

IMO - Selection 2018

Third exam - 26 May 2018

Duration: 4.5 hours

Difficulty: The problems are ordered by difficulty.

Points: Each problem is worth 7 points.

7. Let n be a positive integer. A sequence of $3n$ letters is called *Romanian* if the letters I , M and O appear exactly n times each. Define a *swap* to be the transposition of two adjacent letters. Prove that for any Romanian sequence X , there exists a Romanian sequence Y such that Y cannot be obtained from X using fewer than $\frac{3n^2}{2}$ swaps.

8. Determine all the integers $n \geq 2$ such that for every integer $0 \leq i, j \leq n$:

$$i + j \equiv \binom{n}{i} + \binom{n}{j} \pmod{2}.$$

9. Let a, b, c, d be real numbers. Prove that

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq \frac{9}{16}(a - b)(b - c)(c - d)(d - a).$$

Good Luck!