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A key source of pointing jitter is due to reaction wheels (RWs) mass imbalance about the wheel spin axis. Although these effects are often characterized through experimentation in order to validate requirements, it is of interest to include jitter in a computer simulation of the spacecraft in the early stages of spacecraft development. An estimate of jitter amplitude may be found by modeling wheel imbalance torques as an external disturbance on the spacecraft. In this case, reaction wheel mass imbalances are lumped into static and dynamic imbalance parameters, allowing jitter force and torque to be simply proportional to wheel speed squared. A physically realistic dynamic model may be obtained by defining mass imbalances in terms of a RW center of mass location and inertia tensor. The fully-coupled dynamic model allows momentum and energy validation of the system. This is often critical when modeling additional complex dynamical behavior such as flexible dynamics and fuel slosh. Furthermore, it is necessary to use the fully-coupled model in instances where the relative mass properties of the spacecraft with respect to the RWs cause the simplified jitter model to be inaccurate. This paper presents a generalized approach to reaction wheel imbalance modeling of a rigid hub with N reaction wheels. A discussion is included to convert from manufacturer specifications on RW imbalances to the introduced parameters. In addition, a back-substitution method is introduced to increase the computational efficiency of a computer simulation.

I Introduction

Momentum exchange devices are a fundamental component of most spacecraft for both coarse attitude control and precision pointing. Most modern spacecraft include three or more reaction wheels (RWs), which consist of a flywheel attached to a motor and bearing fixed to the spacecraft. A challenge to using RWs is that they may induce jitter due to mass imbalances in the RW. Characterization and mitigation of RW induced jitter on a spacecraft is important to many missions due to the increasingly rigorous attitude stability requirements and the necessity of avoiding excitation of the spacecraft's structural modes. Excessive vibration of a spacecraft may be detrimental to its instruments and operation. Additionally, many instruments require rigorous attitude stability in order to effectively operate or collect data. Optical instruments in particular often require attitude stability of less than one arc-second per second in order to avoid optical smear or similar effects.[1,2]

RW induced vibration on a spacecraft is usually characterized through experimentation prior to flight in order to validate requirements. Empirical models of RWs allow imbalance parameters to be extracted.[3,4] In addition to experimental demonstration of RW performance on an integrated spacecraft, it is of interest to use an analytic model of a RW for simulation in the early stages of spacecraft development. A simplified model of RW jitter involves including forces and torques resulting from RW imbalance as external disturbances.[5–7] This method is attractive due to its non-computationally expensive formulation – force and torque of jitter are simply proportional to wheel speed squared. Furthermore, the simplified formulation allows a model to be constructed directly from the typical RW manufacturer imbalance specifications: static imbalance and dynamic imbalance. This allows RW mass imbalances to be implemented as lumped parameters instead of using specific terms such as RW center of mass location and inertia tensor.[7] Previous literature puts emphasis on empirical modeling of RW jitter and the effects of RW jitter within context of spacecraft flexible dynamics.[8–10]

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The simplified "lumped parameter" method of modeling RW jitter is inaccurate due to the nonconservative nature of adding a system-internal forcing effect as an external disturbance.[11] Since angular momentum is not conserved in this model, a time varying bias in angular velocity is observed. The magnitude of the bias is dependent on the relative magnitude of the spacecraft inertia versus the reaction wheel imbalance and the wheel speed. For analysis purposes this does not necessarily present a problem. The overall effect of the angular velocity bias is quite small for spacecraft that have small wheel imbalance to spacecraft inertia ratios and the amplitude of RW induced jitter may be computed by subtracting a polynomial fit of appropriate order from the resulting angular velocity. For spacecraft with poorly balanced reaction wheels or small wheel mass/imbalance to spacecraft inertia ratios this approach may become problematic. Additionally, it is undesirable to run this model in a simulation involving pointing accuracy assessment, power assessment, flexible structures, propellant slosh, etc. due to momentum and energy validation being unavailable.

This paper presents a general derivation of equations of motion for a spacecraft with N imbalanced reaction wheels. A Newtonian/Eulerian formulation approach is employed. Special consideration is given to the computational speed of the solution. To avoid inverting a large system mass matrix, the equations of motion are written such that rigid body and RW jitter modes can be solved for sequentially. This provides an elegant analytical one-way decoupling of the equations of motion using a minimal coordinate set, and avoids the kinematic complexities of general N-th order solution such as in Reference 12. Since the spacecraft hub is considered to be rigid, flexible dynamics are not considered. The body of the paper gives detail on the mathematical model, numerical simulation, and draws conclusions on the results.

II Problem Statement

An offset in the center of mass of the RW from the spin axis, denoted static imbalance, results in an internal force on the spacecraft. Asymmetric distribution of mass about the RW spin axis, denoted dynamic imbalance, produces an internal torque on the spacecraft. Figure 1 explains these imbalances geometrically. I_p is a line that is coincident with the center mass of the RW and defines a principal axis of the RW. The static imbalance results in a center of mass offset of the RW but does not change the direction of the principal axes. The dynamic imbalance is result of the principal axes not being aligned with the spin axis. Deflection of the RW wheel bearing due to static and dynamic imbalances further affects the vibrational modes of the system, however, this effect is beyond the scope of this work and is not being considered.

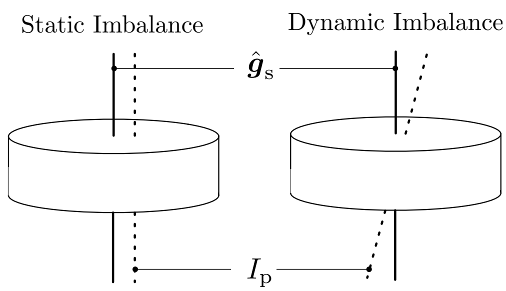

Figure 1: Reaction wheel static and dynamic imbalance.

When deriving the equations of motion (EOMs) for a spacecraft with N reaction wheels, an important assumption is made that the reaction wheels are symmetric and results in the EOMs to be simplified to a convenient and compact form.[11] However, if the reaction wheels are imbalanced the EOMs have to be re-derived to account for the fully-coupled dynamics between the RWs and the spacecraft. This paper follows a development path using Newtonian and Eulerian mechanics using a formulation that uses a minimal coordinate description.[11]

Figure 2 shows the frame and variable definitions used for this problem. The formulation involves a rigid hub with its center of mass location labeled as point B_c, and N RWs with their center of mass locations labeled as W_{c_i}. The frames being used for this formulation are the body frame, $\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$, the motor frame of the i^{th} RW, $\mathcal{M}_i : \{\hat{\boldsymbol{m}}_{s_i}, \hat{\boldsymbol{m}}_{2_i}, \hat{\boldsymbol{m}}_{3_i}\}$, and the wheel frame of the i^{th} RW, $\mathcal{W}_i : \{\hat{\boldsymbol{g}}_{s_i}, \hat{\boldsymbol{w}}_{2_i}, \hat{\boldsymbol{w}}_{3_i}\}$. The dynamics are modeled with respect to the $\mathcal{B}$ frame which can be oriented in any direction. The $\mathcal{W}_i$ frame is oriented such that the $\hat{\boldsymbol{g}}_{s_i}$ axis is aligned with the spin axis of the RW, the $\hat{\boldsymbol{w}}_{2_i}$ axis is perpendicular to $\hat{\boldsymbol{g}}_{s_i}$ and points to W_{c_i}. The $\hat{\boldsymbol{w}}_{3_i}$ completes the right hand rule. The $\mathcal{M}_i$ frame is defined as being equal to the $\mathcal{W}_i$ frame at the beginning of the simulation and therefore the $\mathcal{W}_i$ and $\mathcal{M}_i$ frames are offset by an angle, θ_i, about the $\hat{\boldsymbol{m}}_{s_i} = \hat{\boldsymbol{g}}_{s_i}$ axes. These are the necessary frame and variable definitions needed for this formulation.

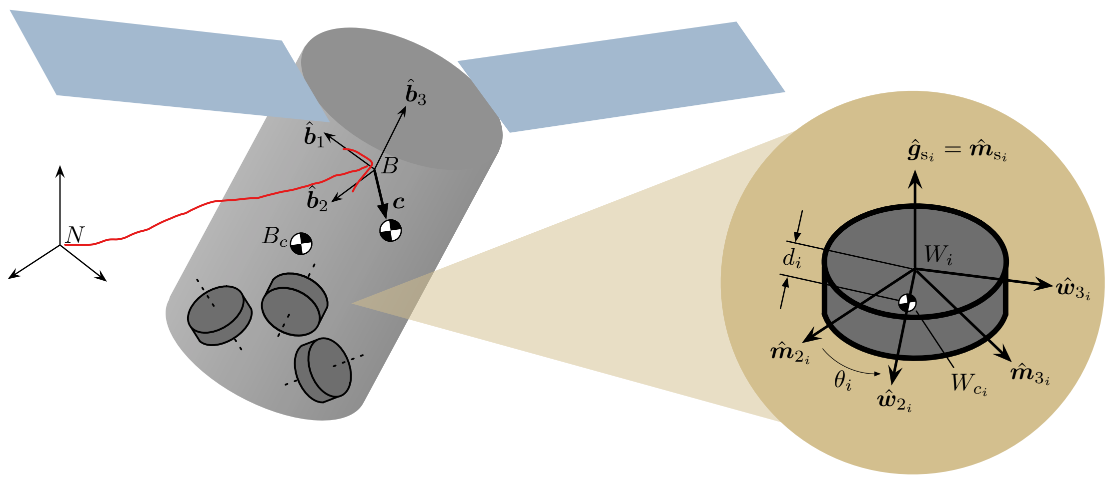

Figure 2: Reference frame and variable definitions.

A few more key variables in Figure 2 need to be defined. Point B is the origin of the $\mathcal{B}$ frame and and is a general body-fixed point that does not have to be identical to the spacecraft center of mass. Point W_i is the origin of the $\mathcal{W}_i$ frame and can also have any location relative to point B. Point C is the center of mass of the spacecraft including the RWs and vector $\boldsymbol{c}$ points from point B to point C. Variable d_i is the center of mass offset of the RW, or the distance from the spin axis, $\hat{\boldsymbol{g}}_{\mathrm{s}_i}$ to W_{c_i}. These variable and frame definitions are leveraged throughout the paper to derive the EOMs.

III Equations of Motion

The system under consideration is an $N + 6$ degrees-of-freedom (DOF) system with the following second order terms: inertial acceleration $\ddot{\boldsymbol{r}}_{B/N}$, angular acceleration $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$, and the acceleration of each RW $\dot{\Omega}_1, \ldots, \dot{\Omega}_N$. Thus, a total of $N + 6$ equations must be developed in order to solve for all second order terms. Section III.A describes the derivation of the translational EOM and represents 3 DOF, section III.B describes the rotational motion and represents 3 DOF, and section III.C describes the motor torque equation and represents N DOF.

III.A Translational Motion

For the dynamical system considered the center of mass of the spacecraft is not constant with respect to the body frame. This results in the necessity to track the center of mass of the spacecraft and its corresponding acceleration. Following a similar derivation as seen in Reference 13, the derivation begins with Newton's first law for the center of mass of the spacecraft seen in Eq. (1).

$$\ddot{\boldsymbol{r}}_{C/N} = \frac{\boldsymbol{F}}{m_{\text{sc}}} \quad = \infty \tag{1}$$

$\boldsymbol{F}$ is the sum of the external forces on the spacecraft which has a mass labeled as m_{sc}. The notation being used for this work can be seen in Reference 11. For example, the vector $\boldsymbol{v}_{B/A}$ is a vector that points from point A to B. The inertial time derivative of $\boldsymbol{v}_{B/A}$ is denoted by $\dot{\boldsymbol{v}}_{B/A}$ and the time derivative taken with respect to the body frame is $\boldsymbol{v}'_{B/A}$.

Ultimately the acceleration of the body frame or point B is desired, which is expressed through

$$\ddot{\boldsymbol{r}}_{B/N} = \ddot{\boldsymbol{r}}_{C/N} - \ddot{\boldsymbol{c}} \tag{2}$$

where the center of mass equation is rewritten to yield

$$\boldsymbol{c} = \frac{1}{m_{\text{sc}}}(m_{\text{hub}}\boldsymbol{r}_{B_c/B} + \sum_{i=1}^{N} m_{\text{rw}_i}\boldsymbol{r}_{W_{c_i}/B}) \tag{3}$$

$r_{B/N} + c''$

Taking the first and second body frame time derivatives of point $\boldsymbol{c}$ results in

$$\boldsymbol{c}' = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} \boldsymbol{r}'_{W_{c_i}/B} \tag{4}$$

$$\boldsymbol{c}'' = \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} \boldsymbol{r}''_{W_{c_i}/B} \tag{5}$$

Takeing the first and second body frame time derivatives of $\boldsymbol{r}_{W_{c_i}/B}$ results in

$$\boldsymbol{r}_{W_{c_i}/B} = \boldsymbol{r}_{W_i/B} + \boldsymbol{r}_{W_{c_i}/W_i} = \boldsymbol{r}_{W_i/B} + d_i \hat{\boldsymbol{w}}_{2_i} \tag{6}$$

$$\boldsymbol{r}'_{W_{c_i}/B} = d\hat{\boldsymbol{w}}'_{2_i} = \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \times d_i \boldsymbol{w}_{2_i} = \Omega_i \hat{\boldsymbol{g}}_{s_i} \times d_i \boldsymbol{w}_{2_i} = d_i \Omega_i \hat{\boldsymbol{w}}_{3_i} \tag{7}$$

$$\boldsymbol{r}''_{W_{c_i}/B} = \Omega_i \hat{\boldsymbol{g}}_{s_i} \times d_i \Omega_i \hat{\boldsymbol{w}}_{3_i} = d_i \dot{\Omega}_i \hat{\boldsymbol{w}}_{3_i} - d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} \tag{8}$$

Using the transport theorem[11] the inertial and body-relative time derivatives of $\boldsymbol{c}$ are related through

$$\ddot{\boldsymbol{c}} = \boldsymbol{c}'' + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}' + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{c}) \tag{9}$$

Substituting Eqns. (8) and (9) into Eq. (2) and grouping second order terms on the left-hand side yields the translational equation of motion.

$\ddot{r}_{B/N}$

$$\ddot{\boldsymbol{r}}_{B/N} - [\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i} \dot{\Omega}_i = \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}} \sum_{i=1}^{N} m_{\text{rw}_i} d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i} \tag{10}$$

Equation (10) shows that the translational acceleration, $\ddot{\boldsymbol{r}}_{B/N}$, is coupled with the rotational acceleration, $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$, and the wheel accelerations, $\dot{\Omega}_i$. This is a result of the fact that the reaction wheels are unbalanced and therefore change the center of mass location of the spacecraft.[11]

III.B Rotational Motion

The rotational motion equation of the spacecraft also needs to be modified. This derivation starts with the angular momentum of the spacecraft about point B:

$$\boldsymbol{H}_{\text{sc},B} = \boldsymbol{H}_{\text{hub},B} + \sum_{i=1}^{N} \boldsymbol{H}_{\text{rw}_i,B} \tag{11}$$

The EOM for the rotational motion is found using the definition of the inertial time derivative of angular momentum when the body fixed coordinate frame origin is not coincident with the center of mass of the body.[11]

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \boldsymbol{L}_B + m_{\text{sc}} \ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} \tag{12}$$

The inertial derivative of the spacecraft angular momentum is expressed as

$$\dot{\boldsymbol{H}}_{\text{sc},B} = \dot{\boldsymbol{H}}_{\text{hub},B} + \sum_{i=1}^{N} \dot{\boldsymbol{H}}_{\text{rw}_i,B} \tag{13}$$

Thus, in order to use Eq. (12), each derivative on the right-hand side of Eq. (13) needs to be evaluated. The equation for finding the angular momentum about a point not coincident with the center of mass of that object[11] is utilized and the following definitions are found

$$\boldsymbol{H}_{\text{hub},B} = \boldsymbol{H}_{\text{hub},B_c} + m_{\text{hub}} \boldsymbol{r}_{B_c/B} \times \dot{\boldsymbol{r}}_{B_c/B} \tag{14}$$

$$\boldsymbol{H}_{\text{rw}_i,B} = \boldsymbol{H}_{\text{rw}_i,W_{c_i}} + m_{\text{rw}_i} \boldsymbol{r}_{W_{c_i}/B} \times \dot{\boldsymbol{r}}_{W_{c_i}/B} \tag{15}$$

where the angular momentum of the hub and reaction wheel about their respective center of masses are

$$\boldsymbol{H}_{\text{hub},B_c} = [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{16}$$

$$\boldsymbol{H}_{\text{rw}_i,W_{c_i}} = [I_{\text{rw}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}} = [I_{\text{rw}_i,W_{c_i}}](\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i \hat{\boldsymbol{g}}_{\text{s}_i}) \tag{17}$$

Cons

Taking the inertial time derivative of hub's angular momentum yields

$$\dot{\boldsymbol{H}}_{\text{hub},B} = [I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times \ddot{\boldsymbol{r}}_{B_c/B} \tag{18}$$

and knowing that $\boldsymbol{r}_{B_c/B}$ is fixed with respect to the body the following are defined

$$\dot{\boldsymbol{r}}_{B_c/B} = \boldsymbol{r}'_{B_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B} \tag{19}$$

$$\ddot{\boldsymbol{r}}_{B_c/B} = \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B}) \tag{20}$$

Substitute Eq. (20) into Eq. (18) yields

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{hub},B} = {}& [I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times (\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B}) + m_{\text{hub}}\boldsymbol{r}_{B_c/B} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{B_c/B}))\end{aligned} \tag{21}$$

Employing the Jacobi triple-product identity, $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} + \boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c})$, on the right-hand side of Eq. (21) and using the parallel axis theorem $[I_{\text{hub},B}] = [I_{\text{hub},B_c}] + m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T$, the hub angular momentum derivative is finally written after much algebra as

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{hub},B} = {}&[I_{\text{hub},B_c}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B_c}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ m_{\text{hub}}[\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_{\text{hub}}[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{r}}_{B_c/B}][\tilde{\boldsymbol{r}}_{B_c/B}]^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ = {}&[I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}\end{aligned} \tag{22}$$

Following an equivalent derivation procedure, the inertial time derivative of reaction wheel angular momentum about point B is

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{rw}_i,B} = {}&[I_{\text{rw}_i,W_{c_i}}]'(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{s_i}) + [I_{\text{rw}_i,W_{c_i}}](\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{s_i}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [I_{\text{rw}_i,W_{c_i}}](\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{s_i}) \\ &+ m_{\text{rw}_i}\boldsymbol{r}_{W_{c_i}/B} \times \ddot{\boldsymbol{r}}_{W_{c_i}/B}\end{aligned} \tag{23}$$

The body relative inertia tensor derivative $[I_{\text{rw}_i,W_{c_i}}]'$ needs to be defined. For this general RW model, the inertia matrix of the RW in the $\mathcal{W}_i$ frame is defined as

$$[I_{\text{rw}_i/W_{c_i}}] = {}^{\mathcal{W}_i}\begin{bmatrix} J_{11_i} & J_{12_i} & J_{13_i} \\ J_{12_i} & J_{22_i} & J_{23_i} \\ J_{13_i} & J_{23_i} & J_{33_i} \end{bmatrix} \tag{24}$$

The definition of $[I_{\text{rw}_i/W_{c_i}}]$ allows for any RW inertia matrix to be considered. Section IV describes the characterization of the dynamic imbalance of the RW by defining parameters in $[I_{\text{rw}_i/W_{c_i}}]$.

In order to take the body frame derivative of $[I_{\text{rw}_i/W_{c_i}}]$, Eq. (24) is rewritten in a general form using outer product expansions.

$$\begin{aligned}[I_{\text{rw}_i/W_{c_i}}] = {}&J_{11_i}\hat{\boldsymbol{g}}_{s_i}\hat{\boldsymbol{g}}_{s_i}^T + J_{12_i}\hat{\boldsymbol{g}}_{s_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{g}}_{s_i}\hat{\boldsymbol{w}}_{3_i}^T \\ &+ J_{12_i}\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{g}}_{s_i}^T + J_{22_i}\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{23_i}\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{3_i}^T \\ &+ J_{13_i}\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{g}}_{s_i}^T + J_{23_i}\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{33_i}\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{3_i}^T\end{aligned} \tag{25}$$

The body frame derivatives of wheel frame basis vectors are

$$\hat{\boldsymbol{g}}'_{s_i} = \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \times \hat{\boldsymbol{g}}_{s_i} = \Omega_i\hat{\boldsymbol{g}}_{s_i} \times \hat{\boldsymbol{g}}_{s_i} = \boldsymbol{0} \tag{26}$$

$$\hat{\boldsymbol{w}}'_{2_i} = \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \times \hat{\boldsymbol{w}}_{2_i} = \Omega_i\hat{\boldsymbol{g}}_{s_i} \times \hat{\boldsymbol{w}}_{2_i} = \Omega_i\hat{\boldsymbol{w}}_{3_i} \tag{27}$$

$$\hat{\boldsymbol{w}}'_{3_i} = \boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{B}} \times \hat{\boldsymbol{w}}_{3_i} = \Omega_i\hat{\boldsymbol{g}}_{s_i} \times \hat{\boldsymbol{w}}_{3_i} = -\Omega_i\hat{\boldsymbol{w}}_{2_i} \tag{28}$$

Taking the body frame derivative and using Eqns. (26)-(28) to simplify yields

$$\begin{aligned}[I_{\text{rw}_i/W_{c_i}}]' = {}&J_{12_i}\Omega_i\hat{\boldsymbol{g}}_{s_i}\hat{\boldsymbol{w}}_{3_i}^T - J_{13_i}\Omega_i\hat{\boldsymbol{g}}_{s_i}\hat{\boldsymbol{w}}_{2_i}^T \\ &+ J_{12_i}\Omega_i\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{g}}_{s_i}^T + J_{22_i}\Omega_i\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{22_i}\Omega_i\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{3_i}^T + J_{23_i}\Omega_i\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{3_i}^T - J_{23_i}\Omega_i\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{2_i}^T \\ &- J_{13_i}\Omega_i\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{g}}_{s_i}^T - J_{23_i}\Omega_i\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{23_i}\Omega_i\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{3_i}^T - J_{33_i}\Omega_i\hat{\boldsymbol{w}}_{2_i}\hat{\boldsymbol{w}}_{3_i}^T - J_{33_i}\Omega_i\hat{\boldsymbol{w}}_{3_i}\hat{\boldsymbol{w}}_{2_i}^T\end{aligned} \tag{29}$$

Eq. (29) is expressed in the wheel frame as

$$[I_{\text{rw}_i/W_{c_i}}]' = {}^{\mathcal{W}_i}\begin{bmatrix} 0 & -J_{13_i} & J_{12_i} \\ -J_{13_i} & -2J_{23_i} & J_{22_i}-J_{33_i} \\ J_{12_i} & J_{22_i}-J_{33_i} & 2J_{23_i} \end{bmatrix}\Omega_i \tag{30}$$

The remaining term in Eq. (23) that needs to be defined is $\ddot{\boldsymbol{r}}_{W_{c_i}/B}$. Following its definition, the time derivatives are:

$$\boldsymbol{r}_{W_{c_i}/B} = \boldsymbol{r}_{W_i/B} + d_i\hat{\boldsymbol{w}}_{2_i} \tag{31}$$

$$\dot{\boldsymbol{r}}_{W_{c_i}/B} = \boldsymbol{r}'_{W_i/B} + d_i\hat{\boldsymbol{w}}'_{2_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{r}_{W_i/B} + d_i\hat{\boldsymbol{w}}_{2_i}) = d_i\Omega_i\hat{\boldsymbol{w}}_{3_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{r}_{W_i/B} + d_i\hat{\boldsymbol{w}}_{2_i}) \tag{32}$$

$$\ddot{\boldsymbol{r}}_{W_{c_i}/B} = d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{W_{c_i}/B} + 2\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times d_i\Omega_i\hat{\boldsymbol{w}}_{3_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{W_{c_i}/B}) \tag{33}$$

Substituting Eq. (33) into Eq. (23) and applying the triple product identity and parallel axis theorem $[I_{\text{rw}_i,B}] = [I_{\text{rw}_i,W_{c_i}}] + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T$ results in

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{rw}_i,B} =& [I_{\text{rw}_i,B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{rw}_i,B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{rw}_i,B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} + [I_{\text{rw}_i,W_{c_i}}]\dot{\Omega}_i\hat{\boldsymbol{g}}_{s_i} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} \\ &+ m_{\text{rw}_i}\boldsymbol{r}_{W_{c_i}/B} \times (d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}) + m_{\text{rw}_i}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{r}_{W_{c_i}/B} \times \boldsymbol{r}'_{W_{c_i}/B})\end{aligned} \tag{34}$$

Note that taking the body time derivative of the parallel axis theorem equation yields

$$[I_{\text{rw}_i,B}]' = [I_{\text{rw}_i,W_{c_i}}]' + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}'_{W_{c_i}/B}][\tilde{\boldsymbol{r}}_{W_{c_i}/B}]^T + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}][\tilde{\boldsymbol{r}}'_{W_{c_i}/B}]^T \tag{35}$$

Now the definition of the inertial time derivatives of the hub's angular momentum and reaction wheels' angular momentum, Eqs. (22) and (34) respectively, are substituted into Eq. (13)

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{sc},B} =& [I_{\text{hub},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{hub},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{rw}_i,B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{rw}_i,B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} + [I_{\text{rw}_i,W_{c_i}}]\dot{\Omega}_i\hat{\boldsymbol{g}}_{s_i} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} \\ &+ m_{\text{rw}_i}\boldsymbol{r}_{W_{c_i}/B} \times (d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}) + m_{\text{rw}_i}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{r}_{W_{c_i}/B} \times \boldsymbol{r}'_{W_{c_i}/B})\Big]\end{aligned} \tag{36}$$

Noting that $[I_{\text{sc},B}] = [I_{\text{hub},B}] + \sum\limits_{i=1}^{N}[I_{\text{rw}_i,B}]$, Eq. (36) is simplified to

$$\begin{aligned}\dot{\boldsymbol{H}}_{\text{sc},B} =& [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} + [I_{\text{rw}_i,W_{c_i}}]\dot{\Omega}_i\hat{\boldsymbol{g}}_{s_i} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big) \\ &+ m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}](d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i})\Big]\end{aligned} \tag{37}$$

Eq. (37) is substituted into Eq. (12) to yield

$$\begin{aligned}\boldsymbol{L}_B + m_{\text{sc}}\ddot{\boldsymbol{r}}_{B/N} \times \boldsymbol{c} =& [I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &+ \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} + [I_{\text{rw}_i,W_{c_i}}]\dot{\Omega}_i\hat{\boldsymbol{g}}_{s_i} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big) \\ &+ m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}](d_i\dot{\Omega}_i\hat{\boldsymbol{w}}_{3_i} - d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i})\Big]\end{aligned} \tag{38}$$

Grouping second order terms on the left-hand side yields the rotational EOM.

$\dot{\omega}$

$$\begin{aligned}
m_{\text{sc}}[\tilde{\boldsymbol{c}}]\ddot{\boldsymbol{r}}_{B/N}+[I_{\text{sc},B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}+\sum_{i=1}^{N}\Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i}+m_{\text{rw}_i}d_i[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{3_i}\Big)\dot{\Omega}_i \\
=\sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}-[I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i}-[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i}+m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big)\Big] \\
-[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}-[I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}+\boldsymbol{L}_B
\end{aligned} \tag{39}$$

Eq. (39) shows that the rotational EOM is coupled with the other second order variables. Similar to the translational EOM, this coupling is due to the fact that the center of mass of the spacecraft is not coincident with point B. The motor torque equation is the remaining necessary EOM to describe the motion of the spacecraft and is defined in the following section.

III.C Motor Torque Equation

The motor torque equation is used to relate body rate derivative $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and wheel speed derivative $\dot{\Omega}_i$. The motor torque u_{s_i} is the spin axis component of wheel torque about point W_i. The transverse torques acting on the wheel $\tau_{w_{2_i}}$ and $\tau_{w_{3_i}}$ are structural torques on the wheel and do not contribute to the motor torque equation.

$$\boldsymbol{L}_{W_i}={}^{\mathcal{W}_i}\begin{bmatrix}u_{s_i}\\ \tau_{w_{2_i}}\\ \tau_{w_{2_i}}\end{bmatrix} \tag{40}$$

Torque about point W_i relates to torque about W_{c_i} by[11]

$$\boldsymbol{L}_{W_i}=\boldsymbol{L}_{W_{c_i}}+\boldsymbol{r}_{W_{c_i}/W_i}\times m_{\text{rw}_i}\ddot{\boldsymbol{r}}_{W_{c_i}/N} \tag{41}$$

Euler's equation[11] applied as follows.

$$\boldsymbol{L}_{W_{c_i}}=\dot{\boldsymbol{H}}_{\text{rw}_i,W_{c_i}} \tag{42}$$

The RW angular momentum about W_{c_i} is expressed as

$$\boldsymbol{H}_{\text{rw}_i,W_{c_i}}=[I_{\text{rw}_i,W_{c_i}}]\boldsymbol{\omega}_{\mathcal{W}_i/\mathcal{N}}=[I_{\text{rw}_i,W_{c_i}}](\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}+\Omega_i\hat{\boldsymbol{g}}_{s_i}) \tag{43}$$

To aid in the simplification of the motor torque equation, $[I_{\text{rw}_i,W_{c_i}}]$ is expressed as an outer product sum as in Eq. (25) and distributed into Eq. (43).

$$\begin{aligned}
\boldsymbol{H}_{\text{rw}_i,W_{c_i}}&=J_{11_i}\hat{\boldsymbol{g}}_{s_i}(\omega_{s_i}+\Omega_i)+J_{12_i}\hat{\boldsymbol{g}}_{s_i}\omega_{w_{2_i}}+J_{13_i}\hat{\boldsymbol{g}}_{s_i}\omega_{w_{3_i}} \\
&+J_{12_i}\hat{\boldsymbol{w}}_{2_i}(\omega_{s_i}+\Omega_i)+J_{22_i}\hat{\boldsymbol{w}}_{2_i}\omega_{w_{2_i}}+J_{23_i}\hat{\boldsymbol{w}}_{2_i}\omega_{w_{3_i}} \\
&+J_{13_i}\hat{\boldsymbol{w}}_{3_i}(\omega_{s_i}+\Omega_i)+J_{23_i}\hat{\boldsymbol{w}}_{3_i}\omega_{w_{2_i}}+J_{33_i}\hat{\boldsymbol{w}}_{3_i}\omega_{w_{3_i}}
\end{aligned} \tag{44}$$

Note that the $\mathcal{W}_i$ frame components of $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ and their corresponding derivatives are defined as

$$\omega_{s_i}=\hat{\boldsymbol{g}}_{s_i}^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{45}$$

$$\omega_{w_{2_i}}=\hat{\boldsymbol{w}}_{2_i}^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{46}$$

$$\omega_{w_{3_i}}=\hat{\boldsymbol{w}}_{3_i}^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{47}$$

$$\dot{\omega}_{s_i}=\hat{\boldsymbol{g}}_{s_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \tag{48}$$

$$\dot{\omega}_{w_{2_i}}=\hat{\boldsymbol{w}}_{2_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}+\Omega_i\omega_{w_{3_i}} \tag{49}$$

$$\dot{\omega}_{w_{3_i}}=\hat{\boldsymbol{w}}_{3_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}-\Omega_i\omega_{w_{2_i}} \tag{50}$$

Grouping like terms in Eq. (44) yields

$$\boldsymbol{H}_{\text{rw}_i,W_{c_i}}=(J_{11_i}\omega_{s_i}+J_{11_i}\Omega_i+J_{12_i}\omega_{w_{2_i}}+J_{13_i}\omega_{w_{3_i}})\hat{\boldsymbol{g}}_{s_i}+p_i\hat{\boldsymbol{w}}_{2_i}+q_i\hat{\boldsymbol{w}}_{3_i} \tag{51}$$

where p_i and q_i are scalar components defined as

$$p_i = J_{12_i}\omega_{s_i} + J_{12_i}\Omega_i + J_{22_i}\omega_{w_{2_i}} + J_{23_i}\omega_{w_{3_i}} \tag{52}$$

$$q_i = J_{13_i}\omega_{s_i} + J_{13_i}\Omega_i + J_{23_i}\omega_{w_{2_i}} + J_{33_i}\omega_{w_{3_i}} \tag{53}$$

Taking the inertial derivative of wheel angular momentum about W_c gives

$$\begin{aligned} \dot{\boldsymbol{H}}_{\text{rw}_i, W_{c_i}} =& (J_{11_i}\dot{\omega}_{s_i} + J_{11_i}\dot{\Omega}_i + J_{12_i}\dot{\omega}_{w_{2_i}} + J_{13_i}\dot{\omega}_{w_{3_i}})\hat{\boldsymbol{g}}_{\text{s}_i} + \dot{p}_i\hat{\boldsymbol{w}}_{2_i} + \dot{q}_i\hat{\boldsymbol{w}}_{3_i} \\ &+ (J_{11_i}\omega_{s_i} + J_{11_i}\Omega_i + J_{12_i}\omega_{w_{2_i}} + J_{13_i}\omega_{w_{3_i}})\dot{\hat{\boldsymbol{g}}}_{\text{s}_i} + p_i\dot{\hat{\boldsymbol{w}}}_{2_i} + q_i\dot{\hat{\boldsymbol{w}}}_{3_i} \end{aligned} \tag{54}$$

where the inertial derivatives of the $\mathcal{W}_i$ frame basis vectors are determined by evaluating the cross product in wheel frame components such as

$$\dot{\hat{\boldsymbol{g}}}_{\text{s}_i} = \boldsymbol{\omega}_{B/N} \times \hat{\boldsymbol{g}}_{\text{s}_i} = \omega_{w_{3_i}}\hat{\boldsymbol{w}}_{2_i} - \omega_{w_{2_i}}\hat{\boldsymbol{w}}_{3_i} \tag{55}$$

Similarly, $\dot{\hat{\boldsymbol{w}}}_{2_i}$ and $\dot{\hat{\boldsymbol{w}}}_{3_i}$ are found to be

$$\dot{\hat{\boldsymbol{w}}}_{2_i} = -\omega_{w_{3_i}}\hat{\boldsymbol{g}}_{\text{s}_i} + (\omega_{s_i} + \Omega_i)\hat{\boldsymbol{w}}_{3_i} \tag{56}$$

$$\dot{\hat{\boldsymbol{w}}}_{3_i} = \omega_{w_{2_i}}\hat{\boldsymbol{g}}_{\text{s}_i} - (\omega_{s_i} + \Omega_i)\hat{\boldsymbol{w}}_{2_i} \tag{57}$$

Substituting Eqns. (55)-(57) into Eq. (54) and grouping like terms yields

$$\begin{aligned} \dot{\boldsymbol{H}}_{\text{rw}_i, W_{c_i}} =& \big[(J_{11_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + J_{11_i}\dot{\Omega}_i + \omega_{s_i}(J_{13_i}\omega_{w_{2_i}} - J_{12_i}\omega_{w_{3_i}}) \\ &+ \omega_{w_{3_i}}\omega_{w_{2_i}}(J_{33_i} - J_{22_i}) + J_{23_i}(\omega_{w_{2_i}}^2 - \omega_{w_{3_i}}^2)\big]\hat{\boldsymbol{g}}_{\text{s}_i} + P_i\hat{\boldsymbol{w}}_{2_i} + Q_i\hat{\boldsymbol{w}}_{3_i} \end{aligned} \tag{58}$$

Scalar quantities, P_i and Q_i are the coefficients for $\hat{\boldsymbol{w}}_{2_i}$ and $\hat{\boldsymbol{w}}_{3_i}$ respectively. Since only the coefficient of $\hat{\boldsymbol{g}}_{\text{s}_i}$ relates directly to the motor torque equation as in Eqns. (40)-(41), specifying P_i and Q_i is unnecessary as they do not contribute to u_{s_i}.

The next step is to define the remaining terms in Eq. (41). This begins by determining the second inertial derivative of $\ddot{\boldsymbol{r}}_{W_{c_i}/N}$.

$$\boldsymbol{r}_{W_{c_i}/N} = \boldsymbol{r}_{B/N} + \boldsymbol{r}_{W_i/B} + \boldsymbol{r}_{W_{c_i}/W_i} = \boldsymbol{r}_{B/N} + \boldsymbol{r}_{W_i/B} + d_i\hat{\boldsymbol{w}}_{2_i} \tag{59}$$

$$\dot{\boldsymbol{r}}_{W_{c_i}/N} = \dot{\boldsymbol{r}}_{B/N} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{W_i/B} + (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \times d_i\hat{\boldsymbol{w}}_{2_i} \tag{60}$$

$$\begin{aligned} \ddot{\boldsymbol{r}}_{W_{c_i}/N} =& \ddot{\boldsymbol{r}}_{B/N} + \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{W_i/B} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}_{W_i/B}) + (\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_i\hat{\boldsymbol{g}}_{\text{s}_i}) \times d_i\hat{\boldsymbol{w}}_{2_i} \\ &+ (\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \times d_i\Omega_i\hat{\boldsymbol{w}}_{3_i} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times [(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \times d_i\hat{\boldsymbol{w}}_{2_i}] \end{aligned} \tag{61}$$

Each cross product in Eq. (61) is evaluated using wheel frame components. For example,

$$(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \Omega_i\hat{\boldsymbol{g}}_{\text{s}_i}) \times d_i\hat{\boldsymbol{w}}_{2_i} = -d_i\omega_{w_{3_i}}\hat{\boldsymbol{g}}_{\text{s}_i} + d_i(\omega_{s_i} + \Omega_i)\hat{\boldsymbol{w}}_{3_i} \tag{62}$$

Repeating this procedure several times yields the following expression for the right hand term of Eq. (41) (R_i is the coefficient in front of $\hat{\boldsymbol{w}}_{3_i}$ and does need to be defined because only the $\hat{\boldsymbol{g}}_{\text{s}_i}$ component is desired):

$$\begin{aligned} \boldsymbol{r}_{W_{c_i}/W_i} \times m_{\text{rw}_i}\ddot{\boldsymbol{r}}_{W_{c_i}/N} =& m_{\text{rw}_i}d_i\Big[\hat{\boldsymbol{w}}_{3_i}^T\ddot{\boldsymbol{r}}_{B/N} - \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} \\ &+ d_i(\hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_i) + d_i\omega_{w_{2_i}}\omega_{w_{3_i}}\Big]\hat{\boldsymbol{g}}_{\text{s}_i} - R_i\hat{\boldsymbol{w}}_{3_i} \end{aligned} \tag{63}$$

The motor torque equation is obtained by summing the $\hat{\boldsymbol{g}}_{\text{s}_i}$ components of Eq. (58) and Eq. (63)

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$$\begin{aligned} u_{s_i} =& (J_{11_i}\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + J_{11_i}\dot{\Omega}_i + \omega_{s_i}(J_{13_i}\omega_{w_{2_i}} - J_{12_i}\omega_{w_{3_i}}) + \omega_{w_{3_i}}\omega_{w_{2_i}}(J_{33_i} - J_{22_i}) \\ &+ J_{23_i}(\omega_{w_{2_i}}^2 - \omega_{w_{3_i}}^2) + m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T\ddot{\boldsymbol{r}}_{B/N} - m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_{\text{rw}_i}d_i\hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} \\ &+ m_{\text{rw}_i}d_i^2(\hat{\boldsymbol{g}}_{\text{s}_i}^T\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_i) + m_{\text{rw}_i}d_i^2\omega_{w_{2_i}}\omega_{w_{3_i}} \end{aligned} \tag{64}$$

Grouping second order terms on the left-hand side yields the simplified motor torque equation.

$$
\begin{aligned}
\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\ddot{\boldsymbol{r}}_{B/N} + \big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{\text{s}_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \big[J_{11_i} + m_{\text{rw}_i} d_i^2\big]\dot{\Omega}_i \\
= J_{23_i}(\omega_{w_{3_i}}^2 - \omega_{w_{2_i}}^2) + \omega_{s_i}(J_{12_i}\omega_{w_{3_i}} - J_{13_i}\omega_{w_{2_i}}) + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) \\
- m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \qquad (65)
\end{aligned}
$$

The balanced motor torque equation may be obtained by zeroing out all imbalance terms (d_i, J_{12_i}, J_{13_i}, J_{23_i}) and making the assumption $J_{22_i} = J_{33_i}$. Under these conditions, Eq. (65) may be simplified to

$$
u_{s_i} = J_{11_i}\big(\hat{\boldsymbol{g}}_{\text{s}_i}^T \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \dot{\Omega}_i\big) \qquad (66)
$$

Eq. (66) is equivalent to the balanced motor torque equation found in Reference.[11]

This concludes the necessary derivations for the EOMs that are needed to describe the fully-coupled jitter model for imbalanced RWs. Since the simplified RW jitter model[5] assumes an external force and torque on the spacecraft, the EOMs for the fully-coupled model and the simplified RW jitter model are significantly different. However, due to the coupled nature of the EOMs, the similar terms in the simplified model compared to the fully-coupled model are not readily apparent in EOMs presented thus far. In the following section a back-substitution method is introduced to increase the computational efficiency of a computer simulation for this model and as a result the similar terms become apparent.

III.D Back-Substitution Method

The equations presented in the previous sections result in $N + 6$ coupled differential equations. Therefore, if the EOMs were placed into state space form, a system mass matrix of size $N + 6$ would need to be inverted to numerically integrate the EOM. This can result in a computationally expensive simulation. In this section, the EOMs are manipulated to increase the efficiency. This manipulation involves inverting an $N \times N$ matrix for the reaction wheel motion EOM, inverting the rotational motion equation (3×3), and then back solving for the reaction wheel and translational motions. This derivation can be seen in the following sections.

Since the translational motion is coupled with both the rotational motion and the motor torque equation, the translational motion equation, Eq. (10), is placed into the motor torque equation

$$
\begin{aligned}
\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big([\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} - \frac{1}{m_{\text{sc}}}\sum_{j=1}^{N} m_{\text{rw}_j} d_j \hat{\boldsymbol{w}}_{3_j}\dot{\Omega}_j + \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{j=1}^{N} m_{\text{rw}_j} d_j \Omega_j^2 \hat{\boldsymbol{w}}_{2_j}\Big) \\
+ \Big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \big[J_{11_i} + m_{\text{rw}_i} d_i^2\big]\dot{\Omega}_i \\
= J_{23_i}(\omega_{w_{3_i}}^2 - \omega_{w_{2_i}}^2) + \omega_{s_i}(J_{12_i}\omega_{w_{3_i}} - J_{13_i}\omega_{w_{2_i}}) + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) \\
- m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \qquad (67)
\end{aligned}
$$

The second order reaction wheel terms are isolated on the left hand side of the equation

$$
\begin{aligned}
\big[J_{11_i} + m_{\text{rw}_i} d_i^2 - \frac{m_{\text{rw}_i}^2 d_i^2}{m_{\text{sc}}}\big]\dot{\Omega}_i - \frac{1}{m_{\text{sc}}}\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\sum_{j=1;j\neq i}^{N} m_{\text{rw}_j} d_j \hat{\boldsymbol{w}}_{3_j}\dot{\Omega}_j \\
= -\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big([\tilde{\boldsymbol{c}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{j=1;j\neq i}^{N} m_{\text{rw}_j} d_j \Omega_j^2 \hat{\boldsymbol{w}}_{2_j}\Big) \\
- \Big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{r}}_{W_i/B}]\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
+ J_{23_i}(\omega_{w_{3_i}}^2 - \omega_{w_{2_i}}^2) + \omega_{s_i}(J_{12_i}\omega_{w_{3_i}} - J_{13_i}\omega_{w_{2_i}}) + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) \\
- m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i} \qquad (68)
\end{aligned}
$$

The $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ terms on the right hand side of the equation are combined

$$
\begin{aligned}
&\Big[J_{11_i} + m_{\text{rw}_i} d_i^2 - \frac{m_{\text{rw}_i}^2 d_i^2}{m_{\text{sc}}}\Big]\dot{\Omega}_i - \frac{1}{m_{\text{sc}}}\big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big] \sum_{j=1;j\neq i}^{N} m_{\text{rw}_j} d_j \hat{\boldsymbol{w}}_{3_j} \dot{\Omega}_j \\
&\quad = -\Big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T + \big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big([\tilde{\boldsymbol{c}}] - [\tilde{\boldsymbol{r}}_{W_i/B}]\Big)\Big]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\
&\quad - \big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big(\ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{j=1;j\neq i}^{N} m_{\text{rw}_j} d_j \Omega_j^2 \hat{\boldsymbol{w}}_{2_j}\Big) \\
&\quad + J_{23_i}(\omega_{w_{3_i}}^2 - \omega_{w_{2_i}}^2) + \omega_{s_i}(J_{12_i}\omega_{w_{3_i}} - J_{13_i}\omega_{w_{2_i}}) + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) \\
&\qquad - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}
\end{aligned}
\tag{69}
$$

Eq. (69) can now be written in a simplified form

$$
[A]\begin{bmatrix}\dot{\Omega}_1\\ \cdot\\ \cdot\\ \dot{\Omega}_N\end{bmatrix} = [F]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{v}
\tag{70}
$$

Where $[A]$ in an $N \times N$ matrix with the following components

$$
a_{i,i} = J_{11_i} + m_{\text{rw}_i} d_i^2 - \frac{m_{\text{rw}_i}^2 d_i^2}{m_{\text{sc}}}
\tag{71a}
$$

$$
a_{i,j} = -\frac{m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T}{m_{\text{sc}}} m_{\text{rw}_j} d_j \hat{\boldsymbol{w}}_{3_j}
\tag{71b}
$$

$[F]$ is an $N \times 3$ matrix with its row elements defined as

1 × 3 3×3

$$
\boldsymbol{f}_i^T = -\Big[(J_{11_i} + m_{\text{rw}_i} d_i^2)\hat{\boldsymbol{g}}_{s_i}^T + J_{12_i}\hat{\boldsymbol{w}}_{2_i}^T + J_{13_i}\hat{\boldsymbol{w}}_{3_i}^T + \big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big([\tilde{\boldsymbol{c}}] - [\tilde{\boldsymbol{r}}_{W_i/B}]\Big)\Big]
\tag{72}
$$

1 × 3

$\boldsymbol{v}$ is an $N \times 1$ vector with the following elements

$$
\begin{aligned}
&v_i - \big[m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T\big]\Big(\ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{j=1;j\neq i}^{N} m_{\text{rw}_j} d_j \Omega_j^2 \hat{\boldsymbol{w}}_{2_j}\Big) \\
&\quad + J_{23_i}(\omega_{w_{3_i}}^2 - \omega_{w_{2_i}}^2) + \omega_{s_i}(J_{12_i}\omega_{w_{3_i}} - J_{13_i}\omega_{w_{2_i}}) + \omega_{w_{2_i}}\omega_{w_{3_i}}(J_{22_i} - J_{33_i} - m_{\text{rw}_i} d_i^2) \\
&\qquad - m_{\text{rw}_i} d_i \hat{\boldsymbol{w}}_{3_i}^T [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{r}_{W_i/B} + u_{s_i}
\end{aligned}
\tag{73}
$$

Eq. (70) can now be solved by inverting matrix $[A]$ ($[E] = [A]^{-1}$).

$$
\begin{bmatrix}\dot{\Omega}_1\\ \cdot\\ \cdot\\ \dot{\Omega}_N\end{bmatrix} = [E][F]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [E]\boldsymbol{v}
\tag{74}
$$

Since the rotation EOM, Eq. (39), has $\dot{\Omega}_i$ terms, it is more convenient to use the expression for $\dot{\Omega}_i$ as

$$
\dot{\Omega}_i = \boldsymbol{e}_i^T[F]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{e}_i^T\boldsymbol{v}
\tag{75}
$$

Where the subcomponents of $[E]$ are defined as 1 × 3 3×1 1 × 1

$$
[E] = \begin{bmatrix}\boldsymbol{e}_1^T\\ \cdot\\ \cdot\\ \boldsymbol{e}_N^T\end{bmatrix}
\tag{76}
$$

The modified Euler's equation has both $\dot{\Omega}_i$ and $\ddot{\boldsymbol{r}}_{B/N}$ terms. To decouple these equations, both the translational equation and the motor torque equation need to be substituted into Euler's equation. The translational equation is substituted into Euler's equation and the $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and $\dot{\Omega}_i$ terms are combined which results in

$$\begin{aligned}\Big([I_{\text{sc},B}] + m_{\text{sc}}[\tilde{\boldsymbol{c}}][\tilde{\boldsymbol{c}}]\Big)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + m_{\text{sc}}[\tilde{\boldsymbol{c}}]\Big(\ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c} + \frac{1}{m_{\text{sc}}}\sum_{i=1}^{N} m_{\text{rw}_i} d_i \Omega_i^2 \hat{\boldsymbol{w}}_{2_i}\Big) \\ + \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i\Big([\tilde{\boldsymbol{r}}_{W_{c_i}/B}] - [\tilde{\boldsymbol{c}}]\Big)\hat{\boldsymbol{w}}_{3_i}\Big]\dot{\Omega}_i \\ = \sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big)\Big] \\ - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + \boldsymbol{L}_B \end{aligned} \tag{77}$$

Replacing the $\dot{\Omega}_i$ term with Eq. (75), combining the resulting $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ terms and isolating them on the left hand side yields

$$\begin{aligned}\Big([I_{\text{sc},B}] + m_{\text{sc}}[\tilde{\boldsymbol{c}}][\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i\Big([\tilde{\boldsymbol{r}}_{W_{c_i}/B}] - [\tilde{\boldsymbol{c}}]\Big)\hat{\boldsymbol{w}}_{3_i}\Big]\boldsymbol{e}_i^T[F]\Big)\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} \\ = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\Big(\ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}\Big) \\ + \sum_{i=1}^{N}\Big[m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i} - [I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big) \\ - m_{\text{rw}_i} d_i\Omega_i^2[\tilde{\boldsymbol{c}}]\hat{\boldsymbol{w}}_{2_i} - \Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i\Big([\tilde{\boldsymbol{r}}_{W_{c_i}/B}] - [\tilde{\boldsymbol{c}}]\Big)\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{e}_i^T\boldsymbol{v}\Big] + \boldsymbol{L}_B \end{aligned} \tag{78}$$

For simplification purposes, Eq. (78) is written in the following form

$$[I_{\text{LHS}}]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\tau}_{\text{RHS}} \tag{79}$$

3x1

Where $[I_{\text{LHS}}]$ is a 3×3 matrix defined as

3x3 3x1 3x 3x1 1x4

$$[I_{\text{LHS}}] = [I_{\text{sc},B}] + m_{\text{sc}}[\tilde{\boldsymbol{c}}][\tilde{\boldsymbol{c}}] + \sum_{i=1}^{N}\Big[[I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i\Big([\tilde{\boldsymbol{r}}_{W_{c_i}/B}] - [\tilde{\boldsymbol{c}}]\Big)\hat{\boldsymbol{w}}_{3_i}\Big]\boldsymbol{e}_i^T[F] \tag{80}$$

4x3

and $\boldsymbol{\tau}_{RHS}$ is a 3×1 vector with the following definition

3x1 3x1 33 3x1 3x3 3x1 3x 1

$$\begin{aligned}\boldsymbol{\tau}_{\text{RHS}} = -[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I_{\text{sc},B}]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - [I_{\text{sc},B}]'\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} - m_{\text{sc}}[\tilde{\boldsymbol{c}}]\Big(\ddot{\boldsymbol{r}}_{C/N} - 2[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}' - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\boldsymbol{c}\Big) \\ + \sum_{i=1}^{N}\Big[m_{\text{rw}_i} d_i\Omega_i^2[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{2_i} - \underline{[I_{\text{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i}} - [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}]\Big([I_{\text{rw}_i,W_{c_i}}]\Omega_i\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\boldsymbol{r}'_{W_{c_i}/B}\Big) \\ - \underline{m_{\text{rw}_i} d_i\Omega_i^2[\tilde{\boldsymbol{c}}]\hat{\boldsymbol{w}}_{2_i}} - \Big([I_{\text{rw}_i,W_{c_i}}]\hat{\boldsymbol{g}}_{s_i} + m_{\text{rw}_i} d_i\Big([\tilde{\boldsymbol{r}}_{W_{c_i}/B}] - [\tilde{\boldsymbol{c}}]\Big)\hat{\boldsymbol{w}}_{3_i}\Big)\boldsymbol{e}_i^T\boldsymbol{v}\Big] + \boldsymbol{L}_B \end{aligned} \tag{81}$$

At this point, Eq. (78), is the rotational motion equation that has been decoupled from the other second order state variables, and Eq. (81) is the equivalent torque on the rotational motion due to the fully-coupled model. This now gives insight into the similarities and differences between the simplified model and the fully-coupled model. The term $m_{\text{rw}_i}[\tilde{\boldsymbol{r}}_{W_{c_i}/B}]d_i\Omega_i^2\hat{\boldsymbol{w}}_{2_i}$ is an internal torque due to the center of mass offset of the RW and is analogous to the external torque due to static imbalance in the simplified model. The term $[I_{\text{rw}_i,W_{c_i}}]'\Omega_i$ is analogous to the dynamic imbalance of the simplified model. The remaining terms in Eq. (81) are the terms that are missing in the simplified model which results in angular momentum not being conserved. This comparison is further expounded upon in the next section.

Now Eq. (79) can be solved for $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$. It is important to note that there are two remaining steps required to implement these equations into a simulation. $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ is placed into the simplified motor torque equation, Eq. (75), to solve for $\dot{\Omega}_i$. The solutions for $\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}$ and $\dot{\Omega}_i$ are placed into the translational motion equation to solve for $\ddot{\boldsymbol{r}}_{B/N}$. This concludes the necessary steps needed to implement imbalanced reaction wheel dynamics into a computer simulation. The recommended coordinate frames for this simulation are to solve everything in the body frame, $\mathcal{B}$, and before integration, place the translational motion in the inertial frame, $\mathcal{N}$. However, this formulation is general, and any coordinate frames can be chosen as needed.

IV Imbalance Parameter Adaptation

IV.A Simplified Imbalance Model

The well-used method to specify reaction wheel imbalance is to lump sources of imbalance into scalar parameters. The simplified reaction wheel imbalance model directly utilizes such specifications to model jitter as an external torque.[5,7] Static imbalance, U_s, typically given in units of g·cm, specifies the proportionality of the square of wheel speed to the magnitude of disturbance force caused by an offset in center of mass from the geometric center of the reaction wheel. That is,

$$\boldsymbol{F}_{s_i} = U_{s_i}\Omega_i^2\hat{\boldsymbol{u}}_i \tag{82}$$

where $\hat{\boldsymbol{u}}_i$ is an arbitrary unit vector normal to the wheel spin axis and $\boldsymbol{F}_{s_i}$ is the resulting force on the spacecraft. If the reaction wheel is not coincident with the spacecraft center of mass, torque on the spacecraft resulting from the static imbalance force is given by the simplified model as

$$\boldsymbol{L}_{s_i} = \boldsymbol{r}_{W_i/B} \times \boldsymbol{F}_{s_i} = U_{s_i}\Omega_i^2[\tilde{\boldsymbol{r}}_{W_i/B}]\hat{\boldsymbol{u}}_i \tag{83}$$

Note that the simplified model uses the approximation $\boldsymbol{r}_{W_{c_i}/B} \approx \boldsymbol{r}_{W_i/B}$ since d_i is usually very small and $\boldsymbol{r}_{W_i/B} \neq \boldsymbol{0}$.

Dynamic imbalance U_d, typically given in units g·cm^2, specifies the proportionality of the square of wheel speed to the magnitude of disturbance torque caused by off diagonal terms in the reaction wheel inertia tensor. That is,

$$\boldsymbol{L}_{d_i} = U_{d_i}\Omega_i^2\hat{\boldsymbol{v}}_i \tag{84}$$

where $\hat{\boldsymbol{v}}_i$ is an arbitrary unit vector normal to the wheel spin axis and $\boldsymbol{L}_{d_i}$ is the resulting torque on the spacecraft. Note that $\hat{\boldsymbol{u}}_i$ and $\hat{\boldsymbol{v}}_i$ are only required to be normal to their corresponding spin axis. This is because the lumped parameters U_{s_i} and U_{d_i} do not contain any information on orientation/location of mass imbalances about $\hat{\boldsymbol{g}}_{\mathrm{s}_i}$. Additionally, the initial value of the wheel angle parameter is arbitrarily chosen, which further emphasizes the arbitrariness of the vectors $\hat{\boldsymbol{u}}_i$ and $\hat{\boldsymbol{v}}_i$ since they relate to the body frame through wheel angle θ_i.

IV.B Imbalance Parameter Adaptation

To relate the simplified model to the fully-coupled model developed within this paper, Eq. (81) is analyzed to identify terms that directly contribute to torque on the spacecraft. Noticing the presence of wheel speed squared and the cross product of wheel location in the term

$$m_{\mathrm{rw}_i} d_i \Omega_i^2 [\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{2_i}$$

it is equated to the simplified static imbalance model to yield

$$U_{s_i}\Omega_i^2[\tilde{\boldsymbol{r}}_{W/B_i}]\hat{\boldsymbol{u}}_i = m_{\mathrm{rw}_i} d_i \Omega_i^2 [\tilde{\boldsymbol{r}}_{W_{c_i}/B}]\hat{\boldsymbol{w}}_{2_i} \tag{85}$$

Rearranging this equation for U_{s_i} and making the approximation $\boldsymbol{r}_{W_{c_i}/B} \approx \boldsymbol{r}_{W_i/B}$ yields an expression for d_i

$$d_i = \frac{U_{s_i}}{m_{\mathrm{rw}_i}} \tag{86}$$

For the dynamic imbalance, the presence of wheel speed multiplied by $[I_{\mathrm{rw}_i,W_{c_i}}]'$ term results in an inertia value times the wheel speed squared

$$[I_{\mathrm{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i}$$

Equating this term to the simplified dynamic imbalance model yields

$$U_{d_i}\Omega_i^2\hat{\boldsymbol{v}}_i = [I_{\mathrm{rw}_i,W_{c_i}}]'\Omega_i\hat{\boldsymbol{g}}_{s_i} = \Omega_i^2 \; {}^{\mathcal{W}}\!\begin{bmatrix} 0 \\ -J_{13} \\ J_{12} \end{bmatrix} \tag{87}$$

Rearranging this equation for U_{d_i} yields Eq. (88) and agrees with the relationship found in Reference 7.

$$U_{d_i} = \sqrt{J_{13_i}^2 + J_{12_i}^2} \tag{88}$$

Thus, the fully-coupled model is under-constrained with respect to the implementation of the simplified model, and some combination of J_{12} and J_{13} must be selected for each wheel such that Eq. (88) is satisfied. Since the unit vector $\hat{\boldsymbol{v}}_i$ is arbitrary (as well as $\hat{\boldsymbol{w}}_{2_i}$ and $\hat{\boldsymbol{w}}_{3_i}$ due to the arbitrariness of initial wheel angle), the following definitions are chosen

$$J_{13_i} = U_{d_i} \tag{89a}$$

$$J_{12_i} = 0 \tag{89b}$$

To complete the discussion of characterizing RW imbalance from manufactures' specifications, the full inertia matrix needs to be defined. The balanced reaction wheel inertia tensor is

$$[I_{\text{rw}_i,W_{c_i}}] = {}^{\mathcal{P}_i}\begin{bmatrix} J_{s_i} & 0 & 0 \\ 0 & J_{t_i} & 0 \\ 0 & 0 & J_{t_i} \end{bmatrix} \tag{90}$$

where $\mathcal{P}_i$ is the principal axes frame of the RW. J_{s_i} and J_{t_i} are the spin axis inertia and transverse axis inertia of the RW, respectively. For there to only be J_{13_i} terms present in the $\mathcal{W}_i$ representation of the RW's inertia tensor, the rotation matrix between $\mathcal{W}_i$ and $\mathcal{P}_i$, labeled as $[\mathcal{W}_i\mathcal{P}_i]$ must be a single axis rotation about the $\hat{\boldsymbol{w}}_{2_i}$ axis:

$$[\mathcal{W}_i\mathcal{P}_i] = \begin{bmatrix} \cos(\beta_i) & 0 & -\sin(\beta_i) \\ 0 & 1 & 0 \\ \sin(\beta_i) & 0 & \cos(\beta_i) \end{bmatrix} \tag{91}$$

where β_i is the angle of rotation. Transforming $[I_{\text{rw}_i,W_{c_i}}]$ from the $\mathcal{P}_i$ frame to the $\mathcal{W}_i$ frame using Eq. (91) and using small angle approximations yields

$$[I_{\text{rw}_i,W_{c_i}}] = {}^{\mathcal{W}_i}\begin{bmatrix} J_{s_i} & 0 & (J_{s_i} - J_{t_i})\beta_i \\ 0 & J_{t_i} & 0 \\ (J_{s_i} - J_{t_i})\beta_i & 0 & J_{t_i} \end{bmatrix} \tag{92}$$

However, from Eq. (89a), $[I_{\text{rw}_i/W_{c_i}}]$ can be written in the following form

$$[I_{\text{rw}_i/W_{c_i}}] = {}^{\mathcal{W}_i}\begin{bmatrix} J_{s_i} & 0 & U_{d_i} \\ 0 & J_{t_i} & 0 \\ U_{d_i} & 0 & J_{t_i} \end{bmatrix} \tag{93}$$

This gives the following relationship between the rotation angle, β_i, and U_{d_i}

$$\beta_i = \frac{U_{d_i}}{J_{s_i} - J_{t_i}} \tag{94}$$

This concludes the necessary steps to relate manufactures' specifications of RW imbalances to parameters needed for the fully-coupled jitter model. In addition, the simplified description of $[I_{\text{rw}_i/W_{c_i}}]$ seen in Eq. (93) simplifies the EOMs developed in the previous sections due to $J_{12_i} = J_{13_i} = 0$. In addition Eqs. (86), (89a) and (93) allow a direct comparison of the results of the simplified model to the fully-coupled model which is discussed in the following section.

V Numeric Simulations

Numeric simulations are provided to demonstrate the fully-coupled imbalanced reaction wheel model developed within this paper. Angular momentum is calculated to confirm that when no external disturbances are present angular momentum is conserved, and system energy is calculated to show that when no external disturbances or reaction wheel motor torques are present, energy is conserved. The fully-coupled model is directly compared to the simplified model using the formulation developed in Section IV.B. Simulation parameters used are given in Table 1.

Table 1: Simulation parameters for the fully-coupled model. Note that wheel parameters apply to all wheels unless otherwise specified.

Parameter	Notation	Value	Units
Number of reaction wheels	N	3	-
Total spacecraft mass	m_{sc}	680	kg
Hub mass	m_{hub}	644	kg
Wheel mass	m_{rw}	12	kg
Hub inertia tensor about hub center of mass	$[I_{\text{hub},B_c}]$	${}^{\mathcal{B}}\begin{bmatrix} 550 & 0.1045 & -0.0840 \\ 0.1045 & 650 & 0.0001 \\ -0.0840 & 0.0001 & 650 \end{bmatrix}$	kg·m^2
Hub C.O.M. location w.r.t. B	$\boldsymbol{r}_{B_c/B}$	${}^{\mathcal{B}}[1 \quad -2 \quad 10]^T$	cm
Wheel orientation matrix	$[G_s]$	${}^{\mathcal{B}}\begin{bmatrix} 0.7887 & -0.2113 & -0.5774 \\ -0.2113 & 0.7887 & -0.5774 \\ 0.5774 & 0.5774 & 0.5774 \end{bmatrix}$	-
Wheel static imbalance	U_s	0.48	g·cm
Wheel static imbalance	U_d	15.4	g·cm^2
Wheel C.O.M. offset (derived from U_s)	d	0.4	μm
Wheel inertia tensor about wheel C.O.M. (derived from U_d)	$[I_{\text{rw},W_c}]$	${}^{\mathcal{W}}\begin{bmatrix} 1.5915 & 0 & 1.54\text{E}{-6} \\ 0 & 0.8594 & 0 \\ 1.54\text{E}{-6} & 0 & 0.8594 \end{bmatrix}$	kg·cm^2
Wheel 1 location vector	$\boldsymbol{r}_{W_1/B}$	${}^{\mathcal{B}}[0.6309 \quad -0.1691 \quad 0.4619]^T$	
Wheel 2 location vector	$\boldsymbol{r}_{W_2/B}$	${}^{\mathcal{B}}[-0.1691 \quad 0.6309 \quad 0.4619]^T$	
Wheel 3 location vector	$\boldsymbol{r}_{W_3/B}$	${}^{\mathcal{B}}[-0.4619 \quad -0.4619 \quad 0.4619]^T$	m
Initial position	$\boldsymbol{r}_{B/N}$	${}^{\mathcal{N}}[0 \quad 0 \quad 0]^T$	m
Initial velocity	$\boldsymbol{v}_{B/N}$	${}^{\mathcal{N}}[0 \quad 0 \quad 0]^T$	m/s
Initial attitude MRP	$\boldsymbol{\sigma}_{\mathcal{B}/\mathcal{N}}$	$[0 \quad 0 \quad 0]^T$	-
Initial angular velocity	$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$	${}^{\mathcal{B}}[0 \quad 0 \quad 0]^T$	deg/s
Initial wheel speeds	Ω	-558, -73, 242	RPM
Initial wheel angles	θ	43, 179, 346	deg
Commanded wheel torques	u_s	200, -500, 350	mN·m

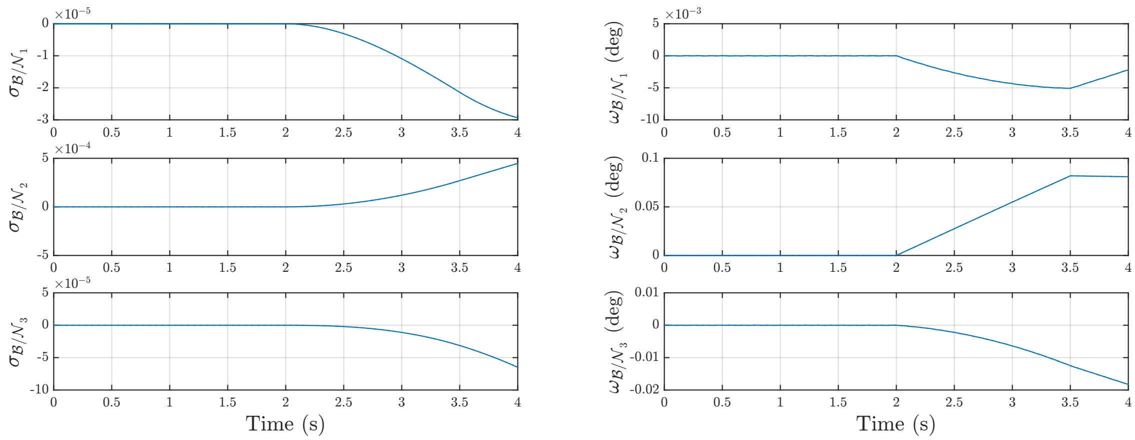
V.A Spacecraft with N Imbalanced Reaction Wheels

The first simulation that is included simulates three RWs. The purpose of this simulation is to show the effect of RW jitter on a spacecraft that is initially inertially fixed, and therefore the only perturbations to the spacecraft will be due to the RW jitter. Accordingly, the spacecraft has no external forces present, has zero initial velocity, and zero initial angular velocity. The RWs are initially spinning with specified values seen in Table 1. Also, to give further confirmation in the model, the motor torque in each RW has a nonzero time history and can be seen in Figure 6(c). Note that the wheel orientation matrix $[G_s]$ (which is useful for many controls applications[11]) is formulated such that each column contains the spin axis unit vector for the ith wheel, $\hat{\boldsymbol{g}}_{s_i}$, and has dimension $3 \times N$.

$$[G_s] = \begin{bmatrix} \hat{\boldsymbol{g}}_{s_1} & \cdots & \hat{\boldsymbol{g}}_{s_N} \end{bmatrix} \tag{95}$$

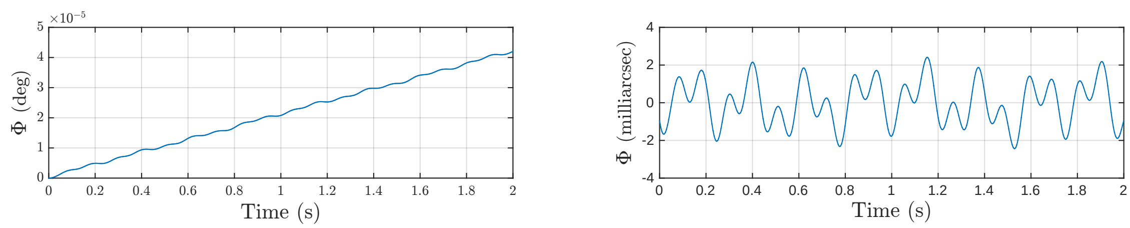
Figures 3-7 show simulation results for the fully-coupled RW imbalance model with $N = 3$ wheels. In Figure 3, the attitude of the spacecraft is shown to be drifting due to the imbalance in the RWs. The impact of jitter is visible in the spacecraft's body rates. Figure 4(a) shows the evolution of the principal angle where the jitter is visible. Whereas in Figure 4(b) shows the principal angle with the drift subtracted out, so that only the jitter is visible. This shows that the RW jitter results in over 0.1 deg of drift over 2 seconds and a jitter amplitude of around 6 arcseconds. These parameters are important to consider when doing analysis of RW jitter.

In addition, the translational position and velocity can be seen in Figure 5 and shows that there is a non-zero effect due to RW jitter on the position and velocity of the spacecraft. The wheel positions and wheel rates seen in Figs. 6(a)



a) Attitude MRP of the spacecraft for the fully-coupled simulation **b)** Body rates of the spacecraft for the fully-coupled simulation

Figure 3: Attitude and body rates of spacecraft



a) Principal angle plotted versus time for the fully-coupled simulation **b)** Principal angle jitter for the fully-coupled simulation

Figure 4: Principal angle and jitter plots

and 6(b) agree with the time history of the motor torque seen in Figure 6(c).

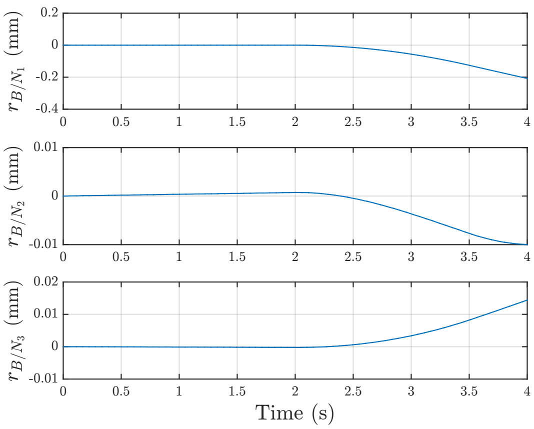
Figure 7 shows the change in energy and momentum during the simulation. Energy is plotted for a 2 second duration because the motor torque is zero during this time and the change in energy only includes integration error. The angular momentum is plotted for the entire simulation, and shows that angular momentum is conserved. These checks give confidence in the formulation and also highlight the difference between this model and the simplified RW jitter model. The energy and angular momentum of the spacecraft is not conserved for the RW jitter model (plots not shown), which is expected based on the formulation. For numerical simulations of a spacecraft, angular momentum and energy conservation is an important check to validate EOMs and for long simulation times the error in the simplified model will grow. This need for validation checks and error propagations are important characteristics to consider between both models. A second simulation is included the in the next section to directly compare results from the two models.

V.B Comparison of Fully-Coupled and Simplified Models

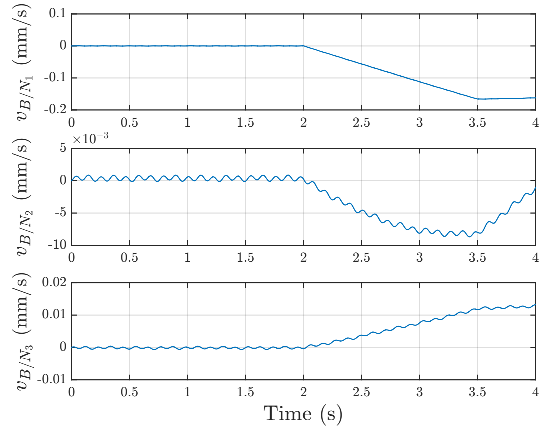
The fully-coupled model is compared to the simplified model. These simulations involve similar initial conditions as seen in Table 1, except only one RW is included for simplicity. Figure 8 shows principal angle jitter of the spacecraft (drift subtracted out) for each model. This result gives confidence that the imbalance parameter adaptation is accurate for converting manufacturers' specifications on RW imbalances to the parameters needed for the fully-coupled simulation. However, this also shows that there is a noticeable difference between the two simulations which is a result of the fully-coupled simulation modeling the RW jitter as in internal rather than an external force and torque on the spacecraft.

VI Conclusion/Future Work

Previous work related to modeling jitter due to RW imbalances models the effect as an external force and torque on the spacecraft. In reality, this effect is an internal force and torque on the spacecraft and thus requires a different formulation. The work presented in this paper develops the general fully-coupled model of RW imbalances. The fully-coupled model allows for momentum and energy validation to be implemented in a simulation. A back-substitution

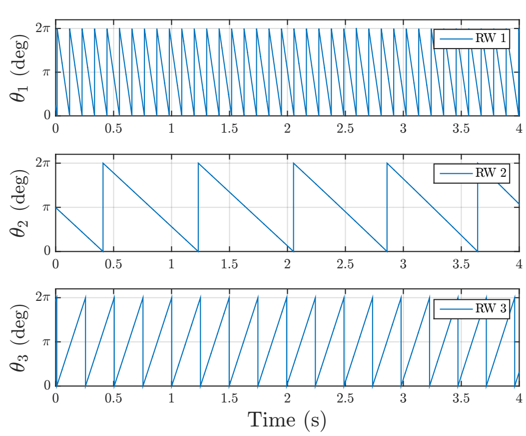


a) Inertial position of the spacecraft for the fully-coupled simulation

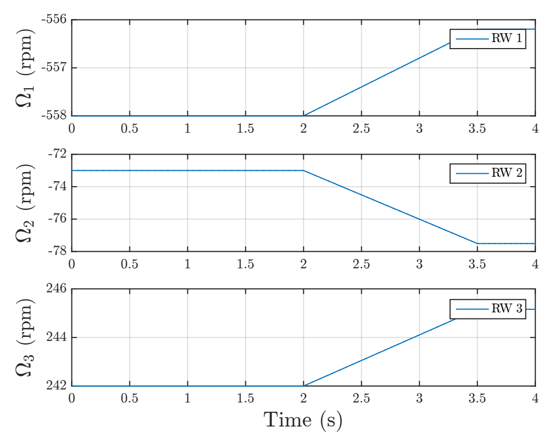


b) Inertial velocity of the spacecraft for the fully-coupled simulation

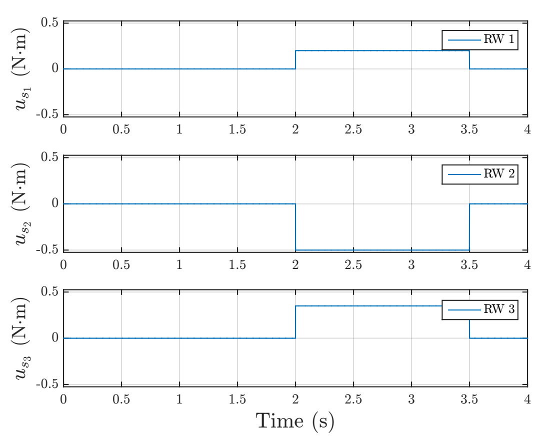
Figure 5: Position and velocity of the spacecraft



a) Wheel angles for the fully-coupled simulation

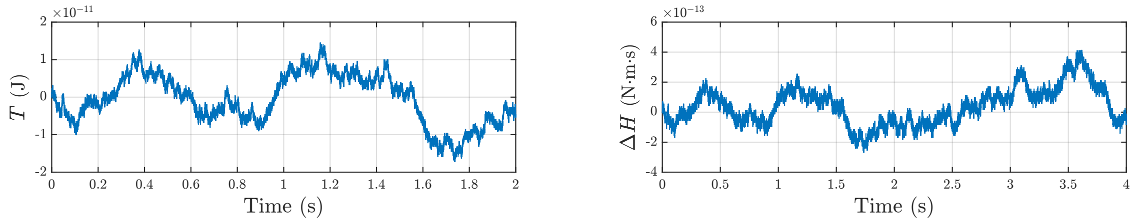


b) Wheel speeds for the fully-coupled simulation



c) Open-loop wheel motor torques for the fully-coupled simulation

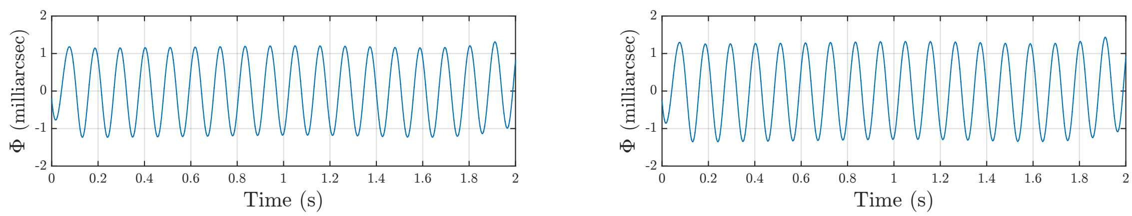
Figure 6: Wheel angle, wheel speed, and motor torque of RWs



a) System energy Δ for the fully-coupled simulation

b) System angular momentum Δ for the fully-coupled simulation

Figure 7: Change in energy and momentum of the spacecraft



a) Principal angle jitter using the the fully-coupled model.

b) Principal angle jitter using the the simplified model.

Figure 8: Comparison of principal angle jitter results for the fully-coupled and simplified models.

method is employed to avoid the computational penalty of inverting a large system mass matrix. Additionally, a discussion is included that aids in converting manufacturers' specifications of RW imbalances to the parameters needed for the fully-coupled simulation.

Energy is shown to be conserved when the motor torques are zero, and momentum is conserved throughout the length of the simulations. This provides validation of the fully-coupled model and highlights drawbacks to the simplified model. A comparison between the fully-coupled model and the simplified model shows that the imbalance parameter adaptation is successful because the fully-coupled and simplified models give similar high-level results. However, the simplified model is not valid in terms of conservation of energy and conservation of angular momentum. This is undesirable when including additional complex dynamical models such as flexible dynamics or fuel slosh and causes error propagation to be a concern for lengthy simulation times.

The fully-coupled model presented does not include higher order effects such as bearing friction, bearing instabilities, and structural vibration.[10] Including these effects will be considered for future work. Additionally, it is of interest to derive the same equations for devices such as control moment gyros (CMGs) and dual-gimbal CMGs.

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