Geometric Modeling

2015

Shape Interpolation, Parametrization

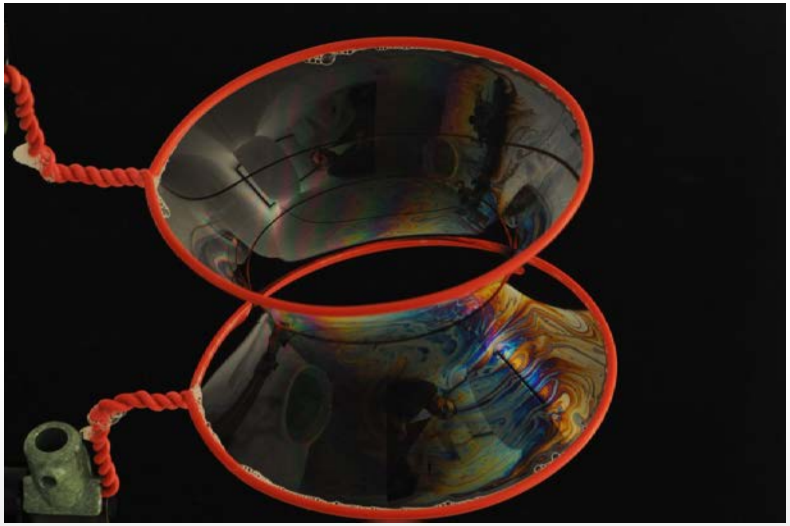
TU Delft

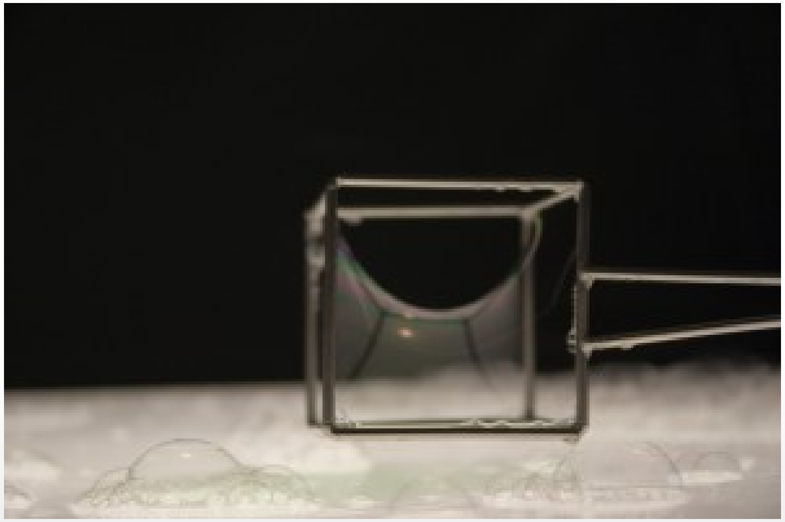
Last Lecture

Examples

Minimal Surfaces

- Surfaces with vanishing mean curvature are called *minimal surfaces*
- They are saddle shaped at every point
- Solution of Plateau’s problem (soap films, minimal area)
  - Soap bubbles have constant mean curvature



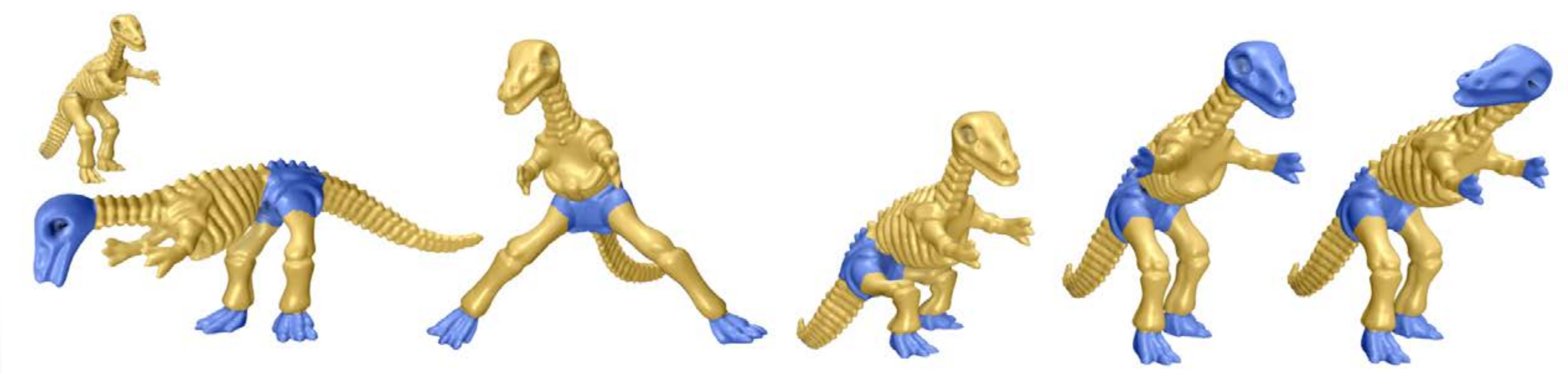


Deformation Energies

General deformation energies

- A deformation energy measures the “energy” stored in a deformation (or the “cost” of a deformation)

$$E : \underbrace{{S_h}^3}_{\text{Displacement}} \mapsto \underbrace{\mathbb{R}}_{\text{Energy}}$$



Constraints

To deform the object the user sets constraints

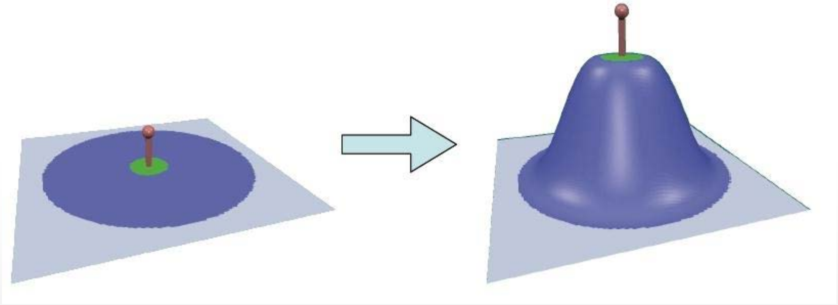
- Hard constraints:

$$Au = a$$

- Soft constraints:

$$E_C(u) = \frac{1}{2}\|Au - a\|^2$$

- A is a rectangular matrix, a is a vector
- Use masses for irregular meshes



Quadratic Program

Soft constraints

- Minimize weighted sum of deformation energy E_L and constraints energy E_C over all displacements $u \in {S_h}^3$
  - $\lambda \in \mathbb{R}_{>0}$

$$E(u) = E_L(u) + \lambda E_C(u)$$

- Necessary condition for a minimum u^* is $\nabla E(u^*) = 0$
- Since E is quadratic and positive definite, this is also a sufficient condition

$$\nabla E(u) = (SM^{-1}S + \lambda A^T A)u - \lambda A^T a$$

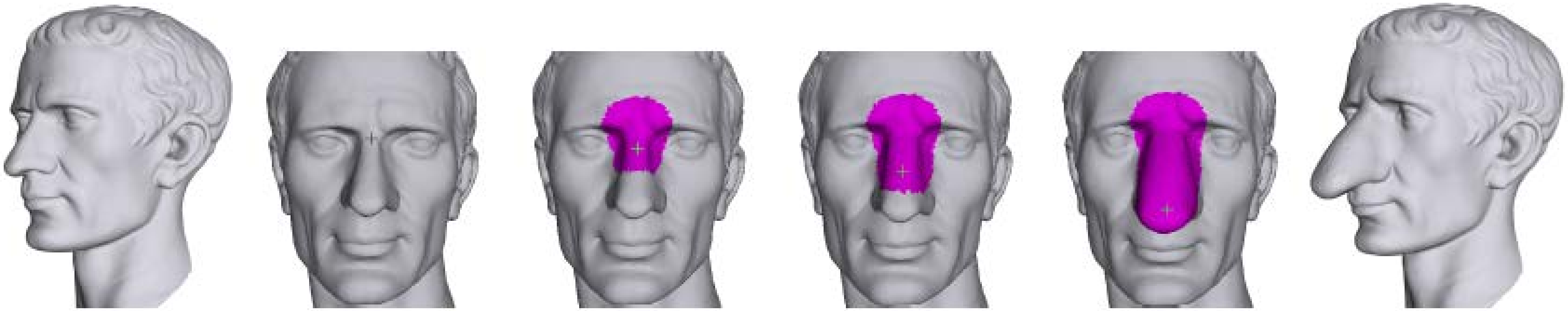
Computing the Deformation

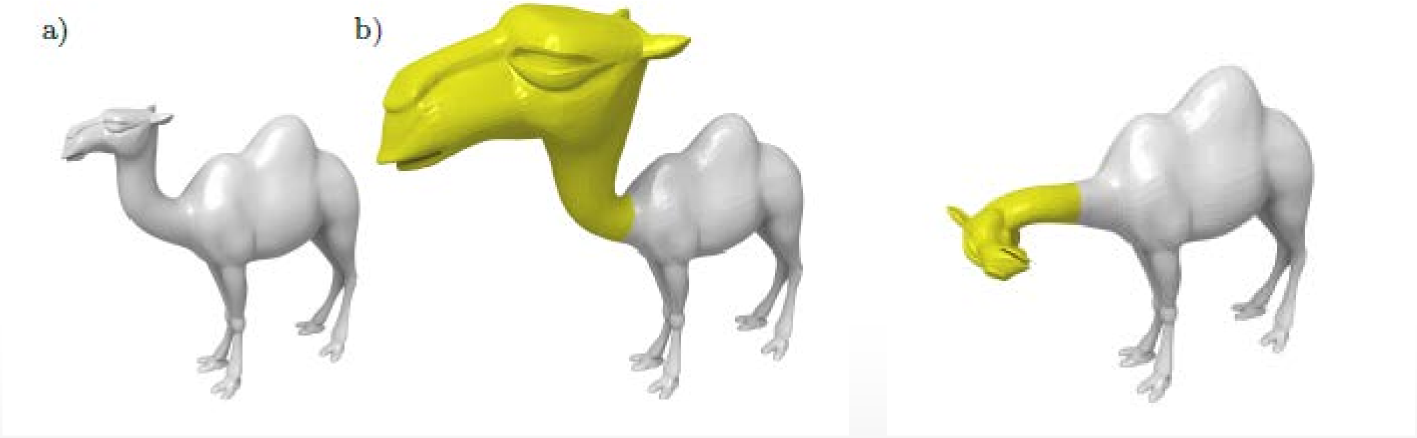
Linear system

- To compute the deformation, the linear system
$$(SM^{-1}S + \lambda A^T A)u = \lambda A^T a$$
has to be solved

- The matrix $(SM^{-1}S + \lambda A^T A)$ is
  - sparse
  - symmetric, positive definite
- An efficient solver is a sparse Cholesky decomposition
- Since changing the positions of the handles only changes the right-hand side, the factorization can be re-used and interactive modeling is possible

Brushes





Brushes

Gradient-Based Editing

- Modify the gradients of the embedding x:

$$Gx \xrightarrow[\text{gradients}]{\text{editing of}} \tilde{g}$$

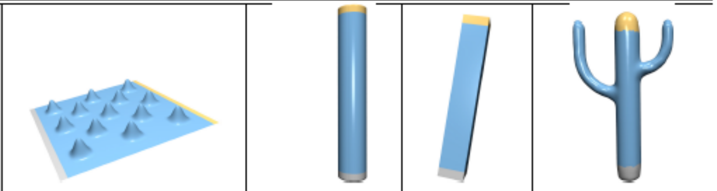
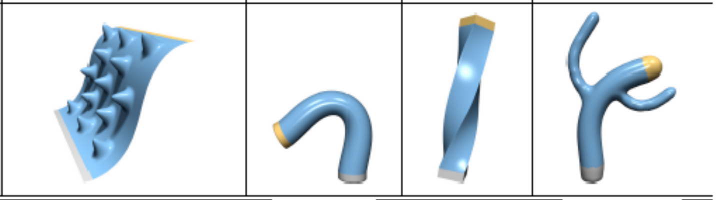
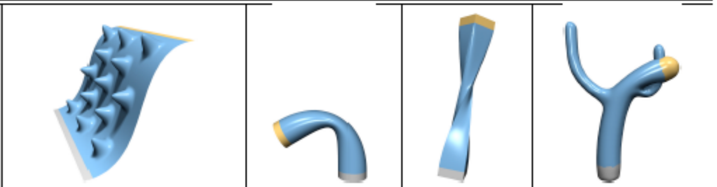
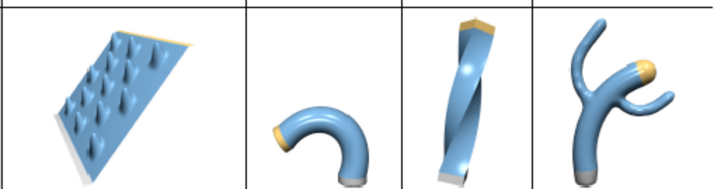
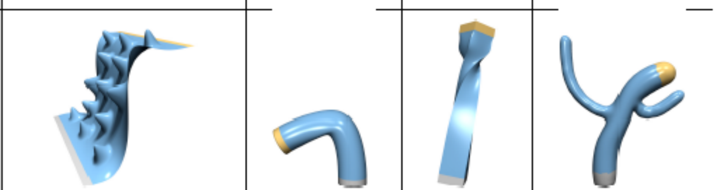
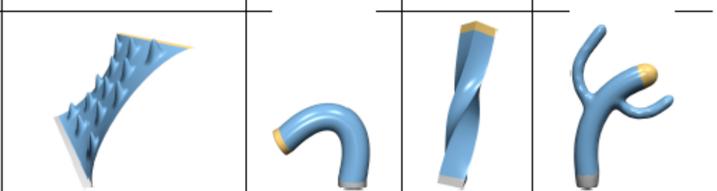
- Find the displacement u such that the gradients of $\tilde{x} = x + u$ best match $\tilde{g}$
- Poisson reconstruction: Minimize

$$E_{PR}(\tilde{x}) = \frac{1}{2}\int_M \|\nabla\tilde{x} - \tilde{g}\|^2 dA$$

- Euler-Lagrange equation $\nabla E(\tilde{x}) = 0$ is

$$S\tilde{x} = G^T M_V \tilde{g}$$

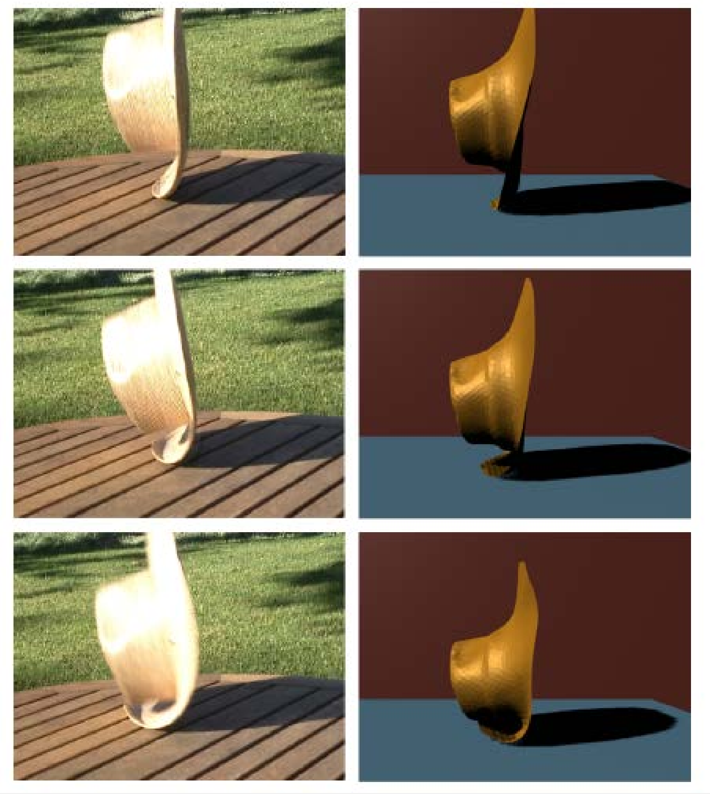
Limitation: Large Deformations

Approach	Pure Translation	120° bend	135° twist	70° bend
Original model				
Non-linear prism-based modeling [12]				
Thin shells [10] + deformation transfer [14]				
Gradient-based editing [72]				
Laplacian-based editing with implicit optimization [60]				
Rotation invariant coordinates [42]				

Non-Linear Energies

Discrete thin shell energy [Grinspun et al. 03]:

$$\underbrace{E_{TS} = E_F}_{\text{Flexural Energy}} + \lambda(\underbrace{E_A + E_L}_{\text{Membrane Energy}})$$



Discrete Shells

Discrete thin shell energy [Grinspun et al. 03]:

$$E_{TS} = \underbrace{E_F}_{\text{Flexural Energy}} + \lambda\,\underbrace{(E_A + E_L)}_{\text{Membrane Energy}}$$

- Flexural energy:

$$E_F = \frac{3}{2}\sum_i \frac{\overbrace{\|\bar{e}_i\|^2}^{\text{length of } e_i}}{\underbrace{\bar{A}_{e_i}}_{\text{area of star } e_i}}\left(\underbrace{\theta_{e_i}}_{\text{dihedral angle}} - \bar{\theta}_{e_i}\right)^2$$

- Membrane energy:

$$E_L = \frac{1}{2}\sum_i \frac{1}{\|\bar{e}_i\|}(\|e_i\| - \|\bar{e}_i\|)^2$$

$$E_A = \frac{1}{2}\sum_i \frac{1}{\bar{A}_i}(A_i - \bar{A}_i)^2$$

Non-Linear Deformations

Further Reading

Survey on linear editing schemes

BOTSCH, M. AND SORKINE, O. 2008. On linear variational surface deformation methods. *IEEE Transactions on Visualization and Computer Graphics 14*, 1, 213–230.

Interactive non-linear editing

JACOBSON, A., BARAN, I., KAVAN, L., POPOVIĆ, J., AND SORKINE, O. 2012. Fast automatic skinning transformations. *ACM Trans. Graph. 31*, 4, 77:1–77:10.

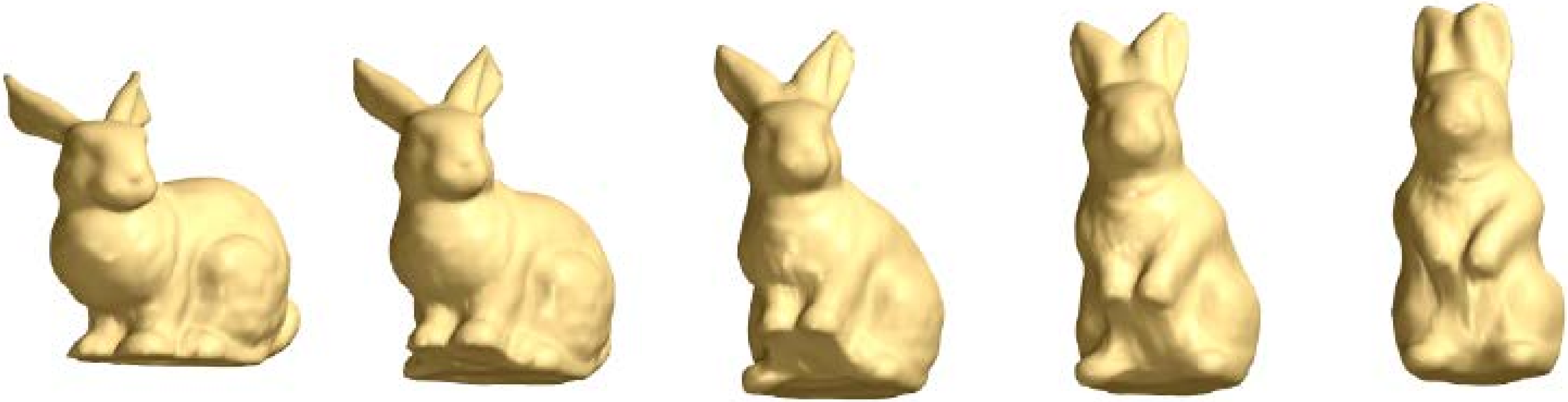
HILDEBRANDT, K., SCHULZ, C., VON TYCOWICZ, C., AND POLTHIER, K. 2011. Interactive surface modeling using modal analysis. *ACM Trans. Graph. 30*, 5, 119:1–119:11.

Shape Interpolation

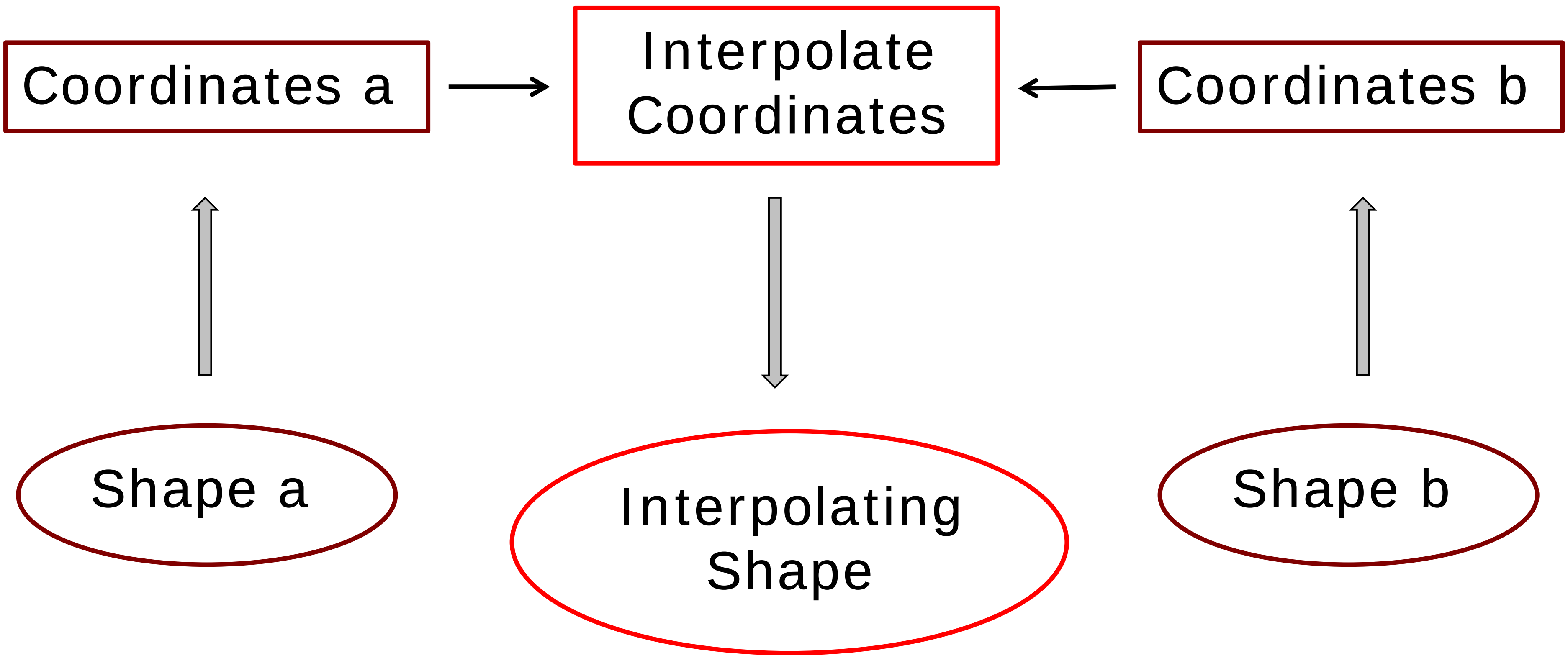
Shape Interpolation

Blending of bunny and rabbit

- Both shapes have the same mesh (hence a correspondence between the shapes is given)



General Framework



Interpolation

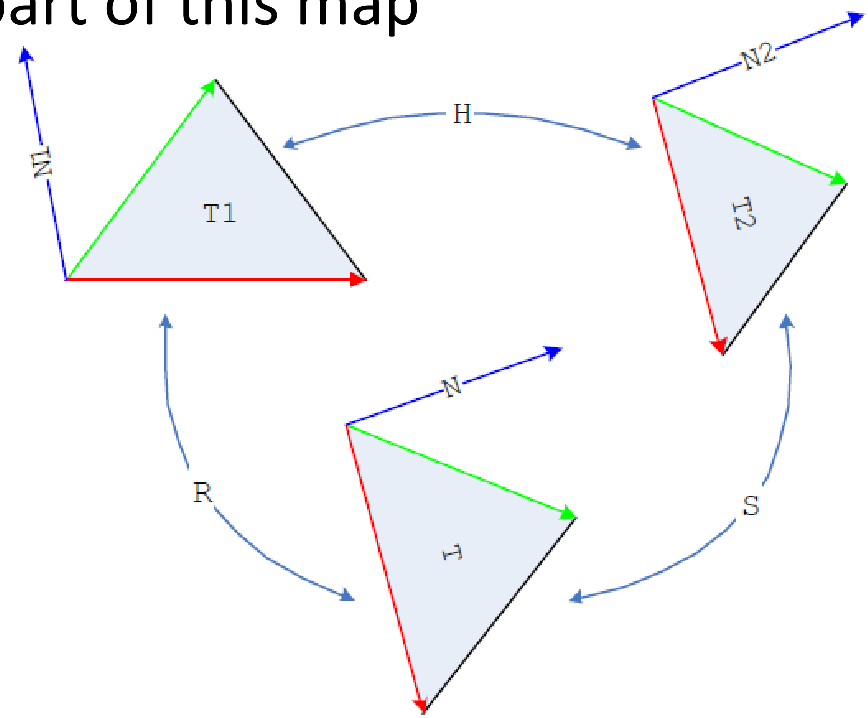
Interpolation between two triangles

- For every pair T_1, T_2 of corresponding triangles on the two shapes, there is a unique affine map of $\mathbb{R}^3$ mapping T_1 to T_2 and the normal N_1 to the normal N_2
- Let H denote the linear part of this map
- Polar decomposition:

  $H = R\ S$

  R is a rotation and

  S is symmetric



SVD & Polar Decomposition

Singular value decomposition (SVD)

- Let H be an arbitrary real matrix (may be rectangular)
- Then H can be written as:
  - $H = U D V^T$
  - The matrices U, V are orthogonal
  - D is a diagonal matrix (might contain zeros)

Polar decomposition from SVD

- $H = U DV^T = UV^TV DV^T = R S$
  - $R = UV^T$
  - $S = V DV^T$

Polar Decomposition

Alternative Computation (when H has full rank)

- Compute eigenvalues and –vectors of $H^T H$
$$H^T H = W^T \Lambda W$$
  - W: orthognal matrix listing the eigenvectors
  - Λ: diagonal matrix listing the eigenvalues
- Set $S = W^T \sqrt{\Lambda}\, W$
  - $\sqrt{\Lambda}$ diagonal matrix listing the square roots of the eigenvalues
- Set $R = HS^{-1}$
  - Then $H = RS$

Why is R orthogonal?

- $R^T R = (HS^{-1})^T HS^{-1} = \cdots = Id$

Reminder: Rotation Axis and Angle

Rotation axis and angle of a rotation matrix

- The eigenvector with eigenvalue 1 points in direction of the rotation axis
- The angle θ of a rotation matrix R satisfies

$$\mathrm{trace}(R) = 1 + 2\cos(\theta)$$

Hence $\theta = \cos^{-1}\left(\frac{\mathrm{trace}(R)-1}{2}\right)$

Interpolate Triangles

Interpolate Rotations

- To interpolate the rotation R,
  we fix the rotation axis and linearly
  interpolate the angle
- To interpolate H:

$$H(t) = \underbrace{R(t)}_{\text{Interpolate the rotation over [0,1]}}((1-t)Id + t\,S)$$

- Interpolate the translation by barycentric interpolation

Poisson Shape Interpolation

Interpolating the surfaces

- Using $H(t)$, we obtain an interpolation of all pairs triangles of the two surfaces, but the interpolating triangles are not connected (triangle soup)
- Problem: Find are connected surface that best matches the unconnected triangle soup

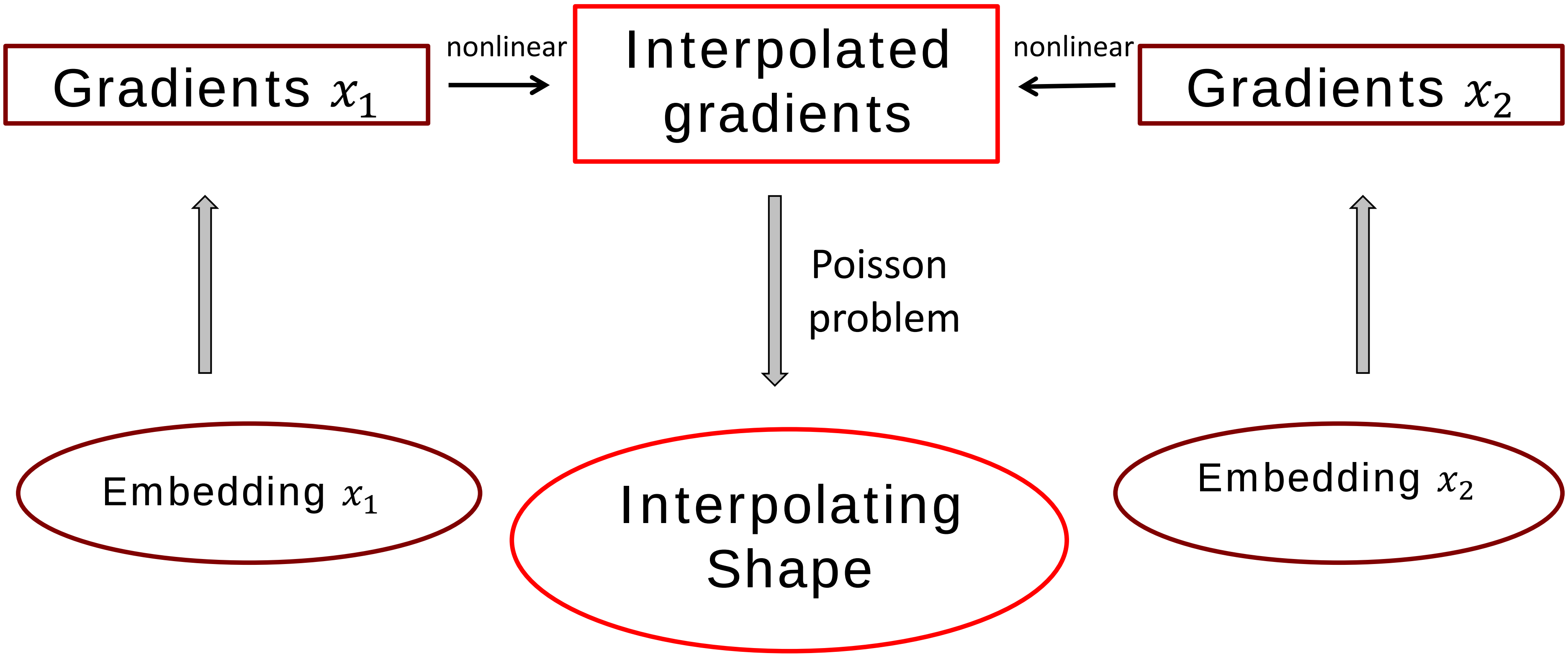
- Idea: Use Poisson reconstruction
  - Compute gradients of the interpoating triangles and construct a surface that best matches the gradients

Poisson Shape Interpolation

Representing the shape as functions

- One triangle mesh M and two embeddings x_1 and x_2
  - M can be one of the shapes

Poisson Shape Interpolation



[Xu, Zhang, Wang, Bao 2005]

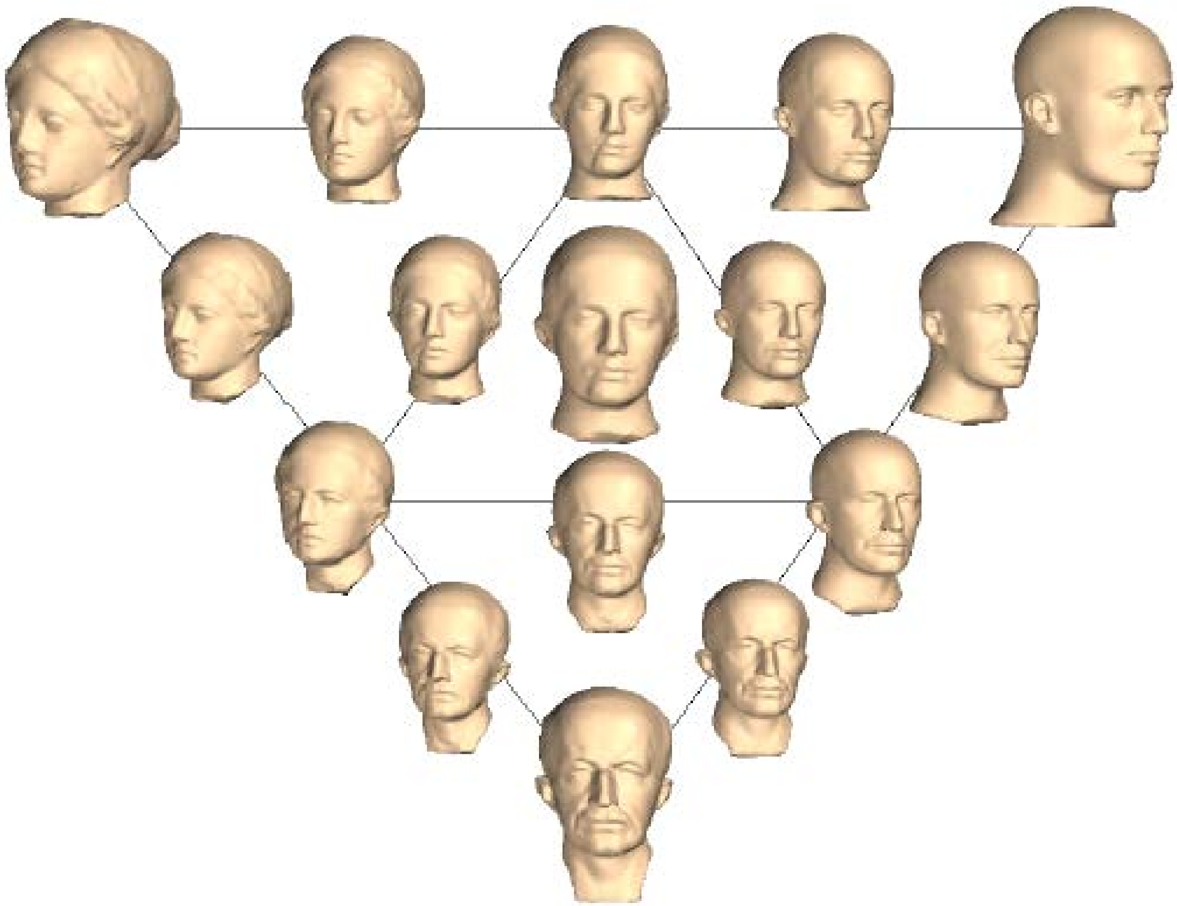
Poisson Shape Interpolation

Poisson reconstruction

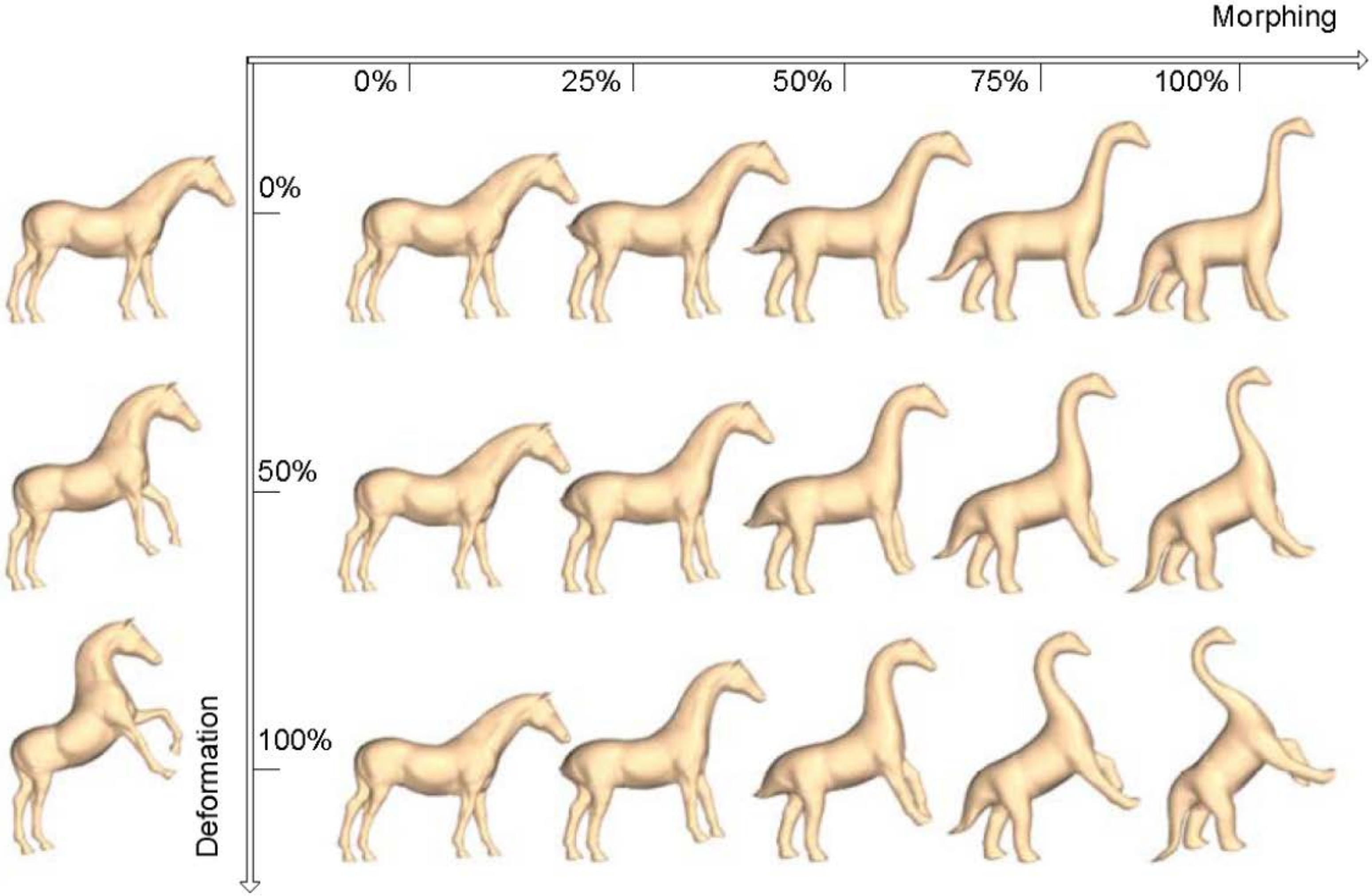
- For any t, we can reconstruct a shape $x(t)$ that best matches the gradients $g(t)$
- Minimize
$$E_{PR}(\mathrm{x}) = \frac{1}{2}\int_M \|\nabla x - g\|^2 dA$$
- Euler-Lagrange equation $\nabla E(x) = 0$ is
$$Sx = G^T M_V g$$
- This means that the $x(t)$ satisfy
$$Sx(t) = G^T M_V g(t)$$

Results

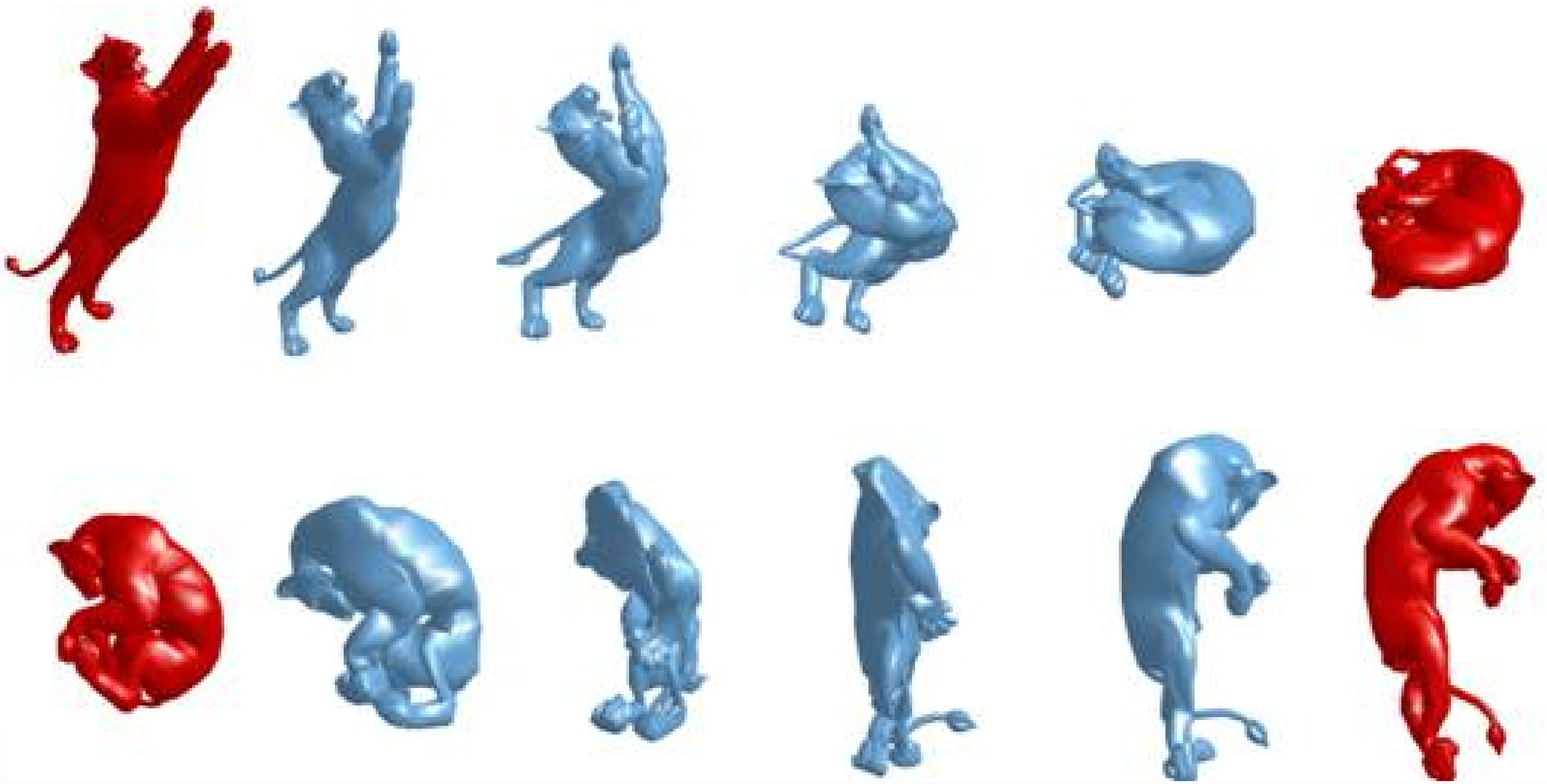
Results



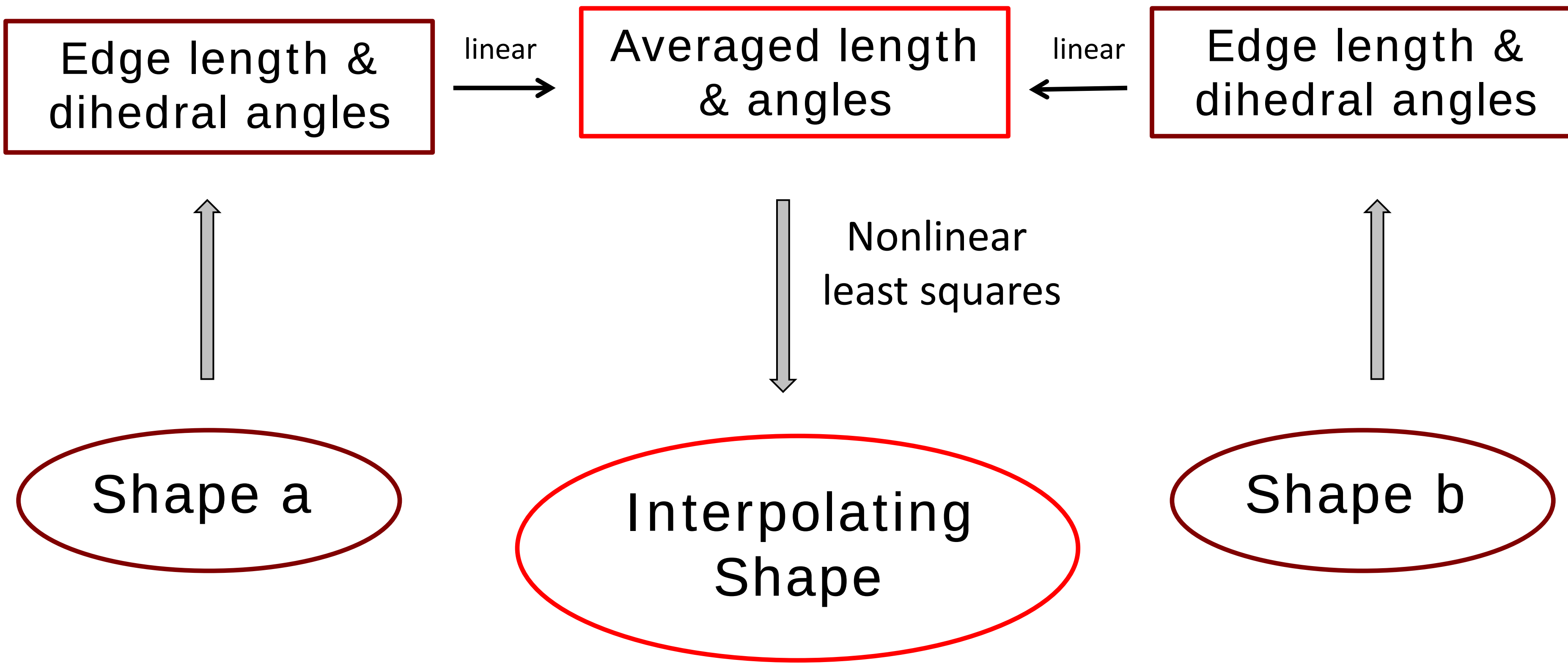
Deformation Transfer



Limitations

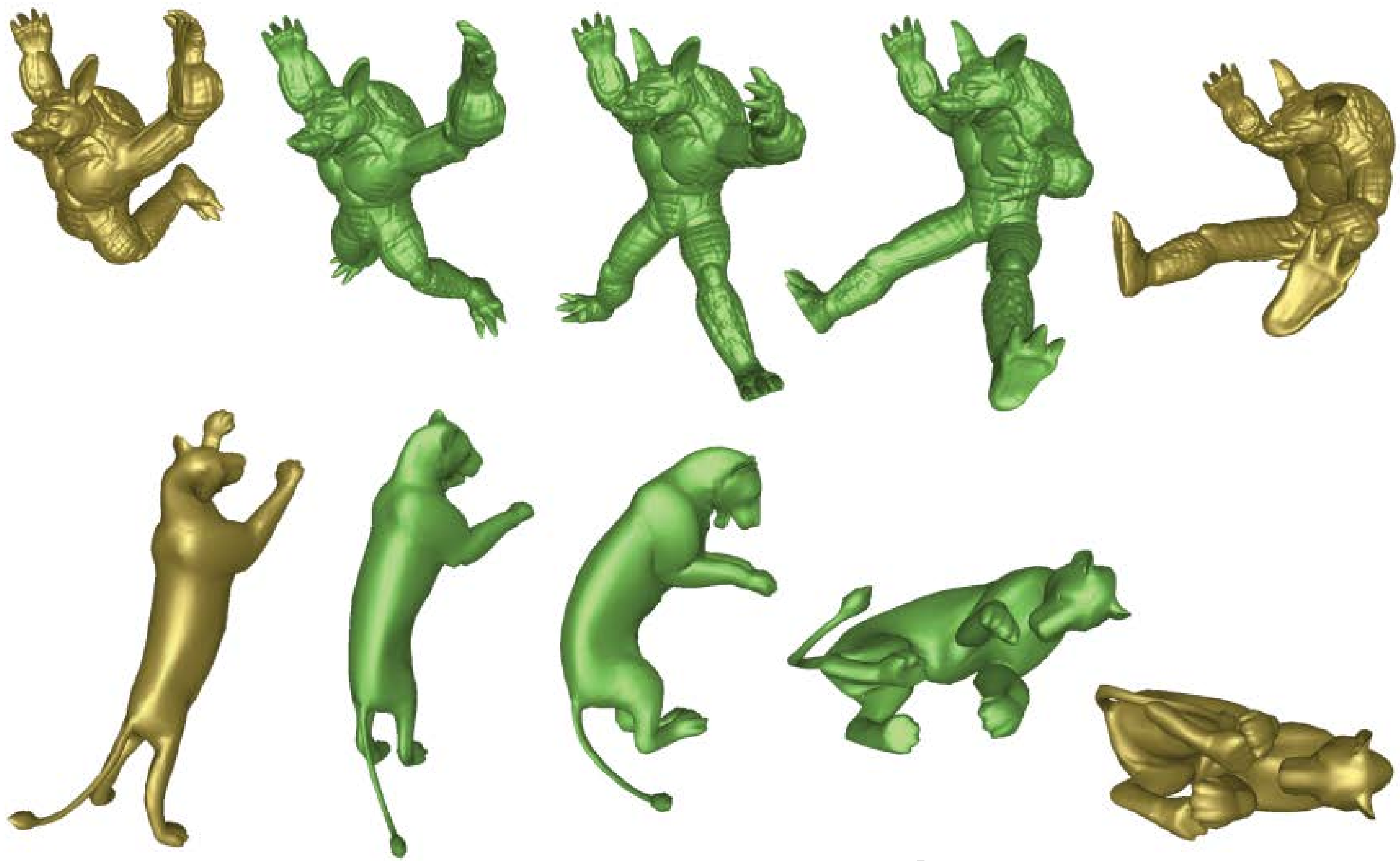


Dihedral Angles and Edge Length



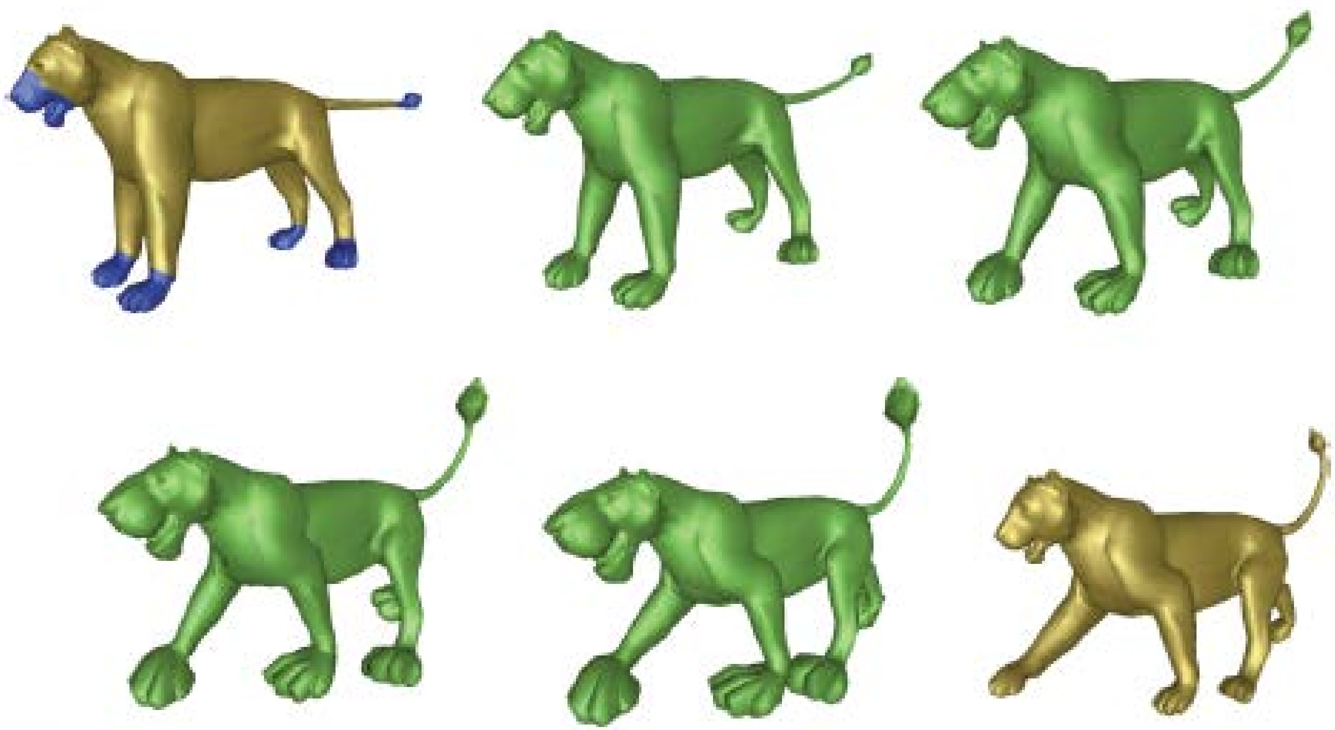
[Winkler, Driesenberg, Hormann, Alexa 2010]

Results



[Winkler, Driesenberg, Hormann, Alexa 2010]

Results



[Winkler, Driesenberg, Hormann, Alexa 2010]

Extrapolation

[Winkler, Driesenberg, Hormann, Alexa 2010]

Real-Time Nonlinear Shape Interpolation

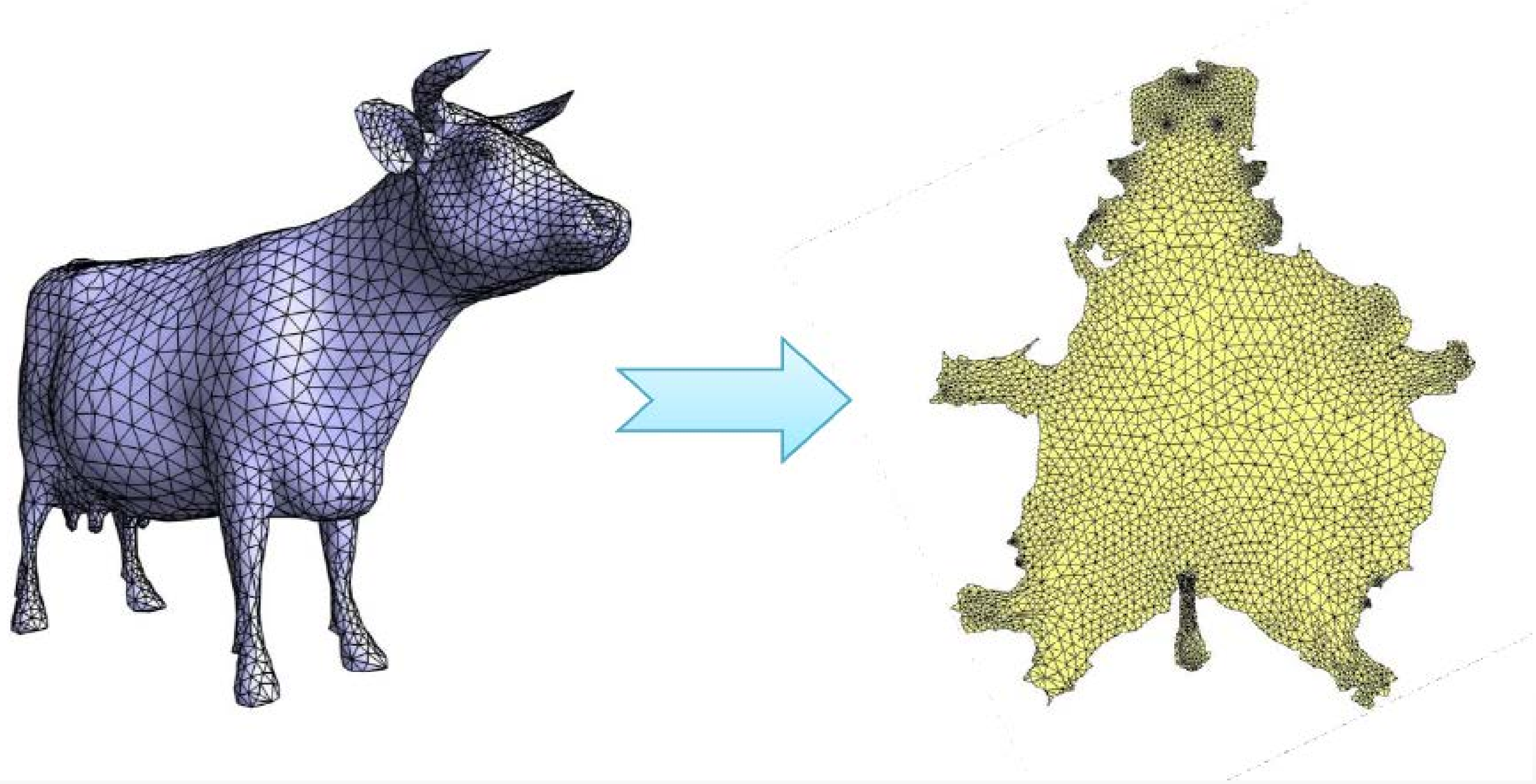
Real-time Nonlinear Shape Interpolation

(no audio)

Real-Time Nonlinear Shape Interpolation
Christoph von Tycowicz, Christian Schulz, Hans-Peter Seidel, Klaus Hildebrandt
ACM Transactions on Graphics 34(3) 2015

Parametrization

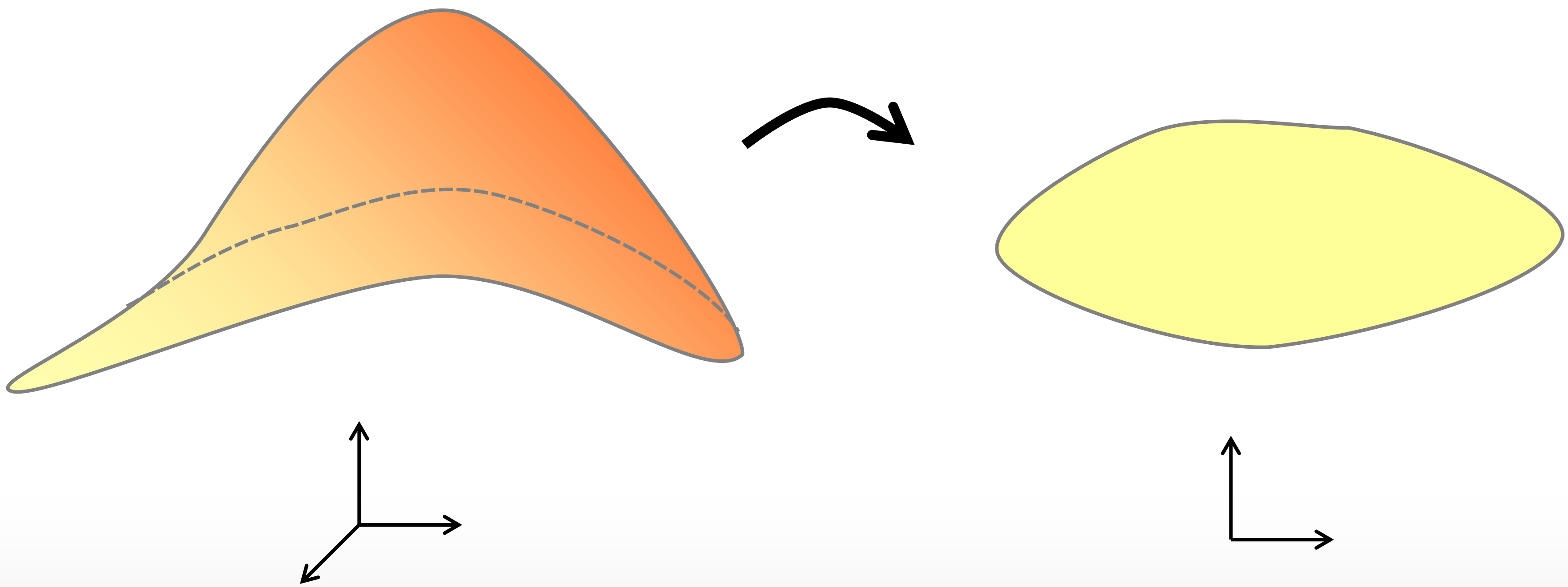
Surface Parameterization



Surface Parameterization

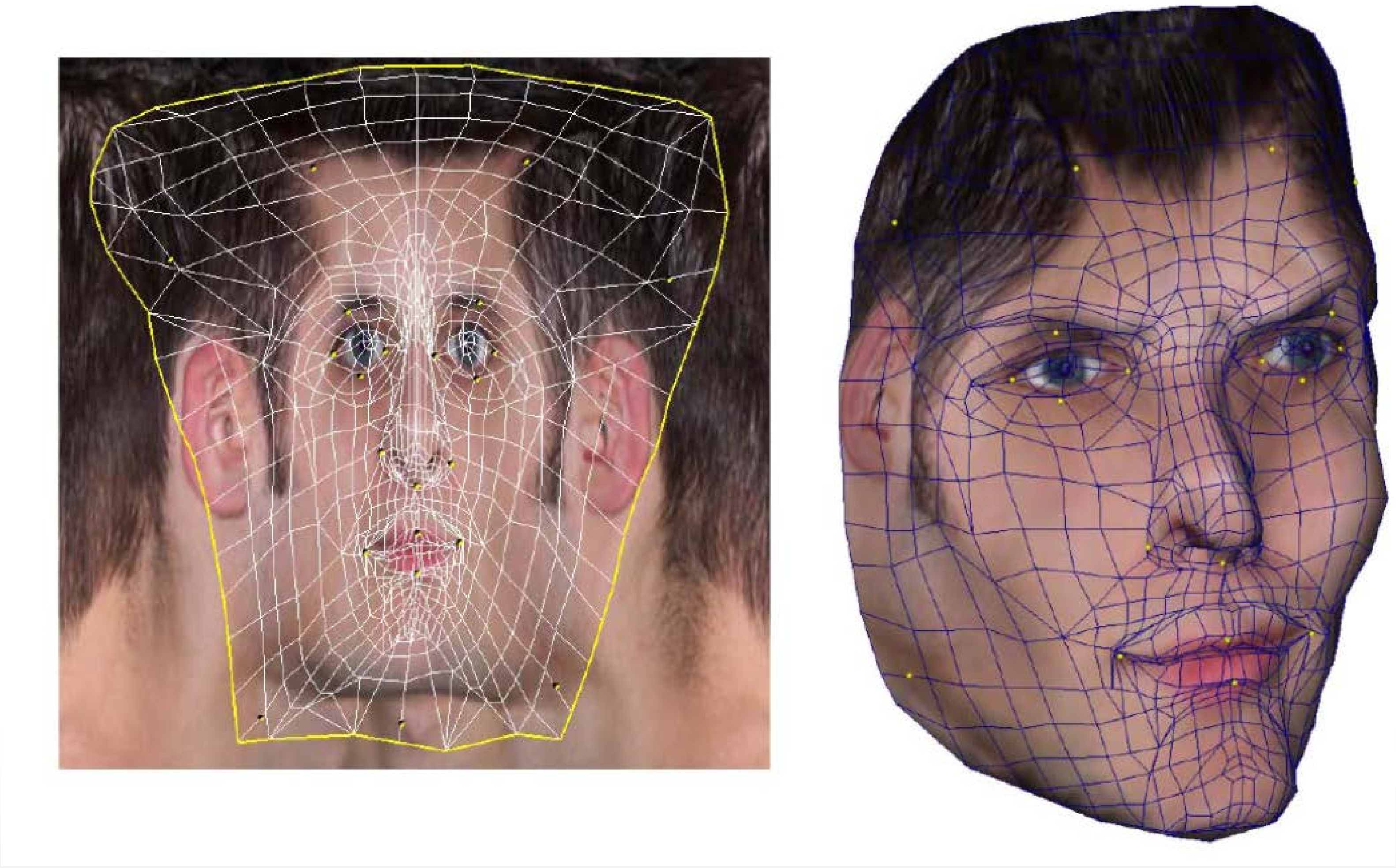
Problem

- Given a surface (mesh), construct a (good, useful) bijective map from the surface to the plane

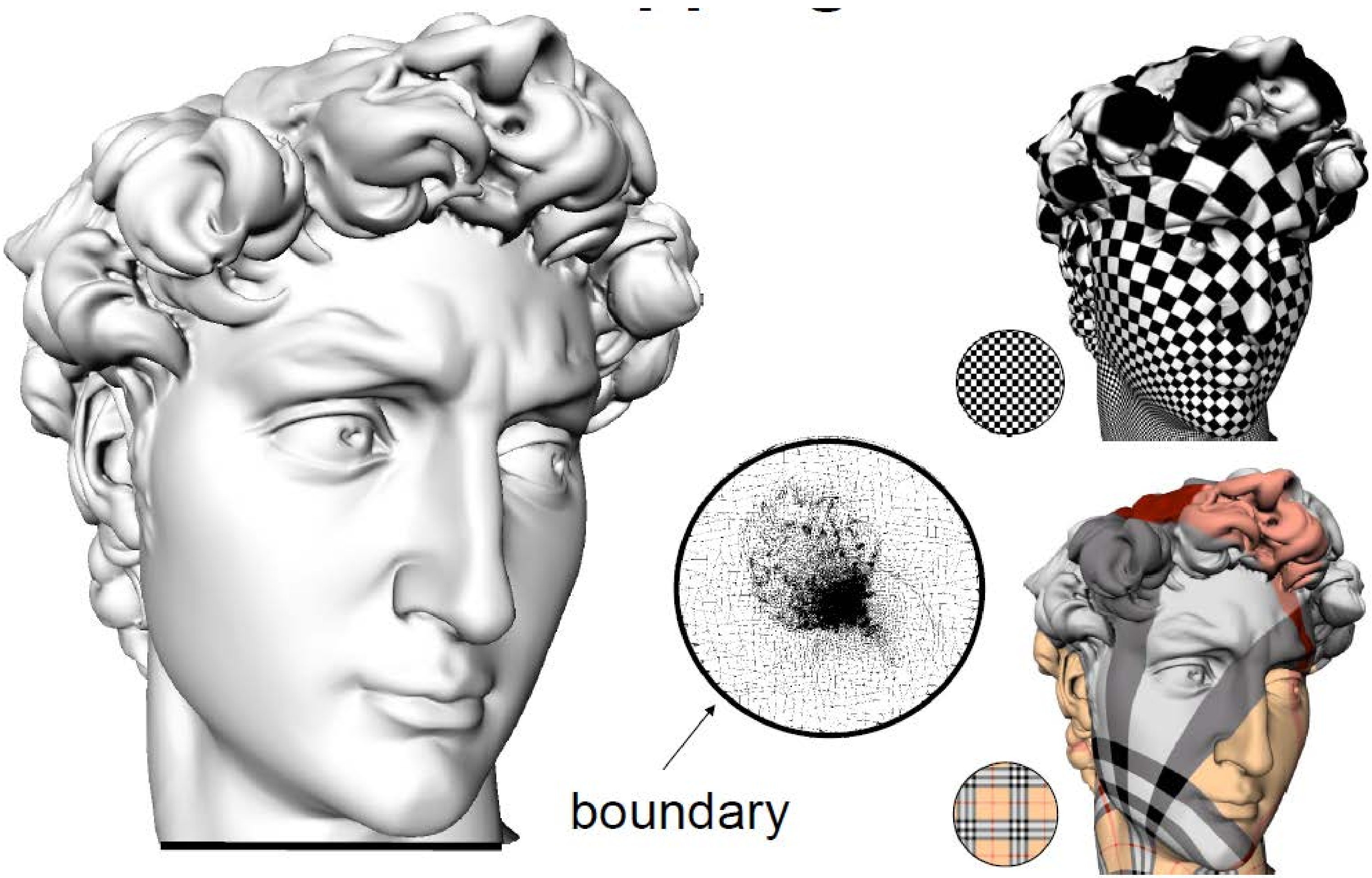


Motivation

Texture mapping



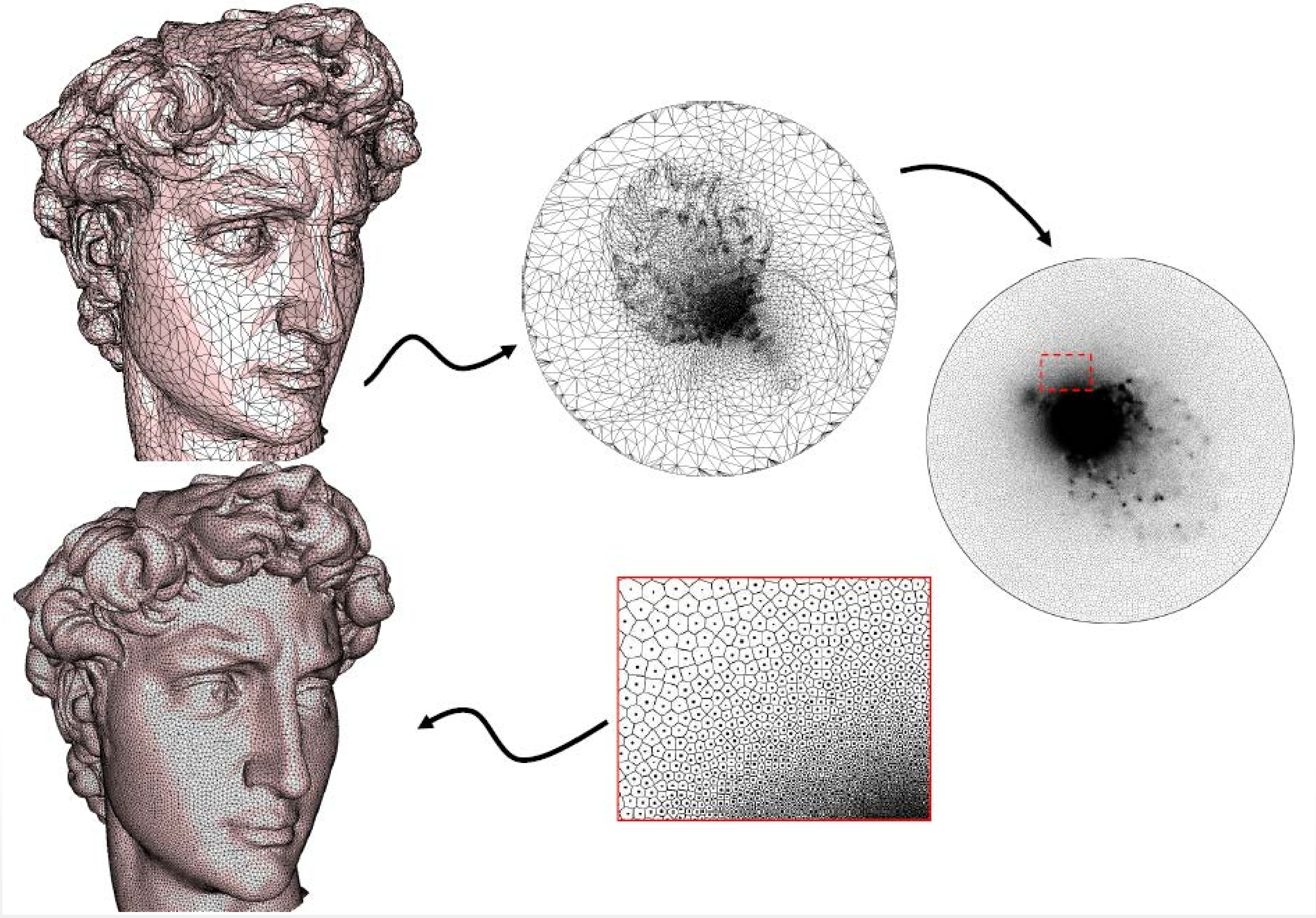
Motivation



Motivation

Operations in 2D are often simpler

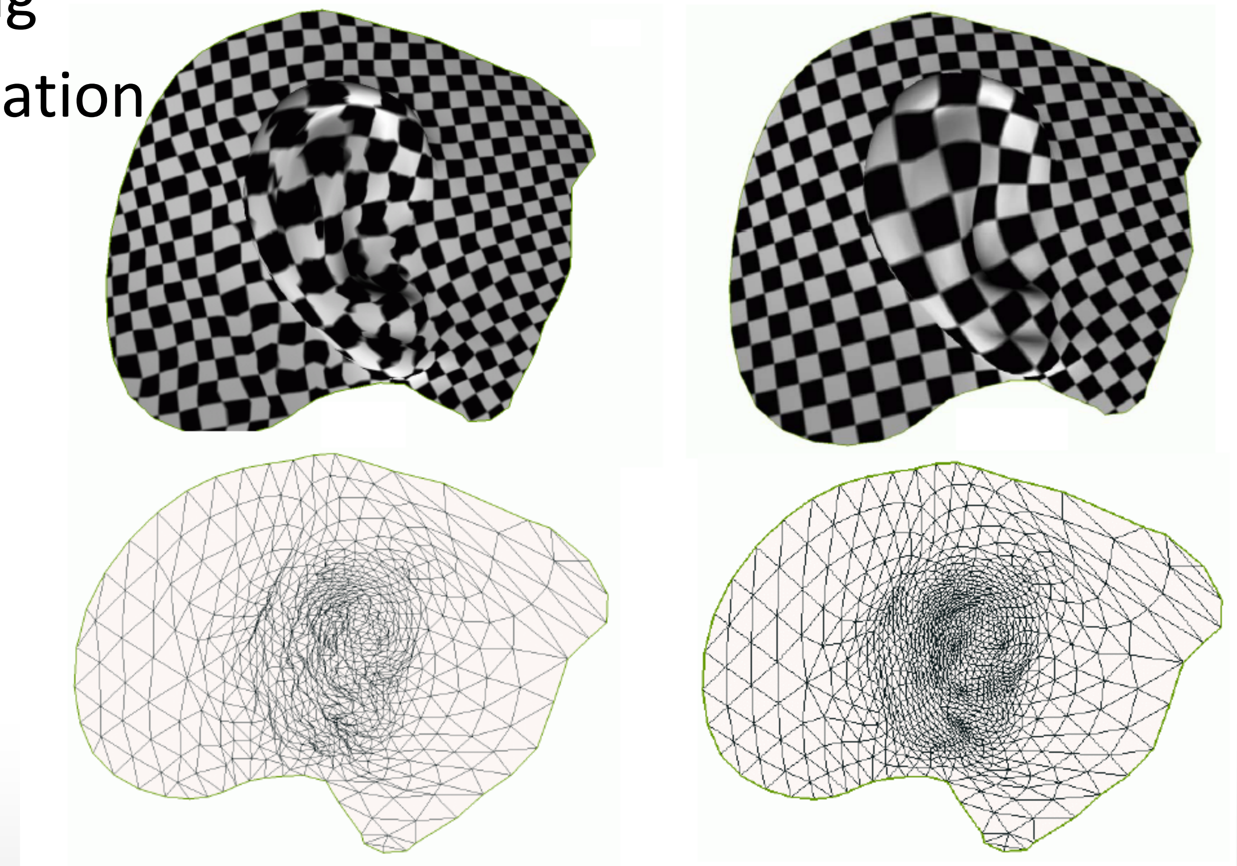
- Example: Remeshing ("good" parametrization is essential)



Parametrization

Desirable properties

- Low distortion
- Bijective mapping
- Efficient computation



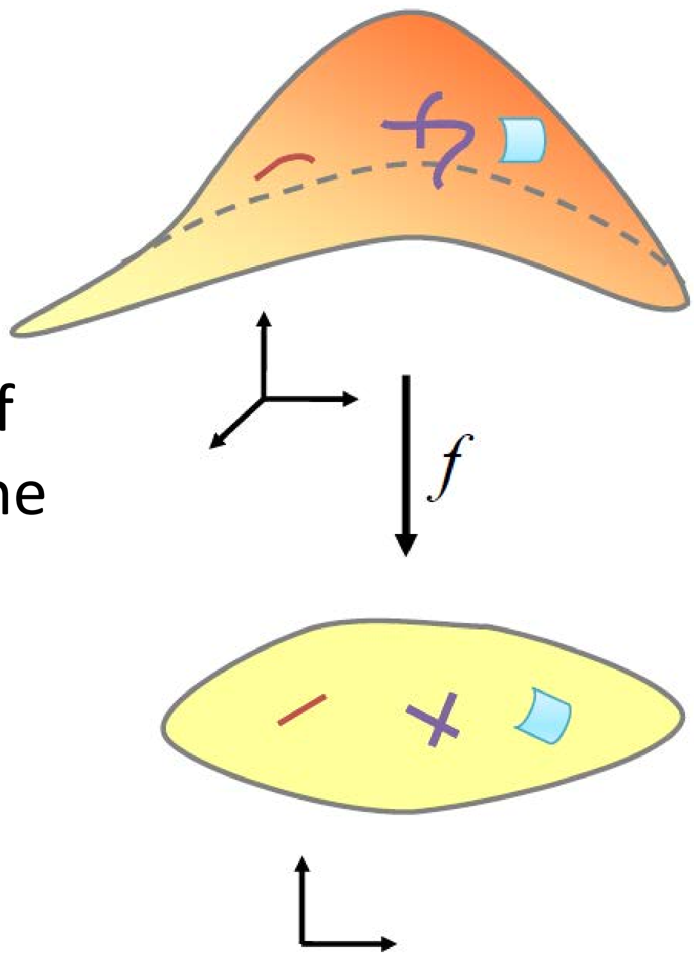
Parametrization

Setting

- Consider a map $f: M \mapsto \Omega \subset \mathbb{R}^2$

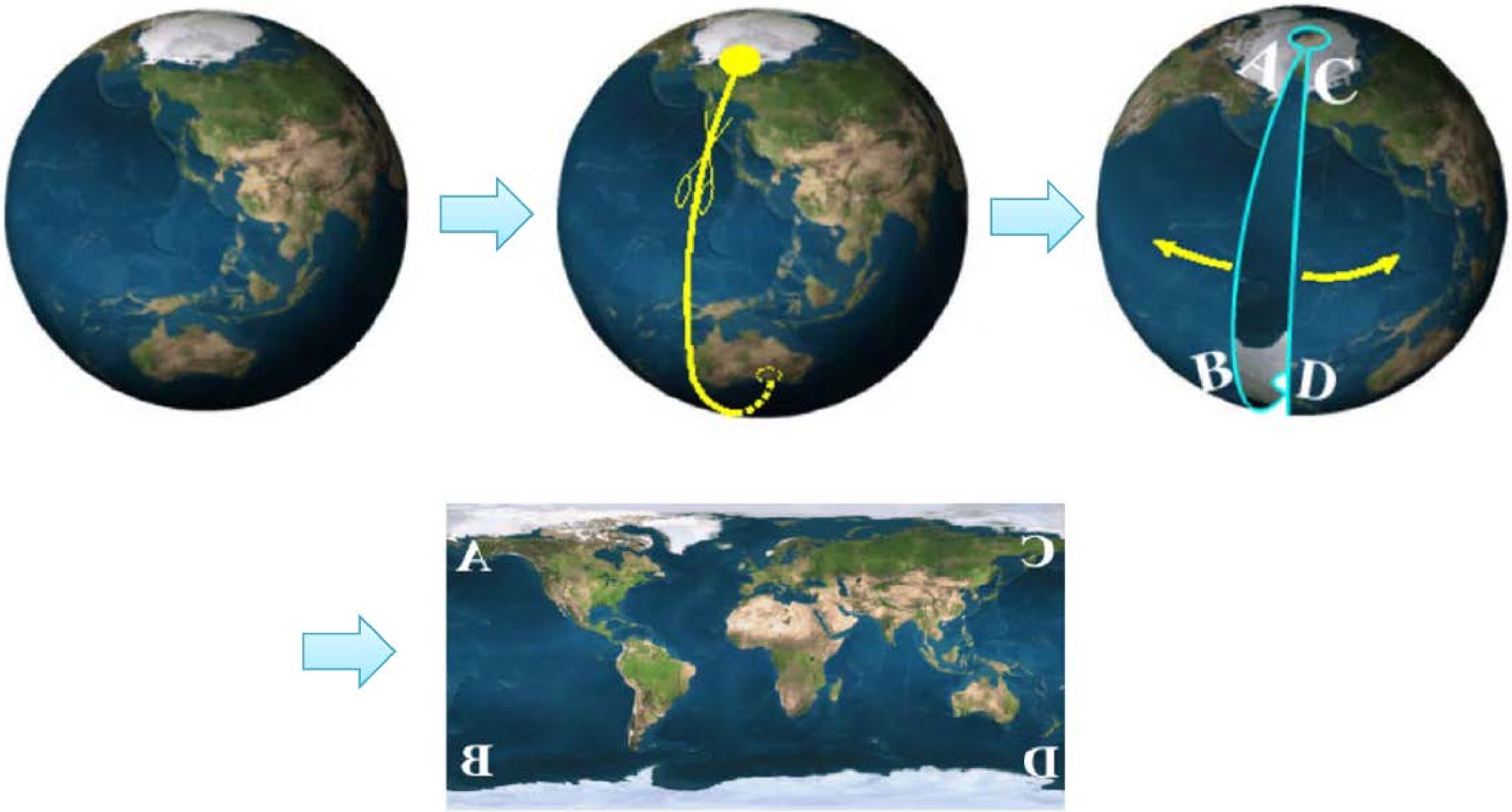
Properties of the map

- The map is *isometric* if the length of every curve c in M is the same as the length of the image $f(c)$ in Ω



Isometries

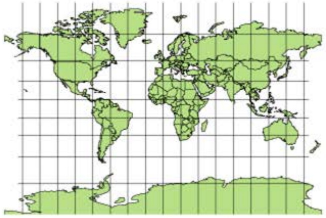
Is there always an isometry?



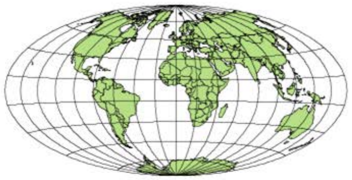
Maps of the Earth

Mollweide-Projektion

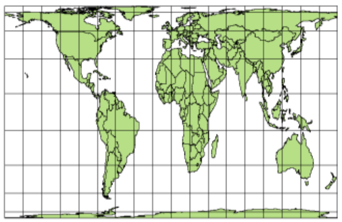
Mercator-Projektion



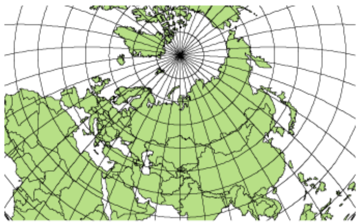
Zylinderprojektion nach Miller



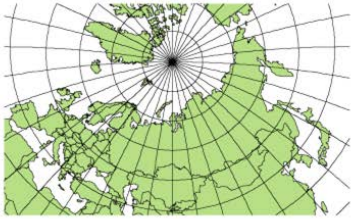
Hammer-Aitoff-Projektion



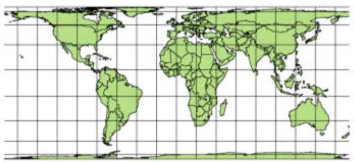
Peters-Projektion



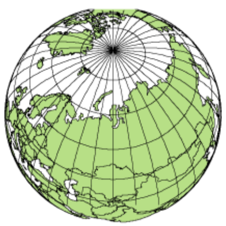
Längentreue Azimuthalprojektion



Stereographische Projektion

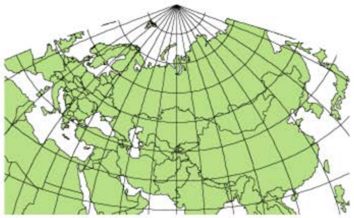


Behrmann-Projektion

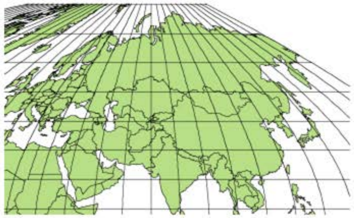


Senkrechte Umgebungsperspektive

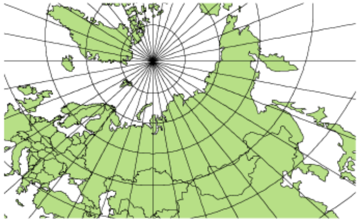
Robinson-Projektion



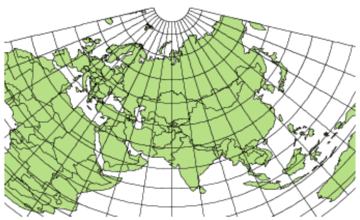
Hotine Oblique Mercator-Projektion



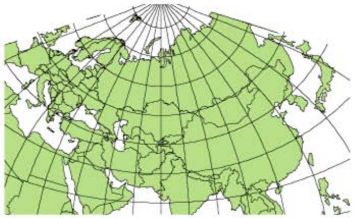
Sinusoidale Projektion



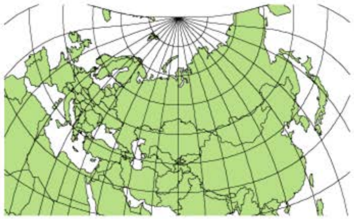
Gnomonische Projektion



Flächentreue Kegelprojektion



Transverse Mercator-Projektion



Cassini-Soldner-Projektion

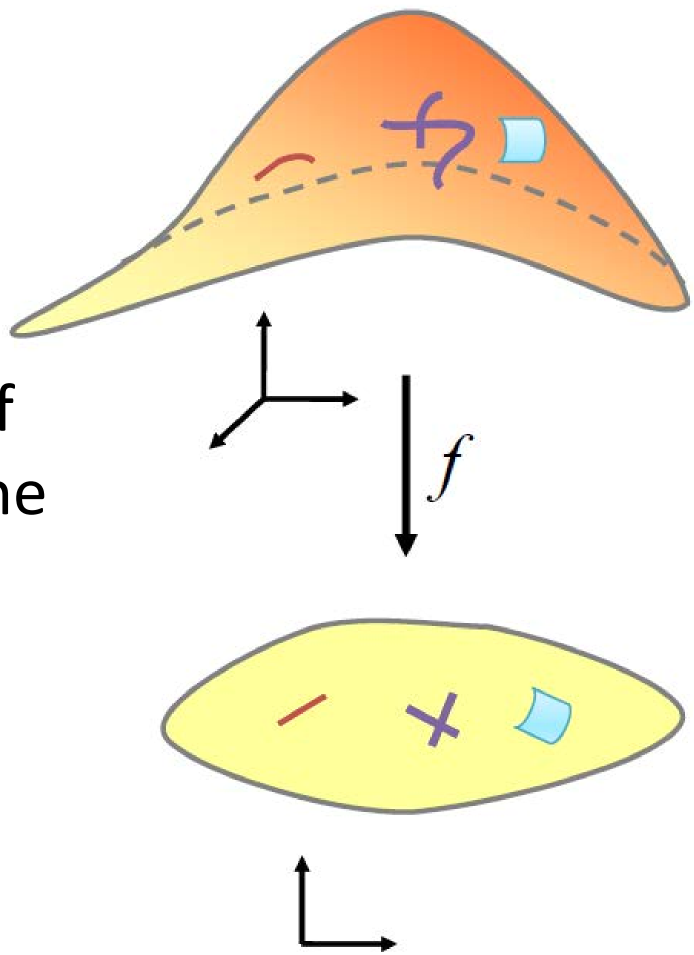
Parametrization

Setting

- Consider a map $f: M \mapsto \Omega \subset \mathbb{R}^2$

Properties of the map

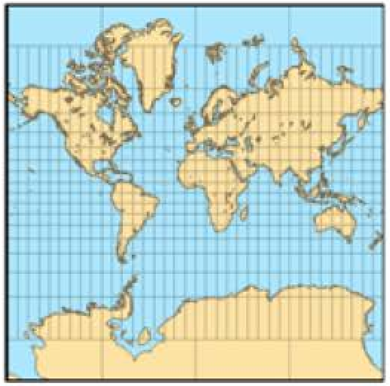
- The map is *isometric* if the length of every curve c in M is the same as the length of the image $f(c)$ in Ω
- The map is *conformal* if the angles between any pair of curves are preserved
- The map is *equiareal* if the area of every domain in M is the same as the area of its image



Maps of the Earth

orthographic

stereographic



Mercator

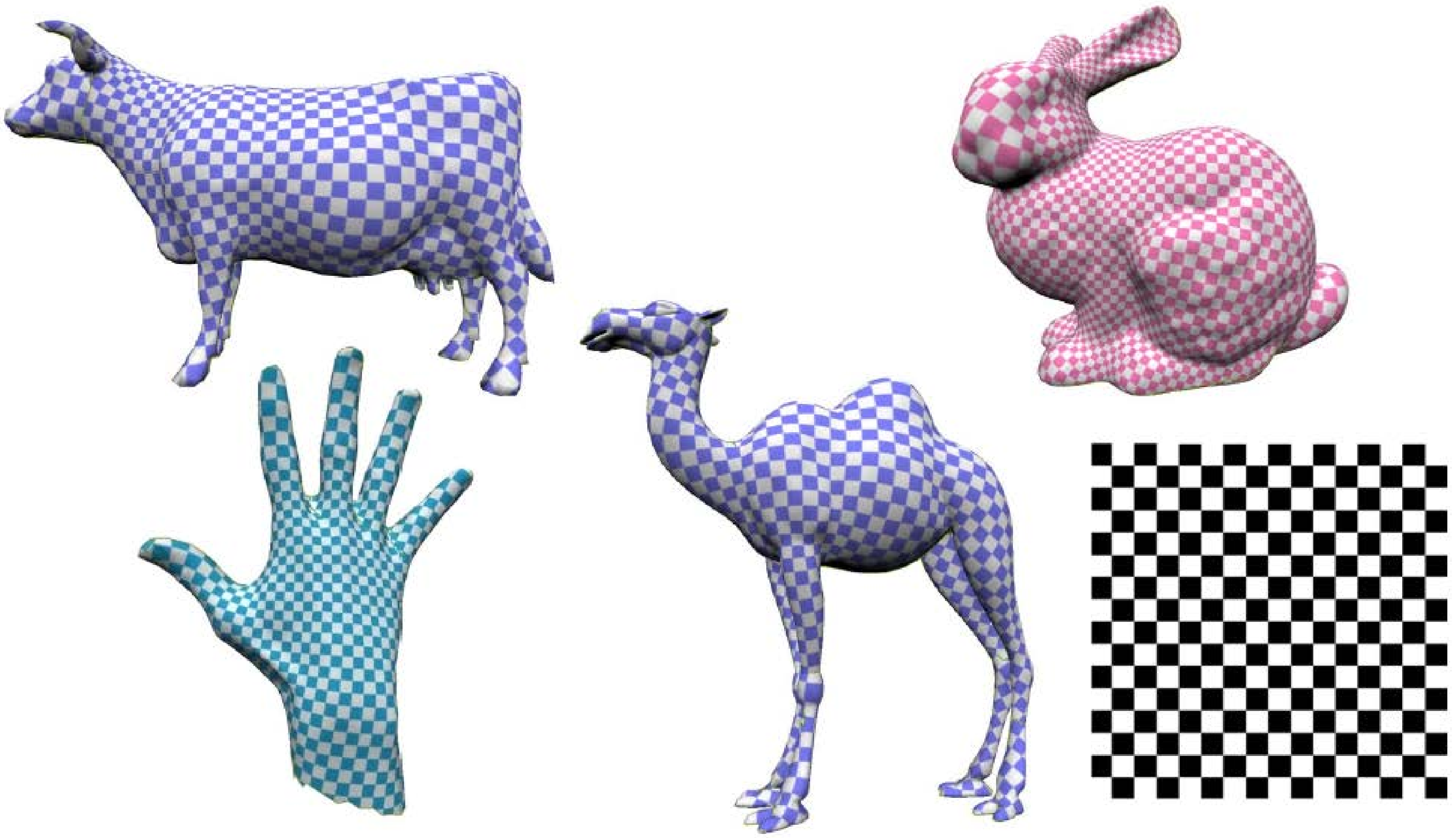
Lambert

preserves angles = **conformal**

preserves area = **equiareal**

Examples

Conformal maps



Characterization of the Properties

Notation

- $f(p) = \binom{u(p)}{v(p)}$, $N(p)$is the surface normal at $p \in M$

Properties

- f is an *isometry* if
  - $\langle \nabla u(p), \nabla v(p) \rangle = 0$ and $\|\nabla u(p)\| = \|\nabla v(p)\| = 1$

  holds for all p
  - This means the gradients $\nabla u(p), \nabla v(p)$ are orthonormal at all p
  - There is an orthogonal (e.g. rotation) that maps the standard basis in $\mathbb{R}^2$ to $\nabla u(p), \nabla v(p)$

Characterization of the Properties

Properties

- f is *conformal* if
  - $\nabla u(p) + N(p) \times \nabla v(p) = 0$ holds for all p
  - Then, $\langle \nabla u(p), \nabla v(p) \rangle = 0$ and $\|\nabla u(p)\| = \|\nabla v(p)\|$
  - There is a map that combines a rotation and scaling and maps the standard basis in $\mathbb{R}^2$ to $\nabla u(p), \nabla v(p)$
  - Remark: f is *anti-conformal* if $\nabla u(p) - N(p) \times \nabla v(p) = 0$ holds for all p

Characterization of the Properties

Properties

- f is *conformal* if
  - $\nabla u(p) + N(p) \times \nabla v(p) = 0$ holds for all p
  - Then, $\langle \nabla u(p), \nabla v(p) \rangle = 0$ and $\|\nabla u(p)\| = \|\nabla v(p)\|$
  - There is a map that combines a rotation and scaling and maps the standard basis in $\mathbb{R}^2$ to $\nabla u(p), \nabla v(p)$
  - Remark: f is *anti-conformal* if $\nabla u(p) - N(p) \times \nabla v(p) = 0$ holds for all p
- f is *equiareal* if
  - $\|\nabla u(p) \times \nabla v(p)\| = 1$
  - This means that the parallelogram formed by the gradients has unit area.

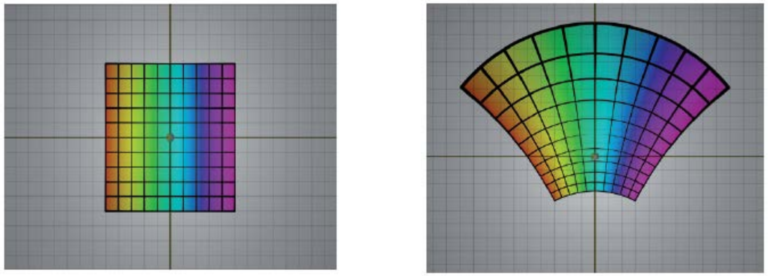
Examples

Isometries of domains in the plane

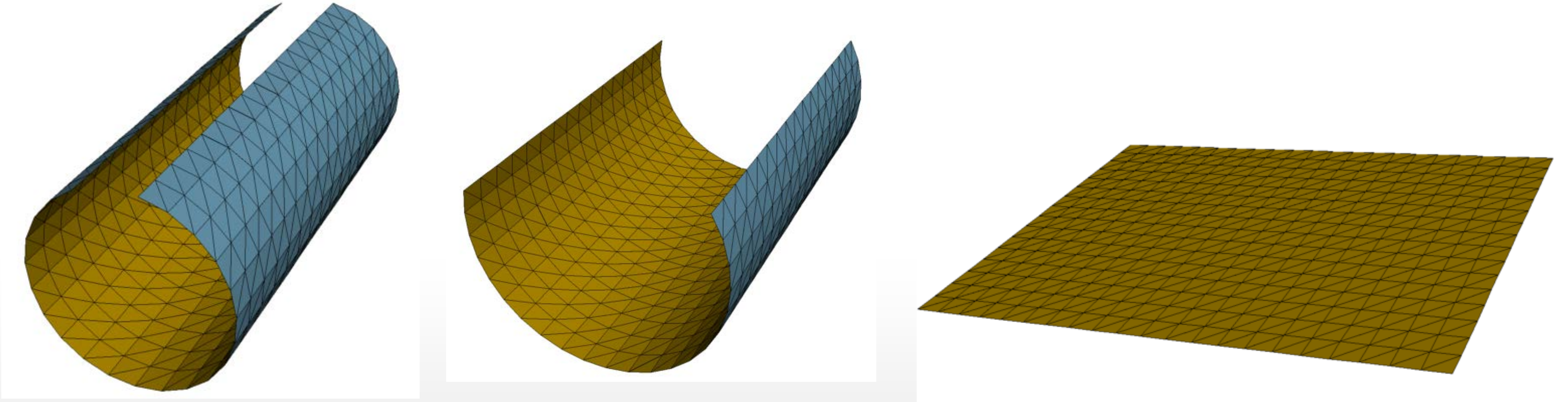
- Rotations, reflections, translation (and concatenations of these)

Conformal maps of domains in the plane

- Möbius transformations



Example: Isometry



Isometry = Conformal + Equiareal

Theorem

- The map f is an isometry if and only if it is conformal (or anti-conformal) and equiareal

Remarks

- Isometries are ideal, but exist only for special surfaces.
- There are always conformal and equiareal maps, however, they are not linear polynomials.
- In practice we construct approximations of conformal/ equiareal maps or mixtures

Approximating Conformal Maps

Least Squares Conformal Maps (LSCM)

- Idea: Penalize deviation from conformality in a least squares sense while fixing the maps on the boundary
- Minimize

$$E_{LSCM}(\mathrm{u}, \mathrm{v}) = \frac{1}{2} \int_M \|\nabla u(p) + N(p) \times \nabla v(p)\|^2 dA$$

over all pairs of maps $u, v: M \mapsto \mathbb{R}$ agree with prescribed functions on the boundary of M

- Discretization use linear polynomials as before

Dirichlet Energy and LSCM

Use Dirichlet energy instead of LSCM

- We have

$$E_{LSCM}(\mathrm{u},\mathrm{v}) = E_D(\mathrm{u},\mathrm{v}) + \mathrm{Area}(\mathrm{u},\mathrm{v})$$

- For fixed boundary, $\mathrm{Area}(\mathrm{u},\mathrm{v})$ does not change when u and v are varied. Then, E_{LSCM} and E_D have the same minimizer

Discrete Problem

Computation

- Energy: (Remember: $f = \binom{u}{v}$)

$$E_D(f) = f^T S f$$

- Constraints:

$$Af = a$$

- Use Lagrange multipliers λ:

$$\begin{bmatrix} S & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} f \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

- Boundary conditions can be modified interactively
- Other types of boundary conditions are possible

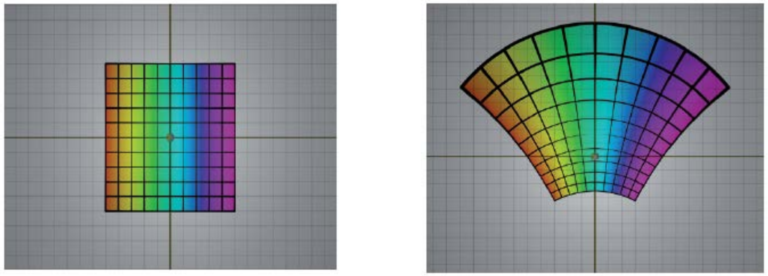
Examples

Isometries of domains in the plane

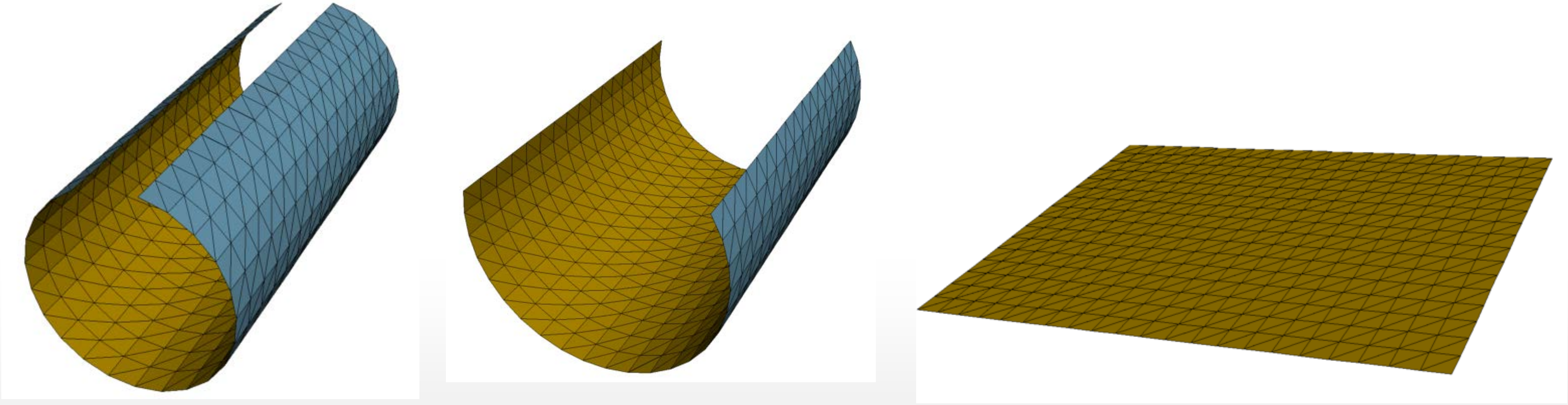
- Affine transformations (rotation + translation)

Conformal maps of domains in the plane

- Möbius transformations

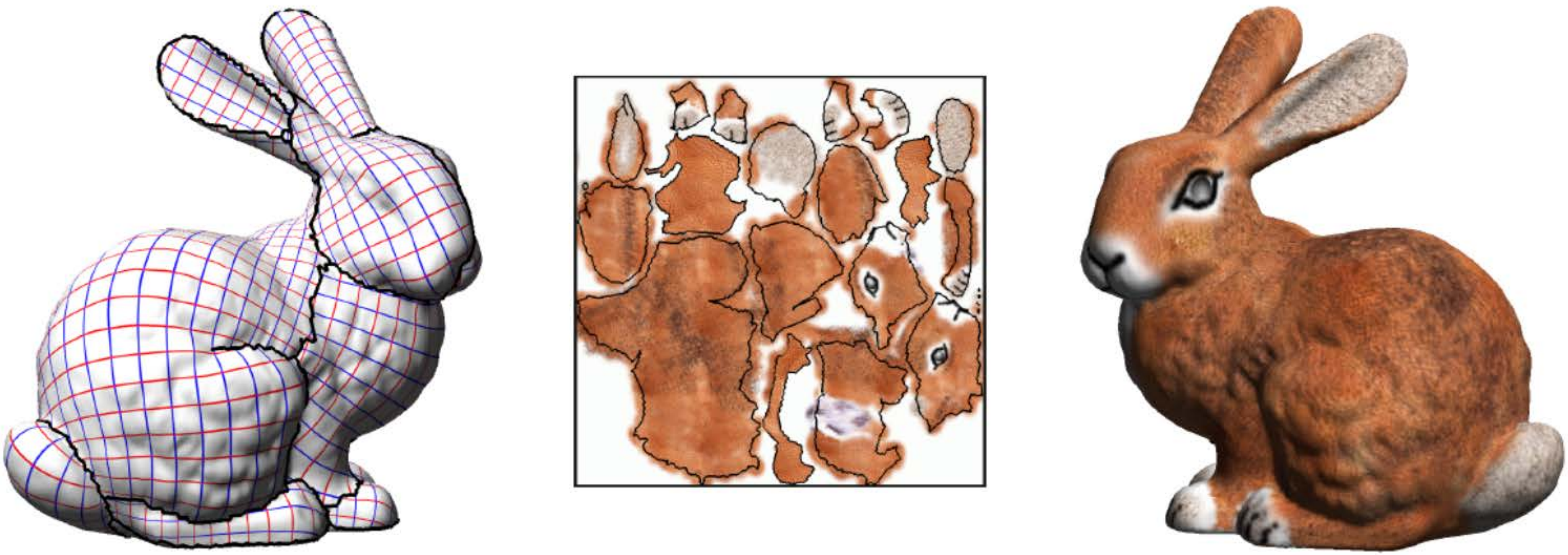


Example: Isometry



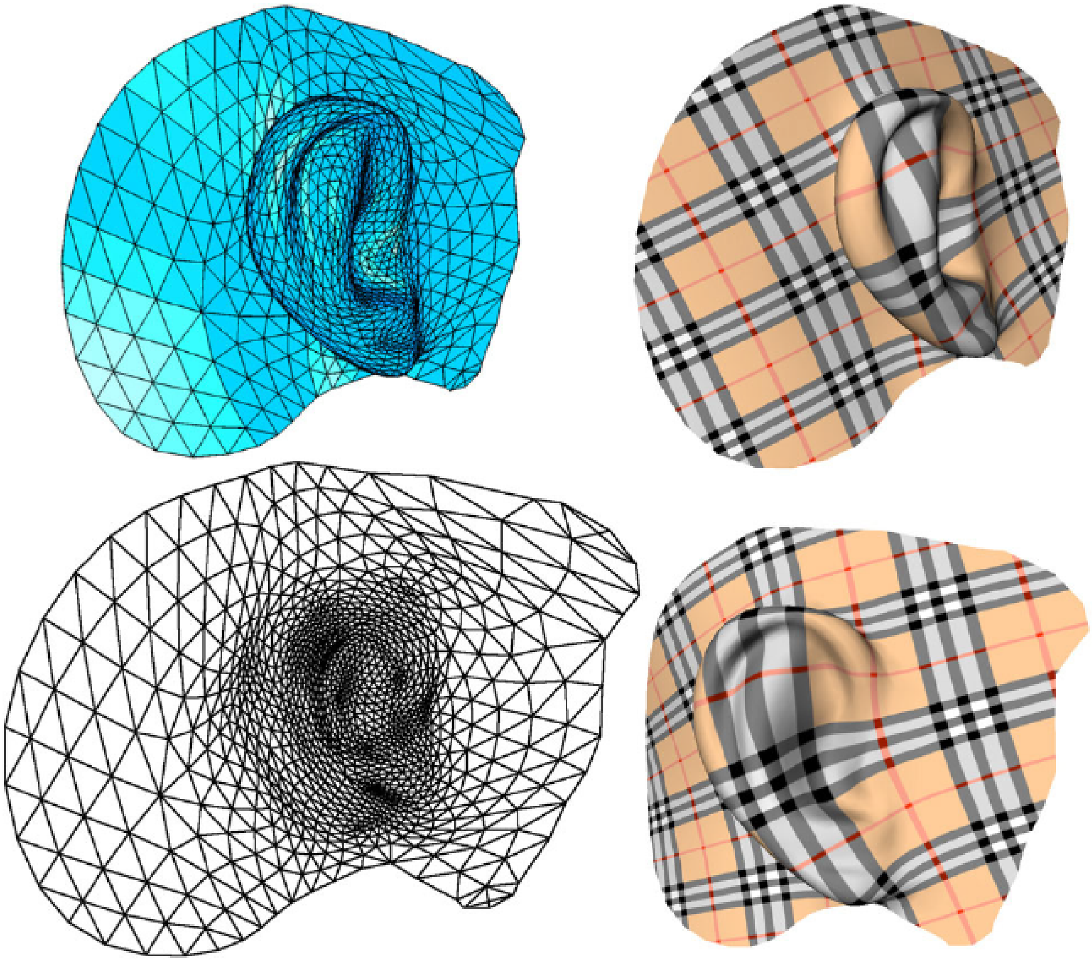
Examples

Texture mapping



Lévy, Petitjean, Ray, and Maillot: *Least squares conformal maps for automatic texture atlas generation*, SIGGRAPH 2002

Examples



Intrinsic parameterizations of surface meshes
M Desbrun, M Meyer, P Alliez
Computer Graphics Forum 21 (3), 209-218

Limitations

Boundary conditions

- Quality of results depends strongly on choice of boundary conditions. How to find good boundary conditions?
- Some approaches:
  - Neumann boundary conditions [Desbrun et al. 2002]
  - Spectral conformal maps [Mullen et al. 2008]
- Nonlinear methods
  - Circle pattern [Kharevych et al. 2005]
  - Angle-based flattening [Sheffer et al. 2005]
  - Ricci Flow [Springborn et al. 2008]

Global Conformal Maps

Seamless parametrizations of surfaces with genus>0

- Global conformal maps [Gu and Yau 2003]