Final round 2022

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Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let k be a circle with centre M and let AB be a diameter of k. Furthermore, let C be a point on k such that $AC = AM$. Let D be the point on the line AC such that $CD = AB$ and C lies between A and D. Let E be the second intersection of the circumcircle of BCD with line AB and F be the intersection of the lines ED and BC. The line AF cuts the segment BD in X. Determine the ratio BX/XD.

Answer: $BX/XD = 1/6$.

First solution (Florian): Let $s = AM$. According to the conditions in the exercise we get:

$$s = AM = MB = \frac{1}{2}AB = AC = \frac{1}{2}CD = \frac{1}{3}AD$$

Using the power of the point A with respect to the circle $EBDC$

$$AE = \frac{AC \cdot AD}{AB} = \frac{s \cdot 3s}{2s} = \frac{3s}{2}$$

and therefore

$$EB = AB - AE = \frac{s}{2}.$$

Since F is an inner point of the triangle ABD, applying Ceva's theorem gives

$$1 = \frac{BX}{XD} \cdot \frac{DC}{CA} \cdot \frac{AE}{EB} = \frac{BX}{XD} \cdot \frac{2s}{s} \cdot \frac{3s/2}{s/2} = 6 \cdot \frac{BX}{XD} \quad \Longrightarrow \quad \frac{BX}{XD} = \frac{1}{6}.$$

Second solution: Using Thales' theorem over the circles k and $EBCD$ gives

$$90^\circ = \angle ACB = \angle DCB = \angle DEB.$$

Therefore, DE and BC are altitudes of the triangle ABD. Thus, F is the orthocenter of ABD. It follows that $EBDC$, $AEXD$ and $ABXC$ are cyclic quadrilaterals (in particular $X \in k$).

By power of the points A, B, D with respect to the circles $EBDC$, $AEXD$, $ABXC$ (individually) we get

$$AE \cdot AB = AC \cdot AD,$$

$$BE \cdot BA = BX \cdot BD,$$

$$DX \cdot DB = DC \cdot DA.$$

Up to this point, we have not used any of the conditions about the lengths. Like in the first solution, we define $s = AM$ and get

$$s = AM = MB = \frac{1}{2}AB = AC = \frac{1}{2}CD = \frac{1}{3}AD.$$

Finally

$$\frac{BX}{XD} = \frac{BX \cdot BD}{DX \cdot DB} = \frac{BE \cdot BA}{DC \cdot DA} = \frac{AB^2 - AE \cdot AB}{DC \cdot DA} = \frac{AB^2 - AC \cdot AD}{DC \cdot DA} = \frac{(2s)^2 - s \cdot 3s}{2s \cdot 3s} = \frac{1}{6}.$$

Marking scheme:

Solution 1

(a) Stating $\frac{BX}{XD} = \frac{1}{6}$ .... (1 point)

(b) Proving $EB = \frac{s}{2}$ or $AE = \frac{3s}{2}$, for $s = AM$ or any distance such as MB, $\frac{1}{2}AB$, AC, $\frac{1}{2}CD$, $\frac{1}{3}AD$ .... (3 points)

(c) Using Ceva's theorem on point F with respect to triangle ABD .... (2 points)

(d) Conclude .... (1 point)

Note: It is not necessary to mention right angles, altitudes or the fact that X lies on k.

Solution 2

(a) Stating $\frac{BX}{XD} = \frac{1}{6}$ .... (1 point)

(b) Proving $AB \perp DE$ and $AD \perp BC$ .... (1 point)

(c) Concluding that F is the orthocenter of ABD .... (1 point)

(d) Proving that $AEXD$ and $ABXC$ are cyclic .... (1 point)

(e) Proving the following 3 identities .... (2 points)

$$AE \cdot AB = AC \cdot AD$$

$$BE \cdot BA = BX \cdot BD$$

$$DX \cdot DB = DC \cdot DA$$

Proving 1 of the last two identities is worth .... (1 point)

(f) Conclude .... (1 point)

Note: To prove (b), (c), (d) and (e) it is not required to use any of the given conditions about the lengths. The fact $X \in k$ must be proven explicitly. If it is not the case, then ***at most 4 points*** *are awarded.*

2. Let n be a positive integer. Prove that the numbers

$$1^1,\ 3^3,\ 5^5,\ \ldots,\ (2^n-1)^{2^n-1}$$

all give different remainders when divided by 2^n.

Solution (Johann): Let's prove the statement by induction. The base case $n=1$ is trivial. Let's prove the inductive step. Given that $1^1,\ 3^3,\ 5^5,\ \ldots,\ (2^n-1)^{2^n-1}$ have different residues $\pmod 2)^n$, we want to show that $1^1,\ 3^3,\ 5^5,\ \ldots,\ (2^{n+1}-1)^{2^{n+1}-1}$ give different results mod 2^{n+1}. Let's split the 2^n elements in two sets A and B, where $A=\{1^1,\ 3^3,\ 5^5,\ \ldots,\ (2^n-1)^{2^n-1}\}$ and $B=\{(2^n+1)^{2^n+1},\ (2^n+3)^{2^n+3},\ (2^n+5)^{2^n+5},\ \ldots,\ (2^n+(2^n-1))^{2^n+(2^n-1)}\}$. Let's closely examine the elements in B mod 2^{n+1}. By Euler-Fermat (we can use it in this case because $\gcd(2^n+k,2^{n+1})=1$) and the binomial expansion, we have:

$$(2^n+k)^{2^n+k}\equiv(2^n+k)^k\equiv\sum_{s=0}^{k}\binom{k}{s}\cdot 2^{ns}\cdot k^{k-s}\equiv 2^n\cdot k^k+k^k\equiv 2^n+k^k \pmod{2^{n+1}}$$

In other words, $B\equiv\{1^1+2^n,\ 3^3+2^n,\ 5^5+2^n,\ \ldots,\ (2^n-1)^{2^n-1}+2^n\} \pmod{2^{n+1}}$. Now, by the inductive hypothesis, the elements in A are distinct $\pmod{2^n}$ and so are the elements in B. Finally, since the elements in $B \pmod{2^{n+1}}$ are simply obtained by adding 2^n to the elements in A, the 2^n numbers give different remainders $\pmod{2^{n+1}}$, concluding the inductive step. Hence, the initial statement is true for all $n\in\mathbb{N}$.

Marking scheme:

Partial solutions .... (≤ 4 points)

The following additive partial points can be obtained:

(a) Any attempt to use induction or to pair up the numbers (even in a way which might not work, eg. $\{k^k,(2^{n+1}-k)^{2^{n+1}-k}\}$) .... (1 point)

(b) Any attempt of considering pairs of the form $\{k^k,(2^n+k)^{2^n+k}\}$ or splitting the numbers into the two sets A and B .... (1 point)

(c) Observing $(2^n+k)^{2^n}\equiv 1 \pmod{2^{n+1}}$ .... (1 point)

(d) Proving that $(2^n+k)^{2^n+k}\equiv 2^n+k^k \pmod{2^{n+1}}$ .... (1 point)

Complete solutions .... (≥ 6 points)

At most one point can be deducted if one has not explicitly stated one of the following elements:

- Forgetting the base case .... (-1 point)
- Not mentioning $\gcd(2^n+k,2^{n+1})=1$ for Euler-Fermat .... (-1 point)

3. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that both

- $f(f(m)f(n)) = mn$
- $f(2022a + 1) = 2022a + 1$

hold for all positive integers m, n and a.

Solution (Arnaud): Soit f une solution du problème. On commence par poser $n = 1$ dans la première équation et on obtient $f(f(n)f(1)) = n$. On en déduit que f est **bijective**. En utilisant la surjectivité, il existe x tel que $f(x) = 1$. Avec $m = n = x$ dans la première équation, on obtient $f(1) = x^2$. Poser $m = n = 1$ donne $f(f(1)^2) = 1 = f(x)$, et donc, par injectivité, $x = f(1)^2$. On conclut que $f(1) = x^2 = f(1)^4$ et ainsi

$$f(1) = 1.$$

Avec $n = 1$ on en déduit que

$$f(f(m)) = m.$$

On peut à présent remplacer m et n par $f(m)$ et $f(n)$, respectivement, pour obtenir que

$$f(mn) = f(m)f(n).$$

Cette équation est l'équation mutliplicative de Cauchy sur les nombres entiers positifs. L'exemple du script nous apprend que f est déterminée par les valeurs $f(p)$ où p est un nombre premier. Il suffit donc de déterminer $f(p)$ pour chaque nombre premier p.

Claim 1: $f(p) = p$ pour tout p qui ne divise pas 2022, c'est-à-dire pour $p \notin \{2, 3, 337\}$.

Proof. Si p ne divise pas 2022, alors $\gcd(p, 2022) = 1$ et donc p a un inverse modulo 2022. Soit donc y un nombre entier positif tel que $py \equiv 1 \pmod{2022}$. La deuxième condition implique que $f(py) = py$. La propriété de Cauchy implique que $f(py) = f(p)f(y)$. Ainsi $f(p)f(y) = py$. De même, $f(p)f(y + 2022) = p(y + 2022)$. Observer que $\gcd(y + 2022, y) = 1$, car $\gcd(y, 2022) = 1$. Donc $\gcd(py, p(y + 2022)) = p$. Mais on doit avoir $f(p) \mid \gcd(py, p(y + 2022))$, donc $f(p) = p$, car $f(p) \neq 1$ par injectivité. $\square$

Claim 2: $f(p)$ est un nombre premier pour tous les nombres premiers p.

Proof. Supposons que $f(p) = xy$ avec $x, y \neq 1$. Alors $p = f(f(p)) = f(xy) = f(x)f(y)$, mais $f(x), f(y) \neq 1$ par injectivité. Contradiction. $\square$

Si $f(p) = q$, alors $f(q) = p$, car $f(f(p)) = p$. Ainsi, tous les nombres premiers qui ne sont pas envoyés sur eux-même par f vont par paires (p, q) avec $f(p) = q$ et $f(q) = p$.

On prétend qu'il existe ainsi quatre fonctions solutions déterminées par:

(a) $f(p) = p$ pour tous nombres premiers p, i.e. f est l'identité,

(b) $f(2) = 3$, $f(3) = 2$ et $f(p) = p$ pour tous nombre premiers $p \notin \{2, 3\}$,

(c) $f(2) = 337$, $f(337) = 2$ et $f(p) = p$ pour tous nombre premiers $p \notin \{2, 337\}$,

(d) $f(3) = 337$, $f(337) = 3$ et $f(p) = p$ pour tous nombre premiers $p \notin \{3, 337\}$.

Il s'agit maintenant de vérifier que ces fonctions sont bien solutions. L'identité est clairement une solution. On vérifie que la fonction (b) est bien solution. L'argument est identique pour les fonctions (c) et (d). La fonction (b) satisfait $f(f(p)) = p$ pour tous les nombres premiers p et $f(p)$ est un nombre premier pour tous les nombres premiers p. Comme la fonction est définie multiplicativement à partir des valeurs qu'elle prend sur les nombres premiers, elle satisfait

$f(mn) = f(m)f(n)$ et donc $f(f(n)) = n$ (en utilisant que $f(f(p)) = p$). Elle satisfait donc la première condition. Soit maintenant un nombre x tel que $x \equiv 1 \pmod{2022}$. On doit montrer que $f(x) = x$. En effet, les facteurs premiers de x sont premiers avec 2022 et donc sont envoyés sur eux-mêmes par f. Ainsi, $f(x) = x$. On en conclut que la fonction (b) satisfait bien les conditions du problème.

Remarque: Le Claim 1 peut aussi être démontré de la manière suivante. Si $f(p) = x$, alors $f(p^k) = x^k$. Si p est premier avec 2022, alors il existe k tel que $p^k \equiv 1 \pmod{2022}$. La deuxième condition implique $p^k = f(p^k) = x^k$ et donc $p = x = f(p)$.

Marking scheme:

Solutions partielles .... (≤ 6 points)

Les points partiels suivants peuvent être obtenus:

(a) Obtenir $f(mn) = f(m)f(n)$ .... (2 points)

(b) Montrer que $f(p) = p$ pour $p \notin \{2, 3, 337\}$ .... (2 points)

(c) Montrer que $f(p) = q$, $f(q) = p$ et $f(r) = r$ pour $\{p, q, r\} = \{2, 3, 337\}$ .... (2 points)

Si (a) n'a pas été obtenu, alors le point suivant peut-être accordé:

- Montrer que $f(1) = 1$ ou $f(f(n)) = n$ .... (1 point)

Si les points (b) et (c) n'ont pas été obtenus, alors le point suivant peut-être accordé:

- Montrer que $f(p)$ est premier pour tout premier p .... (1 point)

Si le point (c) n'a pas été obtenu, alors le point suivant peut-être accordé:

- Énoncer au moins une solution exotique .... (1 point)

Solutions complètes .... (≥ 6 points)

Au plus un point peut être retiré:

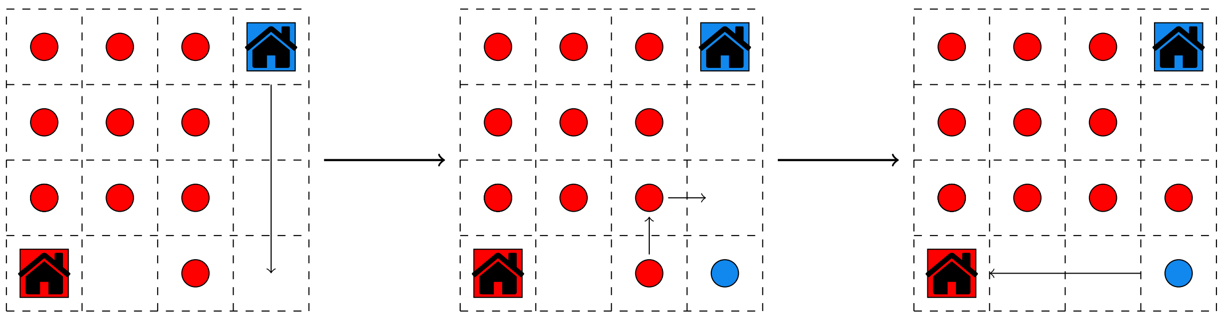
- Manque de vérification d'au moins une solution exotique .... (-1 point)
- Mauvaise factorisation de 2022 .... (-1 point)

4. Let $n \geq 2$ be an integer. Switzerland and Liechtenstein are performing their annual festive show. There is a field subdivided into $n \times n$ squares, in which the bottom-left square contains a red house with k Swiss gymnasts, and the top-right square contains a blue house with k Liechtensteiner gymnasts. Every other square only has enough space for a single gymnast at a time. Each second either a Swiss gymnast or a Liechtensteiner gymnast moves. The Swiss gymnasts moves to either the square immediately above or to the right and the Liechtensteiner gymnasts moves either to the square immediately below or to the left. The goal is to move all the Swiss gymnasts to the blue house and all the Liechtensteiner gymnasts to the red house, with the caveat that a gymnast cannot enter a house until all the gymnasts of the other nationality have left. Determine the largest k in terms of n for which this is possible.

Answer: $(n-1)^2$

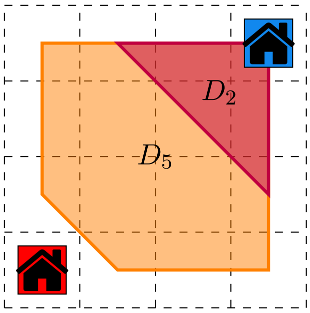
Solution (David, Raphael and Tanish): Firstly, note that this value is indeed possible: simply have the gymnasts of one nation all move out to occupy a $(n-1) \times (n-1)$ square, have the other nation's gymnasts walk around the edge into the now-empty house and then the first nation can do likewise.

The proof that this value is optimal is less straightforward. Firstly, note that a justification along of the lines of "all the Swiss gymnasts should move out before the Liechtensteiner gymnasts can move in, and the latter group need at least $2n-3$ squares free (not including the two houses) to create a path for them" is critically flawed. The reason for this is illustrated in the below diagram:



As shown, a continuous open path is not necessary for a gymnast to cross the field. It is therefore erroneous to argue that such a pathway being necessary is a complete proof and further justification is required - namely, that not having an open path will eventually lead to progress becoming impossible.

Assume by absurdity that the exchange is possible for $k > (n-1)^2$ gymnasts. Assume WLOG that all the Swiss gymnasts leave their house before the blue house is empty. Consider now the state of the board at the moment that the last Swiss gymnasts exits the house. All the Swiss gymnasts must be on the field but cannot yet enter the blue house. The red house is now open for the first time; so at most $2n-4$ spaces on the board are not occupied by a red gymnast, call such spaces *emblue*. We define the *region* D_j as being the set of squares from which one can access the blue house making at most j moves up or to the right.



If the region D_j has $j-1$ or fewer emblues, we say it is *active*. Clearly, D_{2n-3} is initially active; furthermore, if a region becomes active it will stay active (as the number of Swiss gymnasts in a region cannot decrease, at least not until all the Liechtensteiner gymnasts have left their own house).

Consider the active region with smallest index $D_{\mathbf{min}}$, and let the initial value of **min** (at the moment the last Swiss gymnast leaves their house) be i_0. We know that $D_{\mathbf{min}}$ contains at most $\mathbf{min}-1$ emblues by definition. Furthermore, if it contained $\leq \mathbf{min}-2$ emblues then it would follow that $D_{\mathbf{min}-1}$ was also active, contradicting the minimality of $D_{\mathbf{min}}$. So this region must contain exactly $\mathbf{min}-1$ emblues. None of these emblues can lie on the diagonal $D_{\mathbf{min}} \setminus D_{\mathbf{min}-1}$ because as mentioned, this would lead to $D_{\mathbf{min}-1}$ being active (by virtue of having $\leq \mathbf{min}-2$ emblues) as well. So this diagonal must contain no emblues and only Swiss gymnasts, and we will refer to it as the *fence* - as long as the fence remains unbroken, no blue gymnast inside $D_{\mathbf{min}}$ is leaving. We refer to any emblue that is on the side of the fence containing the red house (the complementary of $D_{\mathbf{min}}$) as being *free*. Clearly, freedom is a necessary trait to gain access to the red house. Initially, not all the blue gymnasts are free - those in the blue house must not be free.

Now, the fence must open at some point as otherwise the blue gymnasts who are not yet free cannot get to the red house. At this moment, a Swiss gymnast will cross from $D_{i_0} \setminus D_{i_0-1}$ into D_{i_0-1}. This means that the number of emblues in D_{i_0-1} is now $\leq i_0-2$ and so the value of **min** decreases - let us say the new value is i_1. As proven above, D_{i_1} contains exactly i_1-1 emblues and the diagonal $D_{i_1} \setminus D_{i_1-1}$ is the new fence; now, the blue gymnasts in the complementary of D_{i_1} are free. So all the gymnasts who were previously free are still free, and at most i_0-i_1 new gymnasts are free - the number of emblues in $D_{i_0} \setminus D_{i_1}$ is exactly i_0-i_1, and this is an upper bound on how many blue gymnasts can have been freed when the fence moved. Initially, if some m gymnasts are free, we must have that D_{i_0} is at most $2n-3-m$ as the number of emblue spaces that are not free is at most $2n-4-m$ (recalling that D_{i_0} must contain exactly i_0-1 emblues that are not free). It follows that for $2n-4$ gymnasts to become free, the value of **min** must be reduced by at least i_0-1, and so D_1 will become active at this point - making exiting the blue house impossible.

We therefore have $(n-1)^2 < k \leq 2n-4$. This simplifies to $(n-2)^2 \leq -1$, which is never true, contradiction.

Remark: You might wonder why the fence is defined in this particular way, and not simply as the "closest full diagonal to the blue house" or something along those lines. The reason for this is that, for example, the closest full diagonal to the red house does not always move in one direction - breaking the closest full diagonal does not always create a full diagonal that is closer still. In order to avoid any side cases like this, we want to study some phenomenon that is a true monovariant and only moves in one direction.

Marking scheme:

Partial solutions .... ($\leq$ 4 points)

The following additive partial points can be obtained, up to a maximum of 4 points.:

- Finding the correct value $(n-1)^2$ (an explanation is not necessary) .... (1 point)
- Any attempt to construct a useful monovariant related to distance .... (1 point)
- Showing that by pigeonhole, for $k > (n-1)^2$, that a diagonal is always full of the gymnasts of the first house to become empty .... (1 point)
- A proof that the aforementioned monovariant is indeed monotone that is nearly complete but suffers from a nontrivial flaw .... (2 points)

Complete solutions .... ($\geq$ 5 points)

Typically, solutions will not work for small values of n, so the edge cases mentioned below refers to these small-value cases. At most 2 points may be deducted for the following reasons:

- Failing to verify less than 5 edge cases .... (-1 point)
- Failing to verify finitely many edge cases .... (-2 points)
- Failing to justify why $(n-1)^2$ can be acheived .... (-1 points)

5. For an integer $a \geq 2$, denote by $\delta(a)$ the second largest divisor of a. Let $(a_n)_{n\geq 1}$ be a sequence of integers such that $a_1 \geq 2$ and

$$a_{n+1} = a_n + \delta(a_n)$$

for all $n \geq 1$. Prove that there exists a positive integer k such that a_k is divisible by 3^{2022}.

Solution: Let $p_i = a_i/\delta(a_i)$ be the smallest divisor of a_i not equal to 1. We then have

$$a_{i+1} = a_i + \frac{a_i}{p_i} = (p_i + 1)\frac{a_i}{p_i}.$$

So, if p_1 is odd, then a_{i+1} will be even. Note that the sequence a_n fulfills the condition exactly when there exists a k, such that $v_3(a_k) \geq 2022$.

We differentiate between the three cases:

- **(a_i even)**: In this case we have that $a_{i+1} = \frac{3}{2}a_i$. So here $v_3(a_{i+1}) = v_3(a_i) + 1$
- **(a_i is odd and not divisible by 3)**: In this case $v_3(a_{i+1}) = v_3((p_i + 1)\frac{a_i}{p_i}) \geq v_3(a_i)$ and then by the above case $v_3(a_{i+2}) \geq v_3(a_i) + 1$.
- **(a_i is odd and divisble by 3)**: In this case we have $a_{i+1} = \frac{4}{3}a_i$ and so a_{i+1} is even. Thus, $a_{i+2} = \frac{3}{2}a_{i+1} = 2a_i$ and $a_{i+3} = \frac{3}{2}a_{i+2} = 3a_i$. So, here we get $v_3(a_{i+3}) \geq v_3(a_i) + 1$.

So, if we start with the number a_1, we have a subsequence $(a_{i_l})_{l\geq 1}$, with $v_3(a_{i_{l+1}}) \geq v_3(a_{i_l}) + 1$. By that we have

$$3^{2022} \mid a_{i_{2022}}.$$

Marking scheme:

The following additive partial points can be obtained:

- The idea of comparing $v_3(a_{i+1})$ with $v_3(a_i)$ .... (1 points)
- Note that for the smallest prime divisor p_i of a_i, we have $a_{i+1} = p_{i+1}\frac{a_i}{p_i}$. .... (1 points)
- Look at the a_i even case .... (1 points)
- Look at case a_i odd and not divisible by 3 .... (1 points)
- Look at case a_i odd and divisible by 3 .... (2 points)
- Conclude .... (1 points)

6. Let $n \geq 3$ be an integer. Annalena has infinitely many cowbells in each of n different colours. Given an integer $m \geq n+1$ and a group of m cows standing in a circle, she is tasked with tying one cowbell around the neck of every cow so that every group of $n+1$ consecutive cows have cowbells of all the possible n colours. Prove that there are only finitely many values of m for which this is not possible and determine the largest such m in terms of n.

Answer: $n^2 - n - 1$.

Solution: Let us first give a construction which works for all numbers $m \geq n^2 - n$:

Write $m = kn + r$ for some integer $k \geq n-1$ and $r \in \{0, 1, \ldots, n-1\}$. Note that $k - r \geq 0$. If we call the colours $1, 2, \ldots, n$, we arrange them as follows:

$$\underbrace{1,1,2,3,\ldots,n,1,1,2,3,\ldots,n,\ldots 1,1,2,3,\ldots n}_{r \text{ times}},\underbrace{1,2,3\ldots,n,1,2,3,\ldots,n,\ldots,1,2,3,\ldots,n}_{k-r \text{ times}}$$

Note that $r(n+1)+(k-r)n = kn+r = m$. It is easy to see that this construction indeed works.

We will now prove that it is not possible to accomplish the task for $m = n^2 - n - 1$. Observe that $n^2 - n - 1 = n(n-1) - 1$. By pigeonhole principle, one of the n colours appears at most $n-2$ times, say blue. Pick an arbitrary cow with a blue bell. There exist now $n^2 - n - 2 = (n+1)(n-2)$ other cows among which only at most $n-3$ wear a blue cowbell. If we partition the remaining cows into intervals of length $n+1$, we can see that in at least one of them, there is no cow with a blue cowbell.

Marking scheme:

The marking is split into two parts. Part A is the construction for all $m \geq n^2 - n$ and part B is the proof that $m = n^2 - n - 1$ doesn't allow a valid construction. The points of part A are always additive with the points of part B.

Part A: .... (3 points)

The following are non-additive.

(a.1) Finding constructions for two or more coprime m given a fixed value of n .... (1 point)

(a.2) Proving that the task is impossible for only finitely many values of m .... (2 points)

(a.3) Giving a construction for all $m \geq n^2 - n$ .... (3 points)

Part B: .... (4 points)

The first point is non-additive with the other two.

(b.1) Claiming that the answer is $n^2 - n - 1$ .... (1 point)

(b.2) Observing that for $m = n^2 - n - 1$, one colour appears at most $n-2$ times .... (2 points)

(b.3) Proving that if there is a colour which appears less than $\frac{m}{n+1}$ times, there exists no valid construction .... (2 points)

7. Let $n > 6$ be a perfect number. Let $p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ be the prime factorisation of n where we assume that $p_1 < p_2 < \ldots < p_k$ and $a_i > 0$ for all $i = 1, \ldots, k$. Prove that a_1 is even.

Remark: An integer $n \geq 2$ is called a perfect number *if the sum of its positive divisors, excluding n itself, is equal to n. For example, 6 is perfect, as its positive divisors are $\{1, 2, 3, 6\}$ and* $1 + 2 + 3 = 6$.

Solution: Since n is perfect, we can write

$$2n = \sum_{1 \leq d | n} d = \sum_{0 \leq b_i \leq a_i} p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k} = \prod_{i=1}^{k} (1 + p_i + \cdots + p_i^{a_i}).$$

Now assuming a_1 is odd, we find that

$$(1 + p_1 + \cdots + p_1^{a_1}) \equiv 1 - 1 + \cdots - 1 \equiv 0 \pmod{p_1 + 1}$$

and therefore $p_1 + 1 \mid 2n$.

If $p_1 > 2$, then $p_1 + 1 \mid n$, but since p_1 is the smallest prime divisor of n, no prime divisor of $p_1 + 1$ can divide n, leading to a contradiction.

We conclude that $p_1 = 2 \mid n$. Since $p_1 + 1 = 3 \mid 2n$, we also get $3 \mid n$. But now note that since $n > 6$, the integers $1, n/2, n/3, n/6$ are distinct, proper divisors of n which sum to $n + 1 > n$, contradicting the fact that n is perfect. We conclude that a_1 must be even.

Note: The case where $2 \mid n$ can also be solved as follows. The Euler-Euclid Theorem says that if n is an even perfect number, then there exists a prime p such that $n = 2^{p-1}(2^p - 1)$. Since $n > 6$, then $p > 2$ is odd and so $a_1 = p - 1$ is even.

Marking Scheme:

- Any attempt to sort the divisors of n according to powers of p_1 or writing down the formula $2n = \prod_{i=1}^{k}(1 + p_i + \cdots + p_i^{a_i})$ .... (1 point)
- Stating that $1 + p_1 + \ldots + p_1^{a_1}$ divides $2n$ .... (1 point)
- Proving that if a_1 is odd, then $p_1 + 1 \mid 2n$ .... (2 points)
- In the case that n is even: proving that a_1 is even directly, or getting a contradiction assuming that a_1 is odd .... (2 points)
- In the case that n is odd: proving that a_1 is even directly, or getting a contradiction assuming that a_1 is odd .... (1 point)

8. Let ABC be a triangle and let P be a point in the interior of the side BC. Let I_1 and I_2 be the incenters of the triangles APB and APC, respectively. Let X be the closest point to A on the line AP such that XI_1 is perpendicular to XI_2. Prove that the distance AX is independent of the choice of P.

Solution: Let ω_1 and ω_2 be the two incircles, centered at I_1 and I_2 respectively. We first introduce the points R_1 and S_1 on ω_1 such that XR_1 and XS_1 are tangent to ω_1, with the condition that S_1 is on the line AP. Similarly, we introduce the points R_2 and S_2 on ω_2 such that XR_2 and XS_2 are tangent to ω_2, with the condition that S_2 is on the line AP. Finally, let T_1 and T_2 be the contact points of ω_1 and ω_2 on the line BC. Noting that I_1, I_2 are on the angle bisectors of $\angle R_1XP$, $\angle PXR_2$, respectively, we find that

$$\angle R_1XR_2 = \angle R_1XP + \angle PXR_2 = 2 \cdot (\angle I_1XP + \angle PXI_2) = 2 \cdot 90^\circ = 180^\circ,$$

so that R_1R_2 is a common tangent of ω_1 and ω_2. Note that, by reflection over I_1I_2, we have $R_1R_2 = T_1T_2$. Moreover, as tangents from a point have the same length, we can observe that

$$R_1R_2 = R_1X + XR_2 = XS_1 + XS_2 = S_2S_1 + 2 \cdot XS_2$$

and

$$T_1T_2 = T_1P + PT_2 = PS_1 + PS_2 = S_2S_1 + 2 \cdot PS_1,$$

so that $XS_2 = PS_1$, and we can calculate

$$AX = AS_2 - XS_2 = AS_2 - PS_1.$$

However, these last lengths can be computed as distances from a vertex of a triangle to a contact point of its incircle. Hence,

$$AX = AS_2 - PS_1 = \frac{AB + AP - BP}{2} - \frac{AP + PC - AC}{2} = \frac{AB + AC - BC}{2},$$

which is independent from the choice of P.

Marking Scheme:

The marking is split into two parts. Part A is the identification of X as being on the other common tangent to the incircles and part B is the computation of AX. The points of part A are always additive with the points of part B.

Part A: .... (3 points)

The following are non-additive.

(a.1) Proving that $\angle I_1PI_2 = 90°$ .... (0 points)

(a.2) Claiming that $AX = (AB + AC - BC)/2$, the distance to the contact point .... (1 point)

(a.3) Proving that I_1PI_2X lie on a circle .... (1 point)

(a.4) Claiming that X is on another common tangent of ω_1 and ω_2 .... (2 points)

(a.5) Proving that X is on another common tangent of ω_1 and ω_2 .... (3 points)

Part B: .... (4 points)

The following are additive.

(b.1) Proving that $XS_1 = PS_2$, or an equivalent equality .... (1 point)

(b.2) Computing PS_1 in terms of the sides of ABP .... (1 point)

(b.3) Computing AS_2 in terms of the sides of APC .... (1 point)

(b.4) Conclude .... (1 point)