Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 20682, 3 pages
doi:10.1155/2007/20682

Research Article

Logarithm of the Discrete Fourier Transform

Michael Aristidou and Jason Hanson

Received 1 June 2007; Accepted 7 August 2007

Recommended by Jonathan M. Borwein

The discrete Fourier transform defines a unitary matrix operator. The logarithm of this operator is computed, along with the projection maps onto its eigenspaces. A geometric interpretation of the discrete Fourier transform is also given.

Recall that the *discrete Fourier transform* (DFT) in dimension n is the complex-linear transformation $F:\mathbb{C}^n\rightarrow\mathbb{C}^n$, with $F(\mathbf{u})=\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}=(\hat{u}_0,\dots,\hat{u}_{n-1})$ has components $\hat{u}_j=(1/\sqrt{n})\sum_{k=0}^{n-1}u_j\,e^{-2\pi jk/n}$, where $i\doteq\sqrt{-1}$. In the standard basis for $\mathbb{C}^n$, the DFT can be represented as an $n\times n$ matrix $F=(F_{jk})$, where

$$F_{jk}=\frac{1}{\sqrt{n}}e^{-2\pi jk/n}. \tag{1}$$

Our choice of normalization factor ensures that F is unitary: $F^\dagger\circ F=I$, where I is the identity transformation, and $F^\dagger$ is the Hermitian conjugate of F, that is, $(F^\dagger)_{jk}=\overline{F}_{kj}$.

Also recall that the exponential of a matrix M is given by the infinite series $(M)\doteq\sum_{p=0}^{\infty}(1/p!)M^p$ (provided it converges). Thus a complex-linear map $f:\mathbb{C}^n\rightarrow\mathbb{C}^n$ is a *logarithm* for F if $F=\exp(f)$, and we write $f=\log(F)$. For more information on the exponential and logarithm maps see, [1, Chapter 4].

Theorem 1. *In any dimension n, we may take $\log(F)=(1/2)i\pi G_1+i\pi G_2-(1/2)i\pi G_3$, where $G_1\doteq(1/4)(I-iF-F^2+iF^3)$, $G_2\doteq(1/4)(I-F+F^2-F^3)$, $G_3\doteq(1/4)(I+iF-F^2-iF^3)$. The G_j are projections which satisfy $G_j\circ G_k=0$ for $j\neq k$. Moreover, the image of G_k has dimension q_k, where if $n\equiv 0\,(mod\,4)$, then $q_1=n/4-1$ and $q_2=q_3=n/4$; if*

$n \equiv 1\,(mod\,4)$, then $q_1 = q_2 = q_3 = (n-1)/4$; if $n \equiv 2\,(mod\,4)$, then $q_1 = q_3 = (n-2)/4$ and $q_2 = (n+2)/4$; and if $n \equiv 3\,(mod\,4)$, then $q_1 = (n-3)/4$ and $q_2 = q_3 = (n+1)/4$.

Corollary 2. *$\log(F) = (1/4)i\pi(I-(1+i)F+F^2-(1-i)F^3)$.*

Corollary 3. *$F = (I-(1-i)G_1)(I-2G_2)(I-(1+i)G_3)$, where G_1, G_2, G_3 are as in Theorem 1 , and all factors commute.*

Corollary 3 has a nice geometric interpretation. Since each G_j is a projection, we have $G_j^p = G_j$ for $p > 0$; and thus for any $\theta \in \mathbb{R}$, $\exp(i\theta G_j) = I + (e^{i\theta}-1)G_j$, using the definition of the matrix exponential and the Taylor series expansion of $e^{i\theta}$. In particular, for vectors $\mathbf{u} \in \text{image}\,(G_j)$, we have $\exp(i\theta G_j)\mathbf{u} = e^{i\theta}\mathbf{u}$; and for vectors $\mathbf{v} \in \text{image}\,(G_j)^{\perp}$, the orthogonal complement to the image of G_j,we have $\exp(i\theta G_j)\,\mathbf{v} = \mathbf{v}$. Consequently, the factors in Corollary 3 are rotations by $\pi/2$, π, and $-\pi/2$ in the planes image (G_1), image (G_2), and image (G_3), respectively; these planes are orthogonal to each other. Note that F is the identity (and $\log(F)$ is trivial) in the orthogonal complement to the span of image (G_1), image (G_2), and image (G_3).

Before we prove Theorem 1, we first state some known facts concerning the $n \times n$ DFT matrix F:

$$F^4 = I, \tag{2}$$

$$\text{trace}\,(F) = 1-i, 1, 0, -i, \quad \text{according to } n \equiv 0,1,2,3\,(\text{mod } 4), \tag{3}$$

$$\text{trace}\,(F^2) = 2, 1 \quad \text{according to } \; n \equiv 0,1(\text{mod } 2). \tag{4}$$

Note that as consequences, we have $F^3 = F^{\dagger}$and $\text{trace}\,(F^3) = \overline{\text{trace}\,(F)}$. We remark that (2) and (4) are straight-forward to verify from (1). See also [2, page 244], for instance (although our choice in normalization for F is different from the one used there). Equation (3) is a rearrangement of the *Gauss sum*:

$$\sum_{k=0}^{n=1} e^{ik^2/n} = \epsilon_n\sqrt{n}, \quad \text{where } \epsilon_n = 1+i, 1, 0, i, \quad \text{according to } n \equiv 0,1,2,3\,(\text{mod } 4), \tag{5}$$

which Gauss proved in the 1800's [2, page 76]. Proofs can be found in [3, page 177–180], and in ([4], p. 214–216).

Proof of Theorem 1. As F is unitary, it is diagonalizable; and by (2), its eigenvalues are ± 1, $\pm i$ (the fourth roots of unity). The spectral theorem for a diagonalizable matrix (see [1, page 517] thus implies that $F = G_0 + iG_1 - G_2 - iG_3$, where $G_0, \dots, G_3$ are eigenspace projections with $G_j \circ G_k = 0$ if $j \neq k$, and $G_0 + \cdots + G_3 = I$. For given $\theta_0, \dots, \theta_3 \in \mathbb{R}$, define $f \doteq i\theta_0 G_0 + \cdots + i\theta_3 G_3$. Since $\exp(f) = e^{i\theta_0}G_0 + \cdots + e^{i\theta_3}G_3$, it follows that if we take $\theta_0 = 0$, $\theta_1 = (1/2)\pi$, $\theta_2 = \pi$, and $\theta_3 = -(1/2)\pi$, then $\exp(f) = F$. Although there exist explicit formulae for the projections [1, page 529], it is perhaps more expedient to observe that $(F-I)^p = [(i\theta_1 - 1)G_1 + (i\theta_2 - 1)G_2 + (i\theta_3 - 1)G_3]^p = (i\theta_1 - 1)^p G_1 + (i\theta_2 - 1)^p G_2 + (i\theta_3 - 1)^p G_3$ and in particular, for $p = 1,2,3$. Thus, we need only to solve the (Vandermonde) linear system

$$\begin{pmatrix} (i\theta_1 - 1) & (i\theta_2 - 1) & (i\theta_3 - 1) \\ (i\theta_1 - 1)^2 & (i\theta_2 - 1)^2 & (i\theta_3 - 1)^2 \\ (i\theta_1 - 1)^3 & (i\theta_2 - 1)^3 & (i\theta_3 - 1)^3 \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} F - I \\ (F-I)^2 \\ (F-I)^3 \end{pmatrix}. \tag{6}$$

Inverting this equation yields the formulae for G_1, G_2, G_3 given in the statement of the theorem. For the stated projection image dimensions, recall that the image of the projection P has dimension given by trace (P); thus $q_j = \text{trace}\,(G_j)$, which can be computed using (3) and (4). □

As with the logarithm for complex numbers, the logarithm of a matrix is not unique. For example, if we had taken $\theta_3 = 3\pi/2$ in the above, then we would have obtained the formula $\log(F) = (1/4)i\pi(3I - (1-i)F - F^2 - (1+i)F^3)$ instead of that in Corollary 2.

References

[1] C. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 2000.

[2] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, vol. 84 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1990.

[3] T. Nagell, *Introduction to Number Theory*, Chelsea, New York, NY, USA, 1981.

[4] D. W. Kammler, *A First Course in Fourier Analysis*, Prentice-Hall, Upper Saddle River, NJ, USA, 2000.

Michael Aristidou: Digipen Institute of Technology, Redmond, WA 98052, USA
Email address: maristidou@digipen.edu

Jason Hanson: Digipen Institute of Technology, Redmond, WA 98052, USA
Email address: jhanson@digipen.edu