Geometric Modeling

2015

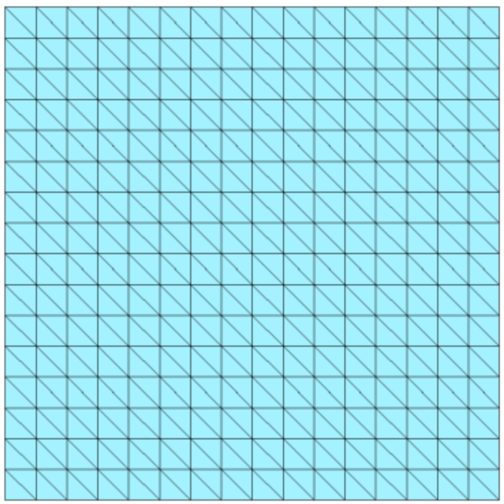
Shape Deformation

TU Delft

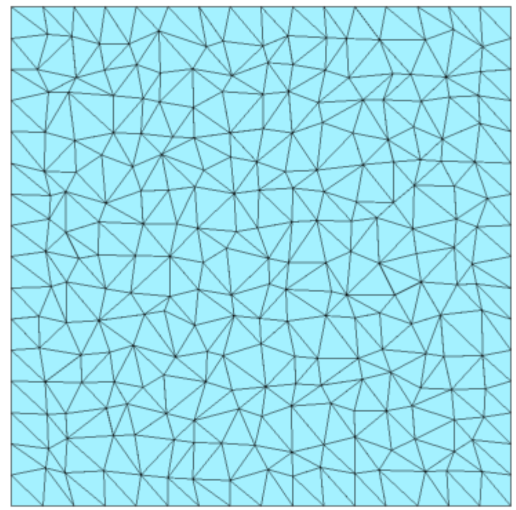
Last Lecture

Mesh Analysis

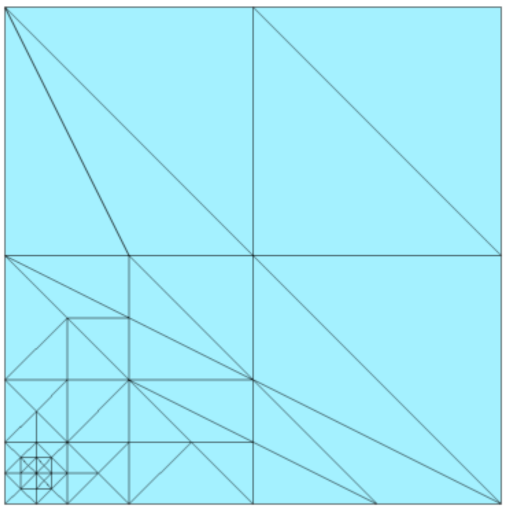
Types of meshes



regular



irregular

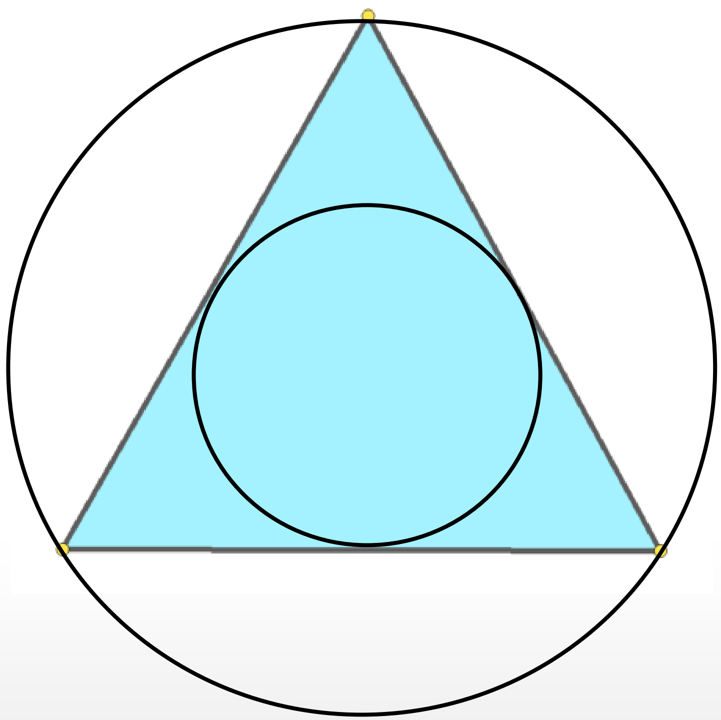


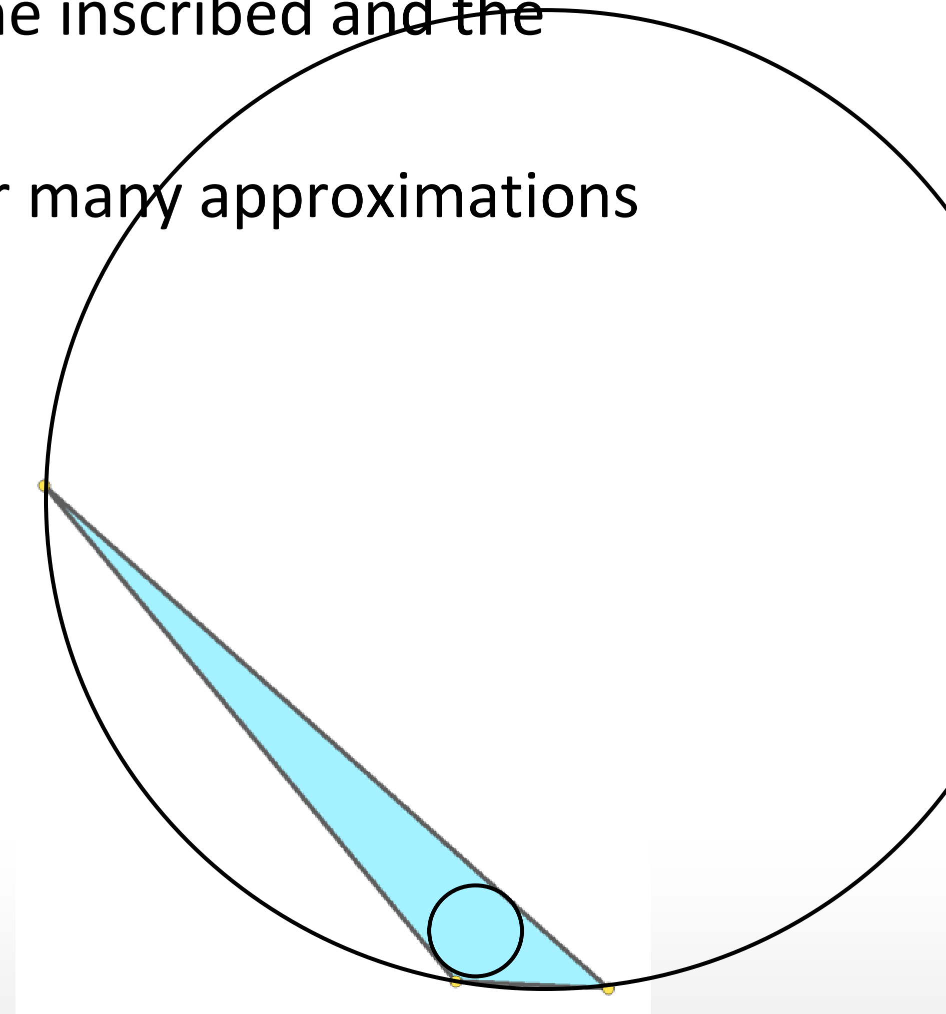
adaptive

Shape Regularity

Shape regularity of a triangle

- Ratio of the diameters of the inscribed and the circumscribed circle
- Appears in error bounds for many approximations

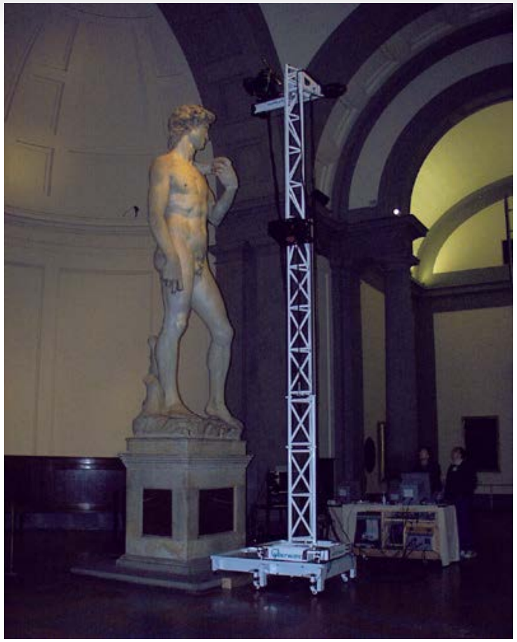


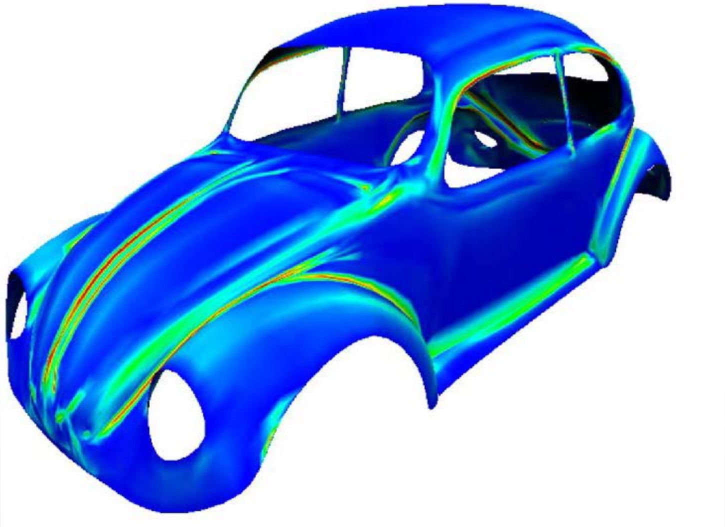


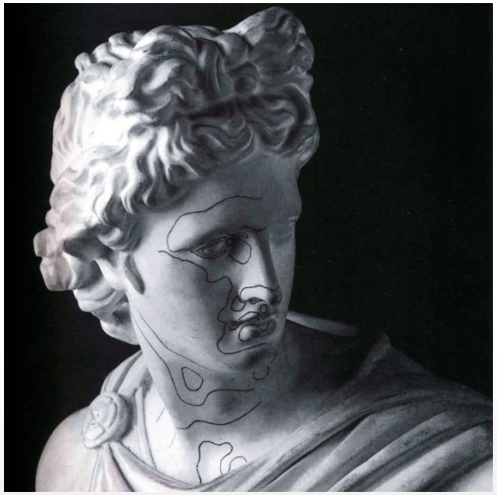
Surface Analysis

Properties of a surface

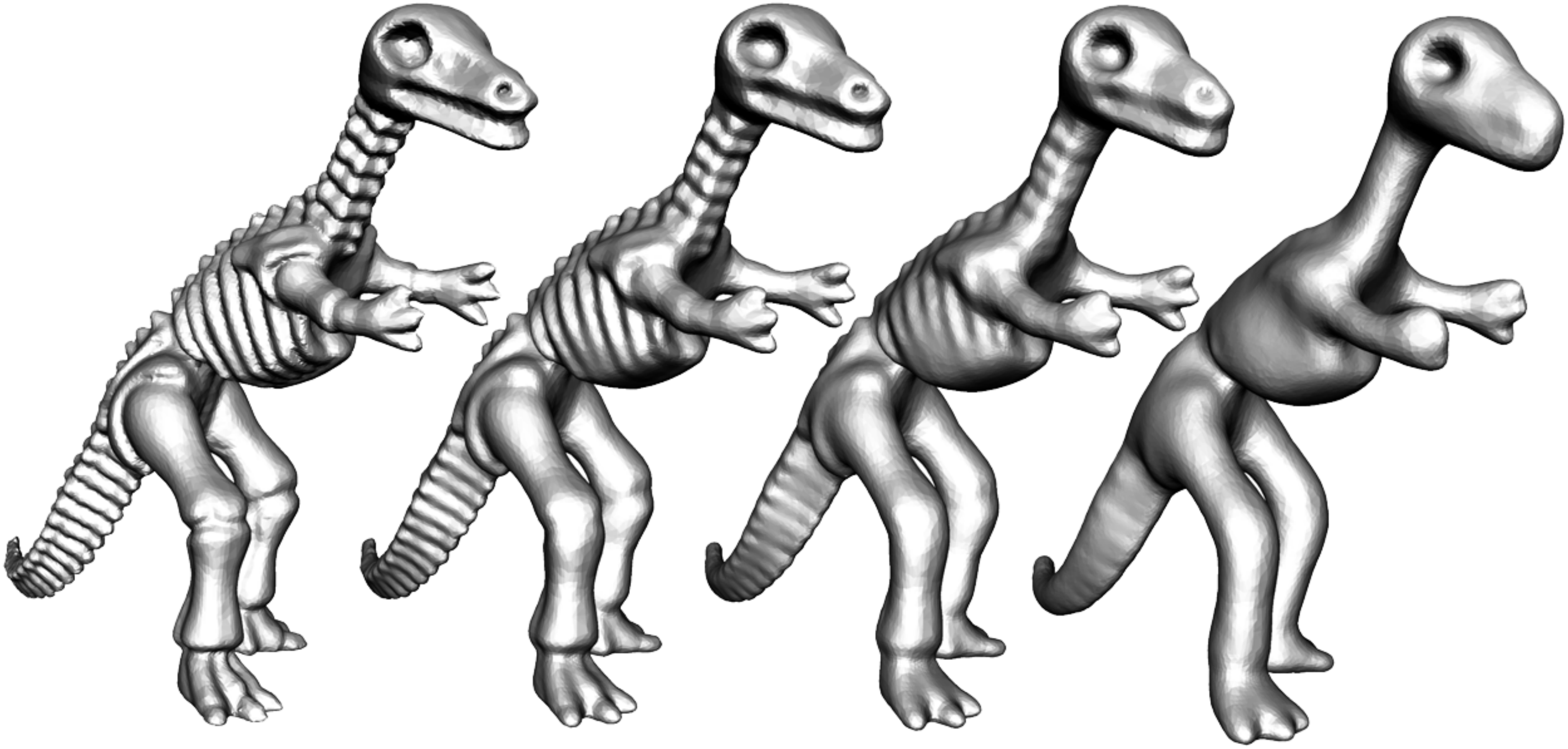
- Area of a surface
- Enclosed volume
  - What is the area/volume of David?
- Special geometric lines on a surface
- Curvatures







Smoothing

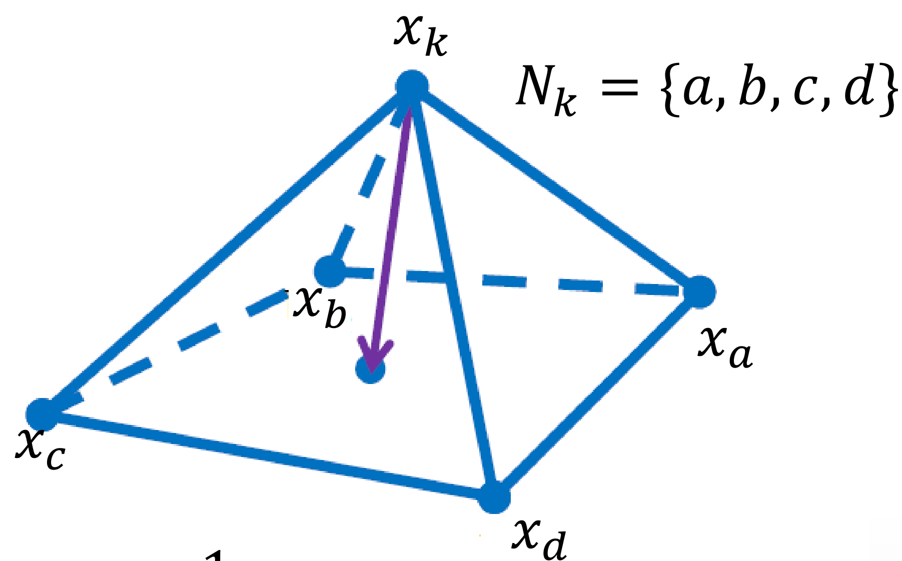


Iterated Averaging (Laplace Smoothing)

For surface meshes

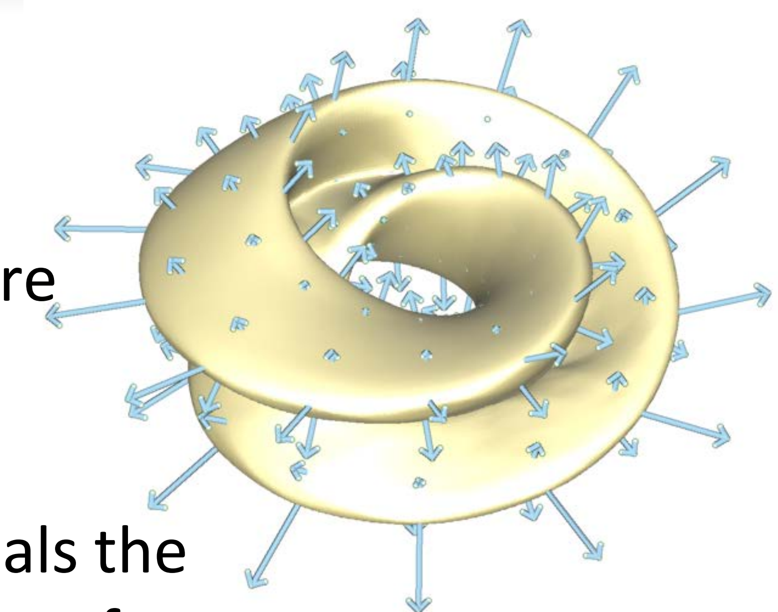
- Iterate: Move every vertex towards the average of its neighbors
- $x_k \leftarrow x_k + \tau \left(\frac{1}{|N_k|} \sum_{l \in N_k} x_l - x_k \right)$

$$\frac{x_{k-1} + x_{k+1}}{2} - x_k$$



$$\frac{1}{|N_k|} \sum_{l \in N_k} x_l - x_k$$

Mean Curvature Vector



Mean curvature vector field

- Normal field
- Length equals the mean curvature

Connection to Laplacian

- Mean curvature vector field equals the Laplacian of the embedding of a surface

$$\vec{H} = \Delta x$$

On a mesh

- Discrete mean curvature vector is $\vec{H}_h \in {S_h}^3$

$$\vec{H}_h = Lx$$

Explicit Euler

Explicit Euler

$$\frac{x^{i+1} - x^{i}}{\tau} = -Lx^{i}$$

Algorithm:

Iterate:

1. Set up the Laplace matrix L of the current embedding x
2. Compute $-Lx$
3. Set $x \leftarrow x - \tau Lx$

Implicit Euler

Limitation of explicit scheme:

- Stable only for small time steps

Semi-Implicit Euler

$$\frac{x^{i+1} - x^{i}}{\tau} = -L^{i} x^{i+1}$$

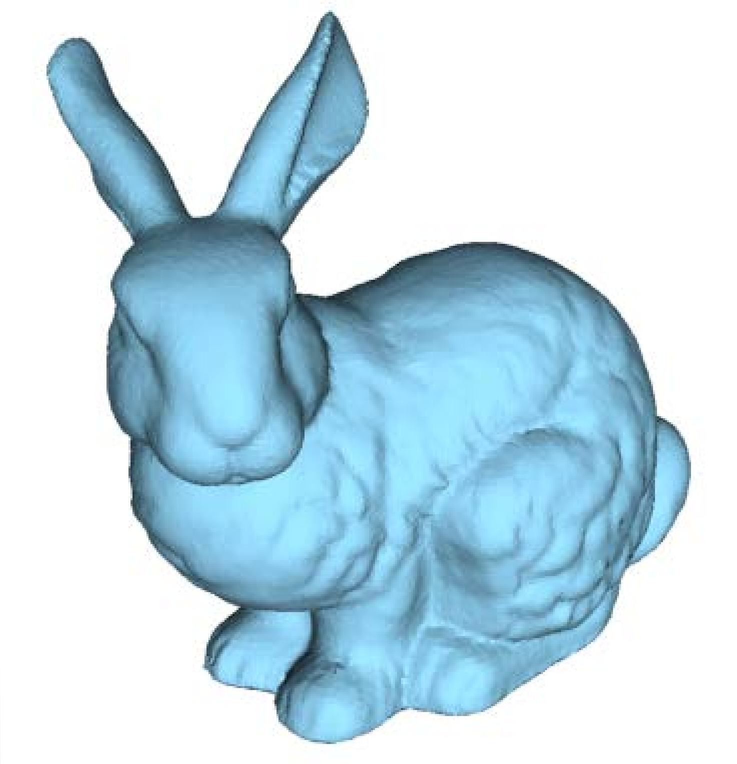
Algorithm:

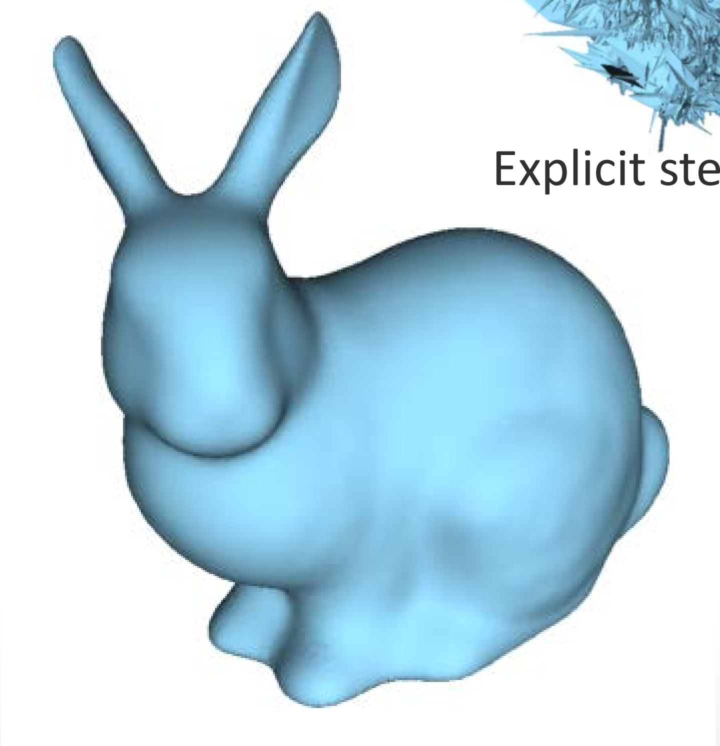
Iterate:

1. Set up the matrices M, S of the current embedding x
2. Solve linear System: $(M + \tau S)x^{i+1} = Mx^{i}$

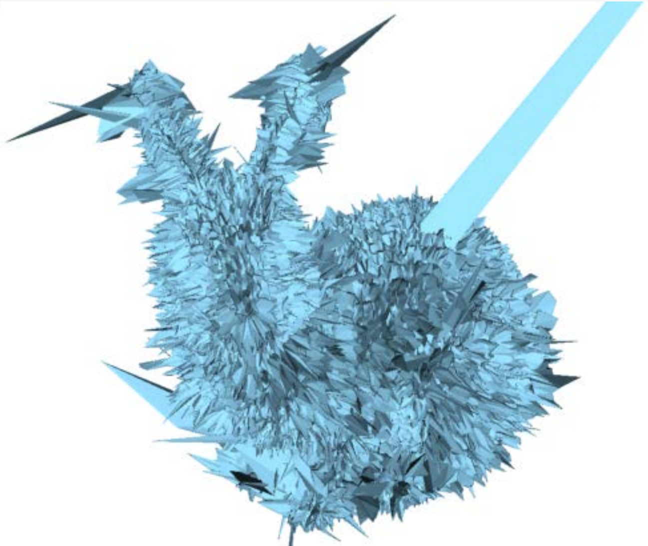
Implicit Scheme

Large time steps are possible





One implicit step



Explicit step of same size

Constrained Smoothing

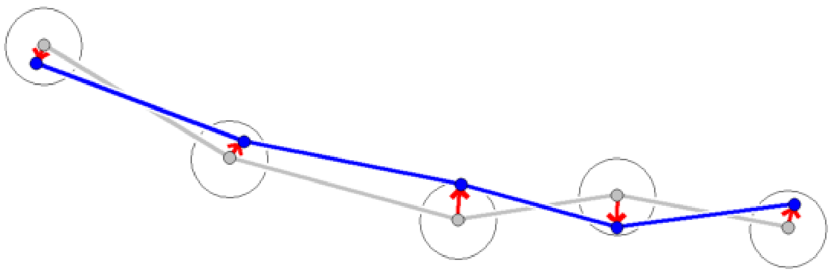
Fairness Energy

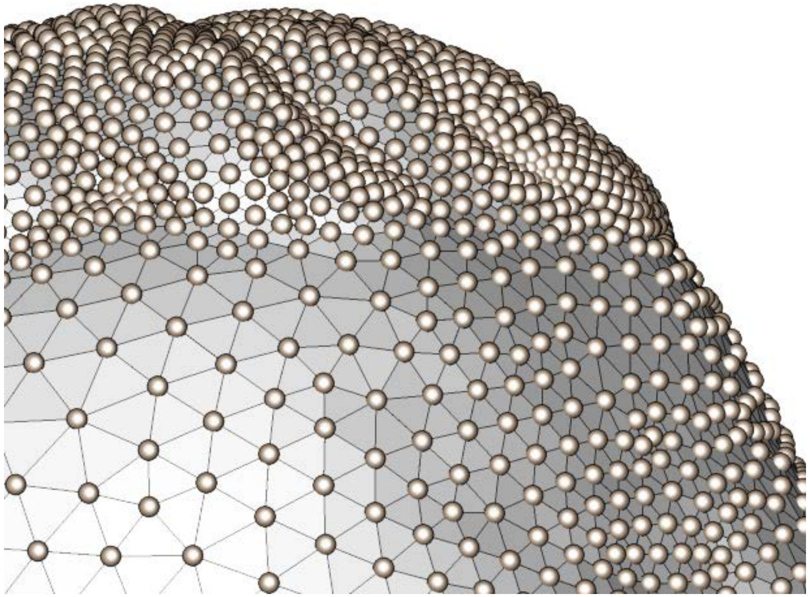
$$E(\mathrm{x}) = \frac{1}{2}\int_M \|\Delta \mathrm{x}\|^2 dA$$

- Matrix representation

$$E(\mathrm{x}) = \frac{1}{2} x^T S M^{-1} S x$$

Constrained Smoothing





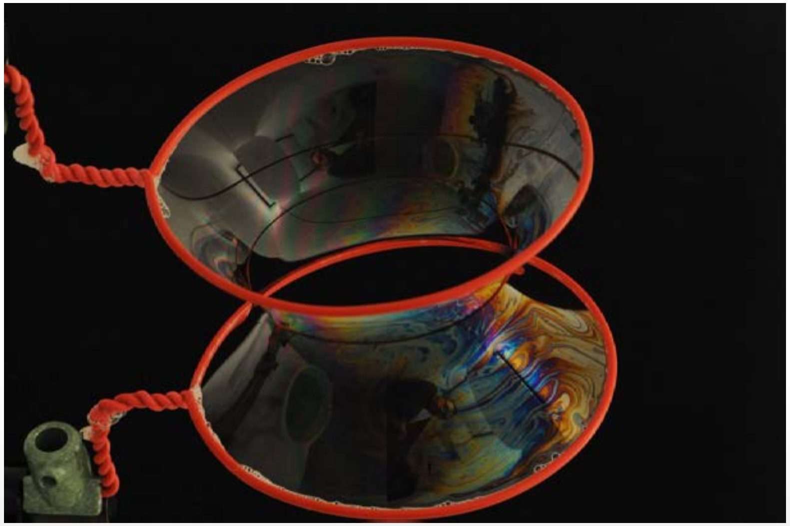
Minimizes E over the feasible set

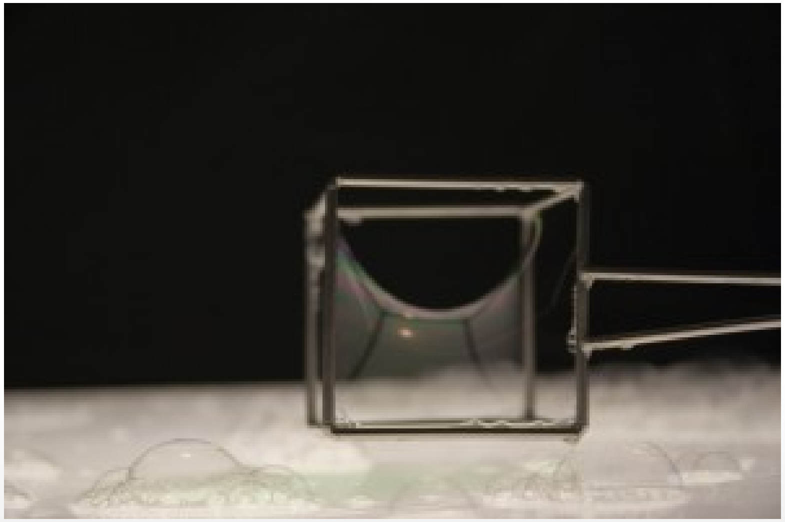
Minimal Surfaces

Examples

Minimal Surfaces

- Surfaces with vanishing mean curvature are called *minimal surfaces*
- They are saddle shaped at every point
- Solution of Plateau's problem (soap films, minimal area)
  - Soap bubbles have constant mean curvature

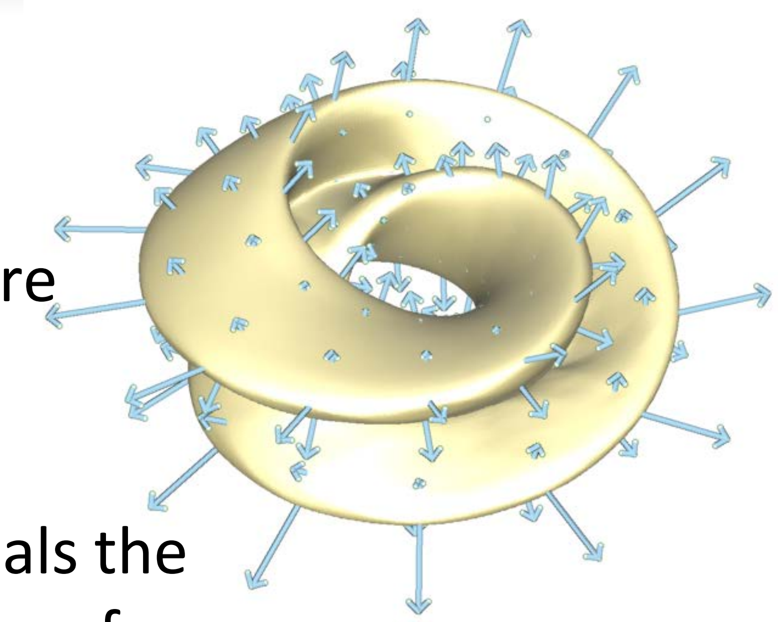




Mean Curvature Vector

Mean curvature vector field

- Normal field
- Length equals the mean curvature



Connection to Laplacian

- Mean curvature vector field equals the Laplacian of the embedding of a surface

$$\vec{H} = \Delta x$$

On a mesh

- Discrete mean curvature vector is $\vec{H}_h \in {S_h}^3$

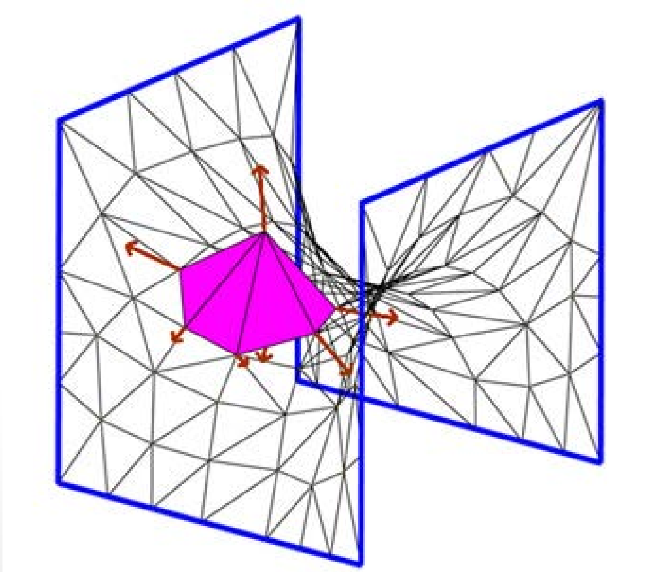
$$\vec{H}_h = Lx$$

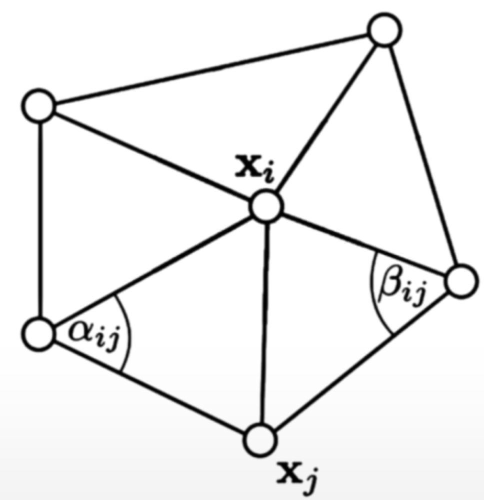
Discrete Mean Curvature Vector

Discrete Mean Curvature Vector

$$\vec{H}_h(x_i) = \frac{3}{2\mathrm{area}\big(\mathrm{star}(x_i)\big)} \sum_{x_j \in link(x_i)} \big(\cot\alpha_{ij} + \cot\beta_{ij}\big)\,(x_i - x_j)$$

Remark: $\mathrm{d\,area}(\mathrm{x})(\mathrm{v}) = \big\langle \vec{H}_h, v \big\rangle_M = x^T S v$

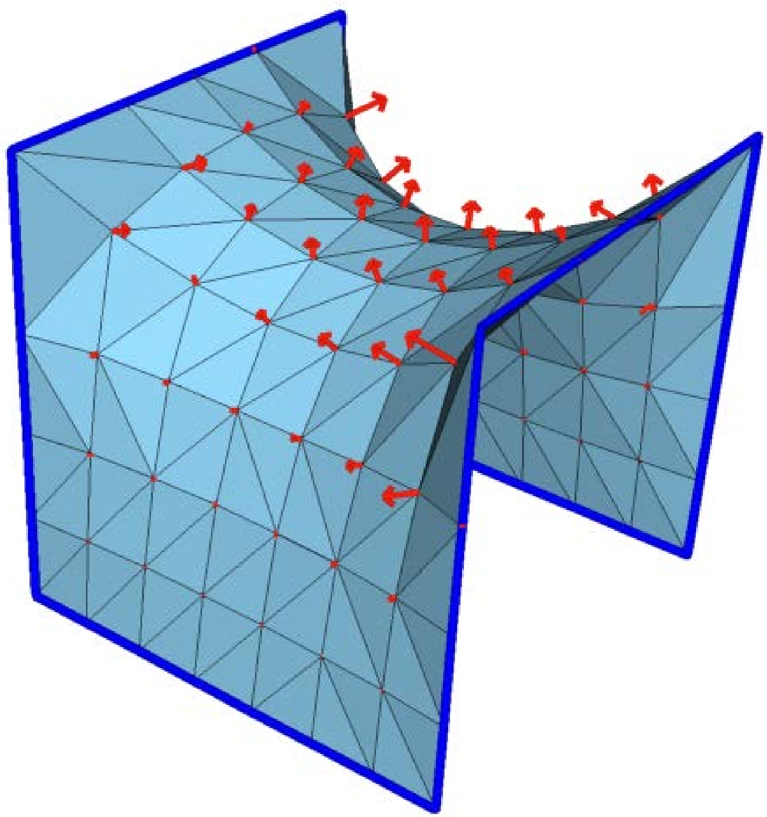




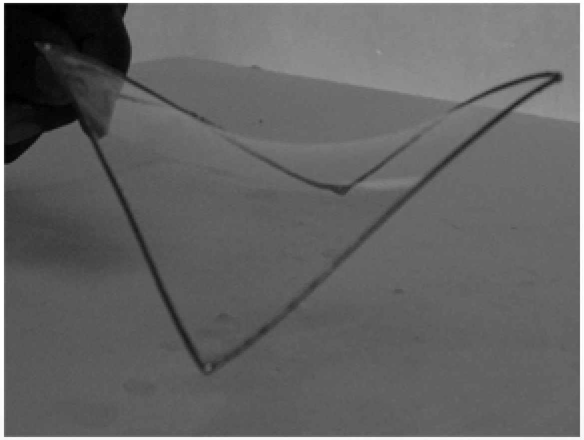
Minimal Surfaces

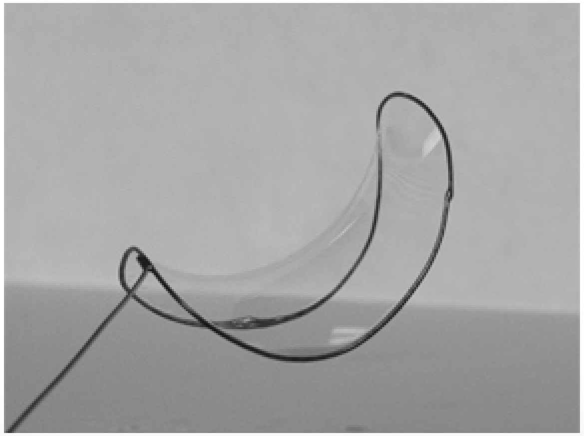
Variation of the area functional

- Minimal surface are critical points of the area functional
- Mean curvature flow is the gradient descent of the area

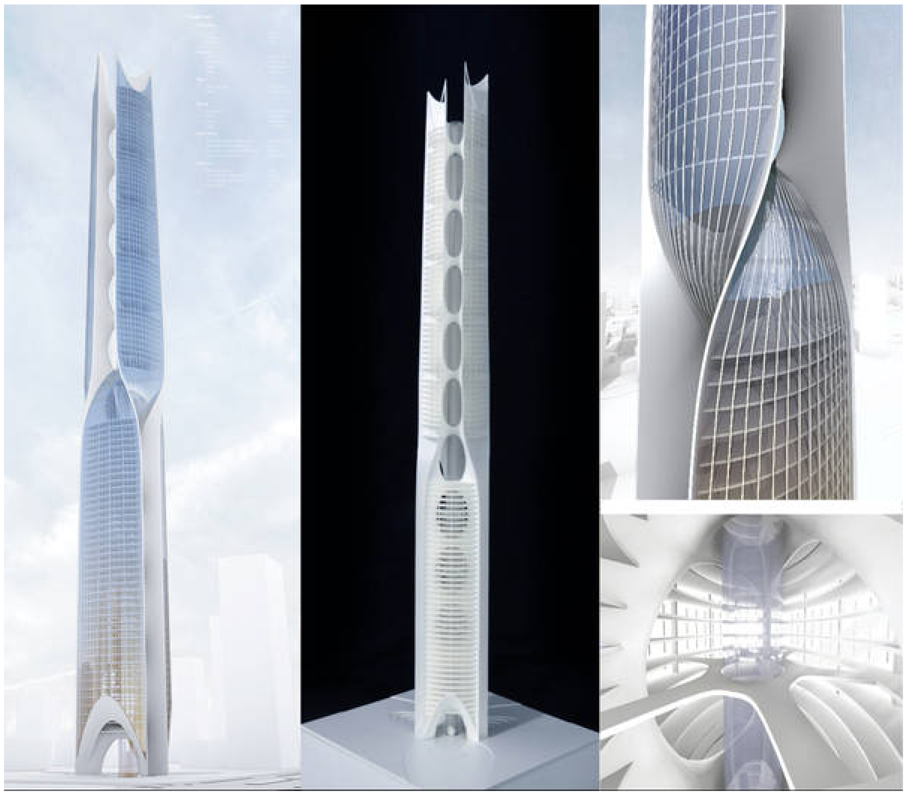


Minimal Surface in Architecture





Minimal Surface in Architecture

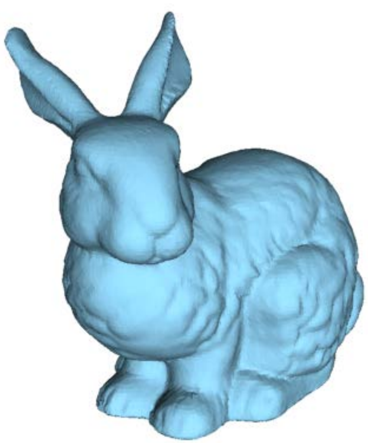


Deformation-Based Editing

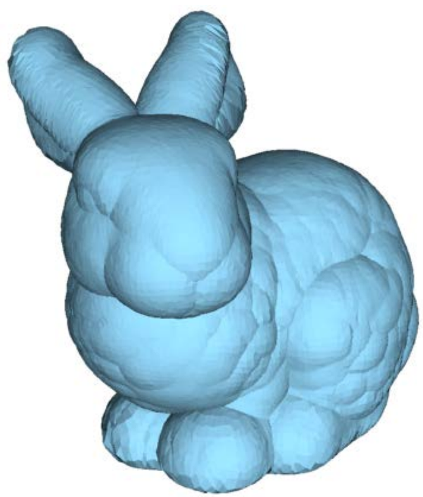
Displacement Vector

Notation

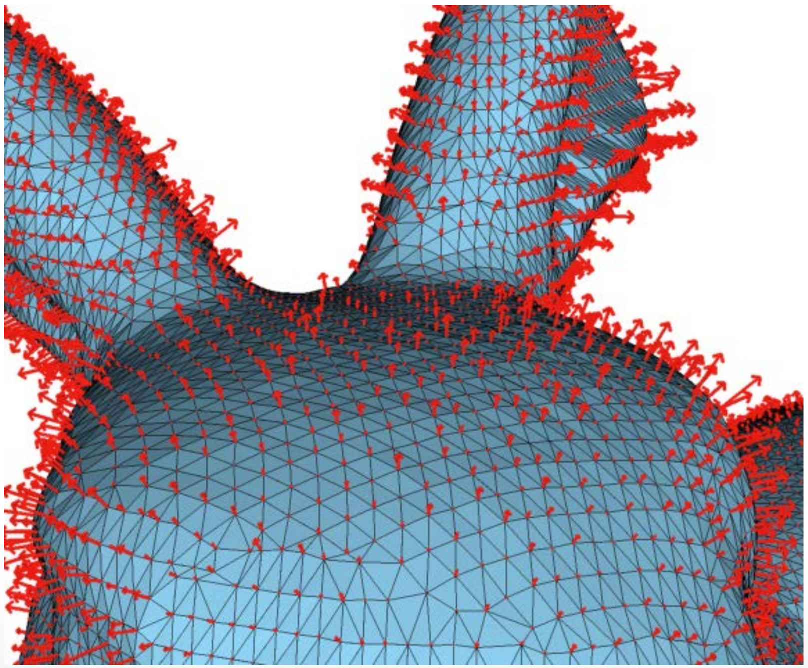
- Denote by $x \in {S_h}^3$ the map that maps every vertex to its positions in $\mathbb{R}^3$ and by $u \in {S_h}^3$ a displacement of the surface



x



$x + u$



Deformation Energies

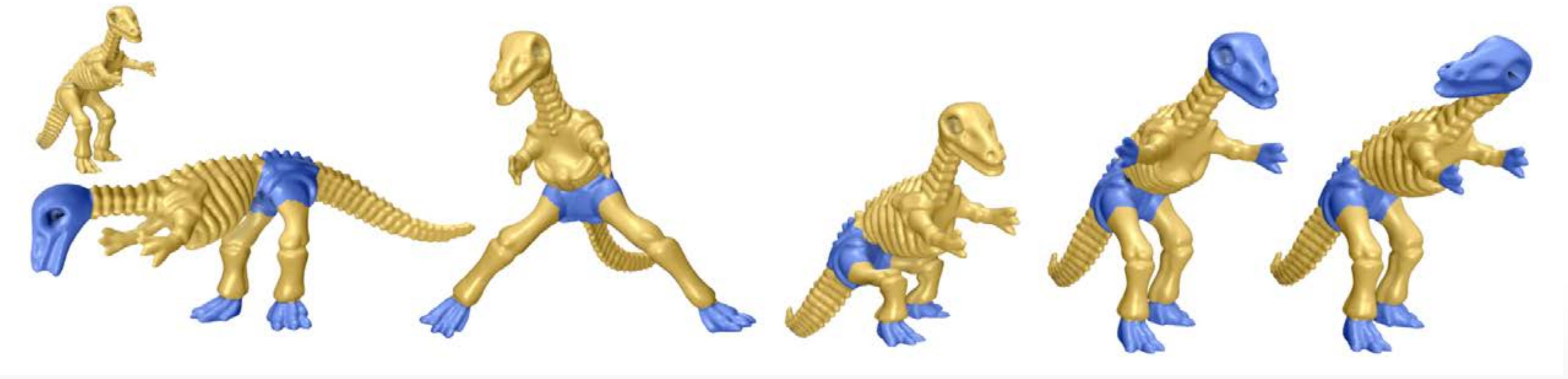
General deformation energies

- A deformation energy measures the “energy” stored in a deformation (or the “cost” of a deformation)

$$E : {S_h}^3 \mapsto \mathbb{R}$$

Displacement

Energy



Quadratic Deformation Energies

Gradient-based deformation energy

$$E_D(u) = \frac{1}{2}\int_M \|\nabla u\|^2 dA$$

- Matrix representation

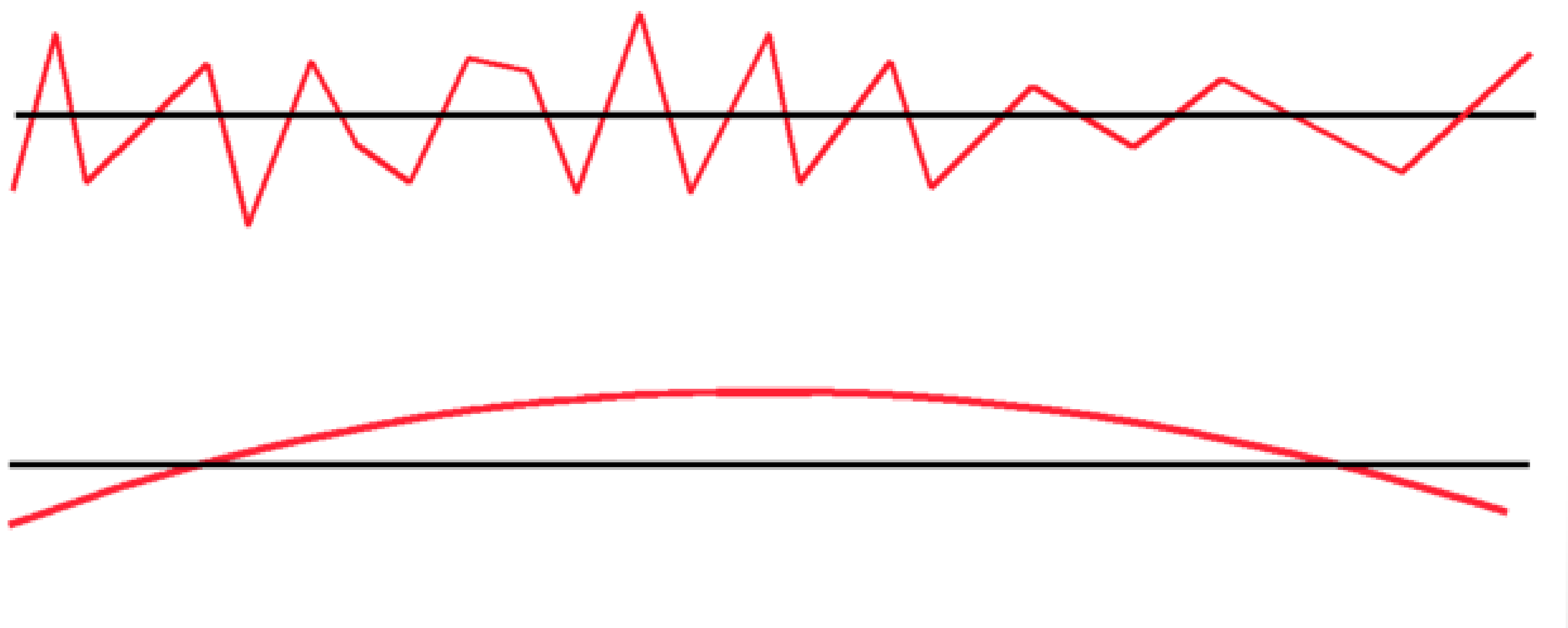
$$E_D(u) = \frac{1}{2} u^T S u$$

- This energy is also called the Dirichlet energy of u

Dirchlet vs. L^2

Dirchlet energy measures magnitude of gradient

- Smoother functions have smaller value
- First function has larger Dirichlet energy than second



Quadratic Deformation Energies

Laplace-based deformation energy

$$E_L(u) = \frac{1}{2}\int_M \|\Delta u\|^2 dA$$

- Matrix representation

$$E_L(u) = \frac{1}{2}u^T L^T M L u = \frac{1}{2}u^T S M^{-1} M M^{-1} S u$$
$$= \frac{1}{2}u^T S M^{-1} S u$$

Displacements

Remember:

- The displacesments u are vector fields
- Denote the x, y, z-coordinate functions by u_x, u_y, u_z
- The deformation energy of a displacesment is the sum of the energy of the 3 coordinate functions

$$E_D(u) = \frac{1}{2}\left(u_x{}^T S u_x + u_y{}^T S u_y + u_z{}^T S u_z\right)$$

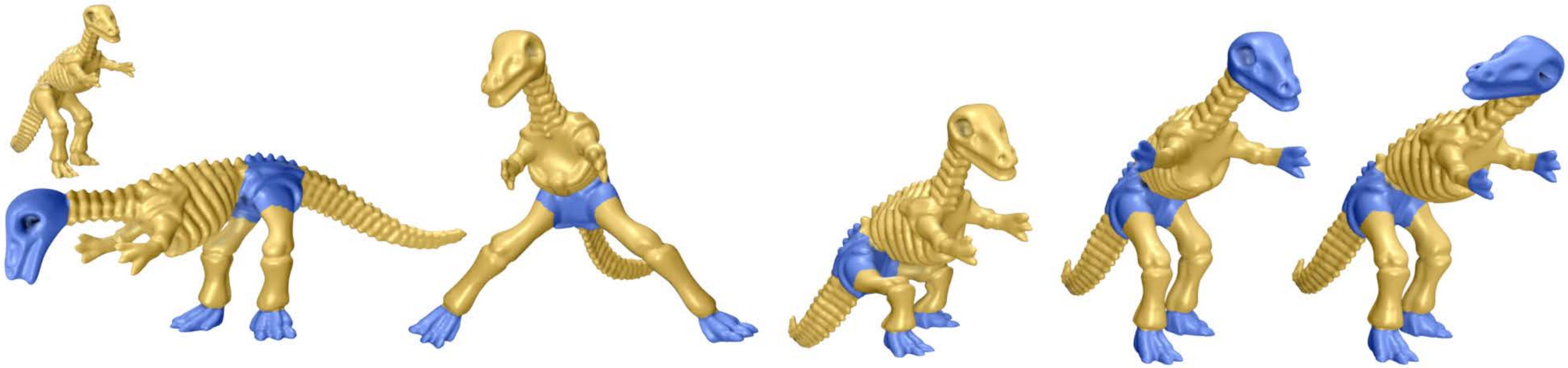
For brevity we write: $\frac{1}{2} u^T S u$

- The same notation is used for $E_L(u)$

Modeling metaphor

Handles (global deformations)

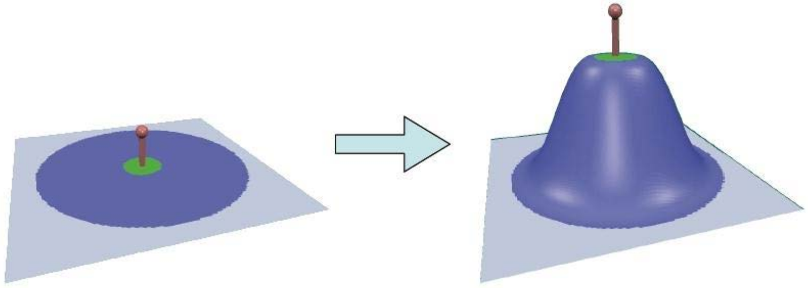
- Handles (blue)



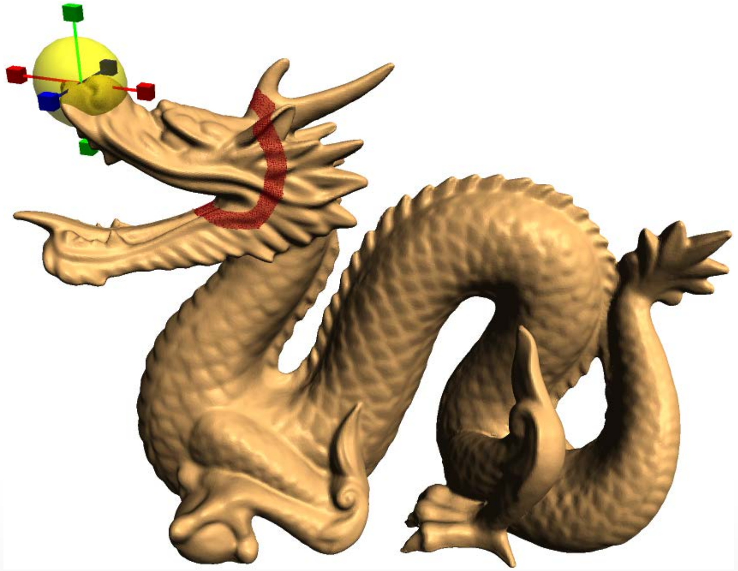
Modeling metaphor

Region of interest (local deformations)

- Support region (blue)
- Fixed vertices (gray)
- Handle regions (green)



Botsch et al. 2004



O. Sorkine et al. 2004

Constraints

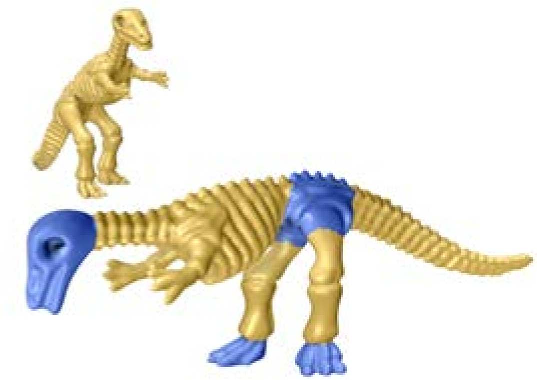
To deform the object the user sets constraints

- Hard constraints:

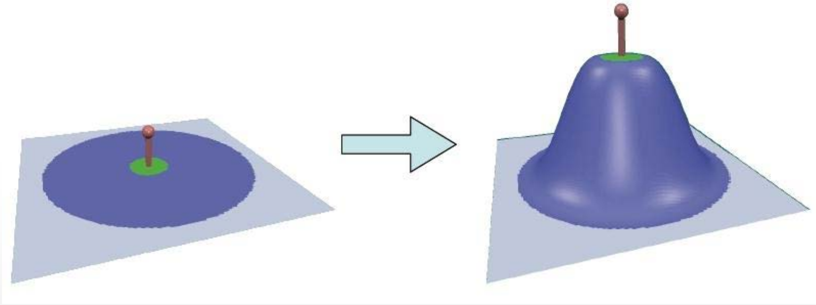
$$Au = a$$

- Soft constraints:

$$E_C(u) = \frac{1}{2}\|Au - a\|^2$$



- A is a rectangular matrix, a is a vector
- Use masses for irregular meshes



Quadratic Program

Soft constraints

- Minimize weighted sum of deformation energy E_L and constraints energy E_C over all displacements $u \in {S_h}^3$
  - $\lambda \in \mathbb{R}_{>0}$

$$E(u) = E_L(u) + \lambda E_C(u)$$

- Necessary condition for a minimum u^* is $\nabla E(u^*) = 0$
- Since E is quadratic and positive definite, this is also a sufficient condition

$$\nabla E(u) = (SM^{-1}S + \lambda A^T A)u - \lambda A^T a$$

Computing the Deformation

Linear system

- To compute the deformation, the linear system
$$(SM^{-1}S + \lambda A^T A)u = \lambda A^T a$$
has to be solved

- The matrix $(SM^{-1}S + \lambda A^T A)$ is
  - sparse
  - symmetric, positive definite
- An efficient solver is a sparse Cholesky decomposition
- Since changing the positions of the handles only changes the right-hand side, the factorization can be re-used and interactive modeling is possible

Quadratic Program

Hard constraints

- Use Lagrange multipliers l
- The displacements are the solution of

$$\begin{bmatrix} SM^{-1}S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} u \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

The matrix $\begin{bmatrix} SM^{-1}S & A^T \\ A & 0 \end{bmatrix}$ is symmetric and positive definite

- Since changing the positions of the handles only changes the right-hand side, the factorization can be re-used and interactive modeling is possible

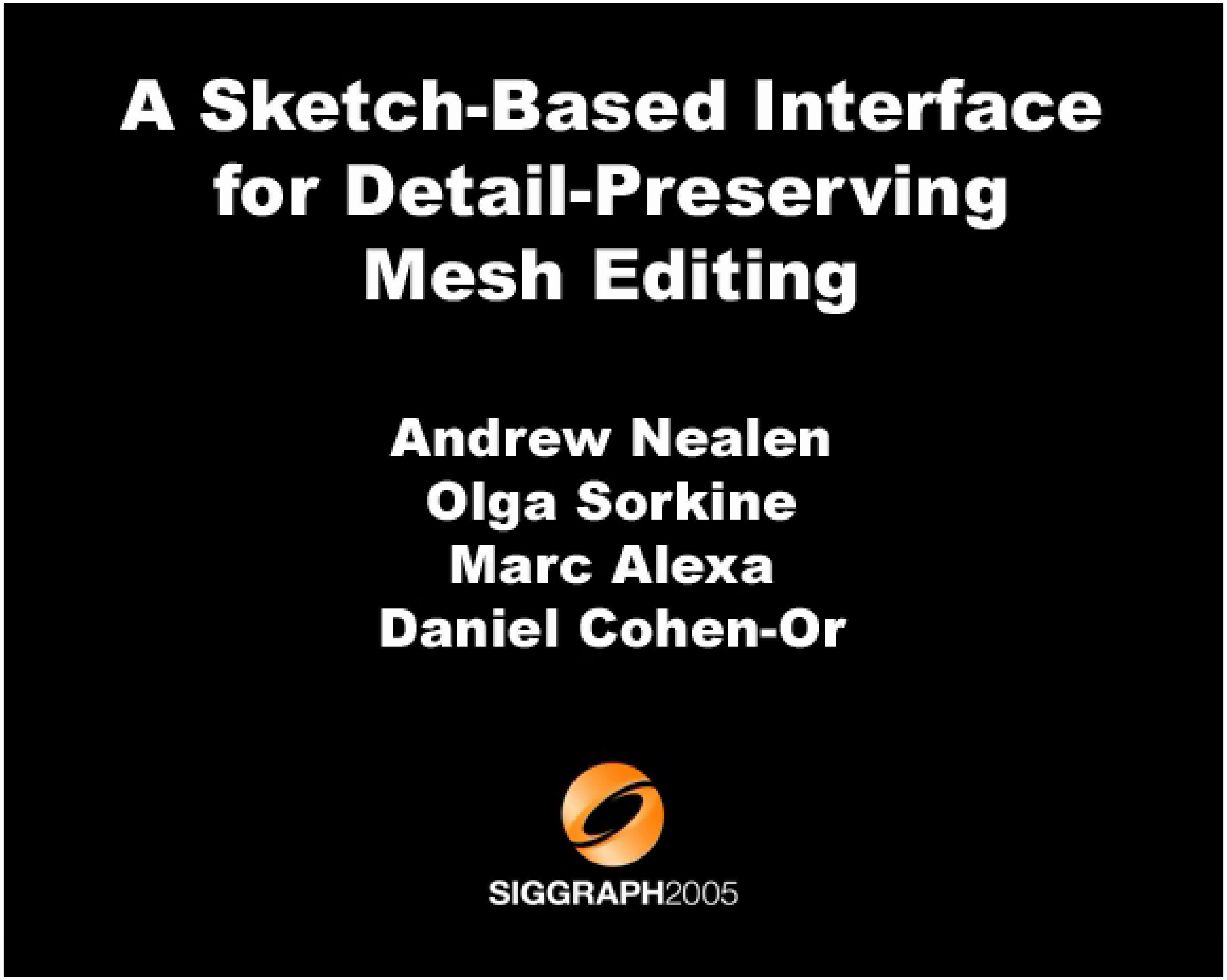
Laplacian Surface Editing

Laplacian Mesh Editing

A short editing session
with the *Octopus*

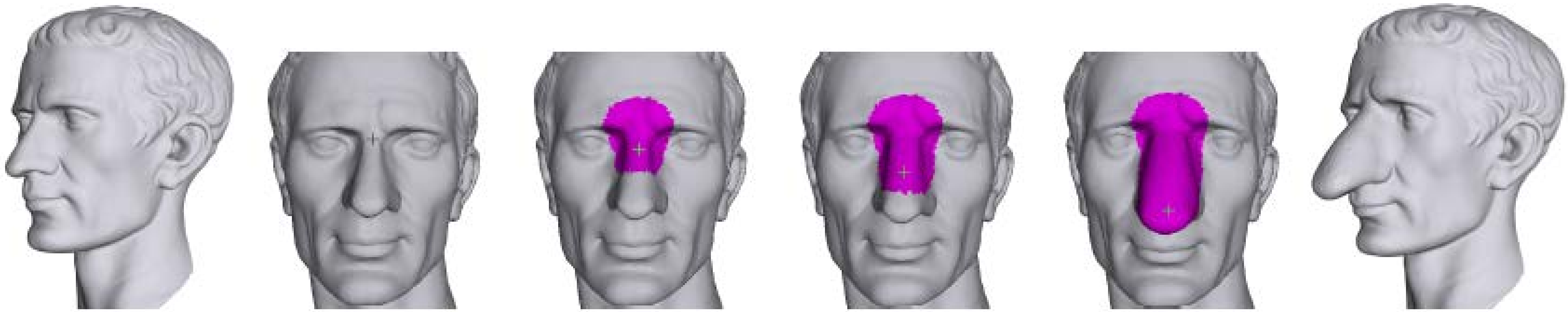
O. Sorkine, D. Cohen-Or, Y. Lipman, M. Alexa, C. Rössl, and H.-P. Seidel. 2004. Laplacian surface editing. In *Proceedings of the 2004 Eurographics/ACM SIGGRAPH Symposium on Geometry Processing*

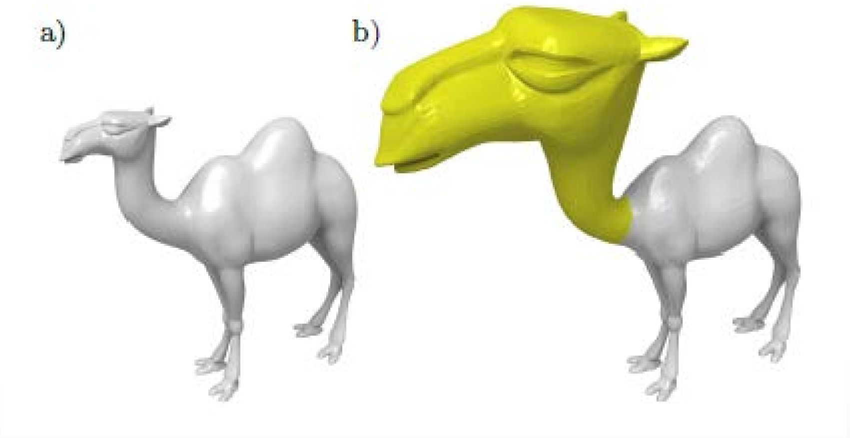
Sketching

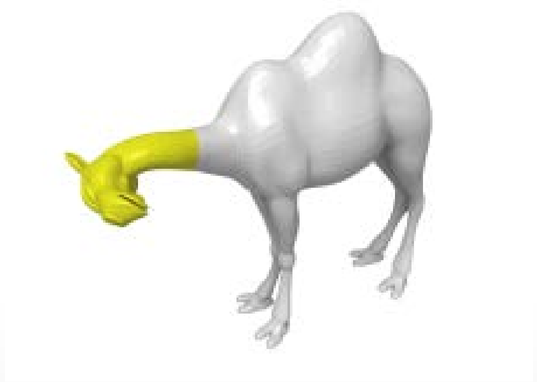


Andrew Nealen, Olga Sorkine, Marc Alexa, and Daniel Cohen-Or. 2005. A sketch-based interface for detail-preserving mesh editing. *ACM Trans. Graph.* 24, 3 (2005)

Brushes







Brushes

Gradient-Based Editing

- Modify the gradients of the embedding x:

$$Gx \xrightarrow[\text{gradients}]{\text{editing of}} \tilde{g}$$

- Find the displacement u such that the gradients of $\tilde{x} = x + u$ best match $\tilde{g}$
- Poisson reconstruction: Minimize

$$E_{PR}(\tilde{x}) = \frac{1}{2}\int_M \|\nabla\tilde{x} - \tilde{g}\|^2 dA$$

- Euler-Lagrange equation $\nabla E(\tilde{x}) = 0$ is

$$S\tilde{x} = G^T M_V \tilde{g}$$