

SMO - Final round 2018

Second exam - 17 January 2018

Duration: 4 hours

Difficulty: The problems are ordered by difficulty.

Points: Each problem is worth 7 points.

6. Let k be the incircle of triangle ABC with center I . The circle k touches the sides BC, CA and AB at the points D, E and F respectively. Let G be the intersection point of the segment AI with the circle k . We assume that the lines BE and FG are parallel. Prove that $BD = EF$.

7. Let n be a natural integer and let k be the number of ways to write n as the sum of one or more consecutive natural integers. Prove that k is equal to the number of odd positive divisors of n .

Example: 9 has three positive odd divisors and $9 = 9$, $9 = 4 + 5$, $9 = 2 + 3 + 4$.

8. Let a, b, c, d and e be positive real numbers. Determine the largest value that the following expression can take:

$$\frac{ab + bc + cd + de}{2a^2 + b^2 + 2c^2 + d^2 + 2e^2}.$$

9. Let n be a natural integer and G be the set of points (x, y) in the plane such that x and y are integers with $1 \leq x, y \leq n$. A subset of G is called *parallelogramfree* if it does not contain four non-collinear points that are the vertices of a parallelogram. How many points at most can a parallelogramfree subset contain ?

10. Let $p \geq 2$ be a prime number. Arnaud and Louis alternatively choose an index $i \in \{0, 1, \dots, p-1\}$ that has not already been chosen and a digit $a_i \in \{0, 1, \dots, 9\}$. Arnaud starts. Once every index has been chosen, they compute the following sum:

$$a_0 + a_1 \cdot 10 + \dots + a_{p-1} \cdot 10^{p-1} = \sum_{i=0}^{p-1} a_i \cdot 10^i.$$

If the sum is divisible by p , Arnaud wins. Otherwise Louis wins. Prove that Arnaud has a winning strategy.

