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Final Round 2024

First Exam – Solutions

Zürich
March 2, 2024

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

1. If a and b are positive integers, we say that a *almost divides* b if a divides at least one of $b-1$ and $b+1$. We call a positive integer n *almost prime* if the following holds: for any positive integers a, b such that n almost divides ab, we have that n almost divides at least one of a and b. Determine all almost prime numbers.

Solution

Answer: The almost prime numbers are 1, 2, 3, 4 and 6.

Note that a almost divides b if and only if $b \equiv \pm 1\ (a)$.

We first check that those are indeed almost primes. Let $n \in \{1, 2, 3, 4, 6\}$. As $\varphi(n) \leq 2$, we have that $(a, n) = 1$ is equivalent to $a \equiv \pm 1\ (n)$. Then

$$ab \equiv \pm 1\ (n) \iff (ab, n) = 1 \iff (a, n) = (b, n) = 1 \iff a \equiv \pm 1\ (n) \text{ and } b \equiv \pm 1\ (n)$$

shows that n is almost prime.

Now let $n \notin \{1, 2, 3, 4, 6\}$. We claim the existence of a positive integer a such that $(a, n) = 1$ but $a \not\equiv \pm 1\ (n)$.

- If $n \equiv 0\ (4)$, let $n = 4k$ with $k \geq 2$. Then $a = 2k + 1$ works as $1 < a < n - 1$ and $(a, n) = (2k + 1, 4k) = (2k + 1, 2) = 1$.
- If $n \equiv 2\ (4)$, let $n = 4k + 2$ with $k \geq 2$. Then $a = 2k + 3$ works as $1 < a < n - 1$ and $(a, n) = (2k + 3, 4k + 2) = (2k + 3, 4) = 1$.
- If $n \equiv 1\ (2)$, let $n = 2k + 1$ with $k \geq 2$. Then $a = k + 1$ works as $1 < a < n - 1$ and $(a, n) = (k + 1, 2k + 1) = (k + 1, 1) = 1$.

As $(a, n) = 1$, we know that a has a multiplicative inverse b modulo n, such that $ab \equiv 1\ (n)$. Note that if $b \equiv \pm 1\ (n)$, then $a \equiv \pm 1\ (n)$, a contradiction. Hence $a \not\equiv \pm 1\ (n)$ and $b \not\equiv \pm 1\ (n)$, proving that n is not almost prime.

Marking Scheme

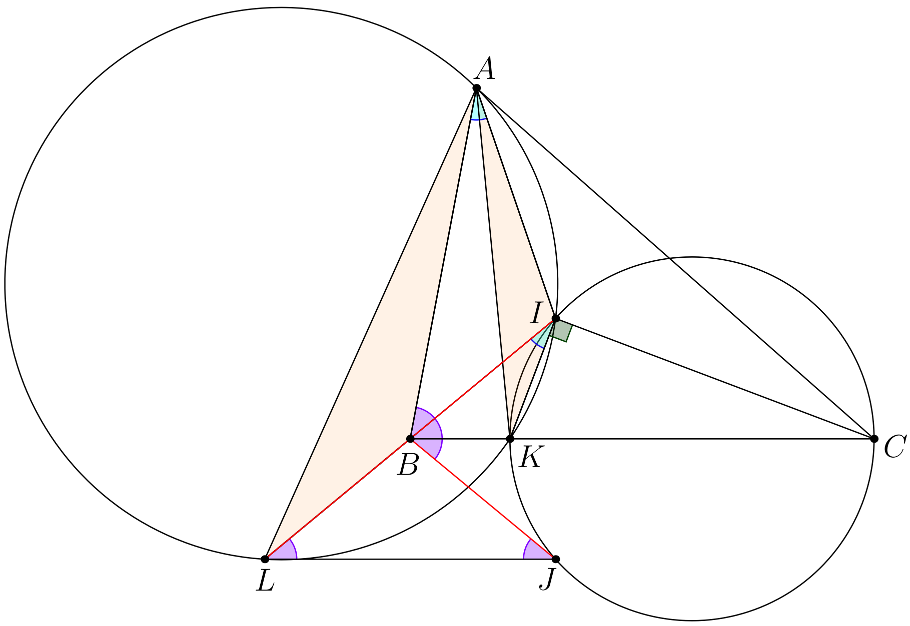
- 2P: Check that $n = 1, 2, 3, 4, 6$ are almost prime.
- 3P: If $n \neq 1, 2, 3, 4, 6$, prove that there exists $a \not\equiv \pm 1\ (n)$ such that $(a, n) = 1$.
- 1P: Prove that we can then pick $b \not\equiv \pm 1$ such that $ab \equiv \pm 1\ (n)$.
- 1P: Conclude.

Remarks:

- Finding four almost primes is worth 1P. Finding three is worth 0P.
- If nothing or the first bullet point is awarded, noting that a almost divides b if and only if $b \equiv \pm 1\ (a)$ is worth 1P.
- Proving the existence of $a \not\equiv \pm 1\ (n)$ such that $(a, n) = 1$ for $n \neq 1, 2, 3, 4, 6$ is subdivided into three cases, and each is worth 1P. In particular, doing it for odd $n \geq 5$ is worth 1P out of the three.

2. Let ABC be a triangle with incenter I, and let J be the reflection of I with respect to line BC. Let K be the second intersection of line BC with the circumcircle of triangle CIJ, and L be the second intersection of line BI with the circumcircle of triangle AIK. Prove that the lines BC and JL are parallel.

Solution



Let α, β, γ be the angles of triangle ABC. By angle chasing on triangles AIB and BIC, we have $\angle AIB = 90^\circ + \frac{\gamma}{2}$ and $\angle BIC = 90^\circ + \frac{\alpha}{2}$. As J is the reflection of I over BC, $\angle KIC = \angle CJK$, so that

$$\angle KIC = \frac{\angle KIC + \angle CJK}{2} = 90^\circ,$$

and $\angle BIK = \angle BIC - \angle KIC = (90^\circ + \frac{\alpha}{2}) - 90^\circ = \frac{\alpha}{2} = \angle BAI$.

Note that

$$\angle LAB = \angle LAK - \angle BAK = \angle LIK - \angle BAK = \angle BAI - \angle BAK = \angle KAI$$

and

$$\angle BLA = \angle ILA = \angle IKA,$$

so $\triangle ABL \sim \triangle AIK$. We have $\angle KBI = \frac{\beta}{2} = \angle IBA$ and $\angle BIK = \frac{\alpha}{2} = \angle BAI$, so we also have $\triangle KBI \sim \triangle IBA$. Using both similarities we have

$$\frac{BL}{AB} = \frac{IK}{AI} = \frac{BI}{AB},$$

so $BL = BI$.

Hence triangle BIJ is isosceles with base IJ. We can now conclude as

$$\angle JLB = 90^\circ - \frac{\angle LBJ}{2} = \frac{\angle JBI}{2} = \frac{\beta}{2} = \angle CBI,$$

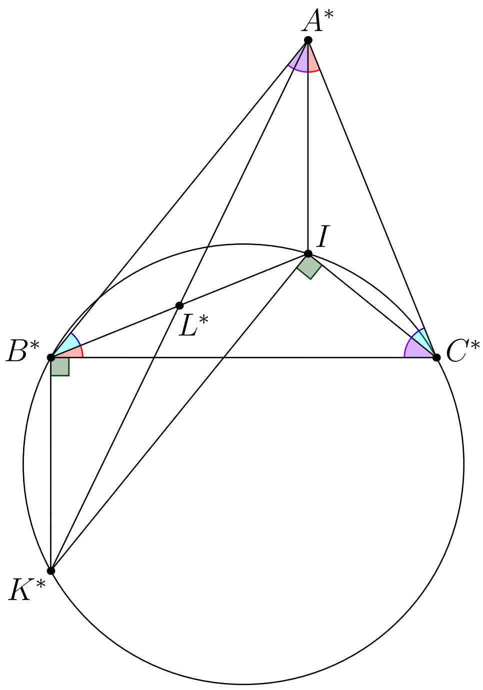
is equivalent to $BC \parallel JL$.

Solution 2: Let L' be the reflection of I over B, and K' the intersection of line BC and the circumcircle of triangle AIL' on the arc IL' that does not contain A. Then it is a well-known property of the symmedian that AK' is the A-symmedian of triangle ALI, as $\angle K'BI = \angle IBA$. Then

$$\angle BIK' = \angle L'IK' = \angle L'AK' = \angle BAI = \frac{\alpha}{2},$$

and $\angle K'IC = \angle BIC - \angle BIK' = (90^\circ + \frac{\alpha}{2}) - \frac{\alpha}{2} = 90^\circ$. This implies that $K'ICJ$ is cyclic, as $\angle K'IC + \angle CJK' = 2 \cdot \angle K'IC = 180^\circ$, so $K' = K$, $L' = L$, and from $BL = BI$ we conclude as in Solution 1.

Solution 3:

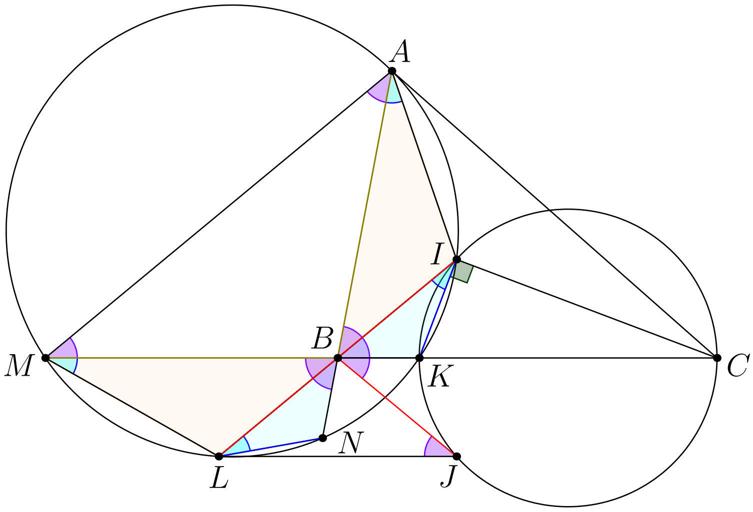


Invert around I. Then $\frac{\alpha}{2} = \angle IB^*A^* = \angle IC^*A^*$, $\frac{\beta}{2} = \angle IC^*B^* = \angle IA^*B^*$ and $\frac{\gamma}{2} = \angle IA^*C^* = \angle IB^*C^*$. From these angles, it follows that I is the orthocenter of $A^*B^*C^*$. Also K^* lies on the circumcircle of triangle B^*C^*I so that $\angle K^*IC^* = 90^\circ$ (as $\angle KIC = 90^\circ$, see Solution 1). Lastly L^* is the intersection of lines A^*K^* and B^*I. Then

$$K^*B^* \perp B^*C^* \perp IA^* \text{ and } K^*I \perp IC^* \perp B^*A^*,$$

so $K^*IA^*B^*$ is a parallelogram, and $IB^* = 2 \cdot IL^*$. Reverting back to the original problem, $IL = 2 \cdot IB$, so $BL = BI$ and we conclude as in Solution 1.

Marking Scheme



- 1P: Prove that $\angle KIC = 90°$ or $\angle KIJ = \frac{\gamma}{2}$.
- 1P: Prove that $\angle BIK = \frac{\alpha}{2}$.
- 1P: Prove that $\triangle KBI \sim \triangle IBA$, define $M = BC \cap (AIK)$, or define $N = AB \cap (AIK)$.
- 1P: Prove that $\angle LAB = \angle KAI$, $BM = BA$, or $LN = IK$.
- 1P: Prove that $\triangle ABL \sim \triangle AIK$, $\triangle MLB \cong \triangle AIB$ or $\triangle LNB \cong \triangle IKB$.
- 1P: Deduce that $BL = BI$.
- 1P: Conclude from $BL = BI$.

Marking scheme (Solution 2):

- 2P: Define L' as the reflection of I over B **and** $K' = BC \cap (AIL')$.
- 2P: Prove that AK' is the A-symmedian of triangle ALI.
- 1P: Prove that $\angle K'IC = 90°$.
- 1P: Deduce that $K' = K$ and $L' = L$, in particular $BL = BI$.
- 1P: Conclude from $BL = BI$.

Marking scheme (Solution 3):

- 1P: Invert around I.
- 1P: Prove that $IA^* \perp B^*C^*$.
- 1P: Prove that $IC^* \perp B^*A^*$.
- 1P: Prove that $K^*B^* \parallel IA^*$.
- 1P: Prove that $K^*I \parallel B^*A^*$.
- 1P: Deduce that $IB^* = 2 \cdot IL^*$.
- 1P: Conclude from $IB^* = 2 \cdot IL^*$.

3. Suppose that a, b, c, d are positive real numbers satisfying $ab^2 + ac^2 \geq 5bcd$. Determine the smallest possible value of

$$(a^2 + b^2 + c^2 + d^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}\right).$$

Solution

Looking at the big expression we observe that it is homogeneous of degree 0. So let's try to change the side condition to be homogeneous of degree 0 as well. In other words let's work with $\frac{ab^2+ac^2}{bcd} = \frac{a}{d}\left(\frac{b}{c} + \frac{c}{b}\right) \geq 5$. Let's try and make the fractions used in the side condition appear: Cauchy-Schwarz gives us

$$(a^2 + b^2 + c^2 + d^2)\left(\frac{1}{d^2} + \frac{1}{c^2} + \frac{1}{b^2} + \frac{1}{a^2}\right) \geq \left(\frac{a}{d} + \frac{b}{c} + \frac{c}{b} + \frac{d}{a}\right)^2.$$

Finally to make use of the side condition, we apply AM-GM on the two terms $\frac{a}{d}$ and $\left(\frac{b}{c} + \frac{c}{b} + \frac{d}{a}\right)$, to arrive at

$$\left(\frac{a}{d} + \frac{b}{c} + \frac{c}{b} + \frac{d}{a}\right)^2 \geq 4 \cdot \frac{a}{d} \cdot \left(\frac{b}{c} + \frac{c}{b} + \frac{d}{a}\right) \geq 4 \cdot (5 + 1) = 24.$$

We still have to make sure that this lower bound is achievable, so we have to check that there does exist an equality case.
We must have $ad = \lambda = bc$, $\frac{a}{d} \cdot \left(\frac{b}{c} + \frac{c}{b}\right) = 5$, and $\frac{a}{d} = \frac{b}{c} + \frac{c}{b} + \frac{d}{a}$. Using the second and third equation we get

$$\frac{a}{d} = 6\frac{d}{a} \Leftrightarrow a = \sqrt{6}d$$

This implies that $\lambda = \sqrt{6}d^2$. Lastly plugging this, together with $c = \frac{\lambda}{b}$ into the second equation gives us

$$\sqrt{6} \cdot \left(\frac{b^2}{\sqrt{6}d^2} + \frac{\sqrt{6}d^2}{b^2}\right) = \sqrt{6} \cdot 5 \Leftrightarrow b^4 + 6d^4 = 5b^2d^2 \Leftrightarrow (b^2 - 3d^2)(b^2 - 2d^2) = 0.$$

Plugging in $d = 1$, we find a solution with $b = \sqrt{3}$ or $b = \sqrt{2}$. All in all we arrive at the equality cases $(a, b, c, d) = (\sqrt{6}, \sqrt{2}, \sqrt{3}, 1)$ and $(a, b, c, d) = (\sqrt{6}, \sqrt{3}, \sqrt{2}, 1)$, as well as their scalar multiples.

Marking Scheme

The following points are additive.

- 1P: Rewrite the side condition **and** apply it to the expression in a useful way.
- 3P: Use CS with a, d and b, c opposed to each other.
- 1P: Plugging in $5\frac{d}{a}$ instead of $\frac{b}{c} + \frac{c}{b}$ and arguing that the RHS doesn't increase.
- 1P: Applying AM-GM and conclude.
- 1P: Find an equality case.

4. Determine the maximal length L of a sequence $a_1, \dots, a_L$ of positive integers satisfying both of the following properties:

- every term in the sequence is less than or equal to 2^{2024}, and
- there does not exist a consecutive subsequence $a_i, a_{i+1}, \dots, a_j$ (where $1 \le i \le j \le L$) with a choice of signs $s_i, s_{i+1}, \dots, s_j \in \{1, -1\}$ for which

$$s_i a_i + s_{i+1} a_{i+1} + \cdots + s_j a_j = 0.$$

Solution

Answer: $2^{2025} - 1$

Let a substring refer to any subsequence of consecutive elements. The problem, whilst phrased as such, is not really a combinatorial game at all. We are only trying to find the maximal length of a sequence of numbers (bounded above by 2^{2024}) with no substring that can be split into two disjoint parts whose sum is 0.

The construction is motivated by noticing that if we take a valid sequence, and multiply it by any number, it remains valid. This is particularly pertinent since a valid sequence with elements at most 2^{2023} can now be extended to one for 2^{2024}: firstly double it, then insert a 1, and then put a doubled version of it again. We know that substrings not containing the 1 are by definition unable to give 0, and substrings containing the 1 cannot give 0 after the insertion of signs as their total sum is odd.

In particular, we can inductively create the following construction:

$$\begin{aligned} L_0 &= 1; \\ L_1 &= 2, 1, 2; \\ L_2 &= 4, 2, 4, 1, 4, 2, 4; \\ &\vdots \\ L_{2024} &= 2 \cdot L_{2023}, 1, 2 \cdot L_{2023}. \end{aligned}$$

Another motivation for this construction can simply be to try playing around with sequences only containing powers of 2, as it is clear that in this case, if each substring contains its smallest element only once, they cannot give 0.

This gives a lower bound of $2^{2025} - 1$. For the upper bound, consider a sequence of length greater than this. We will now imagine that we first puts $+$ and $-$ symbols in front of every number and then remove all but a substring. First, put a $+$ in front of the first element. Then, for each subsequent element, put any symbol ensuring that the partial sum thus far is between 0 and $2^{2025} - 1$. This is always possible as if the partial sum until this point is greater than or equal to the new element, we can take it away, and otherwise adding it will at always give less than the element doubled, so at most $2^{2025} - 1$. Now, by pigeonhole, two of the $2^{2025} + 1$ partial sums we had during this process (including the empty sum, which gives 0) are equal so we simply only keep the elements between them, which gives us a substring with symbols summing to 0 as desired.

Solution 2 (Marco V): The construction is the same. For the upper bound, we instead opt for an inductive approach. Our induction hypothesis is that $2^{n+1}-1$ works for sequences bounded by 2^n. If we can reduce a sequence bounded by 2^n of length 2^{n+1} to a sequence bounded by 2^n of even numbers of length 2^n, we are done. As before, we know that dividing by a constant does not affect whether the sequence is valid or not, so we can divide by 2 and obtain a sequence that we know doesn't work by our inductive hypothesis. Therefore, this reduction of a sequence to one half as long but containing only even numbers will constitute our inductive step.

Consider an arbitrary sequence of length 2^{n+1} and label the odd numbers in our sequence $x_1, \ldots, x_k$. For every even number, call it *Todd* if it has an odd number of odd numbers before it and *Steven* if it has an even number of odd numbers before it. Our idea will be to combine pairs of consecutive odd numbers, along with all the even numbers between them, into a single even number. This can obviously be done systematically by first taking the difference of the two odd numbers, and then the difference with every even number one by one, flipping signs whenever necessary. Since the difference of two numbers is smaller than the larger one, we clearly have that the resultant sum is smaller than 2^{2024}.

In order to ensure that we have at least 2^{2024} numbers at the end of this, we need to keep at least half the even numbers. If we have more Stevens, then we combine the first and second odd number, the third and fourth, and so forth all the away to the end. This removes all the Todds and odd numbers, and we get one even number for every two odd numbers. At the end of this, we have the Stevens, and the new evens we created, possibly one odd number left (if there were an odd number of odds in total) and the Todds after it. However, before this odd number we have at least

$$\left\lfloor \frac{k-1}{2} \right\rfloor + \left\lceil \frac{2^{2025}-k}{2} \right\rceil$$

even numbers, with the first term coming from the new even numbers we created and the second term being a lower bound on the the number of Stevens. Similarly, if we have more Todds then we combine the second and third odd numbers, and so forth. This almost gives the same bound - we may use neither the first or last odd number, so we have

$$\left\lfloor \frac{k-2}{2} \right\rfloor + \left\lceil \frac{2^{2025}-k}{2} \right\rceil$$

instead (which only happens when k is even).

Now, we have to make sure this is at least 2^{2024}. We simply have a few cases to check here. If k is odd then it is easy to see that we get our desired bound (we have the first expression, and we lose 0.5 from half the odds in the first term and gain 0.5 from half the evens in the second.) If k is even, however, we may get one less since we lose 1 in the second case (an even number of odds). However, this only happens when the number of Todds and Stevens is equal, meaning we can just opt to keep the Stevens instead, and we do not lose the 1 anymore.

Marking Scheme

The construction is worth 2 points. The upper bound is worth 4 points.

Construction:

- 1P: Noting that a valid list multiplied by a constant is still a valid list *and* trying to construct a list bounded by 2^n by using (at least two) lists that are valid for 2^{n-1}. The former does not have to be applied to the latter (for instance, they might take a list for 2^{n-1} and shift it by 2^{n-1} instead of multiplying by 2) but both are needed for the point.
- 2P: Providing any valid construction of length $2^{2025} - 1$.

Upper bound:

- 1P: Noting that the sum of every substring with signs is the difference of the sum of two substrings either containing the first element or empty.
- 3P: The correct argument but obtaining $2^{2025} + 1$ instead because the empty sum was neglected.
- 4P: Any valid proof of the upper bound.

Final Round 2024

Second Exam – Solutions

Zürich
March 3, 2024

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

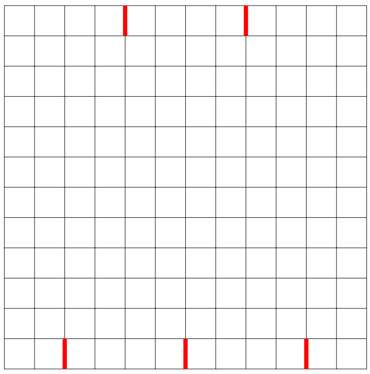
5. The icy ballroom of the White Witch is shaped like a square, and the floor is covered by $n \times n$ identical square tiles. Additionally, between some pairs of adjacent tiles there are magic edges. The White Witch's loyal servant Edmund is tasked with cleaning the ballroom. He starts in one of the corner tiles and may move up, down, left and right. However, as the floor is slippery, he will slide in that direction until he hits a wall or a magic edge. On the upside, he is wearing special shoes that clean each tile he passes over.

Determine the minimum number of magic edges that need to be placed for Edmund to be able to clean all the tiles.

Solution

Answer: $\lfloor \frac{n-1}{2} \rfloor$

Firstly, note the following construction for $\lfloor \frac{n-1}{2} \rfloor$:



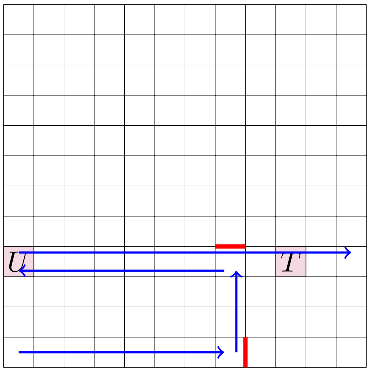
More generally, we place magic edges in between columns $2k$ and $2k+1$, alternating between the top and the bottom row. It is easy to see this works as one can easily reach some square in any column, either by approaching the edges from the left or from the right after a bit of zigzagging up and down to approach the edge, and then move vertically to cover the column. Indeed, to be able to clean the whole room one needs to be able to access every column and so this motivates the construction.

Now, we provide three proofs that this is the minimum number of magic edges required.

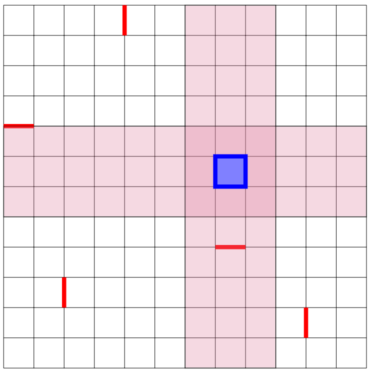
Solution 1 (Tanish). For all central squares, we need to pass over them either horizontally or vertically, so immediately prior to passing over them, Edmund stopped in either their row or column. In other words, we have to be able to stop in either all rows or all columns, as if there is at least one of both we cannot stop in, their intersection is inaccessible. Suppose WLOG we can stop somewhere in all columns. The only way to stop in a central column when entering from another column is to hit a magic edge. Each magic edge allows you to stop in at most two central columns, so we need at least $\lceil \frac{n-2}{2} \rceil = \lfloor \frac{n-1}{2} \rfloor$ magic edges. $\square$

The next two solutions require the following observation: for any tile T in the central $(n-2) \times (n-2)$ square, consider the last tile U Edmund stopped immediately prior to visiting it for the first time. These two share a row/column ℓ; we claim the first time Edmund stopped in this row or column must have been on a tile S been adjacent to a magic edge, which is vertical if ℓ is a column and vertical if ℓ is a row. If S is inside the central $(n-2) \times (n-2)$ this is clear; if S is on the border then, supposing it is not adjacent to an edge, Edmund can only have approached it from another square in ℓ, contradiction.

Note that in particular the statement "for any tile T in the central $(n-2) \times (n-2)$ square, the last tile U Edmund stopped at immediately prior to visiting it (for the first time) is adjacent to a magic edge" is false. Consider the following case:



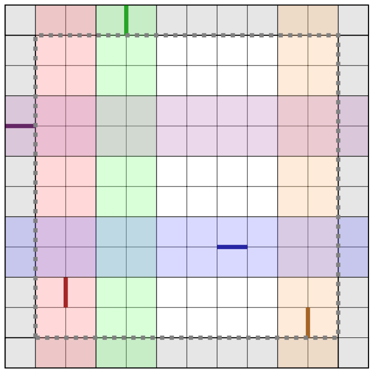
Solution 2 (David). Suppose that we place fewer than $\lfloor \frac{n-1}{2} \rfloor$ magic edges. Note that by the pigeonhole principle, there are both three neighbouring columns with no magic edges between them, and the same is true for the rows. Consider the intersection of the central column and row - this tile is inside the central $(n-2) \times (n-2)$ square and cannot be accessed as it is impossible to stop in that central row or column. $\square$



The blue square is inaccessible.

Solution 3 (Marco). Consider the central $(n-2) \times (n-2)$ square. Again, to access any tile inside this square, one needs to be adjacent to a magic edge when entering the row/column from which we approach the tile, with the edge needing to be oriented parallel to the direction of approach. Each magic edge therefore grants access to at most $2(2n-2)$ of the central tiles. More rigorously: for each magic edge e, consider every tile in the central $(n-2) \times (n-2)$ square that one can pass over immediately after having been adjacent to e. There are $2(2n-2)$ of these, and every tile must be in this form by what we proved earlier.

However, this means that we need at least $\frac{(2n-2)^2}{2(2n-2)}$ magic edges, which gives a lower bound of $\lceil \frac{n-2}{2} \rceil = \lfloor \frac{n-1}{2} \rfloor$. $\square$



The white squares are inaccessible.

Marking Scheme

The construction is worth 3 points. For the lower bound, the markscheme is split into two parts: proving that there is an edge on one of the four lines bounding a central square (2 points), and using this information to bound the number of edges (2 points). Each section except construction is nonadditive within itself, but additive with the other sections.

Construction:

- 1P: Claiming the answer is $\lfloor \frac{n-1}{2} \rfloor$ or equivalent expression.
- 1P: Describing a valid construction (a drawing of a construction for a small case that clearly generalises to larger n suffices).
- 1P: Justifying why the construction works.

Edge-adjacency:

- 1P: Any of the following claims: Edmund must be able to stop in either all columns or all rows; if there are both a row or column with no adjacent magic edge then Edmund cannot reach their intersection; each magic edge only grants access to the two rows/columns to which it is adjacent. Alternatively, stating a statement relating to working backwards from Edmund's position like "(immediately) before passing over a central square, Edmund must be adjacent to a magic edge". If the statement has a minor flaw (e.g., the mistake of stating immediately, not specifying that the magic edge should also be one one of the four lines bounding the square) this will not be penalised.
- 2P: Stating either one of the claims above or a statement without flaws like "before passing over a central square, the first time Edmund entered the column/row from which he approaches the square he is adjacent to a magic edge that is oriented in the direction of approach" with proof.

Counting:

- 1P: Any computation that incorrectly uses an edge-adjacency fact (can have a minor flaw and with or without proof) to obtain an incorrect bound that has a minor flaw too (for example, forgetting Edmund can already access the first and last column).
- 2P: Any computation that correctly uses an edge-adjacency fact (can have a minor flaw and with or without proof) to derive the correct lower bound.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y)f(x-y) \geq f(x)^2 - f(y)^2$$

for every $x, y \in \mathbb{R}$. Assume that the inequality is strict for some $x_0, y_0 \in \mathbb{R}$.

Prove that $f(x) \geq 0$ for every $x \in \mathbb{R}$ or $f(x) \leq 0$ for every $x \in \mathbb{R}$.

Solution

Let f denote a function $\mathbb{R} \to \mathbb{R}$, fulfilling the inequality in the problem statement.
Swap the role of x and y in the given inequality to get $f(x+y)f(y-x) \geq f(y)^2 - f(x)^2$. Adding this to the initial inequality gives us

$$f(x+y)\left(f(x-y) + f(y-x)\right) \geq 0.$$

Note that if one of the two inequalities was strict, then so is this one. So in other words for $x = x_0$ and $y = y_0$ we have the strict inequality

$$f(x_0+y_0)\left(f(x_0-y_0) + f(y_0-x_0)\right) > 0,$$

which implies that neither of the two terms used are equal to 0.
Let $x = \frac{z+t_0}{2}$ and $y = \frac{z-t_0}{2}$ and set $t_0 = x_0 - y_0$. This gives us

$$f(z)\left(f(x_0-y_0) + f(y_0-x_0)\right) \geq 0.$$

To finish we simply have to note that $f(z)$ must have the same sign as $C := f(x_0-y_0)+f(y_0-x_0)$. And as C is nonzero, this finishes the proof.

Solution 2: Plugging in $x = y$, gives us

$$f(2x)f(0) \geq 0,$$

which under our assumption implies that $f(0) = 0$. Plugging in $x = -y$, gives us

$$0 \geq f(x)^2 - f(-x)^2.$$

If we now swap the role of x and $-x$, this implies that $f(x) = \pm f(-x)$. Write $A = f(x+y)$, $B = f(x-y)$ and $C = f(x)^2 - f(y)^2$. Trying out all possible combinations of $(\pm x, \pm y)$ and $(\pm y, \pm x)$ gives us

$$AB \geq C \quad \pm A \cdot \pm B \geq C \quad A \cdot \pm B \geq -C \quad \pm A \cdot B \geq -C,$$

where the signs swap of A and B are independent, but consistent with itself. (That is if e.g. $\pm B = -B$, then we have $\pm B = -B$ in all four inequalities). We now check all four possible signings of $(\pm A, \pm B)$.

(a) $(\pm A, \pm B) = (A, B)$: We have $AB \geq C$ and $AB \geq -C$, giving us $AB \geq 0$.

(b) $(\pm A, \pm B) = (-A, -B)$: We have $AB \geq C$ and $-AB \geq -C \Leftrightarrow AB \leq C$, implying that $AB = C$.

(c) $(\pm A, \pm B) = (-A, B)$: We have $AB \geq C$, $-AB \geq C$, $AB \geq -C \Leftrightarrow -AB \leq C$, and $-AB \geq -C \Leftrightarrow AB \leq C$ implying that $AB = C$ but also $-AB = C$. This can only be the case if $AB = C = 0$.

(d) $(\pm A, \pm B) = (A, -B)$: This case is similar as the one above and results in $AB = C = 0$.

Note that the first case is the only case where strict inequality could hold. In particular for $z_0 = x_0 + y_0, t = x_0 - y_0$ we have $f(z_0) = f(-z_0), f(t_0) = f(-t_0)$ and $f(z_0)f(t_0) > 0$.
So if we pick x, y such that $x + y = z_0$, we can never fall into cases (2) and (3). So by letting $x + y = z_0$ and $x - y = t$, we always have $f(z_0)f(t) \geq 0$. We now finish as in Solution 1.

Solution 3: We do the same 4 cases as in solution 2 we simply give a different argument to finish.

Assume that not all of the $f(x)$ have the same signs.
Let z, t be values, such that $f(z) > 0 > f(t)$. We see that $x = \frac{z+t}{2}$ and $y = \frac{z-t}{2}$ only fit into case (2). That is $f(z) = -f(-z)$ and $f(t) = -f(-t)$. So by our assumption that such z, t exist, the equality $f(x) = f(-x)$ can only occur if $f(x) = 0$. In particular we will always have $f(x) = -f(-x)$, so the case (2) is always true, implying that the given inequality is always an equality, a contradiction to the existence of (x_0, y_0).

Solution 4: Like before we show $f(x) = \pm f(-x)$. We then substitute $x = \frac{s+d}{2}, y = \frac{s-d}{2}$ and let $R = f(\frac{s+d}{2})^2 - f(\frac{s-d}{2})^2$. We get $f(s)f(d) \geq R$. Also by exchanging x with y we get $f(s)f(-d) = \pm f(s)f(d) \geq -R$. Now we assume the problem statement wasn't true. This would imply there exist $s, d \in \mathbb{R}$ s.t $f(s)f(d) < 0$. Now if we had $f(d) = f(-d)$ this would imply $0 > f(s)f(d) \geq R$ but also $0 < -R \leq f(s)f(d)$ which is a contradiction. Since we can find such an s for every d, we get $f(-d) = -f(d)$ for all d. Now we have $-f(s)f(d) \geq -R$, therefore $f(s)f(d) \leq R \geq f(s)f(d)$, which implies $f(s)f(d) = R$ for all s, d. But this is a contradiction to the problem statement, so there are no s, d with $f(s)f(d) < 0$. therefore we have $f(x) \geq 0$ or $f(x) \leq 0$

Marking Scheme

Solution 1

- 3P: Have the inequality $f(x+y)\,(f(x-y)+f(y-x)) \geq 0$.
- 1P: Note that for (x_0, y_0) the above inequality is strict
- 2P: Show that there exists a constant $C \neq 0$, such that $f(z) \cdot C \geq 0$. Here z has to be a free variable.
- 1P: Conclude

Solution 2

- 1P: Show that $f(x) = \pm f(x)$
- 1P: Write out all the versions of the inequality with $(x, y) = (\pm x, \pm y)$ and $(x, y) = (\pm y, \pm x)$. You need all four distinct resulting inequalities to receive this point.
- 1P: Write down the four cases and their respective implications.
- 1P: Show that $f(z_0)f(t_0) > 0$, where z_0, t_0 are defined as in the solution.
- 2P: Show that there exists a constant $C \neq 0$, such that $f(z) \cdot C \geq 0$. Here z has to be a free variable.
- 1P: conclude

Solution 3

- 1P: Show that $f(x) = \pm f(-x)$
- 1P: Write out all the versions of the inequality with $(x, y) = (\pm x, \pm y)$ and $(x, y) = (\pm y, \pm x)$. You need all four distinct resulting inequalities to receive this point.
- 1P: Write down the four cases and their respective implications.
- 1P: Note that (z, t) only fit into case 2, where z, t are such that $f(z) > 0 > f(t)$.
- 2P: Show that we always have $f(x) = -f(-x)$.
- 1P: Conclude, i.e. find a contradiction using (x_0, y_0).

Remarks:

- In partial progress that is not part of Solutions 1, 2 or 3, we give 1 point for any useful summing of inequalities, and 1 point for any useful fixing of $x+y$ and varying of $x-y$ (or vice versa).

Solution 3

- 1P: Show that $f(x) = \pm f(-x)$
- 1P: Getting $f(s)f(d) \geq R$.
- 1P: Getting $f(s)f(-d) = \pm f(s)f(d) \geq R$.
- 1P: Showing $f(-d) - f(d)$.
- 2P: Showing $f(s)f(d) = R$.

- 1P: Concluding.

Solution 4

-

7. Determine all positive integers n satisfying all of the following properties:

- there exist exactly three distinct prime numbers dividing n,
- n is equal to $\binom{m}{3}$ for some positive integer m, and
- $n+1$ is a perfect square.

Solution

The third property allows us to write $n = k^2 - 1$ for some $k \in \mathbb{N}$, and together with the first property we obtain the equation

$$n = \frac{m(m-1)(m-2)}{6} = (k-1)(k+1). \tag{1}$$

Furthermore, let p, q, r be the three distinct prime numbers dividing n, and write $n = p^\alpha q^\beta r^\gamma$.

Certainly such specific conditions must allow a solution! When trying small values of k and m, we easily see that $(k, m) = (11, 10)$ is the smallest solution (that is, $n = 120 = 2^3 \cdot 3 \cdot 5$).

Observing the equation (1), we note that both sides are the product of 'almost coprime' numbers. In particular, $\gcd(k-1, k+1) \mid 2$ and $\gcd(m, m-2) \mid 2$, while $m-1$ is coprime to m and $m-2$.

If $p, q, r > 2$, not both $k-1$ and $k+1$ can have two or more distinct prime factors, so one of them has to be (wlog) p^α (since $k > 2$). However, p^α must also divide exactly one of $m, m-1$ and $m-2$ by the same gcd argument. We conclude that $k-1 \leq m$ must hold. (Otherwise, one of these divisibilities would fail due to the size of the numbers). We get

$$\begin{aligned}
(k-1)(k+1) = \frac{m(m-1)(m-2)}{6} &\geq \frac{(k-1)(k-2)(k-3)}{6} \\
\Rightarrow \quad 6(k+1) \geq (k-2)(k-3) \quad &\Leftrightarrow \quad 0 \geq k^2 - 11k \quad \Leftrightarrow \quad 0 \geq k(k-11).
\end{aligned}$$

Obviously this implies $k \leq 11$, which we already checked. Now let's look at the case where one of p, q, r (wlog r) is equal to 2.

if $r = 2$, both $k-1$ and $k+1$ must be even and we can now argue that $\gcd(\frac{k-1}{2}, \frac{k+1}{2}) = 1$, which again means that one of them is a prime power, which itself divides one of m, $m-1$ or $m-2$. This is also true for a power of 2 since $v_2(\frac{k-1}{2} \cdot \frac{k+1}{2}) = v_2(m(m-1)(m-2)) - 3$ and only one of $v_p(m), v_p(m-1)$ and $v_p(m-2)$ can be greater than 1.

Note that if one of the divisibilities mentioned above were not an actual equality, one element in $\{\frac{k-1}{2}, \frac{k+1}{2}\}$ would be smaller than an element in $\{m-2, m-1, m\}$ by a factor of at least 2, which would again lead to the inequality $k-1 \leq m$, which we already considered. Hence the only other cases left to consider are:

Case 1: $\dfrac{k-1}{2} = m \;\Rightarrow\; m+1 \mid (m-1)(m-2) = (m+1)(m-4) + 6 \;\Rightarrow\; m+1 \mid 6.$

Case 2: $\dfrac{k-1}{2} = m-1 \;\Leftrightarrow\; \dfrac{k+1}{2} = m \;\Rightarrow\; 1 = \dfrac{m-2}{24} \;\Rightarrow\; m = 26, k = 51.$

Case 3: $\dfrac{k-1}{2} = m-2 \;\Leftrightarrow\; \dfrac{k+1}{2} = m-1 \;\Rightarrow\; 1 = \dfrac{m}{24} \;\Rightarrow\; m = 24, k = 45.$

Case 4: $\dfrac{k+1}{2} = m-2 \;\Rightarrow\; m-3 \mid m(m-1) = (m-3)(m+2) + 6 \;\Rightarrow\; m-3 \mid 6.$

In cases 2 and 3 we used the equation $\frac{k-1}{2} \cdot \frac{k+1}{2} = \frac{m(m-1)(m-2)}{24}$ obtained from (1).

Note that Case 1 and Case 4 only lead to values for m which are too small. In contrast, the values from Cases 2 and 3 work and yield the solutions $n = 2024 = 2^3 \cdot 11 \cdot 23$ and $n = 2600 = 2^3 \cdot 5^2 \cdot 13$.

Remark. *It's possible to solve the problem without any case distinction on the parity of n. However, the inequality obtained in that solution is weaker and would require more values to be checked by hand (e.g. up to $m = 27$).*

Marking Scheme

- 1P: Getting $n = \frac{m(m-1)(m-2)}{6}$ and $n = (k-1)(k+1)$.
- 1P Observing that $\gcd(m, m+2) \leq 2$ **and** $\gcd(k-1, k+1) \leq 2$.
- 1P: Noting that one of the factors on the RHS must be a prime power (up to a factor 2)
- 2P: Concluding an inequality of the form $f(k) \leq g(m)$ for some linear functions f and g. (at most 1P can be awarded if such an inequality has not been shown to apply to all cases)
- 1P: Reducing the problem to checking a finite number of values for k or m.
- 1P: Concluding by checking the finitely many remaining values.

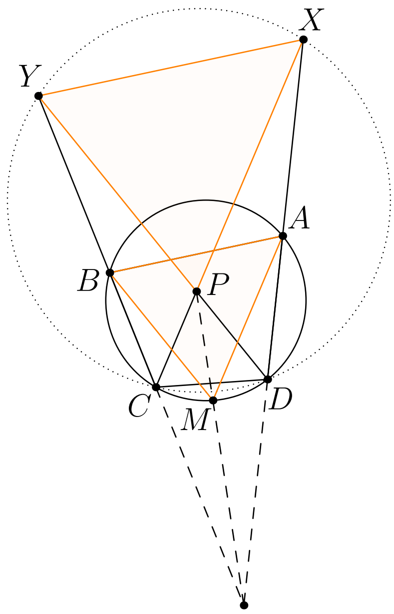
Remark. *In most solutions, at least one of the three properties of n needs proper verification (e.g. computing the prime decomposition of $120, 2024$ and 2600 to show that they have indeed three distinct prime factors), while other properties are implied by the equations used in the solution and don't need to be verified.*

A solution that misses such a verification is awarded at most 6 points.

Remark 2. *The points from the second and third point can be obtained when working only in one of the cases n even / odd. Only one point be obtained from the fourth bullet and no points for the last two bullets.*

8. Let $ABCD$ be a cyclic quadrilateral with $\angle BAD < \angle ADC$. Let M be the midpoint of the arc CD not containing A. Suppose there is a point P inside $ABCD$ such that $\angle ADB = \angle CPD$ and $\angle ADP = \angle PCB$. Prove that the lines AD, PM, BC are concurrent.

Solution



Let $X = PC \cap DA$ and $Y = PD \cap BC$. Also let $\varepsilon = \angle ADB = \angle CPD$, $\varphi = \angle ADP = \angle PCB$ and $\omega = \varepsilon - \varphi$.

First note that $\angle XCY = \angle PCB = \angle ADP = \angle XDY$, and $XYCD$ is cyclic. Then $\angle CBA = 180^\circ - \angle ADC = \angle XYC$, so that $XY \parallel AB$.

Angle chasing also yields

$$\angle CYP = 180^\circ - \angle YPC - \angle PCY = \varepsilon - \varphi = \omega$$

and

$$\angle CBD = \angle BYD + \angle YDB = \omega + (\varepsilon - \varphi) = 2\omega,$$

and by the incenter/excenter lemma, $\angle CBM = \frac{1}{2}\angle CBD = \omega$. This proves that $YP \parallel BM$ as $\angle CYP = \omega = \angle CBM$, and one can similarly prove that $PX \parallel MA$.

Then triangles XYP and ABM are homothetic, so lines AD, PM, BC are concurrent or parallel. However, the condition $\angle BAD < \angle ADC$ prevents the parallel case, and they are concurrent.

Marking Scheme

The marking scheme is split into three parts: first the proof that $PC \parallel AM$ and $PD \parallel BM$, then the introduction of new points to form a third parallel line, and the conclusion from homothetic triangles.

Two parallels (3 points, non-additive):

- 0P: Claim that $PC \parallel AM$ or $PD \parallel BM$.
- 1P: Claim that $PC \parallel AM$ and $PD \parallel BM$.
- 1P: Prove that $\angle BYD = \angle YDB$, $\angle CXA = \angle ACX$ or $\angle(PC, BM) = \angle(AM, PD)$.
- 2P: Prove that $\angle(PC, BM) = \angle(AM, PD)$ and $\angle CPD = \angle AMB$.
- 2P: Prove that $PC \parallel AM$ or $PD \parallel BM$.
- 3P: Prove that $PC \parallel AM$ and $PD \parallel BM$.

Third parallel (2 points, non-additive)

- 1P: Define $X = PC \cap DA$, $Y = PD \cap BC$.
- 1P: Define $U = AM \cap BC$, $V = BM \cap DA$.
- 2P: Define $X = PC \cap DA$, $Y = PD \cap BC$, and show that $XY \parallel AB$.
- 2P: Define $U = AM \cap BC$, $V = BM \cap DA$, and show that $UV \parallel CD$.

Conclusion (2 points, additive)

- 1P: Show that AD, PM, BC are concurrent or parallel.
- 1P: Conclude.

Remarks:

- The point for the conclusion is only awarded to someone that has already proven that AD, PM, BC are concurrent or parallel, and shows that they are not parallel.