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IMO Selection 2021 - Solutions

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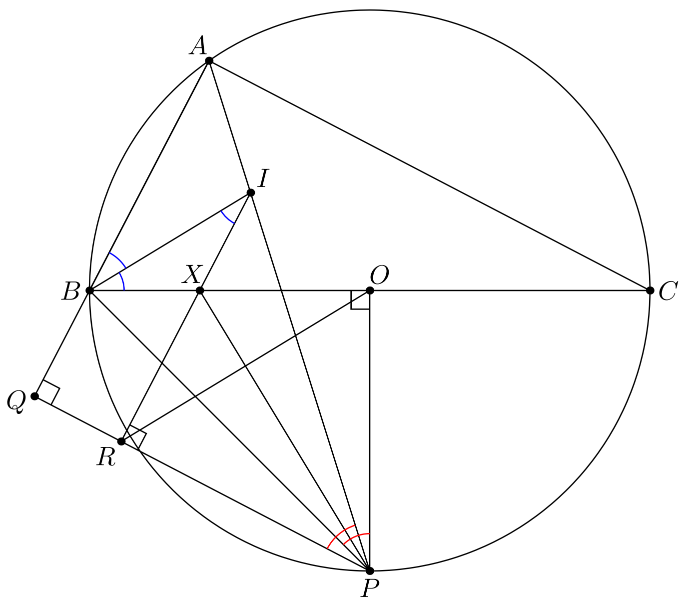
Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let ABC be a triangle where $\angle BAC = 90°$, with circumcenter O and incenter I. The angle bisector of $\angle BAC$ intersects the circumcircle of ABC in A and P. Let Q be the projection of P onto AB, and R be the projection of I onto PQ. Prove that RO bisects CI.

Solution (Raphael): Because O is the midpoint of $\overline{BC}$, it suffices to show that RO is parallel to BI, as the result follows by simply looking at a homothety centered at C.
So first of all by looking at the problem we instantly recognise the point P as the circumcenter of BIC, as this is a well know configuration. So from this we get $|PB| = |PI|$ and $\angle BOP = 90°$. Next up we find AQ to be parallel to IR, as they both are perpendicular to PQ. From this we get $\angle BIR = \angle IBA = \angle IBC$. We now have two same angles at B and I and also $|PB| = |PI|$. This gives us motivation to look at perpendicular bisector of $\overline{BI}$, as it goes through P and $X := RI \cap BC$. If we can show that ROX is also isosceles, then it'll instantly follow that $\angle BOR = \angle OBI$, which then will give us RO parallel to BI as wanted. So now for the fun stuff; more angle chasing.
We easily calculate $\angle APQ = \angle OPB = 45°$. From this we get that $\angle BPQ = \angle OPI$ and thereby that the angle bisector of $\angle IPB$ and $\angle OPR$ coincide. And as the angle bisector of $\angle IPB$ is equal to the perpendicular bisector of BI, we have $\angle XPR = \angle OPR$. Finally using the cyclic quadrilateral $OPRX$ we have $\angle ORX = \angle OPX = \angle XPR = \angle XOR$, which finishes the proof as explained before.



Marking Scheme (additive):

- 1P: note that BPI is isosceles or that $\angle BOP = 90°$ or equivalent (WUM)
- 2P: proof that it suffices to show BI parallel to RO
- 2P: show that the angle bisector of $\angle RPO$, the angle bisector of $\angle BPI$ and the perpendicular bisector of $\overline{BI}$ coincide.
- 1P: $\angle BOR = \angle OBI$
- 1P: finish the proof

alternative Solution e.g. Emily/Ricardo
Let S be the intersection of IC und OR

- 1P: note that BPI is isosceles or that $\angle BOP = 90°$ or equivalent (WUM)
- 2P: note IPC isosceles and say that it suffices to show $\angle ISP = 90°$
- 3P: proof that $PRIS$ is a cyclic quadrilateral
- 1P: finish the proof

2. For each prime p, somewhere in the multiverse there exists a kingdom consisting of p islands numbered from 1 to p with a bridge between any pair of them. When Jana visits a kingdom, coronavirus restrictions mean she must obey the following rule: Directly after visiting island m, she can only cross over to island n if

$$p \,|\, (m^2 - n + 1)(n^2 - m + 1).$$

Show that there are infinitely many kingdoms such that Jana cannot travel to every island in this manner.

Solution (David): Note that the divisibility condition is symmetrical in m and n. In other words: If Jana can cross over from island m to island n, she can also go the other way. Let us define a graph G as follows: Each of the islands $1, 2, \ldots, p$ represents a vertex and we draw an edge between two vertices m and n if and only if

$$p \,|\, (m^2 - n + 1)(n^2 - m + 1).$$

The edges of our graph now exactly represent between which pairs of islands Jana can travel, so it's easy to see that Jana can travel to every island if and only if G is connected.

This insight motivates us to count the number of edges in G, because if we can show that there are less than $p - 1$ edges, G has to be disconnected! By the formula

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v)$$

it is enough to know the number of vertices of a every possible degree. The degree of a fixed vertex m is the number of distinct $n \in \{1, 2, \ldots, p\}$ such that

$$(m^2 - n + 1)(n^2 - m + 1) \equiv 0 \pmod{p}$$

Obviously, there is exactly one choice for n such that $m^2 - n + 1 \equiv 0 \pmod{p}$, namely $n \equiv m^2 + 1$. But what about $n^2 - m + 1 \equiv 0 \pmod{p}$?

Lemma: Let p be an odd prime and a some residue modulo p. Then, $x^2 - a \equiv 0 \pmod{p}$ has

- no solutions in x for exactly $\frac{p-1}{2}$ different values of a.
- exactly one solution in x if and only if $a = 0$.
- exactly two solutions in x for $\frac{p-1}{2}$ different values of a.

Proof of the Lemma: Note that $x^2 \equiv y^2$ is equivalent to $(x - y)(x + y) \equiv 0$ or in other words $y \equiv \pm x \pmod{p}$.

Hence, the squares of $1, \ldots, \frac{p-1}{2}$ are all distinct modulo p and they are equal to the squares of $p - 1, \ldots, \frac{p+1}{2}$, respectively. This gives us $\frac{p-1}{2}$ values of a such that there are two solutions in x. The only residue left is 0, which gives us a unique choice of a with exactly one solution in x. The remaining $\frac{p-1}{2}$ values of a therefore admit no solutions at all.

By letting $a = m - 1$, the Lemma tells us that there are $\frac{p-1}{2}$ different m with two solutions in n. If we assume that the conditions $n^2 - m + 1 \equiv 0 \pmod{p}$ and $m^2 - n + 1 \equiv 0 \pmod{p}$ are never satisfied at the same time (or for $m = n$), we would get

$$|E| = \frac{1}{2} \sum_{v \in V} deg(v) = \frac{1}{2} \cdot \left(\frac{p-1}{2} \cdot 1 + 1 \cdot 2 + \frac{p-1}{2} \cdot 3 \right) = p.$$

Sadly, this is not small enough! We need to reduce the number of edges a bit more. The easiest way of doing this is by assuming that there esists a residue m such that $m^2 - m + 1 \equiv 0 \pmod p$. This corresponds to a *loop* in G (an edge going from m to itself), which is irrelevant for the connectivity of G. A short calculation inspired by Vieta shows that $m^2 - m + 1 \equiv 0 \pmod p$ implies $(1-m)^2 - (1-m) + 1 \equiv 0 \pmod p$, so as long as $m \not\equiv 1 - m \pmod p$, we have two loops, from which we conclude that G is disconnected.

Let us now prove that there are infinitely many primes p such that there exists a residue m satisfying $m^2 - m + 1 \equiv 0 \pmod p$: Let m be the product of the first k primes and let p be a prime factor of $m^2 - m + 1$. In particular, p is not equal to the first k primes and so there are infinitely many such p. If $m \equiv 1 - m \pmod p$ then $m \equiv \frac{p+1}{2} \pmod p$. However, in this case we must have

$$\left(\frac{p+1}{2}\right)^2 - \frac{p+1}{2} + 1 = \frac{p^2+3}{4} \equiv 0 \pmod p,$$

which is equivalent to $p = 3$. This shows that for all such $p > 3$ that we constructed, G is indeed disconnected.

Marking Scheme (additive):

(a) 1P: Introducing G and reformulating the problem into showing that G is not connected

(b) 1P: Realizing that showing that G has fewer than $p-1$ edges for infinitely many p is enough.

(c) 1P: Showing that the degree of every vertex is at most 3 (or any other idea allowing to bound the number of edges)

(d) 1P: Proving that G contains at most p edges

(e) 1P: Noting that if $m^2 - m + 1 \equiv 0 \pmod p$, the number of edges can be reduced by 1

(f) 1P: Showing that the number of edges can even be reduced by 2

(g) 1P: Proving that infinitely many primes allow an appropriate m

Remark: It might look promising to try and reduce the number of edges by finding m and n such that p divides both $(m^2 - n + 1)$ and $(n^2 - m + 1)$, or in other words, choosing p such that $(m^2+1)^2 + 1 \equiv m \pmod p$. However, in general this only reduces the number of edges by one and the case $p = 11, m = 4$ illustrates that this idea alone is not sufficient.

Alternative arguments (inspired by Joel/Mathys): Instead of introducing a simple undirected graph, we introduce a not necessarily simple directed graph where the edge points from m to n if $p \mid m^2 - n + 1$ and from n to m if $p \mid n^2 - m + 1$. The directions are actually completely irrelevant for Jana's travel plans but they allow us to count the edges more elegantly:

For every m there is *exactly* one value of n such that $m^2 - n + 1 \equiv 0 \pmod p$. This means that each of the p vertices has one edge pointing away from it, and therefore we have p edges in total.

To show that the graph is not connected, we will explicitly find two vertices of G which will have loops: Let $p \equiv 1 \pmod 3$, in particular, p is odd and so $p - 1$ is divisible by 6. Take a primitive root a modulo p. For $m = a^{\frac{p-1}{6}}$ we have

$$m^6 - 1 \equiv 0 \pmod p \;\Leftrightarrow\; (m^2 - m + 1)(m+1)(m^3 - 1) \equiv 0 \pmod p \;\Rightarrow\; m^2 - m + 1 \equiv 0 \pmod p$$

because, since a is a primitive root, $m + 1$ and $m^3 - 1$ cannot be divisible by p. But the same calculation can be done with $m = a^{\frac{5(p-1)}{6}}$, so we found two loops for infinitely many p (Dirichlet) and we're done!

3. Let p be an odd prime. Arnaud has hung up $N \geq 1$ towels to dry on a washing line, each coloured either purple or yellow. Then, for each $1 \leq n \leq N$, he calculates what fraction of the first n towels are yellow, and writes down these N fractions in their irreducible form on a piece of paper. Julia finds the piece of paper the next day and notices that all of the fractions $\frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}$ are on the paper. Prove that

$$N \geq \frac{p^3 - p}{4}.$$

Solution (Tanish): Firstly, note that the only way to have a fraction with p in the denominator is if the number of towels being counted is a multiple of p. Let $a_1, a_2, \ldots, a_{p-1}$ denote the numerators of the fractions in the order they first appear in Arnaud's list, and $b_1, b_2, \ldots, b_{p-1}$ the number of towels they correspond to.

Now observe that if we flip the colour of every single towel from purple to yellow and vice versa, then all the fractions will change from $\frac{a_i}{p}$ to $\frac{p-a_i}{p}$. Since all the fractions appear, this means that for every configuration of a given length, there is a configuration of the same length where the order of 1 and $p-1$ are swapped in the list of the a_i, and so we can assume 1 appears before $p-1$.

Suppose $p-1 = a_j$, and let $\lambda := \min(a_j, a_{j+1}, \ldots, a_{p-1})$. We now make two case distinctions:

- $\lambda \geq \frac{p-1}{2}$.
  In this case, let a_k be the last of the numerators $1, 2, \ldots, \lambda - 1$ to appear in the list; it is clear that $k \geq \lambda - 1$. Furthermore, note that this means at least $p(\lambda - 1)$ towels are being taken into consideration (each new fraction needs at least p towels between it and the previous fraction) and so $b_k \geq p \cdot (\lambda - 1)$. The number of purple towels up to this point is therefore at least $\frac{p-a_k}{p} \cdot p \cdot (\lambda - 1) = (p - a_k) \cdot (\lambda - 1) \geq \frac{p+1}{2} \cdot \frac{p-1}{2} = \frac{p^2-1}{4}$. It follows that since $a_j = p - 1$ appears after this point, the purple towels by that point must only be $\frac{1}{p}$ of b_j but there will be at least $\frac{p^2-1}{4}$ of them, so $\frac{p^3-p}{4} \leq b_j \leq N$ as desired.

- $\lambda < \frac{p-1}{2}$.
  Define a_k as in the previous case, and note that we still have at least $(p - a_k) \cdot (\lambda - 1)$ purple towels in the first b_k towels. This implies that b_j is at least $p \cdot (p - a_k) \cdot (\lambda - 1)$; so we have at least $(p-1) \cdot (p - a_k) \cdot (\lambda - 1)$ yellow towels in the first b_k towels. Now let $\lambda = a_l$; we see that since a_l comes after b_j, the yellow towels must only be $\frac{\lambda}{p}$ of b_l but there will be at least $(p-1) \cdot (p - a_k) \cdot (\lambda - 1)$ of them, so $N \geq b_l \geq \frac{p \cdot (p-1) \cdot (p-a_k) \cdot (\lambda - 1)}{\lambda} \geq p \cdot (p-1) \cdot \frac{p+1}{2} \cdot \frac{1}{2} = \frac{p^3-p}{4}$ as desired.

Remark: The intuition behind this line of thinking is as follows: first, you test the case where $a_i = i$ and you notice the symmetry by testing the case $a_i = p - i$, allowing you to make the WLOG statement. Whilst observing these two cases, you notice the "issue" appears to be around the middle of the a_i as it is by this point that too many purple towels have accumulated: even if you only have yellow towels from that point on, you will need too many before their total proportion is $\frac{p-1}{p}$. This encourages you to immediately conclude in the case where there are too many "small fractions" before $\frac{p-1}{p}$ and so now the question is what happens in the other case, where there are many large fractions at the start? In this case, the issue will become the large number of purple towels you have to add after $\frac{p-1}{p}$ in order to reduce the proportion to that of the smallest fraction that appears afterwards, and so you work with this instead. This problem is a nice exercise in formalising observations like this mathematically.

Marking Scheme (additive):
Here, the use of the word "fraction" implies a fraction with denominator p.
If students reformulate the problem (for example, algebraically) they will not receive points for having changed the problem statement, but any progress made in alternative formulations will receive the equivalent points from the combinatorial version.

- 1P: Any evidence that the student has calculated $N \geq \frac{p^3-p}{4}$ in the case where the fractions appear in either ascending or descending order *or* introducing the two inequalities $a_i b_i < a_{i+1}b_{i+1}$, $a_{i+1}b_{i+1} - a_i b_i \leq (b_{i+1} - b_i)p$
- 1P: Placing one of the two extremal fractions somewhere in the list *and* considering either the maximum or minimum of the sets of fractions either before or after it.
- 1P: Noting that (for any particular k) at least one of the fractions $\frac{1}{p}, \ldots, \frac{k}{p}$ will appear after at least kp towels have been counted in total, or a symmetric observation based on whether the student is counting from first fraction to last or last to first, and if their extremal value is 1 or $p-1$
- 1P: Considering the first $\frac{p+1}{2}$ fractions.
- 1P: Solving the case where all the fractions $\frac{1}{p}, \ldots, \frac{p-1}{2p}$ appear before $\frac{p-1}{p}$.
- 2P: Solving the case where some fraction $\in \{\frac{1}{p}, \ldots, \frac{p-1}{2p}\}$ appears after $\frac{p-1}{p}$.

Students do not need to justify why N is an integer value.

The following observations are not worth any points:

- Noting that the total number of towels when Arnaud wrote down one of the fractions must be a multiple of p.
- Noticing the symmetry when swapping purple and yellow towels.
- Using said symmetry to say "WLOG $\frac{1}{p}$ appears before $\frac{p-1}{p}$..." or vice versa.

4. Let n be a positive integer. The islands of the MO-Archipelago are arranged in a regular equilateral triangular unitary grid to form a big equilateral triangle of side length n. The beloved Governor Henning is in charge of building bridges between every pair of islands that are a distance of 1 apart. For every island i, Henning chooses two real numbers x_i and y_i that satisfy $x_i^2 + y_i^2 = 1$. The cost of a bridge between islands i and j is then given by $1 + x_i x_j + y_i y_j$. Determine the minimal amount of money needed to build all the bridges.

Answer: The minimal cost is $\frac{3n(n+1)}{4}$.

Solution (Henning): Let us start with the observation that one can choose $\frac{n(n+1)}{2}$ unit triangles such that each bridge is contained exactly once in one of the triangles. For each triangle one then has the following inequality:

$$(x_1 + x_2 + x_3)^2 + (y_1 + y_2 + y_3)^2 \geq 0$$
$$6 + 2(x_1x_2 + x_2x_3 + x_3x_1) + 2(y_1y_2 + y_2y_3 + y_3y_1) \geq -(x_1^2 + x_2^2 + x_3^2) - (y_1^2 + y_2^2 + y_3^2) + 6 = 3$$

where x_i, y_i belong to the three islands in the unit triangle. Summing now over all unit triangles we arrive at the lower bound of

$$\frac{3n(n+1)}{4}$$

as the cost for all the bridges.

We now show that this lower bound can be reached. One can find that the equality case for one of the unit triangles is fulfilled by choosing

$$((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \left((1, 0), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) \right)$$

It is now important to remark that the equality case can be achieved setting each of the (x_i, y_i) in the big triangle to one of three above vectors. Calling the first vector the type 1 vector, the second and third vector type 2 and 3 respectively, the problem turns into a colouring exercise.

If we start to fill the lowest row of the big triangle according to the $1, 2, 3, 1, 2...$ pattern. The row above with $3, 1, 2, 3, 1..$ and so on, we see that the islands are now coloured in a way that each of the chosen unit triangles has exactly one island of each type, hence the equality case is fulfilled for all of the unit triangles simultaneously.

Marking Scheme

(a) 2P: Finding the minimum for one unit triangle.

(b) 1P: Showing that the equality case can be achieved for a unit triangle.

(c) 1P: Marking the upside unit triangles.

(d) 1P: Realising that any construction or lower bound on an upside unit triangle can easily generalised to the whole problem.

(e) 2P: Showing that the equality case cam be achieved for all unit triangles simultaneously

5. Let n be a positive integer. Some of the squares of a $3n \times 3n$ board are marked. For any marked square T, we denote by $\ell(T)$ the number of marked squares in the same row to the left of T and by $d(T)$ the number of marked squares in the same column below T. Determine the maximal number of marked squares given that $\ell(T) + d(T)$ is even for every marked square T.

Answer: The maximal number of marked squares is $6n^2$.

Solution (Valentin): Suppose that $l(T) + d(T)$ is even for all marked squares T. We first proof an essential lemma.

Lemma.

There are at most k rows with an odd number of marked squares within the left-most k columns.

Proof. We proceed by induction. For $k = 1$, assume that there is more than 1 marked squares in the left-most column and denote the second lowest one by T. We have $l(T) + d(T) = 1$ is odd, a contradiction. It follows that the lemma holds for $k = 1$.

For $k > 1$ assume the lemma holds for $k - 1$, which implies that there are at most $k - 1$ rows that have an odd number of marked squares in the first $k - 1$ columns. Denote by a the number of these rows that also have a marked square in column k. Since the parity of $l(T)$ alternates among the marked squares in each column, it follows that there are at most $a + 1$ marked squares in the k-th column with $l(T)$ even. The number of rows that have an odd number of marked squares in the first k columns is therefore bounded above by $(k - 1 - a) + (a + 1) = k$. This concludes the inductive step. □

From the lemma, and the fact that the parity of $l(T)$ alternates among the marked squares in a given column, it follows immediately that there are at most $2k - 1$ marked squares in the k-th column from the left. Similarly there are at most $2k - 1$ marked squares in the k-th row from the bottom by symmetry of the argument.

Applying these bounds to the left-most n columns and the bottom-most n rows, we find that the total number of marked squares is bounded from above by

$$2 \cdot \Big(1 + 3 + 5 + \cdots + (2n - 1)\Big) + (2n)^2 = 6n^2.$$

This is indeed the maximal number of marked squared and can be obtained via the following construction illustrated here for $n = 3$:

		★	★	★	★	★	★	★
	★	★	★	★	★	★	★	★
★	★	★	★	★	★	★	★	★
	★	★	★	★	★	★	★	★
		★	★	★	★	★	★	★
			★	★	★	★	★	★
				★	★	★	★	★
					★	★	★	
						★		

Marking Scheme (additive):

(a) 1P: Claiming that in the k-th column/row there are at most $2k - 1$ marked squares

(b) 3P: Proving that in the k-th column/row there are at most $2k - 1$ marked squares

  (b1) 1P: Using or stating that the parities of $l(T)$ and $d(T)$ alternate among the marked squares in a given line

(c) 2P: Giving a general construction with $6n^2$ marked squares

(d) 1P: Finishing assuming (b)

6. We call a positive integer *silly* if the sum of its positive divisors is a square. Prove that there are infinitely many silly numbers.

Solution (Louis): Let $\sigma(n)$ denote the sum of all positive divisors of the integer n, and we will order the prime numbers $2 = p_1 < p_2 < p_3 < \ldots$. An important observation is that $\sigma(n)$ is multiplicative, in the sense that if n_1, n_2 are coprime, then $\sigma(n_1 n_2) = \sigma(n_1)\sigma(n_2)$. More generally, we have the identity

$$\sigma(p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}) = \prod_{i=1}^{k} (1 + p_i + \ldots + p_i^{\alpha_i}).$$

We could first try to solve the exercise in the following way: For any integer $k > 2$, consider $S = \{p_1, \ldots, p_k\}$ the set of the k smallest prime numbers. We notice that all prime factors of the numbers $\sigma(p_i)$ are smaller that p_k (since either $\sigma(p_i) < p_k$ or $\sigma(p_k) = p_k + 1$ is not prime). Therefore there are 2^k subsets A of S, but if we consider the product of all elements in A, the sum of its divisors will only be divisible by primes among $p_1, \ldots, p_{k-1}$ so if we consider the parity of the exponent of each of these primes there are only 2^{k-1} results possible, so by the pigeonhole principle there must exist two distinct subsets A and A' such that all the exponents have the same parity. Therefore the product of all the elements in the symmetric difference $A \Delta A'$ will be a silly number (the symmetric difference of two sets is defined as the set of all elements that belong to exactly one of the sets).

This contains most of the ideas for the solution, but the problem is that if we found sets A, A' for k, we will also find these same sets for $k + 1$ so this argument is not enough to guarantee the existence of infinitely many silly numbers and we need a more subtle argument.

Assume that we already have l silly numbers $n_1, \ldots, n_l$. We will prove that there exist a silly number distinct of all these numbers. Let p_α be the largest prime factor of one of these l silly numbers and let γ be the largest exponent in the prime factor decompositions of these numbers. Finally let p_β be a prime number larger than $\sigma(p_\alpha^{\gamma+1})$ and consider the set

$$S = \{p_1^{\gamma+1}, \ldots, p_\alpha^{\gamma+1}, p_{\alpha+1}, \ldots, p_\beta\}.$$

Now by construction we can prove again that for each $a \in S$ the prime factors of $\sigma(a)$ are smaller that p_β so we can make the same argument as before, but now the element we will construct will be distinct from all the $n_1, \ldots, n_l$, since either it will be divisible by a prime number which does not divide any of these numbers, or one of the prime numbers will appear with a higher multiplicity than in any of the l silly numbers.

Marking Scheme (additive):

(a) 1P: Notice that $\sigma(p_1), \ldots, \sigma(p_k)$ are only divisible by $p_1, \ldots, p_{k-1}$.

(b) 2P: Use pigeonhole to construct a silly number as the product of the elements of a certain subset.

(c) 4P: Finish the solution.

7. Let n be a positive integer. Call a sequence of positive integers $a_1, a_2, \ldots, a_n$ *tame* if it satisfies

$$1 \cdot a_1 \leq 2 \cdot a_2 \leq \ldots \leq n \cdot a_n.$$

Determine the number of tame permutations of $1, 2, \ldots, n$.

Answer: The answer is the nth Fibonacci number.

Solution 1, induction (Tanish): We prove that the number of possibilities is the n^{th} Fibonacci number F_n by induction ($F_0 = 1, F_1 = 1$). For the base case, observe that there is 1 way to do it for $n = 1$ and 2 ways for $n = 2$ (all the arrangements work in both cases).
Now suppose this holds true up to $n-1$ and now consider the arrangements for n. Where can we place n in the sequence? Clearly, it can always go in the final position and placing the rest of the numbers is doable in F_{n-1} ways. If it goes in the penultimate position, then the last number must be $n-1$ (since no other number k would satisfy $n(n-1) \leq kn$) and placing the rest is doable in F_{n-2} ways. We now prove n cannot go anywhere else.
Let us place n as a_k, $k < n-1$. Now consider the smallest element of $a_{k+1}, \ldots a_n$. This element is at most k and so must go in the last position (as it will not satisfy the inequality if it is placed any earlier.) However, we now cannot place anyone in the penultimate position, as the smallest number that can go there is $k+1$ but $(k+1)(n-1) > kn$. You can also obtain a contradiction by trying to place $n-1$, as the earliest it can appear is as a_{k+1}.

Solution 2, bijection (Tanish): We prove that the problem is strictly equivalent to writing n as a sum of 1s and 2s, which is a well known recurrence problem left to the reader :)
Consider a permutation that works that is not the identity, and let i be the first index such that $a_i \neq i$. It follows that $a_j = i$ for some $j > i$. Now, where can we place j? If j appears before a_j, it has to be a_i, as otherwise the inequality is not satisfied. But if $j \neq i+1$ then we are not able to take any value for a_{j-1}, as the smallest number that can go there is $i+1$ but $(i+1)(j-1) > ij$. So either $j = i+1$ and we have swapped around two consecutive terms, or j appears after a_j. In the latter case, just take the largest value in $a_i, a_{i+1}, \ldots a_{j-1}$. This value is at least $j+1$ but this immediately contradicts the inequality again. In the former case, the placement of j and $j+1$ do not actually affect our ability to place anything afterwards, so we can look at the next index not mapped to itself and concluded it has been swapped with the one immediately following it by the same reasoning, and continue onwards in this manner. Therefore the only changes we can make from the identity permutation are swapping two consecutive values, and this is equivalent to writing n as a sum of 1s and 2s as desired; a 1 is an element mapped to itself and a 2 is a swap of two consecutive elements.

Marking Scheme:
This first section is additive.

(a) 1P: Stating the answer is F_n or that it satisfies the recurrence relation $f(n) = f(n-1) + f(n-2)$.

(b) 1P: Assuming we are not dealing with the identity, in order to either consider the placement of n or the smallest/largest index not mapped to itself

(c) 2P: Showing that this index has to have been swapped (if you write the permutation as a product of disjoint cycles, it must belong to a 2-cycle).

(d) 1P: Showing that if it was swapped with an index that is more than 2 away, we have a contradiction.

(e) 2P: Concluding (recursive formula if they considered n; looking at next index not mapped to itself $\rightarrow$ equivalence to a problem whose solution is Fibonacci, etc.) If the student does a solution by induction and forgets to note the base cases ($f(1) = 1, f(2) = 2$) they will not get these points, **even** if they say that the solution is the Fibonacci numbers.

The second item should *not* be awarded if the student is considering the placement of 1.
If the student obtains $\leq$ 1P from the first additive section, they are eligible to receive **one** of the following items:

(a) 1P: Considering the placement of 1.

(b) 2P: Showing 1 has to have been swapped with 2 or sent to itself.

(c) 1P: Stating the set of plausible sequences is the identity and all variations of the identity where an index is moved by at most 1 place.

(d) 1P: Attempting to show that when the permutation is written as a product of disjoint cycles, all the cycles are of length 2.

Finally, the following points may be awarded *instead* of the points previously received **if** they have fewer than this many points.

(a) 4P: Any actual proof that there are no cycles of length $>$2

(b) 5P: Any actual proof that there are no cycles of length $>$2 and stating the answer is F_n or satisfies the recurrence relation $f(n) = f(n-1) + f(n-2)$.

In other words, the student receives max(best item from section 3, section 1 + {best item from section 2 if section 1 $\leq$ 1 }).

8. Let ABC be a triangle such that $BC = CA$. Let D be a point inside the segment AB such that $AD < DB$. Let P and Q be two points inside the segments BC and CA respectively such that $\angle DPB = \angle DQA = 90°$. Let the perpendicular bisector of PQ intersect the segment CQ at E. The circumcircles of ABC and PQC intersect at C and F. Suppose that P, E, F are collinear. Prove that $\angle ACB = 90°$.

Solution (Horace): Let M be the midpoint of AB. then

$$\angle CMD = \angle CPD = 180° - \angle CQD = 90°,$$

giving that $CPMDQ$ is an inscribed pentagon. Since $MCP = QCM$, M lies on the perpendicular bisector of PQ. Redifining F as the intersection of PE and the circumcenter of PQC, we see that F is the reflection of C through the perpendicular bisector fo PQ. Now for $FCAB$ to lie on a circle, the perpendicular bisector of CF, which is no different that the perpendicular bissector of PQ, should go through the circumcenter of ABC. This would give that M, as the intersection of the perpendicular bisector of PQ and AB, should be the circumcenter of ABC. This is only possible if $\angle ACB = 90°$. □

Solution (Matthew): The reflection of C through the perpendicular bisector of PQ clearly lies on the circumcircle of PQC. It also clearly lies on PE. In particular, this reflection must be F, giving us that $CPQF$ is an isoceles trapezoid. Therefore

$$PF = CQ.$$

Now

$$\angle BFP = \angle BFC - \angle PFC = \angle BAC - \angle ACF = \angle ABC - \angle ABF = \angle FBP.$$

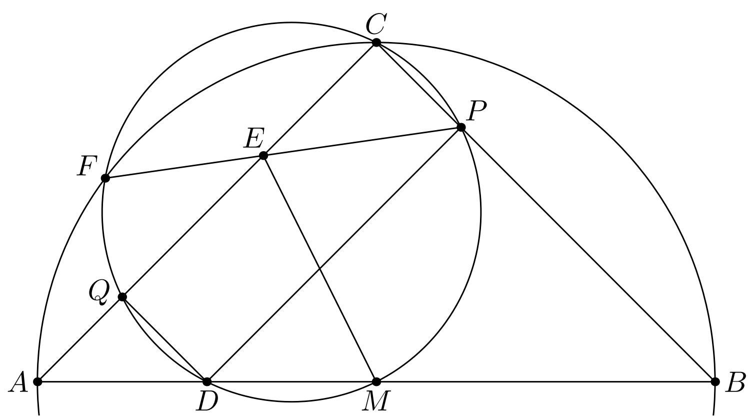
So

$$BP = PF = CQ.$$

Similarly (or from $AC = BC$) $CP = AQ$. This gives, using the similarity of AQD and BPD,

$$\frac{AD}{PC} = \frac{AD}{AQ} = \frac{DB}{PB}.$$

Therefore DP and AC are parallel and $\angle ACB = 90°$ □



Marking Scheme (solution 1) General idea: an incomplete proof has a maximum of 5 point. A complete proof with mistakes get 6 points.

(a) 1P: Introducing M, in whatever meaningful definition (for example, circumcenter of ABC, midpoint of AB, intersection of AB with the circumcircle of PQC, intersection of the perpendicular bisector of CF with the angle bisector in C...). No points should be given for a letter on a drawing.

(b) 1P: Finding that $CPDQ$ are cocyclic. This point is also given if the student prove points (c) or (d).

(c) 2P: Finding that $CPMQ$ is an inscribed quadrilateral, or analogous statement depending on the definition of M. This point is also given if the student prove point (d).

(d) 1P: Proving that the perpendicular bisector of CF intersect AB in its midpoint.

(e) 2P: Finishing.

Marking Scheme (solution 2)

(a) 2P: Showing that FPB or AQF are isoceles. This point is also given if one shows that $BP = CQ$ or $AQ = CP$.

(b) 2P: $BP = CQ$ or $AQ = CP$

(c) 1P: Stating that AQD and BPD are similar.

(d) 2P: Finishing.

9. Find all polynomials P with real coefficients having no repeated roots, such that for any complex number z, the equation $zP(z) = 1$ holds if and only if $P(z-1)P(z+1) = 0$.

First solution (David, inspired by Arnaud): Assume that we have a polynomial P that satisfies the desired conditions. We begin by examining some easy cases:

If P is a constant polynomial, that is, $P(z) = c$ for some $c \in \mathbb{R}$, the condition becomes $cz = 1$ if and only if $c^2 = 0$. If $c = 0$, then $c^2 = 0$ is always satisfied while $cz = 1$ cannot hold, contradiction. If however $c \neq 0$, we can plug in $z = \frac{1}{c}$ to obtain a contradiction. Therefore, there are no solutions among constant polynomials.

What about polynomials of the form $P(z) = mz + b$ where $m \neq 0$? The condition becomes $mz^2 + bz = 1$ if and only if $(m(z-1)+b)(m(z+1)+b) = 0$, so the two quadratic polynomials $mz^2 + bz - 1$ and $m^2z^2 + 2bmz + (b^2 - m^2)$ have the same roots, and this means that they are multiples of each other. By comparing the leading coefficients we see that the multiplying factor is m, and by comparing the other coefficients we get that $bm = 2bm$ and $-m = b^2 - m^2$. Since $m \neq 0$, the first of these two equations gives us $b = 0$ and the second $m = 1$. This shows that the only linear polynomial that can satisfy these conditions is $P(z) = z$, which is obviously a solution.

Let's now assume that $\deg(P) = n \geq 2$.

Consider the polynomial $Q(z) = (z+1)P(z+1) - (z-1)P(z-1)$. It's easy to see that $\deg(Q) \leq \deg(P)$ (since the highest degree terms of $(z+1)P(z+1)$ and $(z-1)P(z-1)$ cancel out). We show that Q also has the same roots as P:

If $P(r) = 0$ for some complex number r, by plugging in $z = r \pm 1$ in the condition of the problem, we see that $(r-1)P(r-1) = 1 = (r+1)P(r+1)$ and therefore $Q(r) = 0$. This shows that all roots P are also roots of Q. Since P has no repeated roots and the degree of Q is not greater than the degree of P, we deduce that Q is just a multiple of P.

Let $P(z) = \sum_{k=0}^{n} a_k z^k$. This gives us

$$Q(z) = \sum_{k=0}^{n} a_k \big((z+1)^{k+1} - (z-1)^{k+1} \big). \qquad (\star)$$

In order to compare Q to P, we would like to write it in the form $Q(z) = \sum_{k=0}^{n} b_k z^k$. To compute b_n, note that the only terms in $(\star)$ that contain z^n come from $k = n-1$ and $k = n$. we get

$$b_n z^n = a_{n-1}(z^n - z^n) + a_n \big((n+1)z^n - (-(n+1)z^n \big) = 2(n+1)a_n z^n,$$

so we must have $Q = 2(n+1)P$. Now we compute b_{n-1}. Similarly to above, we see that the only relevant terms in $(\star)$ come from $k = n-2$, $k = n-1$ or $k = n$. We get

$$\begin{aligned} b_{n-1}z^{n-1} &= a_{n-2}(z^{n-1} - z^{n-1}) + a_{n-1}\big(nz^{n-1} - (-n)z^{n-1} \big) + a_n \left(\binom{n+1}{2} z^{n-1} - \binom{n+1}{2} z^{n-1} \right) \\ &= 2na_{n-1}z^{n-1}. \end{aligned}$$

But if $Q = 2(n+1)P$, we must have $b_{n-1} = 2(n+1)a_{n-1}$, contradiction! This shows that there cannot be any polynomial of degree ≥ 2 satisfying the desired conditions.

Second solution using complex numbers (David):

We give a different argument for the case $n \geq 2$:

Note that the problem statement is equivalent to the statement that the two polynomials $zP(z)-1$ and $P(z-1)P(z+1)$ have the same set of roots. However, the polynomial $zP(z)-1$ has degree $n+1$ and thus at most $n+1$ different roots. On the other hand, for each of the distinct roots $r_1, r_2, \ldots, r_n$ of P, the numbers $r_1+1, \ldots, r_n+1$ are roots of $P(z-1)$ and the numbers $r_1-1, \ldots, r_n-1$ are roots of $P(z+1)$. Write $r_k = x_k + y_k i$. For any fixed y we note the following:

If there are m roots of P with imaginary part y, then there are at least $m+1$ distinct numbers among $\{r_1-1, \ldots, r_n-1, r_1+1, \ldots, r_n+1\}$ with imaginary part y. This is because if WLOG $r_1, \ldots r_m$ all have imaginary part y and real parts $x_1 < \ldots < x_m$ (the strict inequality holds because we cannot have repeated roots), then the $m+1$ numbers $r_1-1, r_1+1, \ldots, r_m+1$ all have different real part and so they must be different.

Since we cannot have more than $n+1$ roots, we conclude that $y_1 = \ldots = y_n$. In fact, they are all equal to zero because we know that if $x_k + y_k i$ is a root of a polynomial with real coefficients, then so is $x_k - y_k i$. But since all y_k are equal, we must have $y_k = -y_k$.

Since $y_1 = \ldots = y_n$, the $x_k = r_k$ are pairwise distinct and we can order them $x_1 < \cdots < x_n$. Analogously to above, we have $x_1 - 1 < x_1 + 1 < \cdots < x_n + 1$, so those have to be our $n+1$ roots. Clearly, $x_k - 1$ is the k-th smallest root, and by comparing to our chain of inequalities above, we must have $x_k - 1 = x_{k-1} + 1$, or $x_k = x_{k-1} + 2$.

All in all, we leared that all the roots of $P(z)$ are real and form an arithmetic progression of difference 2. We can therefore write:

$$P(z) = c \cdot (z-a+2) \cdot (z-a+4) \cdot \ldots \cdot (z-a+2n)$$

for some constants $a, c \in \mathbb{R}$ where $c \neq 0$. Hence, we can also write:

$$zP(z) - 1 = c \cdot z \cdot (z-a+2) \cdot \ldots \cdot (z-a+2n) - 1$$

The conditions in the problem statement now imply that the roots of the polynomial $zP(z)-1$ are $\{a-1, a-3, \ldots a-(2n+1)\}$. Plugging in $z = a-1$ into the equation above, we get

$$1 = c \cdot (a-1) \cdot 1 \cdot 3 \cdot \ldots \cdot (2n-1). \tag{$\star$}$$

On the other hand, plugging in $z = a-(2n+1)$ gives

$$1 = c \cdot (a-(2n+1)) \cdot (-(2n-1)) \cdot \ldots \cdot (-3) \cdot (-1). \tag{$\star\star$}$$

By taking absolute values of ($\star$) and ($\star\star$) and using $c \neq 0$, we see that $|a-1| = |a-(2n+1)|$, so $a = n+1$, which allows us to find $c > 0$ using ($\star$). But now we can plug $z = a-3 = n-2$ into the equation for $zP(z)-1$ above to obtain

$$1 = c \cdot (n-2) \cdot (-1) \cdot 1 \cdot \ldots \cdot (2n-3).$$

Since $n \geq 2$, each factor on the RHS except for the -1 is non-negative, which means that this equation cannot hold! We conclude that there is no such P for $n \geq 2$.

Marking Scheme (First solution)

(a) 1P: Treating both cases $n = 0$ and $n = 1$

(b) 2P: Introducing Q

(c) 1P: Arguing that Q is a multiple of P

(d) 1P: Computing the leading coefficient of Q

(e) 2P: Computing the second coefficient of Q and finishing

Marking Scheme (Second solution)

(a) 1P: Treating both cases $n = 0$ and $n = 1$

(b) 1P: Noting that $P(z-1)P(z+1)$ has at most $n+1$ different roots

(c) 2P: Claiming that the roots of P form an arithmetic progression of difference 2

(d) 1P: Proving the aforementioned claim rigorously

(e) 2P: Finishing

10. Prove that there are infinitely many positive integers n such that

$$n^2 + 1 \mid n!$$

holds.

Solution 1 (Raphi): So we want all the factors of $n^2 + 1$ to be less than n. The best way to ensure that this is true, would be by somehow factorising $n^2 + 1$. Aha, looks like a job for the factorising master Sophie-Germain.

Choose $n = 2k^2$. We then have

$$n^2 + 1 = 4k^4 + 4k^2 + 1 - 4k^2 = (2k^2 + 1)^2 - (2k)^2 = (2k^2 + 2k + 1)(2k^2 - 2k + 1)$$

We further get that

$$\gcd(2k^2 + 2k + 1, 2k^2 - 2k + 1) = \gcd(4k, 2k^2 - 2k + 1) = \gcd(k, 2k^2 - 2k + 1) = \gcd(k, 1) = 1$$

So it suffices to prove that both factors them self divide $n!$, as $a \mid x, b \mid x, (a, b) = 1 \Rightarrow ab \mid x$.

As $2k^2 - 2k + 1 < 2k^2 = n$, it surely divides $n!$. Now lastly we want $2k^2 + 2k + 1$ to divide $n!$.

Lemma 1 *If* $2k^2 + 2k + 1$ *isn't a prime then* $2k^2 + 2k + 1 \mid n!$.

Proof. Assume $2k^2 + 2k + 1 = pq$, with $p \leq q$. It's quite easy to see that $p, q < n$.
If $p < q$, then we have

$$2k^2 + 2k + 1 = pq \mid 1 \cdots (p-1) \cdot p \cdots (q-1) \cdot q \cdots n = n!$$

Else if $p = q$, then $p, q < \frac{n}{2}$

$$2k^2 + 2k + 1 = pq \mid p \cdot 2p \mid n!$$

□

We now note that if $k \equiv 1 \pmod 5$, then $2k^2 + 2k + 1 \equiv 0 \pmod 5$. So choosing $k = 5r + 1$, for $r \in \mathbb{N}$ gives $5 \mid 2k^2 + 2k + 1$, which makes it a non-prime. We done. Hooray.

Solution 2 (Yanta'ish, Ema, Elia): Let's pick two distinct primes $p > q$, with $p, q \equiv 1 \pmod 4$. There now exist natural numbers k, l, with $k^2 \equiv -1 \pmod p$ and $l^2 \equiv -1 \pmod q$. By CRT we can find an $n^2 \equiv -1 \pmod{pq}$, where $0 < n < pq$. The edge cases are exlcuded, as obviously $0^2 \not\equiv -1$. As $n^2 \equiv (pq - n)^2$, we can choose $n = \max(n, pq - n)$,w hilst still having the property $n^2 \equiv -1 \pmod p q$ and $0 < n < pq$. From this we gained the further inequality $n \geq \frac{pq}{2}$. Using $n < pq$, we get $\frac{n^2+1}{pq} < n$, which gives us $\frac{n^2+1}{pq} \mid n!$. Now to prove the wanted $n^2 + 1 \mid n!$, it suffices to prove $\frac{n^2+1}{pq} \mid \frac{n!}{pq}$.
We have $n \geq \frac{pq}{2} > 2p > 2q, p > q$, where p, q are big enough. So we have

$$\frac{n^2 + 1}{pq} \mid \frac{\frac{n^2+1}{pq} \cdot 2p \cdot 2q}{pq} \mid \frac{n!}{pq}$$

Summing up we have now constructed a number n, fulfilling the condition in the problem statement. Now to construct another n', we just pick the primes p, q bigger than any other n constructed. As $n' > p > q$ by the above inequality, the new n' will be different from all the other n constructed so far.

Marking scheme (Solution 1)

(a) 1P: The idea of applying Sophie Germain

(b) 2P: Actually using Sophie Germain

(c) 1P: Showing $\gcd(2k^2+2k+1, 2k^2-2k+1) = 1$

(d) 1P: Producing infinitely many k such that $2k^2+2k+1$ is not prime

(e) 2P: Proving that if $2k^2+2k+1$ is not prime, then $n^2+1 \mid n!$

A minor mistake, like not mentioning that $2k^2-2k+1 < n$, is penalised by a deduction of one point on a complete solution.

Marking scheme (Solution 2)

(a) 1P: Constructing a number n such that n^2+1 has at least two different prime factors

(b) 1P: Show that we may choose $p, q < n$

(c) 1P: Find a useful upper bound on n

(d) 1P: Show that we may choose $\frac{n^2+1}{pq} < n$

(e) 2P: Deal with the cases $\frac{n^2+1}{pq} \in \{p, q\}$

(f) 1P: Show that there are infinitely many n satisfying the restrictions of the construction and conclude

11. Find all even functions $g\colon \mathbb{R} \to \mathbb{R}$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}$

$$g(f(x)+y) = g(x) + g(y) + yf(x+f(x)).$$

Answer: The functions $g(x) = 0$ and $g(x) = x^2$ are the only solutions.

Solution (Arnaud):

First observe that $g(x) = 0$ and $g(x) = x^2$ are solutions. Indeed, both are even functions. For $g(x) = 0$, one can take $f(x) = 0$ (or any function f such that $f(x + f(x)) = 0$). For $g(x) = x^2$, on can take $f(x) = x$, because $(x + y)^2 = x^2 + y^2 + 2xy$.

We now prove that these are the only solutions. Let g be a solution of the problem and f be an associated function. We start with a bunch of substitutions. Let $y = 0$ which gives

$$g(f(x)) = g(x) + g(0). \tag{1}$$

Let further $y = -f(x)$ in the original equation and use the parity of g to get $g(0) = g(x) + g(f(x)) - f(x)f(x + f(x))$. If we plug (1) in, we get

$$2g(x) = f(x)f(x + f(x)). \tag{2}$$

We replace y by $f(y)$ in the original equation and use (1) to obtain $g(f(x) + f(y)) = g(x) + g(y) + g(0) + f(y)f(x + f(x))$. The symmetry between x and y implies

$$f(y)f(x + f(x)) = f(x)f(y + f(y)). \tag{3}$$

If $f \equiv 0$, then $g \equiv 0$. So, if we assume that g is not the constant 0 function (which we know is a solution), then there exists a such that $f(a) \neq 0$. Let $c = f(a + f(a))/f(a)$. We have, with $y = a$ in (3), $f(x + f(x)) = cf(x)$ and using (2) we obtain

$$2g(x) = cf(x)^2. \tag{4}$$

Since we assumed g is not identically 0, it holds $c \neq 0$. We plug (4) in the original equation and use $f(x + f(x)) = cf(x)$ to obtain, after simplifying by c,

$$f(f(x) + y)^2 = f(x)^2 + f(y)^2 + 2yf(x). \tag{5}$$

If one lets $x = y$ in (5), then one obtains

$$f(x)(f(x)(c^2 - 2) - 2x) = 0. \tag{6}$$

If $c^2 = 2$, then $f(x) = 0$ for all $x \neq 0$. In particular, one must have $a = 0$ and $f(0) = f(a) \neq 0$. But then $c = f(f(0))/f(0) = 0$ since $f(0) \neq 0$. Contradiction. Hence $c^2 \neq 2$, and for every x,

$$f(x) = 0 \quad \text{or} \quad f(x) = 2x/(c^2 - 2).$$

In particular, $f(0) = 0$. Let $b := 2/(c^2 - 2) \neq 0$ such that $f(x) = bx$ if $f(x) \neq 0$. Since $f(a) \neq 0$, we have $f(a) = ab$ and $a \neq 0$. Assume there is $t \neq 0$ such that $f(t) = 0$, then, with $x = a$ and $y = t$ in (5), we get

$$f(ab + t)^2 = a^2b^2 + 2tab.$$

Because $t \neq 0$, we must have $f(ab + t) = 0$ and thus $ab + 2t = 0$. So there is at most one $t \neq 0$, such that $f(t) = 0$, namely $t = -ab/2$. But if such a t exists, then $f(ab + t) = 0$ and $ab + t \neq t$

and $ab + t \neq 0$, contradiction. So, $f(x) = bx$ for all x and $g(x) = cb^2/2 \cdot x^2$ for all x by (4). We plug this in the original equation and deduce that $b = 1$ and $c = 2$ (using that $c \neq 0$). Therefore, we proved that if g is not identically 0, then $g(x) = x^2$ for every x.

Marking scheme:

There are two main milestones in the solution which are graded as follows:

(a) 3P: Get to relation (4)

(b) 5P: Prove that (if $g \not\equiv 0$), for every x, $f(x) = 0$ or $f(x) = bx$ for some $b \neq 0$.

The following deduction applies to a full solution:

(c) -1P: Not checking explicitly the solutions (and giving for both solutions g an example of an associated function f)

The following partial points towards (a) can be obtained:

(a.1) 1P: Relation (2)

(a.2) 1P: Relation (3)

The following partial points towards (b) can be obtained (not additive with any points awarded for (a)):

(b.1) 4P: Relation (6)

12. Let ABC be an acute triangle, and I its incenter. Let A_1 be the intersection of AI and BC, and C_1 the intersection of CI and AB. Furthermore, let M and N be the midpoints of AI and CI, respectively. Inside the triangles AC_1I and A_1CI we choose points K and L such that $\angle AKI = \angle CLI = \angle AIC$, $\angle AKM = \angle ICA$ and $\angle CLN = \angle IAC$. Prove that the radii of the circumcircles of the triangles KIL and ABC are equal.

Solution (Marco): Let M_A and M_C be the second intersections of the circle (ABC) with AI and CI, respectively. By the incenter/excenter lemma, $M_AI = M_AB$ and $M_CI = M_CB$, so M_AM_C is the perpendicular bisector of BI. Hence, the reflection of the circle (ABC) over M_AM_C is the circle (M_AIM_C), and they have the same radius. This proves that if K and L are on the circle (M_AIM_C), we are done.

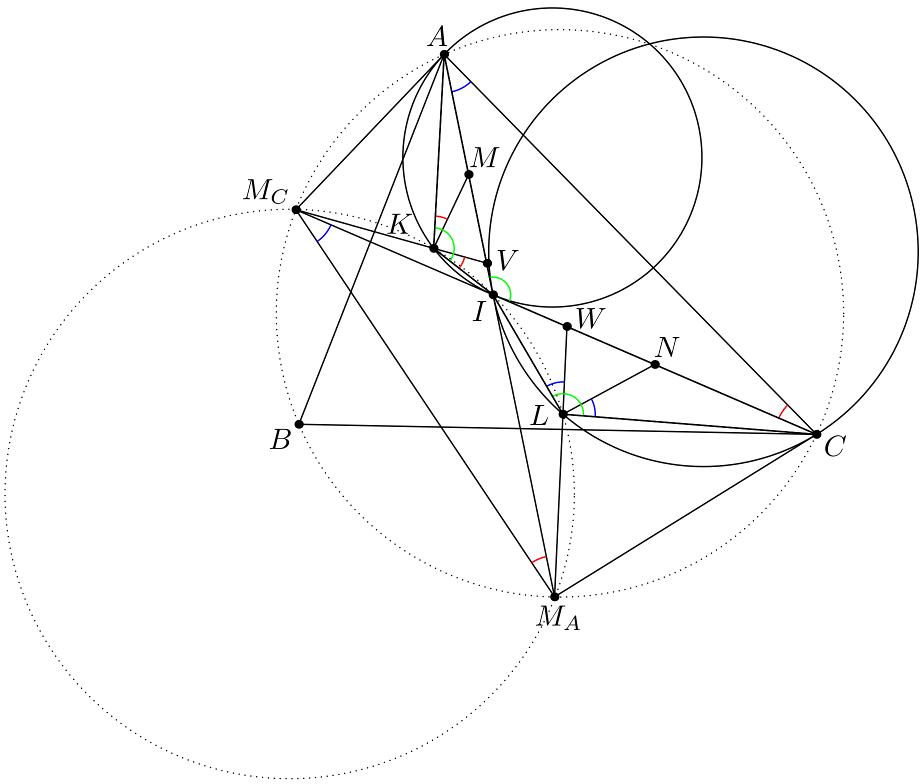
Moreover, by the incenter/excenter lemma,

$$\angle M_CAI = \angle M_CIA = 180^\circ - \angle CIA = 180^\circ - \angle IKA$$

implies that both M_CA and M_CI are tangent to the circle (AKI). By the symmedian lemma, M_CK is the K-symmedian of triangle AKI, so that if V is the intersection of M_CK and AI, then $\angle IKV = \angle MKA$. Thus,

$$\begin{aligned}
\angle M_CKI + \angle IM_AM_C &= 180^\circ - \angle IKV + \angle AM_AM_C \\
&= 180^\circ - \angle MKA + \angle ACM_C \\
&= 180^\circ - \angle ACI + \angle ACM_C \\
&= 180^\circ,
\end{aligned}$$

and K is on the circle (M_AIM_C), as desired. Similarly, introducing W, the intersection of M_AL and CI, we see that L is also on the circle (M_AIM_C), proving that $(KIL) = (M_AIM_C)$ and finishing the problem.



Marking Scheme (additive):

(a) 1P: Introducing M_A and M_C in a sensible way, but only intersecting the circles (ABC) and (KIL) gives 0P.

(b) 1P: Claim that $(KIL) = (M_C I M_A)$, or prove that $(M_C I M_A)$ has the same radius as (ABC).

(c) 2P: Prove that both $M_C A$ and $M_C I$ are tangent to (AKI), or, similarly, that both $M_A C$ and $M_A I$ are tangent to (CLI).

(d) 1P: Prove that $M_C K$ is the K-symmedian of AKI, or, similarly, that $M_A L$ is the L-symmedian of CLI.

(e) 2P: Conclude.