# Geometric Modeling

2015

Laplace-Beltrami Operator

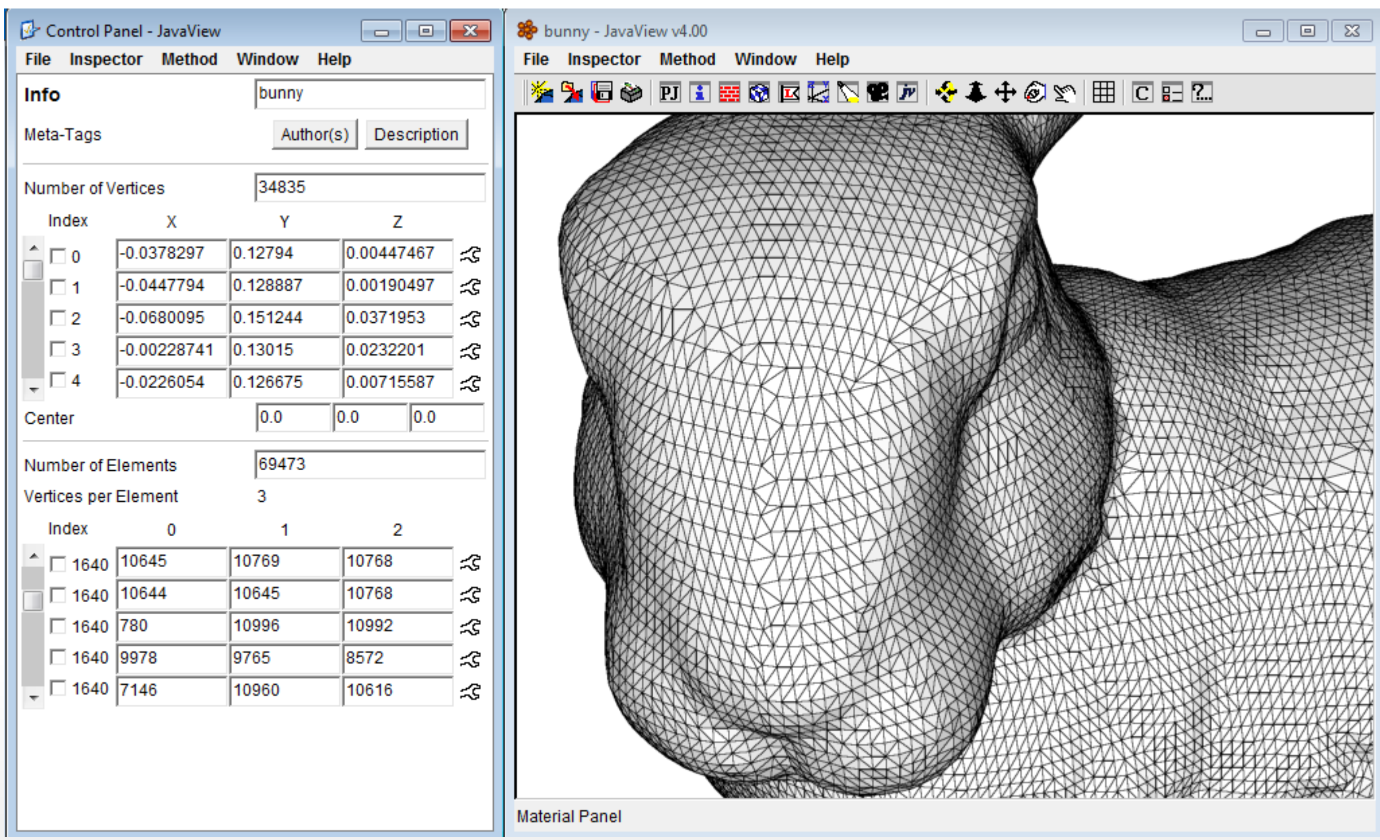
TU Delft

# Last Lecture

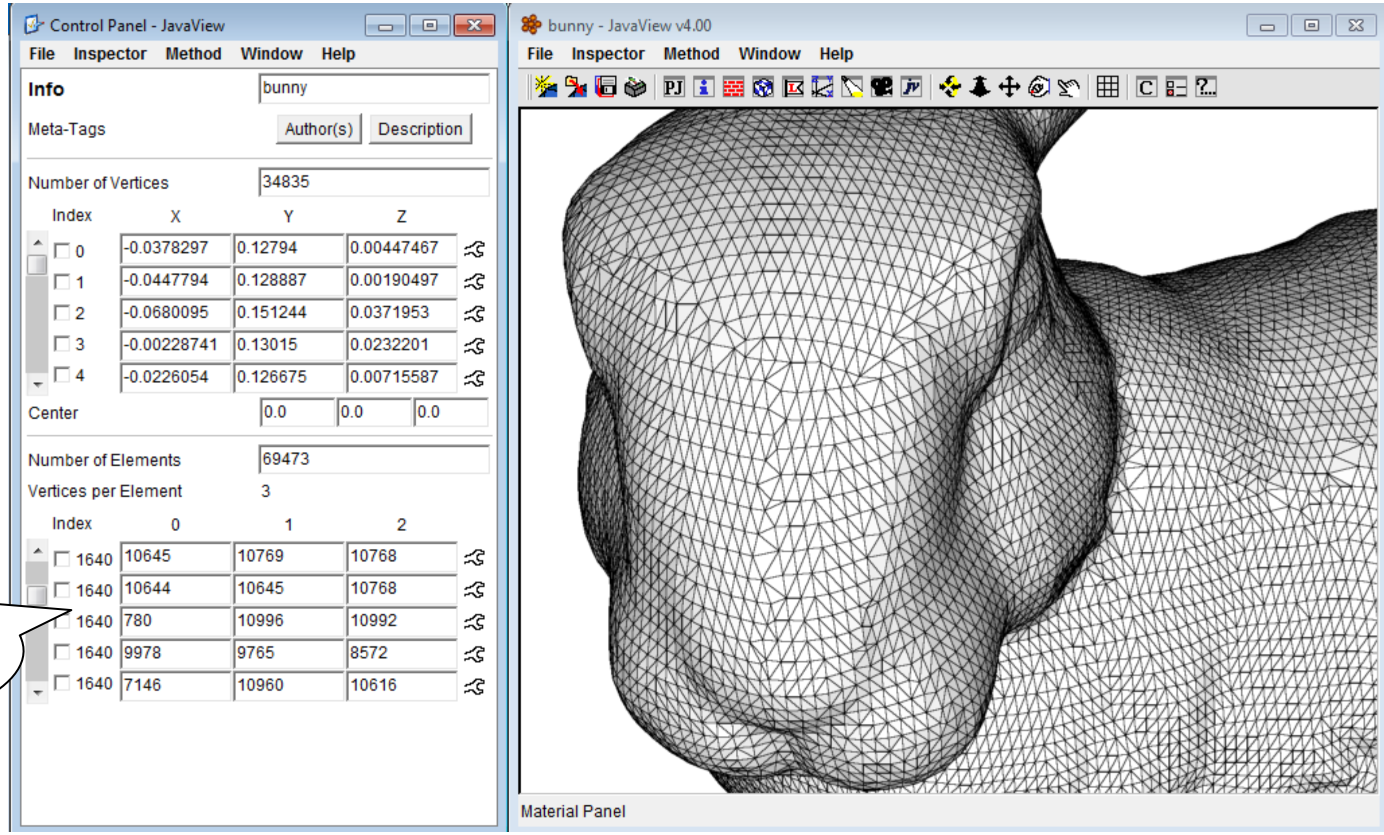
# Triangle Meshes

## Representation of a triangle mesh in $\mathbb{R}^3$

- Vertices: a finite list $\{v_1, \dots, v_n\}$ of points in $\mathbb{R}^3$
- Faces: a list of triples, e.g. $\{\{2,34,7\}, \dots, \{14,7,5\}\}$



# Topology

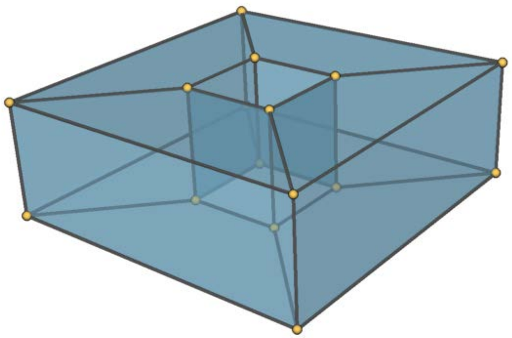



Topology!

The face array contains information about the surface that is independent of the choice of vertex positions

# Topology

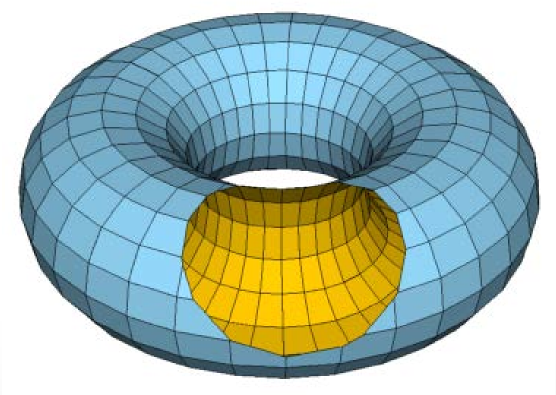
## Genus of a surface and Euler characteristic

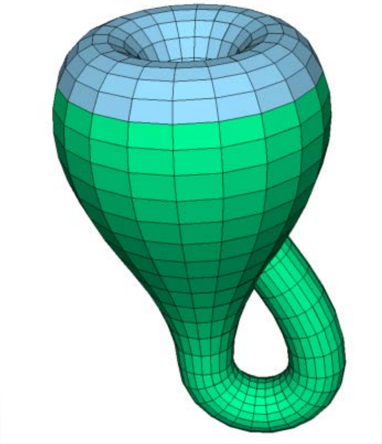


$$V - E + F = 2(1 - g)$$

$$16 - 32 + 16 = 2(1 - 1)$$

## Orientable and non-orientable surfaces



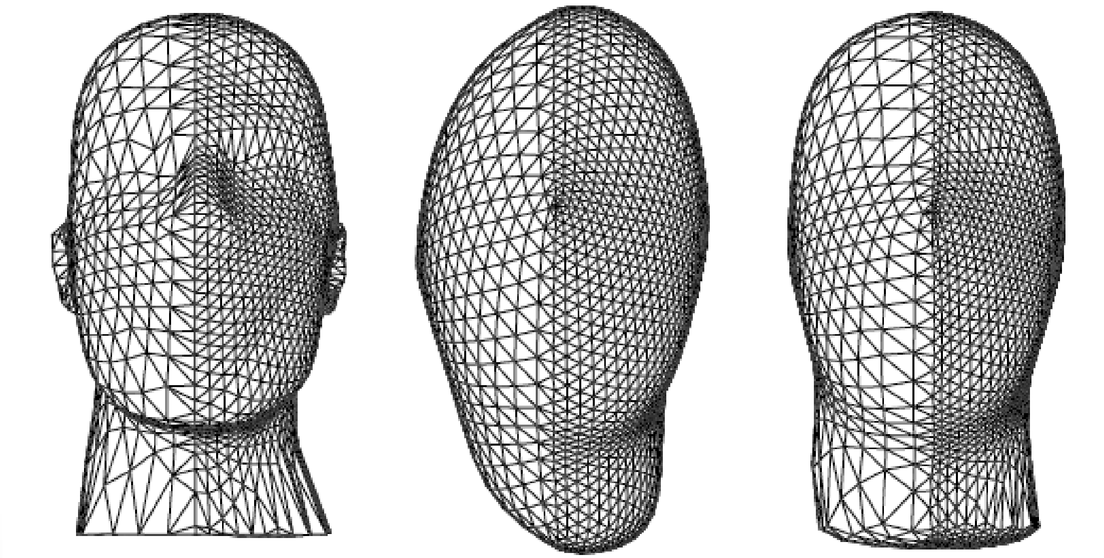


# Functions on Triangles

# Motivation

## Why notions like gradient and Laplace operator?

- Formulate methods mesh independ
  - Method should work for irregular meshes
  - Similar results after simplification, remeshing,...



Symmetric but irregular mesh

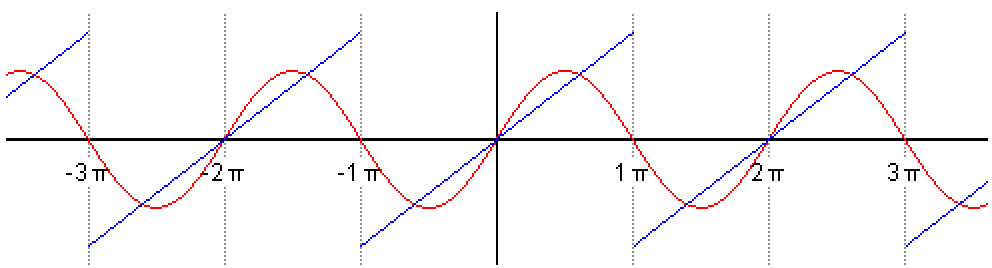
Lost symmetry

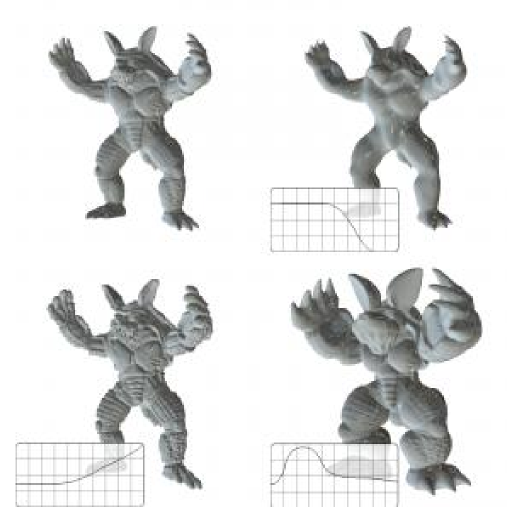
Preserved symmetry

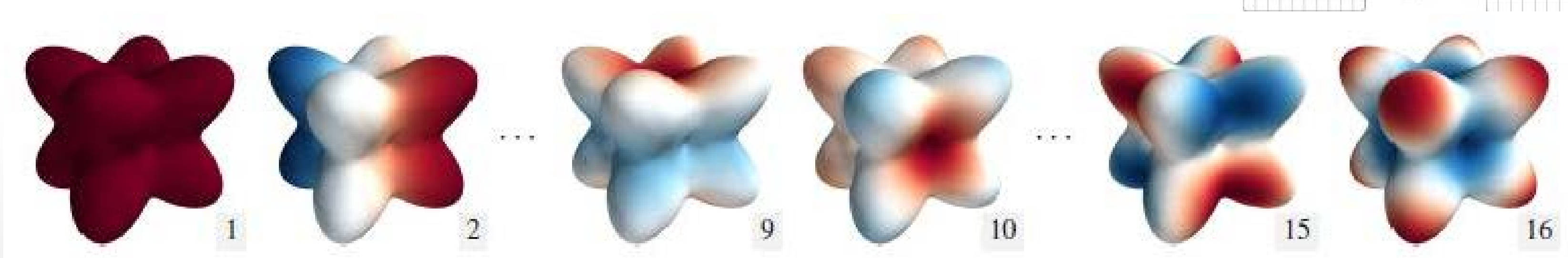
# Motivation

## Why notions like gradient and Laplace operator?

- Transfer concepts and ideas from other areas
  - Signal processing, image processing, physics, geometry
  - Learn language in which concepts are formulated
- Example: Fourier analysis on meshes



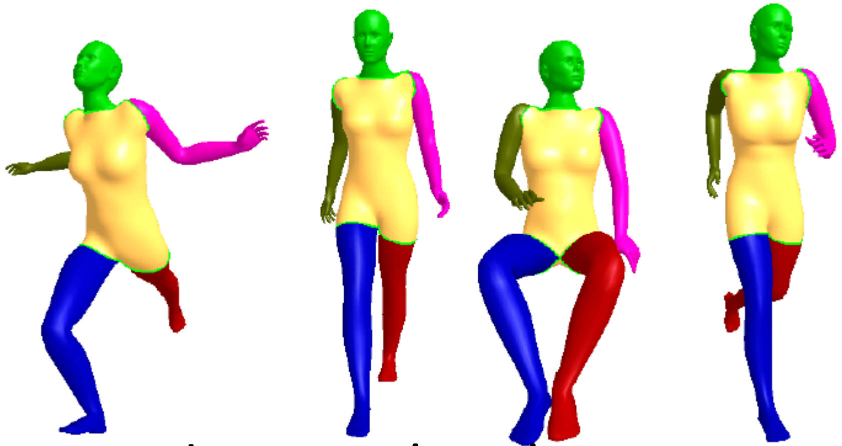





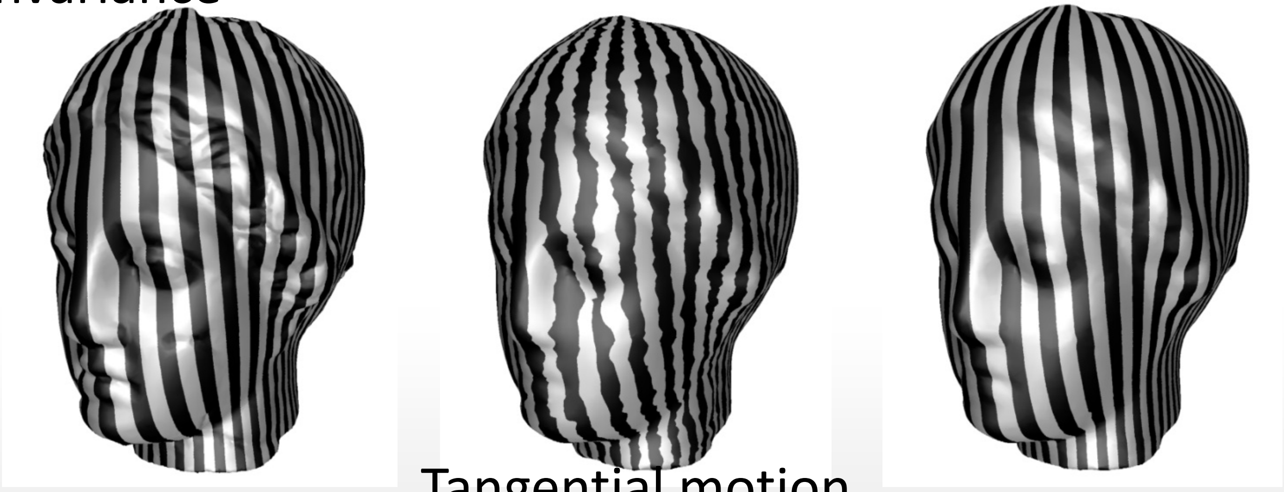
# Motivation

**Why notions like gradient and Laplace operator?**

- Transfer concepts and ideas from other areas



Isometry invariance

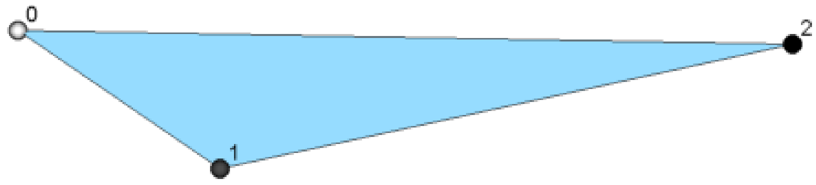


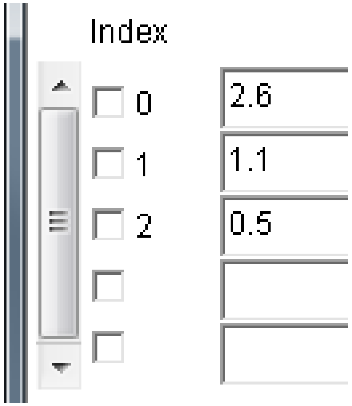
Tangential motion

# Functions on Triangles

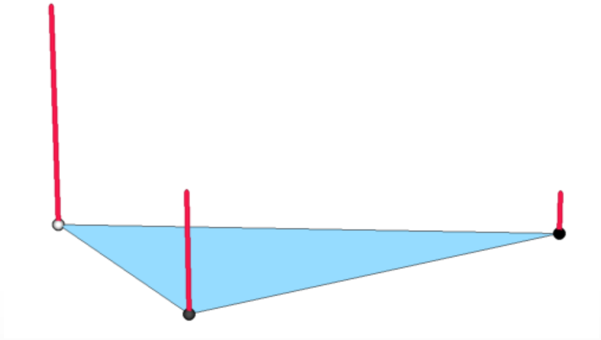
## Discrete Function

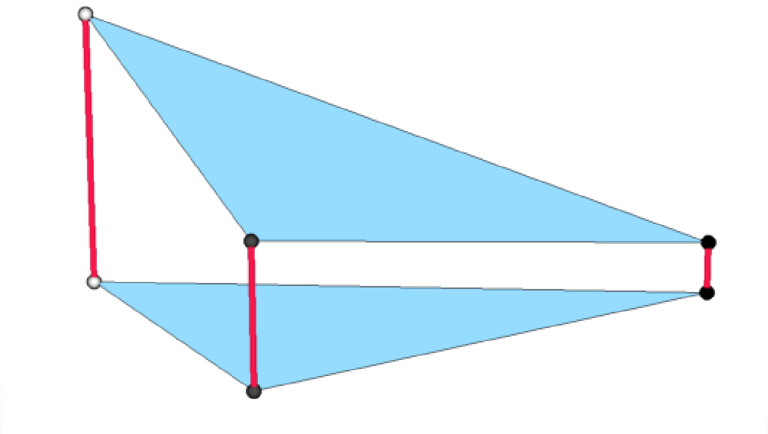
- A function value at every vertex




- This specifies a linear polynomial (linear function + constant) on the triangle





Graph of the function

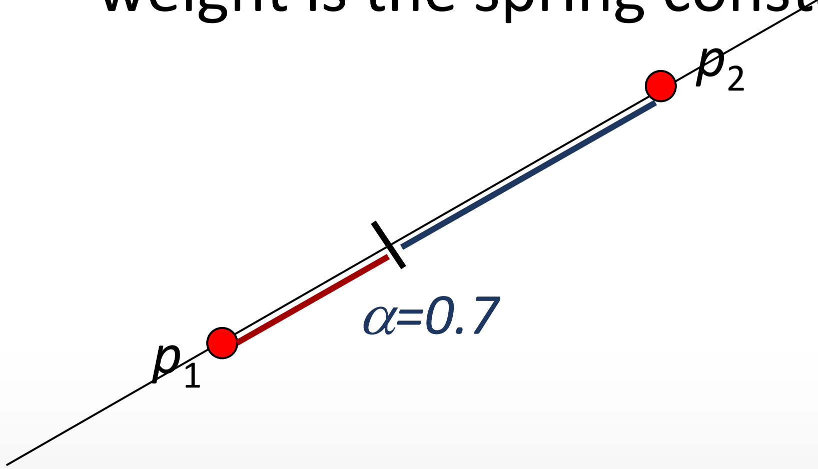
# Barycentric combinations

- A barycentric combination of a set $\{p_1, p_2, \ldots, p_m\} \subset \mathrm{V}$ is a sum

$$\sum_{i=1}^{m} \alpha_i \, p_i$$

with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{m} \alpha_i = 1$.

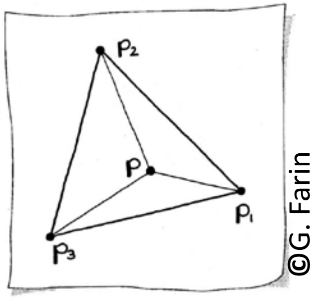
- Example: Think of springs pulling at the point, where the weight is the spring constant




$$\alpha p_1 + (1-\alpha) p_2 = p_2 + \alpha (p_1 - p_2)$$

# Triangles

- Consider a (non-degenerate) triangle $\mathrm{T}$ spanned by $\{p_1, p_2, p_3\}$.
- Set $p = \sum_{i=1}^{3} \alpha_i\, p_i$ and $\sum_{i=1}^{3} \alpha_i = 1$.



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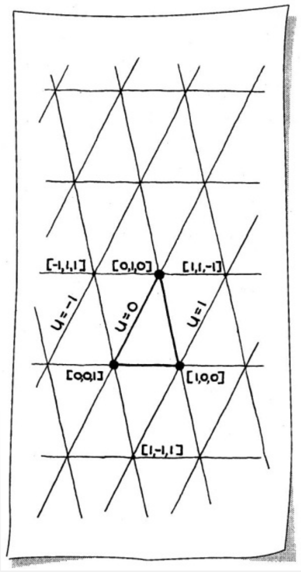
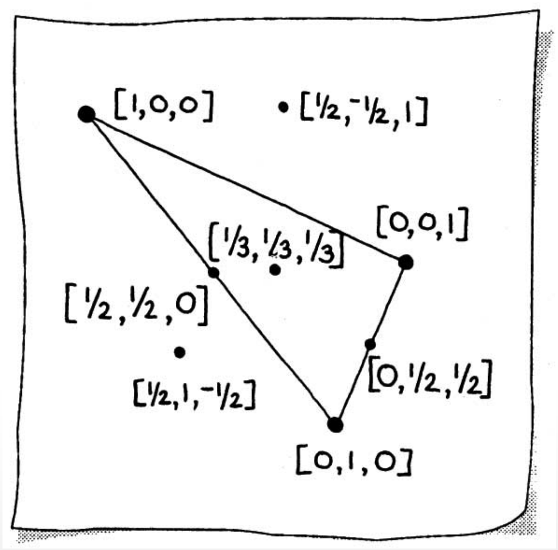
## Properties

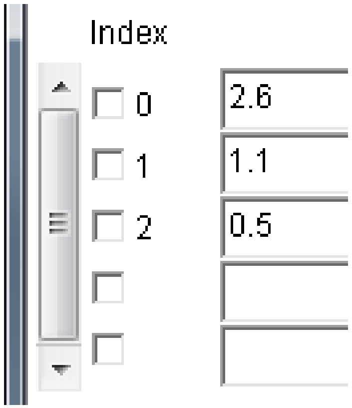
- The barycenter is the point $p$ with $\alpha_1 = \alpha_2 = \alpha_3$.
- $p \in \mathrm{T}$ if and only if all $\alpha_i \geq 0$.



# Evaluate Linear Polynomial

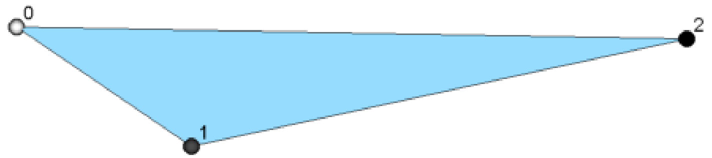
## Use barycentric coordinates




- Point $p$ in the triangle $(p_1, p_2, p_3)$ :

$$p = \sum_{i=1}^{3} \propto_i \; p_i$$

- Function values $u_1, u_2, u_3$

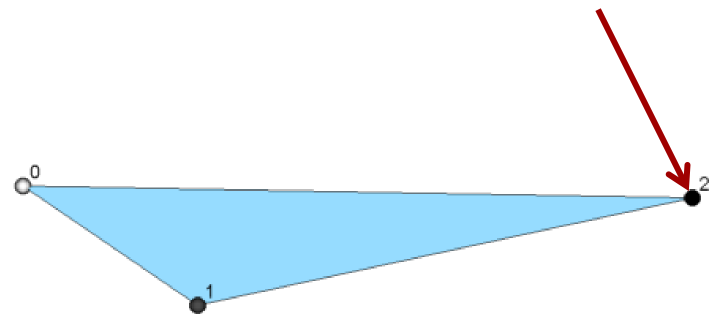


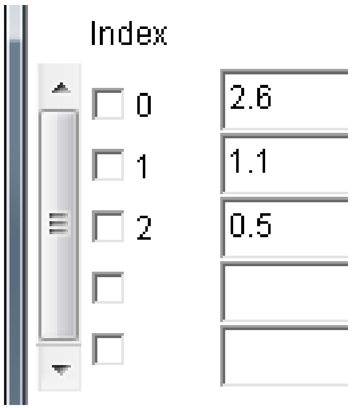

- Linear polynomial $u$ at $p$ has the function value

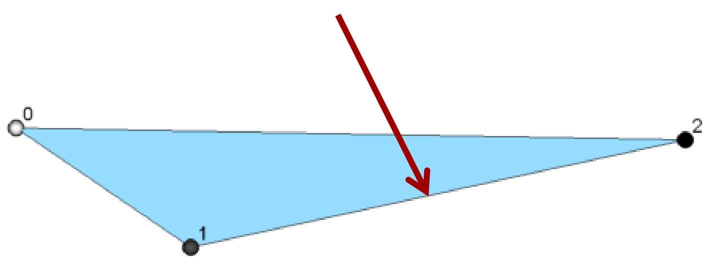
$$u(p) = \sum_{i=1}^{3} \propto_i \; u_i$$

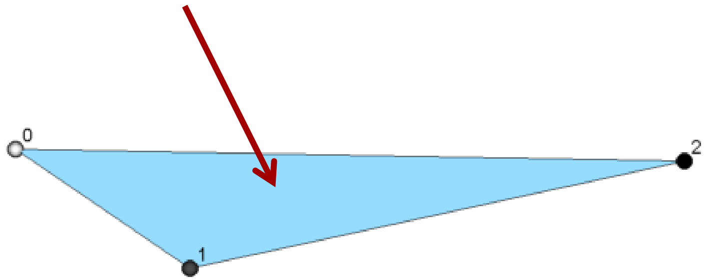
# Evaluate Linear Polynomial

## What is the function value?



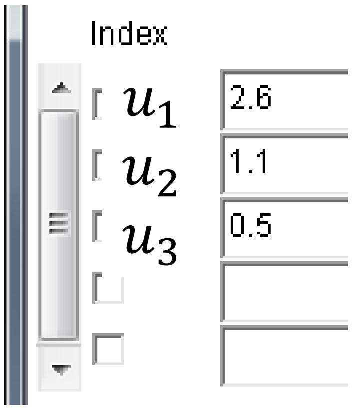



# Vector Space of Linear Polynomial

## Vector space

- The linear polynomials over a triangle form a vector space
  - Add functions by adding function values at vertices

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$



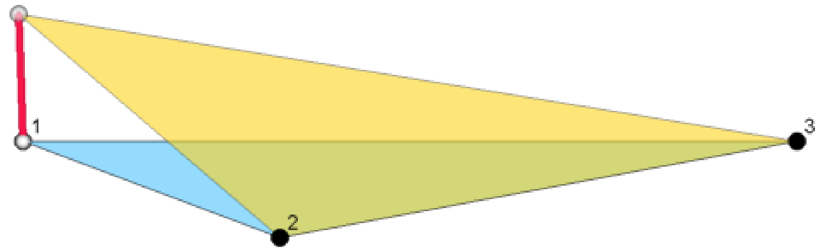

  - Multiply with scalar $a \in \mathbb{R}$

$$a \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} a\,u_1 \\ a\,u_2 \\ a\,u_3 \end{pmatrix}$$

# Lagrange Basis Functions

## Lagrange basis functions

- The linear polynomials $\varphi_i$ that take the value 1 at $p_i$ and 0 at all other vertices




$\varphi_1$

- Any linear polynomial $u$ can be written as

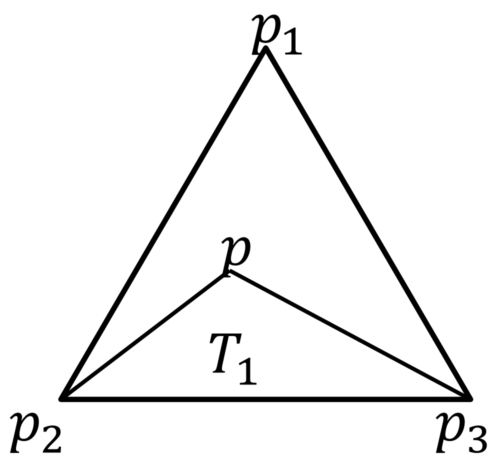
$$u(p) = \sum_{i=1}^{3} u_i \, \varphi_i(p)$$

where $u_i$ are the function values of $u$ at the vertices

# Lagrange Basis Functions

**A geometric formula**

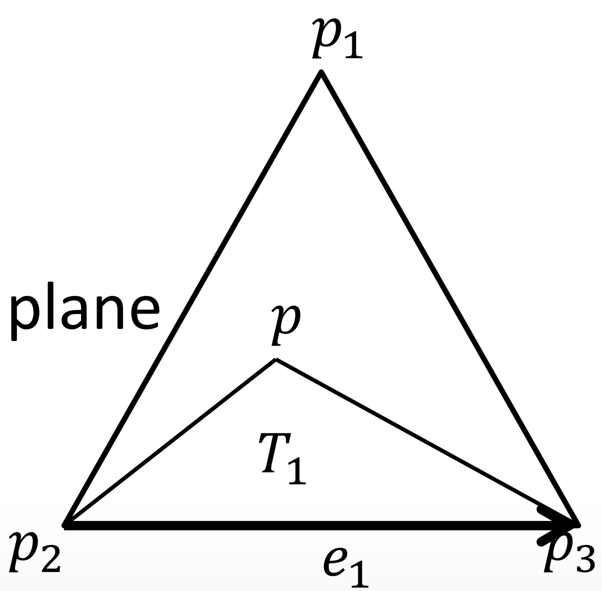
$$\varphi_i(p) = \frac{\text{area}(T_i(p))}{\text{area }(T)}$$




- The area of $T_i$ is given by

$$\text{area}(T_1(p)) = \frac{1}{2}\langle p - p_2, R^{90°} e_1 \rangle$$

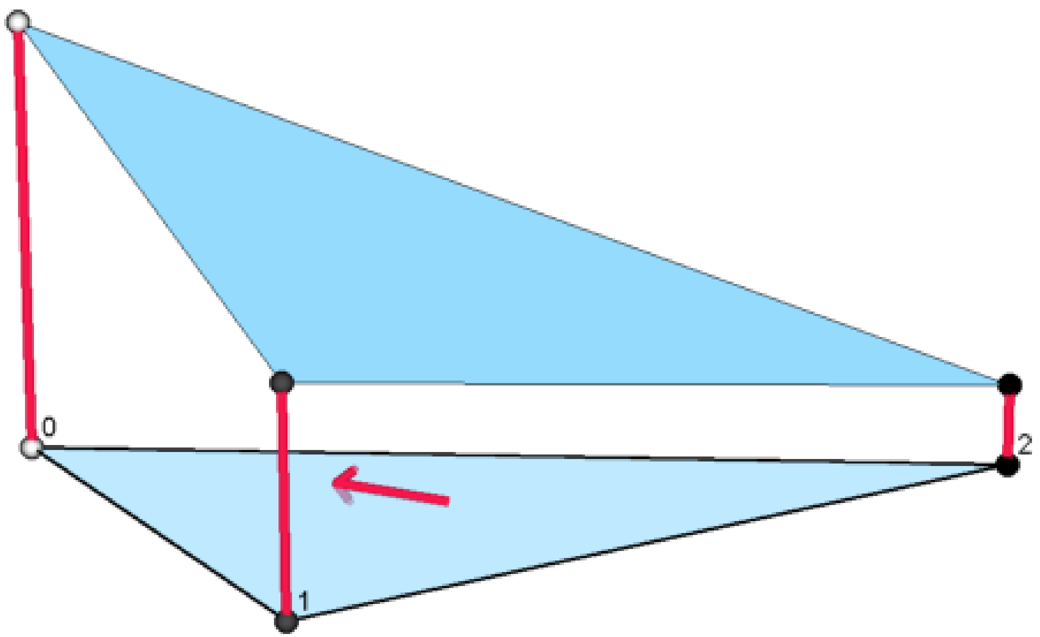
where $R^{90°}$ is the 90° rotation in the plane of the triangle that rotates the edge into the triangle

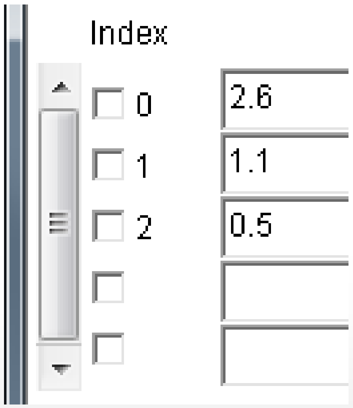


# Gradients of Linear Polynomials

## Gradient of a linear polynomial

- Vector in the plane of the triangle that points into the direction of steepest ascent
- The same vector at all points of the triangle



# Gradients of Linear Polynomials

## We only need the gradients of the basis functions

- What is the gradient of a linear polynomial over a triangle?

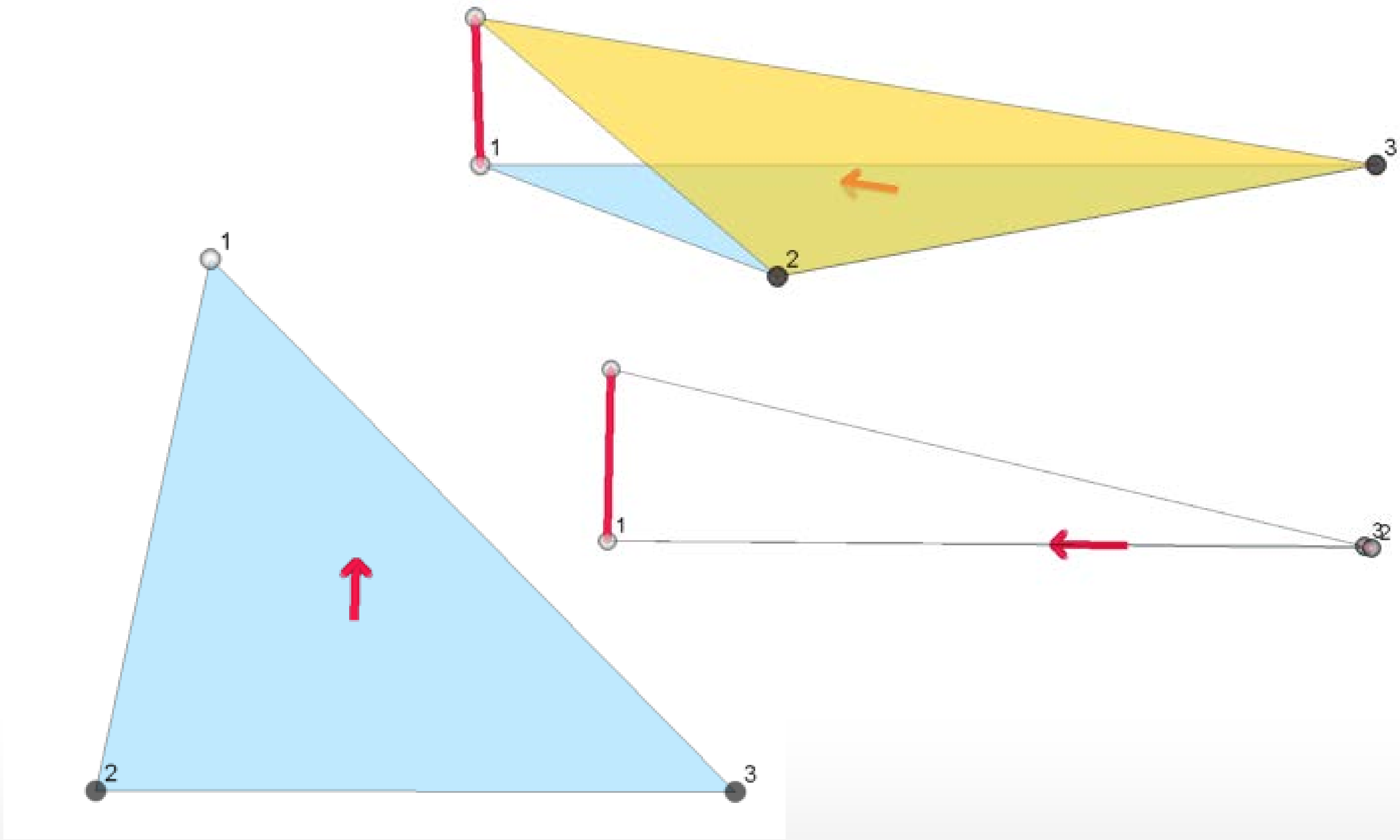
$$u(p) = \sum_{i=1}^{3} u_i \, \varphi_i(p)$$

$$\Longrightarrow \qquad \nabla u(p) = \sum_{i=1}^{3} u_i \, \nabla \varphi_i(p)$$

- We only need the gradients of the basis functions

# Gradients of Linear Polynomials

## Gradient of basis functions

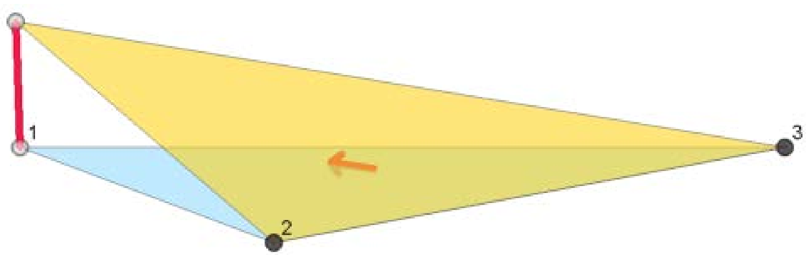


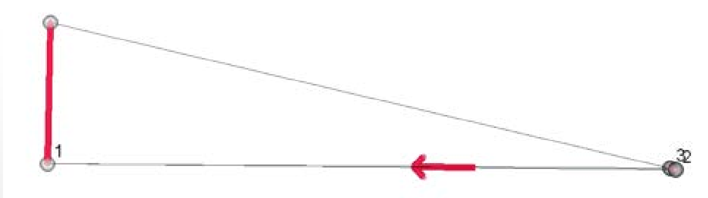
# Gradients of Linear Polynomials

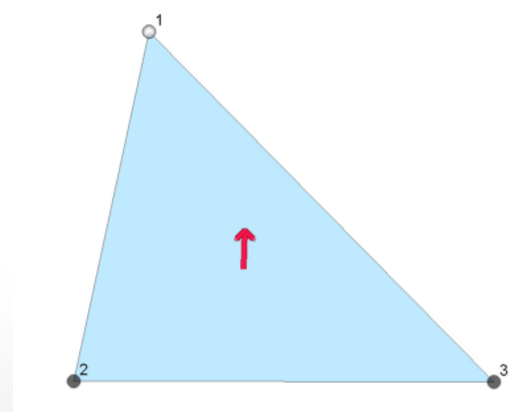
## Gradient of basis functions

Observations:

- Orthogonal to opposite edge
- Length of gradient: 1/height (over opposite edge )



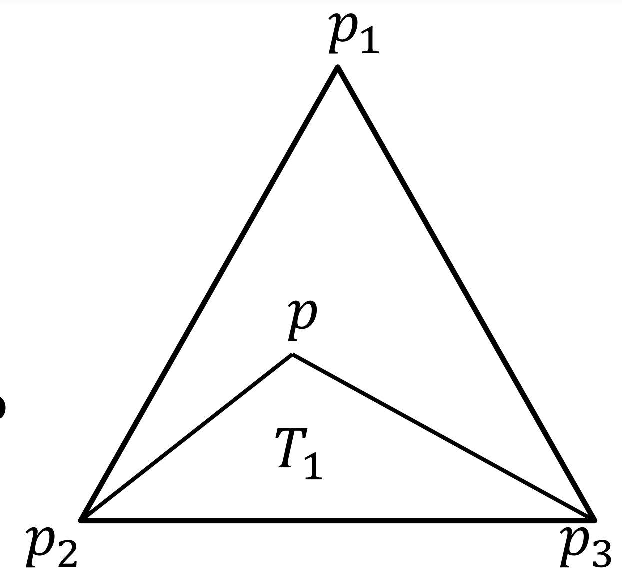



# Gradient

## Gradient of linear functions

- What is the gradient of $\varphi_i$?
- $\varphi_i(p) = \dfrac{\mathrm{area}(T_i(p))}{\mathrm{area}\,(T)}$
- What is the gradient of $\mathrm{area}(T_i(p))$?



# Gradient

**Consider $T_1$**

$$\text{area}(T_1(p)) = \frac{1}{2}\langle p - p_2, R^{90°} e_1 \rangle$$

- The derivative satisfies

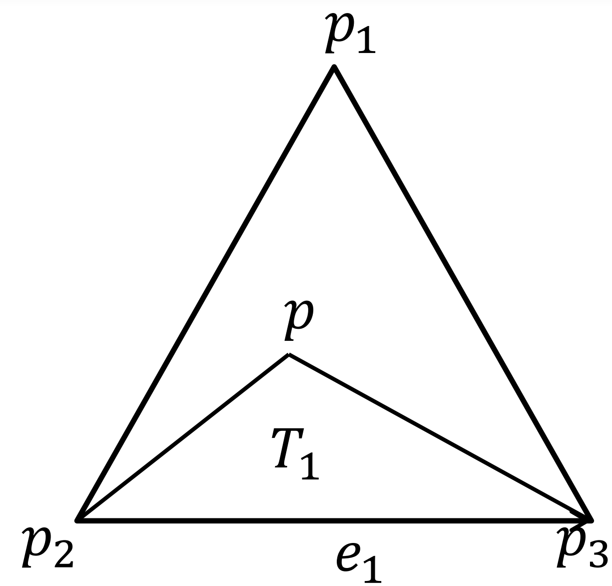
$$\text{d area}(T_1(p))(v) = \frac{1}{2}\langle v, R^{90°} e_1 \rangle$$

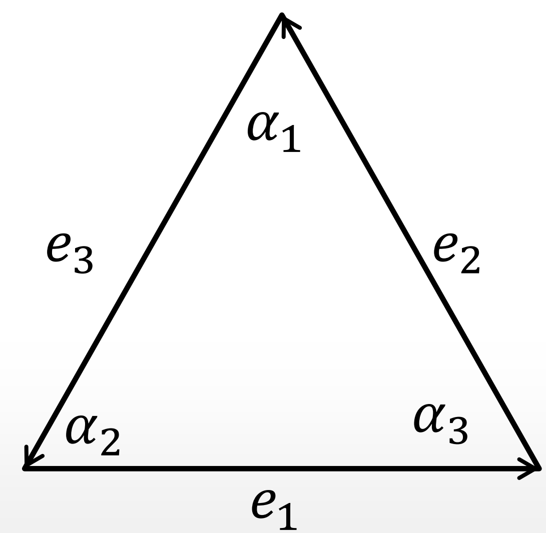
- Hence the gradient is

$$\nabla \text{area}(T_1(p))(v) = \frac{1}{2} R^{90°} e_1$$

- We have

$$R^{90°} e_1 = \cot(\alpha_2)\, e_2 - \cot(\alpha_3)\, e_3$$



# Gradient

## Gradient of linear functions

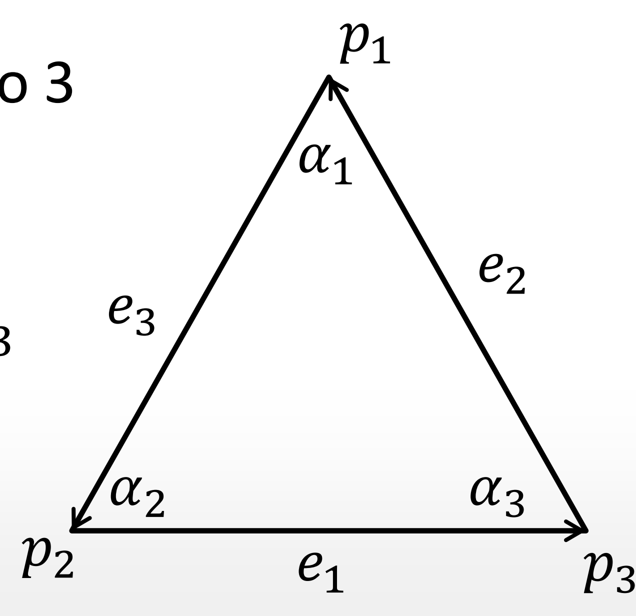
$$\nabla u(p) = \sum_{i=1}^{3} u_i \, \nabla \varphi_i(p) = \frac{1}{2\mathrm{area}(T)} \sum_{i=1}^{3} u_i \, R^{90^\circ} e_i$$

$$= \frac{1}{2\mathrm{area}(T)} \sum_{i=1}^{3} u_i \left(\cot(\alpha_{i+1})\, e_{i+1} - \cot(\alpha_{i+2})\, e_{i+2}\right)$$

Indices $i+1$ and $i+2$ are to be read modulo 3

We used the formula

$$R^{90^\circ} e_1 = \cot(\alpha_2)\, e_2 - \cot(\alpha_3)\, e_3$$



# Gradient

$$\nabla u(p) = \sum_{i=1}^{3} u_i \, \nabla \varphi_i(p) = \frac{1}{2\mathrm{area}(T)} \sum_{i=1}^{3} u_i \, R^{90^\circ} e_i$$

## Gradient is a linear map

- Gradient maps function values to constant vectors

$$\nabla : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

is linear: $\nabla(u+v) = \nabla u + \nabla v, \nabla(a\,u) = a \nabla u$

## Matrix representation

$$\frac{1}{2\mathrm{area}(T)} \begin{pmatrix} R^{90^\circ} e_1 & R^{90^\circ} e_2 & R^{90^\circ} e_3 \end{pmatrix}$$

# Functions on Meshes

# Function Spaces on Meshes

## A function space

- Consider a mesh $M$
- Consider the set $S_h$ of functions that are
  - Continuous
  - Linear polynomial in every triangle

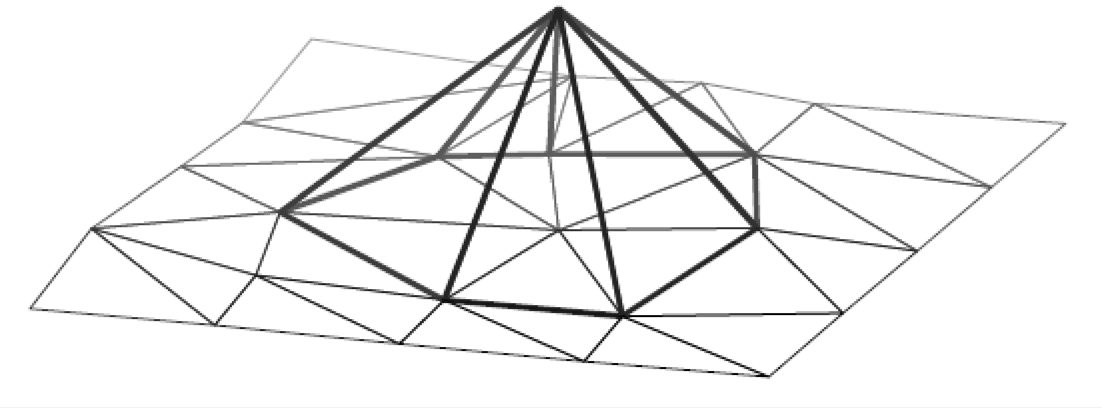
## Linear space

- $S_h$ is a subspace of the vector space of all functions on $M$
  - Check: $u + \lambda v \in S_h$ for any $u, v \in S_h$ and $\lambda \in \mathbb{R}$

# Function Spaces on Meshes

## What is a basis in this space?

- In a triangle a linear function is determined by the function values at the vertices
- Continuity
  - For neighboring triangles the values at the common vertices must agree
- A function in $S_h$ is determined by its values at the vertices (and linear interpolation in the triangles)

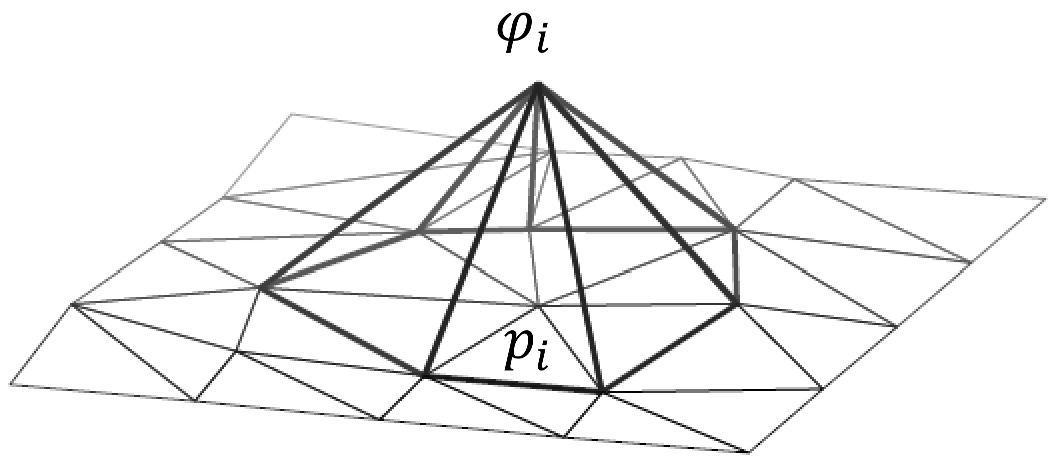


# Function Spaces on Meshes

## Lagrange basis (or nodal basis)

- A basis is given by the family of functions $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ that satisfy

$$\varphi_i(p_j) = \begin{cases} 1 & for\ i = j \\ 0 & for\ i \neq j \end{cases}$$



# Function Spaces on Meshes

## Function representation

- Any $u \in S_h$ can be represented in the Lagrange basis

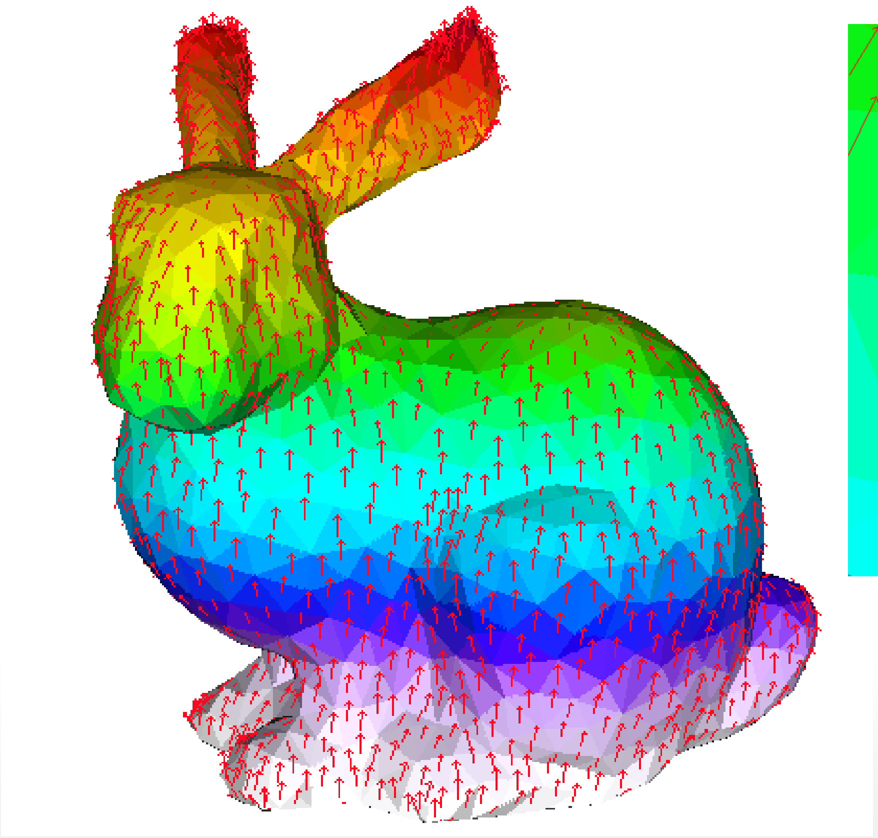
$$u(x) = \sum_{i=1}^{n} u_i \, \varphi_i(x)$$

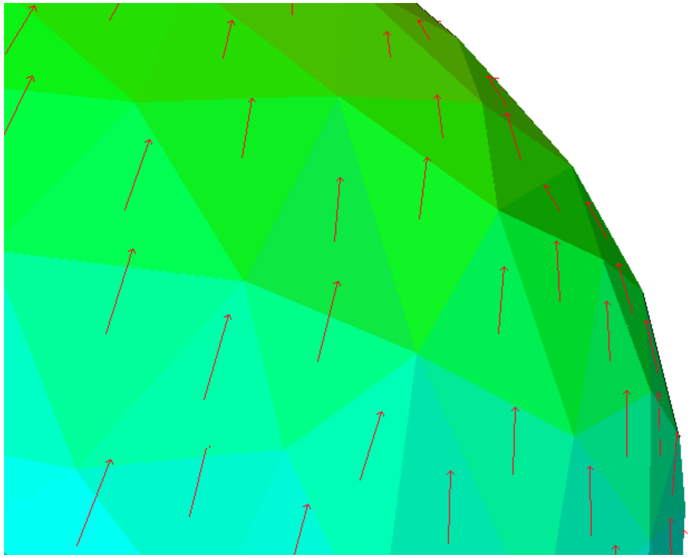
  - $u_i$ is the function value of $u$ at the vertex $p_i$
  - $x$ is an arbitrary point in $M$

# Gradient

**What is the gradient of a function in $S_h$?**

- A constant tangential vector in every triangle
- Denote space of piecewise constant vector fields by $V_h$





# Gradient matrix

## Gradient

- Linear map from functions to vector fields

$$G: S_h \mapsto V_h$$

## Matrix representation of $G$

- $m \times n \;\; (3\#T \times \#V)$ matrix
- Assembled from the elementary matrices:

$$\frac{1}{2\text{area}(T)} \begin{pmatrix} R^{90^\circ} e_1 & R^{90^\circ} e_2 & R^{90^\circ} e_3 \end{pmatrix}$$

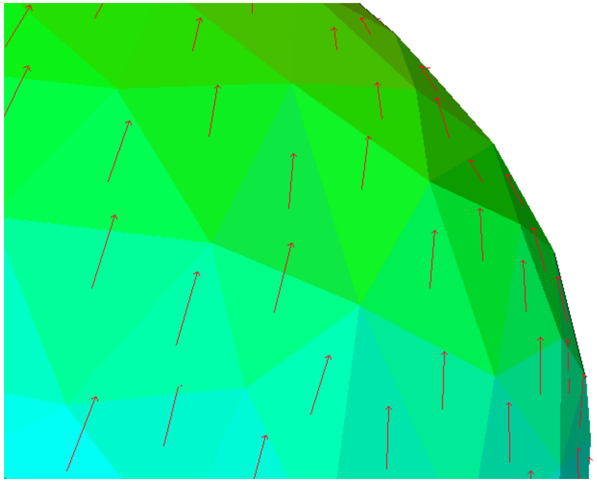
# So Far

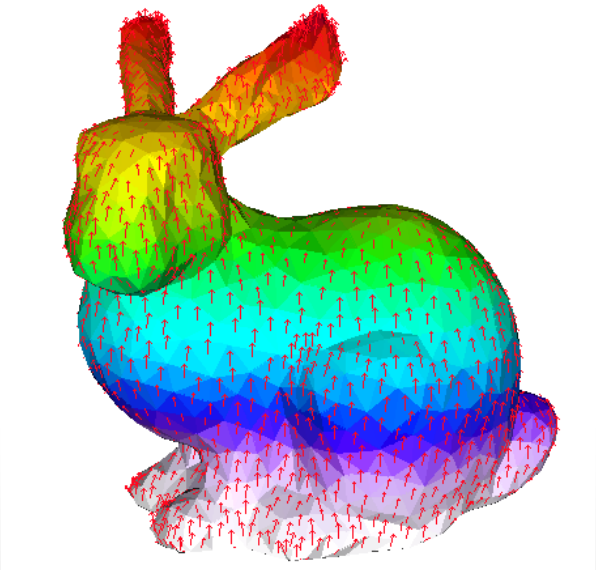
## Two vector spaces

- Functions: real value per vertex
- Vector fields: vector per triangle (vector in the plane of the triangle)

## Gradient

- Linear map from functions to vector fields





# Remainder: Norm on $\mathbb{R}^n$

## Norm on a vector space

- Length of a vector
- Distance between points

## On $\mathbb{R}^n$

- Norm

$$\|\mathrm{v}\|^2 = \sum_{i=1}^{n} |\mathrm{v}_i|^2 = \begin{pmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_n \end{pmatrix} \begin{pmatrix} \mathrm{v}_1 \\ \cdots \\ \mathrm{v}_n \end{pmatrix}$$

## Distance

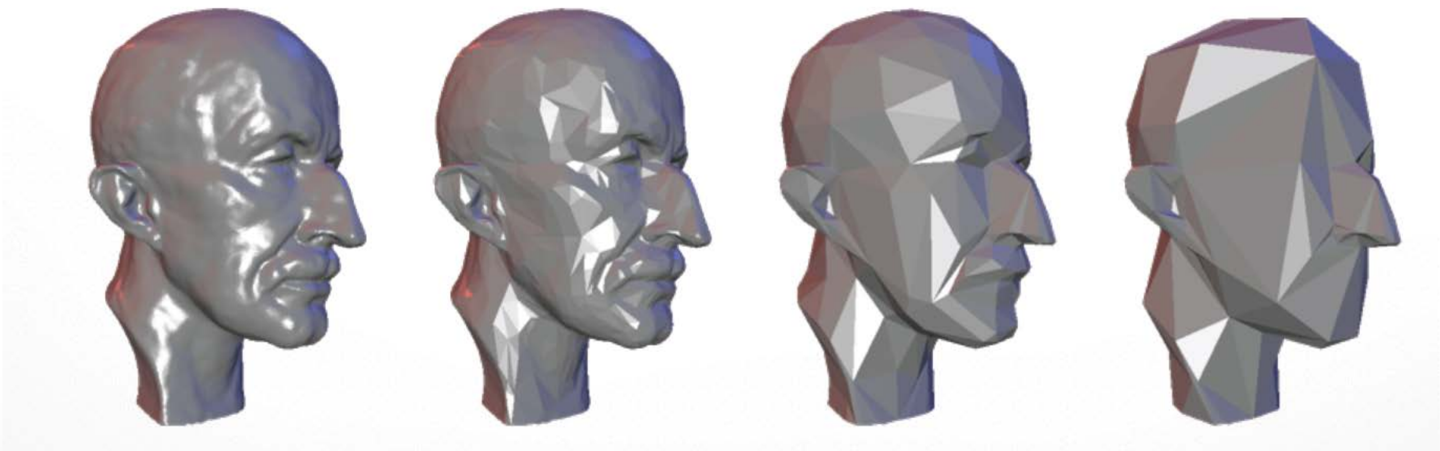
- Distance of two points *p* and *q*

$$d(p,q) = \|p - q\|$$

# Norms for Meshes

## Norm of a function, vector field

- Why? Use for problem modeling
  - Example: Find shortest function that satisfies...
- Why not use norm on $\mathbb{R}^n$?
  - Different results when remeshing, coarse or refine



# Remainder: Scalar product on $\mathbb{R}^n$

## Scalar product

- Measure  length, but also angles between vectors

## On $\mathbb{R}^n$

- Scalar product of u and v

$$\langle \mathrm{v}, \mathrm{w} \rangle = \sum_{i=1}^{n} \mathrm{v}_i \mathrm{w}_i = \begin{pmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_n \end{pmatrix} \begin{pmatrix} \mathrm{w}_1 \\ \cdots \\ \mathrm{w}_n \end{pmatrix}$$

## Relation to norm

$$||\mathrm{v}||^2 = \langle \mathrm{v}, \mathrm{v} \rangle$$

# Piecewise-Constant Vectorfields

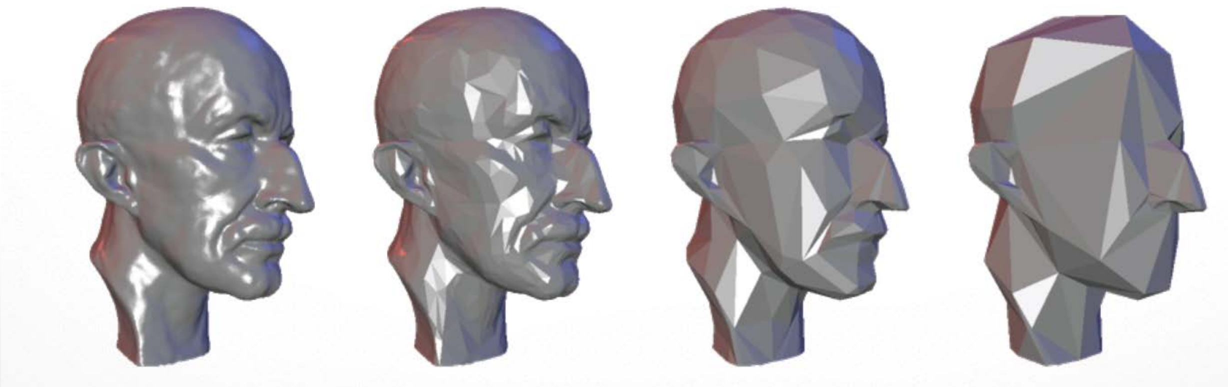
## $L^2$-product of piecewise-constant vector fields

$$\int_M \langle V(x), W(x)\rangle dx = \sum_{T\in M} \int_T \langle V(x), W(x)\rangle dx$$

$$= \sum_{T\in M} A_T \langle V_T, W_T\rangle$$

Notation: $V(x)$ −vector at point $x$, $V_T$ − vector in triangle $T$, $A_T$ −area of triangle $T$

- Benefit of integral formulation: mesh independence



# Piecewise-Constant Vectorfields

## $L^2$-Norm of vector field

$$\|V\|_{L^2} = \sqrt{\sum_{T \in M} A_T \langle V_T , V_T \rangle}$$

# Linear Polynomials

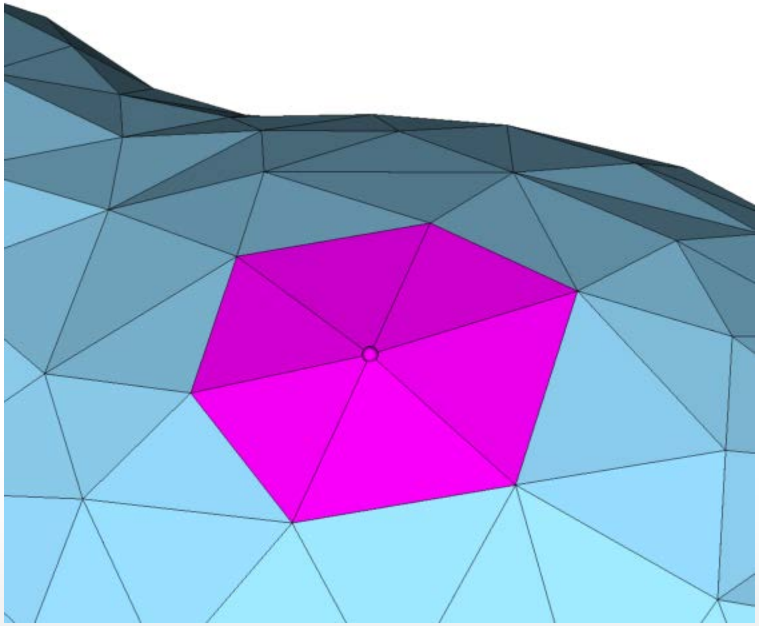
**$L^2$-scalar product**

$$\int_M u(x)\, v(x)\, dx \cong \sum_{p_i \in M} A_{p_i} u_i v_i$$

Notation: $A_{p_i}$ –a third of the sum of areas of all triangle adjacent to $p_i$

**$L^2$-norm**

$$\|u\|_{L^2} = \sqrt{\sum_{p_i \in M} A_{p_i} u_i v_i}$$



# Mass Matrix

## Matrix representation

- Often called the mass matrix
- We denote the matrix by $M$

$$\sum_{p_i \in M} A_{p_i} u_i v_i = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \begin{pmatrix} A_{p_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{p_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

# Piecewise-Constant Vectorfields

## Matrix representation

- We denote the matrix by $M_V$

$$\begin{pmatrix} V_{1,x} & V_{1,y} & \cdots & V_{m,z} \end{pmatrix} \begin{pmatrix} A_{T_1} & 0 & \cdots & 0 \\ 0 & A_{T_1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_{T_m} \end{pmatrix} \begin{pmatrix} W_{1,x} \\ W_{1,y} \\ \vdots \\ W_{m,z} \end{pmatrix}$$

$V_{1,x}$ – $x$-coordinate of $V_1$

# So Far

## Two vector spaces

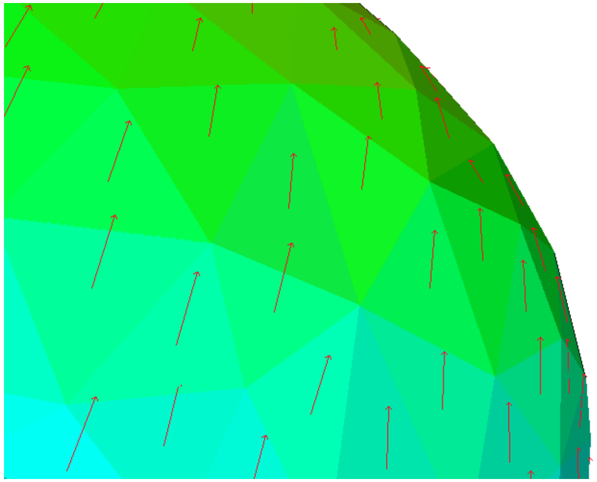
- $S_h$ and $V_h$

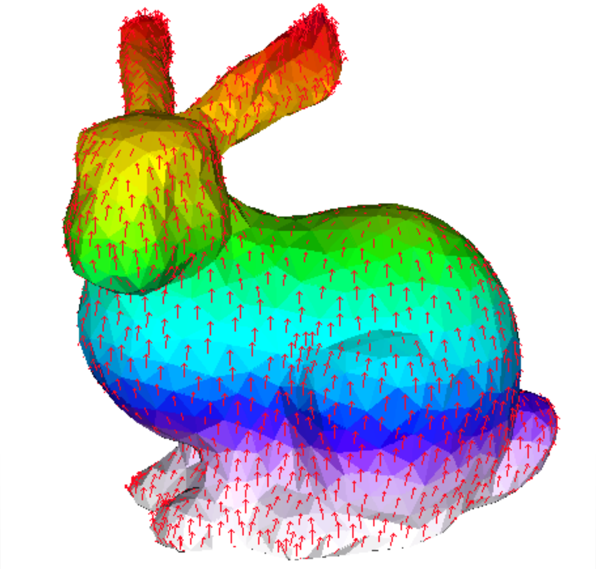
## Gradient

- $G: S_h \mapsto V_h$

## Norms, scalar products

- $M, M_V$





# Stiffness Matrix

## A bilinear form

- Combine gradient and scalar product for vectors
- Consider $u, v \in S_h$

$$u, v \rightarrow \int_M \langle \nabla u , \nabla v \rangle dx$$

- Vanishes for constant functions

## Dirichlet Energy

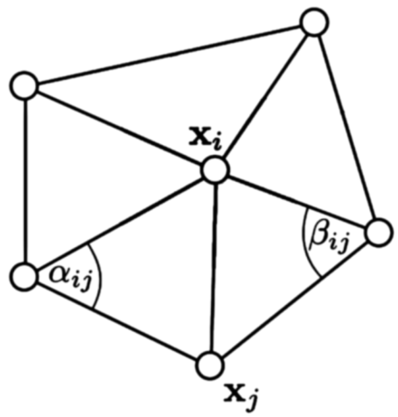
$$E_D(u) = \frac{1}{2} \int_M \langle \nabla u , \nabla u \rangle dx$$

- $\sqrt{E_D(u)}$ is almost a norm, almost because it vanishes for constant functions

# Stiffness Matrix

## Dirichlet Energy

$$u \to \frac{1}{2}\int_M \langle \nabla u\,, \nabla u\rangle dx$$




## Matrix representation

$$S = G^T M_V G$$

$$E_D(\mathrm{u}) = \frac{1}{2} u^T S u$$

- Matrix S explicitly:

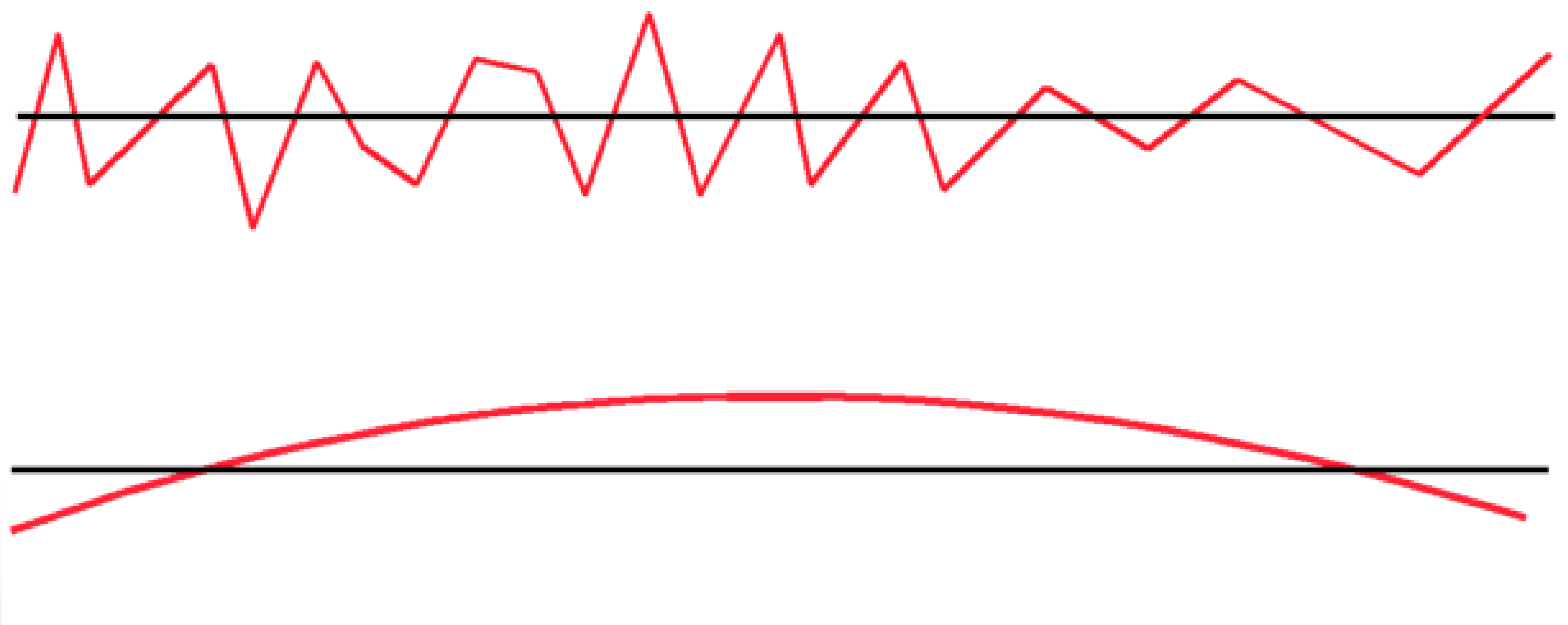
$$s_{ij} = -\frac{1}{2}\left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right) \text{ for } i \neq j$$

$$s_{ii} = -\sum_{j=1}^{n} s_{ij}$$

# Dirchlet vs. $L^2$

## $L^2$ measures magnitude of function values

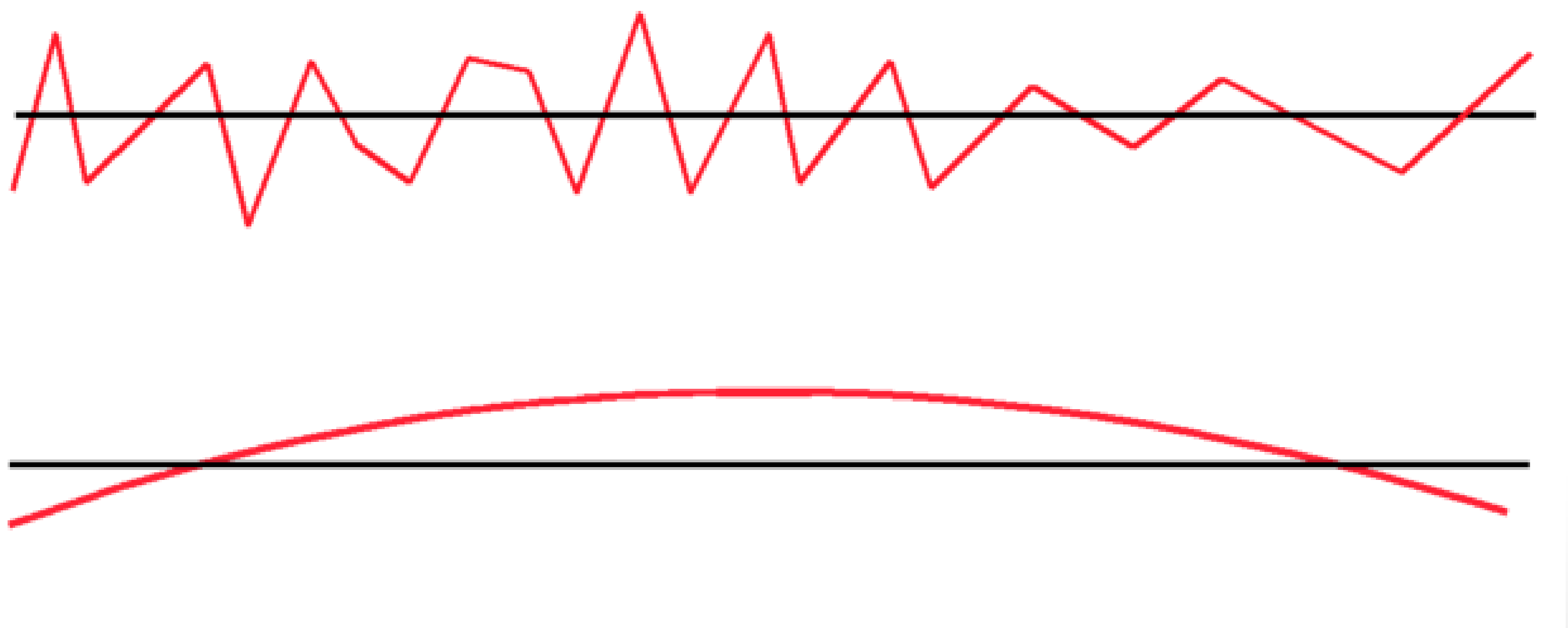
- Both functions have approximately the same value



# Dirchlet vs. $L^2$

**Dirchlet energy measures magnitude of gradient**

- Smoother functions have smaller value
- First function has larger Dirichlet energy than second



# Discrete Laplace-Beltrami Operator

## Laplace Matrix

- We call the matrix $L = M^{-1}S$ the Laplace matrix

## Remarks

- Maps functions to functions
- Continous analog for $\mathbb{R}^2$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \quad \text{or} \quad \Delta u$$

- The constant functions are in the kernel of $L$

# Overview Matrices

## Matrices

$M, M_V$:

- diagonal matrices
- positive entries (areas)

$G$:

- rectangular matrix ($m \times n$)
- sparse

$S = G^T M_V G$:

- symmetric ($\mathrm{n} \times n$)
- sparse
- non-negative:
  $u^T S u \geq 0$ for all $u$
- kernel are the constants

$L = M^{-1} S$

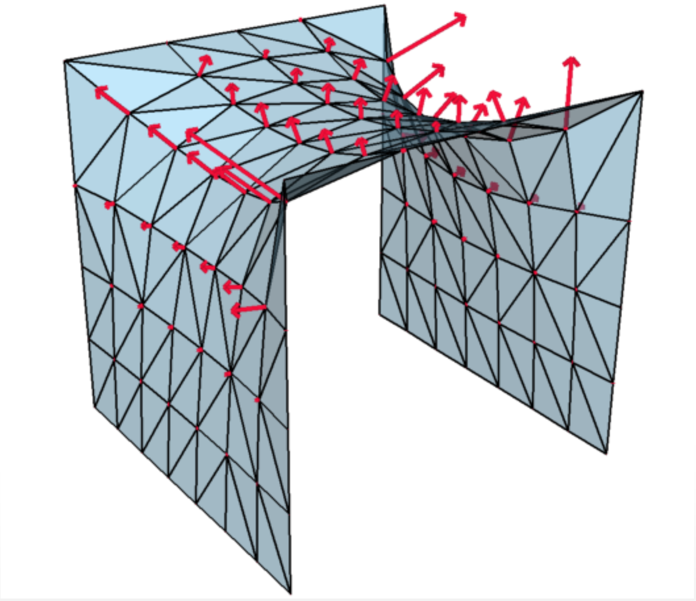
- not symmetric ($\mathrm{n} \times n$)
- sparse
- non-negative eigenvalues
- kernel are the constants

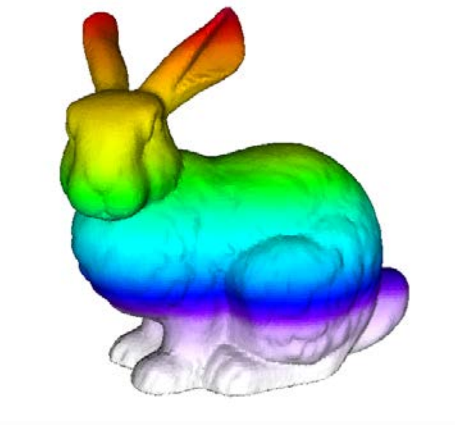
# Deformation-Based Editing

# Function Spaces on Meshes

## Examples

- x-coordinate of every point of the surface
- The embedding is a vector-valued function in ${S_h}^3$
- Displacements of the vertices are ${S_h}^3$

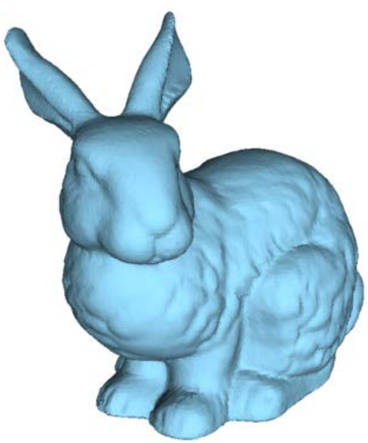




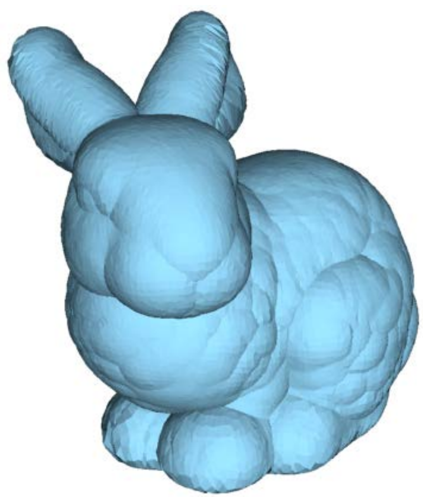
# Displacement Vector

## Notation

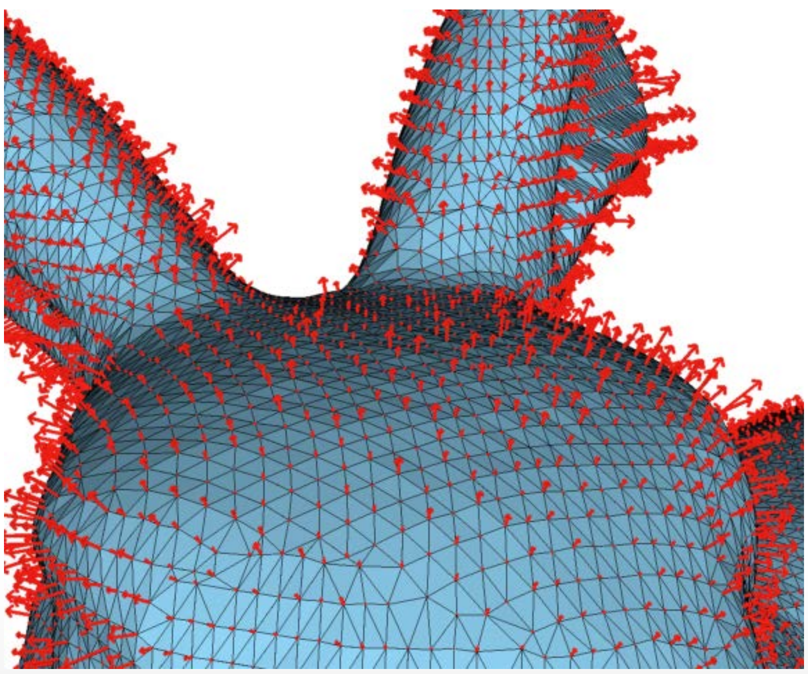
- Denote by $x \in {S_h}^3$ the map that maps every vertex to its positions in $\mathbb{R}^3$ and by $u \in {S_h}^3$ a displacement of the surface



$x$



$x + u$



# Deformation Energies

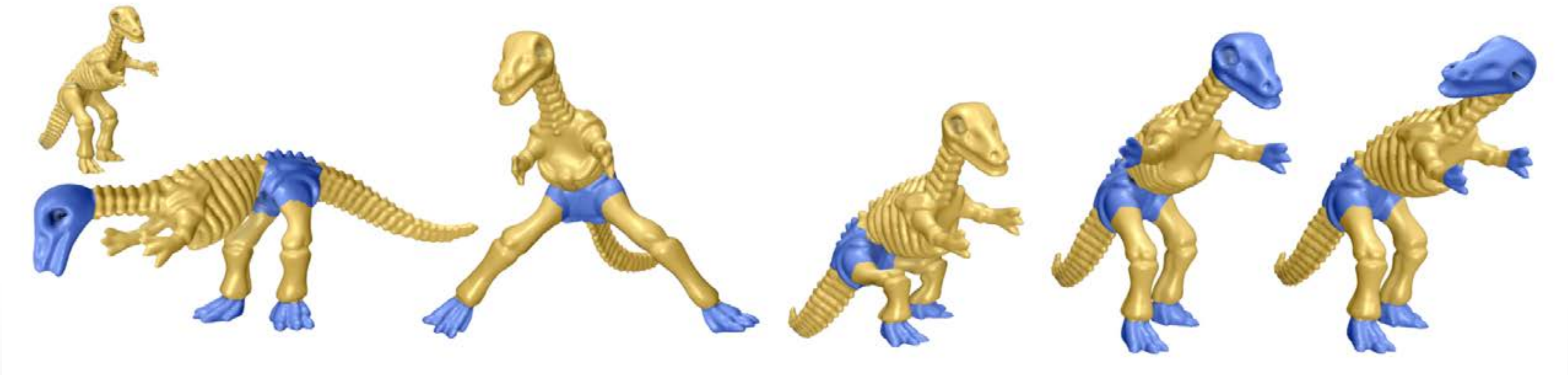
## General deformation energies

- A deformation energy measures the "energy" stored in a deformation (or the "cost" of a deformation)

$$E: {S_h}^3 \mapsto \mathbb{R}$$

Displacement

Energy



# Quadratic Deformation Energies

## Gradient-based deformation energy

$$E_D(\mathrm{u}) = \frac{1}{2}\int_M \|\nabla u\|^2 dA$$

- Matrix representation

$$E_D(\mathrm{u}) = \frac{1}{2} u^T S u$$

- This energy is also called the Dirichlet energy of $u$

# Quadratic Deformation Energies

**Laplace-based deformation energy**

$$E_L(\mathrm{u}) = \frac{1}{2}\int_M \|\Delta u\|^2 dA$$

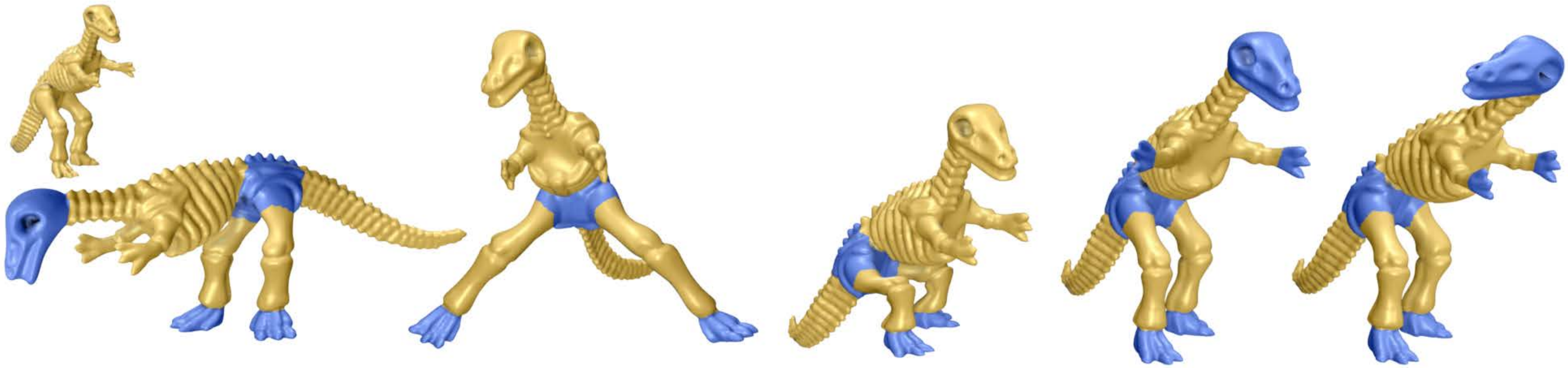
- Matrix representation

$$E_L(\mathrm{u}) = \frac{1}{2}u^T L^T M L u = \frac{1}{2}u^T S M^{-1} M M^{-1} S u$$
$$= \frac{1}{2}u^T S M^{-1} S u$$

# Modeling metaphor

## Handles (global deformations)

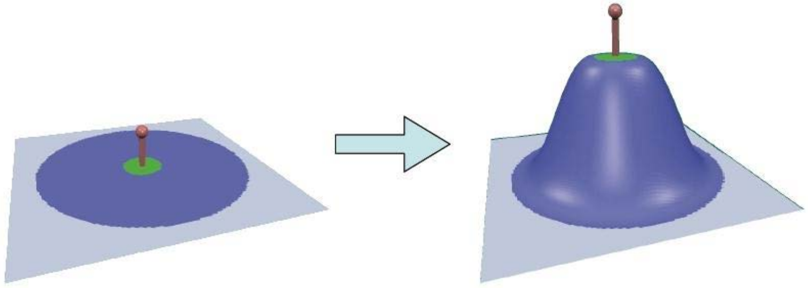
- Handles (blue)



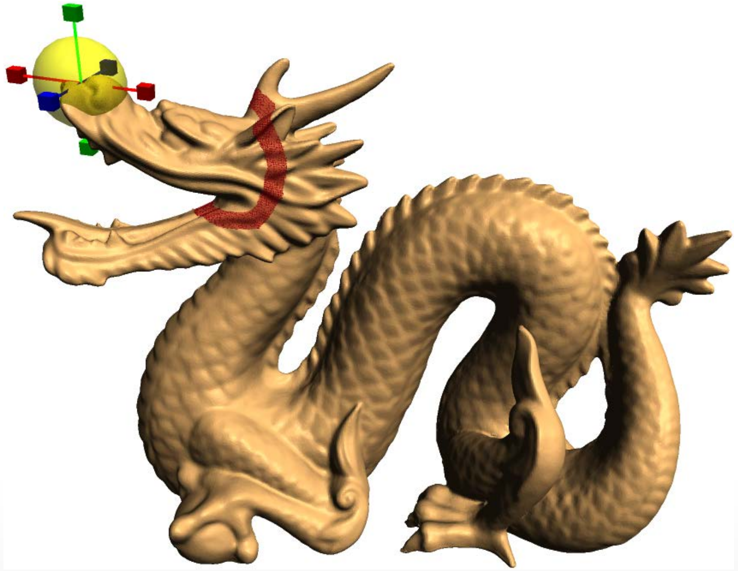
# Modeling metaphor

**Region of interest (local deformations)**

- Support region (blue)
- Fixed vertices (gray)
- Handle regions (green)



Botsch et al. 2004



O. Sorkine et al. 2004

# Constraints

**To deform the object the user sets constraints**

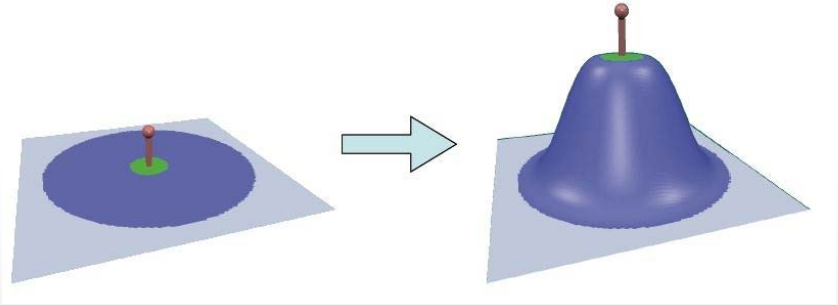
- Hard constraints:

$$Au = a$$

- Soft constraints:

$$E_C(u) = \frac{1}{2}\|Au - a\|^2$$

- $A$ is a rectangular matrix, $a$ is a vector
- Use masses for irregular meshes



# Quadratic Program

## Soft constraints

- Minimize weighted sum deformation energy $E_L$ and constraints energy $E_C$ over all displacements $u \in {S_h}^3$
  - $\lambda \in \mathbb{R}_{>0}$

$$E(u) = E_L(u) + \lambda E_C(u)$$

- Necessary condition for a minimum $u^*$ is $\nabla E(u^*) = 0$
- Since $E$ is quadratic and positive definite, this is also a sufficient condition

$$\nabla E(u) = (SM^{-1}S + \lambda A^T A)u - A^T a$$

# Computing the Deformation

## Linear system

- To compute the deformation, the linear system
$$(SM^{-1}S + \lambda A^T A)u = A^T a$$
has to be solved

- The matrix $(SM^{-1}S + \lambda A^T A)$ is
  - sparse
  - symmetric, positive definite
- An efficient solver is a sparse Cholesky decomposition
- Since changing the positions of the handles only changes the right-hand side, the factorization can be re-used and interactive modeling is possible

# Quadratic Program

## Hard constraints

- Use Lagrange multipliers $\lambda$
- The displacements are the solution of

$$\begin{bmatrix} SM^{-1}S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

The matrix $\begin{bmatrix} SM^{-1}S & A^T \\ A & 0 \end{bmatrix}$ is symmetric and positive definite

- Since changing the positions of the handles only changes the right-hand side, the factorization can be re-used and interactive modeling is possible

# Laplacian Surface Editing

Laplacian Mesh Editing

A short editing session
with the *Octopus*

O. Sorkine, D. Cohen-Or, Y. Lipman, M. Alexa, C. Rössl, and H.-P. Seidel. 2004.
Laplacian surface editing. In *Proceedings of the 2004 Eurographics/ACM SIGGRAPH Symposium on Geometry Processing*