Chapter 1

Part 4

Quantifiers & Proof Technique

QUANTIFIERS

- Most of the statements in mathematics and computer science are not described properly by the propositions.
- Since most of the statements in mathematics and computer science use **variables**, the system of logic must be extended to include **statements with the variables**.

QUANTIFIERS

- Let $P(x)$ is a statement with variable x and A is a set.
- P a **propositional function** or also known as **predicate** if for each x in A, $P(x)$ is a proposition.
- Set A is the **domain of discourse** of P.
- Domain of discourse -> the particular domain of the variable in a propositional function.

QUANTIFIERS

- A **predicate** is a statement that contains variables.

Example:

$P(x) : x > 3$

$Q(x,y) : x = y + 3$

$R(x,y,z) : x + y = z$

Example

$x^2 + 4x$ is an odd integer

(domain of discourse is set of positive numbers).

$x^2 - x - 6 = 0$

(domain of discourse is set of real numbers).

x is rated as Research University in Malaysia

(domain of discourse is set of university in Malaysia).

QUANTIFIERS

A predicate becomes a proposition if the variable(s) contained is(are)

- **Assigned specific value(s)**
- **Quantified**

Example

- $P(x) : x > 3.$

  What are the truth values of $P(4)$ and $P(2)$?

- $Q(x,y) : x = y + 3.$

  What are the truth values of $Q(1,2)$ and $Q(3,0)$?

Propositional functions

- Let P(x) : “x is a multiple of 5”
  - For what values of x is P(x) true?
- Let P(x) : x+1 > x
  - For what values of x is P(x) true?
- Let P(x) : x + 3 =0
  - For what values of x is P(x) true?

QUANTIFIERS

- Two types of quantifiers:
  - **Universal**
  - **Existential**

QUANTIFIERS

- Let A be a propositional function with domain of discourse B. The statement

  for every x, $A(x)$

  is **universally quantified statement**

- Symbol ∀ called a **universal quantifier** is used "**for every**".
- Can be read as "**for all**", "**for any**".

Universal quantifiers

Represented by an upside-down A: $\forall$

- It means "for all"

Example

Let $P(x) = x+1 > x$

We can state the following:

- $\forall x\, P(x)$
- English translation: "for all values of x, P(x) is true"
- English translation: "for all values of x, x+1>x is true"

QUANTIFIERS

- The statement can be written as

$$\forall x \, A(x)$$

- Above statement is true if $A(x)$ is true for every x in B

  (**false if $A(x)$ is false for at least one x in B**).

- OR In order to prove that a universal quantification is **true,**

  it must be shown for **ALL cases**

  In order to prove that a universal quantification is **false**, it must be shown to be false for only **ONE case**

- A value x in the domain of discourse that makes the statement $A(x)$ *false* is called a **counterexample** to the statement.

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Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \geq 0)$$

Domain of discourse is the set of real numbers.

- **This statement is true** because for every real number x, it is true that the square of x is positive or zero.

Example

- Let the universally quantified statement is

$$\forall x\ (x^2 \le 9)$$

  Domain of discourse is a set B = {1, 2, 3, 4}

- When x = 4, the statement produce false value.
- Thus, the above statement is false and the **counterexample is 4**.

QUANTIFIERS

- Easy to prove a universally quantified statement is true or false if the domain of discourse is not too large.
- What happen if the domain of discourse contains a large number of elements?
- For example, a set of integer from 1 to 100, the set of positive integers, the set of real numbers or a set of students in School of Computing. It will be hard to show that every element in the set is *true*.

Use existential quantifier!!

QUANTIFIERS

- Let *A* be a propositional function with domain of discourse *B*. The statement

  There exist x, $A(x)$

  is **existentially quantified statement**
- Symbol ∃ called an **existential quantifier** is used "**there exist**".
- Can be read as "**for some**", "**for at least one**".

QUANTIFIERS

- The statement can be written as

$$\exists x \, A(x)$$

- Above statement is true if $A(x)$ is true for at least one x in B (**false if every x in B makes the statement $A(x)$ false**).

- **Just find one x that makes $A(x)$ true!**

Example

- Let the existentially quantified statement is

$$\exists x \quad \left(\frac{x}{x^2+1} = \frac{2}{5} \right)$$

Domain of discourse is the set of real numbers.

- **Statement is true** because it is **possible to find at least one real number *x*** to make the proposition true.
- For example, if $x = 2$, we obtain the true proposition as below

$$\left(\frac{x}{x^2+1} = \frac{2}{5} \right) = \left(\frac{2}{2^2+1} = \frac{2}{5} \right)$$

Negation of Quantifiers

- Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.

$$\neg (\forall x\, P(x)) \;;\; \exists x \neg P(x)$$

$$\neg (\exists x\, P(x)) \;;\; \forall x \neg P(x)$$

Example

- Let $P(x)$: x is taking Discrete Structure course with the domain of discourse is the set of all students.

$\forall x\, P(x)$

All students are taking Discrete Structure course.

$\exists x\, P(x)$

There is some students who are taking Discrete Structure course.

Example

$$\neg\ (\exists x\, P(x))\ ;\ \ \forall x \neg P(x)$$

$\neg\exists x\ P(x)$

None of the students are taking Discrete Structure course.

$\forall x \neg P(x)$

All students are not taking Discrete Structure course.

Example

$$\neg\,(\forall x\; P(x))\;;\;\; \exists x\, \neg\, P(x)$$

$\neg\forall x\, P(x)$

Not all students are taking Discrete Structure course.

$\exists x\, \neg P(x)$

There is some students who are not taking Discrete Structure course

Translating from English

- Consider "**For every student in this class, that student has studied calculus**"
- Rephrased: "**For every student x in this class, x has studied calculus**"
  - Let C(x) be "**x has studied calculus**"
  - Let S(x) be "**x is a student in this class**"

$$\forall x\, C(x)$$

  - True if the universe of discourse is **all students in this class**

Translating from English

- What about if the universe of discourse is all students (or all people?)

  - $\forall x\, (S(x) \wedge C(x))$ X
    - This is wrong! Why? (because this statement says that all people are students in this class and have studied calculus)

  - $\forall x\, (S(x) \rightarrow C(x))$ √

C(x) : "x has studied calculus"
S(x) : "x is a student in this class"

Translating from English

- Consider:
  - “**Some students have visited Mexico**”
  - Rephrasing: “**There exists a student who has visited Mexico**”

- Let:
  - S(x) be “**x is a student**”
  - M(x) be “**x has visited Mexico**”

    $\exists x\ M(x)$

  - True if the universe of discourse is all students

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Translating from English

- What about if the universe of discourse is all people?

$\exists x \, (S(x) \wedge M(x))$

$\exists x \, (S(x) \rightarrow M(x))$ **This is wrong! Why?**

suppose someone is not student= F->T or F->F, both make the statement true (**refer to truth table *p* → *q***)

S(x) :"x is a student"
M(x): "x has visited Mexico"

Translating from English

- Consider "**Every student in this class has visited Canada or Mexico**"
- Let, S(x) be "**x is a student in this class**"
  M(x) be "**x has visited Mexico**"
  C(x) be "**x has visited Canada**"

$$\forall x\,(M(x) \vee C(x))$$

(**When the universe of discourse is all students in this class**)

$$\forall x\,(S(x) \rightarrow (M(x) \vee C(x)))$$

(**When the universe of discourse is all people or all students**)

Proof Techniques

Mathematical systems consists:

- **Axioms**: assumed to be true.
- **Definitions**: used to create new concepts.
- **Undefined terms**: some terms that are not explicitly defined.
- **Theorem**
  - Statement that can be shown to be true (under certain conditions)
  - Typically stated in one of three ways:
    - As Facts
    - As Implications
    - As Bi-implications

Proof Techniques

Direct Proof (Direct Method)

- Proof of those theorems that can be expressed in the form $\forall x\ (P(x) \rightarrow Q(x))$, *D* is the domain of discourse.
- Select a particular, but arbitrarily chosen, member a of the domain *D.*
- Show that the statement $P(a) \rightarrow Q(a)$ is true. (Assume that $P(a)$ is true).
- Show that $Q(a)$ is true.
- By the rule of Universal Generalization (UG),

  $\forall x\ (P(x) \rightarrow Q(x))$ is true.

Example

"For all integer *x*, if *x* is odd, then x^2 is odd"

Or $P(x)$: x is an odd integer

$Q(x)$: x^2 is an odd integer

$$\forall x(P(x) \rightarrow Q(x))$$

The domain of discourse is set Z of all integer.

Can verify the theorem for certain value of x.

$x=3$, $x^2 = 9$; odd

Or show that the square of an odd number is an odd number

Rephrased: "**if *n* is odd, then n^2 is odd**"

Example

- a is an odd integer

$$\Rightarrow a = 2n + 1$$ *for some integer **n***

$$\Rightarrow a^2 = (2n+1)^2$$

$$\Rightarrow a^2 = 4n^2 + 4n + 1$$

$$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$$

$$\Rightarrow a^2 = 2m + 1$$ *where **m= 2n²** + 2n is an integer*

$$\Rightarrow a^2$$ *is an odd integer*

Proof Techniques

Indirect Proof

- The implication $p \rightarrow q$ is **equivalent** to the implication $(\neg q \rightarrow \neg p)$ **(contrapositive)**
- Therefore, in order to show that $p \rightarrow q$ is true, one can also **show that the implication $(\neg q \rightarrow \neg p)$ is true**.
- To show that $(\neg q \rightarrow \neg p)$ is true, assume that the negation of q is true and prove that the negation of p is true.

Example

***P(n)* : n^2+3 is an odd number**

***Q(n)* : *n* is even number**

$$\forall n(P(n) \rightarrow Q(n))$$

$$P(n) \rightarrow Q(n) \equiv \neg Q(n) \rightarrow \neg P(n)$$

¬ *Q(n)* is true , *n* is not even (*n* is odd), so *n=2k+1*

$$\begin{aligned} n^2 + 3 &= (2k+1)^2 + 3 \\ &= 4k^2 + 4k + 1 + 3 \\ &= 4k^2 + 4k + 4 \\ &= 2(2k^2 + 2k + 2) \end{aligned}$$

Example

$$n^2 + 3 = (2k+1)^2 + 3$$
$$= 4k^2 + 4k + 1 + 3$$
$$= 4k^2 + 4k + 4$$
$$= 2(2k^2 + 2k + 2)$$

$$t = 2k^2 + 2k + 2$$

$$n^2 + 3 = 2t$$

t is integer

n^2+3 is an even integer, thus ¬ *P(n)* is true

Which to use

- When do you use a direct proof versus an indirect proof?
- If it’s not clear from the problem, try direct first, then indirect second
  - If indirect fails, try the other proofs

Example

Prove that "**if n is an integer and *n^3+5* is odd, then *n* is even**"

Via direct proof

- $n^3+5 = 2k+1$ for some integer *k* (definition of odd numbers)
- $n^3 = 2k - 4$
- $$n = \sqrt[3]{2k-4}$$
- Umm…

So direct proof didn't work out.

Next up: indirect proof

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Example

Prove that "**if n is an integer and *n^3+5* is odd, then *n* is even**"

Via indirect proof

- Contrapositive: If *n* is odd, then n^3+5 is even
- Assume *n* is odd, and show that n^3+5 is even
- $n=2k+1$ for some integer k (definition of odd numbers)
- $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$
- As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it is even

Proof Techniques

Proof by Contradiction

Assume that the hypothesis is true and that the conclusion is false and then, arrive at a contradiction.

Proposition "**if *P* then *Q***"
Proof. **Suppose *P* and *~Q***

Since we have a contradiction, it must be that Q is true

Example

Prove that "**there are infinitely many prime numbers**".

Proof:

- Assume there are not infinitely many prime numbers, therefore they are can be listed, i.e. $p_1, p_2, \ldots, p_n$
- Consider the number $q = p_1 \times p_2 \times \ldots \times p_n + 1$.
- q is either prime or not divisible, but not listed above. Therefore, q is a prime. However, it was not listed.
- **Contradiction**! Therefore, there are infinitely many primes numbers.

Example

- For all real numbers x and y, **if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$**.

Proof

- Suppose that the conclusion is false. Then

  $x < 1$ and $y < 1$

  Add these inequalities, $x+y < 1+1 = 2$ (**$x+y < 2$**)
- **Contradiction**
- Thus we conclude that the statement is true.

Example

Suppose $a \in \mathbb{Z}$. **If** $\boldsymbol{a^2}$ **is even, then *a* is even**

Proof

- Contradiction: Suppose a^2 is even and a is not even.
- Then a^2 is even, and a is odd
- Let, $a = 2c + 1$ (odd)

$$a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2\left(2c^2 + 2c\right) + 1 \qquad \text{(odd)}$$

- **Contradiction**
- Thus we conclude that the statement is true.

Exercise

Let $P(x,y)=(x*y)^2 \geq 1$. Given the domain of discourse for x and y is set of integer, Z. Determine the truth value of the following statements. Give the value of x and y that make the statement TRUE or FALSE.

$a)\ \exists x \exists y\ P(x,y)$

$b)\ \forall x \forall y\ P(x,y)$

Thank You