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Selection 2020 - Solution

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Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard with the usual chessboard colouring. A move consists of choosing a 1×1 square and switching the colour of all squares in its row and column (including the chosen square itself). For which n is it possible to get a monochrome chessboard after a finite sequence of moves?

Answer: All even numbers $n = 2k$ are solution.

Solution 1 (Ivan, David, Arnaud)
We will prove that the task is possible for even n and impossible for odd n.
For $n = 2k$, consider all squares that are initially black. We claim that if we make a move on all of those squares, the chessboard will be completely white in the end. Obviously, the order of moves does not matter, so we only have to count the number of times a given 1×1-square has changed its colour. Since a square changes its colour if and only if a move is played on a square in the same row or column, we can simply count the number of initially black squares that share a row or column with a given square.

- Initially, each black square has $k-1$ other black squares in its row and $k-1$ other black squares in its column. Since the square itself is black, the total number of black squares that share a row or column with the given square is $(k-1)+(k-1)+1 = 2k-1$. This means that any black square will change colour $2k-1$ times in total, and since this number is odd, the square will be white in the end.
- Analogously, every initially white square has k initially black squares in its row and k black squares in its column. This means that it changes its colour $2k$ times in total, so it will still be white in the end.

To show that the task is impossible for odd n, we simply consider the first two rows. Initially, there are n black and n white squares in the first two rows. However, any move will change the colour of either $n+1$ or 2 of those squares ($n+1$ if the chosen square is in the first two rows and 2 if it is not). So in any case, the number of squares in the first two rows that change their colour is even. This implies that the parity of the number of black squares in the first two rows will never change. In particular, it will never be possible to reach 0 or $2n$ black squares in the first two rows, so the chessboard will never be monochrome.

Solution 2 (Julia and Tanish)
For n even, proceed as above. For n odd, suppose WLOG that the corner squares are black, and we are aiming to get to a fully black board (this is equivalent to reaching a white board since there is a path between the two, notably by selecting every square once). We note that every move changes the parity of the total number of black squares, as we are flipping an odd number of squares in total (and hence a different number of black and white squares, mod 2). As our initial and final state both have an odd number of black squares, the total number of moves must be even. Now we consider a row with an even number of black squares (showing the existence of such a row is trivial). We count the total number of moves again, this time considering

- The row itself;
- Every other column **without** the square in it belonging to the mentioned row. We call these *wolumns* if a white square was removed and *bolumns* if a black square was removed.

Suppose the row has an odd number of moves applied to it. Then every bolumn must also have an odd number of moves applied to it (so that the black squares in our row stay unchanged) and every wolumn has a even number of moves applied to it (so that the white rows in our row change). The total number of moves, summing over the row, bolumns and wolumns, is therefore odd + (odd $\times$ even) + (even $\times$ odd) = odd.
Similarly, if the row had an even number of moves applied to it, then the bolumns get an even number of moves and the wolumns odd. The total number of moves is now
even + (even $\times$ even) + (odd $\times$ odd) = odd.
In both cases, we have a contradiction.

Marking scheme

(a) Non-additive point:
   - 1P: Find that the correct solution is only for even n without proof
   - 2P: A counterproof for only 1 or 3 modulo 4.

(b) Additive points:
   - 3P: Correct and proven construction for even n
   - 4P: A complete proof that for odd n no solution exists

   Partial points can be awarded for important ideas but incomplete proof (for example by looking at two consecutive rows for the case n odd).

2. Find all positive integers n such that there exists an infinite set A of positive integers with the following property: For all pairwise distinct numbers $a_1, a_2, \ldots, a_n \in A$, the numbers

$$a_1 + a_2 + \ldots + a_n \quad \text{and} \quad a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

are coprime.

Answer: All integers $n > 1$.

Solution 1(Paul)

Clearly there is no such set if $n = 1$. For $n > 1$ we will see that such a set exists, so fix $n > 1$.

Set $x_0 = n$ and recursively define $x_k = (x_0 + \ldots + x_{k-1})! + 1$. Then we choose the set A to be the set of all x_i for $i \geq 1$, note that $x_0 \notin A$. Now suppose $a_1, \ldots, a_n \in A$ are pairwise distinct and that $a_1 \cdots a_n$ and $a_1 + \ldots + a_n$ are not relatively prime. Let p be a prime factor of their greatest common divisor, which must also be a prime factor of $a_1 \cdots a_n$ and consequently a prime factor of some a_{i_0}. We may assume that the a_i are ordered increasingly, then the definition of the sequence $\{x_k\}_{k \in \mathbb{N}}$ implies that $a_k \equiv 1 \pmod{p}$ for $i_0 < k \leq n$. Hence

$$p \mid a_1 + \cdots + a_{i_0 - 1} + (n - i_0).$$

This sum is smaller or equal to $x_0 + x_1 + \cdots + x_{k_0 - 1}$, where k_0 is the unique index such that $x_{k_0} = a_{i_0}$. Therefore $p \mid (x_0 + x_1 + \cdots + x_{k_0 - 1})! = x_{k_0} - 1 = a_{i_0} - 1$, which yields a contradiction.

Solution 2

One can define another possible set A using Dirichlet's theorem on primes in arithmetic progressions, with an idea similar to Solution 1 in mind. Again suppose that $n > 1$.

Take a prime $p_1 > n$. Then we recursively choose primes p_k such that each p_k is congruent to 1 modulo $p_1 \cdots p_{k-1}$ and set $A = \{p_k \mid k \geq 1\}$. We make the additional requirement that $p_1^2 \leq p_2$, the inequalities $p_k \leq \frac{p_{k+1}}{p_1}$ for $k \geq 2$ are built into the definition of the sequence (this will only be important at the end of the solution). Now suppose $q_1, \ldots, q_n \in A$ are distinct and ordered increasingly. Then $q_1, \ldots, q_n$ and $q_1 + \ldots + q_n$ are not relatively prime if and only if some q_{i_0} divides $q_1 + \ldots + q_n$. Since $q_i \equiv 1 \pmod{q_{i_0}}$ for $i_0 < i \leq n$, we get

$$q_{i_0} \mid q_1 + \ldots + q_{i_0 - 1} + (n - i_0).$$

For $1 \leq j < i_0$ we have $q_j \leq \frac{q_{i_0}}{p_1} < \frac{q_{i_0}}{n}$. Hence

$$q_{i_0} \leq q_1 + \ldots + q_{i_0 - 1} + (n - i_0) < \frac{i_0 - 1}{n} q_{i_0} + n \leq q_{i_0},$$

a contradiction.

Marking Scheme For otherwise complete solutions, forgetting to state that $n = 1$ does not work will result in the subtraction of one point.

- 1P: (non-additive) Solving at least one case with $n \geq 3$.
- 1P: (non-additive) attempting to recursively construct a sequence such that new elements have "small" congruence modulo old elements of the sequence.
- +4P: Constructing a valid set A.
- +2P: Deriving a divisibility condition as in the above two solutions.
- +1P: Deriving a contradiction.

3. Sei k ein Kreis mit Mittelpunkt O. Sei AB eine Sehne dieses Kreises mit Mittelpunkt $M \neq O$. Die Tangenten von k an A und B schneiden sich in T. Die Gerade l geht durch T und schneidet k in C und D, mit $CT < DT$ und $BC = BM$.

Beweise, dass der Umkreismittelpunkt des Dreiecks ADM die Spiegelung von O an der Geraden AD ist.

Lösung: Für diese Aufgabe gibt es zwei verschiedene Ansätze um die Aufgabe zu lösen. Die erste Lösung verwendet einen andere Definition der Punkte und Potenzlinien, die zweite Lösung verwendet den Symmedian, welchen wir ab nächsten Jahr auch in der Selektion behandeln.

Solution 1 (français) On souhaite prouver que la réflexion de C par rapport à la droite OM est sur la droite DM. Appelons ce point C'. Il n'y a malheureusement pas de manière simple de prouver un tel résultat. Pour cette raison nous introduisons les points E et E'. E est la seconde intersection de la droite $C'M$ avec le cercle k et E' est la seconde intersection de la droite CM avec le cercle k. Nous voulons prouver que les droites CE, OM ainsi que la tangente à k au point B se coupent en un point. Si cette propriété est vérifiée, il s'ensuit que la droite CE est la droite CD et donc $D = E$.

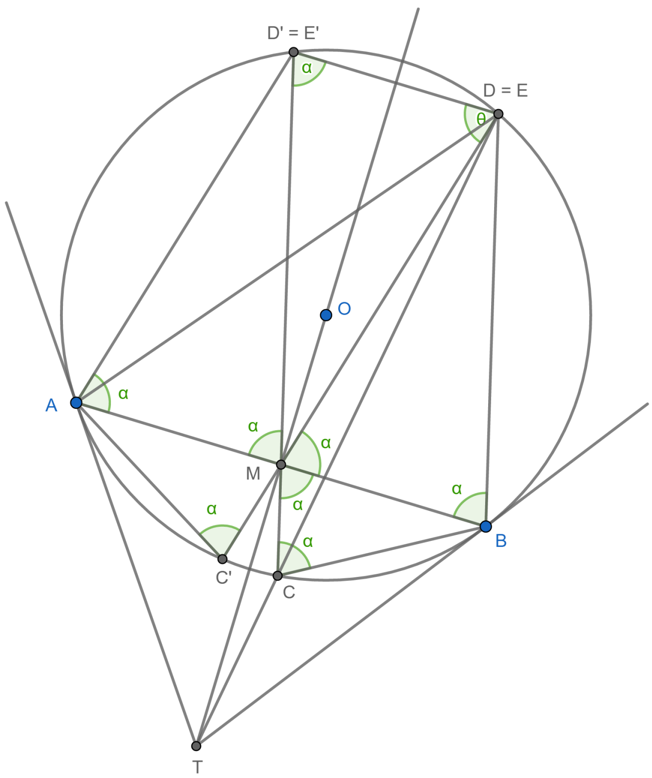


Figure 1: Lösung zur Aufgabe S3

Soit $\alpha = \angle BCM$. Puisque $BC = BM = AM = AC'$ nous avons $\angle CMB = \angle AMC' = \angle AC'M = \angle AC'E = \angle BCE' = \alpha$. De plus nous avons également $\angle ABE = \angle AC'E = \alpha$ et $\angle BAE' = \angle BCE' = \alpha$. Il s'ensuit que $\angle CBE = 180° - 2\alpha + \alpha = 180° - \alpha$ et ainsi $\angle COE = 2\alpha = \angle CME$. Par conséquent $CMOE$ est un quadrilatère inscrit. Soit N le milieu du segment OB. Le point T appartient à la ligne de puissance du cercle de centre N passant par B, M, O et du cercle k ainsi qu'à la ligne de puissance de ce même cercle et du cercle circonscrit à $CMOB$, dont T appartient à la ligne de puissance du cercle circonscrit à $CMOB$ et du cercle

k, qui est la droite CE. Par conséquent E, C, T sont alignés et donc $D = E$. On prouve de la même manière que $E' = D'$, avec D' la réflexion de D par rapport à la droite OM. Par les propriétés des réflexions $MD' = MD$. D'autre part $AD'M$ est un triangle isocèle donc on a $AD' = MD' = MD$. D'autre part $180° - \angle AMD = \angle BMD = \alpha = \angle MAD'$ donc $AMDD'$ est un parallélogramme et puisque O est le centre du cercle circonscrit à $AD'D$, la réflexion de O par rapport à AD est le centre du cercle circonscrit à AMD.

1. Lösung (deutsch) Wir möchten zuerst irgendwie beweisen, dass die Spiegelung von C an der Geraden OM auf der Geraden DM liegt. Nennen wir diesen Punkt C'. Allerdings gibt es keinen einfachen Weg, dies zu beweisen. Aus diesem Grund führen wir die Punkte E und E' ein. E ist der zweite Schnittpunkt des Kreises k mit der Geraden $C'M$ und E' ist der zweite Schnittpunkt k mit der Geraden CM. Wir möchten nun beweisen, dass die Geraden CE, OM, und die Tangente an k durch B sich in einem Punkt schneiden, woraus folgt dass $D = E$.

Sei $\angle BCM = \alpha$. Wegen $BC = BM = AM = AC'$ hat man $\angle CMB = \angle AMC' = \angle AC'M = \angle AC'E = \angle BCE' = \alpha$. Daraus folgt $\angle CBE = 180° - \alpha$ und somit $\angle COE = 2\alpha = \angle CME$. $CMOE$ bildet also ein Sehnenviereck. Sei N der Mittelpunkt von OB. Die Potenzlinien des Kreises mit Mittelpunkt N durch BMO, des Kreises durch $CMOE$ und von k schneiden sich in einem Punkt, T. Also gilt $D = E$ und $D' = E'$. Aus $AD' = MD' = MD$ folgt, dass $\angle MD'D = DD'M = \alpha$. Somit ist $AMD'D$ ein Parallelogramm und da O der Mittelpunkt des Kreises durch ADD' ist, ist die Spiegelung von O an AD der Mittelpunkt des Kreises durch AMD.

2. Lösung Wegen der Eigenschaft des Symmedians folgt $\angle ADM = \angle CDB$. Ausserdem gilt $\angle MAD = \angle BAD = \angle BCD$ und $BC = AM$. Die Dreiecke BCD und MAD sind kongruent zueinander. Somit folgt, dass $R_{AMD} = R_{CBD} = OD = OB$. Aber weil $OM < OB$ ist O nicht der Mittelpunkt des Kreises durch AMD. Die einzige Möglichkeit für einen Mittelpunkt ist somit die Spiegelung von O an der Geraden AD.

Marking Scheme Lösung 1:

- 1P : Feststellen, dass man beweisen möchte C', M und D sind auf einer Geraden, oder dass E' die Spiegelung von D an OM ist oder eine gleichbedeutende Aussage.
- 3P: Beweisen, dass der Punkt C und der Schnittpunkt von MD mit k symmetrisch in MO ist.
- 1P: Daraus folgern, dass $AMDD'$ ein Parallelogramm ist.
- 2P: Den Beweis vollenden.

Marking Scheme Lösung 2:

- 1P: Feststellen, dass DT der Symmedian ist, da T der Schnittpunkt der Tangenten ist.
- 2P: Herausfinden, dass $\angle ADM = \angle CDB$.
- 2P: Beweisen, dass AMD und CBD kongruent sind.
- 2P: Den Beweis vollenden.

-1P: In a partial or complete solution, insufficiently arguing that DT is a symmedian (minimal requirement is either explicitly stating that the reason is that T is the intersection of the tangents, or providing an unambiguous reference)

4. Déterminer tous les entiers positifs impairs n tels que pour toute paire a, b de diviseurs de n premiers entre eux

$$a + b - 1 \mid n.$$

Réponse: Les solutions sont tous les nombres de la forme $n = p^k$ avec p un nombre premier impair et $k \geq 0$.

Solution 1 (Louis)

Si $n = 1$, le seul diviseur est 1 et $1 + 1 - 1 \mid 1$, donc $n = 1$ est une solution. Autrement on peut supposer que n admet au moins un facteur premier p. On vérifie facilement que tout nombre de la forme $n = p^k$ avec p un nombre premier impair est une solution. En effet, la seule manière de trouver deux diviseurs a, b de n premiers entre eux est que l'un d'entre eux soit égal à 1. Disons que $a = 1$. Dans ce cas $a + b - 1 = b$ est effectivement un diviseur de n.

Supposons maintenant que n ne soit pas une puissance d'un nombre premier et soit p le plus petit facteur premier de n. On écrit $n = p^k m$ avec $k \geq 1$ et $m > 1$ premier avec p. Par hypothèse le nombre $p + m - 1$ est un diviseur de n, et puisque p est le plus petit facteur premier de n il s'ensuit que $p - 1$ est premier avec m. Par conséquent $p + m - 1$ doit être une puissance de p. Disons que $p+m-1 = p^r$. Si $k = 1$ on obtient alors $r = 1$ et donc $p+m-1 = p$, ou encore $m = 1$, en contradiction avec le fait que n n'est pas une puissance d'un nombre premier. Autrement on peut refaire le même raisonnement avec p^2 à la place de p. Puisque $p^2 - 1 = (p+1)(p-1)$ et que par hypothèse $p \neq 2$, il s'ensuit que $p + 1$ et $p - 1$ sont tous les deux premiers avec m et donc de nouveau $p^2 + m - 1 = p^s$ avec $s \geq 3$. En prenant la différence entre les deux expressions on obtient $p^2 - p = p^s - p^r$, et cette égalité est vérifiée uniquement pour $s = 2, r = 1$. Ces valeurs de r et s impliquent cependant que $m = 1$, en contradiction avec le fait que n n'est pas une puissance d'un nombre premier.

Solution 2 (David)

Similarly to the solution above, we write $n = p^k m$ for some $k \geq 1$ and $m > 1$, such that p is the smallest prime factor of n and $p \nmid m$. We find again that $p + m - 1$ is a power of p. We can also find that $(p + m - 1) + m - 1$ is a power of p because $p - 2$ is not equal to zero and therefore coprime to m. Now, we have three different powers of p:

$$p < p + m - 1 = p^a < p + 2(m - 1) = p^b$$

Since $m - 1 = p^b - p^a$, we deduce that $p^a \mid m - 1$. But at the same time, $m - 1 = p^a - p$, we get that $p^a \nmid m - 1$, contradiction.

Marking scheme

- 1P: Prouver que tous les nombres de la forme $n = p^k$ sont solution.
- +2P: Prouver que $p + m - 1$ est une puissance de p.
- +3P: Trouver une autre expression qui est aussi une puissance de p.
- +1P: Conclure

2P: Proving that $n = p^k$ is solution and that $n = pm$ with all prime factors of m greater than p is not a solution
-1P: Forgetting the case $n = 1$.

5. Find all polynomials Q with integer coefficients such that every prime number p and any two positive integers a, b with $p \mid ab - 1$ satisfy

$$p \mid Q(a)Q(b) - 1.$$

Answer: The constant polynomials $Q \equiv \pm 1$ and $Q(x) = \pm x^n$ for $n \geq 1$.

Solution 1(Arnaud)

First observe that the only constant solutions are the polynomials $Q \equiv \pm 1$. From now on let Q be a solution of degree $n \geq 1$.

We fix a and take $p > a$. Let $b = a^{-1}$ be the unique inverse of a modulo p such that $1 \leq a^{-1} < p$. Clearly a^{-1} depends on both p and a. But what about $Q(a^{-1})$? Does it still depend on p ? Thinking of Q now as a function $\mathbb{R} \to \mathbb{R}$ that maps integers to integers we can make the following crucial observation. Let $Q(x) = c_n x^n + \ldots + c_0$. Then

$$a^n Q(a^{-1}) \equiv c_n + c_{n-1}a + \ldots + c_0 a^n \pmod{p}.$$

The latter is nothing but a^n times the evaluation of Q on the real number $1/a$ (careful, $1/a \neq a^{-1}$). So if we denote $P(x) := x^n Q(1/x)$, we obtain

$$p \mid Q(a)Q(a^{-1}) - 1 \quad \Rightarrow \quad p \mid a^n Q(a)Q(a^{-1}) - a^n \quad \Rightarrow \quad p \mid Q(a)P(a) - a^n.$$

Now, the term $Q(a)P(a) - a^n$ does not depend on p anymore ! Since the above relation holds for all $p > a$, we must have $Q(a)P(a) = a^n$ for all a. Since P is also a polynomial with integer coefficients, we can treat $Q(a)P(a) = a^n$ as a relation of polynomials with integers coefficients. So, looking at the degree we get

$$n = deg(a^n) = deg(QP) = deg(P) + deg(Q) = deg(P) + n.$$

Hence P has degree zero, or said differently P is constant. This implies $Q(a) = ca^n$ for some constant c.

The initial relation implies, for $a = b = 1$, that $Q(1) = \pm 1$, hence $c = \pm 1$.

Clearly, $Q(x) = \pm x^n$ are solutions for all $n \geq 1$.

Solution 2 (David)

This approach is more algebraic, even analytic. Clearly the only constant solutions are $Q \equiv \pm 1$ and all monomials $Q(x) = \pm x^n$ are solutions. Moreover, observe that Q is a solution if and only if $-Q$ is as well. So we can assume that the leading coefficient of Q is $c > 0$.

If Q is a solution of degree $n \geq 1$ that is not of the form above, then there exists a polynomial R with integer coefficients and of degree at most $n - 1$ such that $Q(x) = cx^n + R(x)$ and ($c \neq 1$ or $R \neq 0$). Since the degree of Q is n, there exists an integer N such that for all $a \geq N$

$$a^{n-1} < Q(a) < a^{n+1}.$$

Moreover since $c \neq 1$ or $R \neq 0$, then up to increasing the value of N, we can assume that for all $a \geq N$ it holds $Q(a) \neq a^n$. In particular if $a \geq N$ is prime, then $Q(a)$ is not a power of a and is therefore divisible by some other prime p. Let b be the inverse of a modulo p. So $p \mid ab - 1$ and $p \mid Q(a)$. This is a contradiction. So the only solutions are the ones given above.

Marking scheme

1P: stating that the set of **all** solutions is $Q \equiv \pm 1$ and $Q(x) = \pm x^n$ and verification

Solution 1

- 1P: trying to connect $Q(a^{-1})$ and $Q(1/a)$ modulo p
- 2P: introducing $P(x) = x^n Q(1/x)$
- 1P: obtaining $p \mid Q(a)P(a) - a^n$
- 1P: concluding that $Q(a) = ca^n$
- 1P: proving $c = \pm 1$ and concluding

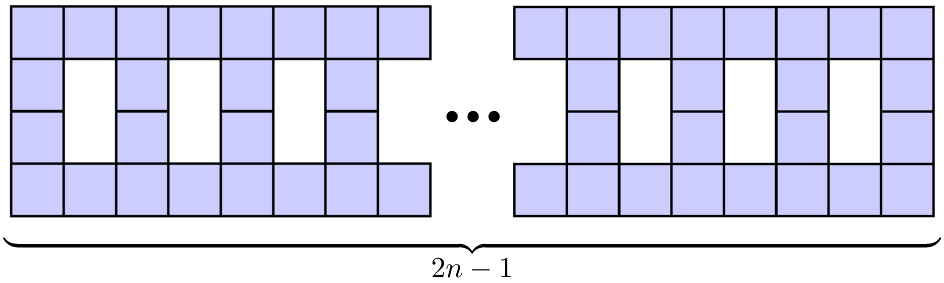
Solution 2

- 1P: the idea of finding p such that there exists $a \not\equiv 0 \pmod p$ with $p \mid Q(a)$
- 1P: the idea that it is actually enough to find a prime q such that $Q(q)$ is not a power of q
- 1P: asymptotic estimations of Q with higher or lower degree terms
- 2P: proving that $Q(q)$ is not a power of q for large prime q
- 1P: conclusion

6. Prove that for every positive integer n, there exists a finite subset of the squares of an infinite chessboard that can be tiled with indistinguishable 1×2 dominoes in exactly n ways.

Solution 1 (Valentin)

Take a $(2n-1) \times 4$ rectangle of squares and remove every second square in row 2 and 3 as illustrated below:



Let us now consider an arbitrary tiling of this subset.

Claim: Of all the dominoes overlapping the first row, exactly one is vertical.

Proof. If all of these dominoes were horizontal, they would cover a total of $2n$ squares in the first row. But since the first row only contains $2n-1$ squares, this is not possible.

If more than one of these dominoes are vertical, choose two of these vertical dominoes D_1 and D_2 such that all dominoes overlapping the first row between the two are horizontal. The total number of squares covered in the first row by the dominoes between D_1 and D_2 is even, however the total number of squares in the first row between D_1 and D_2 is odd, a contradiction.

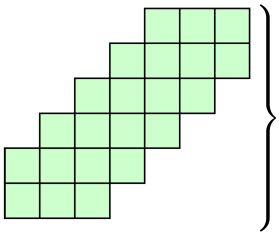
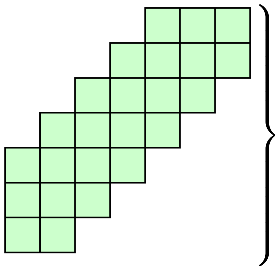
We conclude that exactly one of the dominoes overlapping the first row is vertical. $\square$

Now every of the n choices for which domino in the first row is vertical leads to a uniquely defined tiling of the whole subset since all the dominoes in row 2 and 3 have to be vertical and the bottom row can be uniquely tiled with one vertical domino and $n-1$ horizontal dominoes.

Since every tiling must also exactly have one vertical domino in the first row, we conclude that there are exactly n tilings of this subset.

Solution 2 (Valentin)

For any $n \geq 2$, consider the two subsets

$A_{2n-1} =$  n and $A_{2n} =$  $n+1$

Claim: The subset A_k has exactly k domino tilings for all $k \geq 3$.

Proof. The claim holds for $k = 3, 4$. Now consider $k > 4$.

For k odd, consider D, the domino that covers the bottom left square of A_k. If D is horizontal, there is only one way of covering the lowest square in each other column, namely by a vertical domino. The remaining squares can only be tiled in one way, namely using only horizontal dominoes. If D is vertical then consider D', the domino that covers the bottom left square of the remaining set. If D' is vertical then similarly to the previous case, there is only one way of tiling the remaining squares. If however D' is horizontal, the remaining set is just A_{k-2}. Putting everything together we find that the number of tilings of A_k is the number of tilings of A_{k-2} plus 2 and we are done by induction.

For even k, consider again D, the domino that covers the bottom left square of A_k. If D is vertical, then similarly to the first case for odd k there is only one way of tiling the remaining squares. If D is horizontal, the remaining set is just A_{k-1} and we already know there are $k-1$ ways of tiling these squares. Putting the two together we find that A_k has k tilings. $\square$

This argument solves the problem for all $n \geq 3$. The cases $n = 1, 2$ are trivially true by considering 1×2 and 2×2 rectangles.

Marking Scheme (Additive)

- 3P: Claiming a concrete construction that works for all but finitely many n

Points for bounding the number of tilings:

- 2P: Proving that the construction for n has at most n tilings
- 2P: Proving that the construction for n has at least n tilings

In case the construction distinguishes classes of integers (as in Solution 2), the points are split between them accordingly if the approach leads to a (more or less direct) solution.

- -1P: Finitely many cases missed or missing base case for induction
- -1P: If not both directions of a bijective argument (as in Solution 1) are justified

No points are awarded for merely reducing the problem to primes or other infinite sets like Fibonacci numbers.

7. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le f(x) \le 2x$ for all $x \ge 0$ and such that for all $x, y \in \mathbb{R}$

$$f(x+y) = f(x + f(y)).$$

Answer: The only solutions are $f(x) \equiv x,\ f(x) \equiv 0$ and the sawtooth function given by

$$f(x) = x \text{ on } [0, p) \text{ with period } p$$

or alternatively

$$f(x) = p \cdot \{x/p\} \text{ for all } x \in \mathbb{R}$$

for any positive real number p. Here, $\{x\}$ denotes the fractional part of x

Solution 1 (Valentin) If we let $u = x + y$ we find that

$$f(u) = f(u - y + f(y)) \text{ for all } x, y \in \mathbb{R}$$

and excluding the solution $f(y) \equiv y$, we find that f must have some period $p > 0$. This together with the first condition now implies that $f(x) \ge 0$ for all $x \in \mathbb{R}$. Also excluding the solution $f \equiv 0$, there must exist some $a \in \mathbb{R}$ with $f(a) > 0$.

Let P be the set of all periods of f which are $\le p$. If P is infinite then by pigeonhole there must be two periods $p_1 < p_2$ satisfying $p_2 - p_1 < f(a)/2$. But since the difference of two periods is also a period, f must also have period $p_3 = p_2 - p_1 < f(a)/2$. But now choose $k \in \mathbb{Z}$ such that

$$0 < x_0 = a + k \cdot p_3 < f(a)/2 \text{ and } f(x_0) = f(a) = 2 \cdot (f(a)/2) > 2x_0$$

contradicting the first condition. We conclude that P is finite and in particular, that f has a smallest period p_0. Also, every period of f must be a multiple of the smallest period p_0 since otherwise we could find two periods with difference $< p_0$, contradicting the minimality of p_0.

Now since $|f(y) - y|$ is a period of f, it follows that for all $y \in \mathbb{R}$ there is some $k_y \in \mathbb{Z}$ such that

$$f(y) - y = k_y \cdot p_0$$

and for $0 \le y < p_0$ since we must also have

$$0 \le f(y) = y + k_y \cdot p_0 \le 2y$$

we find that $k_y = 0$ for all y in this interval and conclude that $f(x) = p_0 \cdot \{x/p_0\}$ for all $x \in \mathbb{R}$. It is easy to see that all the claimed functions are indeed solutions by observing that if $f(x) - x$ is a period of f then f must satisfy the original equation.

Comments

Alternatively, one can also lead the assumption that f does not have a smallest period to a contradiction by constructing a decreasing sequence $\{p_k\}$ of periods where $p_{k+1} < p_k/2$ and therefore similarly obtaining a period $p < f(a)/2$.

Also note that it is very easy to make the mistake of only proving that the periods of f are bounded below by a positive constant (which implies that the infimum over all periods is positive). This however is **not** enough to show that f has a minimal period. One also has to use the fact that periods are closed under (distinct) subtraction.

Solution 2 (Tanish)

As above, we shall endeavour to prove that the only solutions are the sawtooth and identity function and $f(x) = 0$. For the rest of the solution, we suppose $f(x) \neq 0$ somewhere, and $f(x) \neq x$ somewhere. It remains to prove the only possible solutions are the sawtooth function.

Lemma 1. *$f(x)$ is periodic.*

Proof. See proof in previous solution. $\square$

Lemma 2. *There exists some $p > 0$ such that $f(x)$ is the identity on $[0, p[$.*

Proof. Suppose no such p exists. As a result, it follows for every $\delta > 0$ there exists $\varepsilon \in [0, \delta]$ such that $f(\varepsilon) \neq \varepsilon$. Now consider the existence of some z such that $f(z) \neq z$, and consider $\varepsilon \in [0, \frac{f(z)}{2}[$. We have from lemma 1 that $|f(\varepsilon) - \varepsilon| < \varepsilon < \frac{f(z)}{2}$ (the first inequality comes from condition 2 in the problem) and as $|f(\varepsilon) - \varepsilon|$ is a period of our function (as noted in Lemma 1) we can find (a la Euclidean division algorithm) some number ζ in the interval $[0, |f(\varepsilon) - \varepsilon|[$ such that $f(\zeta) = f(z)$, but since $\zeta < \frac{f(z)}{2}$, this contradicts the second condition. $\square$

So now we know there is some p such that $f(x) = x$ for $x \in [0, p[$. We can also assume henceforth that this p is maximal (simply take the supremum of the set of possible p).

Lemma 3. *$f(p) = 0$.*

Proof. Suppose otherwise. Firstly, consider the case where $f(p) \neq p$. If $f(p) = 2p$ then as $2p - p = p$ is now a period of the function, $f(0) = 2p$, contradiction. So now, $|f(p) - p| < p$. But as $|f(p) - p|$ is a period of the function, then $f(|f((p) - p|) = f(0) = 0$ which contradicts the fact that f is the identity on $[0, p[$.
We are now left with the case $f(p) = p$. By the aforementioned maximality of p, we have that for all $\delta > 0$, there exists q in $]p, p + \delta[$ such that $f(q) \neq q$.
Consider some such x_i in $]p, p + \frac{1}{i}[$ and similarly y_i in $]p, p + x_i[$ that are not fixed points. Furthermore let $x'_i = |f(x_i) - x_i|$ and $y'_i = |f(y_i) - y_i|$. We have that $p < x'_i, y'_i < p + \frac{1}{i}$, as they are periods of f and so must be greater than p (otherwise f cannot be the identity on $[0, p]$ and less than x_i and y_i respectively (from the second original condition). However, $|x'_i - y'_i| < \frac{1}{i}$ is also a period of f. It suffices to choose i such that $\frac{1}{i} < p$ to find a contradiction. $\square$

We now know f is p-periodic and also that it is the identity on $[0, p[$, which easily allows us to conclude that f is the sawtooth function (modulo p).

Comments

This solution almost exclusively uses ideas from Real Analysis you will see in your first year of university and not what we teach you. This is one of the few olympiad problems that would be difficult for a high-schooler but easier for someone with no olympiad experience at university (provided they are not studying geology). As such, you are not expected to have the ideas that are used in the construction of this solution (in particular, supposing the minimality or maximality of certain variables and using this for a proof by contradiction) but it is useful to employ methods like these, more often seen in combinatorics.

Marking Scheme

Solution 1 (Valentin):

- 1P: Finding three different functions that satisfy the conditions.
- 1P: Proving that if $f(x) \neq x$ then f is periodic.
- 2P: Proving that if $f \neq 0$ and f is periodic then it has a minimal period.
- 1P: Observing that every period of f must be an integer multiple of the minimal period.
- 1P: Proving that f is linear in the period starting at 0.
- 1P: Finishing the problem and checking that all found functions are indeed solutions.

Solution 2 (Tanish):

- 1P: Finding three different functions that satisfy the conditions.
- 2P: Proving that if $f(x) \neq 0$, $f(x) \neq x$ then there exists $p > 0$ with $f(x) = x$ on $[0, p)$ (1P can be given if they just prove periodicity but nothing more)
- 1P: Proving that there exists a maximal such p.
- 2P: Proving that for the maximal such p we have $f(p) = 0$
- 1P: Finishing the problem and checking that all found functions are indeed solutions.

Both Solutions:

- -1P: Using that the infimum of all periods is also a period without justification or similar logical gaps in analytic arguments.

Remark: The description of any solution must clearly define what happens at critical points.

8. Let I be the incenter of a non-isosceles triangle ABC. Let F be the intersection of the perpendicular to AI through I with BC. Let M be the point on the circumcircle of ABC such that $MB = MC$ and such that M is on the same side of the line BC as A. Let N be the second intersection of the line MI with the circumcircle of BIC. Prove that FN is tangent to the circumcircle of BIC.

Solution 1 D'abord, on définit D la deuxième intersection de AI avec le cercle circonscrit du triangle ABC. Par le théorème incenter/excenter, D est le centre du cercle circonscrit de BIC. Puisque $\angle FID = 90$, on déduit que FI est tangent à ce cercle. Ainsi, si l'on prouve $FI = FN$, on aura terminé. De manière équivalente, étant donné que $DI = DN$ (D est le centre du cercle), il suffit de prouver que FD est perpendiculaire à IN. Définissons P comme le point sur IN tel que $\angle DPI = 90$. Notre but: montrer que D, P, F sont alignés. On obtient alors $\angle DPM = 90$, et puisque DM est un diamètre du cercle circonscrit au triangle ABC, P se trouve alors sur ce cercle. Etant donné que $\angle DPI = 90$, le centre du cercle circonscrit de DPI est le milieu du segment ID. Donc, FI est tangent à ce cercle. On a maintenant trois cercles: les cercles circonscrits aux triangles ABC, BIC et PID. Il est connu que les trois lignes de puissance de chaque paire de cercles se croisent en un point. Ce point est F, et puisque DP est la ligne de puissance des cercles circonscrits à DIP et ABC, alors D, P, F sont alignés.

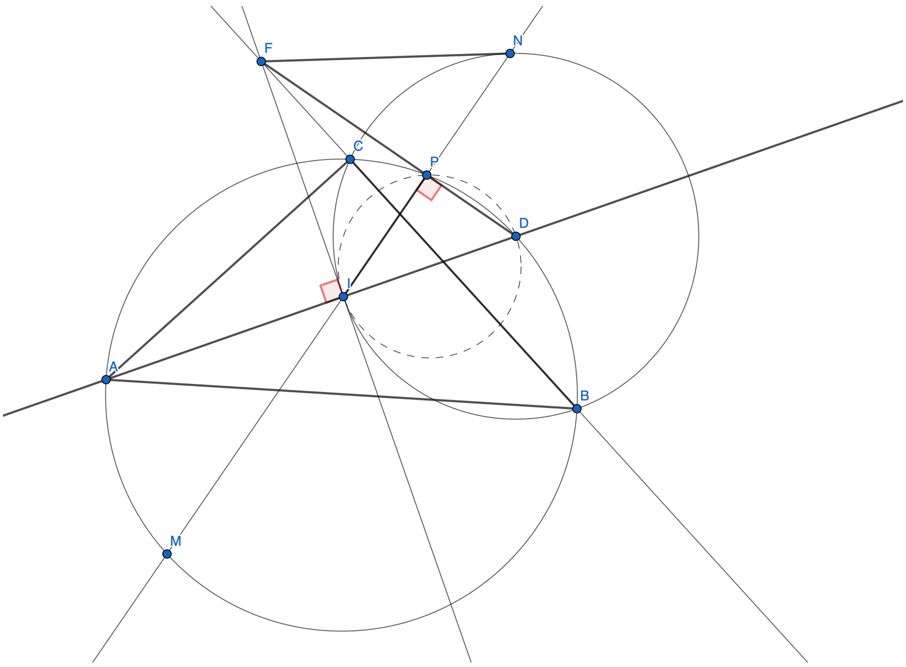


Figure 2: First solution

Marking Scheme

- 1P: Introduce D and explicitely say that D is the center of the circumcircle of BIC.
- 1P: Explicity state that FI is tangent to that circle.
- 1P: Introduce P and prove that P is on the circumcircle of ABC.
- 3P: Prove that D, P, F are collinear.
- 1P: Conclude.

Solution 2 Comme dans la solution précédente, on pose D comme le milieu de l'arc BC. Soit également Q le milieu du segment BC. Nous pouvons voir que DM est un diamètre du cercle circonscrit à ABC. Du plus, $Q \in [DM]$. Donc, $\angle DBM = 90$ et les triangles MQB et MBD sont similaires. Alors $MB^2 = MQ \cdot MD$. Aussi, MB est tangent au cercle circonscrit au triangle BIC. Donc $MB^2 = MI \cdot MN$. Alors $MI \cdot MN = MQ \cdot MD$ et les points D, Q, I, N sont cocyliques. De plus, $\angle FQD = 90 = \angle FID$ et F se trouve également sur ce cercle. Il s'ensuit directement que $\angle FND = 90$ et donc que FN est tangente au cercle circonscrit de BIC.

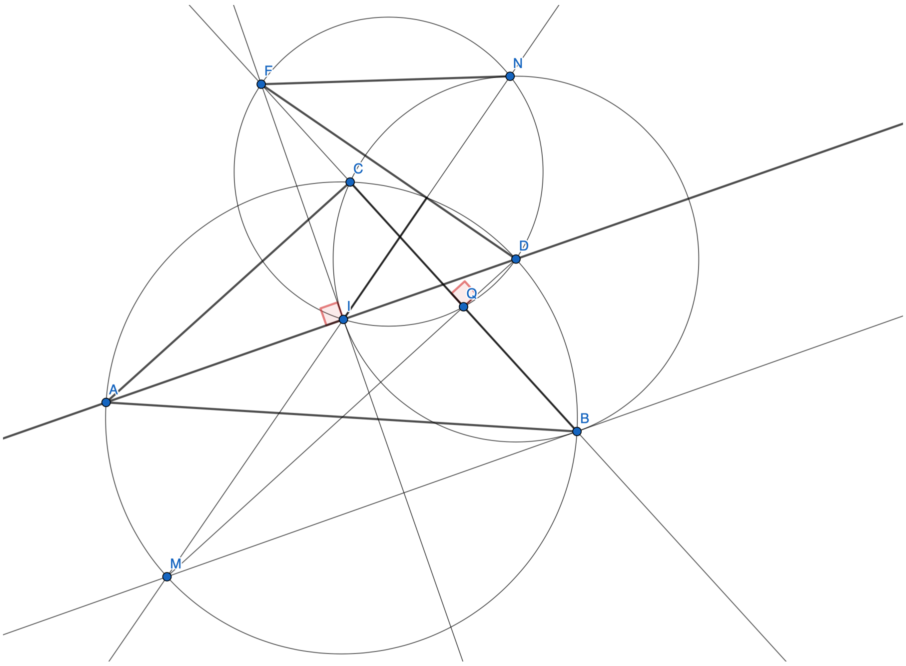


Figure 3: Second solution

Marking Scheme

- 1P: Introduce D and explicitly say that D is the centre of the circumcircle of BIC.
- 1P: Explicitely state that MB (or MC) is tangent to the circumcircle of BIC.
- 2P: Introduce point Q and use similar triangles to obtain $MB^2 = MQ \cdot MD$.
- 1P: Use the power of point M and get $MB^2 = MI \cdot MN$ and prove that N, D, Q, I are concyclic.
- 1P: Prove that F is also on the circle.
- 1P: Conclude.

9. We call a set S of integers *laikable* if for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_n x^n + \ldots + a_1 x + a_0$ are also in S, given it is not the zero polynomial. Find all laikable sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a, b.

Answer: The only solution is $S = \mathbb{Z}$, the set of all integers.

Solution 1: The set $\mathbb{Z}$ of all integers is clearly a solution. We shall prove that any well-rounded set containing all the numbers of the form $2^a - 2^b$ for $a, b > 0$ must be all of $\mathbb{Z}$.

First, note that $0 = 2^1 - 2^1 \in S$ and $2 = 2^2 - 2^1 \in S$. Now, $-1 \in S$, since it is a root of $2x + 2$, ad $1 \in S$, since it is a root of $2x^2 - x - 1$. Also, if $n \in S$ then $-n$ is a root of $x + n$, so it suffices to prove that all positive integers must be in S.

Now, we claim that any positive integer n has a multiple in S. Indeed, suppose that $n = 2^\alpha t$ for $\alpha \geq 0$ and t odd. Then $t \mid 2^{\varphi(t)} - 1$, so $n \mid 2^{\alpha+\varphi(t)+1} - 2^{\alpha+1}$. Moreover $2^{\alpha+\varphi(t)+1} - 2^{\alpha+1} \in S$, so S contains a multiple of every positive integer n.

We will now prove by induction that all positive integers are in S. Suppose that $0, 1, \ldots, n-1 \in S$; furthermore, let N be a multiple of n in S. Consider the base-n expansion of N, say $N = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$. Since $0 \leq a_i < n$ for each a_i, we have that all the a_i are in S. Furthermore $a_0 = 0$ since N is a multiple of n. Therefore, $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n - N = 0$, so n is a root of a polynomial with coefficients in S. This tells us that $n \in S$, completing the induction.

Solution 2: As in the previous solution, we can prove that $0, 1$ and -1 must all be in any well-rounded set S containing all numbers of the form $2^a - 2^b$ for $a, b \in \mathbb{Z}_{>0}$.

We show that, in fact, every integer k with $|k| > 2$ can be expressed as a root of a polynomial whose coefficients are of the form $2^a - 2^b$. Observe that it suffices to consider the case where k is positive, as if k is a root of $a_n x^n + \ldots + a_1 x + a_0 = 0$, then $-k$ is a root of $(-1)^n a_n x^n + \ldots - a_1 x + a_0$.

Note that

$$(2^{a_n} - 2^{b_n})k^n + \ldots + (2^{a_0} - 2^{b_0}) = 0$$

is equivalent to

$$2^{a_n} k^n + \ldots + 2^{a_0} = 2^{b_n} k^n + \ldots + 2^{b_0}.$$

Hence our aim is to show that two numbers of the form $2^{a_n} k^n + \ldots + 2^{a_0}$ are equal, for a fixed value of n. We consider such polynomials where every term $2^{a_i} k^i$ is at most $2k^n$; in other words, where $2 \leq 2^{a_i} \leq 2k^{n-i}$, or, equivalently, $1 \leq a_i \leq 1 + (n - i) \log_2 k$. Therefore, there must be $1 + \lfloor (n - i) \log_2 k \rfloor$ possible choices for a_i satisfying these constraints.

The number of possible polynomials is then

$$\prod_{i=0}^{n} (1 + \lfloor (n - i) \log_2 k \rfloor \geq \prod_{i=0}^{n-1} (n - i) \log_2 k = n!(\log_2 k)^n$$

where the inequality holds as $1 + \lfloor x \rfloor \geq x$.

As there are $(n + 1)$ such terms in the polynomial, each at most $2k^n$, such a polynomial must have value at most $2k^n(n+1)$. However, for large n, we have $n!(\log_2 k)^n > 2k^n(n+1)$. Therefore there are more polynomials than possible values, so some two must be equal, as required.

Marking Scheme -1P: Having a solution which is correct for all but ≤ 10 integers.

Solution 1:

- 0P: Prove that 0, ± 1, ± 2, etc. belong to S.
- 0P: Prove that if $n \in S$, then $-n \in S$.
- 2P: Considering the smallest $n \notin S$ and working with the base-n expansion.
- 2P: Prove that every (positive) odd integer has a multiple in S.
- 1P: Prove that every (positive) integer has a multiple in S.
- 2P: Conclude

Solution 2:

- 1P: Reformulating the problem as finding two polynomials whose value at k is identical for non-trivial reasons.
- 1P: Trying to estimate the number of polynomials and the number of values at k.
- 3P: Estimating the number of polynomials
- 1P: Estimating the number of values these polynomials can take at k
- 1P: Conclude

Remark: Different approaches are possible for solving this problem with approximations. The 4 points for the estimations may be spread differently if the difficulty of the two steps is significantly different from this solution.

10. Let ABC be a triangle with circumcircle k. Let A_1, B_1 and C_1 be points on the interior of the sides BC, CA and AB respectively. Let X be a point on k and denote by Y the second intersection of the circumcircles of BC_1X and CB_1X. Define the points P and Q to be the intersections of BY with B_1A_1 and CY with C_1A_1, respectively. Prove that A lies on the line PQ.

Solution: We first show that the points B_1, Y and C_1 are collinear. Let $\angle B_1YX = \alpha$. Using the cyclic quadrilateral B_1YXC we have $\angle XCB_1 = 180 - \alpha$. Similarly using the cyclic quadrilaterals $CABX$ and C_1BXY, we arrive at $\angle C_1BX = \alpha$ and $\angle XYC_1 = 180 - \alpha$. So indeed B_1, Y and C_1 are collinear.

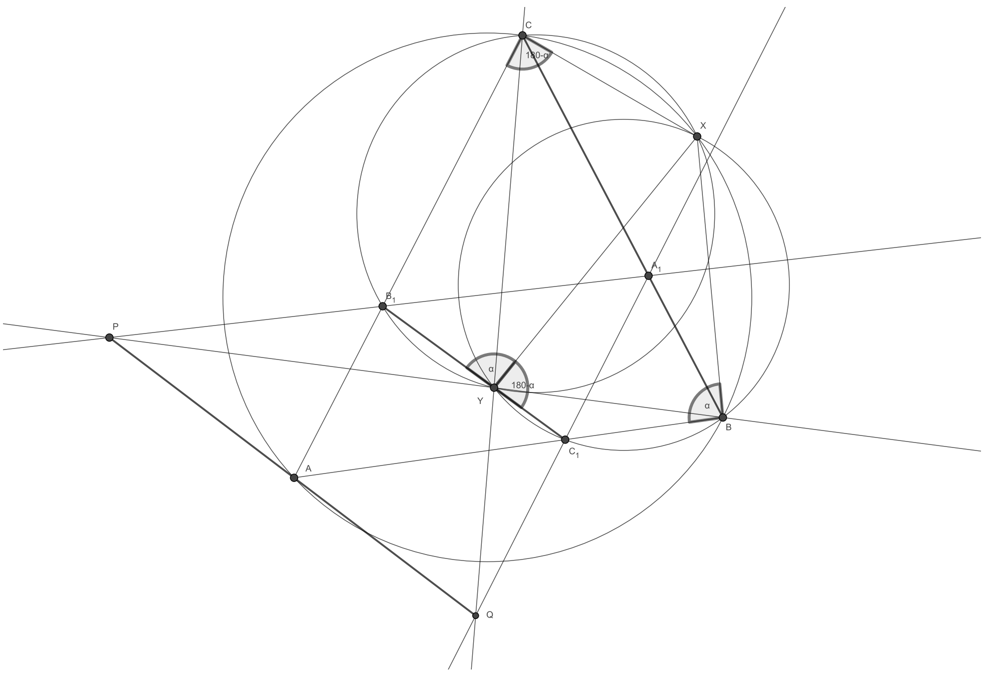


Figure 4: Solution of problem 10

Using Pappus theorem on the two sets of collinear points $\{B_1, Y, C_1\}$ and $\{B, A_1, C\}$, we can now proof that P, Q and A are also collinear.

Marking Scheme

- 1P : Guessing that B_1, Y and C_1 are collinear.
- 3P : Proving it.
- 3P: Use Pappus.

1P can be awarded for someone using Pappus with the correct lines but a wrong order of the points

11. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of non-negative integers satisfying $a_i \leq i$ for every $i \geq 0$ and such that for every positive integer n

$$\binom{n}{a_0} + \binom{n}{a_1} + \cdots + \binom{n}{a_n} = 2^n.$$

Prove that each non-negative integer appears in the sequence.

Solution (Louis): We prove by induction on k that for every k either $a_i = i$ for all $i \leq k$, or there exists a number $l > 0$ with $2l \leq k+1$ such that after reordering, the sequence $a_0, \ldots, a_k$ forms the two sequences $0, 1, \ldots, l-1$ and $0, 1, \ldots k-l$, without repetition. This fact proves the statement, as it implies that every number $N > 0$ must belong to the sequence $a_0, a_1, \ldots a_{2N}$.

Now in order to prove the aforementioned fact, we first notice that it is trivially true for $k = 0, 1$. For the inductive step towards $k+1$, we notice that

- In the first case ($l = 0$, only one sequence) we must have either $a_{k+1} = k+1$ or $a_{k+1} = 0$, as it is necessary that $\binom{k+1}{a_{k+1}} = 1$.
- In the second case we must have either $a_{k+1} = l$ or $a_{k+1} = k - l + 1$, as it is necessary that $\binom{k+1}{a_{k+1}} = \binom{k+1}{l}$

Both of these can be easily proven through three of the key properties of the binomial: $\sum_{i=0}^{n} \binom{n}{i} = 2^n$, $\binom{x}{y} = \binom{x}{x-y+1}$, and if $1 \leq y \neq z \leq \frac{x}{2}$ then $\binom{x}{y} \neq \binom{x}{z}$. These together prove the existence and pseudo-uniqueness of the value that a_{k+1} must take.

We easily conclude that our induction hypothesis is correct.

Marking Scheme:

- 2P: Noting that after reordering, the sequence $a_0, \ldots, a_k$ forms the two sequences $0, 1, \ldots, l-1$ and $0, 1, \ldots k-l$, without repetition (or any equivalent phrasing of this idea).
- +1P: Understanding but not proving that this is as a_{k+1} must "go on the end" of either sequence.
- +3P: Any proof of the aforementioned fact.
- +1P: Conclude that every number appears in the sequence.

If the contestant proves that a_i can be $k+1$ or $0/l$ or $k-l+1$ but does not prove it *must* be one of these, 1 points should be deducted.

No points are to be given for simply stating **any** of the three properties of the binomial that are necessary for the proof.

12. Let a, b, c, d be positive real numbers such that $a + b + c + d = 1$. Prove that

$$\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)^5 \geq 5^5\left(\frac{ac}{27}\right)^2.$$

Solution (David): The goal is to exploit the condition $a + b + c + d$ somehow such that we only get a and c on the right hand side. An observation that can help doing this is the equality case $(a, b, c, d) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})$. One way to achieve our goal is to apply Cauchy-Schwarz:

$$\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)(4(a+b)+(b+c)+4(c+d)+(d+a)) \geq (2a+b+2c+d)^2$$

$$\Leftrightarrow \quad \left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right) \geq \frac{(a+c+1)^2}{5} \tag{1}$$

Where we applied the condition $a + b + c + d = 1$ on both the left hand side and the right hand side. Now, we are already very close to our goal. To get the term ac on the right hand side, we apply AM-GM:

$$a+c+1 = a+c+\frac{1}{3}+\frac{1}{3}+\frac{1}{3} \geq 5\cdot\sqrt[5]{\frac{ac}{27}}.$$

Plugging in this estimation into the inequality (1) above, we get

$$\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right) \geq \frac{(5\cdot\sqrt[5]{\frac{ac}{27}})^2}{5} = 5\left(\sqrt[5]{\frac{ac}{27}}\right)^2.$$

Raising both sides to the fifth power yields

$$\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right)^5 \geq 5^5\left(\frac{ac}{27}\right)^2,$$

which is exactly what we wanted to prove.

Alternative approach (David): We can also simply observe the symmetry between the variables: If we exchange a and c at the same time as b and d, the expressions on both sides don't change. This is the motivation to try to estimate the two terms

$$\left(\frac{a^2}{a+b}+\frac{c^2}{c+d}\right), \quad \left(\frac{b^2}{b+c}+\frac{d^2}{d+a}\right)$$

separately. This can also be done using Cauchy Schwarz:

$$\left(\frac{a^2}{a+b}+\frac{c^2}{c+d}\right)(a+b+c+d) \geq (a+c)^2, \quad \left(\frac{b^2}{b+c}+\frac{d^2}{d+a}\right)(b+c+d+a) \geq (b+d)^2.$$

Using $a + b + c + d = 1$ and setting $t = a + c$, we get the two inequalities

$$\left(\frac{a^2}{a+b}+\frac{c^2}{c+d}\right) \geq t^2, \quad \left(\frac{b^2}{b+c}+\frac{d^2}{d+a}\right) \geq (1-t)^2 = t^2-2t+1,$$

so combined:

$$\left(\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\right) \geq (2t^2-2t+1).$$

Now, the objective is to estimate the right hand side of this inequality to get something nicer. Here it is very useful to know the value of t in the equality case. It turns out that

$$(3t-2)^2 \geq 0 \;\Leftrightarrow\; 9t^2-12t+4 \geq 0 \;\Leftrightarrow\; 10t^2-10t+5 \geq t^2+2t+1 \;\Leftrightarrow\; 2t^2-2t+1 \geq \frac{(t+1)^2}{5}$$

and so we arrive at inequality (1) again and finish in the same way.

Marking Scheme:
Partial points (not additive to the rest):

- 1P: Finding the equality case $(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})$
- 1P: Any useful estimation on the Left Hand side using only a and c

Points for (almost) complete solutions:

- 2P: Making the term $\left(\frac{(a+c+1)^2}{5}\right)^5$ appear
- 3P: Showing that $\left(\frac{(a+c+1)^2}{5}\right)^5$ is greater or equal to the right hand side.
- 2P: Showing that $\left(\frac{(a+c+1)^2}{5}\right)^5$ is smaller or equal to the left hand side.