Chapter 2 (Part 1)

Relations

Relations

- A (binary) relation R from a set X to a set Y is a subset of the Cartesian product $X \times Y$.
- If $(x,y) \in R$, we write

  $x\ R\ y$ (x is related to y)

(Binary) relation from X to Y, where $x \in X$, $y \in Y$, $(x,y) \in X \times Y$ and $R \subseteq X \times Y$

$$x\ R\ y \leftrightarrow (x,y) \in R$$

example

- $A = \{ 1, 2, 3, 4 \}$, $B = \{ p, q, r \}$
- $R = \{ (1, q), (2, r), (3, q), (4, p) \}$
- $R \subseteq A \times B$
- R is the relation from A to B

$1 \; R \; q$ $\qquad$ $3 \not R \; p$

example

$aRb \leftrightarrow a-b \in Z^{even}$

- *Finite set: A*= { 1, 2 } , *B*= { 1, 2, 3 }

  R = { (1, 1), (2, 2), (1, 3) }

- *Infinite set: A=Z* and *B=Z*

  R = { …(-3, -1), (-2, 2), (1, 3), …. }

(note: *Z* is set of integers)

example

- A = { New Delhi, Ottawa, London, Paris, Washington }
- B = { Canada, England, India, France, United States }
- Let $x \in A$, $y \in B$. Define the relation between x and y by "x is the capital of y"

example

- R = { (New Delhi, India),
  (Ottawa, Canada),
  (London, England),
  (Paris, France),
  (Washington, United States) }

example

- "less than" relation from A={0, 1, 2} to B={1, 2, 3}

- Traditional notation:

  0<1, 0<2, 0<3, 1<2, 1<3, 2<3

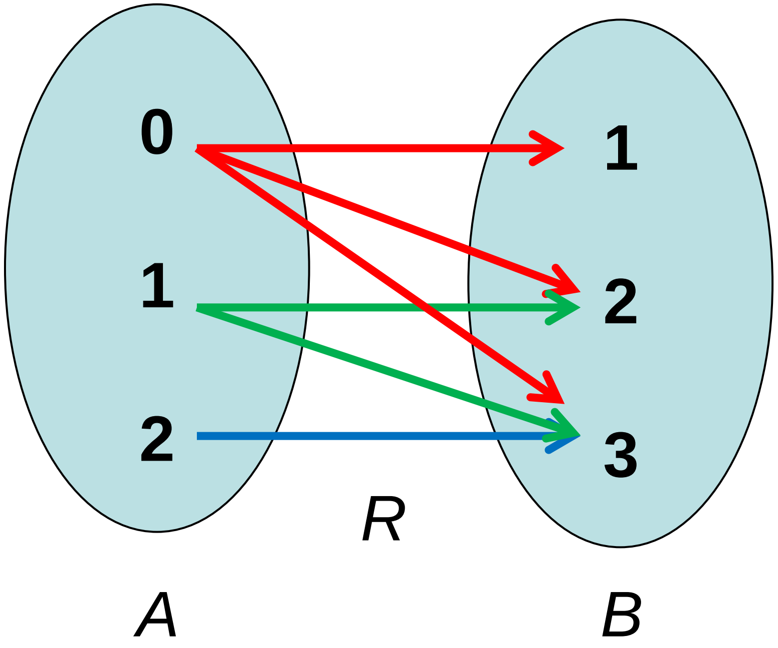
- Set notation

  $A \times B = \{ (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) \}$

  $R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$

example

- Arrow diagrams



Domain and Range

- Let R, a relation from A to B.
- The set, $\{ a \in A \mid (a,b) \in R$ for some $b \in B \}$ is called the domain of R.
- The set, $\{ b \in B \mid (a,b) \in R$ for some $a \in A \}$ is called the range of R.
- In case $A=B$, we call R a(binary) relation on A.

example

- Let R be a relation on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y$, and $x, y \in X$.

- Then,
  $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) \}$

- The domain and range of R are both equal to X.

example

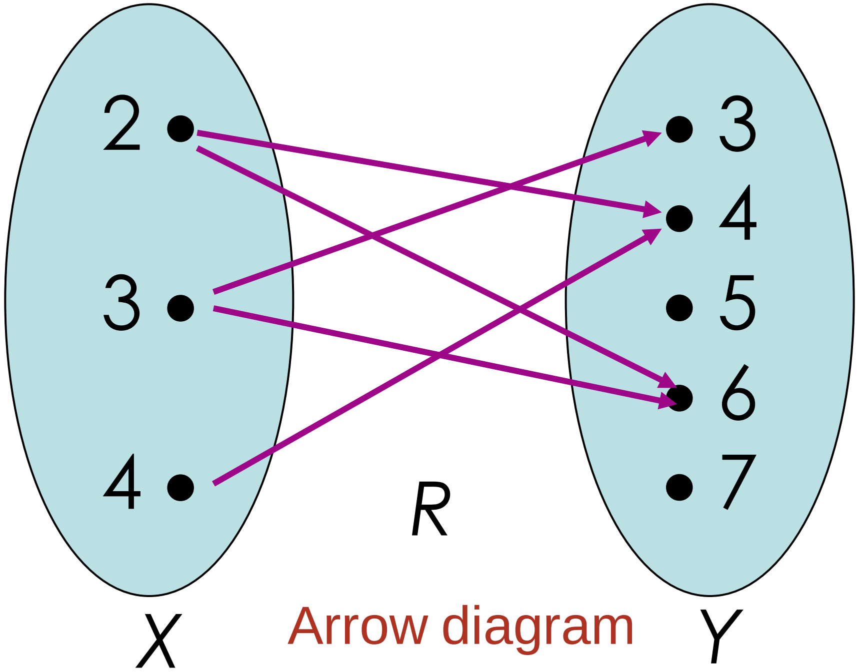
- Let $X = \{ 2, 3, 4 \}$ and $Y = \{ 3, 4, 5, 6, 7 \}$
  If we define a relation R from X to Y by,
  $(x,y) \in R$ if x divides y (with zero remainder)

- We obtain,
  $R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$

  The domain of R is $\{2,3,4\}$
  The range of R is $\{3,4,6\}$

example

$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$



Arrow diagram

Exercise

- Write the relation R as $(x,y) \in R$

(a) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$.

(b) The relation R on {1,2,3,4,5} defined by $(x,y) \in R$ if 3 divides x-y.

Solutions

(a) The relation R on {1, 2, 3, 4} defined by
$(x,y) \in R$ if $x^2 \geq y$.

R = {(1,1), (2,1), (3,1), (4,1), (2,2), (3,2), (4,2), (2,3), (3,3), (4,3), (2,4), (3,4), (4,4)}

(b) The relation R on {1,2,3,4,5} defined by $(x,y) \in R$ if 3 divides x-y.

R ={(1,1), (4,1), (2,2), (2,5), (3,3), (4,1), (4,4), (5,2), (5,5)}

Exercise

- Find range and domain for:

(a) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$.

(b) The relation $R = \{ (1,2), (2,1), (3,3), (1,1), (2,2)\}$ on $X = \{1, 2, 3\}$

Solutions

- Find range and domain for:

(b) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$.

range & domain {1,2,3,4}

(b) The relation $R = \{ (1,2), (2,1), (3,3), (1,1), (2,2)\}$ on $X = \{1, 2, 3\}$

range & domain {1,2,3}

Digraph

- An informative way to picture a relation on a set is to draw its digraph.

- Let R be a relation on a finite set A.

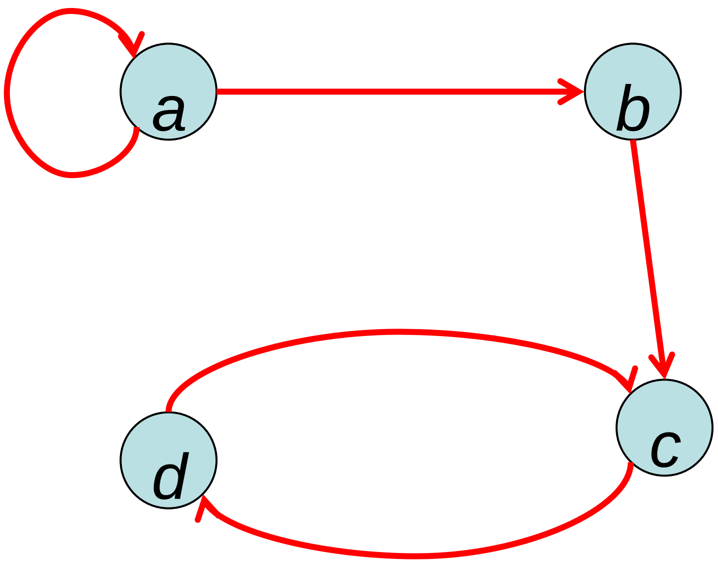
- Draw dots (vertices) to represent the elements of A.

- If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b .

example

- The relation R on $A = \{a, b, c, d\}$,

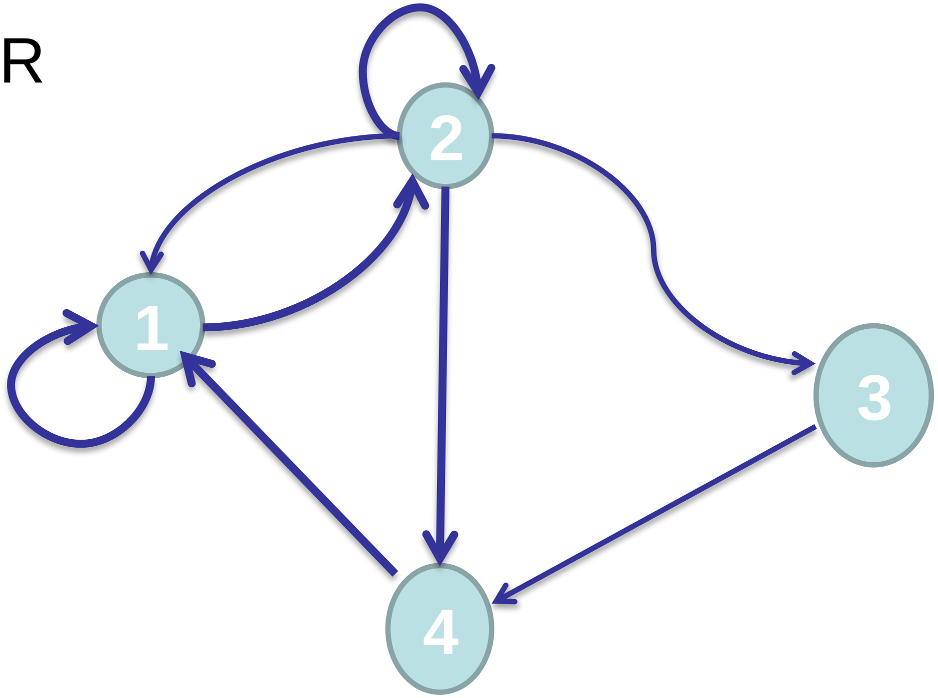
  $R = \{(a, a), (a, b), (c, d), (d, c), (b,c)\}$



example

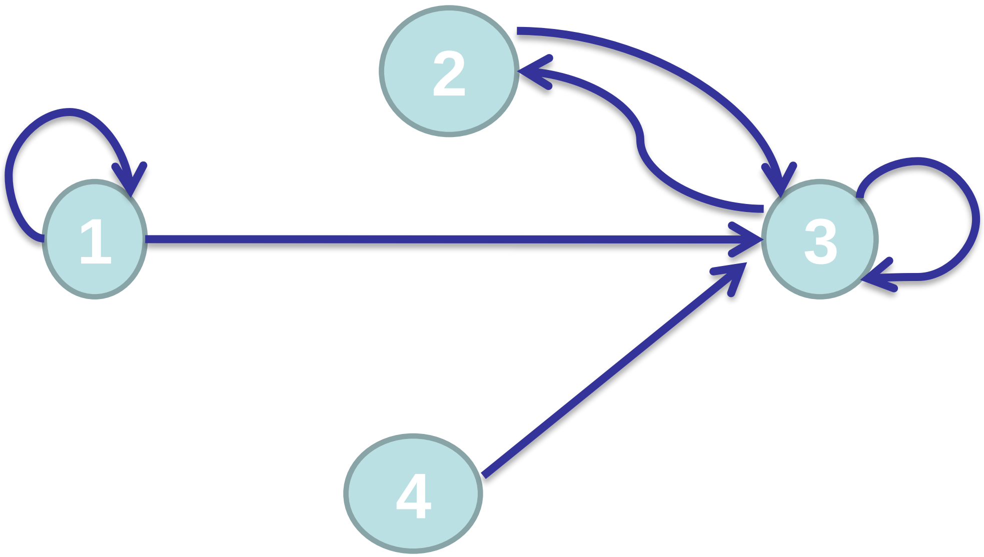
- Let, A = { 1,2,3,4} and R = { (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)}

- Draw the digraph of R



example

Find the relation determined by digraph below.



- Since *a R b* if and only if there is an edge from *a* to *b*, so $R = \{ (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) \}$

exercise

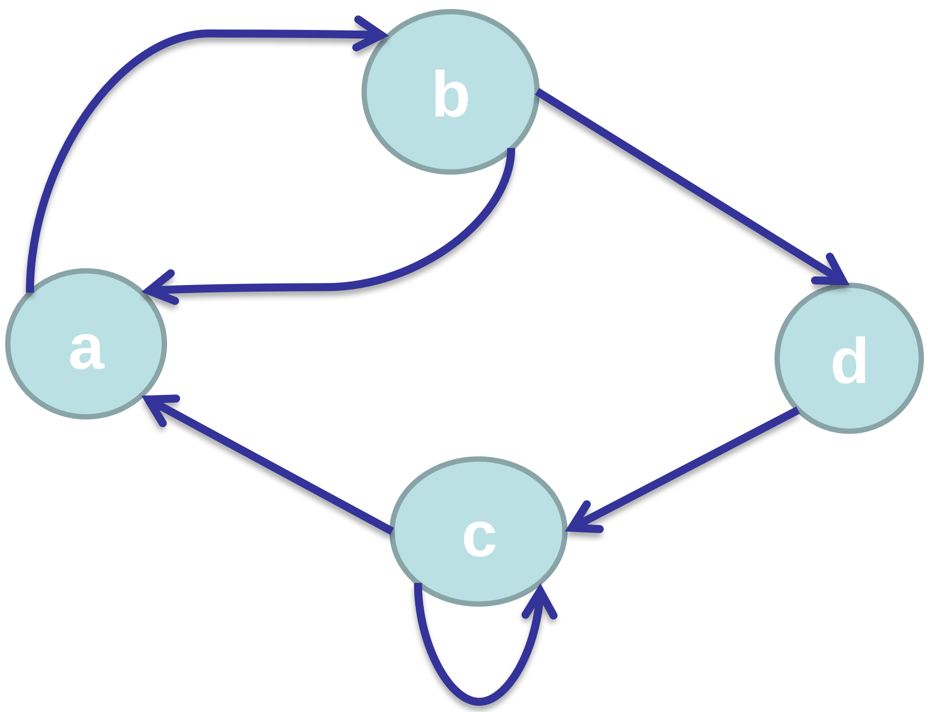
Draw the diagraph of the relation:

(a) R = { (a,c), (b, d), (a,b), (c,d) }

(b) The relation R on {1, 2, 3, 4} defined by $(x,y) \in R$ if $x^2 \geq y$

exercise

Write the relation as a set of ordered pair.



Matrices of Relations

- A matrix is a convenient way to represent a relation *R* from *A* to *B*.
- Label the rows with the elements of *A* (in some arbitrary order)
- Label the columns with the elements of *B* (in some arbitrary order)

Matrices of Relations

- Let $A=\{a_1, a_2,\ldots, a_n\}$ and $B=\{b_1, b_2,\ldots, b_p\}$ be finite nonempty sets.
- Let R be a relation from A into B.
- Let $M_R = [m_{ij}]_{nxp}$ be the Boolean nxp matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

Matrices of Relations

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ : & : & .. & .. & : \\ : & : & .. & .. & : \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

example

- Let $A = \{1, 3, 5\}$ and $B = \{1, 2\}$
- Let R be a relation from A to B and $R = \{(1,1), (3,2), (5,1)\}$
- Then the matrix represent R is

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

example

- The relation,
  $R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$
  from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

or

$$\begin{array}{c c} & \begin{array}{cccc} d & b & a & c \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 1 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

example

- The matrix of the relation R from { 2, 3, 4 } to { 5, 6, 7, 8 } defined by

$$x\ R\ y \quad \text{if } x \text{ divides } y$$

$$\begin{array}{c c}
 & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}$$

example

- Let $A=\{ a, b, c, d \}$
- Let R be a relation on A.
- $R = \{ (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) \}$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

exercise

- Let $A=\{ 1, 2, 3, 4 \}$ and R be a relation on A.
  $R = \{ (1,2),(1,3),(1,4),(2,3),(2,4),(3,4) \}$

(a) What is R (represent)?

(b) What is matrix representation of R?

In Degree and Out Degree

- If R is a relation on a set A and $a \in A$, then the in-degree of a (relative to relation R) is the number of $b \in A$ such that $(b, a) \in R$.

- The out degree of a is the number of $b \in A$ such that $(a, b) \in R$

In Degree and Out Degree

- Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

  “the number of edges terminating at the vertex”

- The out-degree of a vertex is

  “ the number of edges leaving the vertex”

Example

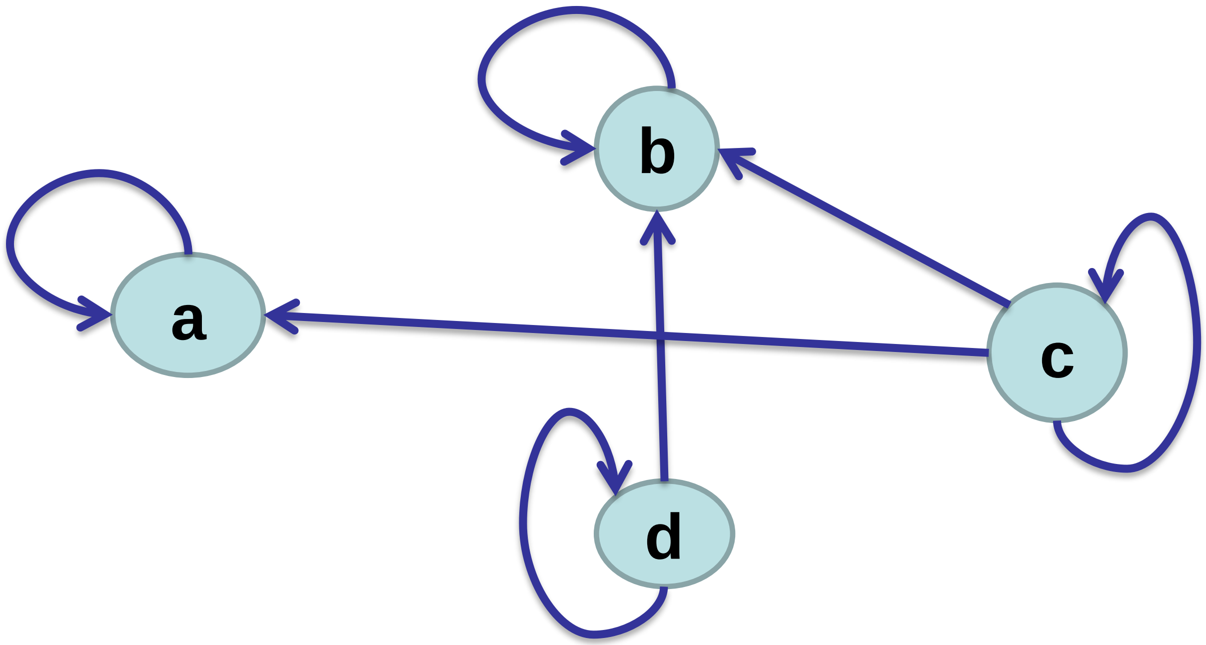
- Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix (given below)

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

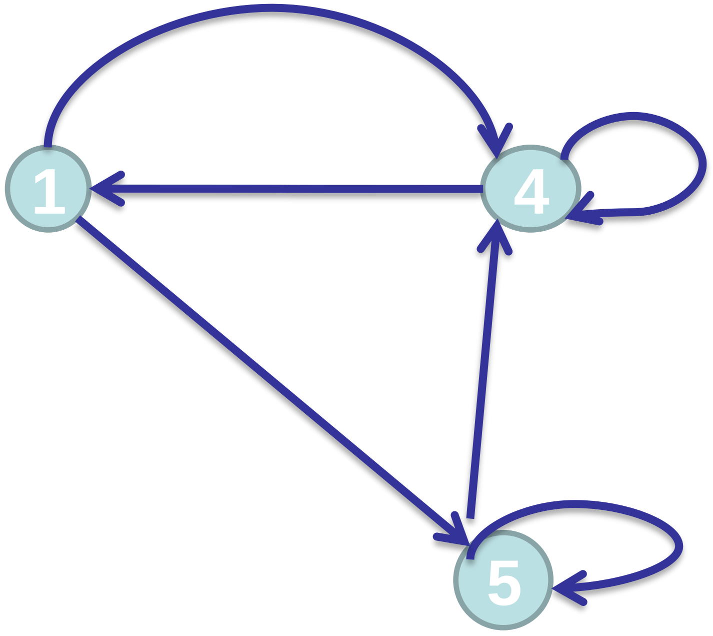
Example

	a	b	c	d
In-degree	2	3	1	1
Out-degree	1	1	3	2



exercise

- Let $A = \{1, 4, 5\}$ and let R be given by the digraph shown below.
- Find M_R and R



example

- An airline services the five cities c_1, c_2, c_3, c_4 and c_5.
- Table below gives the cost (in dollars) of going from c_i to c_j. Thus the cost of going from c_1 to c_3 is \$100, while the cost of going from c_4 to c_2 is \$200

To from	c_1	c_2	c_3	c_4	c_5
c_1		140	100	150	200
c_2	190		200	160	220
c_3	110	180		190	250
c_4	190	200	120		150
c_5	200	100	200	150	

example

- If the relation R on the set of cities $A = \{c_1, c_2, c_3, c_4, c_5\}$: $c_i\ R\ c_j$ if and only if the cost of going from c_i to c_j is defined and less than or equal to \$180.

- Find R.

$$R = \{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2), (c_4,c_3), (c_4,c_5), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$$

exercise

- From the previous example, find the matrices of relations for *R.*

R = $\{(c_1,c_2), (c_1,c_3), (c_1,c_4), (c_2,c_4),(c_3, c_1), (c_3,c_2),$
$(c_4,c_3), (c_4,c_5), (c_4,c_5), (c_5,c_2), (c_5,c_4)\}$

Reflexive Relations

- A relation R on a set X is called reflexive if $(x,x) \in R$ for every $x \in X$.
- That is, if xRx for all $x \in X$.

(R is reflexive if every element $x \in X$ is related to itself)

example

- The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \le y$, $x,y \in X$ is reflexive because for each element $x \in X$, $(x,x) \in R$
- $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$ are each in R.

example

- The relation,

  $R = \{ (a,a), (b,c), (c,b), (d,d) \}$

  on $X=\{a, b, c, d\}$

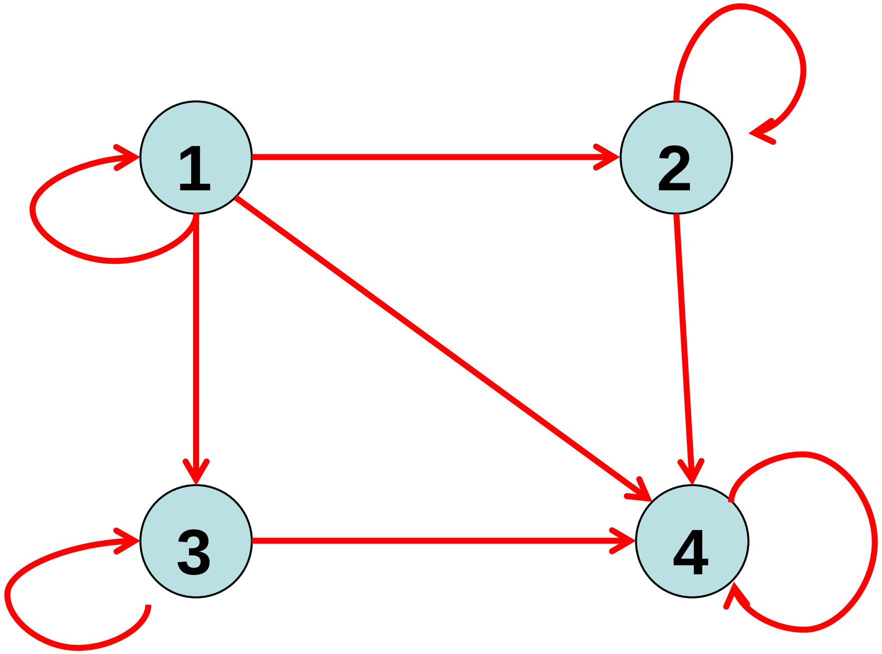
  is not reflexive.

- For example, $b \in X$, but $(b,b) \notin R$

Reflexive Relations

- The digraph of a reflexive relation has a loop at every vertex.

- example



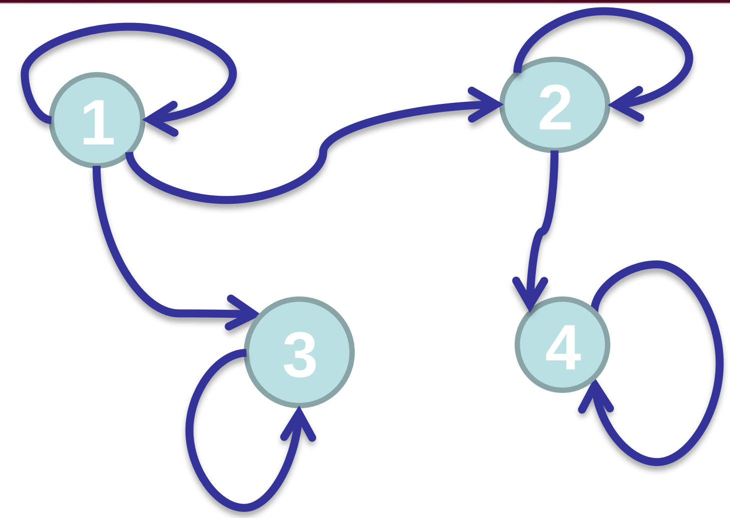
Reflexive Relations

- Irreflexive
  - A relation R on a set A is **irreflexive** if $x \not R x$ or $(x,x) \notin R;\ \forall x : x \in X$

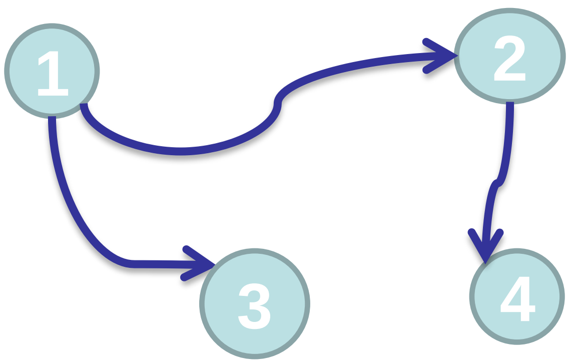
- Not Reflexive
  - A Relation R is **not reflexive** if at least one pair of $(x,x) \notin R,\ \forall x : x \in X$

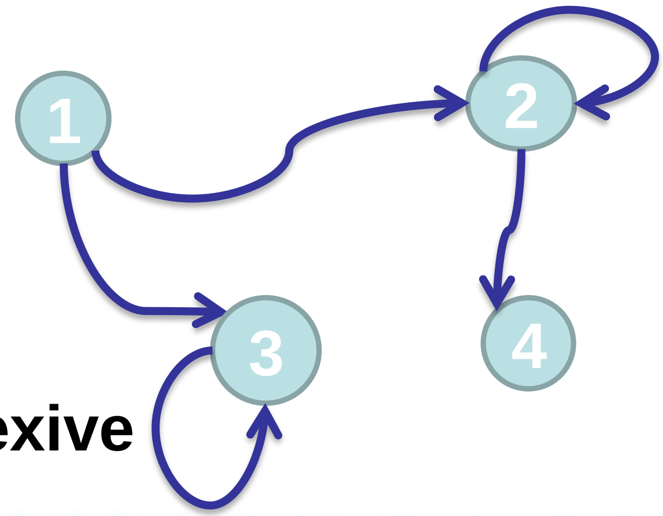
example

Reflexive



Irreflexive





Not reflexive

example

- Consider the following relations on the set {1, 2, 3}

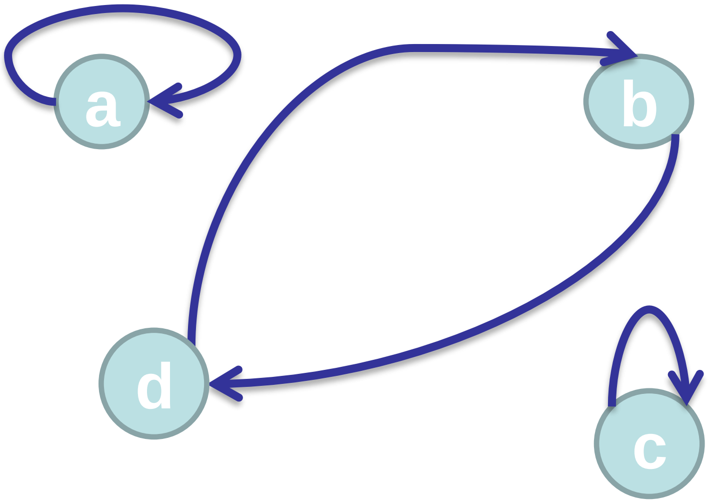
R_1 = { (1,1), (1,2), (2,1), (2,2), (3,1), (3,3) }
R_2 = { (1,1), (1,3), (2,2), (3,1) }
R_3 = { (2,3) }
R_4 = { (1,1) }

Which of them are reflexive?

exercise

(i) Let R be the relation on $X=\{1,2,3,4\}$ defined by $(x,y)\in R$ if $x\leq y$, $x,y\in X$. Determine whether R is a reflexive relation.

(ii) The relation R on $X=\{a,b,c,d\}$ given by the below diagraph. Is R a reflexive relation?



Reflexive Relations

- The relation *R* is reflexive if and only if the matrix of relation has 1's on the main diagonal.

- example

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Reflexive Relations

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

- example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

example

- The relation R is not reflexive.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$b \in X$
$(b,b) \notin R$

exercise

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is reflexive, not reflexive or irreflexive.

(i) R = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) R = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) R ={ (1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

(iv) R ={ (1,2), (1,3), (3,2), (1,4), (4,2), (3,4)}

Symmetric Relations

- A relation R on a set X is called symmetric if for all $x, y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in X,\ (x,y) \in R \rightarrow (y,x) \in R$$

- Let M be the matrix of relation R.

  The relation R is symmetric if and only if for all i and j, the ijth entry of M is equal to the jith entry of M.

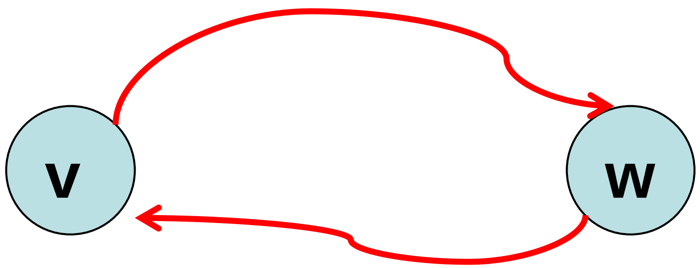
Symmetric Relations

- The matrix of relation M_R is symmetric if $M_R = M_R^T$
- Example

$$M_R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

Symmetric Relations

- The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.



example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$(b,c) \in R$
$(c,b) \in R$

symmetric

example

- The relation R on $X = \{ 1, 2, 3, 4 \}$, defined by

  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$(2,3) \in R$
$(3,2) \notin R$

not symmetric

Antisymmetric Relations

- A relation, *R* on a set *X* is called antisymmetric, if for all $x,y \in X$, if $(x,y) \in R$ and $x \neq y$, then $(y,x) \notin R$.

- A relation *R* on set *X* is antisymmetric if $x \neq y$, whenever xRy, then $y \not R x$. In other word if whenever xRy, then yRx then it implies that $x=y$

$$\forall x,y \in A, (x,y) \in R \wedge x \neq y \rightarrow (y,x) \notin R$$

Or

$$\forall x,y \in A, \ (x,y) \in R \wedge (y,x) \in R \rightarrow x = y$$

Antisymmetric Relations

- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation R satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$.

- If R is antisymmetric relation, then for different vertices i and j there cannot be an edge from vertex i to vertex j and an edge from vertex j to vertex I

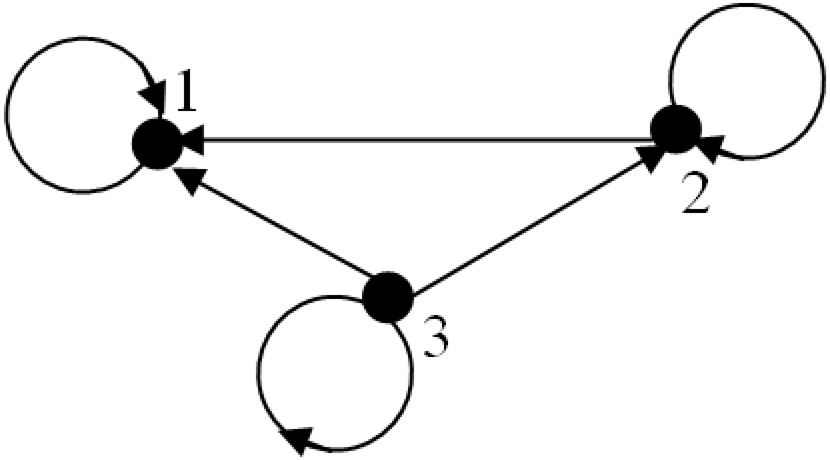
- At least one directed relation and one way

Example

- Let R be a relation on $A = \{1, 2, 3\}$ defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$



example

- The relation R on $X = \{ 1, 2, 3 ,4 \}$ defined by,

  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

$(1,2) \in R$
$(2,1) \notin R$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{array}
$$

antisymmetric

example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$

$(b,c) \in R$
$(c,b) \in R$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

not antisymmetric

example

- The relation

  $R = \{ (a,a), (b,b), (c,c) \}$

  on $X = \{ a, b, c \}$

- R has no members of the form (x,y) *with* $x \neq y$, then R is antisymmetric

$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

example

- Antisymmetric
- Reflexive
- Symmetric

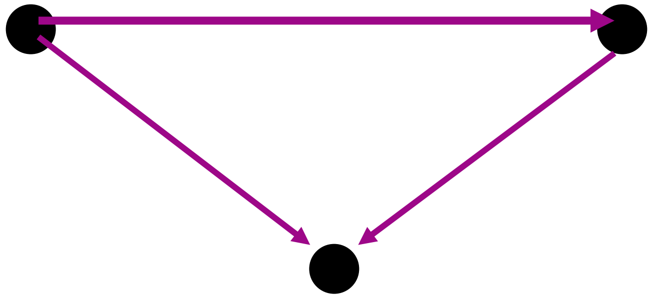
$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

- “Antisymmetric” is not the same as “not symmetric”

Antisymmetric Relations

- The digraph of an antisymmetric relation has at most one directed edge between each pair of vertices.

- Example



Asymmetric

A relation R on set A is asymmetric if whenever aRb, then $b \not R a$.

$$\forall x,y \in X,\ (x,y) \in R \rightarrow (y,x) \notin R$$

In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.

Asymmetric

- The matrix $M_{R} = [m_{ij}]$ of an asymmetric relation R satisfies the property that

  If $m_{ij} = 1$ then $m_{ji} = 0$

  $m_{ii} = 0$ for all i (the main diagonal of matrix M_R consists entirely of 0's or otherwise)
- If R is asymmetric relation, then the digraph of R cannot simultaneously have an edge from vertex i to vertex j and an edge from vertex j to vertex i
- All edges are "one way street"

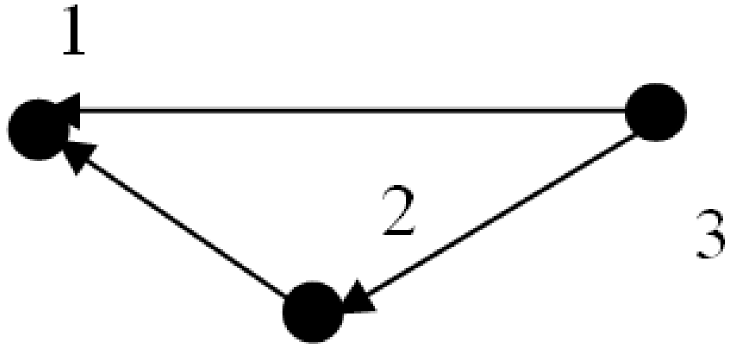
Example

- Let R be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$

$$(3, 1) \in R \text{ but } (1, 3) \notin R$$

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$



Not Symmetric

- Let R be a relation on a set A.
- Then R is called ***not symmetric***, if for all a, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Not Symmetric and Not Antisymmetric

- Let R be a relation on a set A. Then R is called ***not symmetric*** and ***not antisymmetric***, if for all a, $b \in A$, if $(a, b) \in R$, there exist $(b, a) \notin R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \in R$$

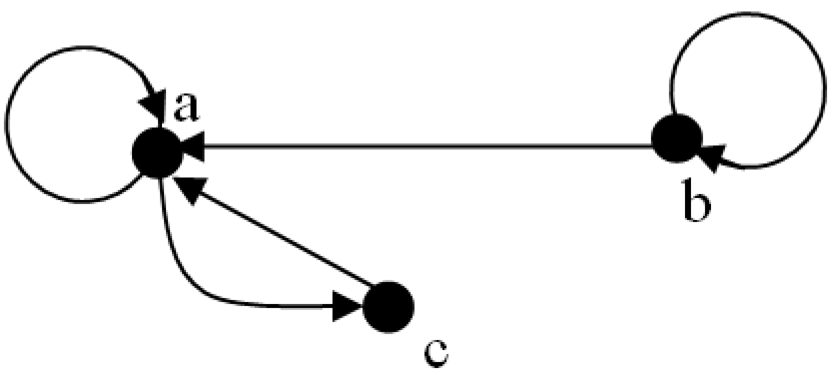
AND

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \notin R$$

Example

- Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$$(a,c), (c,a) \in R \text{ and also } (b,a) \in R \text{ but } (a,b) \notin R$$



Example

1. Let $A=Z$, the set of integers and let $R=\{(a,b)\in A\times A \mid a<b\}$. So that R is the relation "less than".

   Is R symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

   Determine whether R symmetric, asymmetric or antisymmetric.

Example

Question 1

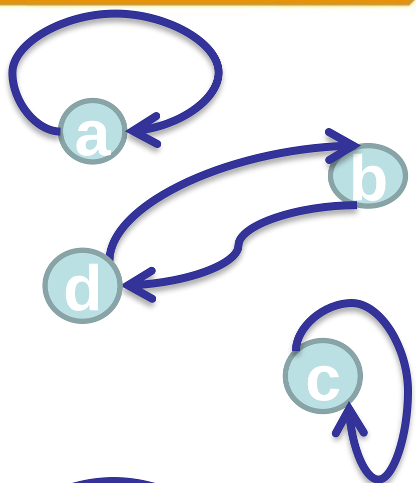
- **Symmetric : If $a<b$, then it is not true that $b<a$, so R is not symmetric**
- **Assymetric : If $a<b$ then $b>a$ (b is greater than a), so R is assymetric**
- **Antisymmetric : If *a*‡*b*, then either $a>b$ or $b>a$, so R is antisymmetric**

Question 2

- **R is not symmetric since $(1,2)\in R$, but $(2,1)\notin$R**
- **R is not asymmetric , since $(2,2)\in R$**
- **R is antisymmetric, since *a*‡*b*, either $(a,b)\notin$R or $(b,a)\notin R$**

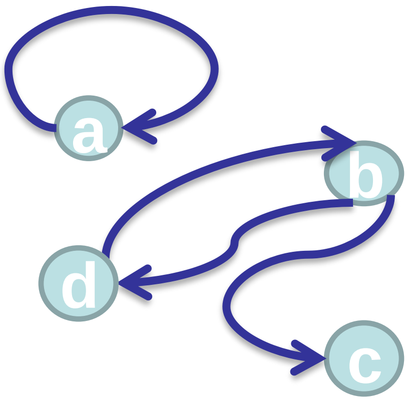
Example

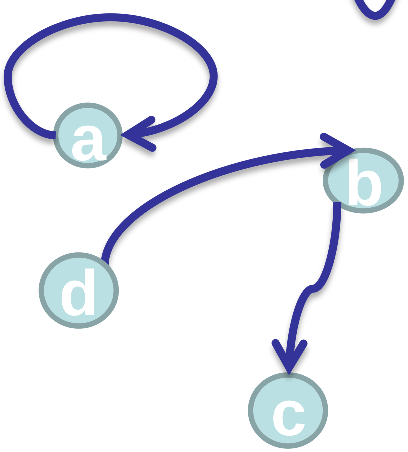
Summary on Symmetric



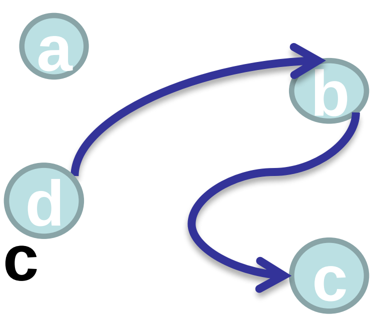
Symmetric

Not Symmetric





Antisymmetric



Asymmetric

Exercise

Let A = { 1,2,3,4). Construct the matrix of relation of *R*. Then, determine whether the relation is symmetric, asymmetric, not symmetric or antisymmetric.

(i) R = { (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4))

(ii) R = { (1,3), (1,1), (3,1), (1,2), (3,3), (4,4)}

(iii) R ={ (1,2), (1,3), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)}

Transitive Relations

- A relation R on a set X is called transitive if for all $x,y,z \in X$,

  if (x,y) and $(y,z) \in R$ then $(x,z) \in R$

- It is often convenient to say what it means for a relation to be not transitive.

- A relation R on X is **not transitive** if there exists x, y, and z in X so that xRy and yRz, but $x \not R z$. If such x,y, and z do not exist, then R is transitive.

Transitive Relations

- Let M_R be the matrix of relation R.
  The relation R is transitive if,

$$M_R \otimes M_R = M_R.$$

$\otimes$ Boolean product.

Boolean Algebra

+	1	0
1	1	1
0	1	0

.	1	0
1	1	0
0	0	0

example

- The relation R on $X = \{ 1, 2, 3, 4 \}$ defined by

  $(x,y) \in R$ if $x \leq y$, $x,y \in X$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

example

- $M_R \otimes M_R = M_R$

transitive

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

example

- The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$ on $X = \{ a, b, c, d \}$ is not transitive.

- (b,c) and $(c,b) \in R$, but $(b,b) \notin R$.

example

- Consider the following relations on the set {1, 2, 3}

  R1 = { (1,1), (1,2), (2,3) }
  R2 = { (1,2), (2,3), (1,3) }

- Which of them is transitive?

example

Let R be a relation on $A=\{1,2,3\}$ is defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$. Find R. Is R a transitive relation?

Solution:

$R=\{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

R is a transitive relation because

$(1,2)$ and $(2,2)\in R$, $(1,2)\in R$

$(1,2)$ and $(2,3)\in R$, $(1,3)\in R$

$(1,3)$ and $(3,3)\in R$, $(1,3)\in R$

$(2,2)$ and $(2,3)\in R$, $(2,3)\in R$

$(2,3)$ and $(3,3)\in R$, $(2,3)\in R$

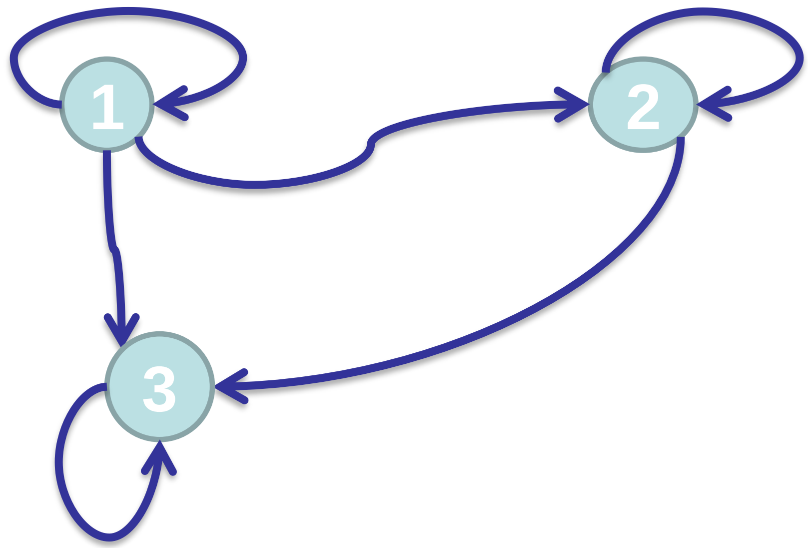
Transitive Relations

In the digraph of *R*, *R* is a transitive relation if and only if there is a directed edge from one vertex *a* to another vertex *b*, an if there exits a directed edge from vertex *b* to vertex *c*, then there must exists a directed edge from *a* to *c*

example

R={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

The diagraph:



example

The relation R on $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$, is a transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and $(2,3)\in R$, $(1,3)\in R$

example

The relation R on $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in A$, is a transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,(1,2) and $(2,3)\in R$, $(1,3)\in R$

Example

The relation R on A={a,b,c,d} IS r={(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)} is not transitive. The matrix of relation M_R,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\boldsymbol{M_R \otimes M_R \neq M_R}$$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Note that,(a,c) and (c,b)$\in R$, (a,b)$\notin R$

Equivalence Relations

- A relation *R* that is reflexive, symmetric and transitive on a set *X* is called an equivalence relation on *X*.

example

Let R={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

example

Let R={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation M_R,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

example

The transpose matrix M_R, M_R^T is equal to M_R, so R is symmetric

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array} \qquad M_R^T = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

The product of Boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

example

- The relation, R={ (1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5) }
  on { 1,2,3,4,5 }

  - Reflexive?
  - Symmetric?
  - Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

example

- Reflexive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Reflexive √

example

- Symmetric?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric √

example

- Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

example

- Reflexive
- Symmetric
- Transitive

$$\left.\begin{array}{l}\text{Reflexive}\\ \text{Symmetric}\\ \text{Transitive}\end{array}\right\}\ \text{Equivalence relation}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

example

- The relation R on $X=\{ 1, 2, 3, 4 \}$, defined by
  $(x,y) \in R$ if $x \leq y,\ x,y \in X$

- Not symmetric
  - $(2,3) \in R$ but $(3,2) \notin R$

- R is not equivalence relation on X.

Partial Orders

- A relation, *R* on a set *X* is called a partial order if *R* is reflexive, antisymmetric, and transitive.

example

Let R be a relation on a set $A=\{1,2,3\}$ defined by $(a,b)\in R$ if $a\leq b$, $a,b\in R$.

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

R is reflexive, antisymmetric and transitive.

So R is a partial order relation.

example

- The relation R defined on the positive integers by

  $(x,y) \in R$ if x divides y (evenly)

  is reflexive, antisymmetric and transitive

- R is a partial order.

example

- The relation R defined on the set of integers by

  $(x,y) \in R \quad \text{if } x \leq y$

  is reflexive, antisymmetric, and transitive.

- R is a partial order.

exercise

The relation R on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x+y \leq 6$

(i) List the elements of R

(ii) Find the domain of R

(iii) Find the range of R

(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

exercise

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if 3 divides $x-y$

(i) List the elements of R
(ii) Find the domain of R
(iii) Find the range of R
(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?

exercise

The relation *R* on the set {1,2,3,4,5} defined by the rule $(x,y) \in R$ if $x=y-1$

(i) List the elements of R
(ii) Find the domain of R
(iii) Find the range of R
(iv) Is the relation of R refelxive, symmetric, assymmetric, antisymmetric, transitive, and/or equivalence relation or partial order?