1. Curves [15 marks]
The dress code for the wooden monkey awards includes a bow tie, described in 2D with the following parametric form: x(t) = 2*sin(t) , y(t) = 5*sin(t)*cos(t) , 0 <= t <= 2π.

a)Can you write the above function in its implicit form [3 marks].

X(t) = 2sin(t) = x, y(t) = 5sin(t)cos(t) = 2sin(t) * 2.5cos(t) = 2.5cos(t)x = y
Therefor, sin(t) = x/2, cos(t) = y/x * 2/5
Since sin(t)^2 + cos(t)62 = 1
$(\frac{x}{2})^2 + (\frac{4}{25})(y/x)^2 = 1$

b)Find the tangent vector [2 marks] and a normal vector [2 marks] to the curve as a function of t.

Let's denote the curve as p(t) = (x(t), y(t)),

We can find the tangent vector as p'(t) = (x'(t), y'(t)), so
p'(t) = (x'(t), y'(t)) = (2cos(t), 5cos(2 t)) and normal vector as (-y'(t), x'(t)), which is (-5cos(2t), 2cos(t))

c)Is the curve symmetric around the X-axis? Y-axis? (proof/counterexample) [2 marks].

The curve is symmetric around X-axis. By substituting x = -x into the implicit function in a) as below, we get the same result as the original function which tells us that the curve is symmetric around X-axis.

$((\frac{-x}{2})^2 + (\frac{4}{25})(y/-x)^2 = (\frac{x}{2})^2 + (\frac{4}{25})(y/x)^2 = 1$

The curve is also symmetric around Y-axis. By substituting y = -y into the implicit function in a) as below, we get the same result as the original function which tells us that the curve is symmetric around Y-axis.

$((\frac{x}{2})^2 + (\frac{4}{25})(-y/x)^2 = (\frac{x}{2})^2 + (\frac{4}{25})(y/x)^2 = 1$

d)What is the enclosed area of the bow-tie [3 marks]?

Since the curve is symmetric around x-axis,

$$\boldsymbol{Area} = \int_{\boldsymbol{0}}^{\boldsymbol{2\pi}} \boldsymbol{y(t)x'(t)\,dt} = \boldsymbol{2} * \int_{\boldsymbol{0}}^{\boldsymbol{\pi}} 5\sin(t)\cos(t)\,2\cos(t)\,\boldsymbol{dt}$$
$$= \boldsymbol{2} * ((-\tfrac{\boldsymbol{10}}{\boldsymbol{3}})\,\boldsymbol{cos^3(\pi)} — \tfrac{\boldsymbol{10}}{\boldsymbol{3}}\boldsymbol{cos^3(0)}) = \boldsymbol{2} * (\tfrac{\boldsymbol{10}}{\boldsymbol{3}} + \tfrac{\boldsymbol{10}}{\boldsymbol{3}}) = \boldsymbol{40/3}$$

e)How can one piecewise linearly approximate the perimeter of the bowtie [3 marks]?

Since the curve is symmetric around x-axis, the equation for finding the arc length can be written as:

$$\boldsymbol{Perimeter} = \int_0^{2\pi} \sqrt{y'(t)^2 + x'(t)^2}\, dt = 2 * \int_0^{\pi} \sqrt{y'(t)^2 + x'(t)^2}\, dt$$
$$= 2 * \int_0^{\pi} \sqrt{(5\cos(2t))^2 + (2\cos(t))^2}\, dt$$

2. Transformations [10 marks]

Two transformations T1 and T2 commute when T1 * T2 = T2 * T1. A point p is a fixed point of a transformation T if and only if Tp = p. For each pair of transformations below, specify whether or not they commute in general. Moreover, if you conclude that they commute, provide a proof, and if you claim the converse, provide a counterexample as proof [2.5 marks for each part].

(a) translation and translation.

This transformation is commutative.

Proof:

Let p = [p1, p2, p3, ..., pn, 1], and T1= $\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & a1 \\ 0 & 1 & 0 & 0 & \cdots & a2 \\ 0 & 0 & 1 & 0 & \cdots & a3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0\cdots & 1 & an \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$,

and T2 = $\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & b1 \\ 0 & 1 & 0 & 0 & \cdots & b2 \\ 0 & 0 & 1 & 0 & \cdots & b3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0\cdots & 1 & bn \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$.

So, Tp = (T1 * T2)p = [p1+a1+b1, p2+a2+b2, p3+a3+b3, ..., pn+an+bn, 1]
= [p1+b1+a1, p2 +b2+a2, p3 +b3+a3, ..., pn +bn+an, 1] = (T2 * T1)p

(b) translation and rotation.

This transformation is not commutative.

Let p = [0, 0, 1], T1 = $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$,T2 = $\begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$, t is the rotation angle.

T1*T2 = $\begin{bmatrix} \cos(t) & -\sin(t) & a \\ \sin(t) & \cos(t) & b \\ 0 & 0 & 1 \end{bmatrix}$, T2*T1 = $\begin{bmatrix} \cos(t) & -\sin(t) & \cos(t)\,a - \sin(t)\,b \\ \sin(t) & \cos(t) & \sin(t)\,a + \cos(t)\,b \\ 0 & 0 & 1 \end{bmatrix}$

Since T1*T2 != T2*T1, this pair of transformation is not commutative.

(c) scaling and rotation, having different fixed points.
This pair of transformation is not commutative, for example,
we have a point p = (1,1), and we rotate a degree of 180 around point (-1,-1), it will goes to (-3,-3), and then we scale it by a factor of 5 around point (0,0), it becomes (-15, -15).

If we first scale the point (1,1) by factor 5 around (0,0), it becomes (5, 5), and we rotate it around (-1, -1), it eventually goes to (-7, -7) which has a different location compared to the first set of transformation.

(d) scaling and scaling, having the same fixed point.
This pair of transformation is commutative.
Let (f1, f2, …, fn) be the fixed point and (p1,p2,…,pn) be the point we are about to scale.
Let A and B be the two factor we are going to scale with.

T1*T2(P) = A*B(p1-f1,p2-f2,…,pn-fn)
= ((A*B)(p1-f1), (A*B)(p2-f2),…, (A*B)(pn-fn))
= ((B*A)(p1-f1), (B*A)(p2-f2),…, (B*A)(pn-fn))
= B*A(p1-f1,p2-f2,…,pn-fn)
= T2*T1(P)

<u>3. Homography [10 marks]</u>

Points (1,0); (0,1); (1,1); (0,0) map to points (6,2); (7,3); (6,3); (7,2) by an Affine transformation.

[8 marks] Derive the Affine transformation. Simply write the steps needed to find it.

Let H denote the affine transformation matrix, and
H = $\begin{bmatrix} a1 & a2 & t1 \\ a3 & a4 & t2 \\ 0 & 0 & 1 \end{bmatrix}$, let A = $\begin{bmatrix} a1 & a2 \\ a3 & a4 \end{bmatrix}$, t = $\begin{bmatrix} t1 \\ t2 \end{bmatrix}$.
We then substitute each original point p to the equation Ap + t = p’, we will then have 8 equations with six unknowns that we can solve for.

At the end, A = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, t = $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

[2 marks] Where does the point (2,5) get mapped to under this transformation?
Ap + t = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

(2,5) will map to (5, 7)

4. Polygons [10 marks]

Given an arbitrary, non-degenerate 2D triangle with vertices v0, v1, and v2, write a procedure for determining if a point q is inside/outside the triangle [4 marks], or on an edge of the triangle [2marks]. The procedure can be described or in pseudo-code, as long as the steps are clear.

Assume the point given is (a, b)
First, compute the equation of line formed by each two vertices.
Let's say they are f0,1(x), f0,2(x), f1,2(x).

Pseudo-code:
#Given point is on the same side as the other vertice, continue to check,
point that is inside the triangle is on the same side as one of the vertices to the line formed by the rest two vertices.

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If (V2[1] > f0,1(V2[0]) && b>= f0,1(a)) or (V2[1] < f0,1(V2[0]) && b =< f0,1(a)):
   If (V1[1] > f0,2(V1[0]) && b>= f0,2(a)) or (V1[1] < f0,2(V1[0]) && b =< f0,2(a)):
      If (V0[1] > f1,2(V0[0]) && b>= f1,2 (a)) or (V0[1] < f1,2 (V0[0]) && b =< f1,2 (a)):

         #The point (a, b) is either inside the triangle or on the edge.
         If( b == f0,1(a)) or b == f0,2(a)) b == f1,2 (a))):
            Return "on the edge"
         Else:
            Return "inside"

Else:
   Return "outside"
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How can one compute the area of a triangle and its centroid (also its center of mass) [4 marks]?

Area

Let v0 = [x0, y0], v1 = [x1, y1], v2 = [x2, y2]

Length of line segment between v0, v1 is $\sqrt{(x1-x0)^2+(y1-y0)^2}$
The slope of this line segment is (y1-y0)/(x1-x0)
The slope of the normal line to the above line segment is -(x1-x0)/ (y1-y0)

Now substitute x = x2, y = y2 to the equation y = -(x1-x0)/ (y1-y0)x+b and we will get

b= y2 + (x1-x0)/ (y1-y0)x2, so the normal line has an equation of y = -(x1-x0)/ (y1-y0)x+ y2 + (x1-x0)/ (y1-y0)x2

We then can find the intersection point between this normal line and the line segment between v0, v1 by equating the two equations.
At the end, we compute the length between the intersection point and v2, and use this length multiply $\sqrt{(x1 - x0)^2 + (y1 - y0)^2}$ and divide it by 2. This is the area of the triangle.

Centroid

It can be found by using the following equation:
Coordinate of the centroid = [(x1 + x2 + x3)/3, (y1 + y2 + y3)/3]