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## Research Article

## Logarithm of the Discrete Fourier Transform

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The discrete Fourier transform defines a unitary matrix operator. The logarithm of this operator is computed, along with the projection maps onto its eigenspaces. A geometric interpretation of the discrete Fourier transform is also given.

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Recall that the *discrete Fourier transform* (DFT) in dimension n is the complex-linear transformation  $F: \mathbb{C}^n \to \mathbb{C}^n$ , with  $F(\mathbf{u}) = \hat{\mathbf{u}}$ , where  $\hat{\mathbf{u}} = (\hat{u}_0, \dots, \hat{u}_{n-1})$  has components  $\hat{u}_j = (1/\sqrt{n}) \sum_{k=0}^{n-1} u_j \, e^{-2\pi j k/n}$ , where  $i = \sqrt{-1}$ . In the standard basis for  $\mathbb{C}^n$ , the DFT can be represented as an  $n \times n$  matrix  $F = (F_{jk})$ , where

$$F_{jk} = \frac{1}{\sqrt{n}} e^{-2\pi jk/n}.\tag{1}$$

Our choice of normalization factor ensures that F is unitary:  $F^{\dagger} \circ F = I$ , where I is the identity transformation, and  $F^{\dagger}$  is the Hermitian conjugate of F, that is,  $(F^{\dagger})_{ik} = \overline{F}_{kj}$ .

Also recall that the exponential of a matrix M is given by the infinite series  $(M) \doteq \sum_{p=0}^{\infty} (1/p!) M^p$  (provided it converges). Thus a complex-linear map  $f : \mathbb{C}^n \to \mathbb{C}^n$  is a *logarithm* for F if  $F = \exp(f)$ , and we write  $f = \log(F)$ . For more information on the exponential and logarithm maps see, [1, Chapter 4].

Theorem 1. In any dimension n, we may take  $\log(F) = (1/2)i\pi G_1 + i\pi G_2 - (1/2)i\pi G_3$ , where  $G_1 \doteq (1/4)(I - iF - F^2 + iF^3)$ ,  $G_2 \doteq (1/4)(I - F + F^2 - F^3)$ ,  $G_3 \doteq (1/4)(I + iF - F^2 - iF^3)$ . The  $G_j$  are projections which satisfy  $G_j \circ G_k = 0$  for  $j \neq k$ . Moreover, the image of  $G_k$  has dimension  $g_k$ , where if  $g_k = 0$  (mod 4), then  $g_k = 0$  and  $g_k = 0$  for  $g_k = 0$  f

 $n \equiv 1 \pmod{4}$ , then  $q_1 = q_2 = q_3 = (n-1)/4$ ; if  $n \equiv 2 \pmod{4}$ , then  $q_1 = q_3 = (n-2)/4$  and  $q_2 = (n+2)/4$ ; and if  $n \equiv 3 \pmod{4}$ , then  $q_1 = (n-3)/4$  and  $q_2 = q_3 = (n+1)/4$ .

Corollary 2.  $log(F) = (1/4)i\pi(I - (1+i)F + F^2 - (1-i)F^3)$ .

COROLLARY 3.  $F = (I - (1 - i)G_1)(I - 2G_2)(I - (1 + i)G_3)$ , where  $G_1, G_2, G_3$  are as in Theorem 1, and all factors commute.

Corollary 3 has a nice geometric interpretation. Since each  $G_j$  is a projection, we have  $G_j^p = G_j$  for p > 0; and thus for any  $\theta \in \mathbb{R}$ ,  $\exp(i\theta G_j) = I + (e^{i\theta} - 1)G_j$ , using the definition of the matrix exponential and the Taylor series expansion of  $e^{i\theta}$ . In particular, for vectors  $\mathbf{u} \in \text{image}(G_j)$ , we have  $\exp(i\theta G_j)\mathbf{u} = e^{i\theta}\mathbf{u}$ ; and for vectors  $\mathbf{v} \in \text{image}(G_j)^{\perp}$ , the orthogonal complement to the image of  $G_j$ , we have  $\exp(i\theta G_j)\mathbf{v} = \mathbf{v}$ . Consequently, the factors in Corollary 3 are rotations by  $\pi/2$ ,  $\pi$ , and  $-\pi/2$  in the planes image  $(G_1)$ , image  $(G_2)$ , and image  $(G_3)$ , respectively; these planes are orthogonal to each other. Note that F is the identity (and  $\log(F)$  is trivial) in the orthogonal complement to the span of image  $(G_1)$ , image  $(G_2)$ , and image  $(G_3)$ .

Before we prove Theorem 1, we first state some known facts concerning the  $n \times n$  DFT matrix F:

$$F^4 = I, (2)$$

trace 
$$(F) = 1 - i, 1, 0, -i,$$
 according to  $n \equiv 0, 1, 2, 3 \pmod{4}$ , (3)

trace 
$$(F^2) = 2, 1$$
 according to  $n \equiv 0, 1 \pmod{2}$ . (4)

Note that as consequences, we have  $F^3 = F^{\dagger}$  and trace  $(F^3) = \overline{\text{trace}(F)}$ . We remark that (2) and (4) are straight-forward to verify from (1). See also [2, page 244], for instance (although our choice in normalization for F is different from the one used there). Equation (3) is a rearrangement of the *Gauss sum*:

$$\sum_{k=0}^{n=1} e^{ik^2/n} = \epsilon_n \sqrt{n}, \quad \text{where } \epsilon_n = 1+i, 1, 0, i, \quad \text{according to } n \equiv 0, 1, 2, 3 \pmod{4}, \tag{5}$$

which Gauss proved in the 1800's [2, page 76]. Proofs can be found in [3, page 177–180], and in ([4], p. 214–216).

*Proof of Theorem 1.* As F is unitary, it is diagonalizable; and by (2), its eigenvalues are  $\pm 1$ ,  $\pm i$  (the fourth roots of unity). The spectral theorem for a diagonalizable matrix (see [1, page 517] thus implies that  $F = G_0 + iG_1 - G_2 - iG_3$ , where  $G_0, \ldots, G_3$  are eigenspace projections with  $G_j \circ G_k = 0$  if  $j \neq k$ , and  $G_0 + \cdots + G_3 = I$ . For given  $\theta_0, \ldots, \theta_3 \in \mathbb{R}$ , define  $f \doteq i\theta_0 G_0 + \cdots + i\theta_3 G_3$ . Since  $\exp(f) = e^{i\theta_0} G_0 + \cdots + e^{i\theta_3} G_3$ , it follows that if we take  $\theta_0 = 0$ ,  $\theta_1 = (1/2)\pi$ ,  $\theta_2 = \pi$ , and  $\theta_3 = -(1/2)\pi$ , then  $\exp(f) = F$ . Although there exist explicit formulae for the projections [1, page 529], it is perhaps more expedient to observe

that  $(F-I)^p = [(i\theta_1 - 1)G_1 + (i\theta_2 - 1)G_2 + (i\theta_3 - 1)G_3]^p = (i\theta_1 - 1)^p G_1 + (i\theta_2 - 1)^p G_2 + (i\theta_3 - 1)G_3]^p = (i\theta_1 - 1)^p G_1 + (i\theta_2 - 1)^p G_2 + (i\theta_3 - 1)G_3$  $(i\theta_3 - 1)^p G_3$  and in particular, for p = 1, 2, 3. Thus, we need only to solve the (Vandermonde) linear system

$$\begin{pmatrix} (i\theta_{1}-1) & (i\theta_{2}-1) & (i\theta_{3}-1) \\ (i\theta_{1}-1)^{2} & (i\theta_{2}-1)^{2} & (i\theta_{3}-1)^{2} \\ (i\theta_{1}-1)^{3} & (i\theta_{2}-1)^{3} & (i\theta_{3}-1)^{3} \end{pmatrix} \begin{pmatrix} G_{1} \\ G_{2} \\ G_{3} \end{pmatrix} = \begin{pmatrix} F-I \\ (F-I)^{2} \\ (F-I)^{3} \end{pmatrix}.$$
(6)

Inverting this equation yields the formulae for  $G_1$ ,  $G_2$ ,  $G_3$  given in the statement of the theorem. For the stated projection image dimensions, recall that the image of the projection P has dimension given by trace (P); thus  $q_i = \text{trace}(G_i)$ , which can be computed using (3) and (4).

As with the logarithm for complex numbers, the logarithm of a matrix is not unique. For example, if we had taken  $\theta_3 = 3\pi/2$  in the above, then we would have obtained the formula  $\log (F) = (1/4)i\pi(3I - (1-i)F - F^2 - (1+i)F^3)$  instead of that in Corollary 2.

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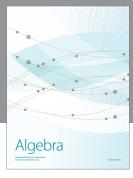
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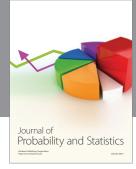
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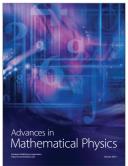






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