IMO - Selection 2018

First exam - 12 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Let $k \geq 0$ be an integer. Determine all polynomials P of degree k with real coefficients such that P has k distinct real roots and for any root a of P

$$P(a+1) = 1.$$

2. Let O be the circumcenter of an acute triangle ABC. The line OA intersects the altitude h_b at P and the altitude h_c at Q. Let H be the orthocenter of ABC. Prove that the circumcenter of PQH lies on the median of ABC through A.

Remark: the altitude h_a is the line through A perpendicular to BC.

3. Along the coast of a round island there are 20 different villages. Each of these villages has 20 fighters and all 400 fighters have different strengths.

Every pair of neighbouring villages A and B organises a competition during which all 20 fighters from village A battles individually against every fighter from the village B. The winner of a battle is always the stronger fighter. We say that the village A is *stronger* than the village B, if during at least k out of the 400 battles, a fighter from village A has won.

It turns out that every village is stronger than its clockwise neighbouring village. Determine the maximal value of k that allows such an outcome.

Good Luck!