ANALYSIS OF THE DE BRUIJN–NEWMAN CONSTANT AND ITS IMPLICATIONS FOR THE RIEMANN HYPOTHESIS

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Abstract. This paper explores the De Bruijn–Newman constant Λ and its implications for the Riemann Hypothesis. We provide a detailed analysis of the integral $H(0, z)$ and the associated series $\Phi(u)$, demonstrating that if $\Lambda = 0$, the Riemann Hypothesis holds. Our results indicate that $H(0, z)$ evaluates to zero, suggesting that $\Lambda \neq 0$, which has implications for the non validity of the Riemann Hypothesis.

1. Introduction

The Riemann Hypothesis, proposed by Bernhard Riemann in 1859, is one of the most significant unsolved problems in mathematics. It posits that all non-trivial zeros of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C},$$

lie on the critical line where the real part of s is $\frac{1}{2}$. Despite extensive numerical evidence supporting this hypothesis, a formal proof or disproof remains elusive.

In the quest to understand and potentially resolve this hypothesis, the De Bruijn–Newman constant Λ plays a crucial role. Introduced by de Bruijn in 1950 and later explored by Newman, Λ provides a way to connect the zeros of the Riemann zeta function with a family of entire functions. If $\Lambda = 0$, the Riemann Hypothesis is true; if $\Lambda > 0$, the hypothesis is false.

This paper aims to analyze the De Bruijn–Newman constant by evaluating the integral $H(0, z)$ and the associated series $\Phi(u)$. We explore whether Λ could be zero and what implications this would have for the Riemann Hypothesis.

2. Theoretical Background

2.1. **The De Bruijn–Newman Constant.** The De Bruijn–Newman constant Λ is associated with the family of entire functions $H(\lambda, z)$ defined by

$$H(\lambda, z) = \int_0^{\infty} e^{\lambda u^2 h} \Phi(u) \cos(zu)\, du,$$

where λ is a real parameter and z is a complex variable. The function $\Phi(u)$ is defined as

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}.$$

The significance of Λ arises from its role in determining the nature of the zeros of $H(\lambda, z)$. If $\Lambda = 0$, then all non-trivial zeros of $H(\lambda, z)$ lie on the real line. If $\Lambda > 0$, then $H(\lambda, z)$ has non-real zeros, indicating that the Riemann Hypothesis is false.

The constant Λ was first introduced by de Bruijn in 1950. De Bruijn demonstrated that there exists a critical value $\lambda = \Lambda$ such that for $\lambda \geq \Lambda$, all zeros of $H(\lambda, z)$ are real. Newman's work further established that Λ is well-defined and lies in the range $-\infty < \Lambda \leq 0$.

2.2. **Connection to the Riemann Hypothesis.** The Riemann Hypothesis concerns the distribution of the zeros of the Riemann zeta function on the critical line $\text{Re}(s) = \frac{1}{2}$. These zeros are closely related to the distribution of prime numbers. The De Bruijn–Newman constant provides a means to connect the zeros of $H(\lambda, z)$ to the Riemann Hypothesis.

If $\Lambda = 0$, then all non-trivial zeros of the zeta function must lie on the critical line, thereby confirming the Riemann Hypothesis. Conversely, if $\Lambda > 0$, it would imply that the Riemann Hypothesis is false, as it would mean that there are non-real zeros for some $\lambda > 0$.

3. Evaluation of $H(0, z)$

3.1. **Integral and Series Analysis.** To determine the value of Λ, we analyze the integral

$$H(0, z) = \int_0^\infty \Phi(u) \cos(zu) \, du.$$

Given

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}},$$

we need to evaluate this integral to understand whether Λ could be zero.

Expanding $\Phi(u)$, we have

$$\Phi(u) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}.$$

To evaluate the series, we use the Euler-Maclaurin formula, which approximates sums by integrals. The formula is given by:

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x) \, dx + \frac{f(n) + f(m)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) + R_p,$$

$$\sum_{i=-\infty}^{\infty} f(i) = \int_{-\infty}^{\infty} f(x) \, dx + \frac{f(\infty) + f(-\infty)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) + R_\infty,$$

where B_{2k} are Bernoulli numbers and R_p is the remainder term.

3.2. **Proof that $R_p \to 0$ as $p \to \infty$ in the Euler-Maclaurin Formula.** Consider the inequality for R_p:

$$|R_p| \leq \frac{2\zeta(p)}{(2\pi)^p} \int_m^n \left|f^{(p)}(x)\right| \, dx$$

We aim to show that $R_p \to 0$ as $p \to \infty$.

Step 1: Behavior of $\zeta(p)$ as $p \to \infty$

Recall that the Riemann zeta function $\zeta(p)$ tends to 1 as $p \to \infty$:

$$\lim_{p \to \infty} \zeta(p) = 1$$

Step 2: Exponential decay of $(2\pi)^p$

The term $(2\pi)^p$ grows exponentially as p increases, so the reciprocal $\frac{1}{(2\pi)^p}$ decays exponentially fast:

$$\lim_{p\to\infty} \frac{1}{(2\pi)^p} = 0$$

Step 3: Behavior of the integral

Next, consider the integral $\int_m^n \left|f^{(p)}(x)\right| dx$. Assume that $f(x)$ is sufficiently smooth, meaning that its higher derivatives exist and are bounded. Therefore, the integral is finite and does not grow unboundedly as p increases.

$$\int_m^n \left|f^{(p)}(x)\right| dx \le M$$

for some constant M.

Step 4: Combine the terms

Now, combining all the terms, we have:

$$|R_p| \le \frac{2\zeta(p)}{(2\pi)^p} \int_m^n \left|f^{(p)}(x)\right| dx \le \frac{2M\zeta(p)}{(2\pi)^p}$$

As $p \to \infty$, $\zeta(p) \to 1$, $\frac{1}{(2\pi)^p} \to 0$, and M is finite. Hence, the entire expression tends to 0:

$$|R_p| \le 0$$

Any value inside the modulus can't be less than 0. Thus, $R_p \to 0$ as $p \to \infty$.

Q.E.D.

3.3. **Evaluation of**

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right)$$

. We are given the function:

$$f(u,n) = \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}},$$

and we aim to prove that every derivative of $f(u,n)$ with respect to n tends to zero as $n \to \pm\infty$.

Step 1: Behavior of the function as $n \to \infty$

The function can be rewritten as:

$$f(u,n) = e^{-\pi n^2 e^{4u}} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right).$$

As $n \to \infty$, the exponential term $e^{-\pi n^2 e^{4u}}$ decays rapidly due to the negative exponent $-\pi n^2 e^{4u}$, which grows very large. Therefore, the whole function tends to zero for large n, regardless of the growth of the polynomial terms n^2 and n^4. Specifically,

$$\lim_{n\to\infty} f(u,n) = 0.$$

Step 2: First derivative with respect to n

We now compute the first derivative of $f(u,n)$ with respect to n. Using the product rule:

$$\frac{d}{dn}f(u,n) = \frac{d}{dn}\left[e^{-\pi n^2 e^{4u}}\right]\cdot\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) + e^{-\pi n^2 e^{4u}}\cdot\frac{d}{dn}\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right).$$

First, differentiate the exponential term:

$$\frac{d}{dn}e^{-\pi n^2 e^{4u}} = -2\pi n e^{4u} e^{-\pi n^2 e^{4u}}.$$

Next, differentiate the polynomial term:

$$\frac{d}{dn}\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) = 8\pi^2 n^3 e^{9u} - 6\pi n e^{5u}.$$

Thus, the first derivative becomes:

$$\frac{d}{dn}f(u,n) = -2\pi n e^{4u} e^{-\pi n^2 e^{4u}}\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) + e^{-\pi n^2 e^{4u}}\left(8\pi^2 n^3 e^{9u} - 6\pi n e^{5u}\right).$$

As $n \to \infty$, both terms tend to zero due to the dominant factor $e^{-\pi n^2 e^{4u}}$, which decays exponentially faster than any polynomial growth of n. Hence:

$$\lim_{n\to\infty}\frac{d}{dn}f(u,n) = 0.$$

Step 3: Higher-order derivatives

Each higher-order derivative will involve the same rapidly decaying exponential term $e^{-\pi n^2 e^{4u}}$ multiplied by polynomial terms in n. As with the first derivative, the exponential decay dominates the growth of the polynomial terms, forcing each higher-order derivative to tend to zero as $n \to \infty$.

Step 4: Behavior as $n \to -\infty$

The behavior of $f(u,n)$ as $n \to -\infty$ is similar. Since the terms involve even powers of n (i.e., n^2 and n^4), the same exponential decay will dominate, and every derivative will tend to zero as $n \to -\infty$.

Conclusion:

Every derivative of the function

$$f(u,n) = \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}$$

with respect to n tends to zero as $n \to \pm\infty$.

Q.E.D. Hence, the series evaluates to

$$\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(0-0)$$

which evaluates to 0.

4. Evaluation of Integral

We aim to evaluate the following integral:

$$I = \int_{-\infty}^{\infty}\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}\,dn.$$

Step 1: Splitting the Integral

We first split the integral into two parts:

$$I = 2\pi^2 e^{9u} \int_{-\infty}^{\infty} n^4 e^{-\pi n^2 e^{4u}} \, dn - 3\pi e^{5u} \int_{-\infty}^{\infty} n^2 e^{-\pi n^2 e^{4u}} \, dn.$$

Step 2: Using Known Integrals

We use the following known integrals to solve each term:

1. $\int_{-\infty}^{\infty} n^4 e^{-\pi n^2 e^{4u}} \, dn = \frac{3\sqrt{\pi}}{4(\pi e^{4u})^{5/2}}$, 2. $\int_{-\infty}^{\infty} n^2 e^{-\pi n^2 e^{4u}} \, dn = \frac{\sqrt{\pi}}{2(\pi e^{4u})^{3/2}}$.

Step 3: Applying the Known Integrals

We now apply these known integrals to the two parts of the split integral:

For the first term, we have:

$$2\pi^2 e^{9u} \int_{-\infty}^{\infty} n^4 e^{-\pi n^2 e^{4u}} \, dn = 2\pi^2 e^{9u} \cdot \frac{3\sqrt{\pi}}{4(\pi e^{4u})^{5/2}} = \frac{3\pi^{3/2} e^{9u} \sqrt{\pi}}{2(\pi e^{4u})^{5/2}} = \frac{3\sqrt{\pi}}{2(\pi e^{4u})^{1/2}}.$$

For the second term, we have:

$$3\pi e^{5u} \int_{-\infty}^{\infty} n^2 e^{-\pi n^2 e^{4u}} \, dn = 3\pi e^{5u} \cdot \frac{\sqrt{\pi}}{2(\pi e^{4u})^{3/2}} = \frac{3\pi^{1/2} e^{5u} \sqrt{\pi}}{2(\pi e^{4u})^{3/2}} = \frac{3\sqrt{\pi}}{2(\pi e^{4u})^{1/2}}.$$

Step 4: Subtracting the Results

Now, subtract the second term from the first:

$$I = \frac{3\sqrt{\pi}}{2(\pi e^{4u})^{1/2}} - \frac{3\sqrt{\pi}}{2(\pi e^{4u})^{1/2}} = 0.$$

Conclusion

Thus, the given integral evaluates to zero:

$$\int_{-\infty}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}} \, dn = 0.$$

5. Evaluation of $\dfrac{f(\infty) + f(-\infty)}{2}$

$$f(n) = \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}.$$

We aim to prove that:

$$\frac{f(\infty) + f(-\infty)}{2} = 0.$$

Step 1: Behavior of $f(n)$ as $n \to \infty$

As $n \to \infty$, the exponential term $e^{-\pi n^2 e^{4u}}$ decays very rapidly due to the large exponent $-\pi n^2 e^{4u}$. This ensures that the entire function $f(n)$ tends to zero as $n \to \infty$, regardless of the polynomial growth of the terms n^4 and n^2 in the expression. Specifically, we have:

$$\lim_{n \to \infty} f(n) = 0.$$

Step 2: Behavior of $f(n)$ as $n \to -\infty$

Similarly, as $n \to -\infty$, the function $f(n)$ also tends to zero. This is because the exponential decay term $e^{-\pi n^2 e^{4u}}$ is symmetric with respect to n, and hence decays rapidly even for negative values of n. Therefore, we have:

$$\lim_{n \to -\infty} f(n) = 0.$$

Step 3: Conclusion

Thus, the value of $f(n)$ at both $n = \infty$ and $n = -\infty$ is zero. Therefore, the average of $f(\infty)$ and $f(-\infty)$ is:

$$\frac{f(\infty) + f(-\infty)}{2} = \frac{0+0}{2} = 0.$$

Hence, the result is:

$$\frac{f(\infty) + f(-\infty)}{2} = 0.$$

Q.E.D. Hence,

$$\sum_{n=-\infty}^{\infty} \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}. = 0+0+0+0 = 0$$

$$\Phi(u) = 0$$

6. Evaluation of De Brujin Newman Constant

$$H(0, z) = \int_0^\infty \Phi(u) \cos(zu)\, du.$$

As, we know that $\Phi(u) = 0$

$$H(0, z) = \int_0^\infty 0 \cos(zu)\, du. = 0$$

Here this terms evaluates to 0 for all real z even for all complex z . So De Brujin Newman constant don't evaluates to 0.

7. Conclusion

In this paper, we analyzed the De Bruijn–Newman constant Λ by evaluating the integral $H(0, z)$. Our findings suggest that $H(0, z) = 0$, for every complex z but for the proof of Riemann Hypothesis it should be zero for only real z which implies that $\Lambda \neq 0$. These results suggest further investigation is needed to fully resolve the implications for the Riemann Hypothesis. But for now i think Riemann Hypothesis is false.

References

[1] N. G. de Bruijn, *The roots of trigonometric integrals*, Duke Mathematical Journal, 1950.

[2] C. M. Newman, *Fourier transforms with only real zeros*, Proceedings of the American Mathematical Society, 1976.

[3] B. Riemann, *On the number of primes less than a given magnitude*, Monthly Notices of the Royal Astronomical Society, 1859.

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