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Mock exam 2019

Mock exam

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. Determine all pairs of positive integers (m, n) such that there exist infinitely many positive integers k such that $k^2 + k(m+n) + mn$ is a perfect square.

2. Let S be a subset of $[0,1]$ consisting of 2019 disjoint intervals $[a_1, b_1], \ldots, [a_{2019}, b_{2019}]$. Assume that for all $d \in [0,1]$ there exist $x, y \in S$ such that $|x-y| = d$ holds. Determine the smallest value that the expression

$$\sum_{k=1}^{2019} (b_k - a_k)$$

can attain.

Remark: An interval $[a,b]$ *is the set of all real numbers* x *such that* $a \leq x \leq b$.

3. Quirin finds himself tied to a wall of a peculiar room. The walls of that room are mirrors and form an acute triangle. To free himself, Quirin shoots a laser beam from his eyes. The beam hits each of the two other walls exactly once and returns to Quirin. If he had not been in the way, the laser beam would have moved along the same path as directly after his shot. Determine all possible positions of Quirin.

Remark: We interpret Quirin as a single point on the edge of the triangle and assume that the laser beam doesn't hit the vertices

4. Let n be a positive integer and $a_0, \ldots, a_n$ be a sequence of numbers with $a_0 = \frac{1}{2}$ and $a_{k+1} = a_k + \frac{a_k^2}{n}$. Prove that

$$\frac{n}{n+1} < a_n < 1.$$

Good Luck!