole Principle, we have either $\binom{(m-1)+n-2}{(m-1)-1}$ edges emanationg out of v_0 with color red or $\binom{m+(n-1)-2}{m-1}$ edges emanating from v_0 with color blue.

Case 1. There are $\binom{(m-1)+n-2}{(m-1)-1}$ edges emanationg out of v_0 with color red. Then, by the induction hypothesis, there exists either a red K_{m-1} or a blue K_n. By adding v_0, then we will get a red K_m.

Case 2. There are $\binom{m+(n-1)-2}{m-1}$ edges emanating from v_0 with color blue. Then, by the induction hypothesis, there exists either a red K_m or a blue K_{n-1}. By adding v_0, we will get a blue K_n.

We are done.

□

Example 2.17. $r(3, 4) \leq 10$.

Fix a vertex, say v_0. By the Pigeonhole Principle, there exists 4 red edges or 6 blue edges emanating out of v_0.

Case 1. There are 4 red edges. If there exists a red edge, then we have a red K_3. Otherwise, all edges are blue, then we have a blue K_4.

Case 2. There are 6 blue edges. By Theorem 2.12, we automatically have a red K_3 or a blud K_3. By adding v_0, we have a blud K_4.

Applying Theorem 2.15 gives a quicker way

$$r(3,4) \leq r(2,4) + r(3,3) = 4 + 6 = 10.$$

However, as we will prove now, $r(3,4) = 9$, so this bound isn't optimal.

Theorem 2.18. $r(4,3) \leq 9$.

Proof. Fix a vertex, say v_0.

Case 1. Suppose that there are at least 4 red edges emanating out of v_0. Let S be the set of vertices that are adjacent to v_0 by a red edge. Since S contains at least four vertices and since $r(2,4) = 4$ (we did not prove that), the 2-coloring of the edges that are within the subgraph induced by S must produce either a red K_2 or a blue K_4 within this subgraph itself. If there is a red K_2 within the subgraph, then we have a red K_3 together with v_0. If there exists a blue K_4 within the subgraph induced by S, then we obviously have already found one in the existent complete graph.

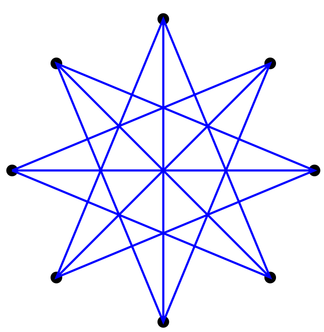
Case 2. Suppose that there are at least 6 blue edges emanating out of v_0. Let T be the set of vertices that are adjacent to v_0 by a blue edge. Since T contains at least 6 vertices and since $r(3,3) = 6$, the 2-coloring of the edges that are within the subgraph induced by T must produce either a red K_3 or a blue K_3 within the subgraph itself. If there is a red K_3 within this subgraph, we have one obviously. If there is a blue K_3 within the subgraph, then, we have a blue K_4 together with v_0.

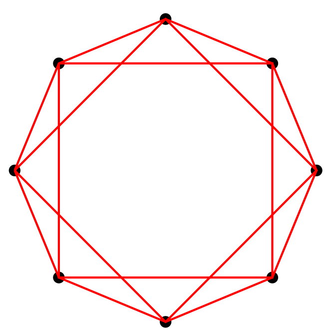
Case 3. Suppose that there are fewer than 4 red edges and fewer than 6 blue edges emanating out of v_0. However, this case can never occur.

We conclude that $r(4,3) \leq 9$.

□

To see why $r(4,3) \geq 9$, we give an example of a red-blue coloring of the edges of K_8.





Theorem 2.19. $r(4,3) = 9$.

Here is one last calculation.

Example 2.20. $r(3,3,3) \leq 17$.

Fix a vertex, say v_0. By the Pigeonhole Principle, among the rest 16 edges emanating out of v_0, there exists 6 edges of the same color.

Without loss of generality, assume we have 6 blue edges emanating out of v_0. If there exists a blue edge among K_6, then blue K_3 exists. If there is no blue edge, then by Theorem 2.12, we are done.

Definition 2.21 (Generalized Ramsey Number)**.** Let $n_1, \dots, n_r \geq 2$ be integers, and let $c_1, \dots, c_r$ be different colors. Define the **Ramsey Number** $r(n_1, \dots, n_r)$ to be the smallest integer d such that, for any way to color the edges of K_d using the colors $c_1, \dots, c_r$, there is, for some i, a K_{n_i} with the color c_i.

Proposition 2.22 (Properties of Ramsey Number)**.**

(a) Ramsey Numbers are symmetric in the inputs, that is, swapping the order of the n_i results in the same value.

(b) If any of the values are 2, then we can remove them the number is unchanged. For example, $r(2, n_2, \dots, n_r) = r(n_2, \dots, n_r)$.

Here is a generalization of the inequality from before (the proof is similar).

Theorem 2.23. *Assume all $n_i \geq 3$. Then,*

$$r(n_1, \dots, n_r) \leq r(n_1 - 1, n_2, \dots, n_r) + r(n_1, n_2 - 1, \dots, n_r) + \cdots + r(n_1, n_2, \dots, n_r - 1) - (r - 2).$$

In the right side, we substract 1 from each input in all possible ways and add the results.

Recall Example 2.20, we will have

$$\begin{aligned} r(3,3,3) &\leq r(2,3,3) + r(3,2,3) + r(3,3,2) - 1 \\ &= r(3,3) + r(3,3) + r(3,3) - 1 \\ &= 6 + 6 + 6 - 1 = 17. \end{aligned}$$

We can also use it to give an explicit upper bound using the multinomial coefficients.

Corollary 2.24.

$$r(n_1, \dots, n_r) \leq \binom{n_1 + \cdots + n_r - r}{n_1 - 1, \dots, n_r - 1}.$$

Now, we are interested in the lower bounds on Ramsey Numbers. We have got some lower bounds by constructing specific examples, but what can be said in general?

Lemma 2.25. *If $n \geq 3$, then $2 \cdot \dfrac{2^{n/2}}{n!} < 1$.*

Proof. We prove by induction.

If $n = 3$, then the left side is $\frac{2}{3}\sqrt{2}$. Its square is $\frac{8}{9}$, which is less than 1, so the inequality holds.

Now, assume that $2 \cdot \dfrac{2^{n/2}}{n!} < 1$. We want to show the same is true for $n + 1$. Since $n \geq 3$, we have $n + 1 > \sqrt{2}$, and so $\dfrac{\sqrt{2}}{n+1} < 1$. Multiply the given inequality by $\dfrac{\sqrt{2}}{n+1}$ to get

$$2 \cdot \frac{2^{(n+1)/2}}{(n+1)!} < \frac{\sqrt{2}}{n+1} < 1.$$

□

The next theorem illustrates how we can use *random examples* to get lower bounds. Constructing an example with specific properties is often hard, but randomly assigning red or blue to edges may do the trick.

Theorem 2.26 (Lower Bound on Ramsey Numbers). *For any $n \geq 3$, we have $r(n,n) \geq 2^{n/2}$.*

Previously, we proved that $r(n,n) \leq \binom{2n-2}{n-1}$. For example, when $n = 4$, we get the inequalities

$$4 \leq r(4,4) \leq 20.$$

They are far off, so it is a little unsatisfying (we know the real value is 18). For $n = 5$, our inequality gives $(\sqrt{2})^5$ as a lower bound, but it rounds up to 6, so

$$6 \leq r(5,5) \leq 70.$$

Again, the range is pretty wide. We can improve the upper bound to 50 since $r(4,5) = 20$ (we did not prove that). In fact, the best known ineuqalities so far are

$$43 \leq r(5,5) \leq 48,$$

but again, the exact value is not known.

3 No Matter How You Slice It: Binomial Theorem

Lecture 9
Wednesday
October 21

Definition 3.1 (Binomial Coefficient)**.** $\binom{n}{k}$ is the number of k-subsets in an n-element set.

Let $f(n,k)=\binom{n}{k}$. How $f(n,k)$ behaves?

Proposition 3.2 (Some Properties of Binomial Coefficient)**.**

(a) $\binom{n}{k}=\frac{n!}{k!(n-k)!}$

(b) $\binom{n}{k}=\binom{n}{n-k}$

(c) $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ *(This is the Pascal's Identity (Theorem 1.28).)*

Pascal's triangle is a triangle which contains the binomial coefficients. Its various properties play a large role in combinatorics.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}
\end{array}
$$

Each entry in the triangle, other than those equal to 1 occuring on the boundary of the triangle, is obtained by adding together two entries in the row above: the one directly above and the one immediately to the left. This is in accordance with Pascal's Identity (Theorem 1.28). For instance, we have

$$\binom{4}{3}=4=3+1=\binom{3}{2}+\binom{3}{3}.$$

Many of the relations involving binomial coefficients can be discovered by careful verification of the Pascal's triangle. The symmetry relation

$$\binom{n}{k}=\binom{n}{n-k}$$

is readily noticed in the triangle.

3.1 Binomial Theorem

The binomial coefficient $\binom{n}{k}$ has many fascinating properties. It comes from one famous theorem, the Binomial Theorem.

Theorem 3.3 (Binomial Theorem)**.** *Let n be a positive integer, then for any x and y,*

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + y^n.$$

In summation notation,

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

Proof. We prove by induction.

For $n = 1$, the formula becomes

$$(x+y)^1 = \sum_{k=0}^{1}\binom{1}{k}x^{1-k}y^k = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0y^1 = x+y$$

and this is forever true.

Now, assume $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$ holds, then we want to show that it also holds for $n+1$. Then,

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)\cdot(x+y)^n \\
&= (x+y)\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\
&= x\cdot\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) + y\cdot\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1} \\
&= \binom{n}{0}x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=0}^{n-1}\binom{n}{k}x^{n-k}y^{k+1} + \binom{n}{n}y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k}x^{n+1-k}y^k + \sum_{k=1}^{n}\binom{n}{k-1}x^{n+1-k}y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]x^{n+1-k}y^k + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^{n+1-k}y^k + y^{n+1}.
\end{aligned}$$

Since $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$, we can rewrite this last equation and obtain

$$(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^{n+1-k}y^k.$$

$\square$

Here is a faster proof. For each k, we note that $x^{n-k}y^k$ appears $\binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$ times.

Theorem 3.4. $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$.

We note that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

this is the number of all subsets of an n-element set.

Another quick observation is that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0.$$

Similarly, we have

$$\sum_{k=0}^{n}\binom{n}{k}2^n=\binom{n}{0}2^0+\binom{n}{1}2^1+\binom{n}{2}2^2+\cdots+\binom{n}{n}2^n=3^n$$

with an application of Theorem 3.3 where $x=2$ and $y=1$.

Theorem 3.5. $\displaystyle\sum_{k=1}^{n}(-1)^{k-1}k\binom{n}{k}=0.$

Proof. Differentiating both sides of Theorem 3.4, we obtain

$$n(1+x)^{n-1}=0+\binom{n}{1}+2\binom{n}{2}x+\cdots+(n-1)\binom{n}{n-1}x^{n-2}+nx^{n-1}$$

Set $x=-1$, we have

$$\binom{n}{1}-2\binom{n}{2}x+\cdots+(-1)^{n-1}n=0.$$

Thus, $\displaystyle\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\cdot k=0.$ ∎

Lecture 10

Friday

October 23

Let's get further. Consider taking the twice derivitive $(1+x)^n$ and adjust the coefficient, then

$$\begin{aligned}\sum_{k=1}^{n}k^2\binom{n}{k}&=\binom{n}{1}+2^2\binom{n}{2}+3^2\binom{n}{3}+\cdots+n^2\binom{n}{n}\\&=\sum_{k=1}^{n}k\binom{n}{k}+\sum_{k=1}^{n}(k^2-k)\binom{n}{k}\\&=n\cdot 2^{n-1}+n(n-1)\cdot 2^{n-2}=n(n+1)2^{n-2}\end{aligned}$$

Repeat this argument, we can obtain $\displaystyle\sum_{k=1}^{n}k^p\binom{n}{k}$ for any positive integer p, but this gets increasingly complicated.

Proposition 3.6. $\displaystyle\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}.$

Proof. Let S be a set with $2n$ elements. Partition S itno two subsets, S_1 and S_2, of n elements each. We note that $\binom{2n}{n}$ is the number of subsets of size n in S.

Each n-subset of S contains k elements of S_1 and $n-k$ elements of S_2, with $0 \le k \le n$. Then,

$$\begin{aligned}\#\text{ of } n\text{-subset} &= \binom{n}{n} + \binom{n}{n-1}\binom{n}{1} + \cdots + \binom{n}{n-k}\binom{n}{k} + \cdots + \binom{n}{0}\binom{n}{n} \\ &= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{k}^2 + \cdots + \binom{n}{n}^2.\end{aligned}$$

Thus, we have two ways to express the number of n-subsets. Hence, the equality holds.

□

Example 3.7. A student walks from her home to school, located 10 blocks east and 14 blocks north from home. The student always goes through the shortest walk, only walk north or east. There are $\binom{24}{10}\binom{14}{14} = \binom{24}{10}$ different walks.

How many walks (24 blocks) are there if the student steps by the friend's home, at 4 blocks east and 5 blocks north? To get to friend's home, there are $\binom{9}{4}\binom{5}{5} = \binom{9}{4}$ possible ways. To get to school from friend's home, there are $\binom{15}{6}\binom{9}{9} = \binom{15}{6}$ ways. Thus, there are $\binom{9}{4}\binom{15}{6}$ possible ways.

3.2 Unimodality of Binomial Coefficients

Consider the nth level of Pascal's triangle

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots \quad \binom{n}{k} \quad \cdots \quad \binom{n}{n}$$

Definition 3.8 (Unimodality)**.** Given a sequence of real numbers $s_0, s_1, s_2, \ldots, s_n$, then the sequence is **unimodal** if there exists $0 \le t \le n$ such that $s_0 \le s_1 \le s_2 \le \cdots \le s_t$ and $s_t \ge s_{t+1} \ge \cdots \ge s_n$.

Theorem 3.9. *Let n be a positive integer. Then, the sequence*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

is unimodal. More precisely, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2}$$

and

$$\binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n};$$

and if n is odd,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2}$$

and

$$\binom{n}{(n+1)/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

Proof. Consider the quotient of successive binomial coefficients in the sequence. Let k be an integer with $1 \leq k \leq n$. Then,

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k}.$$

Case 1. If $\frac{n-k+1}{k} > 1$, then $k < \frac{n+1}{2}$.
If n is even, then $k \leq \frac{n}{2}$. If n is odd, then $k \leq \frac{n-1}{2}$.
Case 2. If $\frac{n-k+1}{k} = 1$, then $k = \frac{n+1}{2}$.
Since $k \in \mathbb{Z}$, this can only happen when n is odd.
Case 3. If $\frac{n-k+1}{k} < 1$, then $k > \frac{n+1}{2}$.
If n is even, then $k \geq \frac{n}{2}$. If n is odd, then $k \geq \frac{n-1}{2}$.

□

Lecture 11
Monday
October 26

Definition 3.10 (Floor, Ceiling)**.** For any real number x, let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x. The integer $\lfloor x \rfloor$ is called the **floor** of x. Similarly, the **ceiling** of x is the smallest integer $\lceil x \rceil$ that is greater than or equal to x.

Corollary 3.11. *For a positive integer n, the largest of the binomial coefficients (at the nth level of the Pascal's triangle) is*

$$\binom{n}{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor}.$$

This is a direct result from Theorem 3.11 and the preceding observations about the floor and ceiling functions.

3.3 Extremal Set Theory

Definition 3.12 (Chain)**.** Let S be a set of n elements. Let $\mathcal{F}$ be a family of subsets of S. We say that $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$, where A_i is a subset of S for all $1 \leq i \leq m$, is a **chain** if $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$.

Given $|S| = n$, how large can a chain be? For $S = \{1, 2, \ldots, n\}$, we can naturally think about a chain to be

$$\mathcal{F} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, \ldots, n\}\}.$$

In this case, for each step, we step in the chain by adding a new element. So, the largest chain has size not greater than n.

Definition 3.13 (Antichain)**.** Let S be a set of n elements. Let $\mathcal{F}$ be a family of subsets of S. We say that $\mathcal{F}$ is an **antichain** if no member in $\mathcal{F}$ contains another member in $\mathcal{F}$.

Given $|S| = n$, how large can an antichain be? For $S = \{1, 2, \ldots, n\}$, we can naturally think about an antichain to be

$$\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \ldots, \{n\}\}.$$

This yields the largest size, which is $|\mathcal{F}| = n$.

However, this is not the largest antichain. Let $\mathcal{F}$ be all pairs of the set S with the assumption that the cardinality of S is sufficiently large. We have $|\mathcal{F}| = \binom{n}{2} > n$ as n is large enough. In this case, each subset has 2 elements, and no subset can contain another. So, this is an antichain. Further, by Corollary 3.11, an antichain contains at most $\binom{n}{\lfloor n/2 \rfloor}$ sets.

Theorem 3.14 (Sperner)**.** *Let S be a set of n elements. Then an antichain $\mathcal{F}$ on S has at most $\binom{n}{\lfloor n/2 \rfloor}$ sets.*

Proof. We count in two different ways the number β of ordered pairs (A, C) such that A is in $\mathcal{F}$, and C is a *maximal chain* containing A. For C, since each maximal chain contains at most one subset in the antichain $\mathcal{F}$, β is at most the number of maximal chains, that is, $\beta \leq n!$. For A in the antichain A, if $|A| = k$, there are at most $k! \cdot (n-k)!$ maximal chains C containing A. Let α_k be the number of subsets in the antichain $\mathcal{F}$ of size k so that $|\mathcal{F}| = \sum_{k=0}^{n} \alpha_k$. Then,

$$\beta = \sum_{k=0}^{n} \alpha_k \cdot k! \cdot (n-k)!$$

and since $\beta \leq n!$, we get

$$\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} \leq 1.$$

By Corollary 3.11, $\binom{n}{k}$ is maximum when $k = \lfloor n/2 \rfloor$. We conclude that

$$\mathcal{F} \leq \binom{n}{\lfloor n/2 \rfloor}.$$

□

Before we proceeds to another theorem in extremal set theory, we first introduce the *partially ordered set*, in short, *poset*.

Definition 3.15 (Partially Ordered Set)**.** A **partially ordered set** or **poset** is a set P and a binary relation $\leq$ such that for all $a, b, c \in P$, we have

(a) (Reflexivity) $a \leq a$.

(b) (Transitivity) $a \leq b$ and $b \leq c$ implies $a \leq c$.

(c) (Anty-symmetry) $a \leq b$ and $b \leq a$ implies $a = b$.

Definition 3.16. A **minimal element** of a partially ordered set is an element a such that no element x satisfies $x < a$. A **maximal element** is an element b such that no element y satisfies $b < y$.

The set of all minimal elements of a partially ordered set forms an antichain, as does the set of all maximal elements.

Theorem 3.17. *Let $(X, \leq)$ be a finite partially ordered set, and let r be the largest size of a chain. Then, X can be partitioned into r but no fewer antichains.*

4 No Overcount: The Sieving Methods

A "sieving method" is a technique that allows us to count or list some things indirectly. We will introduce two sieving methods in this section. Recall that the addition principle gives a formula for counting the number of objects in a union of sets, *provided that the sets do not overlap.* The **Inclusion-Exclusion Principle** gives a formula for the most general of circumstances in which the sets are free to overlap without restriction. We will also derive a generalization of the Inclusion-Exclusion Principle for general partially ordered sets, called **Möbius Inversion**.

Lecture 14 Wednesday November 4

4.1 The Inclusion-Exclusion Principle

Example 4.1 (Math is fun.)**.** Given 9 distinct letters $S = \{\mathrm{M,A,T,H,I,S,F,U,N}\}$. The number of 9-permutations of S is $|S| = 9!$.

Variation 1. How many permutations of S are there without the word MATH occuring as consecutive letters?

Let A_1 be the set of permutations that satisfy our requirement. $|A_1^\complement|$ is our target. Consider treat MATH as a single letter, then $|A_1| = 6!$ clearly. Then, by Proposition 1.13, $|A_1^\complement| = |S| - |A_1| = 9! - 6!$.

Variation 2. How many permutations avoid MATH and avoid IS?

Let A_1 be the set of words with MATH and A_2 be the set of words with IS, with $|A_1| = 6!$ and $|A_2| = 8!$. $|A_1^\complement \cap A_2^\complement|$ is our target in this case.

Are there $9! - 6! - 8!$ permutations avoid MATH and avoid IS? Nope, since we substract too much for the permutations that avoid both MATH and IS, that is $|A_1 \cap A_2|$.

I need to add back words with both MATH and IS once because we substract this amount twice before in both A_1 and A_2. Clearly, $|A_1 \cap A_2| = 5!$. Therefore, $|A_1^\complement \cap A_2^\complement| = 9! - 6! - 8! + 5!$. This is saying that $|A_1^\complement \cap A_2^\complement| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$.

Theorem 4.2. *The number of objects of the set S that have none of the properties $P_1, P_2, \ldots, P_m$ is given by the alternating expression*

$$\left|A_1^\complement \cap A_2^\complement \cap \cdots \cap A_m^\complement\right| = |S| - \sum_{i=1}^{m} |A_i| + \sum_{1\le i<j\le m} |A_i \cap A_j| - \sum_{1\le i<j<k\le m} |A_i \cap A_j \cap A_k| + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|.$$

Proof. □

Example 4.3 (Continuation of Example 4.1)**.** How many permutations are there such that none of the words MATH, IS, and FUN occur as consecutive letters?

Further, let A_3 be the set of words with FUN, thus $|A_3| = 7!$. Also, $|A_1 \cap A_2| = 5!$, $|A_1 \cap A_3| = 4!$, and $|A_2 \cap A_3| = 6!$. Finally, $A_1 \cap A_2 \cap A_3| = 3!$ since these are permutations of three groups of letters MATH, IS, and FUN.

By Theorem 4.2, we have

$$\begin{aligned}\left|A_1^{\complement} \cap A_2^{\complement} \cap A_3^{\complement}\right| &= |S| - (|A_1| + |A_2| + |A_3|) \\ &\quad + (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|) - |A_1 \cap A_2 \cap A_3| \\ &= 9! - 6! - 8! - 7! + 5! + 6! + 4! - 3!.\end{aligned}$$

Lecture 15 Friday November 6

Theorem 4.4 (Inclusion-Exclusion)**.** *The number of objects of S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by*

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{i=1}^{m} |A_i| - \sum_{1 \le i < j \le m} |A_i \cap A_j| + \sum_{1 \le i < j < k \le m} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m+1} |A_1 \cap A_2 \cap \cdots \cap A_m|.$$

Proof. The set $A_1 \cup A_2 \cup \cdots \cup A_m$ consists of all these objects in S which possess at least one of the properties. Also,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |S| - \left|(A_1 \cup A_2 \cup \cdots \cup A_m)^{\complement}\right|.$$

By the DeMorgan's Law,

$$(A_1 \cup A_2 \cup \cdots \cup A_m)^{\complement} = A_1^{\complement} \cap A_2^{\complement} \cap \cdots \cap A_m^{\complement}$$

we have

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |S| - \left|A_1^{\complement} \cap A_2^{\complement} \cap \cdots \cap A_m^{\complement}\right|.$$

Combining this equation with Theorem 4.2, we are done. ∎

Note that the formula we get is also equivalent to

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{k=1}^{m} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

4.2 Combinations with Multisets

Previously, we showed that the number of r-subsets of a set with n *distinct* element is $\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$ and that the number of r-combinations of a multiset with k distinct objects, *each with an infinite repetition number*, equals $\binom{r+k-1}{r}$.

Consider the set $S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$. The number os 12-permutations is $\frac{12!}{3! \cdot 4! \cdot 5!}$. The number of 11-permutations is $\frac{11!}{2! \cdot 4! \cdot 5!} + \frac{11!}{3! \cdot 3! \cdot 5!} + \frac{11!}{3! \cdot 4! \cdot 4!} = \frac{12!}{3! \cdot 4! \cdot 5!}$.

How about combinations? We only have 3 subsets with 1 element. They are $\{a\}$, $\{b\}$, and $\{c\}$. How many 10-subsets are there? We need to consider all the cases with application of the Inclusion-Exclusion Principle.

Example 4.5. Bakery has three kinds of donus: chocolate, cinnanon, and plain. A box contains 12 donuts.

Let S be the number of all 12-combinations of doughnuts with infinite amount each. Then, we have 12 stars and $3 - 1 = 2$ bars, so

$$|S| = \binom{12+2}{2} = \binom{14}{2}.$$

Now, assume at a particular time, there are only 6 chocolate, 6 cinnamon, and 3 plain donuts.

Consider the linear equation $x_1 + x_2 + x_3 = 12$ where x_1 is the number of chocolate doughnuts, x_2 is the number of cinnamon doughnuts, and x_3 is the number of plain doughnuts.

Let S_1 be the set that there are at least 7 chocolate doughnuts. Then, consider the transformation $y_1 = x_1 - 7$, $y_2 = x_2$, and $y_3 = x_3$. Thus, we are interested in the number of integral solutions of $y_1 + y_2 + y_3 = 5$. Thus,

$$|S_1| = \binom{5+2}{2} = \binom{7}{2}.$$

Let S_2 be the set that there are at least 7 cinnamon doughnuts, so $|S_2| = \binom{7}{2}$.

Let S_3 be the set that there are at least 4 plain doughnuts, similarly, we have

$$|S_3| = \binom{8+2}{2} = \binom{10}{2}.$$

Now, the set $S_1 \cap S_2$ denotes that there are at least 7 chocolate doughnuts and at least 7 cinnamon doughnuts, but this case is impossible, so $|S_1 \cap S_2| = 0$. Also, the set $S_2 \cap S_3$ denotes that there are at least 7 cinnamon doughnuts and at least 4 plain doughnuts, so by the Stars and Bars Theorem, we have

$$|S_2 \cap S_3| = \binom{12-7-4+2}{2} = \binom{3}{2}.$$

Similarly, $|S_3 \cap S_1| = \binom{3}{2}$. Still, the last case $|S_1 \cap S_2 \cap S_3| = 0$ since $S_1 \cap S_2$ is an empty set already.

Therefore, by Theorem 4.2,

$$\begin{aligned} \left|S_1^{\complement} \cap S_2^{\complement} \cap S_3^{\complement}\right| &= |S| - |S_1| - |S_2| - |S_3| + |S_1 \cap S_2| + |S_2 \cap S_3| + |S_3 \cap S_1| - |S_1 \cap S_2 \cap S_3| \\ &= \binom{14}{2} - 2 \cdot \binom{7}{2} - \binom{10}{2} + 2 \cdot \binom{3}{2}. \end{aligned}$$

4.3 Derangements

Now, let's n think of a permutation $\{1, 2, \ldots, n\}$ (abbreviated as $[n]$) as the same thing as a bijection $f : [n] \to [n]$ (given a bijection, $f(i)$ is the position in the permutation where i is supposed to appear).

Definition 4.6 (Derangement)**.** A **derangement** of size n n is a permutation such that for all i, i i does not appear in position i. Equivalently, it is a bijection f such that $f(i) \neq i$ for all i.

Theorem 4.7. *Let D_n be the number of derangements of $\{1, 2, \ldots, n\}$. For $n \geq 1$,*

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!}.$$

Proof. It turns out to be easier to count the number of permutations which are *not* derangements and then subtract that from the total number of permutations. For $i = 1, 2, \ldots, n$, let A_i be the set of bijections f such that $f(i) = i$. Then, the set of non-derangements is $A_1 \cup A_2 \cup \cdots \cup A_n$. We need to count $\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}\right|$ for some choice of indices $i_1, i_2, \ldots, i_j$. This is the set of bijections $f : [n] \to [n]$ such that $f(i_1) = i_1, f(i_2) = i_2, \ldots, f(i_j) = i_j$. The remaining information to specify f are its values outside of $i_1, i_2, \ldots, i_j$, which we can interpret as a bijection of $[n] \setminus \{i_1, i_2, \ldots, i_j\}$ to itself. So, there are $(n-j)!$ of them. So, we get

$$\begin{aligned}
|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_j \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}| \\
&= \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \le i_1 < \cdots < i_j \le n} (n-j)! \\
&= \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} (n-j)! \\
&= \sum_{j=1}^{n} (-1)^{j+1} \frac{n!}{j!}.
\end{aligned}$$

Remember that we have to subtract this from $n!$. Finally, we get

$$D_n = n! - \sum_{j=1}^{n} (-1)^{j+1} \frac{n!}{j!} = \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}.$$

□

The problem with the number of derangements is the alternating sign, which makes hard to find the behavior of the quantity as n grows. Note that the binomial coefficients $\binom{n}{i}$ (for fixed i) limit to infinity as n goes to infinity.

However, the alternating sum $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$. For derangements, recall that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. If we plug in $x = -1$ and only take the terms up to $i = n$, then we get the number of derangements divided by $n!$, that is, the percentage of permutations that are derangements. From Calculus, taking the first n terms of a Taylor Expansion is supposed to be a good approximation for a function, so for $n \to \infty$, the proportion of permutations that are derangements is $e^{-1} \approx 0.368$, or roughly 36.8%.

Example 4.8. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. How many permutations are there such that exactly 4 integers are in its natural positions?

There are $\binom{8}{4}$ ways to choose four integers that should be in their natural positions. So, there are other four integers are not in their natural positions. Then, D_4 is the number of derangements of $\{1, 2, \ldots, 8\}$. Thus, there are

$$\binom{8}{4} \cdot 4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

permutations of $\{1, 2, \ldots, 8\}$ that exactly four integers are in their natural positions.

Lecture 16
Monday
November 9

Example 4.9. At a party, 7 gentlemen check their hats. How many ways can the hat be returned such that no gentlemen receives his own hat? Since no one receives his own hat, this is a derangement of 7 hats. Thus, there are D_7 ways that no gentleman receives his own hat.

How about at least one gentleman receives his own hat? We note that the complement of the event that at least one of the gentlemen receives his own hat is no gentleman receives his own hat. There are 7! permutations of these hats. Thus, there are $7! - D_7$ ways that at least one of the gentlemen receives his own hat.

How about at least two gentlemen receive his own hat? Further, we want to determine the ways that exactly one gentleman receives his own hat. There are 7 ways to fix one man who receives his own hat. The rest is the derangement of 6 hats. Thus, there are $7! - D_7 - 7 \cdot D_6$ ways that at least two of the gentlemen receive their own hats.

Example 4.10 (Circular Permutations)**.** How many circular permutations are there is the multiset $S = \{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\}$ such that for each letter type, all letters do not appear consecutively?

Since there only one d, so after d settles, all other seats are labeled. Now, the question turns to be the ways that $\{3 \cdot a, 4 \cdot b, 2 \cdot c\}$ can be seated in seats 1 through 9 in a linear permutation.

Let A be the set of circular permutations with all a's appear consecutively, B be the set of circular permutations with all b's appear consecutively, and C be the set of circular permutations with all c's appear consecutively. That is, for set A, we can treat "aaa" as a single letter, similar idea applies for B and C.

For set A, we treat all a's as a single letter, then it turns into a permutation of 1 aaa, 4 b's, and 2 c's. Thus, there are $|A| = \dfrac{7!}{1! \cdot 4! \cdot 2!} = \dfrac{7!}{4! \cdot 2!}$ permutations such that all a's appear consecutively. Similarly, we have $|B| = \dfrac{(3+1+2)!}{3! \cdot 1! \cdot 2!} = \dfrac{6!}{3! \cdot 2!}$ and $|C| = \dfrac{(3+4+1)!}{3! \cdot 4! \cdot 1!} = \dfrac{8!}{3! \cdot 4!}$.

Going deeper, for the intersection of A and B, we are permutating 1 aaa, 1 $bbbb$, and 2 c's. Then, there are $|A \cap B| = \dfrac{(1+1+2)!}{1! \cdot 1! \cdot 2!} = \dfrac{4!}{2!}$ permutations such that all a's and all b's appear consecutively. Similarly, we have $|B \cap C| = \dfrac{(3+1+1)!}{3! \cdot 1! \cdot 1!} = \dfrac{5!}{3!}$ and $|C \cap A| = \dfrac{(1+4+1)!}{1! \cdot 4! \cdot 1!} = \dfrac{6!}{4!}$. Finally, we have $|A \cap B \cap C| = 3!$ as a permutation of aaa, $bbbb$, and cc.

Thus, there are

$$\frac{9!}{3! \cdot 4! \cdot 2!} - \frac{7!}{4! \cdot 2!} - \frac{6!}{3! \cdot 2!} - \frac{8!}{3! \cdot 4!} + \frac{4!}{2!} + \frac{5!}{3!} + \frac{6!}{4!} - 3!$$

permutations such that for each type of letter, all letters of that type do not appear consecutively.

5 A Function Is Worth Many Numbers: Generating Functions

Let g_n be the number of nonnegative integral solutions of the linear system $x_1 + x_2 + x_3 + x_4 = n$. Then, the general term of the sequence $g_0, g_1, g_2, \ldots, g_n, \ldots$ satisfies

$$g_n = \binom{n+3}{3}.$$

Now, we will develop algebraic methods for solving some counting problems invoving an unknown parameter n. Our methods lead to a function, namely, **generating function**.

5.1 Sequences

Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ denote a sequence of numbers.

Definition 5.1 (Arithmetic Sequence)**.** Each term in **arithmetic sequences** is a constant q more than the previous term.

For $n \geq 1$, let h_n be the nth term of the arithmetic sequence, we have the rule

$$h_n = h_{n-1} + q$$

and the general term is given by

$$h_n = h_0 + n \cdot q \qquad n \geq 0.$$

Definition 5.2 (Geometric Sequence)**.** Each term in **geometric sequences** is a constant multiple q of the previous term.

For $n \geq 1$, let h_n be the nth term of the geometric sequence, we have the rule

$$h_n = q \cdot h_{n-1}$$

and the general term is

$$h_n = h_0 \cdot q^n \qquad n \geq 0.$$

Proposition 5.3 (Sum of Geometric Sequence)**.** *For the common ratio $q \neq 1$, the sum of the first $n+1$ terms of a geometric sequence is given by*

$$\sum_{k=0}^{n} h_0 \cdot q^k = h_0 \cdot \frac{1 - q^{n+1}}{1 - q}.$$

Proof. Let $S = \sum_{k=0}^{n} h_0 \cdot q^k$. We note that

$$\begin{aligned} S &= h_0 + h_0 q + h_0 q^2 + \cdots + h_0 q^n \\ q \cdot S &= \phantom{h_0 + {}} h_0 q + h_0 q^2 + \cdots + h_0 q^n + h_0 q^{n+1} \end{aligned}$$

Then, $S \cdot (1-q) = h_0 - h_0 \cdot q^{n+1}$. We conclude that for $q \neq 1$,

$$S = h_0 \cdot \frac{q^{n+1}-1}{q-1}.$$

$\square$

For $q = 1$, it is immediate that $S = n+1$ since every term is 1.

Further, we can explore about the convergence of geometric series, but this is not the main point for this section. We will encounter the partial sum of geometric series soon.

5.2 Linear Recurrence Relation

Our first application of ordinary generating functions is to solve linear recurrence relations. A sequence of numbers is said to satisfy a **linear recurrence relation of order** d if there are scalars $c_1, c_2, \ldots, c_d$ such that $c_d \neq 0$, and for all $n \geq d$, we have

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_d a_{n-d}.$$

Definition 5.4 (Fibonacci Numbers)**.** The Fibonacci numbers F_n are given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \cdots$. This isn't really telling what the general F_n is, so instead we define that for all $n \geq 2$, we have

$$F_n = F_{n-1} + F_{n-2}.$$

Together with the initial conditions $F_0 = 1$, $F_1 = 1$, this is enough information to calculate any F_n. By definition, the Fibonacci numbers satisfy a linear recurrence relation of order 2.

Before finding the general formula for the Fibonacci sequence, let's first explore a remarkable property.

Corollary 5.5. *The partial sums of the terms of the Fibonacci sequence are*

$$S_n = F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

In particular, the partial sums are one less than a Fibonacci number.

Proof. We prove by induction.

Base step. For $n = 0$, we have $S_n = F_0 = 1$ and $F_{n+2} - 1 = F_2 - 1 = 1 + 1 - 1 = 1$.

Inductive step. Assume the statement holds for $n' < n$, then

$$\begin{aligned} F_0 + F_1 + F_2 + \cdots + F_n &= (F_0 + F_1 + F_2 + \cdots + F_{n-1}) + F_n \\ &= F_{n+1} - 1 + F_n \\ &= F_{n+2} - 1. \end{aligned}$$

This completes our proof.

$\square$

Example 5.6 (Fibonacci Continued)**.** Recall the Fibonacci numbers F_n, defined by

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \qquad n \geq 2. \end{aligned}$$

Assume we know how to solve the homogeneous linear recurrence relation for now. This is similar to the second order differential equation. The characteristic polynomial is $t^2 - t - 1$, with roots $\frac{1 \pm \sqrt{5}}{2}$. Let $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. Then, we have

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Plug in $n = 0, 1$, we get

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ \alpha_1 \cdot r_1 + \alpha_2 \cdot r_2 &= 1 \end{aligned}.$$

Here I omit the calculation process, so

$$\alpha_1 = \frac{1+\sqrt{5}}{2\sqrt{5}} \qquad \alpha_2 = -\frac{1-\sqrt{5}}{2\sqrt{5}}.$$

Therefore,

$$\begin{aligned} F_n &= \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

Here is one last note on the property of Fibonacci numbers.

Lecture 17
Friday
November 13

Theorem 5.7 (Fibonacci and Pascal's Triangle)**.** *The sums of the binomial coefficients along the diagonals of Pascal's Triangle running upward from the left are Fibonacci numbers. More precisely, the nth Fibonacci number F_n satisfies*

$$F_n = \sum_{k=0}^{t} \binom{n-1-k}{k}$$

where $t = \lfloor \frac{n-1}{2} \rfloor$.

Proof. We note that as $k > \lfloor \frac{n-1}{2} \rfloor$, $\binom{n-k-1}{k} = 0$. Let $g_n = \sum_{k=0}^{n} \binom{n-k-1}{k}$. We want to show $g_n = F_n$. We prove by induction.

Base step. For $n = 1$, we have

$$g_1 = \sum_{k=0}^{1} \binom{-k}{k} = \binom{0}{0} + \binom{-1}{1} = 1 = F_1.$$

For $n = 2$, we have
$$g_2 = \sum_{k=0}^{2} \binom{1-k}{k} = \binom{1}{0} + \binom{0}{1} + \binom{-1}{2} = 1 = F_2.$$
We check two base cases because of the floor function we use here.

Inductive step. Assume the statement holds for $n = p - 1$ and $n = p - 2$. Thus,

$$\begin{aligned}
g_p &= g_{p-1} + g_{p-2} \\
&= \sum_{k=0}^{p-1} \binom{p-2-k}{k} + \sum_{k=0}^{p-2} \binom{p-3-k}{k} \\
&= \binom{p-2}{0} + \sum_{k=1}^{p-1} \binom{p-2-k}{k} + \sum_{k=1}^{p-1} \binom{p-2-k}{k-1} \\
&= \binom{p-2}{0} + \sum_{k=1}^{p-1} \binom{p-1-k}{k} + \binom{0}{p-1} \\
&= \sum_{k=0}^{p} \binom{p-1-k}{k}.
\end{aligned}$$

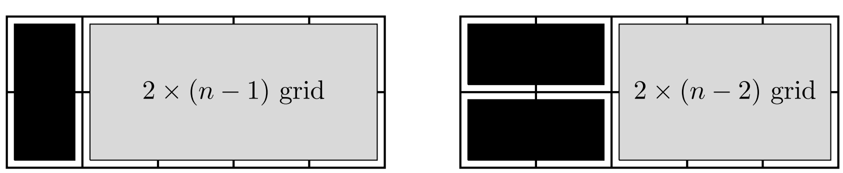
Therefore, $g_n = F_n$.

$\square$

The Fibonacci numbers also occur in the combinatorial problems.

Example 5.8. Let h_n be the number of ways to perfectly tile a $2 \times n$ grid with dominoes (of size 2×1). The grid is perfectly tiled if there is no open square on the board and no 2 dominoes overlap each other.

We start from simple case. For $n = 0$, it seems weird to tile a 2×0 grid, but actually this grid is perfectly tiled automatically and we just need to do nothing, so $h_0 = 1$. For $n = 1$, there is only one way to tile a 2×1 grid by a domino, so $h_1 = 1$. For $n = 2$, there are two ways, either place two dominoes both vertically or both horizontally, thus $h_2 = 2$.



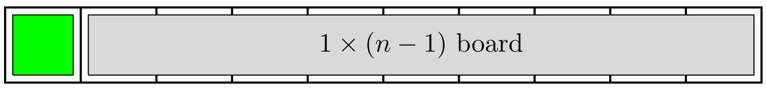
Now, assume we want to tile $2 \times n$ grid by dominoes. We have two cases. If we place 1 domino vertically at the left, then the rest of grid has h_{n-1} ways to be perfectly tiled by the definition of h_n. If we place 2 dominoes horizontally at the left, then the rest of grid has h_{n-2} ways to be perfectly tiled.

Thus, we have $h_n = h_{n-1} + h_{n-2}$ with the initial value $h_1 = 1$ and $h_2 = 2$. We shift our index right by 1. Therefore,
$$h_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+2}.$$

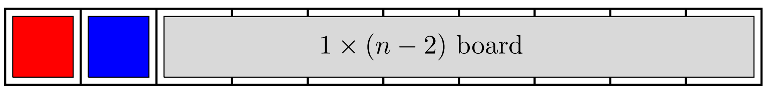
Example 5.9 (Last Example on Tiling)**.** We have an 1-by-n board and color each square red, blue, or green. Let h_n be the number of ways to color the $1 \times n$ board such that no 2 adjacent squares are red.

For $n = 1$, we can color the square red, blue, or green, so $h_1 = 3$. For $n = 2$, if the first square is blue or green, then the second square could be in any color, so there are $2 \times 3 = 6$ ways; if the first square is red, then the second square must be blue or green. So, $h_2 = 6 + 2 = 8$.

If the leftmost square is blue or green, then the rest of the board has h_{n-1} ways to color in each case. Thus, in this case, we have $2 \cdot h_{n-1}$ ways to color the board.



If the leftmost square is red, then we must have a blue or green square on the right and the rest of the board has h_{n-2} ways to color in each case. Thus, in this case, we also have $2 \cdot h_{n-2}$ ways to color the board.



Thus, the recurrence relation is $h_n = 2h_{n-1} + 2h_{n-2}$. The characteristic polynomial of this recurrence relation is $p(r) = r^2 - 2r - 2$. The root of the polynomial is

$$r_1 = 1 + \sqrt{3} \qquad r_2 = 1 - \sqrt{3}.$$

Thus,

$$h_n = c_1 \left(1 + \sqrt{3}\right)^n + c_2 \left(1 - \sqrt{3}\right)^n.$$

For $n = 1$, there are three ways to color the chessboard, so $h_1 = 3$. For $n = 2$, if we color the first square red, then the second square must be either blue or green; if we color the first square blue or green, then the second square can be any color. Thus, $h_2 = 2 + 3 \cdot 2 = 8$.

Now, we want to solve the following linear system

$$\begin{aligned} \left(1 + \sqrt{3}\right) c_1 + \left(1 - \sqrt{3}\right) c_2 = 3 \\ \left(1 + \sqrt{3}\right)^2 c_1 + \left(1 - \sqrt{3}\right)^2 c_2 = 8 \end{aligned}.$$

Thus, we have

$$c_1 = \frac{3 + 2\sqrt{3}}{6} \qquad c_2 = \frac{3 - 2\sqrt{3}}{6}.$$

Hence,

$$h_n = \frac{3 + 2\sqrt{3}}{6} \left(1 + \sqrt{3}\right)^n + \frac{3 - 2\sqrt{3}}{6} \left(1 - \sqrt{3}\right)^n.$$

Lecture 18
Monday
November 16

5.3 Ordinary Generating Functions

Ordinary generating functions are just a way of encoding infinite sequences of numbers as formal power series. We are going to find and interpret operations on ordinary generating functions, which allow us to count the number of possibilities for a problem by means of algebra. In the combinatorial perspective, the mindset is that we think of a_n as counting the number of some kind of "structure" on the set $\{1, 2, \ldots, n\}$.

Definition 5.10 (Ordinary Generating Function)**.** Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Then, the formal power series

$$F(x) = \sum_{n\geq 0} a_n x^n$$

is the **ordinary generating function** of the sequence $(a_n)_{n\geq 0}$.

We first do some simple examples to set up some algebraic lemmas for further.

Example 5.11. The generating function of the infinite sequence

$$1, 1, 1, \ldots, 1, \ldots,$$

each of whose terms equals 1, is

$$g(x) = 1 + x + x^2 + \cdots + x^n + \cdots .$$

The ordinary generating function $F(x)$ is the sum of a geometric series with value

$$F(x) = \frac{1}{1-x}$$

since

$$\sum_{n\geq 0} x^n - x \sum_{n\geq 0} x^n = (1-x)F(x) = 1.$$

Lemma 5.12. *For the finite sequence such that the number* 1 *occurs* n *times*

$$\underbrace{1, 1, 1, \ldots, 1}_{n \text{ times}},$$

the generating function is given by

$$F(x) = \frac{1-x^{n+1}}{1-x}.$$

Example 5.13. Let $S = \{1, 2, \ldots, m\}$ and g_n be the number of subsets of size n. Then, we clearly have

$$g_n = \binom{m}{n}.$$

The ordinary generating function is given by

$$g(x) = \sum_{i=0}^{\infty} \binom{m}{i} \cdot x^i = (1+x)^m$$

with each coefficient of x^n given by $\binom{m}{n}$.

Example 5.14. A bakery has 4 types of donuts with limited amount. Say, we have ≤ 3 Chocolate, ≤ 2 Plain, ≤ 1 Strawberry, and ≤ 2 Cinnamon. The box has n donuts, and let g_n be the number of boxes of n donuts.

Equivalently, we want to find the nonnegative integral solutions of $t_1 + t_2 + t_3 + t_4 = n$ where $0 \leq t_1 \leq 3$ (t_1 counts the number of Chocolate donuts), $0 \leq t_2 \leq 2$ (t_2 counts the number of Plain donuts), $0 \leq t_3 \leq 1$ (t_3 counts the number of Strawberry donuts), and $0 \leq t_4 \leq 2$ (t_4 counts the Cinnamon donuts).

Consider the following function

$$F(x) = (1 + x + x^2 + x^3)(1 + x + x^2)(1 + x)(1 + x + x^2).$$

We note that the coefficients of each term x^n is compose as

$$x^n = x^{t_1} \cdot x^{t_2} \cdot x^{t_3} \cdot x^{t_4}$$

provoded that $n = t_1 + t_2 + t_3 + t_4$ with $0 \leq t_1 \leq 3$, $0 \leq t_2 \leq 2$, $0 \leq t_3 \leq 1$, and $0 \leq t_4 \leq 2$. Thus, the coefficients of x^n of $F(x)$ equals the number of nonnegative integral solutions of the linear system we defined.

Now, we will show how a counting problem can be explicitly solved by means of ordinary generating function.

Example 5.15. The frog population of an infinitely large lake grows fourfold each year. On the first day of each year, 100 frogs are taken out of the lake and shipped into another lake. Assume there were 50 frogs in the lake originally, then let a_n be the number of frogs at the end of nth year. Then, we can get a recurrence relation $a_{n+1} = 4a_n - 100$ for $n \geq 0$ with the initial value $a_0 = 50$. We get

$$\sum_{n\geq 0} a_{n+1}x^{n+1} = \sum_{n\geq 0} 4a_n x^{n+1} - \sum_{n\geq 0} 100x^{n+1}.$$

The left-hand side is almost the ordinary generating function $F(x)$ of the sequence $(a_n)_{n\geq 0}$. Indeed, after replacing $n+1$ by n, the only missing term is a_0. Thus, the left-hand side is $F(x) - a_0$. The first term of the right-hand side is $4xF(x)$, while the second term of the right-hand side is $\dfrac{100x}{1-x}$ by Example 5.11. Thus,

$$F(x) - a_0 = 4xF(x) - \frac{100x}{1-x}.$$

Further calculation results as

$$F(x) = \frac{50}{1-4x} - \frac{100x}{(1-x)(1-4x)}.$$

Decomposition to partial fraction gives

$$\begin{aligned}
\frac{100x}{(1-x)(1-4x)} &= \frac{100}{3} \cdot \frac{1}{1-4x} - \frac{100}{3} \cdot \frac{1}{1-x} \\
&= \frac{100}{3}\left(\sum_{n\geq 0} 4^n x^n - \sum_{n\geq 0} x^n\right) \\
&= \frac{100}{3}\sum_{n\geq 0}(4^n - 1)x^n.
\end{aligned}$$

Thus, we conclude that the coefficient of x^n is

$$a_n = 50 \cdot 4^n - \frac{100}{3}\left(4^n - 1\right).$$

We can also check our result by induction as follows

$$\begin{aligned} a_{n+1} &= 4a_n - 100 \\ &= 4\left(50 \cdot 4^n - \frac{100}{3}\left(4^n - 1\right)\right) - 100 \\ &= 50 \cdot 4^{n+1} - \frac{100}{3}\left(4^{n+1} - 1\right). \end{aligned}$$

Here is one last example of ordinary generating function in a combinatorial interpretation.

Example 5.16. A fruit market has apples, bananas, oranges, and pears. Let h_n be the bags of fruit such that in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is at most 1.

The generating function is given by

$$\begin{aligned} F(x) &= (1 + x^2 + x^4 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x + x^2 + x^3 + x^4)(1 + x) \\ &= \frac{1}{1 - x^2} \cdot \frac{1 - x^5}{1 - x} \cdot \frac{1}{1 - x^5} \cdot (1 + x) \\ &= \frac{1}{(1 - x)^2} = \frac{1}{1 - x} \cdot \frac{1}{1 - x}. \end{aligned}$$

Recall $\dfrac{1}{1 - x} = 1 + x + x^2 + \cdots$, so

$$F(x) = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots).$$

Now, it is easy to see the coefficients of $x^n = x^{t_1} \cdot x^{t_2}$ provided that $t_1, t_2 \geq 0$ and $n = t_1 + t_2$. Thus, by Theorem 1.38,

$$h_n = \binom{n+1}{1} = n + 1.$$

We have dealt with the product of several ordinary generating functions in an intuitive way. Does the product have combinatorial interpretations? Indeed, they have. Now, we go up with rigor.

Theorem 5.17. *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences, and let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$ be their respective generating functions. Define $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, and let $C(x) = \sum_{n\geq 0} c_n x^n$. Then,*

$$A(x) \cdot B(x) = C(x).$$

In other words, the coefficient of x^n in $A(x) \cdot B(x)$ is $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

Proof. When we multiply the infinite sum $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and the infinite sum $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$, we multiply each term of the first sum by each term of the second sum, then add all these products. So, a product is in the form $a_i x^i \cdot b_j x^j$. The exponent of x in this product will be n if and only if $j = n - i$, and the claim follows.

□

The product of generating functions can be thought of as a way of "concatenating" structures. We can think of this as the number of ways of first picking a way to break the set $\{1, 2, \ldots, n\}$ into 2 consecutive pieces $\{1, 2, \ldots, i\}$ and $\{i+1, i+2, \ldots, n\}$ and then putting structure A on the first set and the structure B on the second set.

Example 5.18. A quarter at UC San Diego consists of n days. We want to split the lectures of the class into two pieces: the first part is the theoretical part and the second part is the laboratory part. The theoretical part needs 1 day for a guest lecturer while the laboratory part needs 2 days. How many ways can we plan out this course?

Let $a_n = n$ be the number of ways of picking a day for a guest lecturer for a course with n days and let $b_n = \binom{n}{2}$ be the number of ways of picking two days for a guest lecturer. Then, define $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 0} b_n x^n$. The coefficient of x^n in $A(x) \cdot B(x)$ is the answer we want.

We can find nice expressions by taking derivatives of $\sum_{n\geq 0} x^n = \dfrac{1}{1-x}$ and multiplying by the appropriate powers of x for $A(x)$ take derivative then multiply by x; for $B(x)$ take derivative twice and then multiply by $\frac{1}{2}x^2$.

$$A(x) = \sum_{n\geq 0} n \cdot x^n = \frac{x}{(1-x)^2}$$
$$B(x) = \sum_{n\geq 0} \binom{n}{2} x^n = \frac{x^2}{(1-x)^3}.$$

Then, the product is

$$A(x) \cdot B(x) = \frac{x^3}{(1-x)^5}.$$

By the Binomial Theorem, we have

$$\frac{x^3}{(1-x)^5} = x^3 \sum_{n\geq 0} \binom{-5}{0} (-x)^n = x^3 \sum_{n\geq 0} \binom{n+4}{n} x^n = \sum_{n\geq 3} \binom{n+1}{4} x^n.$$

This shows that there are $\binom{n+1}{4}$ ways that we can plan the course.

Lecture 20

Monday

November 23

5.4 Exponential Generating Functions

Not all recurrence relations can be turned into a closed formula by using an ordinary generating function. Sometimes, a closed formula may not exist. Some other times, it could be that we have to use a different kind of generating function.

Definition 5.19 (Exponential Generating Function)**.** Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Then, the formal power series

$$F(x) = \sum_{n\geq 0} a_n \cdot \frac{x^n}{n!}$$

is called the **exponential generating function** of the sequence (a_n).

When $a_n = 1$ for all $n \geq 0$, we get

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}.$$

Exponential generating functions can be useful in permutations, while ordinary generating functions are useful in combinations.

Example 5.20. Let $S = \{1, 2, \ldots, n\}$. Recall that the number of k-permutations in an n-element set is $\dfrac{n!}{(n-k)!}$. Let a_k be the number of k-permutations of S. Then, the exponential generating function is

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} a_k \cdot \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \cdot k! \cdot \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \cdot x^k \\ &= (1+x)^n. \end{aligned}$$

Thus, the coefficients of each x^k is $\dbinom{n}{k}$.

Now, we introduce some calculation tricks when we are dealing with exponential generating functions.

Theorem 5.21 (The Number e)**.**

(a) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

(b) $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots$

Corollary 5.22.

(a) $\dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots$

(b) $\dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots$

Theorem 5.23. *Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$, where $n_1, n_2, \ldots, n_k$ are nonnegative integers. Let h_n be the number of n-permutations of S. Then the exponential generating function $F(x)$ for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ is given by*

$$F(x) = f_{n_1}(x) \cdot f_{n_2}(x) \cdots f_{n_k}(x)$$

where, for $i = 1, 2, \ldots, k$,

$$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_i}}{n_i!}.$$

Example 5.24 (Coloring Chessboard)**.** We have an 1-by-n chessboard. Color each square red, green, or blue. Let h_n be the number of ways to color the $1 \times n$ chessboard such that red appears even number of times.

Then, h_n is the number of n-permutations of a multiset of three colors, each with an infinite repetition number, in which red occurs an even number of times. Thus, the exponential generating function is

$$\begin{aligned}
F(x) &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(1 + x + \frac{x^2}{2!} + \cdots\right) \cdot \left(1 + x + \frac{x^2}{2!} + \cdots\right) \\
&= \frac{e^x + e^{-x}}{2} \cdot e^x \cdot e^x \\
&= \frac{e^{3x} + e^x}{2} \\
&= \frac{1}{2}\left(\sum_{n \geq 0} 3^n \cdot \frac{x^n}{n!} + \sum_{n \geq 0} \frac{x^n}{n!}\right) = \frac{1}{2} \sum_{n \geq 0} (3^n + 1) \cdot \frac{x^n}{n!}.
\end{aligned}$$

Hence, $h_n = \dfrac{3^n + 1}{2}$.

Example 5.25 (Coloring Chessboard)**.** We have an 1-by-n chessboard and color red, white, green, and blue. Let h_n be the number of colorings such that the number of red is even and the number of white is odd.

Then, the exponential generating function is

$$\begin{aligned}
F(x) &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \cdot \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \cdot \left(1 + x + \frac{x^2}{2!} + \cdots\right)^2 \\
&= \frac{e^x + e^{-2}}{2} \cdot \frac{e^x - e^{-x}}{2} \cdot (e^x)^2 \\
&= \frac{e^{4x} - 1}{4} \\
&= \frac{1}{4} \sum_{n \geq 0} \frac{(4x)^n}{n!} - \frac{1}{4} \\
&= \sum_{n \geq 1} 4^{n-1} \cdot \frac{x^n}{n!}.
\end{aligned}$$

Hence, $h_n = 4^{n-1}$ for all $n \geq 1$ and $h_0 = 0$ since there is no way to color odd occurence of white in an 1×0 board.

Lecture 21
Wednesday
November 25

Here is one last example on chessbard coloring.

Example 5.26. We have an 1-by-n chessboard and we want to color each square red, green, blue, and orange. Let h_n be the number of ways to color the board such that the number red squares and the number of blue squares are even.

The exponential generating function is

$$\begin{aligned}
F(x) &= \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 \cdot \left(1 + x + \frac{x^2}{2!} + \cdots\right)^2 \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 \cdot e^{2x} \\
&= \frac{1}{4}e^{4x} + \frac{1}{2}e^{2x} + \frac{1}{4} \\
&= \frac{1}{4} + \frac{1}{4}\sum_{n\geq 1} \frac{(4x)^n}{n!} + \frac{1}{2} + \frac{1}{2}\sum_{n\geq 1} \frac{(2x)^n}{n!} + \frac{1}{4} \\
&= 1 + \sum_{n\geq 1} (4^{n-1} + 2^{n-1}) \cdot \frac{x^n}{n!}
\end{aligned}$$

so $h_n = 4^{n-1} + 2^{n-1}$ for $n \geq 1$ and $h_0 = 1$ since we can have even occurences of red and blue squares in an 1×0 board.

So, when we need to use the exponential generating function if we cannot determine we are dealing with a combination problem or a permutation problem? When a sequence grows too fast, the ordinary generating function will be problematic.

Example 5.27. Let $a_0 = 1$, and let $a_n = n(a_{n-1} - n + 2)$ for all $n \geq 1$. Find a closed formula for a_n.

Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the ordinary generating function. Then, we can try to find a relation

$$\begin{aligned}
A(x) &= a_0 + \sum_{n\geq 1} a_n x^n \\
&= a_0 + \sum_{n\geq 1} n a_{n-1} x^n + \sum_{n\geq 1} n(2-n)x^n.
\end{aligned}$$

The first sum is simplified as $x \cdot \dfrac{d}{dx}(x \cdot A(x))$, so the relation on $A(x)$ is a differential equation. Also, this sequence is growing so fast, so its ordinary generating function cannot be simplied.

Instead, the exponential generating function is given by

$$\begin{aligned}
F(x) &= a_0 + \sum_{n\geq 1} a_n \cdot \frac{x^n}{n!} \\
&= a_0 + \sum_{n\geq 1} a_{n-1} \cdot \frac{x^n}{(n-1)!} - \sum_{n\geq 2} (n-2) \cdot \frac{x^n}{(n-1)!} \\
&= a_0 + x \cdot F(x) - x\left(\sum_{n\geq 1} (n-1) \cdot \frac{x^{n-1}}{(n-1)!} - \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!}\right) \\
&= a_0 + x \cdot F(x) - x^2 \cdot e^x + x \cdot e^x.
\end{aligned}$$

Hence,

$$F(x) = \frac{a_0 + x \cdot (1-x) \cdot e^x}{1-x} = \frac{a_0}{1-x} + x \cdot e^x = a_0 \sum_{n\geq 0} x^n + \sum_{n\geq 0} \frac{x^{n+1}}{n!}.$$

The coefficient of x^n on the right side is $a_0 + \dfrac{1}{(n-1)!}$, and the coefficient of x^n on the left side is $\dfrac{a_n}{n!}$, so we conclude that $a_n = n! + n$.

Just as we have seen for ordinary generating functions, the product of two exponential generating functions has a very natural combinatorial meaning.

Lemma 5.28. *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences. Let $A(x) = \sum_{n\geq 0} a_n \cdot \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \cdot \frac{x^n}{n!}$, then $A(x)\cdot B(x) = \sum_{n\geq 0} c_n \cdot \frac{x^n}{n!}$ where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$.*

Proof. The coefficient of x^n in $A(x)\cdot B(x)$ is $\sum_{i=0}^{n} \frac{a_i}{i!}\cdot\frac{b_{n-i}}{(n-i)!}$. It is also $\frac{c_n}{n!}$, so $c_n = \sum_{i=0}^{n}\binom{n}{i} a_i b_{n-i}$. □

This gives a variation of "concatenating" structures like we say for the product of ordinary generating functions. If we thinnk a_n and b_n as counting the number of structures (call them type α and β) on the set $\{1, 2, \ldots, n\}$, then c_n above counts the number of ways of choosing a subset S of $\{1, 2, \ldots, n\}$ (not necessarily consecutive) and putting a structure of type α on S and a structure of type β on $\{1, 2, \ldots, n\} \setminus S$.

Theorem 5.29. *Let $A(x) = \sum_{n\geq 0} a_n \cdot \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \cdot \frac{x^n}{n!}$. $A(x)\cdot B(x)$ is the exponential generating function for picking two disjoint subsets S_1 and S_2 of $\{1, 2, \ldots, n\}$ such that $S_1 \cup S_2 = \{1, 2, \ldots, n\}$, then putting a structure of type α on S_1 and a structure of type β on S_2.*

Let a_n be the number of ways putting a structure of type α on the set $\{1, 2, \ldots, n\}$ and assume that $a_0 = 0$. Let h_n be the number of ways of first picking a set partition of $\{1, 2, \ldots, n\}$ and putting a structure of type α on each block.

To be continued...... Lecture 22

Monday

5.5 Homogeneous Linear Recurrence Relation

November 30

Previously, we have learned about linear recurrence relation in the Fibonacci sequence. Now, we will give a formal definition of a certain class of recurrence relations for which there is a general method of solution.

Definition 5.30 (Homogeneous Linear Recurrence Relation)**.** A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ satisfies a **linear recurrence relation of order** k if there exists real numbers $c_1, c_2, \ldots, c_k$ and b_n such that $c_k \neq 0$ and

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + b_n.$$

Further, if $b_n = 0$, then it is called a **homogeneous linear recurrence relation**.

The recursive formula for Fibonacci sequence $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ is a homogeneous linear recurrence relation of order 2. The recurrence relation for derangement $D_n = (n-1)\cdot D_{n-1} + (n-1)\cdot D_{n-2}$ is not a linear recurrence.

In general, if we want to define a sequence using a linear recurrence relation of order k, we need to specify the first k initial values $a_0, a_1, \ldots, a_{k-1}$ to allow us to calculate all of the terms.

When $k = 1$, this is easy to do

$$a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = c_1^3 a_{n-3} = \cdots = c_1^n a_0.$$

When $k = 2$, we have a sequence of numbers $a_0, a_1, a_2, \ldots$ that satisfies a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

whenever $n \geq 2$. Here, c_1 and c_2 are constants and $c_2 \neq 0$. We want to find a closed formula for a_n.

Definition 5.31 (Characteristic Polynomial)**.** The **characteristic polynomial** of this recurrence relation is defined to be

$$t^2 - c_1 t - c_2.$$

The roots of this polynomial are $\dfrac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$. Name them r_1 and r_2. The roots will be imaginary numbers if $c_1^2 + 4c_2 < 0$, but everythign will still work. Thus, we can factor the characteristic polynomial as

$$t^2 - c_1 t - c_2 = (t - r_1)(t - r_2).$$

Then, we get $r_1 + r_2 = c_1$ and $r_1 r_2 = c_2$, so $r_1 \neq 0$ and $r_2 \neq 0$ by our assumption $c_2 \neq 0$.

Theorem 5.32. *If $r_1 \neq r_2$, then there are constants α_1 and α_2 such that*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for all n.

Proof. Define

$$F(x) = \sum_{n \geq 0} a_n x^n.$$

The recurrence relation says that we have an identity

$$\begin{aligned} F(x) &= a_0 + a_1 x + \sum_{n \geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\ &= a_0 + a_1 x + c_1 \sum_{n \geq 2} a_{n-1} x^n + c_2 \sum_{n \geq 2} a_{n-2} x^n. \end{aligned}$$

Remember the recurrence is only valid for $n \geq 2$, so we have to separate out the first two terms. The last two sums are almost the same as $F(x)$ if we re-index them

$$\begin{aligned} &\sum_{n \geq 2} a_{n-1} x^n = \sum_{n \geq 1} a_n x^{n+1} = x \sum_{n \geq 1} a_n x^n = xF(x) - a_0 x \\ &\sum_{n \geq 2} a_{n-2} x^n = \sum_{n \geq 0} a_n x^{n+2} = x^2 F(x). \end{aligned}$$

In particular,

$$F(x) = a_0 + a_1 x + c_1 x F(x) - c_1 a_0 x + c_2 x^2 F(x).$$

We can rewrite this as

$$F(x) = \frac{a_0 + (a_1 - c_1 a_0)x}{1 - c_1 x - c_2 x^2}.$$

We want to factor the denominator. Let $\frac{1}{r_1}$ and $\frac{1}{r_2}$ be two roots of the polynomial $1 - c_1 x - c_2 x^2$, thus $1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$. Now, apply partial fraction decomposition, we have

$$F(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x}$$

for some constants α_1, α_2. But these terms are both geometric series, so we can further write

$$F(x) = \alpha_1 \sum_{n \geq 0} r_1^n x^n + \alpha_2 \sum_{n \geq 0} r_2^n x^n.$$

The coefficients of x^n on the left side is a_n and the coefficient of x^n on the right side is $\alpha_1 r_1^n + \alpha_2 r_2^n$. So we have equality for all n.

□

We just proved the general formula for the second order linear recurrence relation. How about more general one?

Lemma 5.33. *$a_n = r^n$ $(r \neq 0)$ is a solution to the linear homogeneous recurrence relation*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0$$

if and only if r is a root of the polynomial

$$x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k = 0.$$

Proof. Let $a_n = r^n$ be a solution to

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0.$$

Then,

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k = 0.$$

□

Actually, $a_n = d \cdot r^n$, for $d \in \mathbb{R}$, is also a solution to the linear recurrence relation.

Recall from Calculus, $p(x) = x^k - c_1 x^{k-1} - \cdots - c_k = 0$ has $\leq k$ distinct real roots. Let $r_1, r_2, \ldots, r_k$ be k distinct roots. Thus, for any $1 \leq i \leq k$, $c_i r_i^n$ is a solution to the linear recurrence relation. Furthermore, the linearity implies that any linear combination will be a solution to the recurrence relation

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0.$$

Hence, the general solution of the linear recurrence relation is

$$a_n = d_1 r_1^n + d_2 r_2^n + \cdots + d_k r_k^n.$$

where $d_1, d_2, \ldots, d_k \in \mathbb{R}$.

Example 5.34. Solve h_n for $h_n = 5h_{n-1} - 6h_{n-2}$ for $n \geq 2$ with initial values $h_0 = 1$ and $h_1 = -2$.

The characteristic polynomial is $x^2 - 5x + 6 = (x-3)(x-2)$, so the roots are $x_1 = 3$, $x_2 = 2$. The general solution is

$$h_n = c_1 \cdot 3^n + c_2 \cdot 2^n$$

so we want to solve the following linear system

$$c_1 + c_2 = 1$$

$$3c_1 + 2c_2 = -2.$$

Thus, we have $c_1 = -4$ and $c_2 = 5$. Hence, $h_n = -4 \cdot 3^n + 5 \cdot 2^n$.

There is a loose end. If the k roots of the characteristic polynomial are not distinct, then what will happen?

We start from a second order homogeneous linear recurrence relation.

Theorem 5.35. *Let r_1 and r_2 be the roots of characteristic polynomial for the linear recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$. If $r_1 = r_2$, then there are constants α_1 and α_2 such that*

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$$

for all n.

Proof. We can start in the same way as we prove for Theorem 5.32. The only difference is that we are trying to take the partial fraction decomposition of

$$F(x) = \frac{a_0 + (a_1 - \alpha_2 a_0)x}{(1 - r_1 x)^2}.$$

This can still be done, but now it looks like

$$\frac{\beta_1}{1 - r_1 x} + \frac{\beta_2}{(1 - r_1 x)^2}$$

for some constants β_1 and β_2. The first is a geometric series, and the second we have seen: recall that $\dfrac{1}{(1-x)^2} = \sum_{n \geq 0} (n+1)x^n$. So, we get

$$F(x) = \beta_1 \sum_{n \geq 0} r_1^n x^n + \beta_2 \sum_{n \geq 0} (n+1) r_1^n x^n.$$

Comparing coefficients, we get

$$a_n = \beta_1 r_1^n + \beta_2 (n+1) r_1^n = (\beta_1 + \beta_2) r_1^n + \beta_2 n r_1^n.$$

So, $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_2$.

□

Here is one simple calculation example.

Example 5.36. Solve h_n for $h_n = 8h_{n-1} - 16h_{n-2}$ for $n \geq 2$ with initial values $h_0 = -1$ and $h_1 = 0$.

The characteristic polynomial is $x^2 - 8x + 16 = (x-4)^2$, so it has indeed repeated roots $r_1 = r_2 = 4$. Then, the general solution is

$$h_n = c_1 \cdot 4^n + c_2 \cdot n \cdot 4^n.$$

With initial values, we get $c_1 = -1$ and $c_2 = 1$. Hence, $h_n = -4^n + n \cdot 4^n$.

Theorem 5.37. *Let $r_1, r_2, \ldots, r_t$ be distinct roots of the characteristic polynomial of the linear homogeneous recurrence relation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

such that r_i is an s_i-fold root (that is, r_i repeats s_i times) of characteristic polynomial, then the general solution is

$$a_n = \sum_{i=1}^{k} (d_1 + d_2 \cdot n + d_3 \cdot n^2 + \cdots + d_{s_i} \cdot n^{s_i - 1}) r_i^n.$$

Here is a calculation for higher degree recurrence relation.

Example 5.38. Solve $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$ for $n \geq 3$ with initial values $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$.

The characteristic polynomial is $x^3 - x^2 - 9x + 9 = (x-1)(x-3)(x+3)$, so we have general solution $h_n = c_1 + c_2 \cdot 3^n + c_3 \cdot (-3)^n$. Then, we plug in the initial values,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 3c_2 - 3c_3 &= 1 \\ c_1 + 9c_2 + 9c_3 &= 2 \end{aligned}$$

so $c_1 = -\frac{1}{4}$, $c_2 = \frac{1}{3}$, and $c_3 = -\frac{1}{12}$. Hence,

$$h_n = -\frac{1}{4} + \frac{1}{3} \cdot 3^n - \frac{1}{12} \cdot (-3)^n.$$

Lecture 23
Wednesday
December 2

5.6 Nonhomogeneous Linear Recurrence Relations

Recurrence relations that are not homogeneous are, in general, more difficult to find out a solution and require some special techniques on the nonhomogeneous part of the relation.

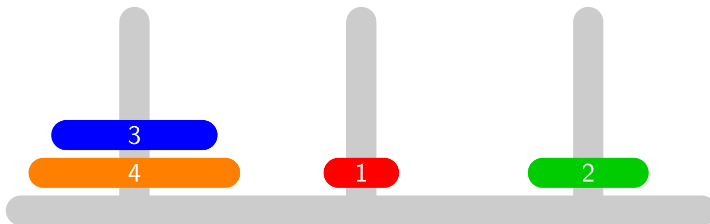
Definition 5.39 (Nonhomogeneous Linear Recurrence Relation)**.** A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ satisfies a **nonhomogeneous linear recurrence relation of order** k if there exists real numbers $c_1, c_2, \ldots, c_k$ such that $c_k \neq 0$ and

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where $f(n)$ is a function of n.

The first example is a famous puzzle.

Example 5.40 (Towers of Hanoi)**.** There are three pegs and n disks of increasing size on one peg, with the largest disk at the bottom. Our goal is to move all disks to another peg. During moving process, we can only move one disk at a time and at no time is there a larger disk on top of a smaller disk.



Let h_n be the minimum number of moves to move n disks to a desired peg. We start with some simple cases. We get $h_0 = 0$, $h_1 = 1$, and $h_2 = 3$. Can we find a recurrence relation that is satisfied by h_n?

To move n disks to another peg, we must first move the top $n-1$ disks to a peg, move the largest disk to the desired peg, and then move the $n-1$ disks to the peg which now contains the largest disk. Thus, we get the relation $h_n = 2h_{n-1} + 1$ for $n \geq 1$ with the initial value $h_0 = 0$.

This is a linear recurrence relation of order 1 with constant coefficients, but it is not homogeneous because of the presence of the quantity 1. To find h_n, we iterate this relation to get

$$\begin{aligned}
h_n &= 2h_{n-1} + 1 \\
&= 2(2h_{n-2} + 1) + 1 \\
&= 2^2(2h_{n-3} + 1) + 2 + 1 \\
&= 2^{n-1}(h_0 + 1) + 2^{n-2} \cdots + 2^2 + 2 + 1 \\
&= 2^{n-1} + \cdots + 2^2 + 2 + 1.
\end{aligned}$$

Therefore, this is a summation of geometric sequences, we get

$$h_n = \frac{2^n - 1}{2 - 1} = 2^n - 1 \qquad (n \geq 0).$$

We can check by induction we get the correct answer.

This is a special way that utilizes the nice property of this recursive relation. In general, we have a technique to deal with nonhomogeneous recurrence relation.

(1) Find the general solution of the corresponding homogeneous relation.

(2) Find a particular solution of the nonhomogeneous relation.

(3) Combine the general solution and the particular solution, and determine the constants of solution terms.

Example 5.41 (Continuation of Example 5.40)**.** Solve $h_n - 2h_{n-1} = 1$ for $n \geq 2$ with $h_1 = 1$.

First, we solve for homogeneous relation $h_n = 2h_{n-1}$. The general solution is $h_n = c \cdot 2^n$.

Second, we want to find a particular solution through guessing based on the right-hand side, which is a constant. So, we guess the particular solution is $h_n = c'$, a constant. Then, $c' - 2c' = 1$, so $c' = -1$.

Therefore, $h_n = c \cdot 2^n - 1$. Initial condition gives $c = 1$, so $h_n = 2^n - 1$.

In general, the main difficulty in this process is to find a particular solution in the second step. There are certain types of particular solutions to try.

(I) If $f(n)$ is a polynomial of degree k in n, then look for a particular solution h_n that is also a polynomial of degree k in n. Thus, we can try

  (i) $h_n = c$ (a constant) if $f(n) = d$ (a constant),

  (ii) $h_n = c_1 n + c_2$ if $f(n) = d_1 n + d_2$,

  (iii) $h_n = c_1 n^2 + c_2 n + c_3$ if $f(n) = d_1 n^2 + d_2 n + d_3$.

(II) If $f(n)$ is an exponential, then look for a particular solution that is also an exponential. Thus, we can try $h_n = c \cdot d^n$ if $f(n) = d^n$.

Example 5.42. Solve $h_n = 3h_{n-1} - 4n$ for $n \geq 1$ with $h_0 = 2$.

It is obvious that the homogeneous solution is $h_n^{(h)} = c_1 \cdot 3^n$.

For a particular solution, we guess $h_n^{(p)} = a \cdot n + b$, so

$$a \cdot n + b - 3\left(a \cdot (n-1) + b\right) = -4n$$

then $(4 - 2a) \cdot n + (3a - 2b) = 0$, so $a = 2$ and $b = 3$.

Thus, the general solution is $h_n = c_1 \cdot 3^n + 2n + 3$. The initial condition gives $c_1 + 3 = 2$, so $c_1 = -1$. Hence, the solution to this nonhomogeneous relation is $h_n = -3^n + 2n + 3$.

Example 5.43. Solve $h_n = 4h_{n-1} + 3 \cdot 2^n$ for $n \geq 1$ with $h_0 = 1$.

Still, it is obvious that the general homogeneous solution is $h_n^{(h)} = c_1 \cdot 4^n$.

Then, for a particular solution, we guess $h_n^{(p)} = c_2 \cdot 2^n$. So, $c_2 \cdot 2^n = 4c_2 \cdot 2^{n-1} + 3 \cdot 2^n$, we get $c_2 = -3$.

Thus, the general solution is $h_n = c_1 \cdot 4^n - 3 \cdot 2^n$. By the initial condition $h_0 = 1$, we have $c_1 - 3 = 1$, so $c_1 = 4$. Hence, teh solution to this nonhomogeneous relation is $h_n = 4 \cdot 4^n - 3 \cdot 2^n$.

Another technique to solve the nonhomogeneous recurrence relations is to use the ordinary generating functions. Here is an example.

Example 5.44 (Alternative Method for Example 5.43)**.** Let $F(x)$ be the ordinary generating function of the sequence (h_n). We have

$$F(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$

and

$$-4x \cdot F(x) = -4h_0 x - 4h_1 x^2 - \cdots - 4h_{n-1} x^n.$$

Then, we get

$$\begin{aligned}F(x) - 4xF(x) &= h_0 + (h_1 - 4h_0)x + (h_2 - 4h_1)x^2 + \cdots + (h_n - 4h_{n-1})x^n + \cdots \\ &= 1 + 3 \cdot 2x + 3 \cdot (2x)^2 + 3 \cdot (2x)^3 + \cdots \\ &= 1 + 3 \cdot \sum_{n=1}^{\infty} (2x)^n \\ &= -2 + 3 \cdot \sum_{n=0}^{\infty} (2x)^n = -2 + \frac{3}{1-2x},\end{aligned}$$

so $F(x) = \dfrac{1+4x}{(1-2x)(1+4x)} = -\dfrac{3}{1-2x} + \dfrac{4}{1-4x}$.

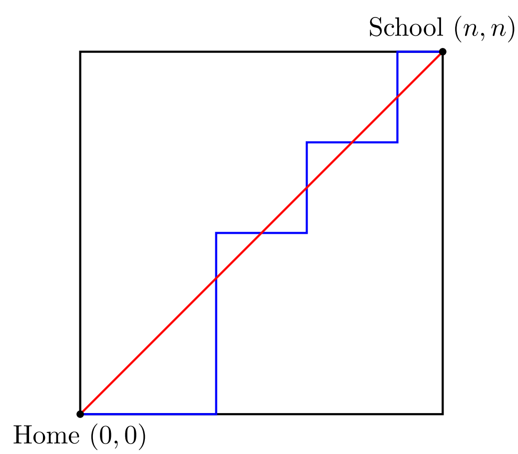
Then, expanding to the infinite geometry series, we get

$$\begin{aligned}F(x) &= -3 \cdot \sum_{n=0}^{\infty} (2x)^n + 4 \cdot \sum_{n=0}^{\infty} (4x)^n \\ &= \sum_{n=0}^{\infty} (-3 \cdot 2^n + 4 \cdot 4^n) \cdot x^n.\end{aligned}$$

Hence, the solution to this nonhomogeneous recurrence relation is $h_n = 4 \cdot 4^n - 3 \cdot 2^n$.

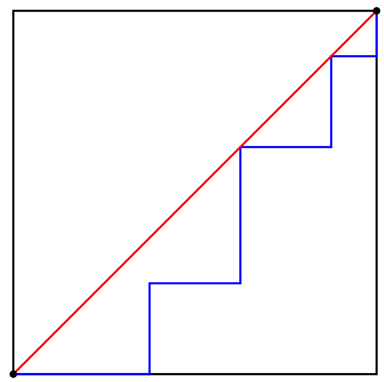
5.7 Catalan Numbers

A student walks from his home to school, located n blocks east and n blocks north from home. He always goes through the shortest walk, which means he only walks north or east. So, there is $\binom{2n}{n} \cdot \binom{n}{n} = \binom{2n}{n}$ different walks.



This is a sample route from home to school, which crosses the diagonal. Lecture 24

Now, we are interested the number of walks from $(0, 0)$ to (n, n) such that it does *not* cross above (or below, it does not matter) the diagonal but it can touch the diagonal. Friday December 4



The number of walks from $(0,0)$ to (n,n) that does not cross above the diagonal is $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is called *Catalan Numbers.*

The first few *Catalan Numbers* are $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, and $C_5 = 42$. It is not immediately clear from the definition that C_n's are integers, but this will be clear after the following theorems proved below.

Theorem 5.45 (Catalan Numbers)**.** *The number of walks from $(0,0)$ to (n,n) that is not crossing the diagonal is $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

Proof. Let A_n be the *acceptable walks* (does not cross above the diagonal) and U_n be the *unacceptable walks* (does cross above the diagonal). So, $A_n = \binom{2n}{n} - U_n$. We want to find U_n, the number of *unacceptable walks.*

Let $a_1, a_2, \ldots, a_{2n}$ of $2n$ terms that can be formed by exactly n $+1$'s (east) and exactly n -1's (north). For an *acceptable walk* that does not cross the diagonal, for every $1 \le k \le 2n$, we need $\sum_{i=1}^{k} a_i \ge 0$, which means at each state, the number of steps toward east is greater than or equal to the number of steps toward north. (Finally, we will get $\sum_{i=1}^{2n} a_i = 0$ always.)

Consider an *unacceptable sequence*, there exists a first $1 \le k \le 2n$ (first time crossing above the diagonal) such that $\sum_{i=1}^{k} a_i < 0$. By our assumption that k is the first time we cross above the diagonal, there are equal numbers of steps toward east $(+1)$ and steps toward north (-1), so $\sum_{i=1}^{k-1} a_i = 0$ and $a_k = -1$. In particular, k is odd. We now reverse the signs of each of the first k terms (steps), that is, we replace a_i by $-a_i$ for each $1 \le i \le k$ and leave unchanged the remaining steps. The resulting sequence (a_i') is a sequence of $n+1$ $+1$'s and $n-1$ -1's. We note that this process is reversible since given a sequence of $n+1$ $+1$'s and $n-1$ -1's, there exists a first step when the number of steps toward east exceeds the number of steps toward north (since there are more $+1$'s than -1's). Reversing the signs of $+1$'s and -1's up to that step results in an unacceptable sequence of n $+1$'s and n -1's. Thus, this is a bijection between the *unacceptable walks* and walks from $(0,0)$ to $(n+1,n-1)$, in short. Hence, the number of *unacceptable walks* is

$$U_n = \frac{(2n)!}{(n+1)!\cdot(n-1)!}.$$

Therefore,

$$\begin{aligned}
A_n &= \frac{(2n)!}{n!\cdot n!} - \frac{(2n)!}{(n+1)!\cdot(n-1)!}\\
&= \frac{(2n)!}{n!\cdot(n-1)!}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= \frac{(2n)!}{n!\cdot(n-1)!}\cdot\frac{1}{n\cdot(n+1)}\\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

We conclude that the number of walks that do not cross above the diagonal is $\dfrac{1}{n+1}\dbinom{2n}{n}$.

□

This shows that the Catalan Numbers are integers, although this also follows from the identity $C_n = \dbinom{2n}{n} - \dbinom{2n}{n+1}$ which comes up in the proof. There are many different interpretations of this number in the study of combinatorics. Here is another example.

Example 5.46. There are $2n$ people in line to get into a theater. Ticket is 50 cents. Among $2n$ people, n people have 50-cent piece and n people have 1-dollar bill. The box office begins with an empty cash register. How many ways we can line up such that anytime a person with 1-dollar bill buys a ticket, the box office has 50-cent piece to return in change?

We note that at each stage, the number of people with 50-cent piece who buy the ticket should be greater than or equal to the number of people with 1-dollar bill who buy the ticket, otherwise, there is no 50-cent piece to return in change. $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ is the number of ways to order the people based on money.

Going further, we also need to order the people with 50-cent piece and 1-dollar bill. In each case, there are $n!$ ways to order them, since people are distinct. Therefore, there are $C_n \cdot n! \cdot n! = \dfrac{(2n)!}{n+1}$ ways to line up people.

For now, we will see how we can use ordinary generating functions to obtain a recurrence relation for C_n. We define $C(x) = \sum_{n\geq 0} C_n \cdot x^n$.

Theorem 5.47. *For $n > 0$, the Catalan Numbers satisfy the following recurrence relation*

$$C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i-1}.$$

Proof. Consider the walks of students going to school from home. On the left-hand side, C_n counts the number of walks that do not cross above the diagonal. Since we are allowed to touch the diagonal, let (i,i) be the coordinate that first touches to the diagonal with $0 \leq i \leq n-1$.

Then, the walk breaks up into two pieces. The part to the right of (i,i) is still required to not cross above the diagonal, so there are C_{n-i-1} possible walks. At the left of (i,i), the number of walks that do not cross above the diagonal is C_i by definition again. So, for each $0 \le i \le n-1$, there are a total of $C_i \cdot C_{n-i-1}$ walks which do not cross above the diagonal. We conclude that the recurrence relation is given by $C_n = \sum_{i=0}^{n-1} C_i \cdot C_{n-i-1}$.

□

Now, we note that the right-hand side of the relation above is the coefficient of x^{n-1} in $C(x)^2$ and get

$$\begin{aligned} C(x) &= 1 + \sum_{n\ge 1} C_n \cdot x^n \\ &= 1 + \sum_{n\ge 1} \left(\sum_{i=0}^{n-1} C_i \cdot C_{n-i-1} \right) x^n \\ &= 1 + x \sum_{n\ge 1} \left(\sum_{i=0}^{n-1} C_i \cdot C_{n-i-1} \right) x^{n-1} \\ &= 1 + x \cdot C(x)^2. \end{aligned}$$

This means that $C(x)$ is a root of the polynomial $x \cdot t^2 - t + 1 = 0$. Then, we get that $C(x)$ is one of the solutions $\frac{1 \pm \sqrt{1-4x}}{2x}$. Since $C(x)$ is a power series, it must be that x divides the numerator, that is, the numerator cannot have a constant term. The constant term of $\sqrt{1-4x}$ is $\binom{1/2}{0} = 1$, so $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. By the Binomial Theorem, we have

$$\sqrt{1-4x} = \sum_{n\ge 0} \binom{1/2}{n} (-4x)^n.$$

Then, assume $n > 0$, we get

$$(-1)^n \cdot 4^n \binom{1/2}{n} = (-1)^n \cdot 4^n \cdot \frac{1}{n!} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdots \left(-\frac{2n-3}{2}\right) = -2^n \cdot \frac{(2n-3)!!}{n!}.$$

Note that $(2n-3)!! \cdot (2n-2)!! = (2n-2)!$, so we can multiply the numerator and denominator by $(2n-2)!!$,

$$-2^n \cdot \frac{(2n-2)!}{n! \cdot (2n-2)!!} = -2 \cdot \frac{(2n-2)!}{n! \cdot (n-1)!} = -\frac{2}{n} \binom{2n-2}{n-1}.$$

Since $\binom{1/2}{0} = 1$, we get

$$\frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x} \sum_{n\ge 1} \frac{2}{n} \binom{2n-2}{n-1} x^n = \sum_{n\ge 1} \frac{1}{n} \binom{2n-2}{n-1} x^{n-1} = \sum_{n\ge 0} \frac{1}{n+1} \binom{2n}{n} x^n.$$

This gives us $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proposition 5.48. *The number of walks from* $(0,0)$ *to* (n,n) *that stays strictly below the diagonal (except at* $(0,0)$ *and* (n,n)*) is* C_{n-1}.

6 Additional Topics on Sequences

We have touched some counting sequences previously. The number of permutations of a set of n elements for each is $0!, 1!, 2!, \ldots, n!, \ldots$ Also, we have investigated the Fibonacci sequence $f_0 = 1, f_1 = 1$ with $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. The explicit formula for f_n is given as

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Now, we will get into the *difference sequence* and sequence of *Stirling Numbers.* Lecture 25 Monday December 7

6.1 Difference Sequences

Definition 6.1 (Difference Sequence)**.** Let (h_n) be a sequence of numbers. We define the **(first order) difference sequence** (Δh_n) given by $\Delta h_n = h_{n+1} - h_n$ for $n \geq 0$.

Here is a straightforward example.

Example 6.2 (Fibonacci Sequence)**.** Let (f_n) be the *Fibonacci sequence.* The difference table for (f_n) is

$$\begin{array}{ccccccccccc} f_0 = 1 & & f_1 = 1 & & f_2 = 2 & & f_3 = 3 & & f_4 = 5 & & f_5 = 8 \\ & \Delta f_0 = 0 & & \Delta f_1 = 1 & & \Delta f_2 = 1 & & \Delta f_3 = 2 & & \Delta f_4 = 3 & \end{array}$$

More generally, we can inductively define the *p-th order difference sequence.*

Definition 6.3 (p-th Order Difference Sequence)**.** Let (h_n) be a sequence of numbers. We define the **second order difference sequence** $(\Delta^2 h_n)$ given by $\Delta^2 h_n = \Delta h_{n+1} - \Delta h_n$.

In general, the p**th order difference sequence** $(\Delta^p h_n)$ is defined by $\Delta^{p-1} h_{n+1} - \Delta^{p-1} h_n$.

Example 6.4. Let $t_n = 2n^2 + 3n + 1$. The difference table for (t_n) is

$$\begin{array}{ccccccccccc} t_0 = 1 & & t_1 = 6 & & t_2 = 15 & & t_3 = 28 & & t_4 = 45 & & t_5 = 66 \\ & \Delta t_0 = 5 & & \Delta t_1 = 9 & & \Delta t_2 = 13 & & \Delta t_3 = 17 & & \Delta t_4 = 21 & \\ & & \Delta^2 t_0 = 4 & & \Delta^2 t_1 = 4 & & \Delta^2 t_2 = 4 & & \Delta^2 t_3 = 4 & & \\ & & & \Delta^3 t_0 = 0 & & \Delta^3 t_1 = 0 & & \Delta^3 t_2 = 0 & & & \end{array}$$

The third order difference sequence in this case consists all 0's and hence so do all higher order difference sequences.

We can relate the difference sequence with derivatives. Given $t_n = 2n^2 + 3n + 1$, we have

$$\frac{dt}{dn} = 4n + 3$$
$$\frac{d^2t}{dn^2} = 4$$
$$\frac{d^3t}{dn^3} = 0.$$

This is a polynomial of degree 2, so the elements in the third order difference sequence are all 0 by differentiating three times. This gives the following property.

Theorem 6.5. *Let the general formula of a sequence (h_n) be a polynomial of degree p, that is,*

$$h_n = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0.$$

Then, $\Delta^{p+1} h_n = 0$ for all $n \geq 0$.

Proof. We prove by induction.

Base case. For $p = 0$, then we have $h_n = a_0$, which is a constant for all $n \geq 0$. Then, $\Delta h_n = h_{n+1} - h_n = a_0 - a_0 = 0$.

Inductive step. Assume the statement holds for $p' < p$. We have

$$\begin{aligned}
\Delta h_n &= h_{n+1} - h_n \\
&= \big(a_p(n+1)^p + a_{p-1}(n+1)^{p-1} + \cdots + a_1(n+1) + a_0\big) \\
&\qquad - \big(a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0\big) \\
&= a_p(n+1)^p - a_p n^p + \text{ polynomial of degree } p-1 \\
&= (a_p n^p + \text{ polynomial of degree } p-1) \\
&\qquad - a_p n^p + \text{ polynomial of degree } p-1 \\
&= \text{polynomial of degree } p-1.
\end{aligned}$$

By induction hypothesis, $\Delta^{p+1} h_n = \Delta^p(\Delta h_n) = 0$.

Hence, we conclude that the $(p+1)$th order difference sequence for a sequence that is a polynomial of degree p is all 0.

□

Theorem 6.6. *The general term of the sequence whose difference table has its 0th diagonal equal to*

$$c_0, c_1, c_2, \ldots, c_p \neq 0, 0, 0, 0, \ldots$$

is a polynomial of degree p satisfying

$$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots + c_p\binom{n}{p}.$$

Further, these constants $c_0, c_1, \ldots, c_p$ are uniquely determined.

Proof. It is intuitive that if $h_n = f_n + g_n$, then $\Delta h_n = \Delta f_n + \Delta g_n$. In general, we have $\Delta^p h_n = \Delta^p f_n + \Delta^p g_n$, for pth order difference. So, the diagonal of difference table is closed under addition.

Now, we start our inductive construction. If the 0th diagonal of difference table is $c_0, 0, 0, \ldots$, then we can imply that the original sequence (zero order difference sequence) is $c_0, c_0, c_0, \ldots$ So, $h_n = c_0 = c_0\binom{n}{0}$ for all n.

If the 0th diagonal of difference table is $0, c_1, 0, 0, \ldots$, then we imply that the first order difference sequence is $c_1, c_1, c_1, \ldots$ and further the original sequence is $0, c_1, 2c_1, 3c_1, \ldots$ We conclude that the general formula is $h_n = c_1 n = c_1 \binom{n}{1}$ for all n.

Similarly, if the 0th diagonal of difference table is $0, 0, c_2, 0, 0, \ldots$, then we imply that the second order difference sequence is all c_2's and the first order difference sequence is given by $\Delta h_n = c_2 \binom{n}{1}$. Therefore, we get the original sequence is given by $h_n = c_2 \binom{n}{2}$ by observation of the difference table.

In general, we can conclude that $h_n = c_p \binom{n}{p}$ for 0th diagonal of difference table as $0, 0, \ldots, 0, c_p, 0, 0, \ldots$.

Hence, we conclude that the 0th diagonal of difference table determines the whole sequence in a way that

$$h_n = c_0 \binom{n}{0} + c_1 \binom{n}{1} + \cdots + c_p \binom{n}{p}.$$

□

Here is a straightforward application of this theorem.

Example 6.7. Let $h_n = 2n^2 - n + 3$, find a closed formula for $\sum_{k=0}^{n} h_k$.

By simple calculation, the 0th diagonal of the difference table is $3, 1, 4, 0, 0, 0, \ldots$. Then, by Theorem 6.6, we have

$$h_n = 2n^2 - n + 3 = 3\binom{n}{0} + 1\binom{n}{1} + 4\binom{n}{2}.$$

Recall that $\sum_{k=0}^{n} \binom{k}{p} = \binom{n+1}{p+1}$, so

$$\begin{aligned} \sum_{k=0}^{n} (2n^2 - n + 3) &= \sum_{k=0}^{n} \left(3\binom{n}{0} + 1\binom{n}{1} + 4\binom{n}{2} \right) \\ &= 3\binom{n+1}{1} + 1\binom{n+1}{2} + 4\binom{n+1}{3}. \end{aligned}$$

Corollary 6.8. *Assume that the sequence (h_n) has a difference table whose 0th diagonal equals $c_0, c_1, c_2, \ldots, c_p \neq 0, 0, 0, 0, \ldots$ Then,*

$$\sum_{k=0}^{n} h_k = c_1 \binom{n}{1} + c_1 \binom{n}{2} + \cdots + c_p \binom{n+1}{p+1}.$$

Lecture 26
Wednesday
December 9

6.2 Stirling Numbers

Consider the sequence (h_n) given by $h_n = n^p$. We can evaluate the sum of pth powers of the first n positive integers. Recall that the 0th diagonal of the difference table determines the whole sequence, in the form,

$$c(p, 0), c(p, 1), c(p, 2), \ldots, c(p, p), 0, 0, 0, \ldots$$

so we can rewrite the power as

$$n^p = c(p, 0)\binom{n}{0} + c(p, 1)\binom{n}{1} + \cdots + c(p, p)\binom{n}{p}.$$

Then,

$$n^p = \sum_{k=0}^{p} c(p,k)\binom{n}{k} = \sum_{k=0}^{p} \frac{c(p,k)}{k!} \cdot \frac{n!}{(n-k)!}.$$

Definition 6.9 (Stirling Numbers of the Second Kind)**.** The **Stirling Numbers of the second kind** $S(p,k)$ is given by

$$S(p,k) = \frac{c(p,k)}{k!}$$

where $c(p,k)$ is the ,th number in the 0th diagonal of the difference table for the sequence n^p.

Since $S(p,0) = \dfrac{c(p,0)}{0!} = c(p,0)$, we have

$$S(p,0) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p \geq 1. \end{cases}$$

Expand n^p, the coefficient of n^p is definitely 1. For the expansion as several binomial coefficients, the coefficient of n^p is $\dfrac{c(p,p)}{p!}$. Thus, we have $S(p,p) = \dfrac{c(p,p)}{p!} = 1$ for $p \geq 0$.

Theorem 6.10 (Recurrence Relation)**.** *For $1 \leq k \leq p-1$, we have*

$$S(p,k) = k \cdot S(p-1,k) + S(p-1,k-1).$$

Proof. By definition, we have $n^p = \sum_{k=0}^{p} S(p,k) \cdot \dfrac{n!}{(n-k)!}$ and $n^{p-1} = \sum_{k=0}^{p-1} S(p-1,k) \cdot \dfrac{n!}{(n-k)!}$.

Thus,

$$\begin{aligned} n^p &= n \cdot n^{p-1} \\ &= n \sum_{k=0}^{p-1} S(p-1,k) \cdot \frac{n!}{(n-k)!} \\ &= \sum_{k=0}^{p-1} S(p-1,k) \cdot n \cdot \frac{n!}{(n-k)!} \\ &= \sum_{k=0}^{p-1} S(p-1,k) \cdot (n-k) \cdot \frac{n!}{(n-k)!} + \sum_{k=0}^{p-1} S(p-1,k) \cdot k \cdot \frac{n!}{(n-k)!} \\ &= \sum_{k=0}^{p-1} S(p-1,k) \cdot \frac{n!}{(n-k-1)!} + \sum_{k=1}^{p-1} S(p-1,k) \cdot k \cdot \frac{n!}{(n-k)!}. \end{aligned}$$

We replace k by $k-1$ in the left summation and then

$$\begin{aligned} n^p &= \sum_{k=1}^{p} S(p-1,k-1) \cdot \frac{n!}{(n-k)!} + \sum_{k=1}^{p-1} S(p-1,k-1) \cdot k \cdot \frac{n!}{(n-k)!} \\ &= S(p-1,p-1) \cdot \frac{n!}{(n-p)!} + \sum_{k=1}^{p-1} (S(p-1,k-1) + k \cdot S(p-1,k)) \cdot \frac{n!}{(n-k)!}. \end{aligned}$$

For each $1 \leq k \leq p-1$, we conclude that

$$S(p,k) = S(p-1,k-1) + k \cdot S(p-1,k).$$

$\square$

Theorem 6.11. *The Stirling Number of the second kind $S(p,k)$ is the number of ways to partition the set $\{1,2,\dots,p\}$ into k non-empty and indistinguishable parts.*

Proof. Let $S^*(p,k)$ denote the number of partitions of a set of p elements into k indistinguishable parts in which no part is empty. It is obvious that $S^*(p,p)=1$ for $p \geq 0$ and $S^*(p,0)=0$ for $p \geq 1$. We want to show that $S^*(p,k)=S(p,k)$ for all k and p with $0 \leq k \leq p$ by showing $S^*(p,k)$ satisfies the same recurrence relation as the *Stirling Numbers of the second kind.*

Consider the set of the first p positive integers $1,2,\dots,p$ as the set to be partitioned. The partitions of $\{1,2,\dots,p\}$ into k nonempty and indistinguishable parts are of two types: (a) those in which p is all alone in a part, and (b) those in which p is not in a part by itself, that is, the part contains at least one more element.

For type (a), if we remove p from the part that contains it, we are left with a partition of $\{1,2,\dots,p-1\}$ into $k-1$ nonempty and indistinguishable parts. Thus, there are $S^*(p-1,k-1)$ partitions of $\{1,2,\dots,p\}$.

For type (b), if we remove p from the part that contains it, then we are left with a partition of $\{1,2,\dots,p-1\}$ into k nonempty and indistinguishable parts. There are k parts, so we conclude that there are $k \cdot S^*(p-1,k)$ partitions of type (b).

Therefore, we conclude that

$$S^*(p,k) = k \cdot S^*(p-1,k) + S^*(p-1,k-1).$$

□

With this combinatorial significance of *Stirling Numbers of the second kind*, we get the following.

Proposition 6.12. *The Stirling Numbers of the second kind satisfy the following relations.*

(a) $S(n,1) = 1$ *for* $n \geq 1$.

(b) $S(n,2) = 2^{n-1} - 1$ *for* $n \geq 2$.

(c) $S(n,n-1) = \binom{n}{2}$ *for* $n \geq 1$.

Proof.

(a) $S(n,1)$ represents the number of ways to partition a set of n elements to only 1 part, so it is obvious that there is only one way to partition into 1 non-empty and indistinguishable part.

(b) We allow one of the two parts to be empty at first. Then, one part is the subset of n elements. So, there are 2^n possible ways to partition.

Now, we remove the condition of allowing the parts to be empty. Since we are overcounted here by 2 ways, either the first part is empty, or the second part is empty.

Further, the order of two parts does not matter, which leaves us

$$S(n,2) = \frac{1}{2} \cdot (2^n - 2) = 2^{n-1} - 1.$$

(c) $S(n, n-1)$ is equivalent to the number of possible ways that we pick 2 elements among n elements, so $S(n, n-1) = \binom{n}{2}$.

$\square$

That's the end of Math 184...