IMO - Selection 2018

Fourth exam - 27 May 2018

Duration: 4.5 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

10. Let ABC be a triangle, M the midpoint of BC and D a point on the line AB such that B lies between A and D. Let E be a point such that E and B are on different sides with respect to the line CD and such that $\angle EDC = \angle ACB$ and $\angle DCE = \angle BAC$. Let F be the intersection point of CE with the parallel line to DE through A. Let Z be the intersection point of AE and DF. Prove that the lines AC, BF and MZ intersect in a point.

11. Determine all the pairs (f, g) of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

- $f(x) \geq 0$,
- $f(x + g(y)) = f(x) + f(y) + 2yg(x) - f(y - g(y))$.

12. David and Linus play the following game: David chooses a subset Q of $\{1, \ldots, 2018\}$. Then Linus chooses a natural number a_1 and computes inductively the numbers $a_2, \ldots, a_{2018}$ with a_{n+1} being the product of all positive divisors of a_n.

Let P be the set of integers $k \in \{1, \ldots, 2018\}$ for which a_k is a perfect square. Linus wins if $P = Q$. Otherwise David wins. Determine which player has a winning strategy.

Good Luck!