Homework -1-

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Exercice 1: which of the following sets are convex

1) A Rectangle, i.e $S = \{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i, \; i = 1 \dots n\}$
→ this set is the intersection of a set of halfspaces characterized by the inequalities $\alpha_i \le x_i \le \beta_i$ which forms a polyhedron so it is convex.

2) The hyperbolic set: $S = \{x \in \mathbb{R}^2_+ \mid x_1 x_2 \ge 1\}$
→ it is convex

- we take two points (x_1, x_2) and (y_1, y_2) from S so $x_1 x_2 \ge 1$ and $y_1 y_2 \ge 1$
- we define the linear combination of x and y as $z = \theta x + (1-\theta) y$ and we evaluate if $z \in S$ in 3 cases:

case ①: $x \ge y \Rightarrow x_1 \ge y_1$ and $x_2 \ge y_2$

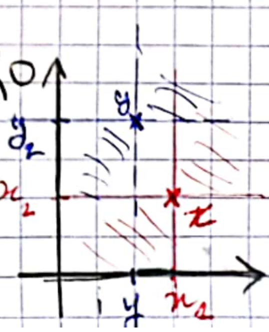
so $z = \theta x + (1-\theta) y \ge y$

$z_1 z_2 \ge y_1 y_2 \ge 1 \Rightarrow z \in S$

case ②: $x \le y \Rightarrow$ similarly to case ① we get

$z_1 z_2 \ge x_1 x_2 \ge 1 \Rightarrow z \in S$

case ③ $x \not\ge y$ so $(y_1 - x_1)(y_2 - x_2) < 0$



so $z_1 z_2 = (\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2)$

$= \theta^2 x_1 x_2 + \theta(1-\theta) x_1 y_2 + \theta(1-\theta) x_2 y_1 + (1-\theta)^2 y_1 y_2$

(2

$$z_1 z_2 = \theta x_1 x_2 + (1-\theta) y_1 y_2 - \theta x_1 x_2 - (1-\theta) y_1 y_2 + \theta^2 x_1 x_2 + (1-\theta)^2 y_1 y_2$$
$$+ \theta(1-\theta) x_1 y_2 + \theta(1-\theta) x_2 y_2$$
$$= \theta x_1 x_2 + (1-\theta) y_1 y_2 - \theta(1-\theta)(x_1 - y_1)(x_2 - y_2)$$

so $z_1 z_2 \geq 1 \rightarrow \underline{z \in S}$

3) The set of points closer to a given point than a given set i.e. $S = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$ $S \subseteq \mathbb{R}^n$

$\rightarrow$ this set is convex

we have: the inequality $\|x - x_0\|_2 \leq \|x - y\|_2$ is equivalent to $\|x - x_0\|_2^2 \leq \|x - y\|_2^2$

$(x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$

$x^T x - 2 x_0^T x + x_0^T x_0 \leq x^T x - 2 y^T x + y^T y$

$2 (y^T - x_0^T) x \leq y^T y - x_0^T x_0$

$2 (y - x_0)^T x \leq y^T y - x_0^T x_0$

This can be written as $Ax \leq b$ $\quad / A = 2(y - x_0)^T$

this is a half space $\rightarrow$ convex $\quad b = y^T y - x_0^T x$

4) The set of points closer to one set than another

$\{x \mid \operatorname{dist}(x, S) \leq \operatorname{dist}(x, T)\}$ $\quad S, T \subseteq \mathbb{R}^n$

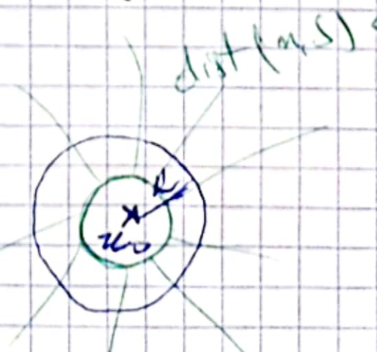
$\rightarrow$ this set is Not Convex $\quad$ and $\operatorname{dist}(x, S) = \inf\{\|x - z\| \mid z \in S\}$

we take the example of S being a circle

$S = \{x \mid \|x - x_0\|_2 = R, R \in \mathbb{R}\}$

$T = \{x_0\}$

we clearly see that this set is Not convex.



5) The set $S = \{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^n$ (3

the set S can be expressed as

$S = \{x \mid x + S_2 \in S_1\} = \{x \mid x + y \in S_1\}$ st $y \in S_2$

$= \{x \mid x \in S_1 - y_2\}$

this is an intersection of multiple convex sets so S is convex.

Exo 2: Determine whether the functions are convex concave or not.

① $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$

we have $\nabla f = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ and $H = \nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\det(H - \lambda I) = \lambda^2 - 1$

λ_1 and λ_2 have different signs so f is neither convex nor concave.

then we have the superlevel set defined by $\{x \mid x_1 x_2 > \alpha$

which is convex so f is quasiconcave!

② $f(x_1, x_2) = 1/x_1 x_2$ on $\mathbb{R}^2_{++}$

$\nabla f = \begin{bmatrix} -\frac{1}{x_1^2 x_2} \\ -\frac{1}{x_1 x_2^2} \end{bmatrix}$ and $H = \nabla^2 f = \frac{1}{x_1 x_2} \begin{bmatrix} 2/x_1^2 & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} \end{bmatrix}$

$\det(H - \lambda I) = \frac{1}{x_1 x_2} \begin{vmatrix} \frac{2}{x_1^2} - \lambda & \frac{1}{x_1 x_2} \\ \frac{1}{x_1 x_2} & \frac{2}{x_2^2} - \lambda \end{vmatrix} = \frac{1}{x_1 x_2} \left[\lambda^2 - 2 \frac{x_1^2 + x_2^2}{x_1^2 x_2^2} \lambda + \frac{3}{x_1^2 x_2^2} \right]$

we have $\lambda_1 \lambda_2 = \frac{3}{x_1^2 x_2^2} > 0$

and $\lambda_1 + \lambda_2 = 2 \frac{x_1^2 + x_2^2}{x_1^2 x_2^2} > 0$

so $\lambda_1, \lambda_2 \in \mathbb{R}_+$

H is positive semidefinite

$f \to$ convex

③ $f(x_1, x_2) = x_1 / x_2$ on $\mathbb{R}^2_{++}$

$$\nabla f = \begin{bmatrix} \frac{1}{x_2} \\ -\frac{x_1}{x_2^2} \end{bmatrix} \quad \text{and} \quad H = \nabla^2 f = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

$$\det(\lambda I - H) = \begin{vmatrix} \lambda & \frac{1}{x_2^2} \\ \frac{1}{x_2^2} & \lambda - \frac{2x_1}{x_2^2} \end{vmatrix} = \lambda^2 - \frac{2x_1}{x_2^3}\lambda - \frac{1}{x_1^2 x_2^2}$$

$\Rightarrow -\frac{1}{x_1^2 x_1^2} < 0$ so λ_1 and λ_2 dont have the same sign.

therefor H is Not convex nor concave.

④ $f(x_1, x_2) = x_1^{\alpha} x^{1-\alpha}$ st $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$

$$\nabla f = \begin{bmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{bmatrix}$$

$$H = \begin{bmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{1-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{1-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-1} \end{bmatrix}$$

$$= \alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{bmatrix} -x_1^{-2} & x_1^{-1} x_2^{-1} \\ x_1^{-1} x_2^{-1} & -x_2^{2} \end{bmatrix}$$

Now $H = -\alpha(1-\alpha) x_1^{\alpha} x_2^{1-\alpha} \begin{bmatrix} x_1^{-1} \\ x_2^{-1} \end{bmatrix} \begin{bmatrix} x_1^{-1} \\ x_2^{-1} \end{bmatrix}^T \leq 0$

so f is concave

Exo3 : show that the following functions are convex

① $f(X) = tr(X^{-1})$, $dom\, f = S^n_{++}$

to prove that is convex, we prove that the composition f with an arbitrary line $Z + tV$ is convex

let $g(t) = f(Z+tV) = tr((Z+tV)^{-1})$

$$= tr\left[\left(Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})Z^{\frac{1}{2}}\right)^{-1}\right]$$

$$= tr\left(Z^{-1}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})^{-1}\right)$$

Now we introduce $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}} = Q\Lambda Q^T$ the eigenvalue decomposition

so $g(t) = tr\left(Z^{-1}(I + tQ\Lambda Q^T)^{-1}\right)$

$$= tr\left(Z^{-1}(Q(I+t\Lambda)Q^T)^{-1}\right)$$

$$= tr\left(Z^{-1}Q(I+t\Lambda)^{-1}Q^T\right)$$

$$= tr\left(Q^TZ^{-1}Q(I+t\Lambda)^{-1}\right)$$

$$= \sum_{i=1}^{n} (Q^TZ^{-1}Q)_{ii}(1+t\lambda_i)^{-1}$$

this is a linear combination of convex functions of the form $(1+t\lambda_i)^{-1}$

(2) $f(X, y) = y^T X^{-1} y$ on $\text{dom} f = S^n_{++} \times \mathbb{R}^n$

- we consider $z \in \mathbb{R}^n$

we will expand the following expression

$$(X^{-1}y - Xz)^2 \geq 0$$

$$(X^{-1}y - Xz)^T (X^{-1}y - Xz) \geq 0$$

$$y^T X^{-1} y + z^T X z \geq 2zy$$

$$y^T X^{-1} y \geq 2zy - z^T X z$$

so $y^T X^{-1} y \geq \sup (2zy - z^T X z)$

so it is convex.

(3) $f(X) = \sum_{i=1}^n \sigma_i(X)$ sum of singular values of X

we note $f(X) = 0$ iff $X = 0$ —— (I)

and $f(tX) = t f(X)$ —— (II)

Then we set $X = U \Sigma V^T = \sum_i \sigma_i u_i v_i^T$ singular value decomposition.

set $Q = UV^T = UIV^T$ so $\sigma_1(Q) = \sigma_{max}(Q) = 1$

then $\langle Q, X \rangle = \langle UV^T, U\Sigma V^T \rangle$

$$= tr(VU^T U \Sigma V^T)$$

$$= tr(V^T V U^T U \Sigma) = tr(\Sigma)$$

$$= \sum_{i=1}^n \sigma_i = f(X)$$

so $\sum_{i=1}^n \sigma_i \leq \sup_{\sigma_1(Q) \leq 1} \langle Q, X \rangle$ —— (1)

on the other hand we have

$$\langle Q, X\rangle = tr(Q^T U \Sigma V^T) = tr(V^T Q^T U \Sigma)$$
$$= \langle U^T Q V, \Sigma \rangle = \sum_{i=1}^{n} \sigma_i (U^T Q V)_{ii}$$
$$= \sum_{i=1}^{n} \sigma_i u_i Q v_i$$

Now $\sup \sum_{i=1}^{n} \sigma_i u_i Q_i v_i < \sup \sum \sigma_i \underbrace{\sigma_{max}(Q)}_{1}$

$$= \sum_{i=1}^{n} \sigma_i$$

so $\sup \langle Q, X\rangle \leq \sum_{i=1}^{n} \sigma_i$ — ②

from ① and ② $\underline{\sum \sigma_i = \sup \langle Q, X\rangle}$

then $f(X+Y) = \sup\langle Q, X+Y\rangle \leq \sup(Q, X) + \sup(Q, Y)$

$$= f(X) + f(Y)$$

$$f(X+Y) \leq f(X) + f(Y) \quad \text{— VI}$$

cauchy ineq.

so f is Norm since it satisfies all three conditions of Norms $\rightarrow$ therefore it is convex.

Exo 4 : $K_{m+} = \{x \in \mathbb{R}^n \mid x_1 > x_2 \cdots x_n \geq 0\}$

① proper cone

- we have K_{m+} is composed of the intersection of half planes delimited by the inequalities (polyhedron) so it is closed
- And, K_{m+} is non empty, and it is pointed iff $x \in K_{m+}$ then $-x \in K_{m+}$ only if $x = 0$

we have $x_1 \geq x_2 \geq \cdots \geq 0$ so for $-x \in K_m$ we need to have $x = 0$

② The dual cone K^*_{m+}

we find the set $y \in \mathbb{R}^n$ st $y^T x \geq 0$ $\forall x$

$y^T x = \sum y_i x_i \geq 0$ iff $y_1, y_1 + y_2, \cdots y_1 + \cdots y_n \geq 0$

so $K^*_{m+} = \{y \mid \sum_{i=0}^{k} y_i \geq 0, k = 1, \cdots n\}$