Swiss Mathematical Competition

First Part - May 4, 2001

Time: 3 hours
Each problem is worth 7 points.

1. In a park there is a square of 2001 $\times$ 2001 trees. What is the largest number of trees that can be cut down, so that from each stump you cannot see another stump?

2. Let a, b and c be the sides of a triangle. Prove the inequality:

$$\sqrt{a+b-c}+\sqrt{c+a-b}+\sqrt{b+c-a} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

   Under what conditions does equality hold?

3. A convex pentagon is given, in which each diagonal is parallel to one side. Prove that the ratio between the lengths of each diagonal and the side which is parallel to it, is the same for every diagonal and determine the value of this ratio.

4. Let $n \in \mathbb{N}$, $n \geq 2$ and $t_1, t_2, \ldots, t_k$ be k different divisors of n.
   An identity $n = t_1 + t_2 + \ldots + t_k$ is called representation of n as sum of divisors, while two such representations are called equal if they differ only in the order of the summands (for example: $20 = 10 + 5 + 4 + 1$ and $20 = 5 + 1 + 10 + 4$ are the same representation of 20 as sum of divisors).
   Let $a(n)$ be the number of different representations of n as sum of divisors.

   Prove or disprove:

   There is an $M \in \mathbb{N}$ with $a(n) \leq M$ for all $n \in \mathbb{N}, n \geq 2$

5. Let a be a sequence of positive integers $a_1 < a_2 < \ldots < a_n$ with the property, that for $i < j$, the decimal representation of a_j does not start with the one of the a_i (for example: 137 and 13729 cannot both appear in the sequence). Prove that:

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{9}$$

Swiss Mathematical Competition

Second Part - May 19, 2001

Time: 3 hours
Each problem is worth 7 points.

1. A function $f : [0, 1] \to \mathbb{R}$ has the following properties:

   (a) $f(x) \geq 0 \quad \forall\, x \in [0, 1]$

   (b) $f(1) = 1$

   (c) $f(x + y) \geq f(x) + f(y) \quad \forall\, x, y, x + y \in [0, 1]$

   Prove that: $f(x) \leq 2x \quad \forall x \in [0, 1]$

2. Let ABC be an acute-angled triangle with circumcenter O. Let S be the circle through A, B and O. The straight lines AC and BC meet S in the additional points P and Q respectively.
   Prove that $CO \perp PQ$.

3. Find the two smallest positive integers n, such that the fractions

   $$\frac{68}{n+70}, \frac{69}{n+71}, \frac{70}{n+72}, \ldots, \frac{133}{n+135}$$

   are irreducible.

4. In Geneva there are 16 secret agents at work. Each of them is spying on at least one other agent, but no two agents are spying on one another. Suppose that we can number every ten agents in a way, that the first one is spying on the second, the second on the third and so on and the tenth is spying on the first.
   Prove that also every 11 agents can be numbered in a way, so that every agent is spying on the next one.

5. Prove that each 1000-element subset $M \subset \{0, 1, \ldots, 2001\}$ has the following property: There exists one element in M which is a power of two or there are two distinct elements in M whoze sum is a power of two.