Second Round 2024

Zürich

December 16, 2023

Duration: 3 hours

Difficulty: The problems of each topic are ordered by difficulty.

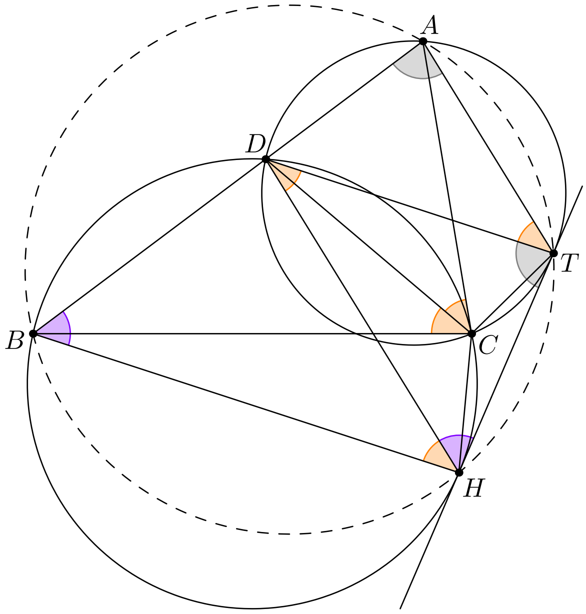
Points: Each problem is worth 7 points.

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

G1) Let ABC be a triangle. The angle bisector of $\angle ACB$ intersects AB in D. Let T and H be points on the circumcircles of CAD and CDB respectively, such that TH is a common tangent to the two circumcircles and C is inside the quadrilateral $BATH$. Show that $BATH$ is cylic.

Solution



Solution 1:

By the tangent chord theorem, $\angle DTH = \angle DAT$ and $\angle THD = \angle HBD$. Also, by the inscribed angle theorem, $\angle DHB = \angle DCB = \angle ACD = \angle ATD$. Hence,

$$\angle HBA + \angle ATH = \angle HBD + \angle ATD + \angle DTH = \angle THD + \angle DHB + \angle DAT = \angle THB + \angle BAT.$$

We deduce that $2 \cdot (\angle HBA + \angle ATH) = \angle HBA + \angle BAT + \angle ATH + \angle THB = 360°$, so $\angle HBA + \angle ATH = 180°$, proving that $BATH$ is cyclic.

Solution 2:

As in Solution 1, $\angle DTH = \angle DAT$, $\angle THD = \angle HBD$, and $\angle DCB = \angle DHB$. We can now compute

$$\begin{aligned}\angle HDT &= 180° - \angle DTH - \angle THD \\ &= 180° - \angle DAT - \angle HBD \\ &= 180° - \angle DAC - \angle CAT - \angle HBC - \angle CBA \\ &= (180° - \angle BAC - \angle CBA) - (\angle CDT + \angle HDC) \\ &= \angle ACB - \angle HDT.\end{aligned}$$

Hence, $\angle HDT = \frac{\angle ACB}{2} = \angle DCB = \angle DHB$. We can now conclude as

$$\begin{aligned}\angle THB + \angle BAT &= \angle THD + \angle DHB + \angle DAT \\ &= \angle THD + \angle HDT + \angle DTH \\ &= 180°.\end{aligned}$$

Marking Scheme

Part A: Tangencies (2 points, additive)

- 1P: Proving that $\angle DTH = \angle DAT$ or that $\angle CTH = \angle CAT$.
- 1P: Proving that $\angle THD = \angle HBD$ or that $\angle THC = \angle HBC$.

Part B: More angle chasing (3 points, non-additive)

- 1P: Proving one of the following: $\angle DHB = \frac{\gamma}{2}$, $\angle ATD = \frac{\gamma}{2}$, $\angle BDC = \alpha + \frac{\gamma}{2}$, $\angle CDA = \beta + \frac{\gamma}{2}$.
- 2P: Proving one of the following: $\angle THB = \angle HBA + \frac{\gamma}{2}$, $\angle ATH = \angle DAT + \frac{\gamma}{2}$, $\angle DHB = \angle ATD$, $\angle HDT = \frac{\gamma}{2}$.
- 3P: Proving $\angle HDT = \angle DHB$, or $\angle HDT = \angle ATD$, or both $\angle THB = \angle HBA + \frac{\gamma}{2}$ and $\angle ATH = \angle DAT + \frac{\gamma}{2}$.

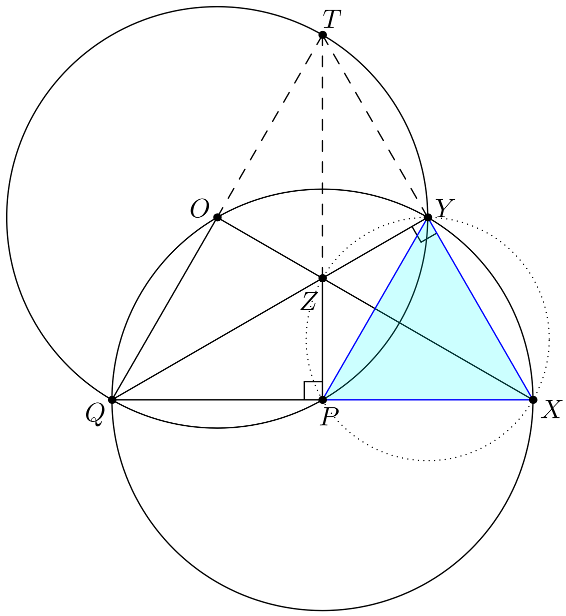
Part C: Concluding (2 points)

- 2P: Concluding.

Remark: Proving $AT \parallel DH$, or $DT \parallel BH$ is at least worth 5 points.

G2) Let points P and Q lie on a circle k_1 with centre O. Let k_2 be the circle which is centered at P and passes through Q. Define X as the second intersection of k_2 and line PQ, and Y as the second intersection of k_2 and k_1. Let Z be the intersection of OX with QY. Prove that if $PZYX$ is cyclic, then PYX is an equilateral triangle.

Solution



By Thales theorem we know that $\angle QYX = 90°$, as QX is a diameter of the circle k_2. Under the assumption of $PZYX$ being cyclic, we then also get that $\angle XPZ$ is also a right angle. Let T be the intersection of PZ and XY. We then have $\angle TPQ = \angle TYQ = 90°$, as two angles on a straight line add up to $180° = 90° + 90°$.
This not only implies that $QTYP$ is cyclic but also that O is the midpoint of the line QT. This follows from Thales theorem, as QT must be a diameter of the circumcircle and O is exactly the centre of this circle.
Taking a step back we realise that Z must be the orthocentre of the triangle QTX; indeed we have

$$\angle YTQ = 180° - \angle YPQ = \angle YPX = \angle YZX = 180° - \angle YZO,$$

implying that $OTYZ$ is cyclic. So $\angle ZOT = 180° - \angle TYZ = 90°$.
We can now finish in multiple ways:

(a) One can either show that $\angle XQT = \angle QTX$ by noting that X is on the perpendicular bisector, which makes the triangle XQT isosceles. This implies that

$$\angle XPY = \angle 180° - \angle YPQ = \angle QTY = \angle PQT = 180° - \angle TYP = \angle PYX.$$

And from $|PY| = |PX|$ (as Y, X both lie on the circle with center P), we know that $\angle PYX = \angle YXP$, implying that PXY must be equilateral.

(b) One can also show that Z is not only the orthocentre of the triangle QTX but also the circumcentre. This implies that the altitudes are also the perpendicular bisectors and thereby all three points Q, T, X lie on the perpendicular bisector of the other two points, implying that QRX is equilateral. So $\angle YXP = \angle TXQ = 60°$. Once again using $\angle PYX = \angle YXP = 60°$, we can conclude that PYX is equilateral.

Solution 2: We can also prove the statement without introducing any new points.

By Thales theorem, we know that $\angle ZYX = 90°$. By the cyclic quadrilateral this also implies that

$$\angle ZPQ = 180° - \angle XPZ = \angle ZYX = 90°.$$

Define the angle α, as the angle at $\angle OXQ$. We then have

$$\angle POQ = 2 \cdot \angle PYQ = 2 \cdot \angle PYZ = 2 \cdot \angle PXZ = 2 \cdot \alpha.$$

As the triangle OPQ is isosceles, as $|OP| = |OQ|$, we have $\angle OPQ = \angle PQO = \frac{180° - \angle QOP}{2} = 90° - \alpha$. Together with $\angle ZPQ = 90°$, we get $\angle ZPO = \angle ZPQ - \angle OPQ = \alpha$, meaning that

$$\angle ZPO = \angle ZYX = \angle ZYP$$

by cyclicity. Comparing angles $\angle YPO$ and $\angle OYP$ in the isosceles triangle OYP yields that

$$\angle YPZ = \angle OYZ,$$

which in turn implies that

$$\angle YXO = \angle OYQ = \angle YXZ = \angle YPZ = \angle OYQ = \angle YQO$$

(because $|OY| = |OQ|$), that is, O belongs to k_2.
We easily conclude by a bunch of lengths equalities :

$$|PQ| = |PY| = |PO| = |YO| = |QO|$$

implies that $\triangle OYP$ and $\triangle OQP$ are equilateral. Thus, angle $\angle XPY$ equals $180° - 2 \cdot 60° = 60°$, finishing the proof.

Marking Scheme

Solution 1:

- +1P for introducing the intersection T of PZ and XY, respectively introducing the antipode of Q
- +2P for realising that T is the antipode of Q, respectively the intersection of PZ and XY (1P for claiming so, +1P for proving it)
- +2P for deducing that Z is the orthocentre of XQR
- +2P for finishing

We deduct 1 point for any minor flaws.

Solution 2:

- +2P for any useful angle chase, leading to angle equalities like $\angle ZPO = \angle ZYP$
- +1P for deducing that $\angle YPZ = \angle OYZ$
- +2P for realizing that O belongs to k_2 (1P for claiming so, +1P for proving it)
- +2P for finishing

We deduct 1 point for any minor flaws.

C1) Let n be a positive integer. Annalena has n different bowls numbered 1 to n and also n apples, $2n$ bananas and $5n$ strawberries. She wants to combine ingredients in each bowl to make fruit salad. The fruit salad is *delicious* if it contains strictly more strawberries than bananas and strictly more bananas than apples. How many ways are there for Annalena to distribute all the fruits to make delicious fruit salad in each of the different bowls?

Remark: A delicious fruit salad is allowed to not contain any apples.

Solution

The answer is $\binom{2n-1}{n} \cdot \binom{3n-1}{2n} = \frac{(2n-1)!(3n-1)!}{(2n)! \cdot n! \cdot (n-1)!^2} = \frac{(3n-1)!}{2n^2 \cdot (n-1)!^3}$. How'd you like them apples?

For a salad to be *delicious* we will need at least one banana and at least two strawberries, as the number of apples is at least 0. So we might as well distribute this necessary amount already, leaving us with $n - 0 = n$ apples, $2n - n = n$ bananas and $5n - 2n = 3n$ strawberries. We shall now distribute the rest of the fruits, starting with apples.

Every time we add an apple to a bowl, we will need to add a banana and a strawberry to it as well, to keep the salad delicious. After all if we only added an apple and no banana, the amount of bananas won't be strictly more than the amount of apples anymore. Similarly, since we added a banana we have to add a strawberry to keep the amount of strawberry strictly bigger than the amount of bananas.
From the script we know that are $\binom{a+b-1}{a}$ ways of distributing a objects into b bowls, where we don't care in which order we distribute the objects and multiple objects can be placed into the same bowl. So there are $\binom{2n-1}{n}$ ways of distributing the n apples together with the n bananas and strawberries into the n bowls.

We are left with $n - n = 0$ apples, $n - n = 0$ bananas and $3n - n = 2n$ strawberries. So the only fruits that we can distribute are the strawberries. Clearly adding more strawberries to an already delicious salad will still leave it delicious. So we can freely distribute the remaining $2n$ strawberries onto the n bowls. We know that there are $\binom{2n+n-1}{n} = \binom{3n-1}{n}$ ways of doing so.

So all in all there are $\binom{2n-1}{n} \cdot \binom{3n-1}{n} = \frac{(3n-1)!}{2n^2 \cdot (n-1)!^3}$ different ways of creating n delicious salads.

Solution 2: Number the n bowls from 1 to n. Distributing a apples onto b bowls is the same thing as drawing a number in $\{1, \ldots, b\}$ for each of the a apples and then putting the apple into the bowl corresponding to that number. So as we showed in the script, there are exactly $\binom{a+b-1}{a}$ ways of distributing a objects onto b bowls.

Let's first distribute all the apples. There are $\binom{2n-1}{n}$ ways to do so.
Next up are bananas. We have to match the number of apples in each bowl and then add one more banana to it, to make the amount of bananas strictly greater than the amount of apples. To match the amount of apples, we need n bananas, and then adding one more banana to each bowl will use up all the remaining bananas. So the distribution of the bananas is forced by the distribution of the apples.
Just as we did it with the apples and bananas, we have to match the amount of bananas and then at least one more strawberry to ensure the salad is delicious. This will force the placement of $2n + n$ of our strawberries, leaving us with $5n - 3n = 2n$ strawberries to freely distribute. There are $\binom{2n+n-1}{2n}$ to do so.

So all in all there are $\binom{2n-1}{n} \cdot \binom{3n-1}{2n}$ ways to make n delicious fruit salads.

Marking Scheme

Marking scheme, sol 1: We award the following partial points

- +2P For noting that there are $\binom{a+b-1}{a}$ ways of distributing a objects onto b bowls and applying it in the context of this problem (with possibly incorrect use case but correct application).
- +1P for handing out 1 banana, 2 strawberries to each bowl.
- +1P Noting that every time we put an apple into a bowl we also have to add a banana and one strawberries.

We deduct 1 point for any minor flaws (e.g. using $n+1$ instead of n) and for lack of explanation of certain steps.
For a complete proof, where one used the wrong formula to calculate the ways of distributing a fruits onto the n bowls, we deduct 3 points.

Marking scheme, sol 2: We award the following partial points

- +2P For noting that there are $\binom{a+b-1}{a}$ ways of distributing a objects onto b bowls (with possibly incorrect use case but correct application).
- +1P for handing out the apples, bananas, strawberries in that order.
- +1P Noting that $2n$ of the bananas and $3n$ of the strawberries are forced.

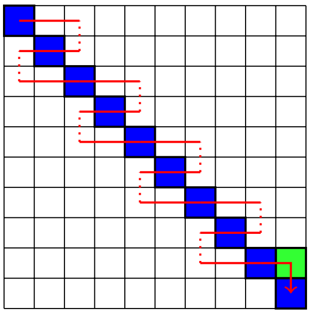
We deduct 1 point for any minor flaws (e.g. using $n+1$ instead of n) and for lack of explanation of certain steps.
For a complete proof, where one used the wrong formula to calculate the ways of distributing a fruits onto the n bowls, we deduct 3 points.

C2) Consider a 2024×2024 grid, where the 2024 squares on one of the two diagonals are coloured blue. Sam writes one of the numbers $1, 2, \ldots, 2024^2$ in each of the squares of the grid in such a way that every number appears exactly once and the squares containing $i-1$ and i share an edge for any $2 \leq i \leq 2024^2$. Prove there are always two blue squares containing values that differ by exactly 2.

Solution

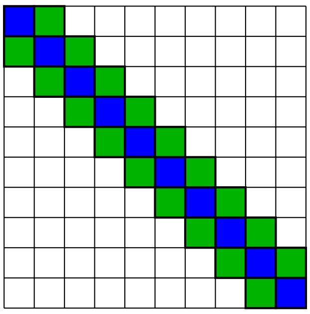
Clearly, the numbering corresponds to a path that visits every square of the grid exactly once. We will proceed with this analogy in mind.

If you try and draw a path that does not satisfy the condition, you will realise there appears to be a lack of "space" to move through. More precisely, if you manage to move in such a manner that you do not re-enter the diagonal immediately after exiting it, you will seemingly be forced either at the start or the end of your path.



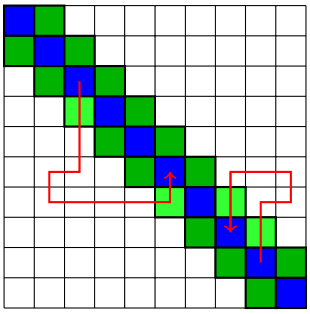
The diagrams we will provide will be of a 10×10 grid, but the only important fact here is that the sidelengths are even. These diagrams generalise to the case outlined by the problem.

Let us formalise this idea. Colour the squares adjacent to the diagonal in green. There are $2023 \cdot 2 = 4046$ green squares.



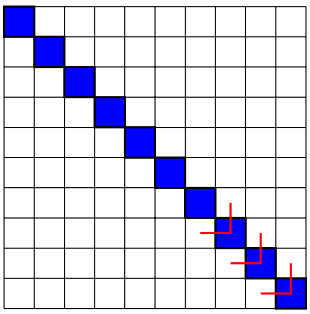
When leaving and entering a blue square, one must pass via a green square. Therefore, if it was possible to find a path that does not satisfy the condition, then between visiting any two blue squares we must visit at least two green squares, as a single green square would have to be both the exit and entry point, and fall exactly between the two blue squares on the path. With 2024 blue squares and 4046 green squares, it follows that between any two consecutive blue squares on the path we must visit exactly two green squares.

Here, we can find a contradiction in two different ways. The first is that the two green squares between a given pair of consecutive blue squares must be on the same side of the diagonal. With an odd number of squares on either side of the diagonal, it is impossible to partition the 4046 green squares into 2023 pairs that are both on the same side of the diagonal, and so such a path is impossible.

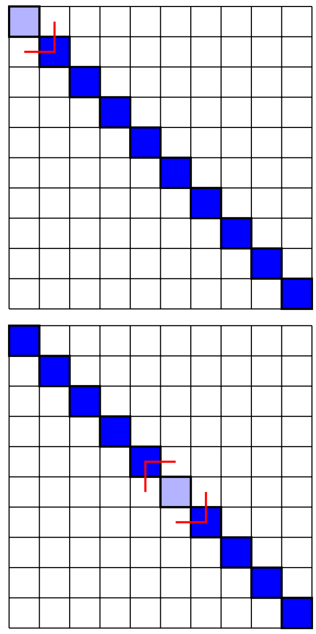


The second is that if we consider a chessboard coloration of the board, then all the squares on the main diagonal have the same colour. As a result, there is no path that starts and ends on the diagonal and visits every square, as with an even number of squares, and switching colour ever time you move to a new square, you must start and end on squares of different colours. This means that there must be either a green square before the first blue square or after the last blue square (since we cannot start and end on both of these) and the remaining green squares (at most 4045) are not enough to have two greens between every two consecutive blues.

There is another way to approach the pigeonhole idea via a more local argument. As before, note that we cannot start and finish on the diagonal. Note that, given a corner we do not start or finish in, we must forcibly move through it in an L shape. This remains true for all adjacent squares on the diagonal from which we do not start or finish.



There are three cases, depending on whether one end point is on the diagonal and whether that end point is the corner, but all three can be solved in a similar fashion by looking at Ls until we reach some point with a clash leading a contradiction.



Marking Scheme

Part A: Setup
The following points are additive with each other.

- 0P: Considering the squares adjacent to the main diagonal.
- 1P: Stating that directly before or directly after a blue square, we have a green square.
- 1P: Noting that the question is equivalent to having exactly one green square between two blue squares (in the path analogy)
- 1P: Noting that there are exactly enough green squares to have two between every pair of consecutive blue squares.

Part B: Contradiction
The following points are non-additive with each other.

- 1P: Claiming there is a green square before the first blue or after the last blue square.
- 0P: Proving the points on the diagonal have the same parity in the path.
- 1P: Proving you cannot start and end on the main diagonal.
- 3P: Proving you cannot start and end on the main diagonal *and* this implies there is a green square before the first blue or after the last blue square.
- 0P: Noting that there are an odd number of squares on either side of the diagonal.
- 2P: Noting that there an odd number of squares on either side of the diagonal *and* one enters and exits on the same side of the diagonal.
- 3P: Noting that there an odd number of squares on either side of the diagonal *and* one enters and exits on the same side of the diagonal *and* this means there is at least one green square not found between two blue squares in the path.

The contestant is awarded the sum of their scores from both parts of the markscheme. They are awarded an additional point for noting the contradiction between the two results means they are finished.

For contestants who adopt the local approach with forced L-moves rather than the global one, the following markscheme is applied.

- 3P: The case where you neither start nor end on the diagonal.
- 2P: The case where you start or end on the diagonal.
- 1P: Proving you cannot start and end on the diagonal.
- 1P: Combining the above to conclude.

Partial points may be awarded if the contestant shows awareness that there is a case that needs to be handled.

Any other complete and valid solutions will be awarded 7 points.
Remark: Equivalent points will be awarded for equivalent statements. For instance, "Proving you cannot start and end on the main diagonal." is equivalent to "Proving the numbers 1 and 2024^2 are not both on the main diagonal".

N1) Determine all triples (a, b, n) of positive integers where a divides n, b divides n, and

$$(a+1)(b+1) = n.$$

Solution

All the possible triples are $(1, 1, 4)$, $(1, 2, 6)$, $(2, 1, 6)$, $(2, 3, 12)$ and $(3, 2, 12)$.

Proof. Let (a, b, n) be such a triple. We know that

$$a \mid n = (a+1)(b+1)$$

and since a and $a+1$ are coprime, we must have $a \mid b+1$. Similarly, we find $b \mid a+1$. Since $a+1$ and $b+1$ are positive, it follows that $a \leq b+1$ and $b \leq a+1$. We now split into two cases. If $a \neq b+1$ and $b \neq a+1$ then we must have $2a \leq b+1$ and $2b \leq a+1$. Combining these inequalities we get

$$4a \leq 2(b+1) = 2b+2 \leq (a+1)+2 = a+3,$$

hence $a \leq 1$ and similarly $b \leq 1$. This leads to the solution $(1, 1, 4)$. In the other case we can assume without loss of generality that $a = b+1$. But now we have

$$a-1 = b \mid n = (a+1)(b+1) = (a+1)a = (a-1)(a+2)+2$$

which implies that $a-1 \mid 2$. This is only true for $a = 2, 3$. In this case we get two more solutions $(2, 1, 6)$ and $(3, 2, 12)$ and of course their symmetries $(1, 2, 6)$ and $(2, 3, 12)$. $\square$

Marking Scheme

- +1P Proving that $a \mid b+1$
- +1P Proving that $b \mid a+1$ (or the previous point + an explicit mention of symmetry)
- +2P Finishing the case where $a = b+1$ (or symmetrical)
- +1P Obtaining a useful inequality
- +1P Noting that if $a \neq b+1$ then $2a \leq b+1$ (or symmetrical)
- +1P Finishing the case where $a \neq b+1$ and $b \neq a+1$

If a contestant forgets some of the symmetrical solutions, at most **1 point** can be deducted and **only if they would otherwise have ≥ 6 points**.

N2) Determine all positive integers n with the property that for each divisor x of n there exists a divisor y of n such that $x + y \mid n$.

Remark: The divisors may be negative.

Solution

The set of natural numbers satisfying this property is the set of even numbers.

Proof. We need to prove two statements here. Firstly, if n is odd, then the property does not hold, and secondly, if n is even, the statement is true.

If n is odd, then note that, any divisor of n is odd. Thus, for any divisors $x, y \mid n$, we get that x, y are odd, and hence $x + y$ is always even, which implies that $x + y$ is **never** a divisor of n. In conclusion, for any n odd, the property does not hold.

If n is even, then by definition, we can write $n = 2m$ for some $m \in \mathbb{N}$. Let x be a divisor of n. The goal here is to construct y such that $x + y \mid n$. We have two subcases based on the parity of x.

- If x is odd, then since x is coprime to 2, $x \mid 2m$ implies $x \mid m$. Hence, $2x \mid 2m = n$, so $2x$ is a divisor of n. Thus, if we take $y = -2x$, we get that $x + y = -x$ is a divisor of n as required.
- If x is even, then $\frac{x}{2}$ is a natural number, and is a divisor of n (since $\frac{x}{2} \mid x \mid n$). Therefore, if we take $y = -\frac{x}{2}$, one gets that $x + y = \frac{x}{2}$ is a divisor of n as required.

In conclusion, either way, we successfully managed to find, given a divisor x of n, a divisor y such that $x + y$ is also a divisor of n. This concludes the proof. $\square$

Remark: In the case n even and x odd, another valid choice for y is $y = x$, since $x + y = 2x \mid n$ in this specific case.

Marking Scheme

- +1P Stating that the set of solutions is the set of even numbers
- +2P Proving that the property does not hold for n odd
  - +1P Stating explicitly that, if n is odd, any divisor of n must be odd
  - +1P Deducing that $x + y$ must be even for any divisors x, y of n
- +3P Proving that the property does hold for n even
  - +1P Proving that the statement holds for n even, x even
  - +1P Stating or justifying explicitly that, if n even, x odd, then $2x$ is a divisor of n
  - +1P Concluding the case n even, x odd
- +1P Conclude