Final Round 2023
Solutions

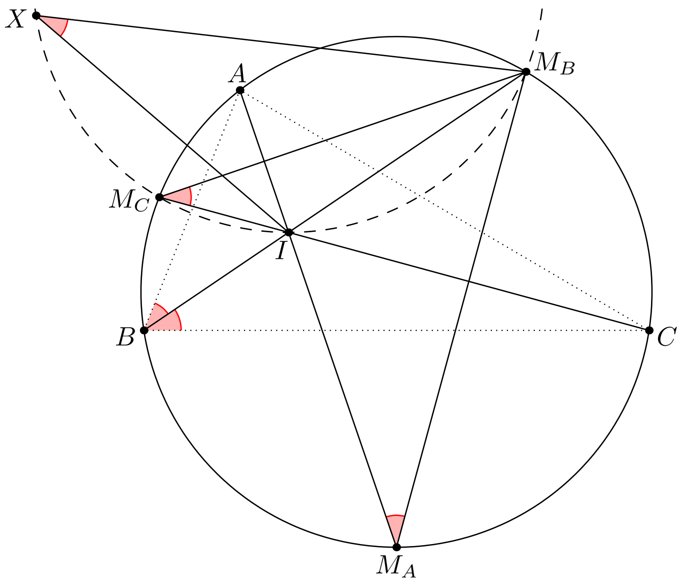
Aarburg
March 10, 2023
First Exam

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

1. Let ABC be an acute triangle with incentre I. On its circumcircle, let M_A, M_B and M_C be the midpoints of minor arcs BC, CA and AB respectively. Prove that the reflection of M_A over the line IM_B lies on the circumcircle of the triangle IM_BM_C.

Solution:



Let X be the reflection of M_A over the line IM_B. We wish to prove that X, M_C, I, M_B lie on a circle. By WUM, observe that A, I, M_A, B, I, M_B and C, I, M_C are collinear. Therefore, by symmetry, we get

$$\angle IXM_B = \angle IM_AM_B = \angle AM_AM_B = \angle ABM_B$$

On the other hand, note that

$$\angle IM_CM_B = \angle CM_CM_B = \angle CBM_B = \angle ABM_B$$

Thus, we get $\angle IXM_B = \angle ABM_B = \angle IM_CM_B$, so by the converse of the inscribed angle theorem, X, M_C, I, M_B lie on a circle, as required.

Marking Scheme: *A)*

- 1P: Justifying at least one of the triples A, I, M_A, B, I, M_B and C, I, M_C are collinear by WUM.
- 2P: Stating that $\angle M_BXI = \angle IM_AM_B$
- 3P: Proving that $\angle IM_AM_B = \angle M_BM_CI$
- 1P: Finishing from $\angle M_BXI = \angle M_BM_CI$

B)

- 1P: Justifying at least one of the pairs A, I, M_A, B, I, M_B and C, I, M_C are collinear by WUM.
- 1P: Stating that $\angle XM_BI = IM_BM_A$.
- 1P: Proving that $M_AM_C \perp IM_B$.
- 1P: Proving that M_A, M_C, X are collinear from $M_AM_C \perp IM_B$.
- 2P: Proving that $\angle M_AM_CI = \angle IM_BM_A$
- 1P: Finishing from $\angle XM_BI + \angle IM_CX = 180°$

Remark: If everything is proven but not $\angle M_AM_CI = \angle IM_BM_A$, the solution is worth 4 points.

2. The wizards Albus and Brian are playing a game on a square of side length $2n+1$ metres surrounded by lava. In the centre of the square there sits a toad. In a turn, a wizard chooses a direction parallel to a side of the square and enchants the toad. This will cause the toad to jump d metres in the chosen direction, where d is initially equal to 1 and increases by 1 after each jump. The wizard who sends the toad into the lava loses. Albus begins and they take turns. Depending on n, determine which wizard has a winning strategy.

Solution:

Brian wins, irrespective of n. Suppose Brian plays with the following strategy: for every move Albus makes, Brian makes a move in the opposite direction. If Brian were to lose, this would mean the toad was at the edge of the board two jumps prior, before Albus's final move, as otherwise Brian's final move could not possibly have sent it off the board. Now, because every two jumps the total displacement is 1, it would take at least $2n$ jumps for the toad to be on the edge at the end of Brian's turn specifically. But this means Albus's next move is of length $2n+1$; which always sends the toad off the board, contradiction.

Marking Scheme:

- 0P: Claiming Brian always wins.
- 2P: Noting the strategy of playing in the opposite direction to the previous move (not necessarily by Brian, could be implemented by either wizard).
- +1P: Claiming Brian always wins **via this strategy**.
- +3P: Proving that Brian will not have to play a move that ends with him on the edge until move $2n$. If this is incomplete, one can obtain the following nonadditive partials:
  (a) 1P: Working two steps backwards from the end to show that if someone is forced to lose, at the end of their previous turn they must be on the edge.
  (b) 2P: Stating that the total displacement every two moves is 1, with this strategy.
- +1P: Concluding with the fact that Albus' next move will lose.

Note that the points do not have to be obtained sequentially.

Remark: Minimising distance to the centre/maximising distance to the edges is strictly equivalent to playing in the opposite direction and will be rewarded as such.

3. Let x, y and $a_0, a_1, a_2, \ldots$ be integers satisfying $a_0 = a_1 = 0$ and

$$a_{n+2} = x \cdot a_{n+1} + y \cdot a_n + 1$$

for all integers $n \geq 0$. Let p be any prime number. Show that $\gcd(a_p, a_{p+1})$ is either equal to 1 or greater than $\sqrt{p}$.

Solution: Assume that $\gcd(a_p, a_{p+1}) \neq 1$ and let q be a prime dividing $\gcd(a_p, a_{p+1})$. Considering the sequence $a_0, a_1, a_2, \ldots$ modulo q, any two consecutive terms still uniquely determine the subsequent term. The fact that

$$a_p \equiv a_{p+1} \equiv a_0 \equiv a_1 \equiv 0 \pmod{q}$$

implies that the sequence is periodic and that p is a period. If d is the minimal period, we must have $d \mid p$, so $d \in \{1, p\}$. If $d = 1$, then the sequence is constant modulo q and $1 = a_2 \equiv a_1 = 0 \pmod{q}$, a contradiction. Therefore $d = p$, and we cannot find $0 \leq i < j < p$ such that $a_i \equiv a_j$ and $a_{i+1} \equiv a_{j+1} \pmod{q}$, since otherwise the minimal period would be smaller than p. But since there are only q^2 distinct pairs of residue classes modulo q, it follows that $p \leq q^2$. Since p is a prime and hence not a square, we conclude that $\sqrt{p} < q \leq \gcd(a_p, a_{p+1})$.

Marking Scheme:

- 1P: Reducing the sequence modulo (a prime divisor of) $\gcd(a_p, a_{p+1})$.
- 1P: Arguing that the reduced sequence is (eventually) periodic.
- 1P: Arguing that the reduced sequence has minimal period dividing p.
- 1P: Proving that the reduced sequence cannot be (eventually) constant.
- 2P: Arguing that the minimal period is at most the modulus squared.
- 1P: Concluding

4. Determine the smallest possible value of the expression

$$\frac{ab+1}{a+b}+\frac{bc+1}{b+c}+\frac{ca+1}{c+a},$$

where $a,b,c\in\mathbb{R}$ satisfy $a+b+c=-1$ and $abc\leq-3$.

Solution: The minimum is 3, which is obtained for $(a,b,c)=(1,1,-3)$ and permutations of this triple.

As abc is negative, the triple (a,b,c) has either exactly one negative number or three negative numbers. Also, since $|abc|\geq 3$, at least one of the three numbers has absolute value greater than 1.
If all of a,b,c were negative, the previous statement would contradict $a+b+c=-1$, hence exactly one of a,b,c is negative.

Wlog let c be the unique negative number. So $a,b>0>c$, as the value 0 isn't possible by $|abc|\geq 3$. Let S be the given sum of fractions. We then have

$$\begin{aligned} S+3&=\sum_{cyc}\frac{ab+1+a+b}{a+b}=\sum_{cyc}\frac{(a+1)(b+1)}{a+b}=\sum_{cyc}-\frac{(a+1)(b+1)}{c+1}\\ &\geq\sum_{cyc}|a+1|=(a+1)+(b+1)-(c+1)=2a+2b+2,\end{aligned}$$

using AM-GM on the three pairs of summands respectively for the inequality. We can do this since $a+1,b+1>0$ and $-(c+1)=a+b>0$, so every summand is positive.
So all we want to do now is show $a+b\geq 2$, to conclude $S\geq 3$. From the two given conditions we have $ab(1+a+b)\geq 3$. If $a+b<2$, then $ab\leq\left(\frac{a+b}{2}\right)^2<1$ and thereby $ab(1+a+b)<3$. So the implication $ab(1+a+b)\geq 3\Rightarrow a+b\geq 2$ is indeed true.

Marking Scheme:

- 1P: Claiming the minimum and giving a triple
- 1P: Proving that exactly one of a,b,c is negative
- 1P: Factorising the expression in a useful way.
- 1P: Applying AM-GM correctly
- 1P: Argue why this can be done
- 2P: Showing $ab(1+a+b)\geq 3\Rightarrow a+b\geq 2$

Final Round 2023
Solutions

Duration: 4 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Aarburg
March 11, 2023
Second Exam

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

5. Let D be the set of real numbers excluding -1. Find all functions $f\colon D \to D$ such that for all $x, y \in D$ satisfying $x \neq 0$ and $y \neq -x$, the equality

$$\Big(f(f(x)) + y\Big) f\left(\frac{y}{x}\right) + f(f(y)) = x$$

holds.

Solution: Plugging in $x = y \neq 0$ yields $f(f(x)) = \alpha x$ for some constant α. This holds for all $x \neq 0, -1$. If α was any real number different from 0 or 1, we would get $f(f(-1/\alpha)) = -1$, which is a contradiction. Hence, only $\alpha = 0$ or $\alpha = 1$ are possible. In the former case $x = 1, y = 0$ gives a contradiction, so $f(f(x)) = x$ for all $x \in D, x \neq 0$. Letting $x = 1, y = f(1)$ gives $f(1) = 0$ and so $f(0) = 1$. Now, for all other x, we can plug in $\frac{1}{x}$ instead of x and $y = 1$ to obtain $f(x) = \frac{1-x}{1+x}$ (this formula now also works for $x = 0$), and we can verify that this is indeed a solution:

$$f(f(x)) = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1 - x - (1-x)}{1 + x + 1 - x} = x$$

and thus

$$(f(f(x)) + y) f\Big(\frac{y}{x}\Big) + f(f(y)) = (x+y) f\Big(\frac{y}{x}\Big) + y = (x+y)\frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} + y = x - y + y = x.$$

Marking Scheme:

- 1 point: Show that $f(f(x)) = \alpha x + \beta$ for some constants α and β.
- 1 point: Show that f must be of the form $f(x) = \frac{1-\alpha x-\beta}{\alpha+\beta+x}$.
- 1 point: Show that $\beta = 0$.
- 2 points: Proof that $\alpha = 1$.
- 1 point: Show that $f(x) = \frac{1-x}{1+x}$ is the only possible solution.
- 1 point: Proof that $f(x) = \frac{1-x}{1+x}$ is indeed a solution

Note that forgetting about cases like $x = 0$, $x = -y$, $x = -1$ etc. where needed will give minus 1 point.

6. Determine all integers $n \geq 3$ such that

$$n! \mid \prod_{\substack{p<q\leq n \\ p,q \text{ prime}}} (p+q).$$

Remark: The expression on the right-hand side denotes the product over all sums of two distinct primes less than or equal to n. For $n = 6$, this is equal to $(2+3)(2+5)(3+5)$.

Solution: For a fixed $n \geq 3$, let us denote the product on the right by $P(n)$ and let r be the largest prime less than or equal to n. We now observe that

$$r \mid n! \mid P(n) = P(r)$$

and therefore there are primes $p < q \leq r$ with $r \mid p+q$. But $0 < p+q < 2r$ implies $p+q = r$. Since $r \geq 3$ is an odd prime, we must have $p = 2$ and $q = r-2$ is an odd prime. But now we also have

$$r-2 \mid n! \mid P(n) = P(r) = P(r-2) \prod_{\substack{s \leq r-2 \\ s \text{ prime}}} (r+s).$$

Hence either $r-2 \mid P(r-2)$ and similar to the above case we find that $r-4$ is prime, or there is a prime $s \leq r-2$ with $r-2 \mid r+s$. Since $r-2 < r+s < 3(r-2)$, we must have $r+s = 2(r-2)$ and again $s = r-4$ is prime. We conclude that all of $r, r-2, r-4$ are prime and they have distinct residues modulo 3, it follows that $r-4 = 3$ and thus $r = 7$. It remains to check the values $n = 7, 8, 9$ and 10.

Observing that

$$7! = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \mid 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 = P(7)$$

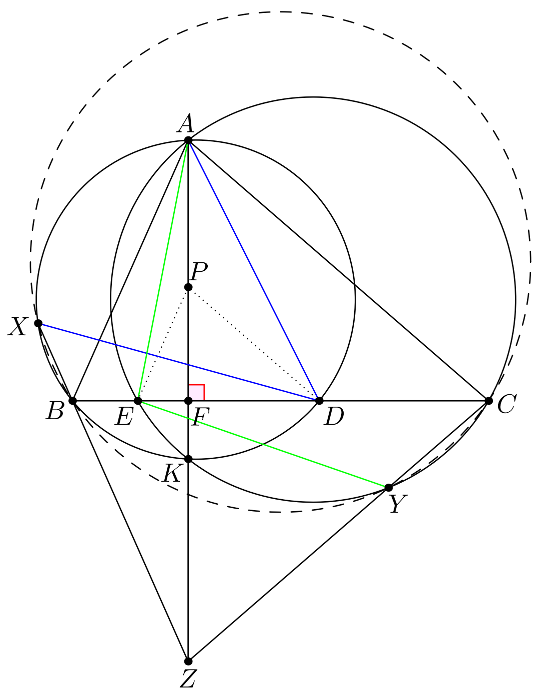
we find that $n = 7$ is a solution. However, since $P(7) = P(8) = P(9) = P(10)$ and since all of $8!, 9!, 10!$ are divisible by 2^7, we conclude that $n = 7$ is the only solution.

Marking Scheme:

- 1P: Considering r, the largest prime at most n.
- 1P: Arguing that $r = p+q$ for some primes $p < q \leq n$.
- 1P: Proving that $r-2$ is prime.
- 2P: Proving that $r-4$ is prime.
- 1P: Deducing that $r = 7$.
- 1P: Explicitly proving divisibility for $n = 7$ and checking that the other remaining cases don't work.

7. In the acute triangle ABC, the point F is the foot of the altitude from A, and P is a point on the segment AF. The lines through P parallel to AC and AB meet BC at D and E respectively. Points $X \neq A$ and $Y \neq A$ lie on the circumcircles of triangles ABD and ACE respectively, such that $DA = DX$ and $EA = EY$. Prove that $BCXY$ is a cyclic quadrilateral.

Solution:



Since PE is parallel to AB and PD is parallel to AC, by the basic proportionality theorem, we get the following ratio equalities

$$\frac{FE}{FB} = \frac{FP}{FA} = \frac{FD}{FC} \iff FC \cdot FE = FB \cdot FD$$

Therefore, the power of F with respect to (ACE) and with respect to (ABD) is the same, so F lies on the power line of both circles. In other words, if K is the intersection of both circles, then K lies on AF.

Now, define $Z = BX \cap CY$. Let's prove that Z is in fact the reflection of A with respect to line BC. To prove this, it suffices to show the angle equalities $\angle ZBC = \angle CBA$ and $\angle ZCB = \angle BCA$. Let's focus on the first equality and point X. We have two cases.

- Case 1: X lies on the opposite side of A with respect to line BC Since $DA = DX$ and B, X, A, D lie on a circle, by WUM, BD is the internal angle bisector of angle XBA, so $\angle XBD = \angle DBA$, implying $\angle ZBC = \angle CBA$, as required.

- Case 2: X lies on the same side as A with respect to line BC Once again, since $DA = DX$ and B, X, A, D lie on a circle, BD is the external angle bisector of angle XBA (by WUM again or by angle chasing), and so $\angle ZBD = \angle DBA$, implying $\angle ZBC = \angle CBA$, as required.

Therefore, Z is indeed the reflection of A with respect to line BC. Therefore, this implies that A, F, K, Z are collinear. To conclude, Z lying on the power line of circles (ABD) and (ACE) yields

$$ZB \cdot ZX = ZK \cdot ZA = ZY \cdot ZC$$

and thus, $BYCX$ is a cyclic quadrilateral, as required.

Remark: Note that the case distinction is not necessary if directed angles are used in the entire proof.

Marking Scheme:

- 1P: Proving that $FE/FB = FD/FC$.
- 1P: Proving that F lies on the power line on (ABD) and (ACE)
- 1P: Proving that BX is the reflection of BA over BC.
- 1P: Proving that CY is the reflection of CA over BC.
- 1P: Proving that $BX \cap CY$ is the reflection of A over BC,
  or that AF, BX, and CY concur.
- 1P: Proving that $BX \cap CY$ is on the power line of (ABD) and (ACE).
- 1P: Concluding using power of a point.

8. Let n be a positive integer. Kimiko starts with n piles of pebbles each containing a single pebble. She can take an equal number of pebbles from two existing piles and combine the removed pebbles to create a new pile. Determine, in terms of n, the smallest number of nonempty piles Kimiko can end up with.

Solution: If n is a power of 2 then there may be only one pile remaining; otherwise, there will be at least two piles remaining, but this can be attained. It is clear why you

can reach one pile if n is a power of 2: the first $n/2$ piles can each receive pebbles from two piles with a pebble, the next $n/4$ piles can inherit two pebbles from two piles with two stones each and so forth, doubling each "generation". Suppose n is not a power of 2 and write $n = c2^k, c > 1$. We claim there is always an pile with a number of pebbles *not* divisible by c. This is clearly true initially. Suppose it is not true at some point, and consider what happens when you next create a pile. If this pile does not receive some pebbles from the existing pile(s) possessing a quantity not divisible by c, then they will maintain the invariant. If instead the pile receives some x pebbles from them, then the pile will have $2x$, but c does not divide $2x$ as c is odd and does not divide x. In particular, one can never reduce to a single pile, as n is of course divisible by c.

Finally, we show that you can always reach just two piles. This can be done via induction: Write $n = 2^k + r, r < 2^k$. If we get to an pile with r pebbles, then the remaining 2^k pebbles can always be consolidated into one pile, irrespective of the initial distribution: we begin by having 2^{k-1} piles receiving 1 pebble from two piles, but not from each other, creating 2^{k-1} piles with 2 pebbles each, and then double each generation like above. To create a pile with r pebbles, we have two cases. If $r < 2^{k-1}$ then simply create two piles with 2^{k-1} pebbles by doubling, and then take away some pebbles from both of them so that they both have r pebbles, and set one of them aside. If $r > 2^{k-1}$ then create an pile with 2^k pebbles and an pile with 2^{k-1}, and again take away some pebbles from both of them so the larger pile has r pebbles. One can formulate an alternative induction argument where you use the fact n can be consolidated into two piles of size $n-1$ and 1, to show that $n+1$ can be consolidated into $n+2$ and 2 and then show via a number-theoretic argument that you can reach $n+1$ and 1 from here if n is odd.

Marking Scheme:

- 0P: Claiming the correct values (1 for powers of 2, 2 otherwise).
- +3P: Showing that a power of 2 can be consolidated into 1 pile and any other number can be consolidated into 2 piles. Students will be awarded for partial progress as follows, nonadditively:
  (a) 0P: Showing that a power of 2 can be consolidated into 1 pile.
  (b) 1P: Showing that for an odd number n, repeatedly doing the doubling operation has forwards- and backwards- uniqueness.
  (c) 2P: Showing that for an odd number n, you can get from $n-2, 2$ to $n-1, 1$.

(d) 1P: Building the largest power of 2 possible **and** using it to create desirable configurations.

- +3P: Showing that any nonpower of 2 cannot be consolidated into 1 pile. Students will be awarded for partial progress as follows, nonadditively:

  (a) 0P: Showing that a number not divisible by 4 cannot be consolidated into 1 pile.

  (b) 1P: Proving that if c is the largest odd divisor of n, it would be impossible to have only piles of size c.

  (c) 2P: Claiming that if c as above defined will never divide all the piles, or an equivalent statement.

- 1P: Completing both parts.