Geometric Modeling

2015

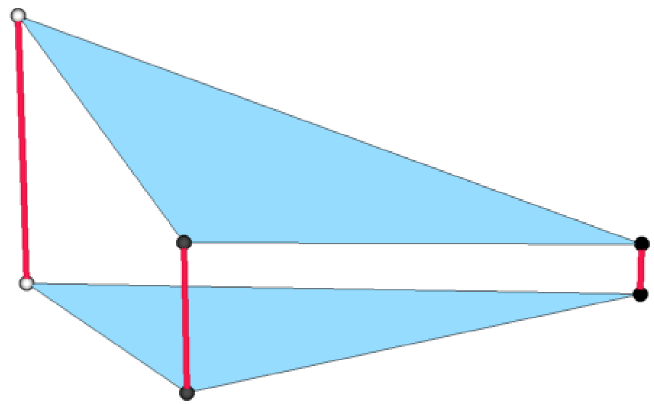
Surface Smoothing

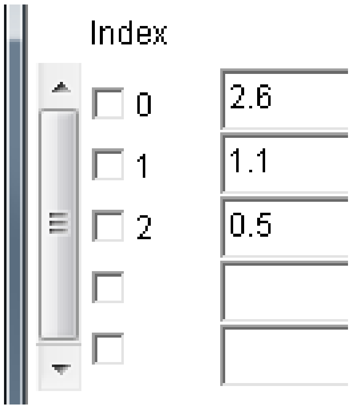
TU Delft

Last Lecture

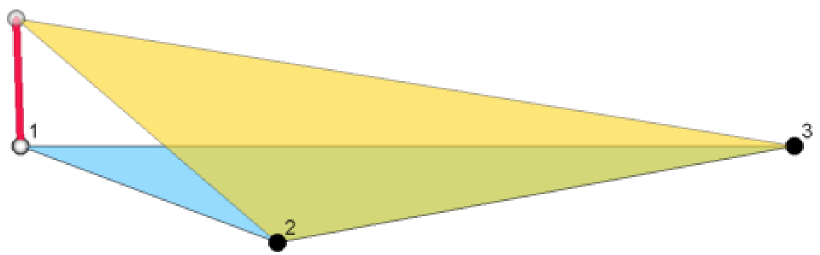
Lagrange Basis Functions

Functions on a mesh





Lagrange basis functions

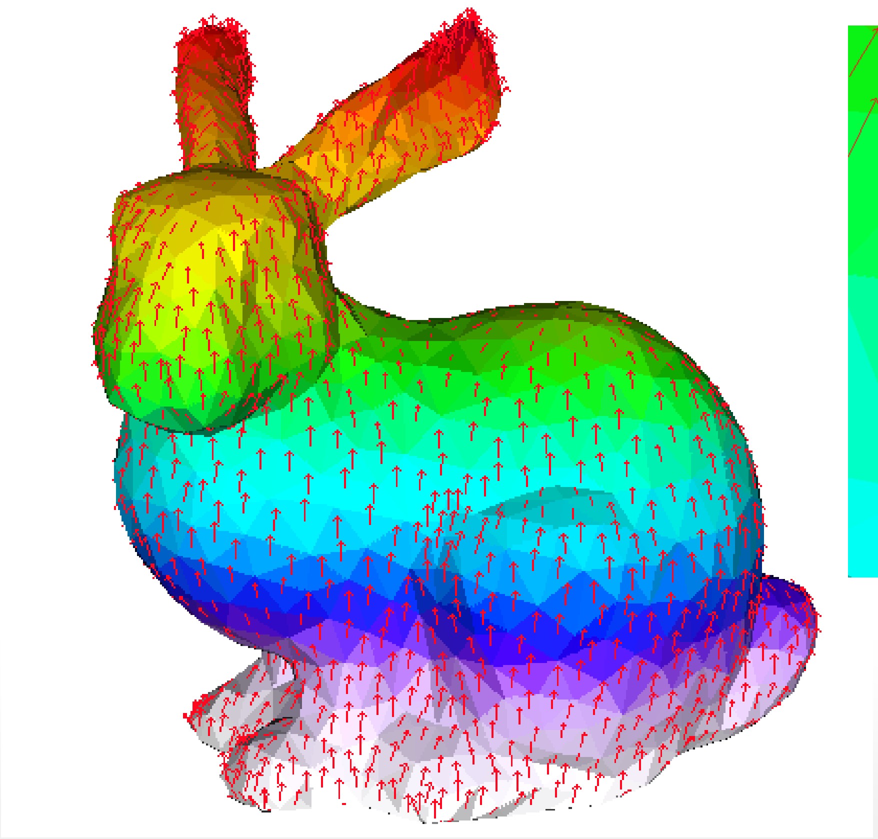


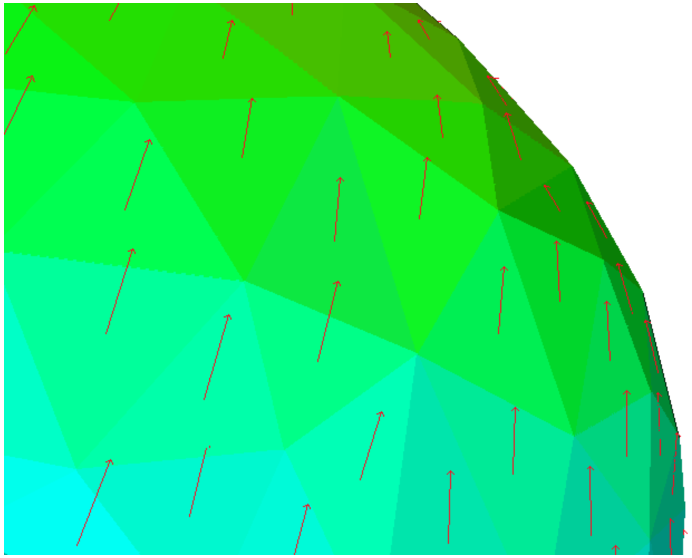
$$u(p) = \sum_{i=1}^{3} u_i \, \varphi_i(p)$$

Gradient

What is the gradient of a function in S_h?

- A constant tangential vector in every triangle
- Denote space of piecewise constant vector fields by V_h





Gradient matrix

Gradient

- Linear map from functions to vector fields

$$G: S_h \mapsto V_h$$

Matrix representation of G

- $m \times n \;\; (3\#T \times \#V)$ matrix
- Assembled from the elementary matrices:

$$\frac{1}{2\mathrm{area}(T)}\left(R^{90^\circ}e_1 \quad R^{90^\circ}e_2 \quad R^{90^\circ}e_3\right)$$

Mass Matrix

Matrix representation

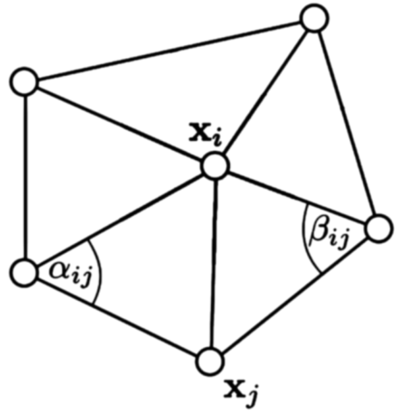
- Often called the mass matrix
- We denote the matrix by M

$$\sum_{p_i \in M} A_{p_i} u_i v_i = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} A_{p_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{p_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Stiffness Matrix

Dirichlet Energy

$$u \to \frac{1}{2}\int_M \langle \nabla u , \nabla u \rangle dx$$



Matrix representation

$$S = G^T M_V G$$

$$E_D(\mathrm{u}) = \frac{1}{2} u^T S u$$

- Matrix S explicitly:

$$s_{ij} = -\frac{1}{2}\left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right) \text{ for } i \neq j$$

$$s_{ii} = -\sum_{j=1}^{n} s_{ij}$$

Discrete Laplace-Beltrami Operator

Laplace Matrix

- We call the matrix $L = M^{-1}S$ the Laplace matrix

Remarks

- Maps functions to functions
- Continous analog for $\mathbb{R}^2$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \quad \text{or} \quad \Delta u$$

- The constant functions are in the kernel of L

Overview Matrices

Matrices

M, M_V:

- diagonal matrices
- positive entries (areas)

G:

- rectangular matrix ($m \times n$)
- sparse

$S = G^T M_V G$:

- symmetric ($\mathrm{n} \times n$)
- sparse
- non-negative:
  $u^T S u \geq 0$ for all u
- kernel are the constants

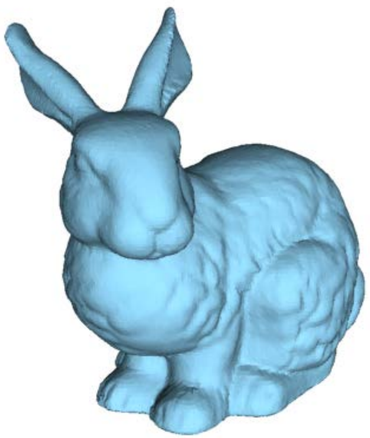
$L = M^{-1} S$

- not symmetric ($\mathrm{n} \times n$)
- sparse
- non-negative eigenvalues
- kernel are the constants

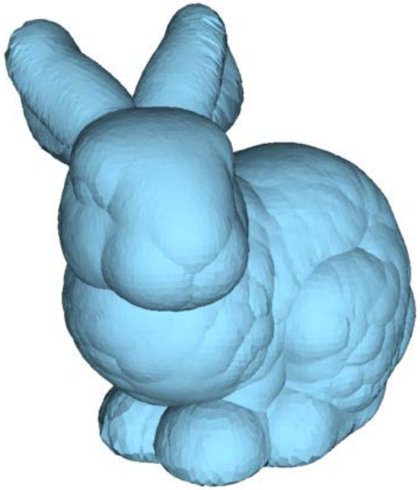
Displacement Vector

Notation

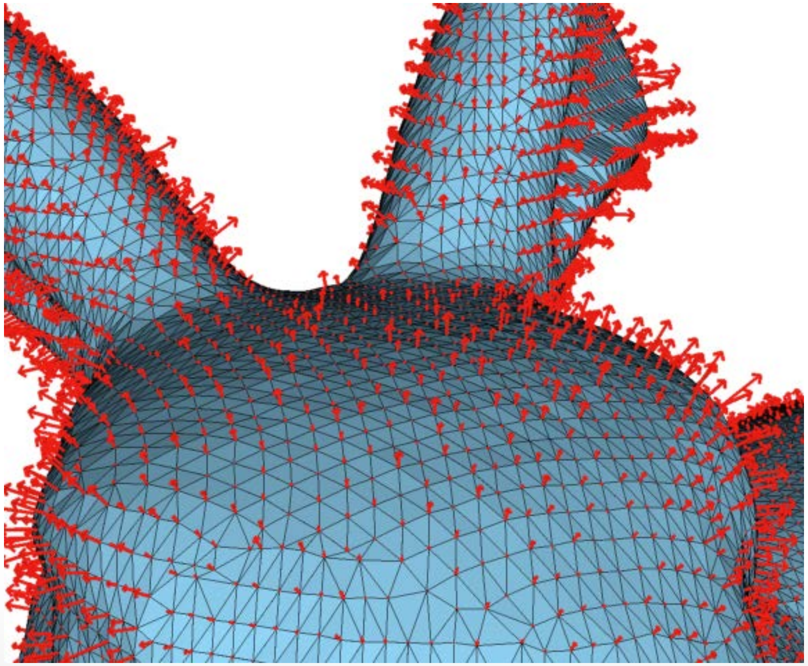
- Denote by $x \in {S_h}^3$ the map that maps every vertex to its positions in $\mathbb{R}^3$ and by $u \in {S_h}^3$ a displacement of the surface



x



$x + u$



Deformation Energies

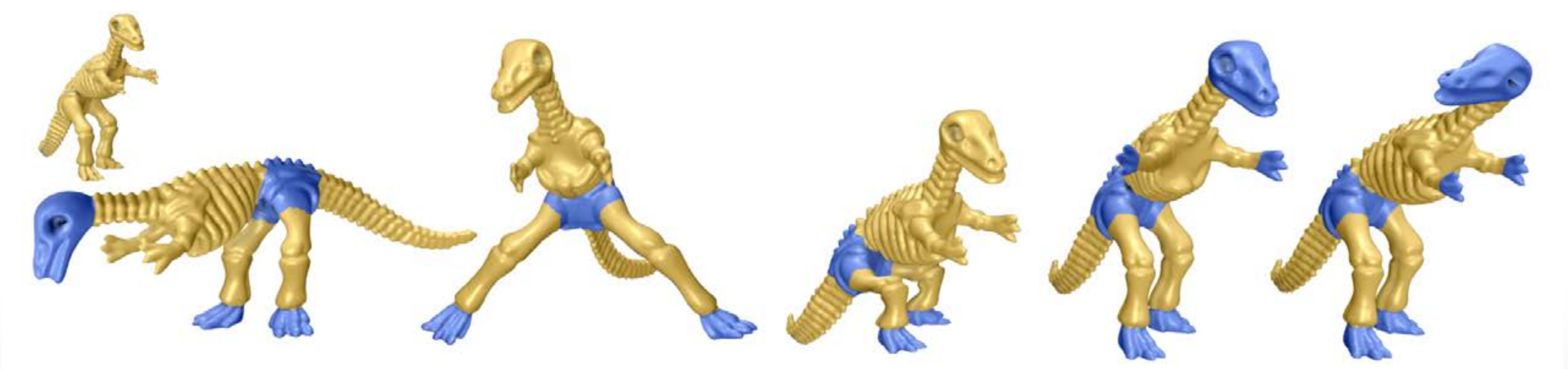
General deformation energies

- A deformation energy measures the “energy” stored in a deformation (or the “cost” of a deformation)

$$E : {S_h}^3 \mapsto \mathbb{R}$$

Displacement

Energy



Mesh and Surface Analysis

Mesh and Surface Analysis

Mesh Analysis

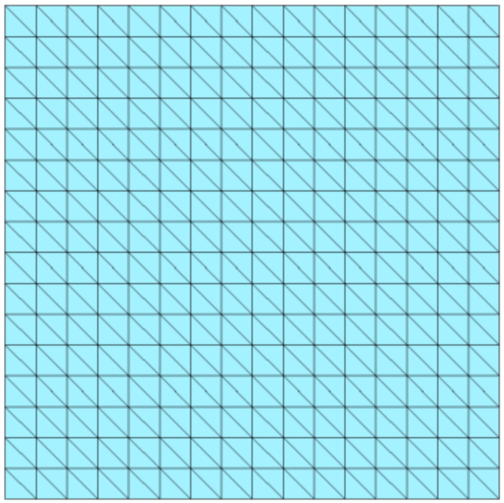
- Properties of a mesh
  - Shapes of triangles
  - Uniform/adaptive mesh

Surface Analysis

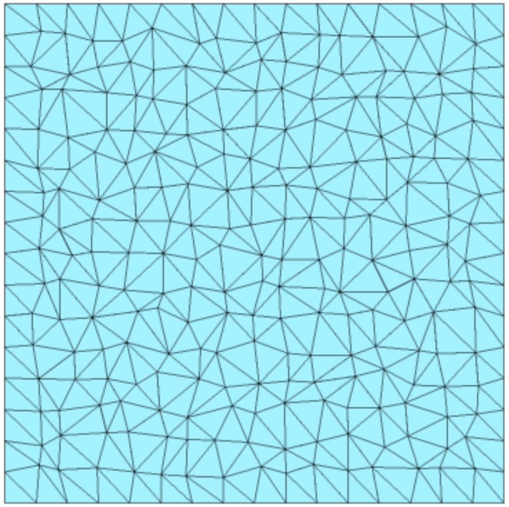
- Properties of the surface described by the mesh
  - Area, enclosed volume
  - Mean curvature (tension in the surface)
  - Visual quality

Mesh Analysis

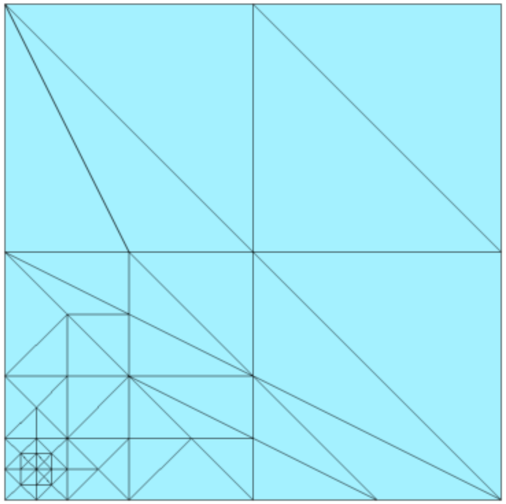
Types of meshes



regular



irregular

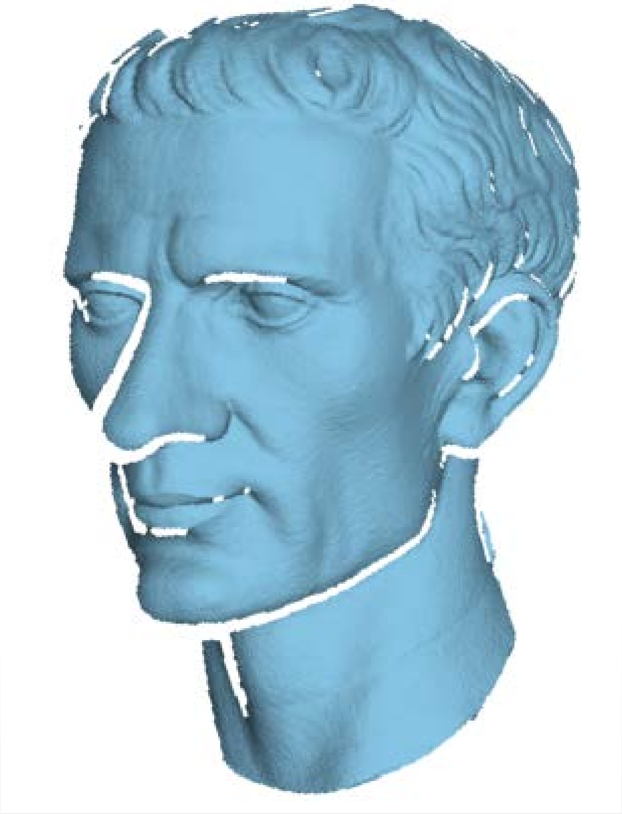


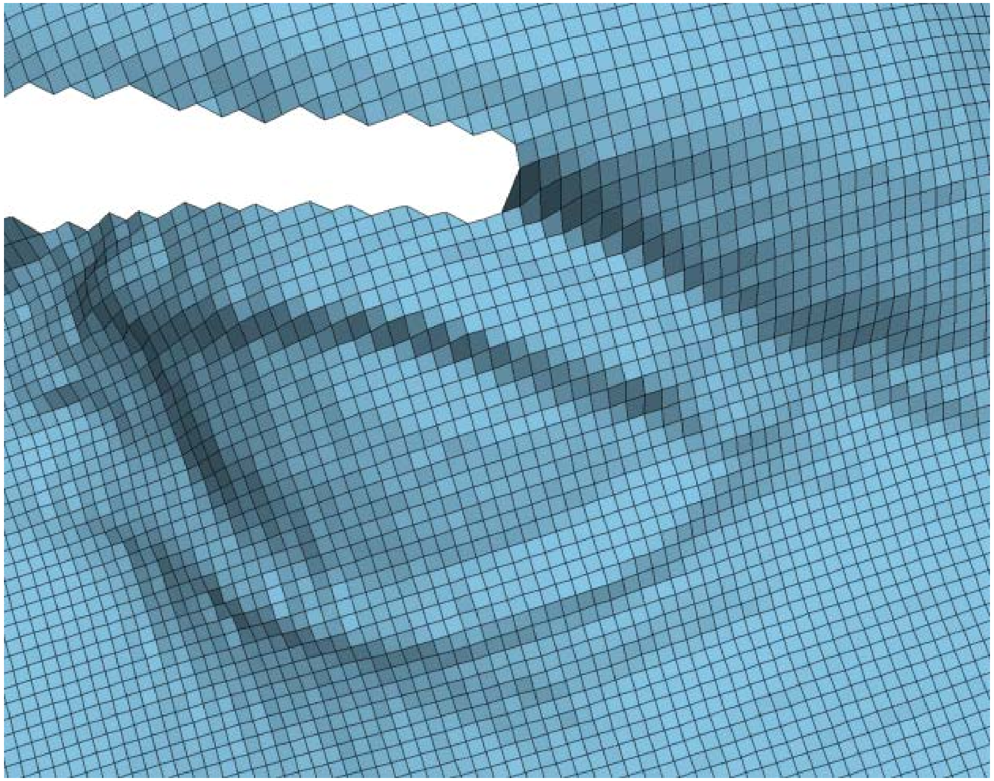
adaptive

Examples

3D-Scanner range image

- Quads
- Noise
- Holes

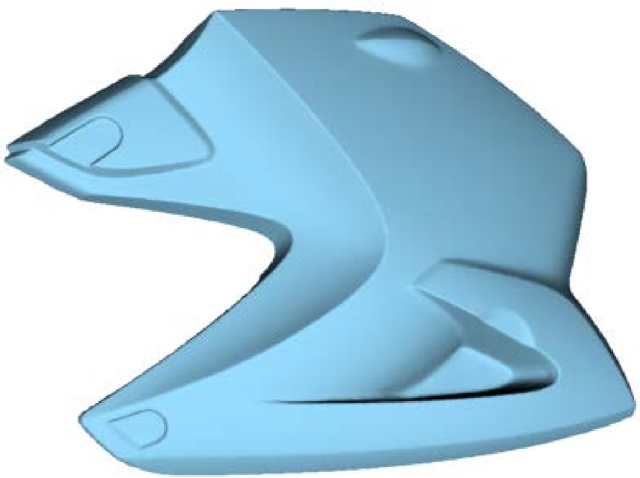


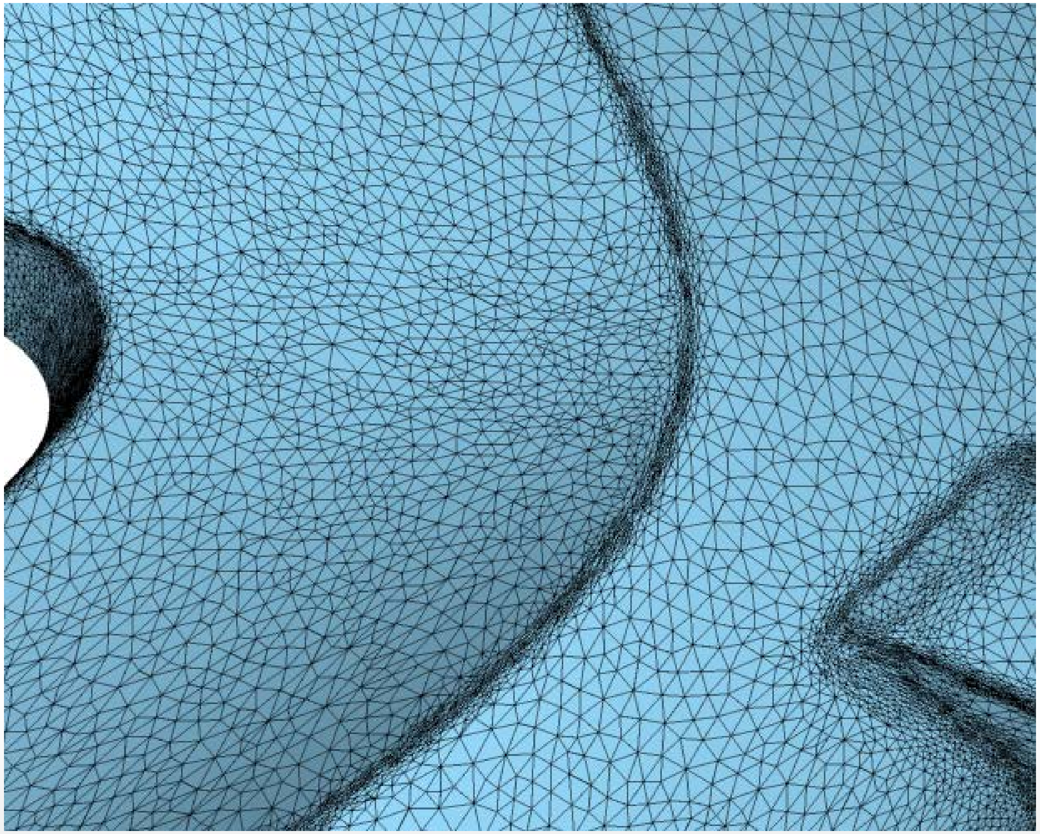


Examples

A simplified 3D-scan

- Irregular
- Adaptive

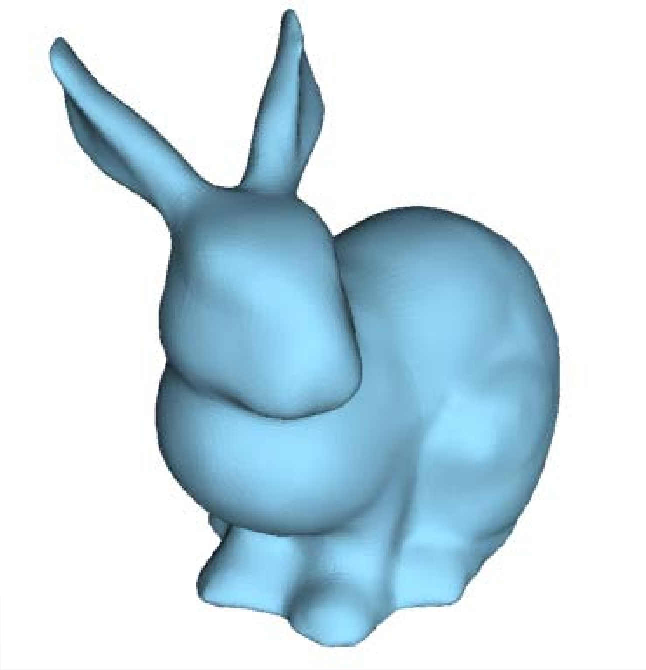


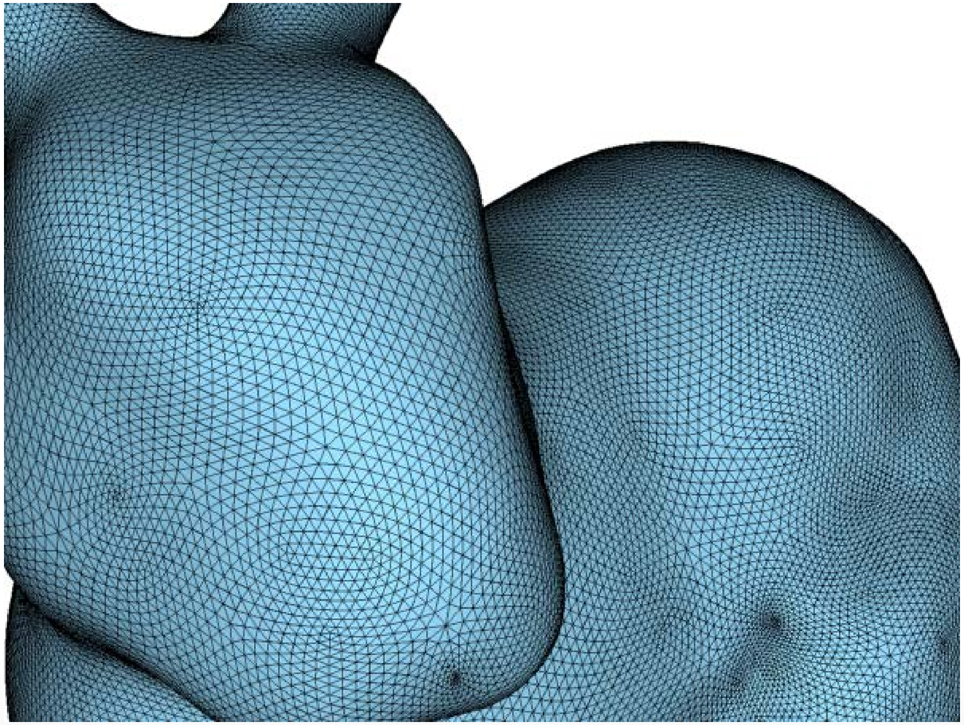


Examples

Subdivision

- Refinement inserts regular vertices (valence 6)
- Inital irregular vertices still present

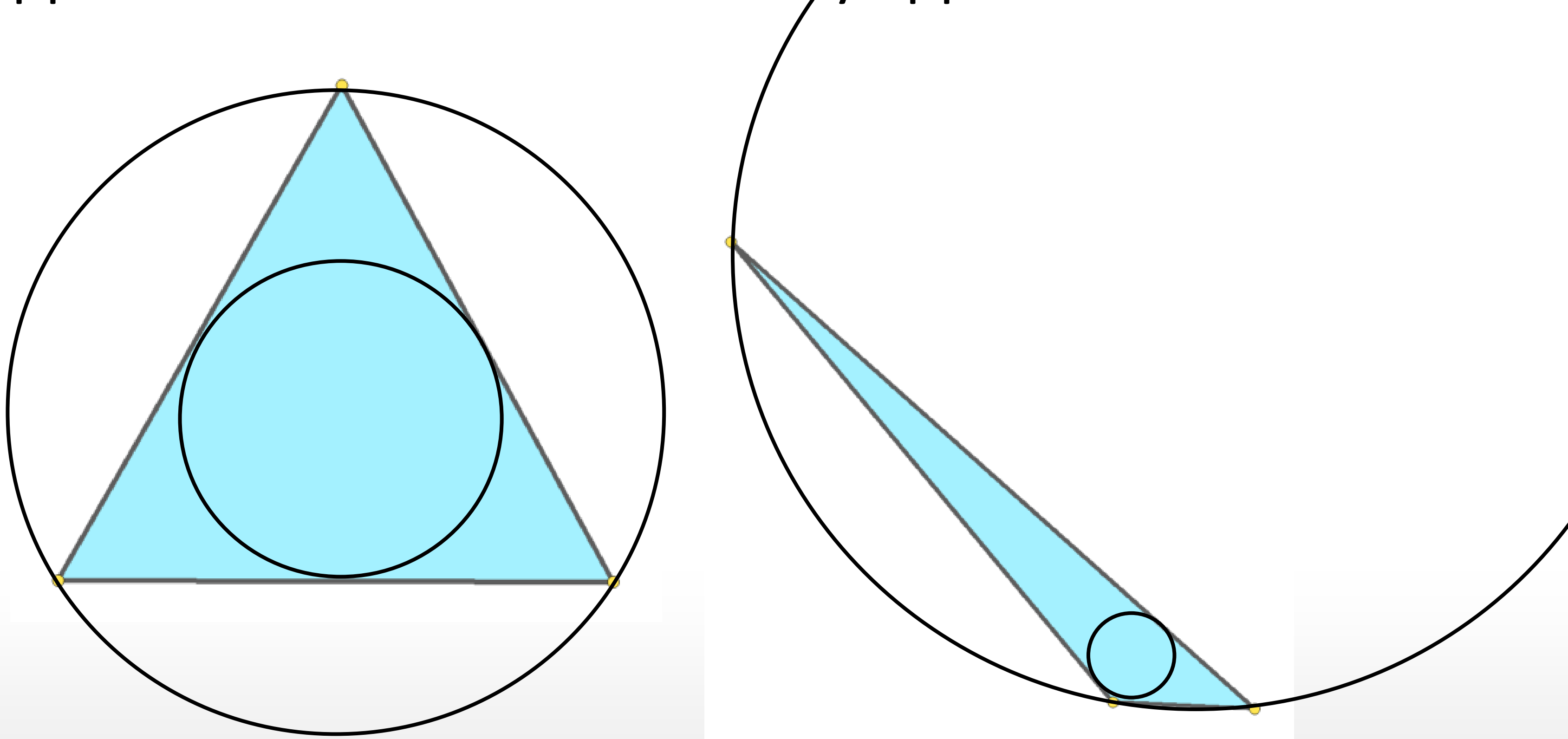




Shape Regularity

Shape regularity of a triangle

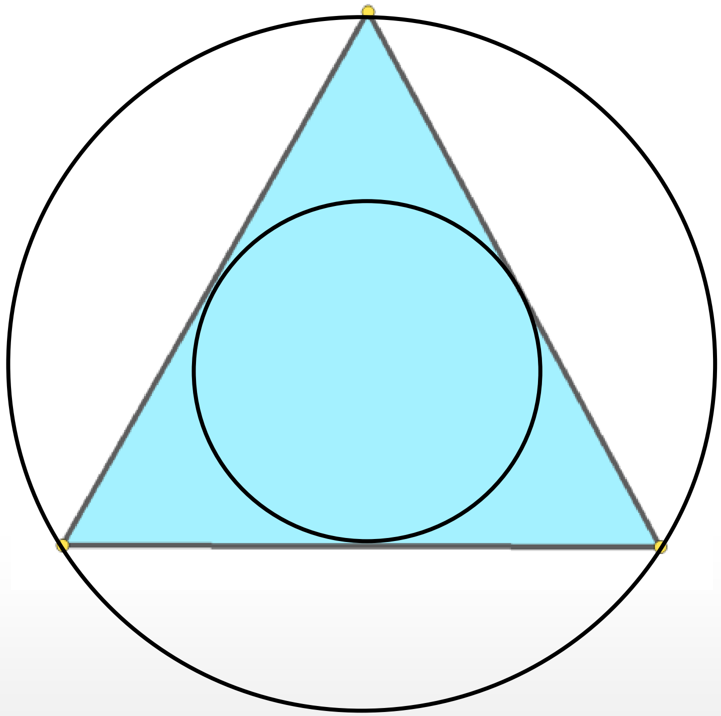
- Ratio of the diameters of the inscribed and the circumscribed circle
- Appears in error bounds for many approximations

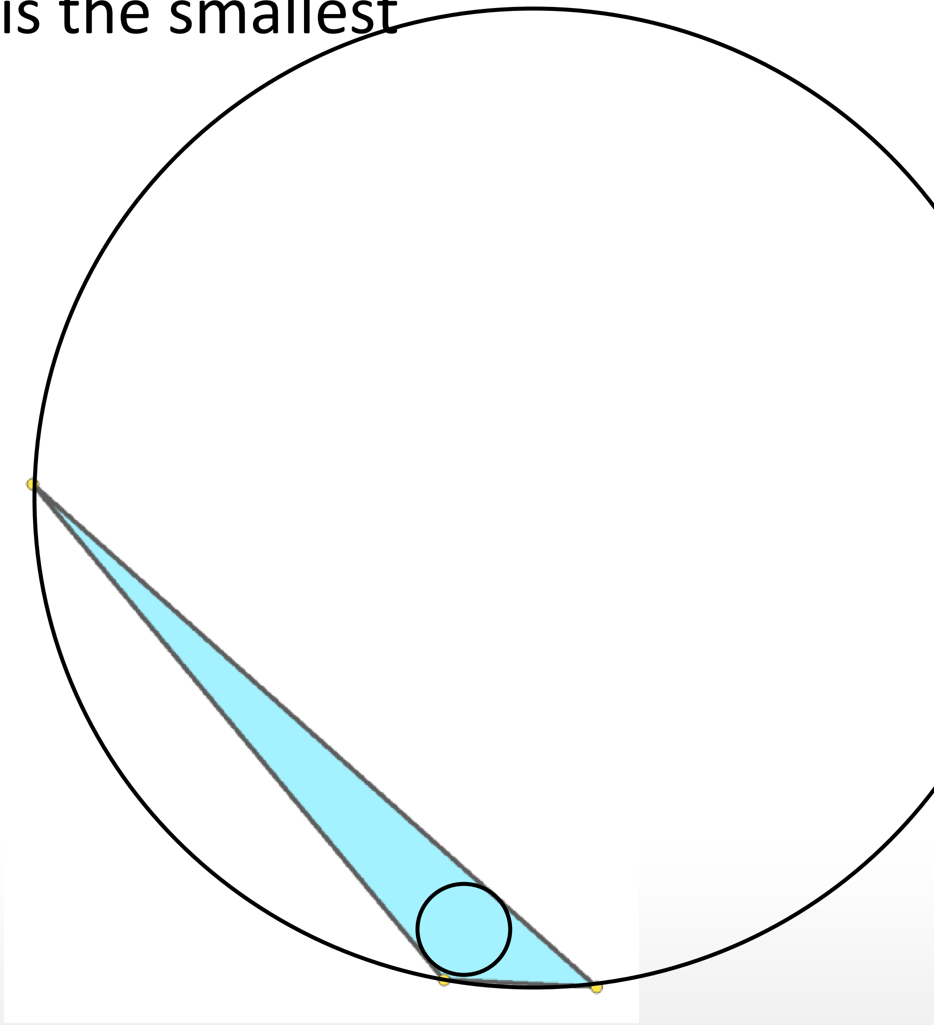


Shape Regularity

Shape regularity of a triangle

- Estimate: $2/\sin(\theta)$, where θ is the smallest angle of the triangle

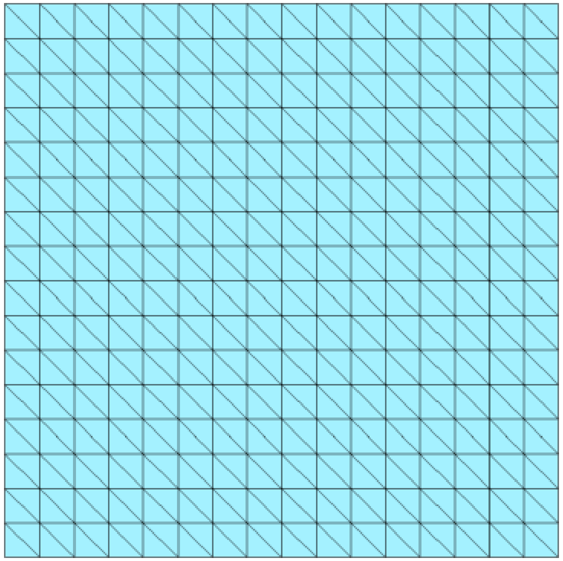


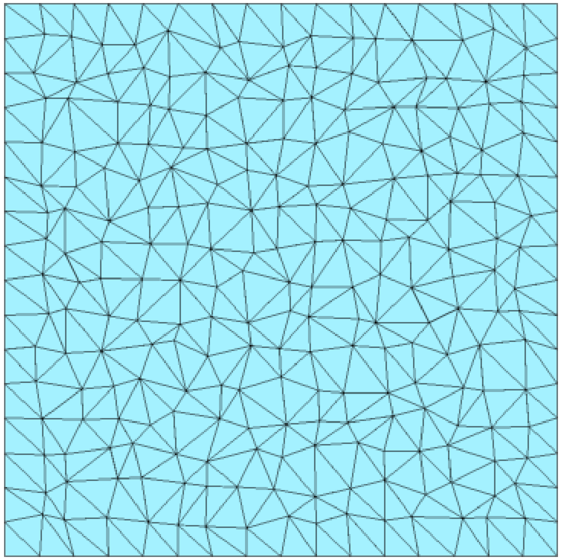


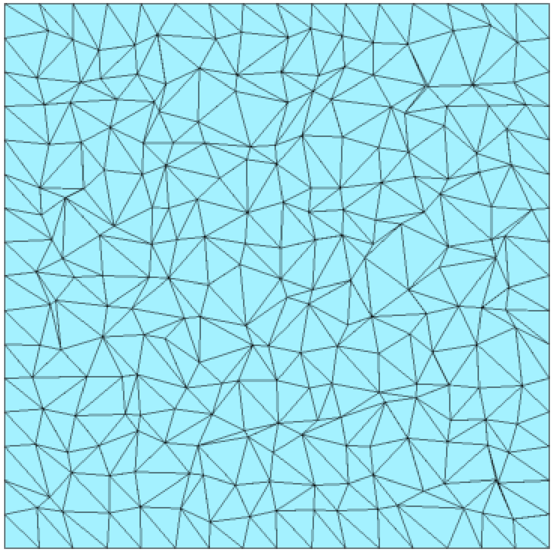
Shape Regularity

Shape regularity of a mesh

- Minimum of the shape regularities of all triangles



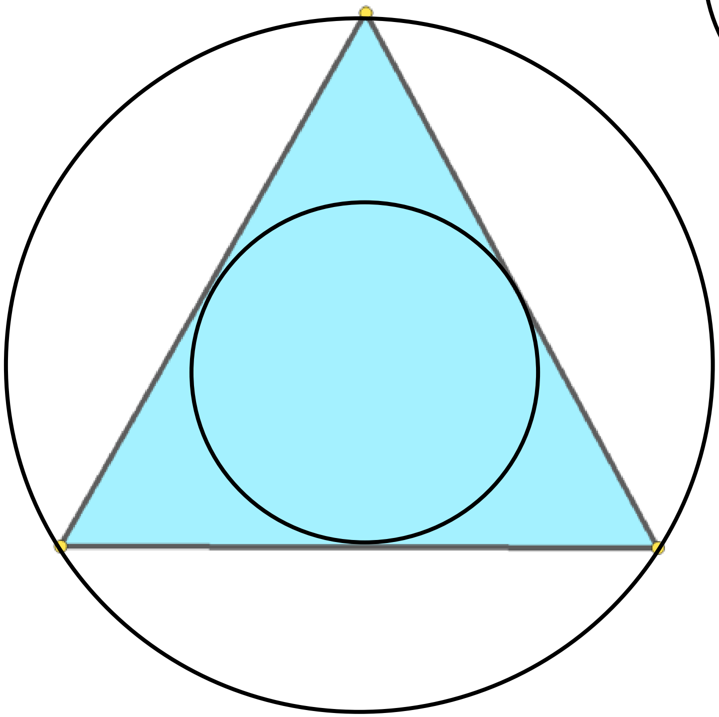


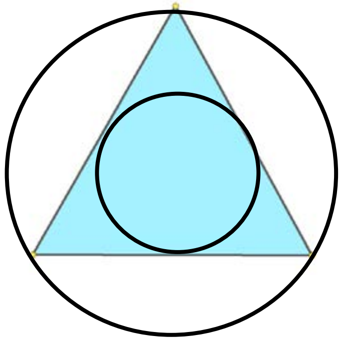


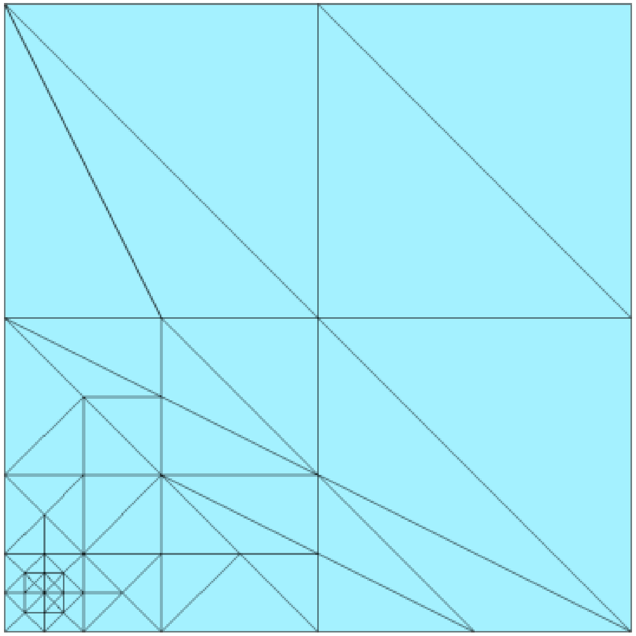
Shape Regularity

Shape regularity

- Scale invariant



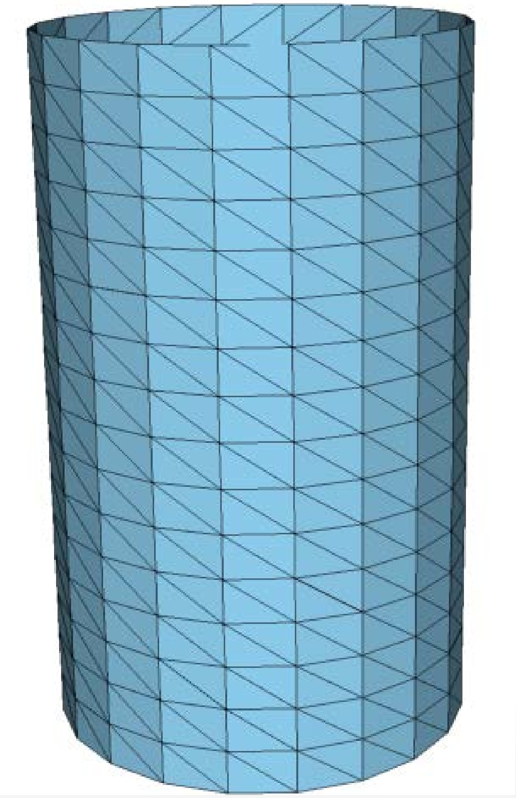


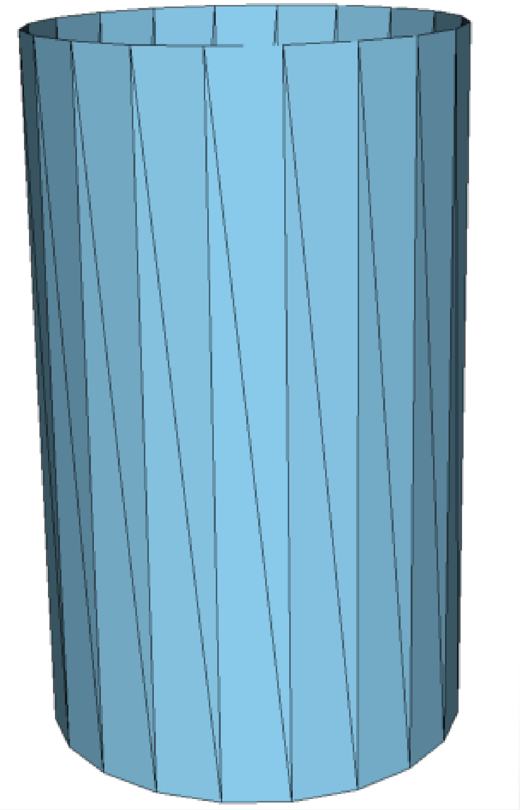


Shape Regularity

However skinny triangles are not always bad

- Two surfaces below approximate the cylinder equally well
- However, the right one has fewer triangles

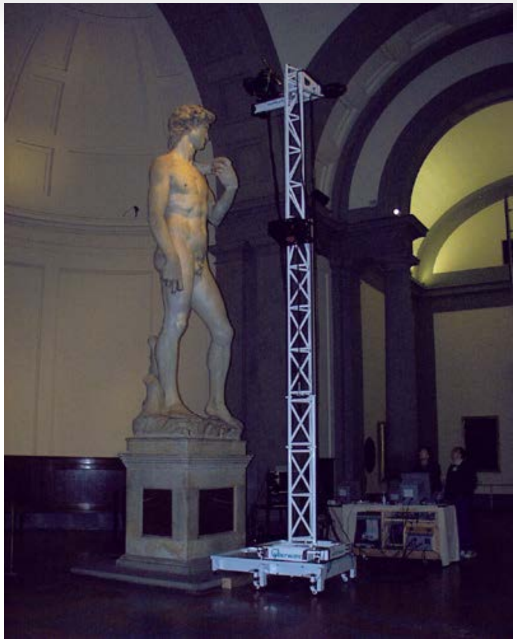


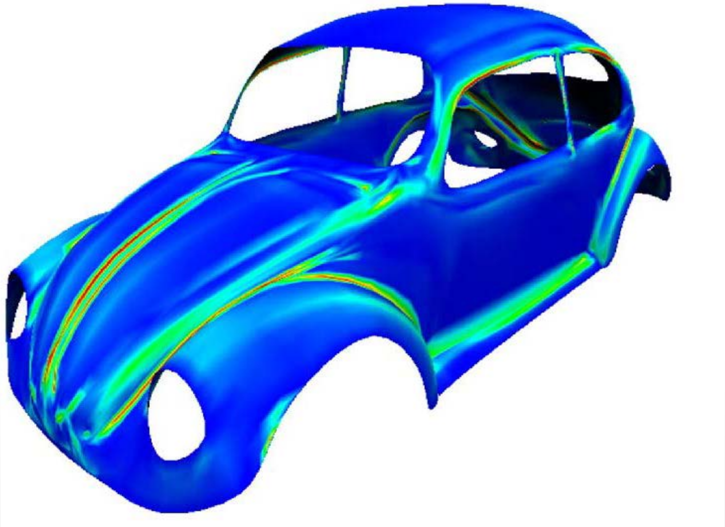


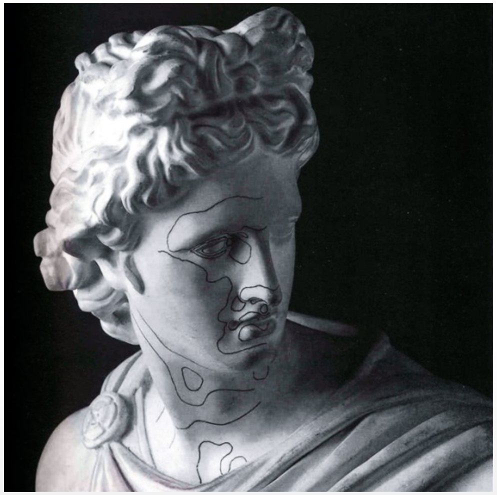
Surface Analysis

Properties of a surface

- Area of a surface
- Enclosed volume
    - What is the area/volume of David?
- Special geometric lines on a surface
- Curvatures





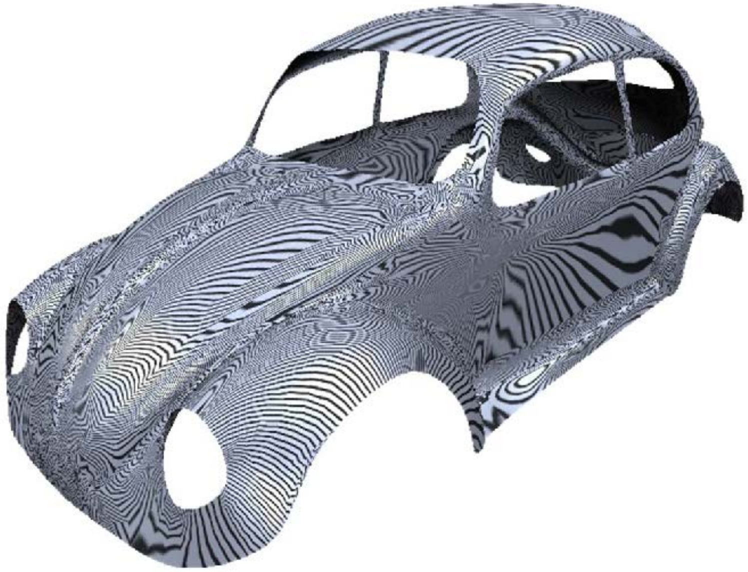


Visual Quality

Analysis of visual quality with reflection lines

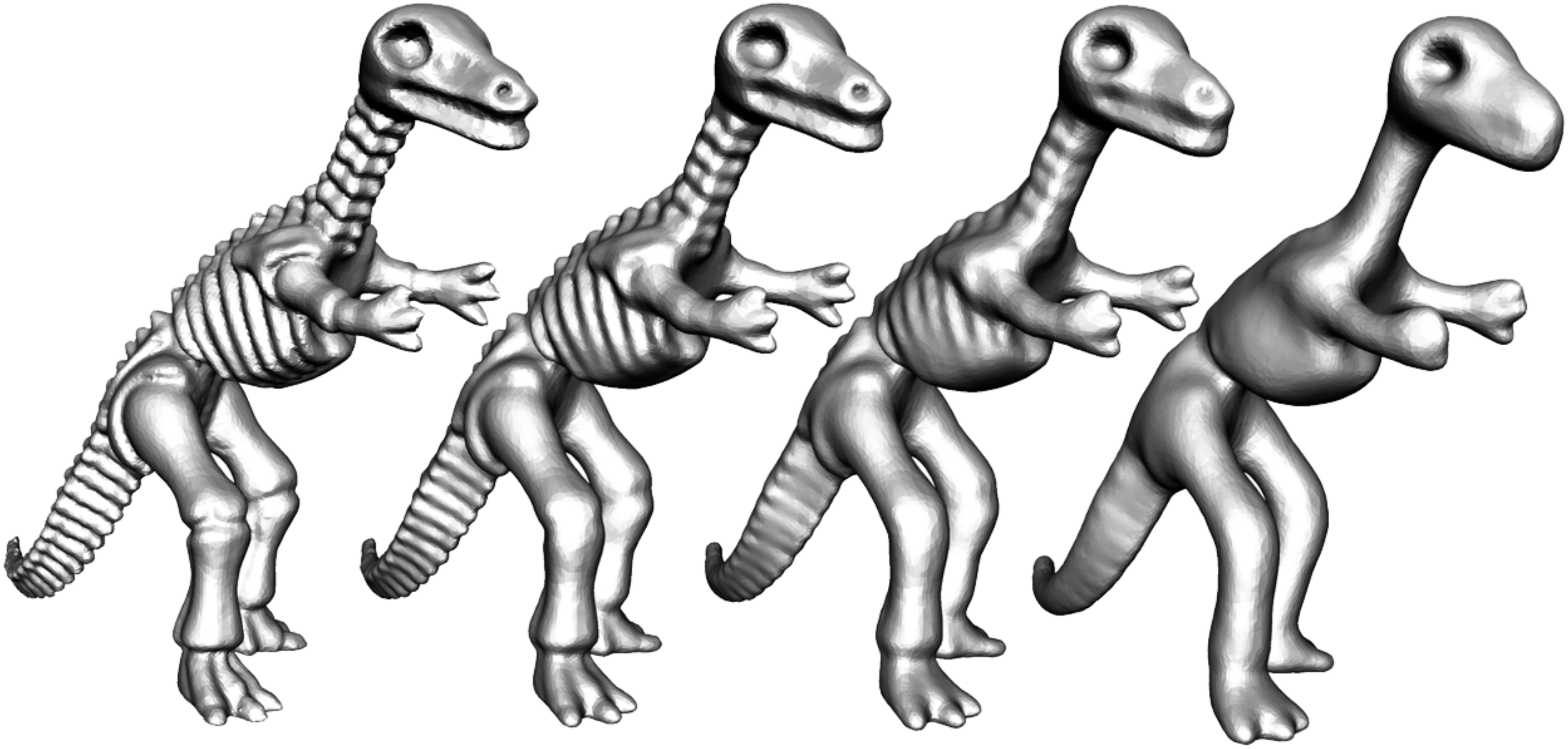
Visual Quality

This can be done easier on digitalized surfaces



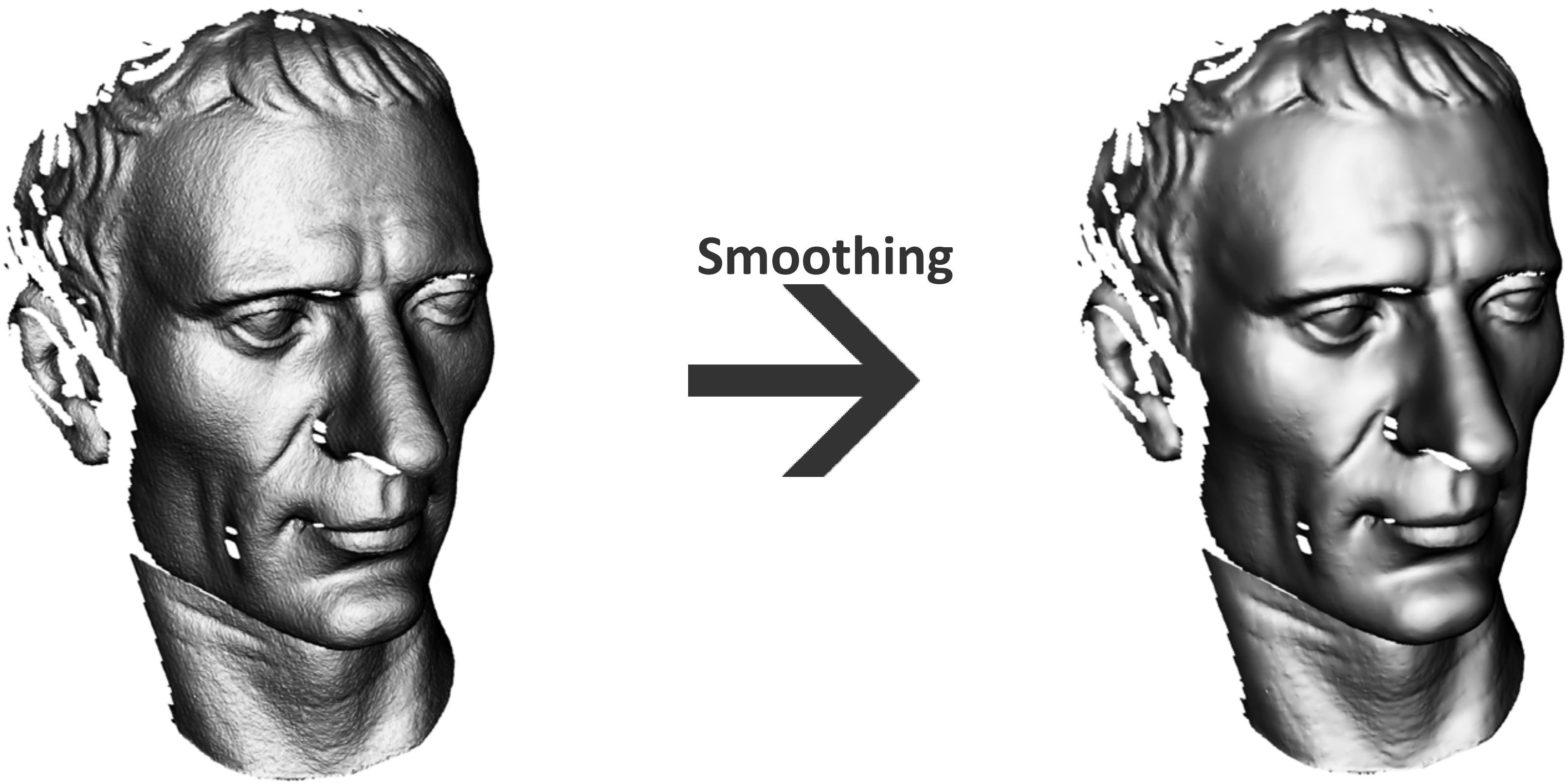
Mesh Smoothing

Smoothing



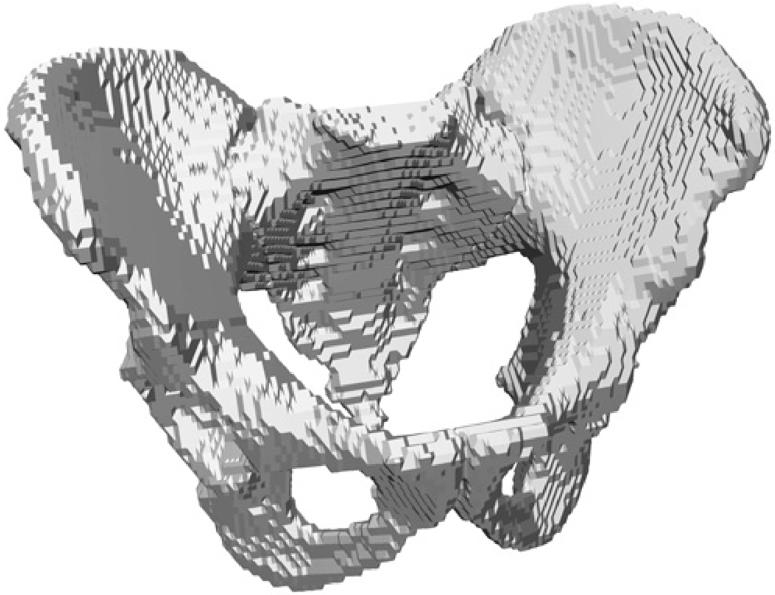
Applications

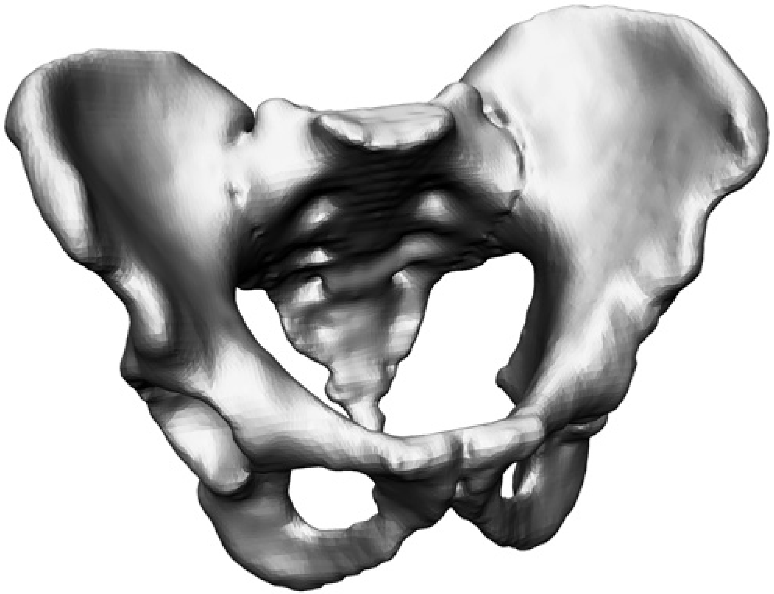
Denoising



Applications

Remove Artifacts



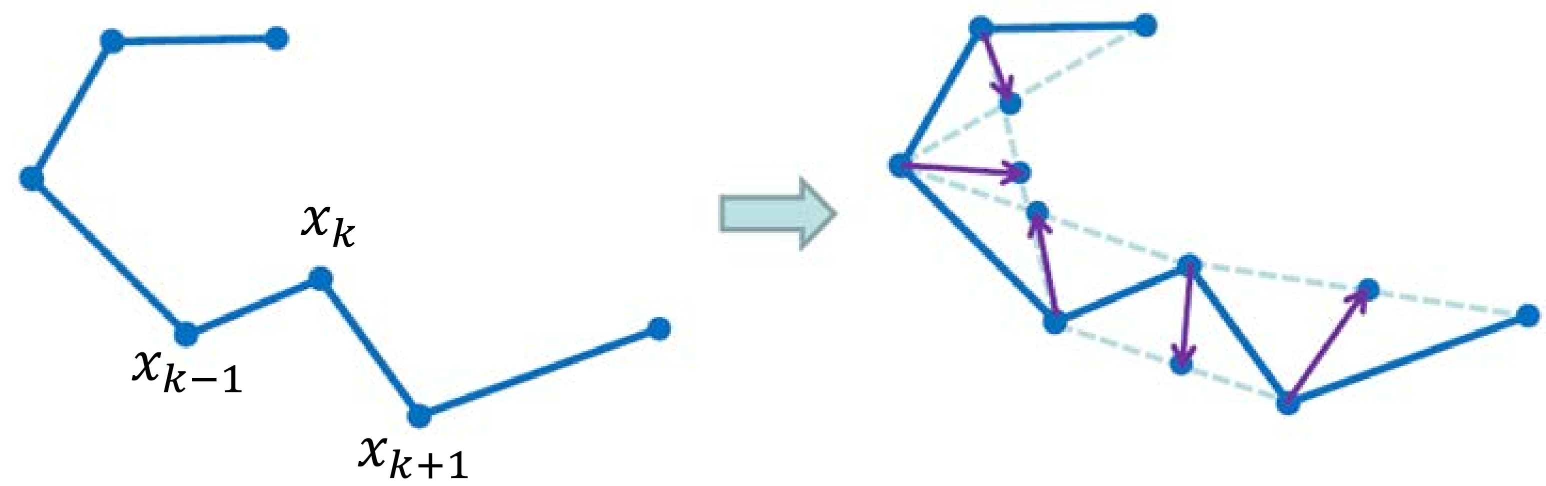


Iterated Averaging (Laplace Smoothing)

Consider curves first

- Compute difference to average of neighbors

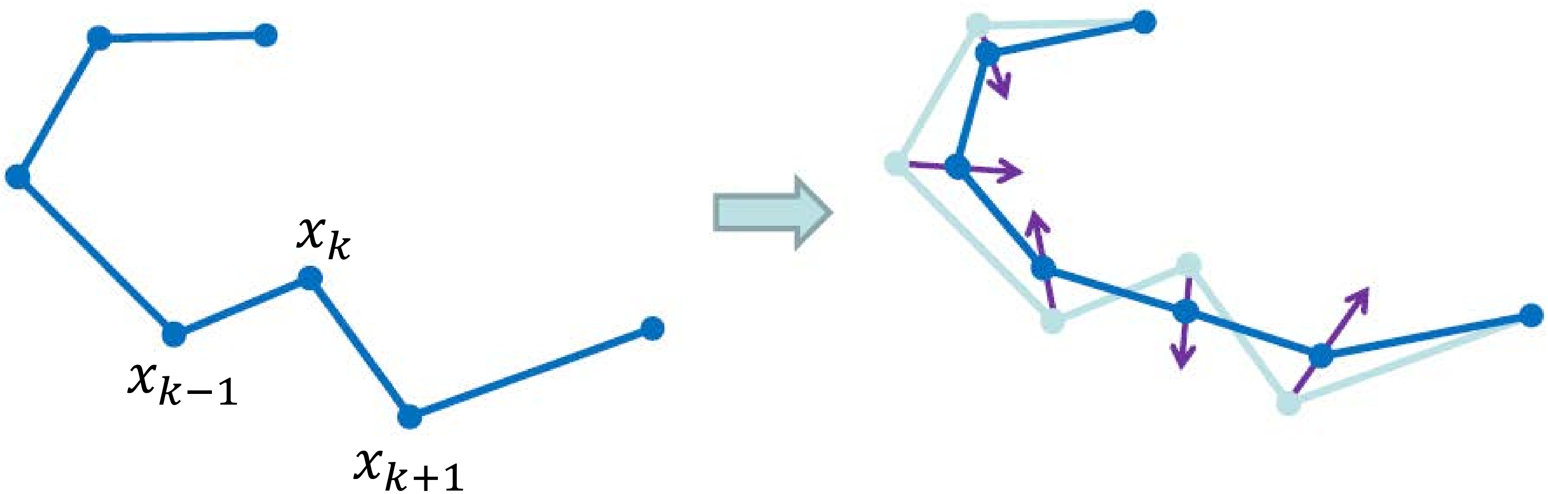
$$\frac{x_{k-1} + x_{k+1}}{2} - x_k$$



Iterated Averaging (Laplace Smoothing)

Smoothing step

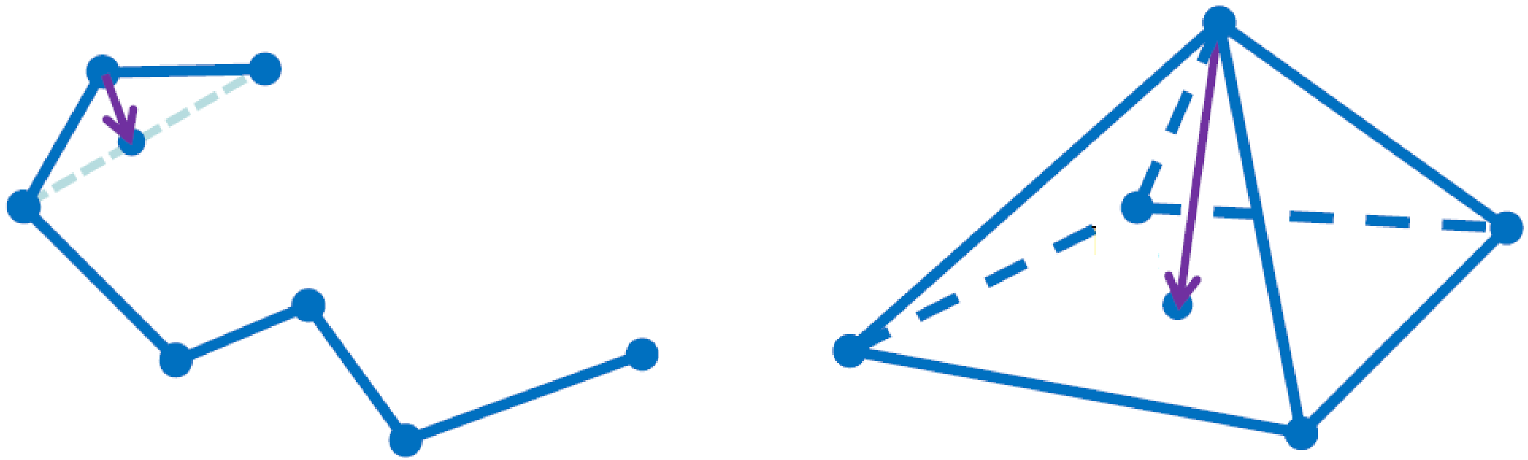
- Iterate: Move every vertex towards to average of its neighbors
- $x_k \leftarrow x_k + \tau\left(\frac{x_{k-1}+x_{k+1}}{2} - x_k\right)$
- $\tau \in (0,1]$ is the stepsize



Iterated Averaging (Laplace Smoothing)

For surface meshes

- Same as for curves

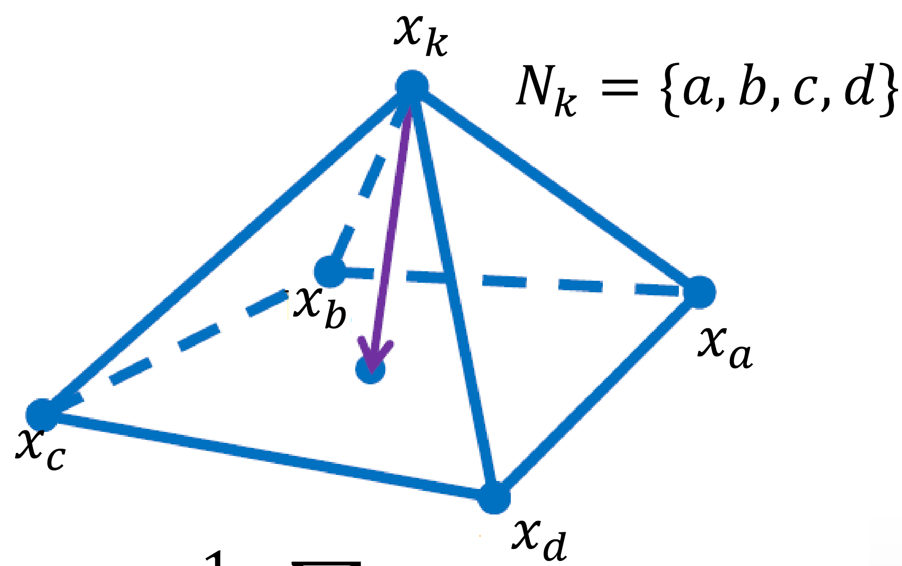


Iterated Averaging (Laplace Smoothing)

For surface meshes

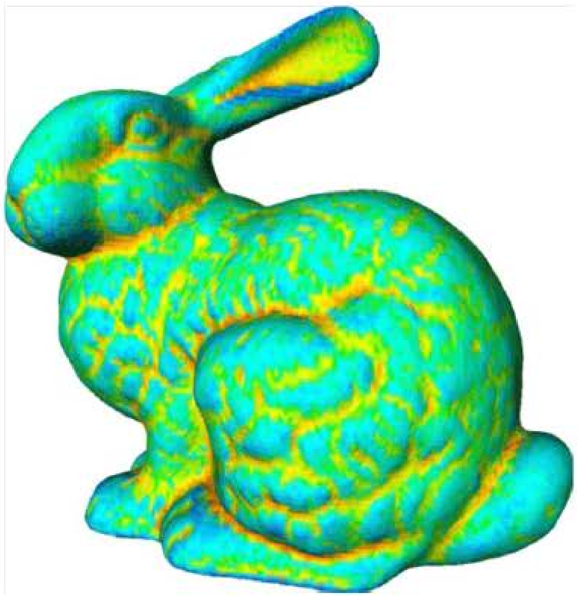
- Iterate: Move every vertex towards the average of its neighbors
- $x_k \leftarrow x_k + \tau\left(\frac{1}{|N_k|}\sum_{l \in N_k} x_l - x_k\right)$

$$\frac{x_{k-1} + x_{k+1}}{2} - x_k$$

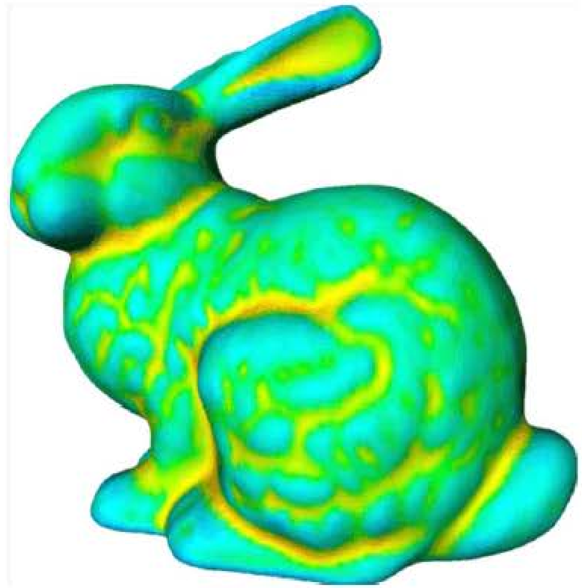


$$\frac{1}{|N_k|}\sum_{l \in N_k} x_l - x_k$$

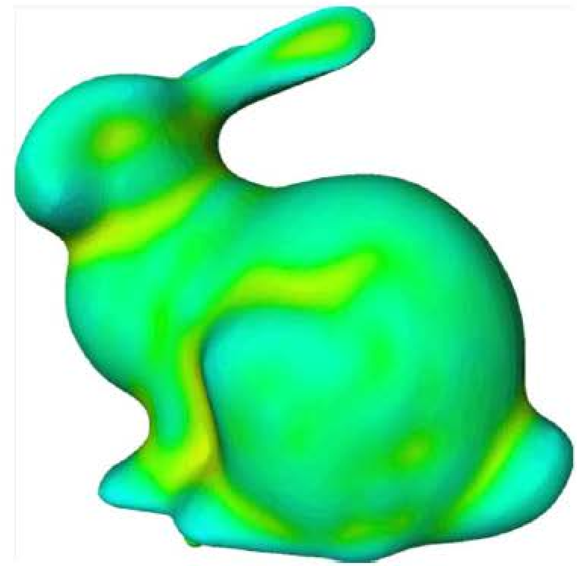
Example



0 Iterations



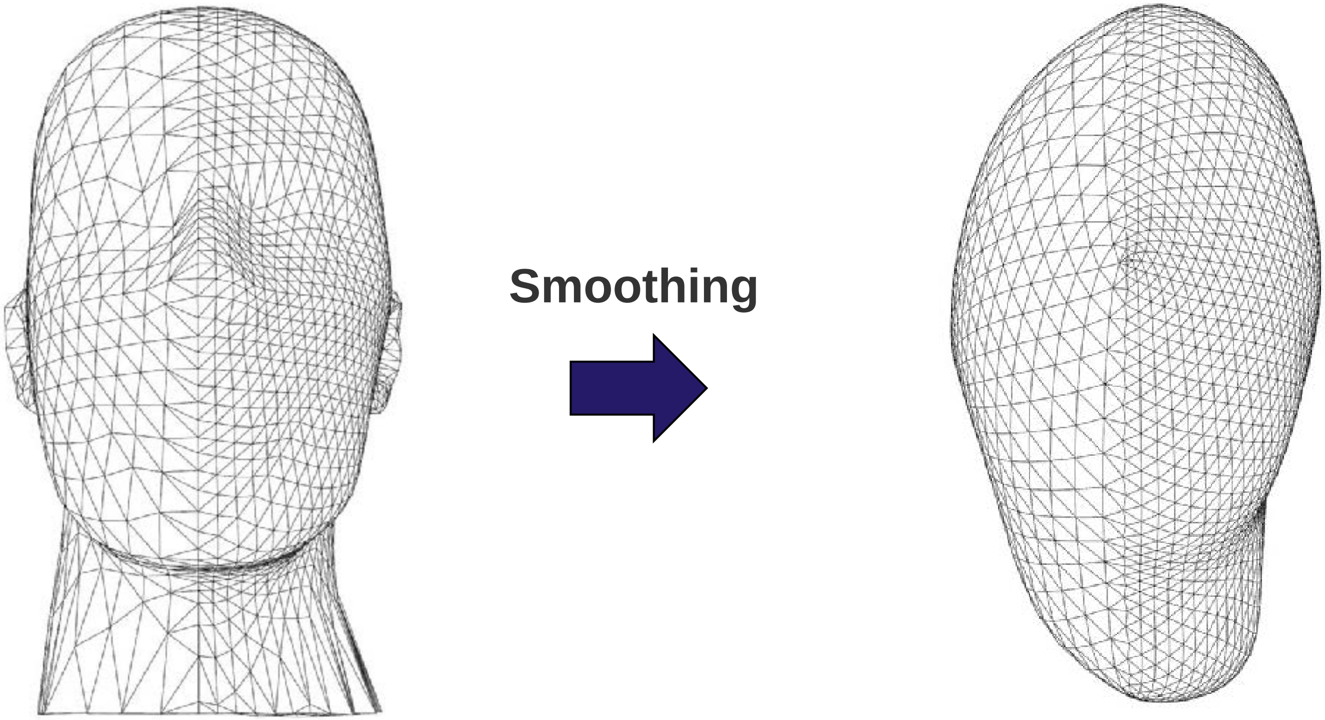
5 Iterations



20 Iterations

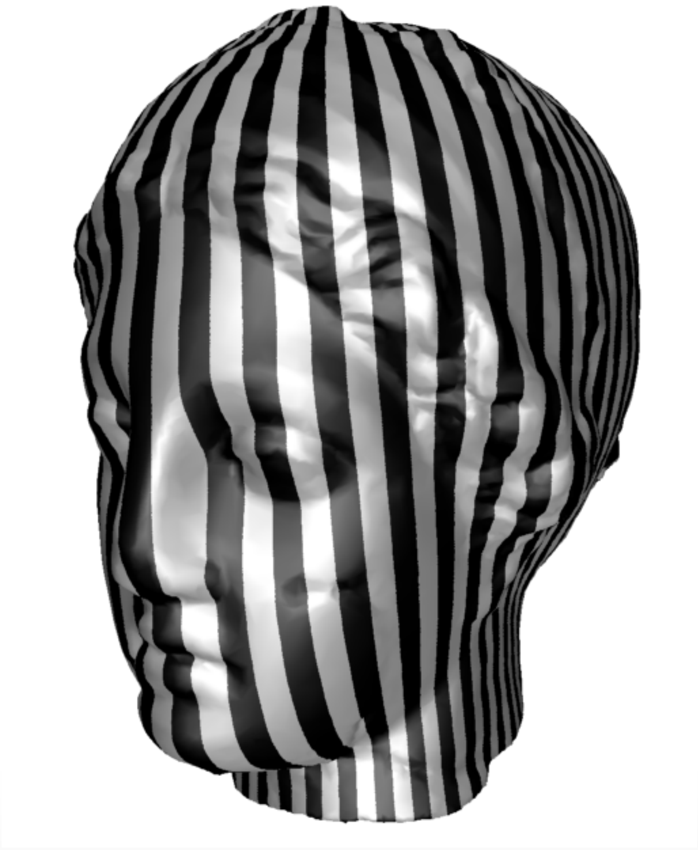
Problems

Irregular meshes

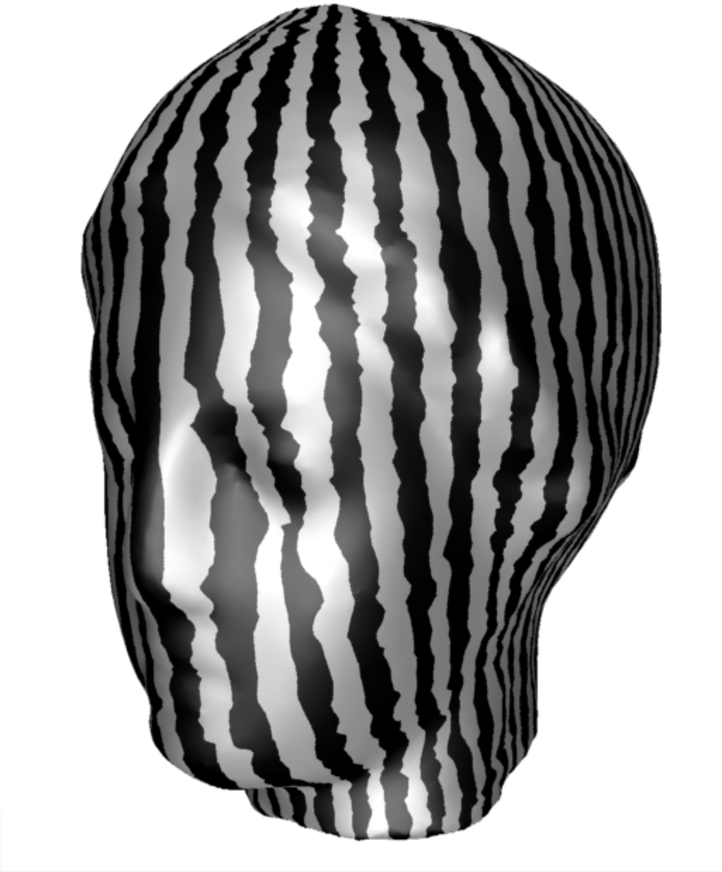


Problems

Tangential Drift



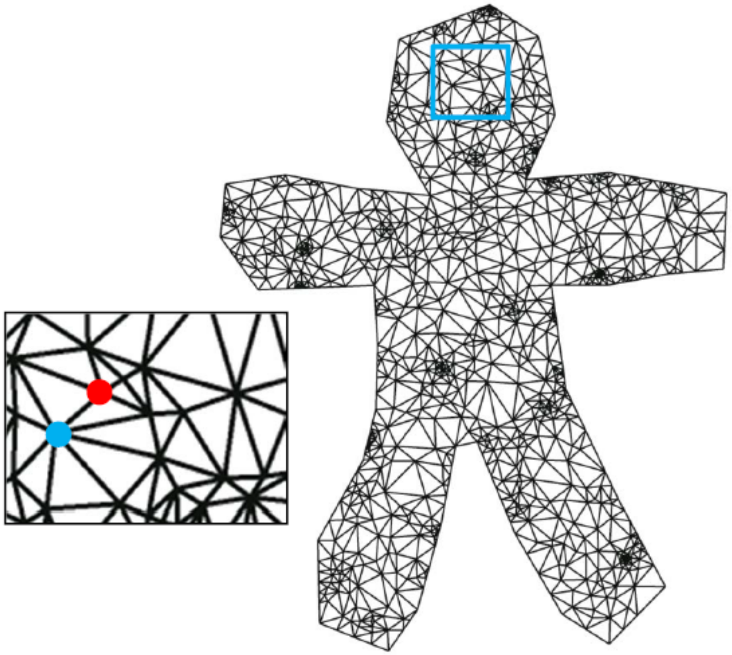
Smoothing



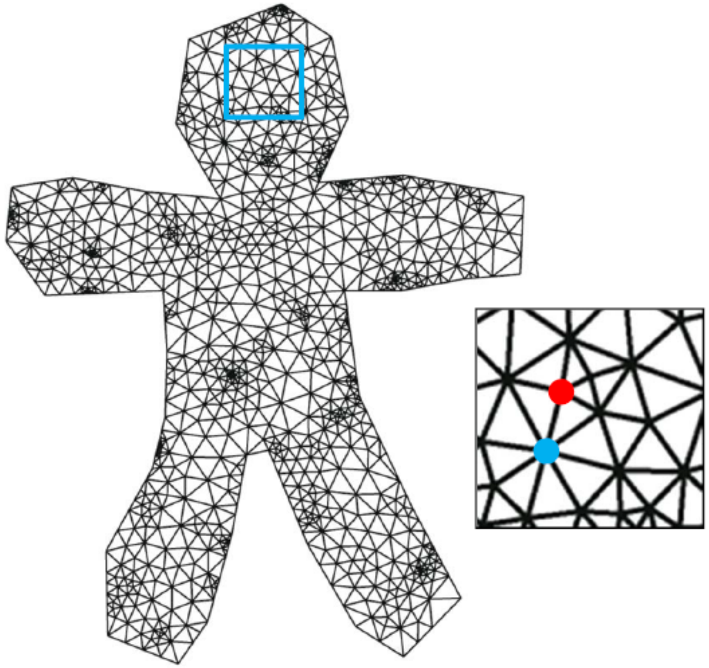
Problems

Tangential Drift

- What happens when smoothing a planar mesh?

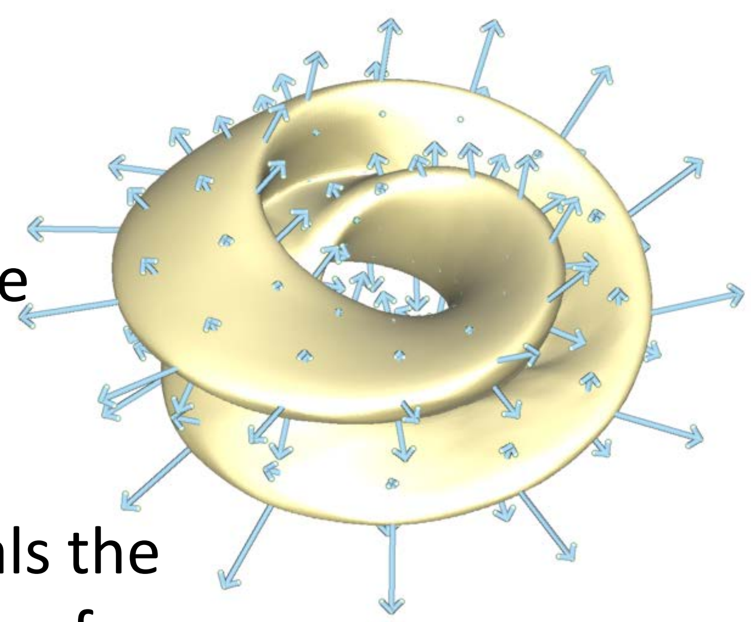


0 Iterations



5 Iterations

Mean Curvature Vector



Mean curvature vector field

- Normal field
- Length equals the mean curvature

Connection to Laplacian

- Mean curvature vector field equals the Laplacian of the embedding of a surface

$$\vec{H} = \Delta x$$

On a mesh

- Discrete mean curvature vector is $\vec{H}_h \in {S_h}^3$

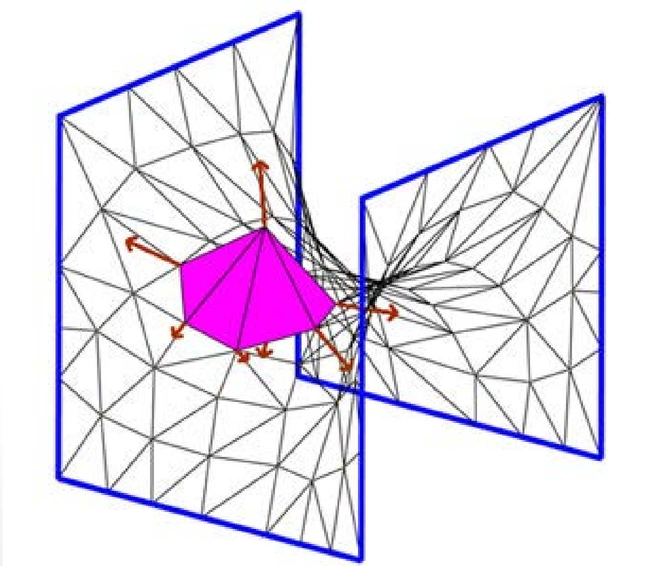
$$\vec{H}_h = Lx$$

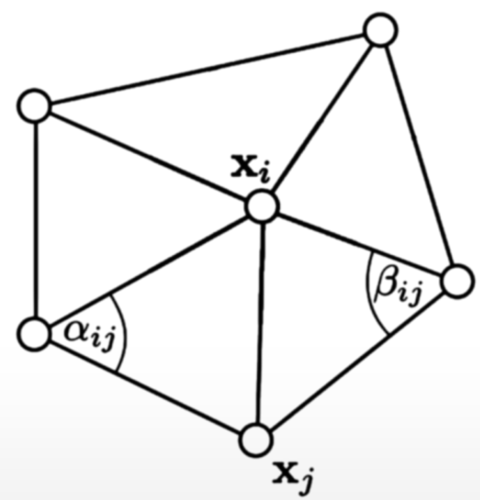
Discrete Mean Curvature Vector

Discrete Mean Curvature Vector

$$\vec{H}_h(x_i) = \frac{3}{2\mathrm{area}\big(\mathrm{star}(x_i)\big)} \sum_{x_j \in link(x_i)} \big(\cot\alpha_{ij} + \cot\beta_{ij}\big)\,(x_i - x_j)$$

Remark: $\mathrm{d\,area}(\mathrm{x})(\mathrm{v}) = \langle \vec{H}_h, v \rangle_M = x^T S v$





Mean Curvature Flow

Mean curvature flow

- Surface flows
- Velocity of every vertex equals the negative of the mean curvature vector

$$\frac{d}{dt}x(t) = -\vec{H}(t)$$

Geometric diffusion

- Mean curvature flow equals a non-linear diffusion
- Laplace operator changes during the evolution

$$\frac{d}{dt}x(t) = -\Delta x(t)$$

Discretization

On a mesh

- Time continuous:

$$\frac{d}{dt}x(t) = -Lx(t)$$

- Matrix L changes during the evolution

Time discretization

- Simplest scheme

$$\frac{d}{dt}x(i\tau) \approx \frac{x^{i+1} - x^{i}}{\tau}$$

Explicit Euler

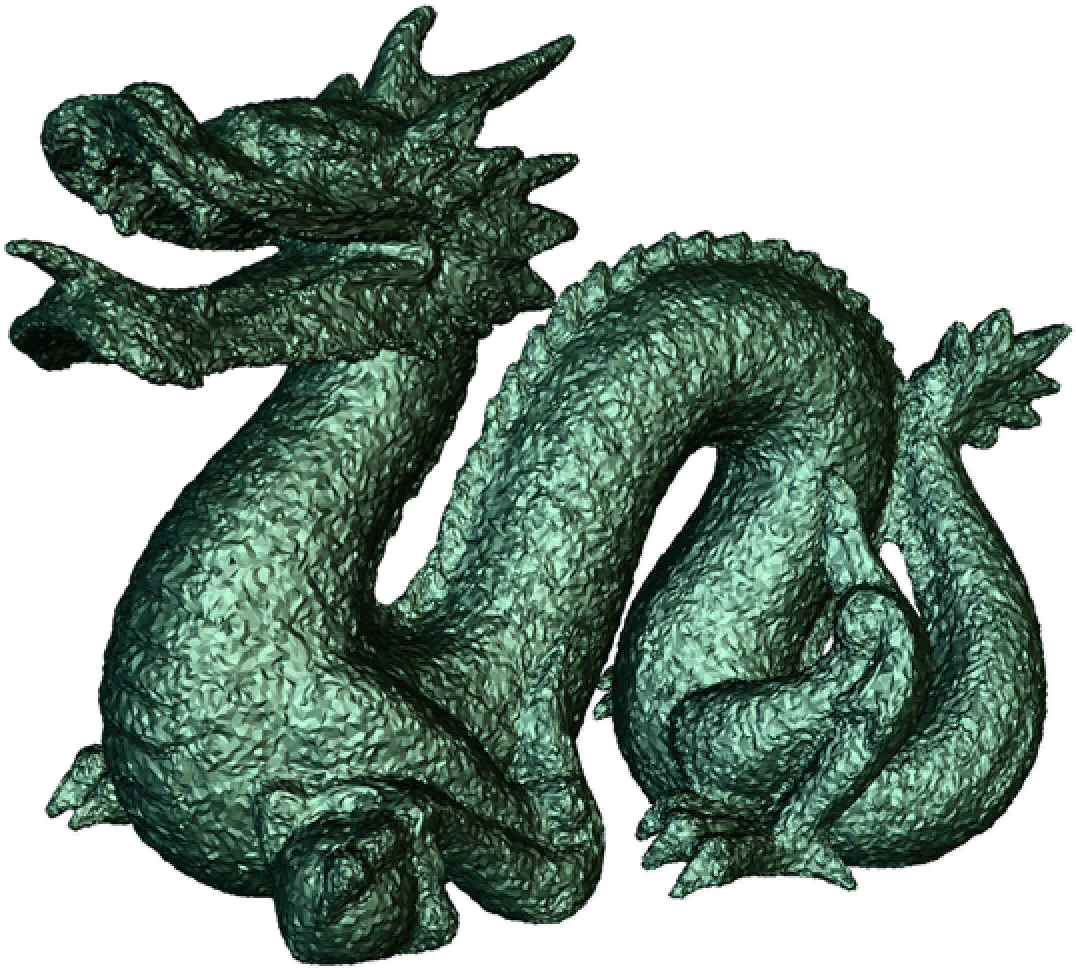
Explicit Euler

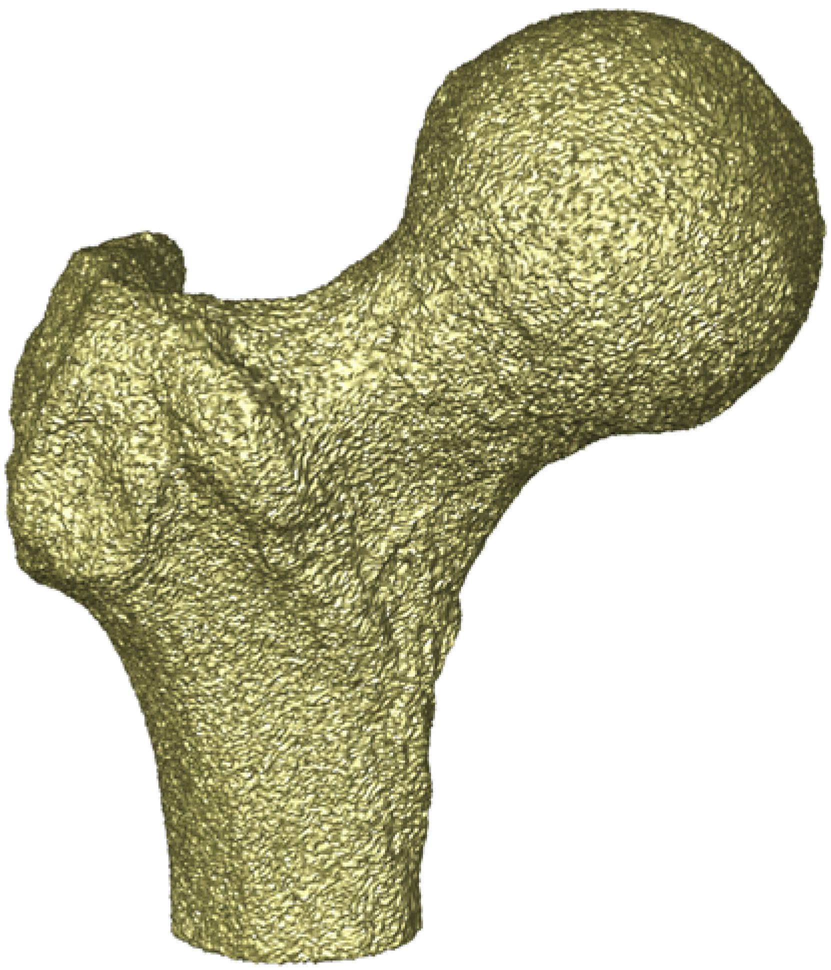
$$\frac{x^{i+1} - x^i}{\tau} = -Lx^i$$

Algorithm:

Iterate:

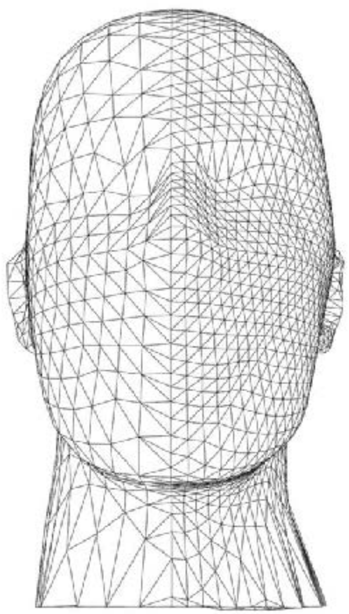
1. Set up the Laplace matrix L of the current embedding x
2. Compute $-Lx$
3. Set $x \leftarrow x - \tau Lx$

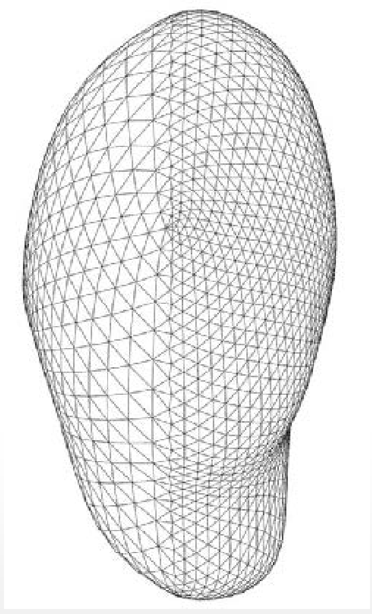


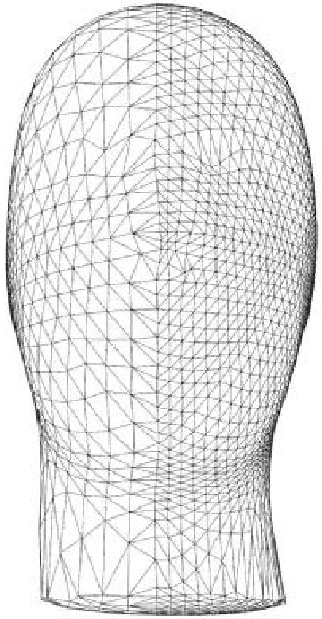


Irregular Meshes

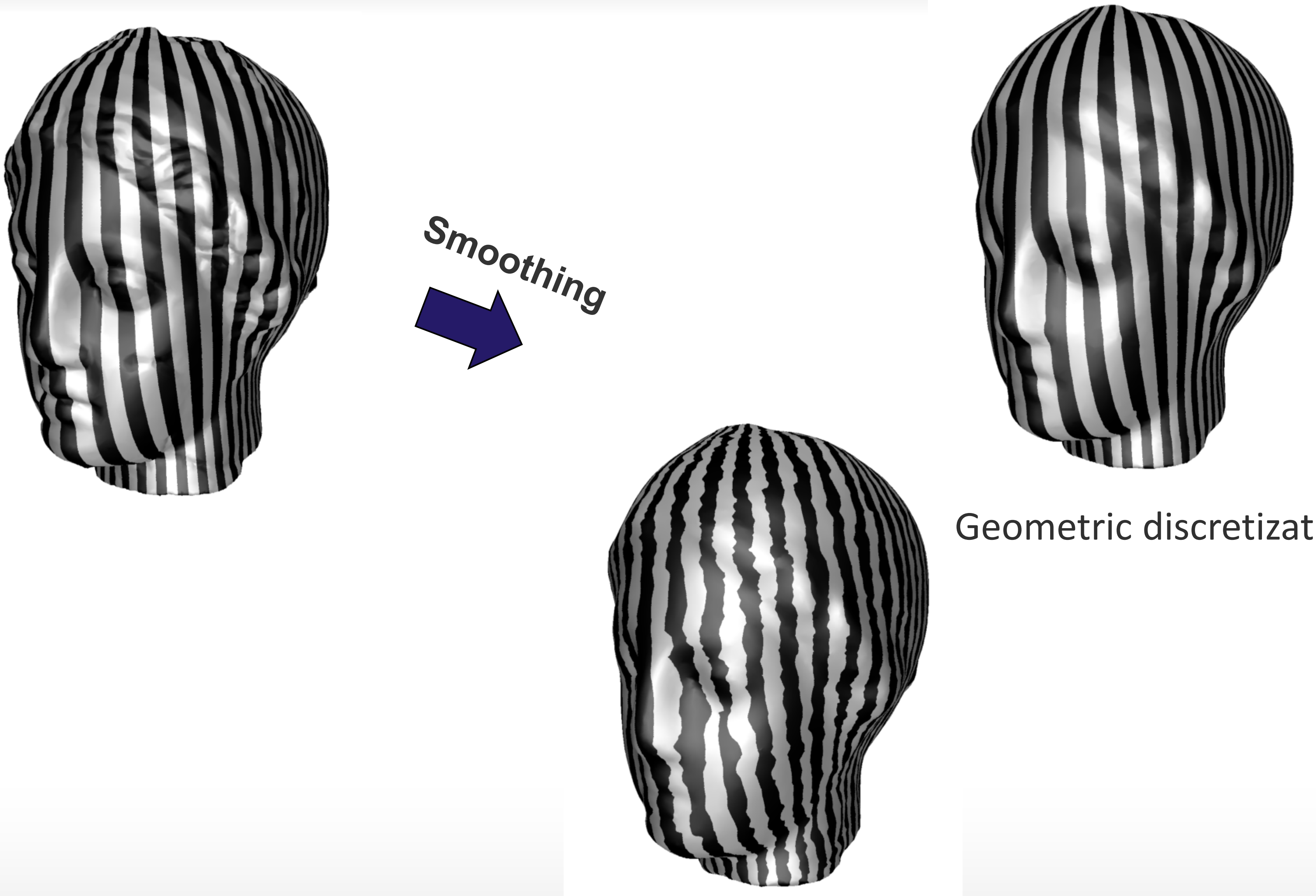
Can process irregular meshes







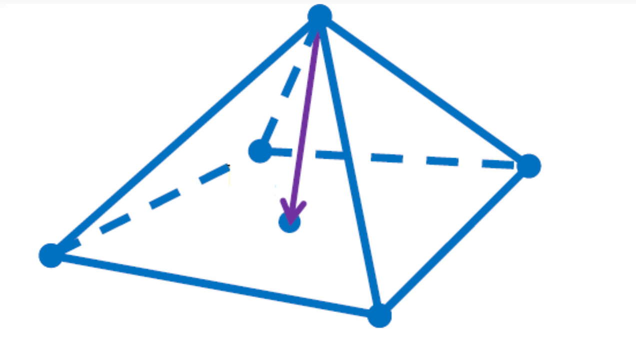
No Tangential Drift



Geometric discretization

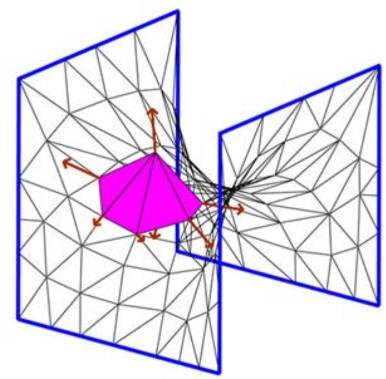
Tangential motion

Relation to Iterated Averaging



Iterated Averaging

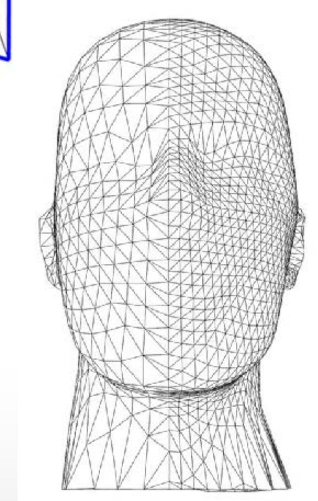
$$x_k \leftarrow x_k + \tau\left(\frac{1}{|N_k|}\sum_{l\in N_k} x_l - x_k\right)$$



Explicit MCF

$$x \quad \leftarrow x \quad - \tau L x$$

- Reminder $L = M^{-1}S$
- Cotangent weights
- M^{-1} to get mesh independence



$$\vec{H}_h(x_i) = \frac{3}{2\mathrm{area}(\mathrm{star}(x_i))}\sum_{x_j\in link(x_i)} (\cot\alpha_{ij} + \cot\beta_{ij})\,(x_i - x_j)$$

Implicit Euler

Limitation of explicit scheme:

- Stable only for small time steps

Semi-Implicit Euler

$$\frac{x^{i+1} - x^{i}}{\tau} = -L^{i} x^{i+1}$$

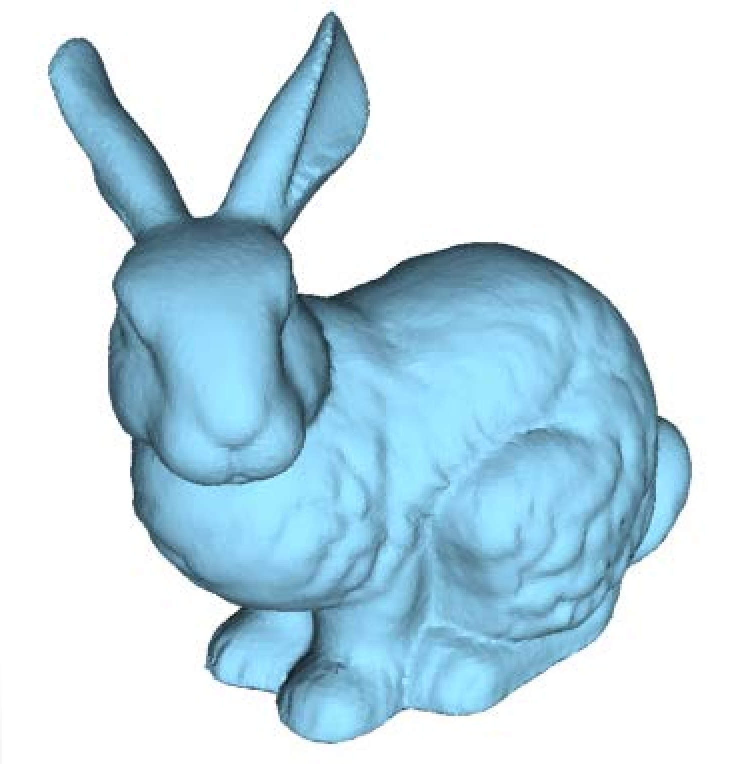
Algorithm:

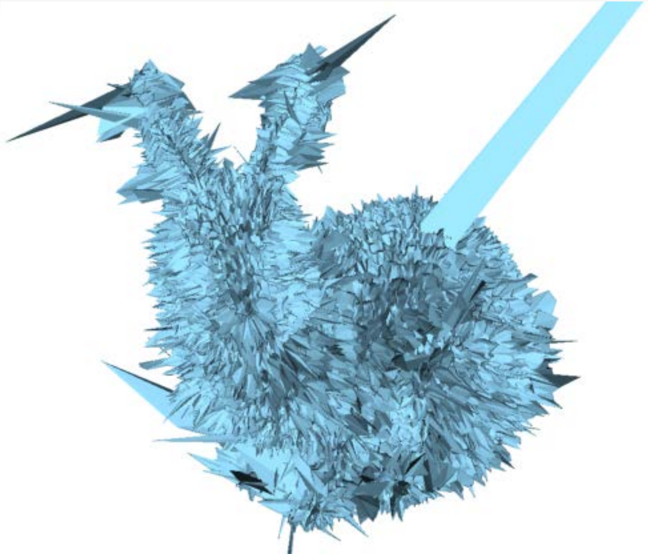
Iterate:

1. Set up the matrices M, S of the current embedding x
2. Solve linear System: $(M + \tau S)x^{i+1} = Mx^{i}$

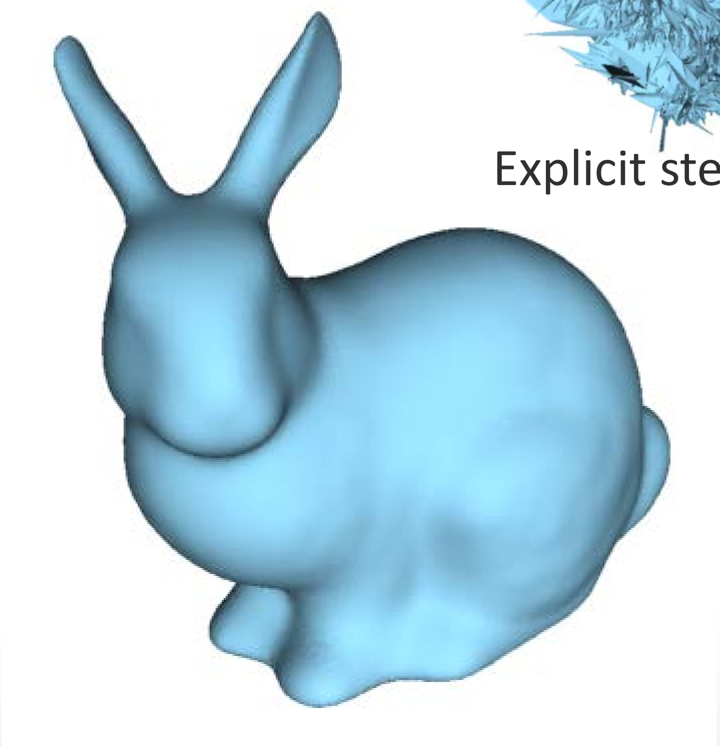
Implicit Scheme

Large time steps are possible





Explicit step of same size



One implicit step

Shrinkage

Problem: Shrinkage and loss of features.

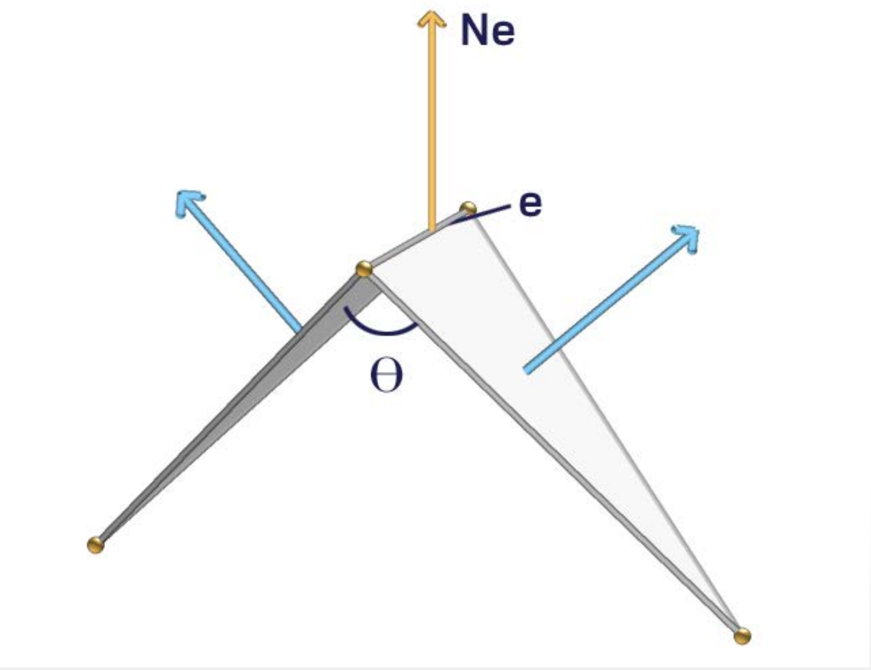
Anisotropic Smoothing

Anisotropic Diffusion:

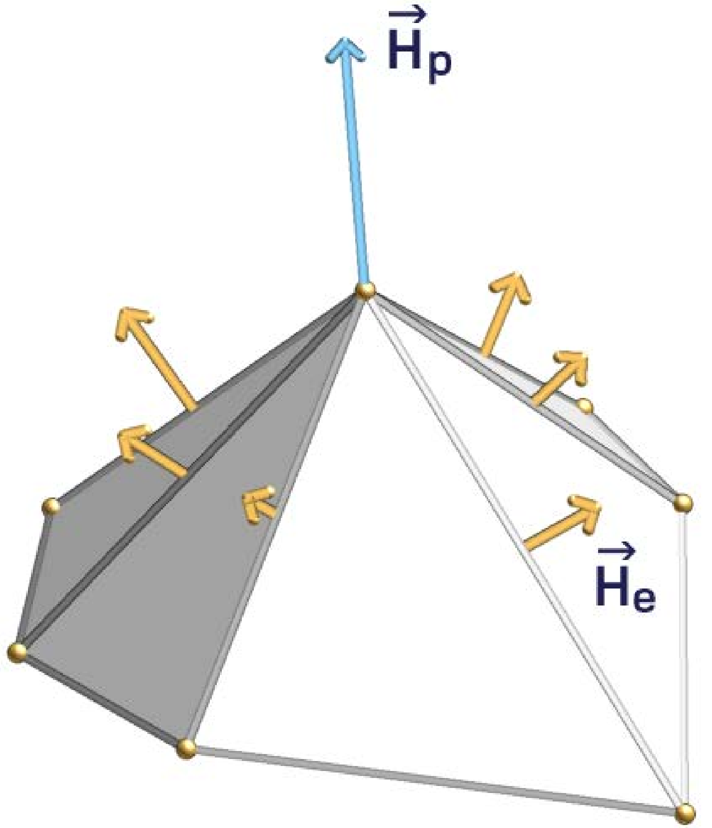
w.r.t. basis of principal curvature directions.

Edge-Based Mean Curvature Vector

→



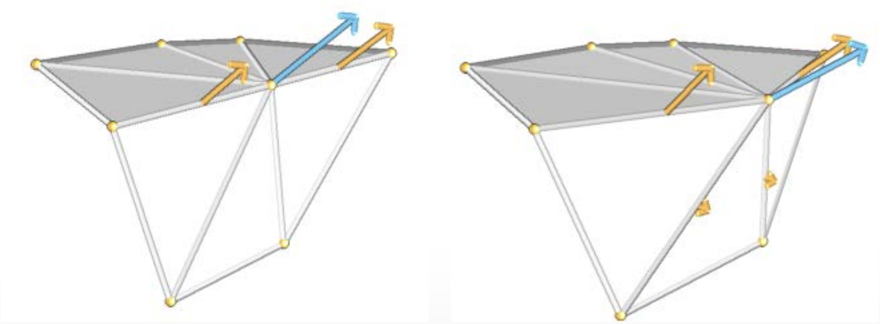
Discrete Mean Curvature Vectors



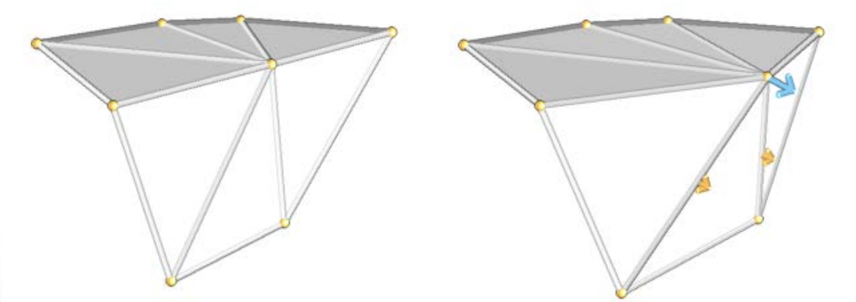
Anisotropic MC Vector

The discrete anisotropic mean curvature vector at an edge is

At a vertex:

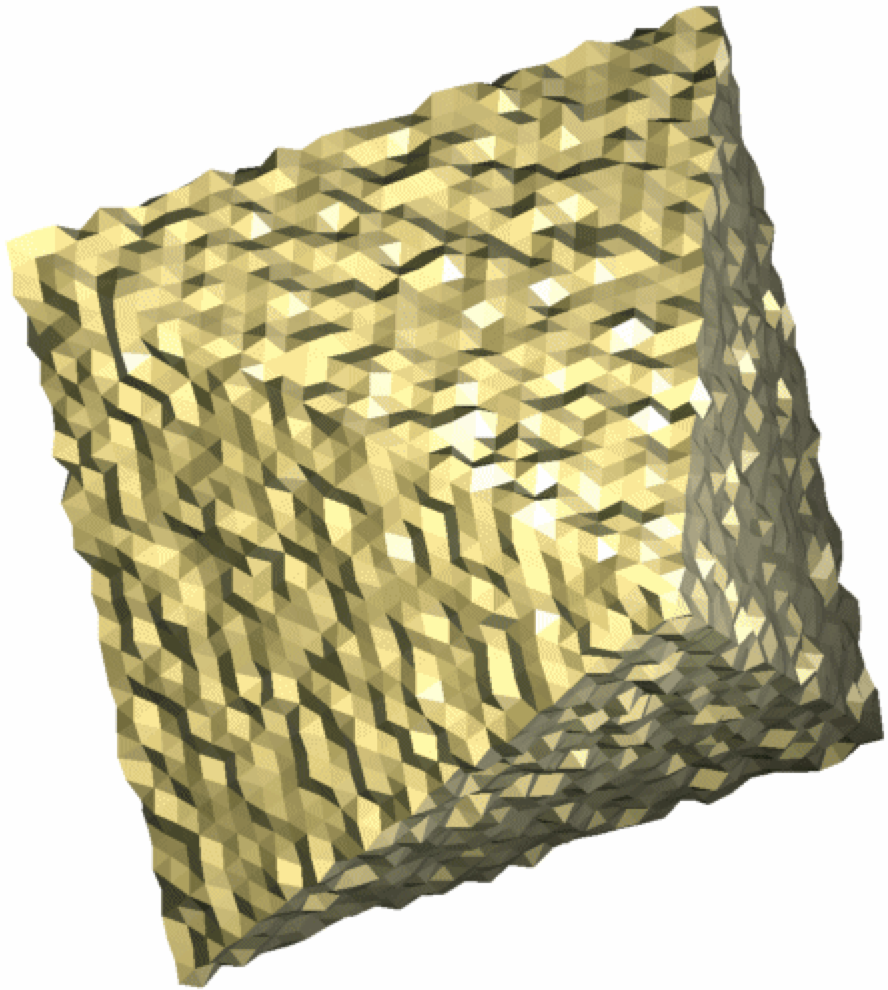


Mean curvature vector

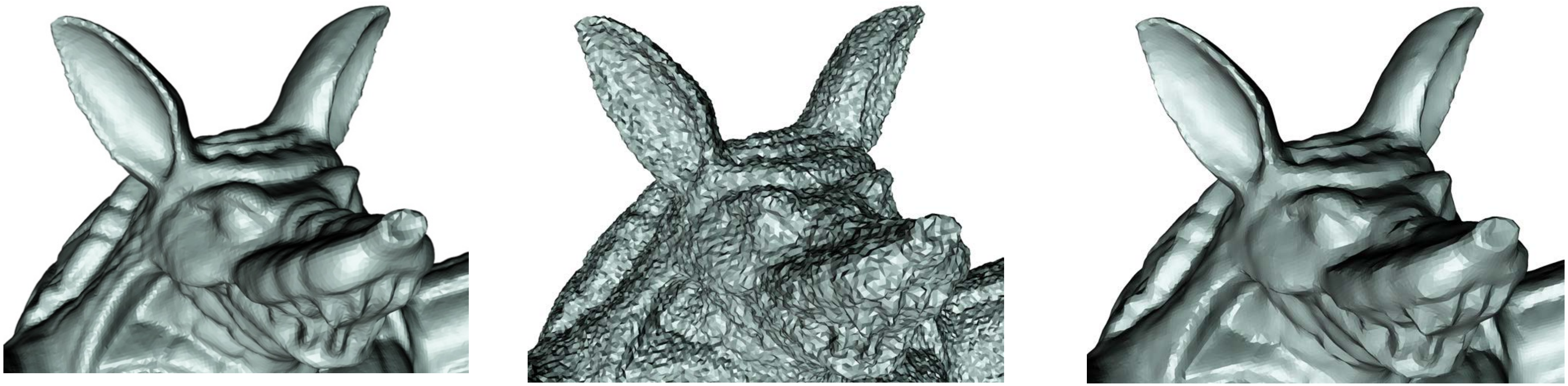


Anisotropic mc vector

Example



Example



Constrained Smoothing

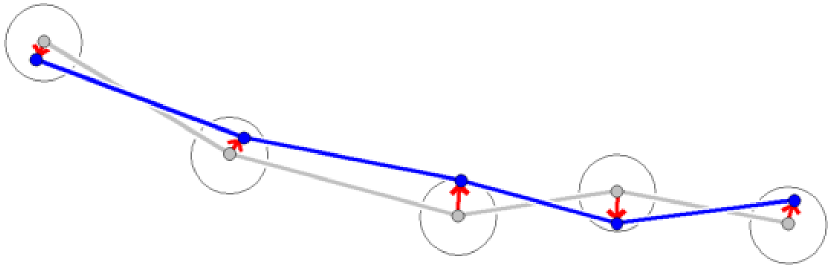
Fairness Energy

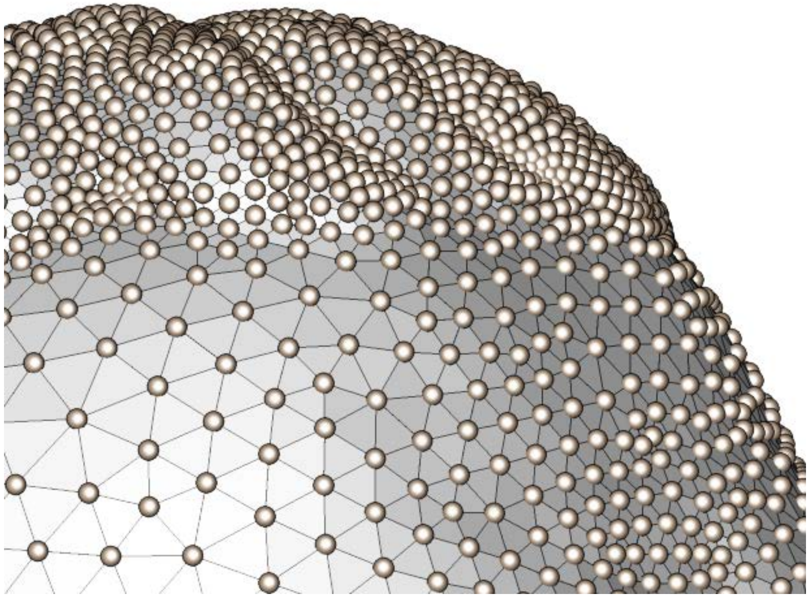
$$E(\mathrm{x}) = \frac{1}{2}\int_M \|\Delta \mathrm{x}\|^2 dA$$

- Matrix representation

$$E(\mathrm{x}) = \frac{1}{2} x^T S M^{-1} S x$$

Constrained Smoothing





Minimizes E over the feasible set

Results

Chinese Lion

1.3m triangles

Size ~ 100mm

Max. deviation: 0.1mm