

CHAPTER 2

Part 3

Recurrence Relation & Recursive Algorithms

Definition

- A **recurrence relation** is an equation that defines a sequence $\{a_n, a_{n+1}, \dots\}$ based on a rule that gives the next term as a function of one or more of the previous term in the sequence for all integers n with $n \geq n_0$, (a_0, a_1, \dots)
- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

Simple Recurrence Relation

The simplest form of a **recurrence relation** is the case where the next term depends only on the immediately previous term.

$$3, 8, 13, 18, 23, \dots$$

The above sequence shows a pattern, given initial condition, $a_1 = 3$:

$$3, 3+5, 8+5, 13+5, 18+5, \dots$$
$$a_1, a_2, a_3, a_4, a_5, \dots$$

generated from an equation:

$$a_n = a_{n-1} + 5, \quad n \geq 2$$

n_{th} term of a sequence

Recurrence relation can be used to compute any n -th term of the sequence, given the initial condition, a_0 :

$$a_n = a_{n-1} + 5, \quad n \geq 2$$

$$3, 8, 13, 18, 23, \textcolor{red}{28}, a_{n-1} + 5, \dots$$

$$a_2 = a_1 + 5, \quad 3 + 5 = 8$$

$$a_3 = a_2 + 5, \quad 8 + 5 = 13$$

$$a_4 = a_3 + 5, \quad 13 + 5 = 18$$

$$\vdots$$

$$a_6 = a_5 + 5, \quad 23 + 5 = 28$$

Example 1

Consider the following sequence:

$$3, 9, 27, 81, 243, \dots$$

The above sequence shows a pattern:

$$3^1, 3^2, 3^3, 3^4, 3^5, \dots$$

$$a_1, a_2, a_3, a_4, a_5, \dots$$

Recurrence relation is defined by:

$$a_n = 3^n, n \geq 1$$

Example 2

Given initial condition, $a_0 = 1$ and recurrence relation:

$$a_n = 1 + 2a_{n-1} , n \geq 1$$

First few sequence are:

$$a_1 = 1 + 2(1) = 3$$

$$a_2 = 1 + 2(3) = 7$$

$$a_3 = 1 + 2(7) = 15$$

1, 3, 7, 15, 31, 63, ...

Example 3

Given initial condition, $a_0 = 1$, $a_1 = 2$ and recurrence relation:

$$a_n = 3(a_{n-1} + a_{n-2}), \quad n \geq 2$$

First few sequence are:

$$a_2 = 3(2 + 1) = 9$$

$$a_3 = 3(9 + 2) = 33$$

$$a_4 = 3(33 + 9) = 126$$

1, 2, 9, 33, 126, 477, 1809, 6858, 26001,...

Exercise

1) $a_n = 6a_{n-1} - 9a_{n-2}$, where $a_0 = 2, a_1 = 3$

2) $a_n = 2a_{n-1} - a_{n-2}$, where $a_0 = 5, a_1 = 3$

Example 4

For a photo shoot, the staff at a company have been arranged such that there are 10 people in the front row and each row has 7 more people than in the row in front of it. Find the recurrence relation and compute number of staff in the first 5 rows.

Example 4 -Solution

Notice that the difference between the number of people in successive rows is a constant amount.

This means that the n_{th} term of this sequence can be found using:

$$a_n = a_{n-1} + 7, \quad n \geq 2 \text{ with } a_1 = 10$$

Example 4 -Solution

Number of staff in the first 5 rows:

$$a_1 = 10 ,$$

$$a_2 = a_1 + 7, \quad 10 + 7 = 17$$

$$a_3 = a_2 + 7, \quad 17 + 7 = 24$$

$$a_4 = a_3 + 7, \quad 24 + 7 = 31$$

$$a_5 = a_4 + 7, \quad 31 + 7 = 38$$

10, 17, 24, 31, 38

Example 5

Find a recurrence relation and initial condition for

1, 5, 17, 53, 161, 485, ...

Solution:

Look at the differences between terms:

4, 12, 36, 108, ...

the difference in the sequence is growing by a factor of 3.

Example 5 -Solution

However the original sequence is not.

$$1(3)=\mathbf{3}, 5(3)=\mathbf{15}, 17(3)=\mathbf{51}, \dots$$

$$1, 5, 17, 53, 161, 485, \dots$$

It appears that we always end up with 2 less than the next term.

So, the recurrence relation is defined by:

$$a_n = 3(a_{n-1}) + 2, \quad n \geq 1, \text{ with initial condition, } a_0 = 1$$

Example 6

A depositor deposits RM 10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years? Let P_n denote the amount in the account after n years.

Example 6 - Solution

Derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}$$

Where, P_n = Current balance and P_{n-1} = Previous year balance and 0.05 is the compounding interest.

Example 6 - Solution

Initial condition, $P_0 = 10,000$. Then,

$$P_1 = 1.05P_0$$

$$P_2 = 1.05P_1 = (1.05)^2P_0$$

$$P_3 = 1.05P_2 = (1.05)^3P_0$$

...

$$P_n = 1.05P_{n-1} = (1.05)^nP_0 ,$$

now we can use this formula to calculate n_{th} term without iteration

Example 6 - Solution

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$$P_{30} = (1.05)^{30}(10,000) = 43,219.42$$

After 30 years, the account contains RM 43,219.42.

Example

Consider the following sequence:

1, 5, 9, 13, 17

Find the recurrence relation that defines the above sequence.

Answer:

$$a_n = a_{n-1} + 4, \quad n \geq 2 \text{ with } a_1 = 1$$

Exercise

A basketball is dropped onto the ground from a height of 15 feet. On each bounce, the ball reaches a maximum height 55% of its previous maximum height.

a) Write a recursive formula, a_n , that completely defines the height reached on the n_{th} bounce, where the first term in the sequence is the height reached on the ball's first bounce.

b) How high does the basketball reach after the 4_{th} bounce? Give your answer to two decimal places.

Recursive Algorithms

- A recursive procedure is a procedure that invokes itself.
- A recursive algorithm is an algorithm that contains a recursive procedure.
- Recursion is a powerful, elegant and natural way to solve a large class of problems.

example

Factorial problem

- If $n \geq 1$,

$$n! = n (n - 1) \dots 2. 1$$

and $0! = 1$

- Notice that, if $n \geq 2$, n factorial can be written,

$$\begin{aligned} n! &= n (n - 1)(n - 2) \dots 2. 1 \\ &= n. (n-1)! \end{aligned}$$

example

- $n=5$
- $5!$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5 \cdot 4!$$

$$4! = 4 \cdot 3!$$

$$3! = 3 \cdot 2!$$

Algorithm

- Input: n , integer ≥ 0
- Output: $n!$
- Factorial (n) {
 if ($n=0$)
 return 1
 return $n * \text{factorial}(n-1)$
}

example

- Fibonacci sequence, f_n

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

1, 1, 2, 3, 5, 8, 13,

Algorithm

- Input: n
- Output: $f(n)$
- $f(n)$ {
 - if ($n=1$ or $n=2$)
return 1
 - return $f(n-1) + f(n-2)$}

Example

Consider the following arithmetic sequence: 1, 3, 5, 7, 9,...

Suppose a_n is the term sequence. The generating rule is $a_n = a_{n-1} + 2$, for $n > 1$. The relevant recursive algorithm can be written as

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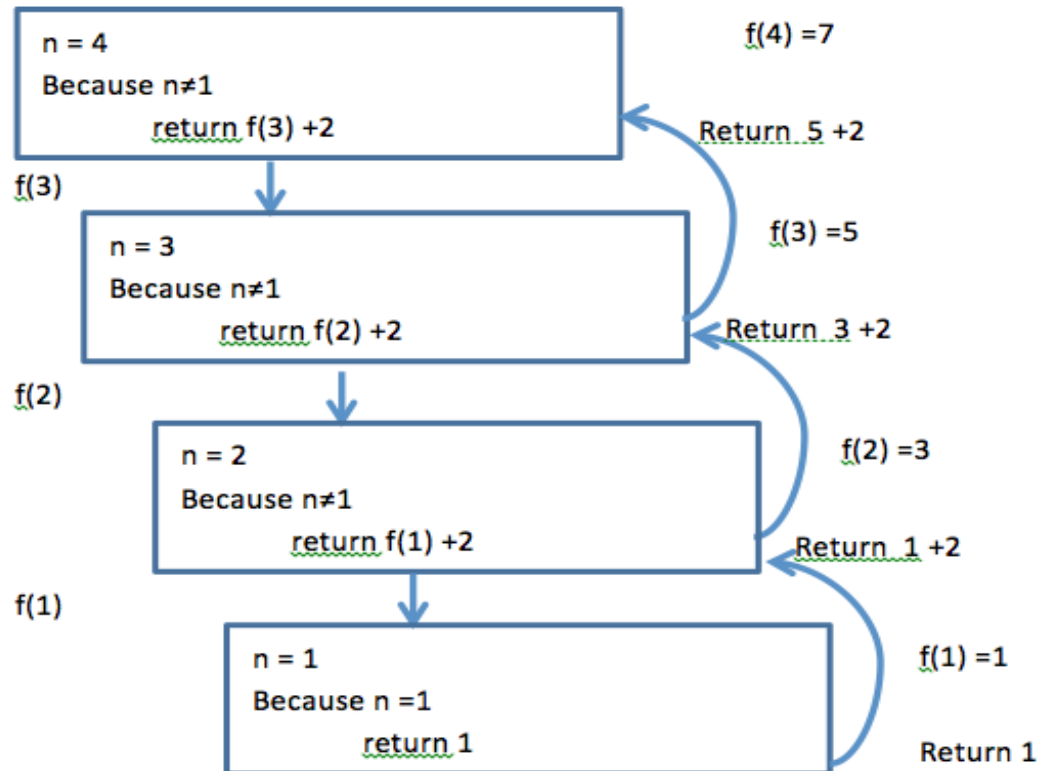
f(n)
{
    if (n = 1 )
        return 1
    return f(n - 1) + 2
}
    
```

Use the above recursive algorithm to trace $n = 4$

Solutions

Trace the output if $n = 4$

f(4)



Answer = 7

exercise

- Given,

$$s_n = 3 + 6 + 9 + \dots + 3n$$

$$s_1 = 3 \quad s_n = s_{n-1} + 3n \quad \text{for } n \geq 2,$$

- Write a recursive algorithm to compute s_n , $n \geq 1$.

Exercise

1. Suppose that the number of bacteria in a colony triples every hour.
 - a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
 - b) If 10 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?
2. Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan of RM9000 at rate of 8% if a payment of RM150 is made each month.
[Hint: Express $B(k)$ in terms of $B(k-1)$; the monthly interest is $(0.07/12) (B(k-1))$]
3. Write a recursive algorithm for computing n^2 where n is a nonnegative integer using the fact that $(n+1)^2 = n^2 + 2n + 1$.