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IMO Selection 2022 - Solutions

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Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let n be a positive integer. Prove that there exists a finite sequence S consisting of only zeros and ones, satisfying the following property: For any positive integer $d \geq 2$, when S is interpreted as a number in base d, the resulting number is non-zero and divisible by n.

Remark: The sequence $S = s_k s_{k-1} \cdots s_1 s_0$ interpreted in base d is the number $\sum_{i=0}^{k} s_i d^i$.

Solution 1 (Valentin): Working in any base $d \geq 2$, by pigeonhole, two of the numbers

$$1,\ 11,\ 111,\ \ldots,\ \overbrace{11\cdots11}^{n+1}$$

have the same residue mod n. Therefore their difference

$$D = \overbrace{11\cdots11}^{t}\overbrace{00\cdots00}^{k} \text{ with } k,t \leq n$$

is divisible by n in base d. But then also

$$n \mid D \cdot d^{n-k} \cdot (1 + d^t + d^{2t} + \cdots + d^{n!-t}) = \overbrace{11\cdots11}^{n!}\overbrace{00\cdots00}^{n}$$

in base d. But the representation of this number is independent of d, hence we have found our sequence of zeros and ones.

Solution 2 (Raphi, David, Linus): It suffices to find a non-empty, finite set $\sigma \in \mathbb{N}$ such that

$$n | \sum_{s\in\sigma} d^s$$

for all $d \geq 2$. Consider the set $\sigma = \{\varphi(n), 2\varphi(n), \dots, n\varphi(n)\}$. For $d \geq 2$, if $(d, n) = 1$:

$$\sum_{s\in\sigma} d^s = \sum_{k=1}^{n} \left(d^{\varphi(n)}\right)^k = n$$

Else, if $g = (d, n) \neq 1$, let $n = xy$ with $(x, y) = (y, d) = 1$ and y maximal:

$$\sum_{s\in\sigma} d^s = \sum_{k=1}^{n} \left(d^{\varphi(n)}\right)^k \equiv \frac{d^{\varphi(n)}(d^{n\varphi(n)} - 1)}{d^{\varphi(n)} - 1} \equiv 0 \mod x$$

which makes sense, as $d^{\varphi(n)} - 1$ is coprime to x, and true since $v_p(x) \leq v_p(n) \leq \varphi(n)$ (ask raphi for an elaborate explanation). And y divides this sum by the first argument as above. Now $n = xy$ divides the sum since $(x, y) = 1$.

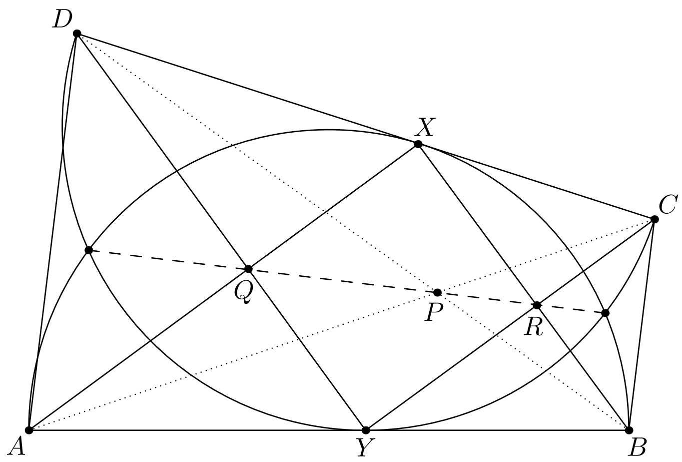
Marking Scheme (additive):

- 1P: Using Pigeonhole Principle on a set of strings
- 2P: For every $d \geq 2$, finding a string of a particular form that is divisible by n is base d (e.g. $111\dots1000\dots0$)
- 2P: Find a string that works for infinitely many $d \geq 2$
- 2P: Find a string that works for all $d \geq 2$

Marking Scheme (additive):

- 1P: Applying Euler-Fermat to the base d
- 1P: Find a string that works for all $d \geq 2$ with $(n, d) = 1$
- 1P: Proving the above string works for all $d \geq 2$ with $(n, d) = 1$
- 1P: Making progress to treat the case $(n, d) \neq 1$ (for example by splitting the prime factors of n according to whether they divide d or not)
- 1P: Note that $\varphi(a) \mid \varphi(n)$ for all $a \mid n$ (or a similar result)
- 1P: Find a string that works for all $d \geq 2$
- 2P: Proving that the above string works for all $d \geq 2$

2. Let $ABCD$ be a convex quadrilateral such that the circle with diameter AB is tangent to the line CD, and the circle with diameter CD is tangent to the line AB. Prove that the two intersection points of these circles and the point $AC \cap BD$ are collinear.



Solution: Let X be the tangency point of CD with the first circle and Y the tangency point of AB with the second circle. Further, let P be the intersection of AC with BD. As we aim to use Pappus's theorem, we also introduce the points $Q = AX \cap DY$ and $R = BX \cap CY$.

We claim that $\triangle AYQ \sim \triangle DXQ \sim \triangle YBR \sim \triangle XCR$. Let $\alpha = \angle YAQ$ and $\beta = \angle QYA$. By the tangent chord theorem, $\angle RXC = \alpha$, and as $\angle AXB = 90°$, we have that $\angle DXQ = 90° - \alpha$. Similarly, by the tangent chord theorem, $\angle XCR = \beta$, and as $\angle CYD = 90°$, we have that $\angle QDX = 90° - \beta$. Observe that

$$180° - \alpha - \beta = \angle AQY = \angle XQD = 180° - (90° - \alpha) - (90° - \beta) = \alpha + \beta,$$

hence $\beta = 90° - \alpha$. The claim follows immediately.

It now follows from the similarities that Q and R are on the power line of the two circles, as $QA \cdot QX = QY \cdot QD$ and $RY \cdot RC = RB \cdot RX$. By Pappus's theorem, Q, P, and R are collinear, so P also lies on the power line. We are now done, as the line through the intersection points of the circles is always their power line.

Marking Scheme :

- 1P: Introducing $Q = AX \cap DY$ and $R = BX \cap CY$.
- 1P: Proving that $AYXD$ is cyclic, or that $\triangle AYQ \sim \triangle DXQ$.
- 1P: Proving that $YBCX$ is cyclic, or that $\triangle YBR \sim \triangle XCR$.
- 1P: Proving that Q is on the power line of the circles.
- 1P: Proving that R is on the power line of the circles.
- 1P: Using Pappus's theorem to prove that Q, P, and R are collinear.
- 1P: Conclude.

Remarks :

- *Proving all the angle equalities needed to show that $AYXD$ and $YBCX$ are cyclic is worth 1P, if neither of these observations are stated.*
- *Proving that $XYMN$ is cyclic, where M and N denote the midpoints of AB and CD, is worth 1P. It is not additive with any other observations needed to show that AYXD and YBCX are cyclic.*

3. A hunter and a rabbit are playing the following game on the cells of an infinite square grid. First, the hunter fixes a colouring of the cells using finitely many colours. After that, the rabbit secretly chooses a cell to start in. Each turn, the rabbit reports the colour of its current cell to the hunter and then secretly moves to an adjacent cell (sharing an edge) that has not been visited before.

The hunter wins if at any point of the game either:

- They can determine with certainty the current cell the rabbit is in.
- The rabbit cannot make a legal move.

Determine if the hunter has a winning strategy.

Solution: The hunter has a winning strategy. The basic idea of the hunter's strategy is to combine multiple different colourings $C_1, C_2, \ldots C_k$ at once (introducing a unique colour for all possible colour combinations), such that each of the colourings potentially gives him a useful piece of information which he can then combine to determine the rabbit's position.

Introduce a coordinate system such that each point with integer x- and y-coordinates is the midpoint of a square cell. We then assign the cell to the coordinates of its midpoint. Here is one example for such colourings that together reveal all information to the hunter.

- The colouring C_1 is obtained by tiling the infinite grid with 3×3 squares and colouring each square in the same way with 9 different colours.
- The colouring C_2 consists of vertical black stripes such that all pairs of consecutive stripes have unique distance. The rest of the cells are white.
- The colouring C_3 consists of horizontal black stripes with unique distance between any two consecutive stripes.
- The colouring C_4 consists of diagonal black stripes with unique distance between any two consecutive stripes.

Using colouring C_1, the hunter will always know whether the rabbit moved up, down, left or right. Colouring C_2 helps the hunter determine the rabbit's x-coordinate: If the rabbit ever hits two consecutive vertical black stripes, the hunter knows the distance of these stripes with the help of C_1 and since consecutive stripes have unique distance, he exactly knows their x-coordinates. Thus, he can keep track of the rabbit's x-coordinate from that point onwards. Similarly, C_3 allows the hunter to find the rabbit's y-coordinate, given that the rabbit hits two consecutive horizontal black stripes.

Hence, the only possible strategy for the rabbit could be to either not cross two consecutive horizontal or vertical black lines. Let's WLOG assume that it does not cross two consecutive horizontal black lines. This means that the rabbit's x-coordinate must be bounded. Since the rabbit can never visit the same cell twice, we conclude that the y-coordinate is unbounded (since otherwise, the rabbit can visit only finitely many cells). Hence, the rabbit will hit two consecutive horizontal stripes and the hunter will know its y-coordinate. Furthermore the rabbit must hit two consecutive diagonal stripes as well, meaning the hunter can uniquely determine the sum (or difference) of its x- and y-coordinates. This means the hunter has enough information to determine both coordinates, thus he wins.

Marking scheme: The following points are additive:

- 1P: Either stating or implicitly using the idea of merging together different colourings.
- 1P: Describing a colouring that allows the hunter to track the directions in which the rabbit moves.
- 2P: Describing a colouring that, given some viable assumptions, allows the hunter to determine one of the rabbit's coordinates (or information of similar value).
- 1P: Describing one or more such colourings and proving that regardless of the rabbit's movement, at least one of the rabbit's coordinates (or similar information) can be uniquely determined.
- 1P: Describing a set of colourings that together generate all information needed.
- 1P: Conclude

4. Given a (simple) graph G with $n \geq 2$ vertices $v_1, v_2, \ldots, v_n$ and $m \geq 1$ edges, Joël and Robert play the following game with m coins:

i) Joël first assigns to each vertex v_i a non-negative integer w_i such that $w_1 + \cdots + w_n = m$.

ii) Robert then chooses a (possibly empty) subset of edges, and for each edge chosen he places a coin on exactly one of its two endpoints, and then removes that edge from the graph. When he is done, the amount of coins on each vertex v_i should not be greater than w_i.

iii) Joël then does the same for all the remaining edges.

iv) Joël wins if the number of coins on each vertex v_i is equal to w_i.

Determine all graphs G for which Joël has a winning strategy.

Solution: We refer to Joël as A and Robert as B; furthermore, the condition of coin placement can just be paraphrased as directing the edges, with the in-degree remaining inferior to $w(i)$, which we will henceforth refer to as a function, for simplicity's sake.

The solution has two key parts: proving A wins on bipartite graphs, and proving B wins otherwise. The former is purely constructive so we just give the construction, and for the latter there are several ways to reason, so we will provide three separate proofs.

If G is bipartite, A simply takes the induced partition into two disconnected sets of vertices and then for one of the two sets, he assigns $w(i) = \deg(v_i)$ and for the other he assigns $w(i) = 0$. This forces how every single edge has to be directed, and so all of the subsequent edge directions are forced (they have to point away from the latter and towards the former set).

If G is not bipartite...

Solution 1 (David):

If G is not bipartite, we colour each vertex u black if $w(u) < \deg(u)$ and white otherwise. There now must be two adjacent vertices u, v of the same colour.

If u, v are both black, B can direct $w(u)$ edges towards u and $w(v)$ edges towards u without using the edge $\{u, v\}$. But then, no matter how A directs $\{v, w\}$, he will lose.

If v, w are both white, we note that no matter how the edge $\{v, w\}$ is directed, there is never equality at both of them.

Solution 2 (Tanish):

A graph is bipartite iff it contains no odd cycles; ergo, we may assume there is an odd cycle; WLOG call it $H = \{v_1v_2, v_2v_3, \ldots, v_{2k+1}v_1\}$. After A has chosen the values $w(1), \ldots, w(n)$, suppose there is indeed a way to direct the entire graph where these values are attained (otherwise B can just do nothing and win). Now, what B can do is take this valid complete directing and apply it to all the edges on $G \setminus H$. We are now left with just H and A's initial values $w(1), \ldots, w(2k+1)$ induce new values $w'(1), \ldots, w'(2k+1)$ such that $w'(1) + \cdots + w'(2k+1) = 2k+1$ once we "take away" what was already assigned. In other words, we can always reduce to the case where we just have an odd cycle, so we just need to solve this case. So now suppose we are just working on the odd cycle H.

For every group of consecutive indices $i, i+1, \ldots$ for all of whom $w' > 0$, we sum these up. In particular, there must be a sequence of consecutive nonzero values of w' whose sum is odd. We have a few possibilities:

- If the only occurrence of this is a single vertex of degree 1, then there must be vertex elsewhere of degree 2 next to a non-zero vertex, by pigeonhole.
- If there is a sequence whose sum is 3 or greater, then this must either contain a 2 next to a 1 or at least 3 1s in a row.

If there is a 2 next to another value > 0, then direct an edge away from the vertex corresponding to the 2. If there are 3 1s in a row, direct both edges away from the vertex corresponding to the middle 1. In both cases, A loses.

Note that there are several simpler ways to prove an odd cycle does not work, but these tend to be quite similar to the argument in Solution 3 and are usually true for bipartite graphs in general. We therefore show this slightly more convoluted but more natural argument.

Solution 3 (Joël):

Consider a vertex v. Let d_v be its degree and z_v be the number of neighbours u of v such that $w(u) = 0$. Now we claim that if $w(v) > z_v$ for some vertex v, then B wins. (Note that this is essentially an if and only if statement, but we only need one side of this implication).

To prove this, B can just assign the $d_v - z_v$ other edges away from v, implying that after A's turn, the in-degree at v is $\leq z_v$ which is itself smaller than $w(v)$, but we need equality between the two.

Furthermore, if two vertices where $w = 0$ are adjacent, then we are trivially done, as directing the edge between them immediately means A loses.

This now gives rise to a new upper bound for the sum of w. Consider a non-bipartite graph G and suppose A can win there. Now we know that $m = w(1) + \cdots + w(n) \leq z_1 + ... + z_n$ (by what we did above). Note however, that the RHS here is just the number of edges connecting a zero and a non-zero vertex. However, since the graph is not bipartite, not every edge can connect a zero and a non-zero vertex, thus we actually get $z_1 + \cdots + z_n < m$, which combined with what we have before gives $m < m$, which is clearly a contradiction.

Solution 4:

Similarly to the previous solution, we look at z_v, the number of neighbours u of v such that $w(u) = 0$; the proof that $w(v) > z_v$ for some vertex v implies victory for B is identical. Furthermore, $w(v) \geq z_v$ as every such edge to a vertex where $w = 0$ must be directed towards v, proving $w(v) = z_v$. This means for any vertex where $w > 0$, the only incoming edges are from the neighbours where $w = 0$. As a result, no edge can connect two vertices where $w = 0$ or two vertices where $w > 0$ and the partition of the vertex set into these two groups yields a bipartite graph.

Marking Scheme: The following points are additive:

- 1P: Claiming A wins for **only** bipartite graphs.
- 1P: Proving A wins for bipartite graphs.

Note that obtaining the second point does not necessarily imply you obtain the first, as the contestant must claim that A does not win on any other graphs to obtain the point.

The following points are additive for each solution category but non-additive between different categories:

Solution 1:

- 1P: Stating or using that if the graph is not bipartite, then any partition of the vertices into two groups will result in two vertices in a group being adjacent.
- 1P: Considering the two groups "$w(u) < \deg(u)$" and "$w(u) = \deg(u)$".
- 1P: Proving if two elements of the group "$w(u) < \deg(u)$" are adjacent, B wins.
- 2P: Proving if two elements of the group "$w(u) = \deg(u)$" are adjacent, B wins.

Although trivial, the contestant should either state "$w(u) > \deg(u)$" is impossible or use the group $w(u) \geq \deg(u)$, and ignoring this edge case is penalised by the deduction of 1 point.

Solution 2:

- 1P: Stating or using that if the graph is not bipartite, there is an odd cycle.
- 1P: Remarking that we can reduce to any subgraph of the initial graph.
- 1P: Reducing to an odd cycle.
- 2P: Solving the case where there is just an odd cycle.

If the contestant incorrectly assumes the graph not being bipartite implies there is a triangle, and solves this case, they should be awarded 3 points out of the 5 on offer.

Solution 3:

- 1P: Stating or using that if the graph is not bipartite, then any partition of the vertices into two groups will have strictly less than m edges connecting the two groups.
- 1P: Proving that $w(v) \leq z_v$ for all vertices v, where z_v is the number of neighbours u with $w(u) = 0$.
- 1P: Observing that the sum of the z_v is the number of edges connecting a zero and a nonzero vertex.
- 1P: Proving that $m \leq z_v$.
- 1P: Proving that $z_v < m$.

Solution 4:

- 1P: Proving that $w(v) \leq z_v$ for all vertices v, where z_v is the number of neighbours u with $w(u) = 0$.
- 1P: Proving that $w(v) \geq z_v$ for all vertices v, where z_v is the number of neighbours u with $w(u) = 0$.
- 1P: Proving that no two vertices where $w = 0$ are adjacent.

- 1P: Proving that no two vertices where $w > 0$ are adjacent.
- 1P: Concluding that the graph is bipartite.

5. Let a, b, c, λ be positive real numbers with $\lambda \geq 1/4$. Show that

$$\frac{a}{\sqrt{b^2+\lambda bc+c^2}}+\frac{b}{\sqrt{c^2+\lambda ca+a^2}}+\frac{c}{\sqrt{a^2+\lambda ab+b^2}} \geq \frac{3}{\sqrt{\lambda+2}}.$$

Solution: Denote the left side of the inequality by LS. By Hölder we have

$$\Big(a(b^2+\lambda bc+c^2)+b(c^2+\lambda ca+a^2)+c(a^2+\lambda ab+b^2)\Big)(LS)^2 \geq (a+b+c)^3.$$

So now it is sufficient to prove

$$\frac{(a+b+c)^3}{a^2b+ab^2+b^2c+bc^2+c^2a+ca^2+3\lambda abc} \geq \frac{9}{\lambda+2}$$

Now for easier notation write $\sum a^2b$ for $a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$. Multiplying by the denominators and cancelling terms on both sides results in the inequality

$$2(a^3+b^3+c^3)+12abc+\lambda\left(a^3+b^3+c^3-21abc+3\sum a^2b\right) \geq 3\sum a^2b.$$

Note that $a^3+b^3+c^3+3\sum a^2b-21abc \geq 0$ by AM-GM. Since the LS is a linear function in λ and the coefficient for λ is positive, the inequality is stricter for smaller λ. So in other words we can now assume $\lambda = 1/4$. Multiplying by 4/9 and rearranging once again we arrive at the final inequality:

$$(a^3+b^3+c^3)+3abc-\sum a^2b \geq 0$$

and this inequality is true by Schur.

Marking Scheme (additive)

- 2P: Using Hölder or equivalent in a sensible way.
- 1P: Rearranging the inequality to be linear in λ.
- 2P: Reduce it to $\lambda = 1/4$.
- 2P: Apply Schur.

6. Let $n \geq 2$ be an integer. Prove that if

$$\frac{n^2 + 4^n + 7^n}{n}$$

is an integer, then it is divisible by 11.

Solution (Valentin): The "Lifting the Exponent" lemma tells us that

$$v_{11}(4^n + 7^n) = v_{11}(4 + 7) + v_{11}(n) = 1 + v_{11}(n)$$

and we therefore have $11 \mid (4^n + 7^n)/n$. Now let p denote the smallest prime factor of n. By assumption we have $p \mid n \mid 4^n + 7^n$, with $p \neq 2, 7$. Now setting $x \equiv 4 \cdot 7^{-1} \mod p$, we find:

$$4^n \equiv -7^n \mod p \implies x^n \equiv -1 \mod p \implies x^{2n} \equiv 1 \mod p$$

If d denotes the order of x mod p, the above implies $d \mid 2n$. By Euler-Fermat we also have $d \mid \varphi(p) = p - 1$. By minimality of p, we have $\gcd(n, p - 1) = 1$, and thus $d \mid 2$. Therefore,

$$x^2 \equiv 1 \mod p \implies 4^2 \equiv 7^2 \mod p \implies p \mid 49 - 16 = 33 \implies p = 3, 11.$$

But since $4^n \equiv 7^n \equiv 1 \mod 3$, we cannot have $p = 3 \mid n$, since otherwise n does not divide $4^n + 7^n$. We conclude that $p = 11 \mid n$ and the result follows.

Marking Scheme (additive)

- 1P: Showing that $11 \mid (4^n + 7^n)/n$.
- 1P: Considering d, the order of $4/7$, mod p for some prime $p \mid n$.
- 1P: Showing that $d \mid 2n$ and $d \mid p - 1$.
- 2P: Showing that $d \mid 2$.
- 1P: Showing that 3 does not divide n.
- 1P: Showing that $11 \mid n$.

7. Let n be a positive integer. Find all polynomials P with real coefficients such that

$$P(x^2 + x - n^2) = P(x)^2 + P(x)$$

for all real numbers x.

Answer: The only solution is the constant zero polynomial $P \equiv 0$.

Solution 1 (Linus): We quickly check that $P \equiv 0$ is indeed a solution. So from here on out let's assume $P \not\equiv 0$. Plugging in $x = n$, we get that $P(n) = 0$, from which it follows that $P(x) = (x - n)^r Q(x)$, for some polynomial $Q \in \mathbb{R}[x]$ with $Q(n) \neq 0$. We now have

$$(x^2 + x - n - n^2)^r Q(x^2 + x - n^2) = (x - n)^{2r} Q(x) + (x - n)^r Q(x).$$

Factoring out and dividing by $(x - n)^r$, we get

$$(x + n + 1)^r Q(x^2 + x - n^2) = (x - n)^r Q(x) + Q(x).$$

But if we plug in $x = n$, we get $0 = Q(n)$, a contradiction. So $P \equiv 0$ is the only solution.

Solution 2 (David): Plugging in $x = -n$ yields $P(-n) = 0$. Next, $x = n - 1$ gives us $0 = P(-n) = P(n-1)^2 + P(n-1)$ and so $P(n-1) \in \{-1, 0\}$. Note that the quadratic equation $x^2 + x - n^2 = n - 1$ certainly has a positive solution x_1, namely

$$x_1 = \frac{-1 + \sqrt{1 + 4(n^2 + n - 1)}}{2} = \frac{\sqrt{4n^2 + 4(n-1) + 1} - 1}{2}.$$

We also observe that $n - 1 < x_1 < n$. If $P(n-1) = -1$, plugging $x = x_1$ into the original equation would yield $-1 = P(n-1) = P(x_1)^2 + P(x_1)$. However, $y^2 + y \geq -1/4$ for all $y \in \mathbb{R}$, contradiction! We conclude that $P(n-1) = 0$ and the equation becomes $0 = P(x_1)^2 + P(x_1)$. Similarly to above, $P(x_1) \in \{-1, 0\}$. This is basically the situation that we faced at the beginning, except that we replaced $n - 1$ by x_1. Hence we try to iterate the argument:

Again, the equation $x^2 + x - n^2 = x_1$ certainly has a solution, since $x_1 > 0$. Let

$$x_2 = \frac{\sqrt{4n^2 + 4x_1 + 1} - 1}{2}$$

be the positive solution. Since $x_1 < n$, we easily get $x_2 < n$ as well. Furthermore,

$$4x_2^2 + 4x_2 + 1 = (2x_2 + 1)^2 = 4n^2 + 4x_1 + 1 \;\Rightarrow\; x_2 > x_1,$$

since if $x_1 \geq x_2$, the right hand side would be strictly larger than the left hand side (as $x_2 < n$). Plugging $x = x_2$ into the original equation yields $P(x_1) = P(x_2)^2 + P(x_2)$. By the same reasoning as above, we conclude that $P(x_1) = -1$ is impossible and hence $P(x_1) = 0$ and $P(x_2) \in \{-1, 0\}$. It is easy to see that we can iterate this argument by replacing x_1 by x_k and x_2 by x_{k+1} in the k-th step. We obtain an arbitrarily long sequence $n - 1 < x_1 < x_2 < \ldots < n$ of distinct zeros of P. Since any non-zero polynomial only has a finite number of roots, we conclude that P must be the constant zero-polynomial. This is indeed a solution.

Solution 3 (Raphi, inspired by David): Let $f(x) = x^2 + x - n^2$ and $g(x) = x^2 + x$. The equation can then be rewritten as

$$P(f(x)) = g(P(x)).$$

Now note that $f(\mathbb{R}) = [-n^2 - \frac{1}{4}, \infty)$ and $g(\mathbb{R}) = [-\frac{1}{4}, \infty)$. So from this we get

$$P(y) \geq -\frac{1}{4}, \text{ for } y \in [-n^2 - \frac{1}{4}, \infty). \tag{1}$$

So if $x \geq -n^2 - \frac{1}{2}$ and $f(x)$ is a root of P, we have that $0 = g(P(x))$, so $P(x) = -1$ or $P(x) = 0$. But because $P(x) \geq -\frac{1}{2}$, $P(x)$ can't be -1 and x is a root of P.

First we solve the case $n = 1$ and then we solve the case $n \geq 2$. Plugging in $n = -1$, we get that $P(-1) = 0$. Now since $f(0) = -1$, it follows from the argument above, that $P(0) = 0$. Similarly, since $f(\frac{1+\sqrt(5)}{2}) = 0$, $\frac{1+\sqrt(5)}{2}$ is also a root of P. Now note that $f(x) > x$, for $x > 1$. such that $a_{2i} \in [-\frac{1}{2}, n)$ and $a_{2i-1} \in [-n-1, -\frac{1}{2}]$

Now by plugging in $x = -n$, we find that $P(-n) = 0$. We will now inductively construct a series $(a_i)_{i \in \mathbb{N}}$ consisting of pairwise distinct roots of P, such that $a_i \geq -n^2 - \frac{1}{4}$. If we succeed to construct such a series, it'll follow immediately, that $P \equiv 0$ is indeed the only solution.
To do that define

$$f_- : (-\infty, -\frac{1}{2}] \to [-n^2 - \frac{1}{4}, \infty) \qquad f_+ : [-\frac{1}{2}, \infty) \to [-n^2 - \frac{1}{4}, \infty)$$
$$x \mapsto f(x) \qquad\qquad x \mapsto f(x).$$

So in other words f_- and f_+ are the restrictions of f to $(-\infty, -\frac{1}{2}]$ and $[-\frac{1}{2}, \infty)$ respectively. Note that f_-/f_+ are strictly decreasing/increasing and thereby bijective.
First we solve the case $n = 1$ and then we solve the case $n \geq 2$. Plugging in $n = 1$, we get that $P(1) = 0$. Now since $0 = f_+^{-1}(1)$, it follows from the argument above, that $P(0) = 0$. Similarly, since $\frac{1+\sqrt(5)}{2} = f_+^{-1}(0)$, $\frac{1+\sqrt(5)}{2}$ is also a root of P. Now note that $f(x) > x$, for $x > 1$. Define $a_1 := \frac{1+\sqrt(5)}{2}$ and then $a_{k+1} = f_+^{-1}(a_k), \forall k \geq 2$. It is easy to check, that all the a_k are well defined and distinct roots of P, finishing the case $n = 1$.
Ok, now to the the case $n \geq 2$. Here we set a further constraint on the series $(a_i)_{i \in \mathbb{N}}$. Namely $a_{2i} \in [-n-1, -\frac{1}{2}]$ and $a_{2i-1} \in [-\frac{1}{2}, n]$ for all $i \in \mathbb{N}$.
Set $a_1 := n$, which is indeed a root of P and in the interval $[-\frac{1}{2}, n]$.
Now assume we already have $a_1, \ldots, a_k$. If k is odd define $a_{k+1} := f_+^{-1}(a_k)$, else set $a_{k+1} := f_-^{-1}(a_k)$. So let's check that a_{k+1} is well defined, lies in $[-n-1, -\frac{1}{2}]$ for k even, lies in $[-\frac{1}{2}, n]$ for k odd and is a root of P.
k even:

- a_{k+1} is well defined, as f_- is bijective and $a_k \in [-\frac{1}{2}, n] \subset [-n^2 - \frac{1}{4}, \infty)$.
- Using the strict monotony of f_-, one can find that $f_-^{-1}([-\frac{1}{2}, n]) = [-n-1, \frac{-1-\sqrt{4n^2-1}}{2}] \subset [-n-1, -\frac{1}{2}]$. So by construction of a_{k+1}, it lies in the interval $[-n-1, -\frac{1}{2}]$
- We have that $g(P(a_{k+1})) = P(f(a_{k+1})) = P(a_k) = 0$. So $P(a_{k+1})$ is a root of $g(x)$ and thereby equal to -1 or 0. But now $a_{k+1} \in [-n-1, -\frac{1}{2}] \subset [-n^2 - \frac{1}{4}, \infty)$, as $n \geq 2$, forcing $P(a_{k+1}) = 0$.

k odd

- a_{k+1} is well defined, as f_+ is bijective and $a_k \in [-n-1, -\frac{1}{2}] \subset [-n^2 - \frac{1}{4}, \infty)$, as $n \geq 2$.
- Using the strict monotony of f_+, one can find that $f_+^{-1}([-n-1, -\frac{1}{2}]) = [\frac{-1+\sqrt{4n^2-4n-3}}{2}, \frac{-1+\sqrt{4n^2-1}}{2}] \subset [-\frac{1}{2}, n]$. So by construction of a_{k+1}, it lies in the interval $[-\frac{1}{2}, n]$

- We have that $g(P(a_{k+1})) = P(f(a_{k+1})) = P(a_k) = 0$. So $P(a_{k+1})$ is a root of $g(x)$ and thereby equal to -1 or 0. But now $a_{k+1} \in [-\frac{1}{2}, n] \subset [-n^2 - \frac{1}{4}, \infty)$, forcing $P(a_{k+1}) = 0$.

Marking Scheme solution 1 (additive)

- 1P: asserting that $P \equiv 0$ is a solution
- 1P: showing that n is a root of P
- 1P: introducing the polynomial $Q(x) = P(x)/(x-n)^r$ for some $r \in \mathbb{N}$ and inserting it in the equation
- 2P: showing that $Q(n) = 0$
- 1P: choosing r maximal or repeatedly dividing P by $(x-n)$ till P is not divisible by $(x-n)$ anymore
- 1P: deriving a contradiction to the maximality of r or the fact that roots of non-zero polynomials have finite multiplicity

Marking Scheme solution 2 (additive)

- 1P: asserting that $P \equiv 0$ is a solution
- 1P: $P(n-1) \in \{-1, 0\}$
- 1P: note that $\mathrm{Im}(P) \subset [-\frac{1}{4}, \infty)$
- 1P: $P(x_0) \in \{-1, 0\}$, where x_0 is a real root of $x^2 + x - n^2 = n - 1$
- 1P: $n - 1 < x_0 < n$
- 1P: $x^2 + x - n^2 = x_{k-1}$ has a real solution x_k
- 1P: $x_{k-1} < x_k < n$ and conclude

Marking Scheme solution 3 (additive)

- 1P: asserting that $P \equiv 0$ is a solution
- 2P: proving $P \equiv 0$ for $n = 1$
- 1P: having the idea of constructing infinitely many roots of P
- 1P: constructing an infinite sequence of roots of P
- 2P: proving that this sequence contains infinitely many different terms

No points will be given for

- proving that either $P \equiv 0$ or all the roots lie in $[-n-1, n]$
- constructing a finite amount of roots by hand

8. Johann and Nicole are playing a game on the coordinate plane. First, Johann draws any polygon $\mathcal{S}$ and then Nicole can shift $\mathcal{S}$ to wherever she wants. Johann wins if there exists a point with coordinates (x, y) in the interior $\mathcal{S}$, where x and y are coprime integers. Otherwise, Nicole wins. Determine who has a winning strategy.

Answer: Nicole always has a winning strategy.

Solution (Valentin): Let S be any shape picked by Johann. Since S is bounded, there is some $n \in \mathbb{N}$ such that S fits inside a $n \times n$ square. It thus suffices to find a pair of integers x, y such that $x+i$ and $y+j$ are not coprime for all $0 \leq i, j \leq n$, allowing Nicole to place the shape inside the square with diagonally opposite vertices (x, y) and $(x+n, y+n)$. Let p_{ij} be distinct primes for $0 \leq i, j \leq n$. By the Chinese Remainder Theorem, there exist integers x, y such that

$$x + i \equiv 0 \mod \prod_{j=0}^{n} p_{ij} \quad \text{for all } 0 \leq i \leq n,$$

$$y + j \equiv 0 \mod \prod_{i=0}^{n} p_{ij} \quad \text{for all } 0 \leq j \leq n.$$

We conclude that p_{ij} divides $x+i$ and $y+j$ for all $0 \leq i, j \leq n$, as desired.

Marking Scheme (additive):

- 1P: Noting that the specific polygon does not matter
- 2P: Reducing to the existence of integers satisfying certain (explicit) properties. 1P may be awarded if the reduction is flawed.
- 1P: Stating the intention to use CRT
- 1P: Any applications of CRT
- 1P: Introducing at least order N coprime integers, where N grows as the order of the area of the polygon.
- 1P: Finishing

9. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that TE is parallel to CD and TF is parallel to AD. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Solution 1: Introduce the points $P = AC \cap TE$ and $Q = AC \cap TF$. As PE is parallel to CD and ED is tangent to the circumcircle of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D, and E lie on a circle. Similarly, the points C, Q, D, and F lie on some other circle.

We now claim that TD is tangent to $(APDE)$ and $(CQFD)$. Note first that these circles are tangent to each other, as $\angle EAD = \angle BCD = 180° - \angle DCF$. To prove the claim, we only need to check that T lies on the power line. However, $\angle DEP = \angle FDC = \angle FQC$ proves that $PQFE$ is cyclic, hence $TP \cdot TE = TQ \cdot TF$, proving the claim.

Using both tangencies, we have

$$180° - \angle TPD = \angle EPD = \angle TDK = \angle TKD$$

and

$$\angle TQD = 180° - \angle FQD = 180° - \angle TDE = \angle TDK = \angle TKD,$$

proving that T, P, Q, D, and K lie on a circle.

We can now prove that BD is parallel to TK as

$$\angle BDE = \angle BCD = \angle TQD = \angle TKD,$$

and BC is parallel to PK as

$$\angle BFE = \angle PQD = \angle PKD.$$

Finally, as $\triangle BCD$ and $\triangle KPT$ have parallel sides, the center of the homothety mapping a triangle to the other has to lie on PC, DT, and BK, proving that these lines intersect at one point.

Solution 2: Introduce the points $P = AC \cap TE$ and $Q = AC \cap TF$. As PE is parallel to CD and ED is tangent to the circumcircle of $ABCD$, we have

$$\angle EPA = \angle DCA = \angle EDA,$$

and so the points A, P, D, and E lie on a circle. Similarly, the points C, Q, D, and F lie on some other circle. Using all three circles available, we have

$$\angle DPE = \angle DAE = \angle DCB = \angle DQT,$$

so we find a fourth circle τ passing through T, P, D and Q.

Note that all the sides of $\triangle TPQ$ and $\triangle DCA$ are parallel, so there exists a homothety h mapping $\triangle TPQ$ to $\triangle CDA$, with center $X = AC \cap TD$. Also note that $h(\tau) = \Omega$, so the point $D' = h(D)$ lies on Ω. Finally, by

$$\angle DCD' = \angle TPD = \angle BAD,$$

the points B and D' are symmetric with respect to the diameter of Ω passing through D. This yields $DB = DD'$, so $h(K) = B$. Hence X lies on BK, finishing the proof.

Marking Scheme Solution 1 (additive):

- 1P: Introduce P (or Q) and show that $APDE$ (or $CQFD$) is cyclic.
- 1P: Show that TD is tangent to $(APDE)$ and $(CQFD)$.
- 1P: Show that P, Q, D, and K lie on a circle.
- 1P: Show that BD is parallel to TK.
- 1P: Show that PK is parallel to BC.
- 1P: Claim that $\triangle BCD$ (resp. $\triangle BAD$) and $\triangle KPT$ (resp. $\triangle KQT$) have parallel sides.
- 1P: Conclude.

Marking Scheme Solution 2 (additive):

- 1P: Introduce P (or Q) and show that $APDE$ (or $CQFD$) is cyclic.
- 1P: Show that T, P, Q, and D lie on a circle.
- 1P: Introduce the homothety mapping $\triangle TPQ$ to $\triangle CDA$.
- 1P: Introduce $D' = h(D)$. The point is not given if one only introduces $D' = DT \cap \Omega$.
- 1P: Show that $DD' = DB$.
- 1P: Show that $h(K) = B$.
- 1P: Conclude.

10. Let $ABCD$ be a parallelogram such that $AC = BC$. A point P is chosen on the extension of the segment AB beyond B. The circumcircle of the triangle ACD meets the segment PD again at Q, and the circumcircle of the triangle APQ meets the segment PC again at R. Prove that the lines CD, AQ, and BR intersect at one point.

Solution (Johann): Define $X = CD \cap BR$ and let $\angle ABC = \alpha$ and $\angle APC = \beta$. Let's prove that A, Q, X are collinear.
First, observe that $ABRC$ is a cyclic quadrilateral. Indeed, this is revealed with a bit of angle chasing:

$$\angle ABC = \angle ADC = \angle ACD = \angle AQD = 180° - \angle AQP = 180° - \angle ARP = \angle ARC.$$

Secondly, note that $QRXC$ is a cyclic quadrilateral. Indeed, this once again follows by some angle chasing. Using $AC = CB$ and that $ABRC$ is cyclic, we get that

$$\angle CRX = 180° - \angle CRB = \angle CAB = \angle ABC = \alpha.$$

Using parallel lines, $\angle XCR = \angle XCP = \angle CPB = \beta$. Therefore, it follows that $\angle CXR = 180° - \angle CRX - \angle XCR = 180° - \alpha - \beta$.
Now, using that $\angle AQC = 180° - \alpha$ and $\angle AQP = 180° - \alpha$, we get

$$\angle CQP = 360° - \angle AQC - \angle AQP = 360° - (180° - \alpha) - (180° - \alpha) = 2\alpha.$$

We are now interested in calculating $\angle CQR$. Observe that, using $\angle ARB = \angle ACB = 180° - 2\alpha$ (since $AC = BC$), we get

$$\angle ARP = \angle ARB + \angle BRP = \angle ACB + \angle CRX = (180° - 2\alpha) + \alpha = 180° - \alpha.$$

Using the previous equality yields

$$\angle RQP = \angle RQP = 180° - \angle APR - \angle ARP = 180° - \beta - (180° - \alpha) = \alpha - \beta.$$

Hence, we obtain $\angle CQR = \angle CQP - \angle RQP = 2\alpha - (\alpha - \beta) = \alpha + \beta$, and this implies $\angle CQR + \angle CXR = (\alpha + \beta) + (180° - \alpha - \beta) = 180°$.

Finally, using that $QRXC$ is cyclic, we reach that

$$\angle AQX = \angle AQC + \angle CQX = \angle AQC + \angle CRX = (180° - \alpha) + \alpha = 180°$$

giving us the required result.

Note that one can alternatively define $X = CD \cap AQ$ or $X = AQ \cap BR$. The sketch of the proof will be the exact same as presented above.

Solution 2: Define X as above and Y as the intersection of DA and RB. We still want to show that X, Q and A are collinear, for this it suffices to show that X lies on the power line of the circles $AQCD$ and $APRQ$. In the same way as before we can show that $ABRC$ is cyclic.
Now we show that the triangles RXC and DXY are similar. This is another angle chase:

$$\angle XRC = 180° - \angle CRB = \angle BAC = \angle CBA = \angle CDA,$$

where we used that $ABRC$ is cyclic, $CA = CB$ and that $ABCD$ is a parallelogram. Now we have two common angles and therefore the triangles are similar.
The next angle chase shows that Y lies on $APRQ$:

$$\angle RPA = 180^\circ - \angle PRB - \angle RBP = 180^\circ - \angle CRX - \angle CXR = \angle RCX = \angle DYX = \angle AYR,$$

where we used parallelity of PB and CX as well as the similar triangles.
The similarity also implies $XR/XD = XC/XY$, or $XR \cdot XY = XC \cdot XD$. This means that X has the same power with respect to the two circles $AQCD$ and $APRQ$ and we are done.

Marking Scheme Solution 1 (additive):

- 3P: Prove that $ABRC$ is a cyclic quadrilateral
- 3P: Prove that $CQRX$ is a cyclic quadrilateral
- 1P: Conclude.

Marking Scheme Solution 2 (additive):

- 1P: Introduce the point Y, the intersection of DA and RB.
- 1P: Prove that Y lies on $APRQ$.
- 2P: Prove that $ABRC$ is a cyclic quadrilateral
- 2P: Prove that RXC and DXY are similar.
- 1P: Conclude.

11. Let $n \geq 2$ be an integer. Each of the squares of an $n \times n$ board contains a bit-coin with 0 on one side, 1 on the other. Initially, all bit-coins in the leftmost column show a 0. A move consists of one of the following:

- Within any row, look at the rightmost two neighbouring bit-coins that display different numbers (if they exist) and flip both of these as well as all bit-coins to their right.
- Within any column, look at the topmost two neighbouring bit-coins that display different numbers (if they exist) and flip both of these as well as all bit-coins above them.

Find the minimal value of k such that there always exists a sequence of moves resulting in at most k bit-coins showing 1.

Answer: $k = \max\{n, 2n-3\}$.

Solution (David): The key idea is the following: We can introduce a weight for each square cell, such that the *weighted* sum S of all displayed numbers is invariant under all possible moves. Here's an example for $n = 6$:

16	8	4	2	1	1
16	8	4	2	1	1
32	16	8	4	2	2
64	32	16	8	4	4
128	64	32	16	8	8
256	128	64	32	16	16

In the special case $n = 2$, we immediately see that the minimal value of k is 2. So from now on, let $n \geq 3$.

A quick calculation shows that on the $n \times n$ board, the largest weight (in the bottom left square) is 2^{2n-4} and the sum of all weights but the ones in the leftmost column is equal to 2^{2n-3}. It's very easy to see that if all but the leftmost bit-coins show a 1, we can perform n moves to flip all the coins on the board and end up with a total value of n. This is not the worst case, as we will see. So from now on we ignore this case and assume that less than $n(n-1)$ coins initially display a 1.

In this case, the value of our invariant S is strictly smaller than 2^{2n-3}. In particular, we can write S as a binary number, using at most $2n-3$ digits.

Lemma 1: There must always be at least m bit-coins showing a 1 where m is the number of ones in the binary representation of S.

Proof of Lemma 1: We do induction on m. If $m = 1$, this implies $S > 0$, so we obviously need at least one coin displaying a 1. Now let $m \geq 2$. Write

$$S = \sum_{i=1}^{m} 2^{a_i}, \text{ where } 0 \leq a_1 < \ldots < a_m \leq 2n - 4.$$

Since all weights are powers of two, there exists one or more coins showing 1 such that their weights add up to 2^{a_m} (we always choose a coin with the largest weight and repeat this until the weights of the chosen coins add up to at least 2^{a_m}. By construction, they have to add up to exactly 2^{a_m}). We flip these coins to 0 (for the sake of this proof), note that the new value of S now has $m - 1$ digits equal to 1 and apply the induction hypothesis to see that at least $m - 1$ bit-coins show a 1. This means that at the start, at least m of the coins displayed a 1. $\square$

Lemma 1 now gives us an easy lower bound for the value of k: If initially all coins but the left row and the top right corner show a 1, the value of S will be $2^{2n-3} - 1 = 1 + 2 + \ldots + 2^{2n-4}$, therefore $m = 2n - 3$ and we will always need to have at least $2n - 3$ bit-coins display a 1, hence $k \geq 2n - 3$ in this case.

To prove that $k \leq 2n - 3$, we define the *target* set $\mathcal{T}$ as the union of cells in the leftmost column and the cells in the topmost row, excluding the top left and top right corner squares. We note that $|\mathcal{T}| = 2n - 3$ and that for each $0 \leq i \leq 2n - 4$, $\mathcal{T}$ contains exactly one cell with weight 2^i. Our goal is to perform moves such that the only bit-coins displaying 1 are in the target set $\mathcal{T}$.

16	8	4	2	1	1
16	8	4	2	1	1
32	16	8	4	2	2
64	32	16	8	4	4
128	64	32	16	8	8
256	128	64	32	16	16

Lemma 2: As long as not all bit-coins displaying 1 are in $\mathcal{T}$, there exists a sequence of moves which strictly increases $S_{\mathcal{T}}$, the weighted sum of bit-coins in $\mathcal{T}$.

Proof of Lemma 2:

a) If the bit-coin in the top-left corner displays a 1, there must be at least one bit-coin in the leftmost column displaying a 0, otherwise S would have to be at least 2^{2n-3}, which is not true (It could technically be equal to 2^{2n-3}, but we already excluded that case). Hence we can apply the second move to the leftmost column, resulting in an increase of $S_{\mathcal{T}}$ by 2^{n-2}.

b) Now assume that the top-left coin displays a 0. If the bit-coin in the top-right displays a 1, we can apply the first move in the top row, increasing either $S_{\mathcal{T}}$ by 1 or flipping the top-left coin to 1 and all the other coins in the top-row to 0, which would decrease $S_{\mathcal{T}}$ by $2^{n-2}-1$. But then, we can apply a) to increase $S_{\mathcal{T}}$ by 2^{n-2}, resulting in a net increase of $S_{\mathcal{T}}$ by 1.

c) If the coins in the top-left and top-right corner both show 0, there is a bit-coin somewhere in the lower right $(n-1)\times(n-1)$ board displaying 1. We can assume such a bit-coin to be in the rightmost column, otherwise we just apply the first move on the row of that bit-coin. Now, since the top-right coin shows 0, we can apply the second move in the rightmost column, resulting in the top-right coin showing 1. By doing the same moves as in b), we increase $S_{\mathcal{T}}$ by 1. □

Lemma 2 implies that after finitely many moves, all bit-coins showing 1 can be moved to the target set $\mathcal{T}$. Since $|\mathcal{T}| = 2n-3$, we conclude that $k \leq 2n-3$, which finishes the proof.

Marking Scheme: The following points are additive

- 1P: Stating the answer $k = 2n-3$ for $n \geq 3$.
- 2P: Constructing the invariant
- 1P: Proving the lower bound via construction.
- 1P: Describing the idea to move all bit-coins showing 1 into a suitable set of squares
- 2P: Proving that this can be achieved.

Remark: The following minor mistakes will result in a deduction of one point. These deductions are however not cumulative:

- *Forgetting to treat $n = 2$ separately,*
- *Forgetting to treat the case of $n(n-1)$ bit-coins showing 1,*
- *Minor flaws in the reasoning of upper/lower bound*

12. Let $\mathbb{R}_{>0}$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$x + f(yf(x) + 1) = xf(x + y) + yf(yf(x))$$

for all positive real numbers x and y.

Answer: The only solution is the function $f(x) = 1/x$.

Solution (Johann): Let f be a solution to the FE. By plugging $y = \frac{x}{f(x)}$, we obtain

$$x + f(x+1) = xf\left(x + \frac{x}{f(x)}\right) + x \iff f(x+1) = xf\left(x + \frac{x}{f(x)}\right). \tag{1}$$

Plug in $x = 1$ in (1) to get

$$f(2) = f\left(1 + \frac{1}{f(1)}\right). \tag{2}$$

Note that if f is injective, one can easily finish as $f(1) = 1$ by the previous equality. By plugging in $x = 1$ in the original equation, we get

$$1 = yf(y) \iff f(y) = \frac{1}{y}, \quad \forall y \in \mathbb{R}_{>0}.$$

which is indeed a solution to the equation:

$$x + f\left(\frac{y}{x} + 1\right) = \frac{x}{x+y} + yf\left(\frac{y}{x}\right) \iff x + \frac{x}{x+y} = \frac{x}{x+y} + x, \quad \forall x, y \in \mathbb{R}_{>0}.$$

Now, it remains to prove that f is injective: Assume that there exist $u, v \in \mathbb{R}_{>0}$, with $u > v$ such that $f(u) = f(v)$. Rewriting the initial equation, we get:

$$x \cdot (1 - f(x+y)) = yf(yf(x)) - f(yf(x) + 1). \tag{3}$$

Substituting $x = u$ and $x = v$ in the latter equation, we get that

$$u \cdot (f(u+y) - 1) = v \cdot (f(v+y) - 1), \quad \forall y \in \mathbb{R}_{>0}. \tag{4}$$

and by replacing $y \to y - v$ and introducing $C = u - v > 0$, we get

$$u \cdot (f(C+y) - 1) = v \cdot (f(y) - 1), \quad \forall y > v. \tag{5}$$

Let y_0 be a fixed constant such that $v < y_0 \leq v + C$ and let's analyse the sequence $(f(y_0 + nC))_{n\in\mathbb{N}}$. If $f(y_0) = 1$, we get that the sequence $(f(y_0 + nC))_{n\in\mathbb{N}}$ is constant and every term is equal to 1.

Now, if $f(y_0) \neq 1$, by (5), observe that

$$\frac{u}{v} = \frac{f(y_0 + nC) - 1}{f(y_0 + (n+1)C) - 1} \quad \forall n \in \mathbb{N}$$

Therefore, by induction or telescopic product, we get

$$f(y_0 + nC) - 1 = \left(\frac{v}{u}\right)^n \cdot (f(y_0) - 1) \tag{6}$$

Hence, by making $n \to \infty$ and using $v < u$, we get that the sequence $(f(y_0 + nC))_{n\in\mathbb{N}}$ converges to 1. From these two cases, we conclude that the sequence $(f(y_0 + nC))_{n\in\mathbb{N}}$ converges to 1 for any $v < y_0$. Now, let $a_n = y_0 + nC$ and let's make the substitution $y = \frac{a_n}{f(x)}$. We get

$$x + f(a_n + 1) = xf\left(x + \frac{a_n}{f(x)}\right) + \frac{a_n}{f(x)} \cdot f(a_n) \tag{7}$$

and let's fix $x = c$. Then, when

$$n \to +\infty, \quad c + f(a_n + 1) \to c + 1$$

because the sequence

$$(f((y_0 + 1) + nC))_{n\in\mathbb{N}} = (f(a_n + 1))_{n\in\mathbb{N}}$$

converges to 1. However, we have

$$cf\left(c + \frac{a_n}{f(c)}\right) + \frac{a_n}{f(c)} \cdot f(a_n) \to +\infty$$

because

$$\frac{a_n}{f(c)} \cdot f(a_n) \to +\infty \text{ and } cf\left(c + \frac{a_n}{f(c)}\right) > 0.$$

Thus, we get a contradiction by (7).

Marking Scheme:

- 1P: Show that $f(x) = 1/x$ is a solution
- 1P: Find $f(2) = f\left(1 + \frac{1}{f(1)}\right)$
- 4P: prove injectivity
  - 2P: prove that the sequence $(f(y_0 + nC))_{n\in\mathbb{N}}$ converges to 1 for any $y_0 > v$
  - 2P: obtain a contradiction
- 1P: finishing assuming injectivity