IMO Selection 2024

First Exam – Solutions

Bern
May 4, 2024

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

1. Let $n > 1$ be an odd integer with smallest prime divisor p. Assuming that any prime divisor q of n also divides n/q, prove that

$$\sqrt{n^{p+1}} \mid 2^{n!} - 1.$$

Solution

It suffices to show that for every prime number q we have

$$v_q(\sqrt{n^{p+1}}) \leq v_q(2^{n!} - 1).$$

We may further assume that $q \mid n$, since otherwise $v_q(\sqrt{n^{p+1}}) = 0$. Now, note that

$$v_q(\sqrt{n^{p+1}}) = \frac{1}{2} v_q(n^{p+1}) = \frac{p+1}{2} v_q(n).$$

Now since $q \mid n$, we clearly have $q - 1 \leq n$ and thus $q - 1 \mid n!$ holds. By Fermat, we know that $q \mid 2^{q-1} - 1$ using that q is odd. Now using LTE, we find that

$$v_q(2^{n!} - 1) = v_q\left((2^{q-1})^{\frac{n!}{q-1}} - 1\right) = v_q(2^{q-1} - 1) + v_q\left(\frac{n!}{q-1}\right) = v_q(2^{q-1} - 1) + v_q(n!),$$

where we used that q does not divide $q - 1$ in the last equality. But by counting powers of q,

$$v_q(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{q^i} \right\rfloor \geq \sum_{i=1}^{v_q(n)} \frac{n}{q^i} = \frac{n}{q^{v_q(n)}} \frac{q^{v_q(n)} - 1}{q - 1} \geq \frac{q^{v_q(n)} - 1}{q - 1} \geq \frac{p^{v_q(n)} - 1}{p - 1},$$

where we used the fact that $q^i \mid n$ for $1 \leq i \leq v_q(n)$ in the first inequality and monotonicity in $q \geq p$ for the second. Now note that

$$\frac{p^{v_q(n)} - 1}{p - 1} = (1 + p) + p^2 + \cdots + p^{v_q(n)-1} \geq (v_q(n) - 1)(p + 1)$$

since $p^i \geq p + 1$ for all $i \geq 2$. It now suffices to show that

$$(v_q(n) - 1)(p + 1) \geq \frac{p+1}{2} v_q(n)$$

which is equivalent to $v_q(n) \geq 2$ and therefore true by assumption.

Marking Scheme

(a) 1P: Any useful statement using p-adic valuations

(b) 2P: Showing that $v_q(2^{n!} - 1) \geq v_q(n!)$

- -1P: Not arguing why $q - 1 \mid n!$

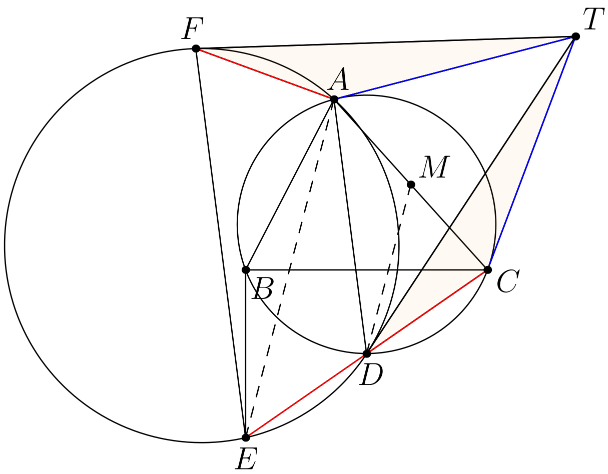
(c) 3P: Showing that $v_q(n!) \geq v_q(n)(p+1)/2$

- 1P: For showing $v_q(n!) \geq \sum_{i=1}^{v_q(n)} \frac{n}{q^i}$, or an equivalent statement
- 1P: For showing $\sum_{i=1}^{v_q(n)} \frac{n}{q^i} \geq \frac{p^{v_q(n)}-1}{p-1}$, or an equivalent statement
- 1P: For showing $\frac{p^{v_q(n)}-1}{p-1} \geq (v_q(n) - 1)(p+1)$, or an equivalent statement

(d) 1P: Concluding

2. Let ABC be a triangle with circumcircle Γ. Let $D \neq A$ be the second intersection of the internal bisector of $\angle BAC$ with Γ. We define E to be the intersection of line CD with the line perpendicular to BC through B, and ω to be the circumcircle of ADE. The line parallel to AD passing through E intersects ω at $F \neq E$. Moreover, let the tangents to Γ at A and C intersect at T. Prove that TF is tangent to ω.

Solution



(David)
We can start by noticing that $DB = DC$ by WUM. Moreover, as EBC is a right triangle, we can conclude that D is the midpoint of EC.
Now, we will prove the following key claim :

Claim : The line TD is tangent to circle ω.

Proof. If we introduce the midpoint M of side AC we can conclude by two observations :

- On one hand, the line DT is the D-symmedian in triangle DCA, so

$$\angle CDT = \angle MDA.$$

- On the other hand, if we focus on triangle ACE we notice that D, M are midpoints of sides, so $AE \parallel MD$ by Thales and we have

$$\angle MDA = \angle EAD.$$

We easily conclude by the tangent chord theorem. Now, we will clearly be done if we can prove $TD = TF$. But notice that

- As $EF \parallel AD$ and these four points are on a circle, they form an isosceles trapezoid and we know that $CD = DE = AF$.
- $TC = TA$ as T is the intersection of the tangents from Γ.

- There are a few ways to show that $\angle TCD = \angle TAF$. For example one could introduce the second intersection $G \neq A$ of FA with Γ, to get the equalities

$$\angle TCD = 180^\circ - \angle DAC = 180^\circ - \angle GDA = 180^\circ - \angle GAT = \angle TAF,$$

  because by construction, $FADE$ is an isosceles trapezoid and so $\angle GAD = \angle ADC \implies GADC$ is also an isosceles trapezoid.

Therefore, we conclude by side-angle-side that triangles TCD and TAF are congruent, yielding in particular that $TD = TF$.

Solution 2 (Mathys):
We first prove TD is tangent to ω in the same manner as the first solution.

Now, notice that we can reformulate the problem in terms of triangle $\triangle AEC$ (with D as the midpoint of side EC), and circumcircles (ACD) and (AED). With this reformulation, we can introduce $G \in \Gamma$ the same way as above (notice that then $CG \parallel AD$) and apply the above claim on triangle $DFG:$ it gives that the tangents to ω at F, D, and the tangent to Γ at A are concurrent.

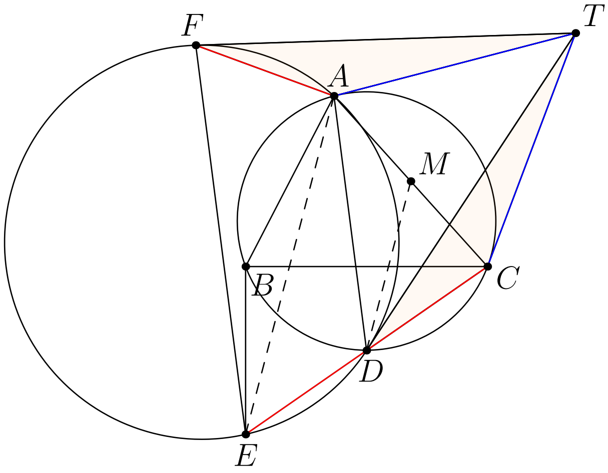
Putting everything together, we indeed get that T (being the intersection of the tangent to Γ at A and the tangent to ω at D) is the desired point of concurrence. In particular TF is tangent to ω.

Solution 3 (Marco):
We first prove $\triangle TAF \cong \triangle TCD$ in the same manner as the first solution.

Now we have that $TD = TF$, but also that $\angle FTA = \angle DTC$, so that

$$\angle FTD = \angle ATC = 180^\circ - 2\beta.$$

As $\triangle TFD$ is isosceles, we have $\angle DFT = \beta = \angle CDA = \angle DEF$, and TF is tangent to ω.


F
T
A
M
B
C
D
E

Marking Scheme

(a) 4P: Proving that $\triangle TCD$ is congruent to $\triangle TAF$.

The following are **additive**.

- 1P: Proving that $AF = CD$.
- 1P: Proving that $\angle TAF = \angle TCD$.

(b) 3P: Finishing

The following are **non-additive**.

- 1P: Proving that $\angle FTD = 180° - 2\beta$ (as in Solution 3).
- 2P: Introducing the midpoint of AC, and involving it in a useful way (either the use of symmedian or the use of Thales, as in Solution 1).
- 3P: Proving that TD is tangent to ω.

Remarks:

- If no other point is granted, it is worth 1P to claim that TD is tangent to ω.
- Proving $\triangle TCD \cong \triangle TAF$ and that TD is tangent to ω without concluding is worth 6P.
- Minor mistakes deduce 1P.

3. Determine all monic polynomials P with integer coefficients such that for all integers a and b, there exists an integer c such that $P(a)P(b) = P(c)$.

Solution

Mathys

The solution is the set of all polynomials P of the form $P(x) = (x - d)^n$ for some $d \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$.

Now assume that $P(x)$ is a non-constant polynomial that fulfils the condition in the problem statement.
By induction we can show that for any set of integers $(b_1, \dots, b_k)$ there exists an integer $c \in \mathbb{Z}$, such that $P(b_1) \cdot P(b_2) \cdots P(b_k) = P(c)$. So specifically we know that for every $b, k \in \mathbb{Z}$ there exists a c_n, such that $P(C) = P(b)^k$. We shall now show that if P isn't of the form $P(x) = (x - d)^n$, then there exists a constant $C > 0$, such that for $c \in \mathbb{Z}, |c| > C$, we have that $P(c)$ never is an n-th power, where $n := \deg(P)$. This will lead to a contradiction for $k = n$ and $|b|$ sufficiently large, as $P(b)$ tends to infinity as $|b|$ tends to infinity.

Write $P(x) = x^n + a_{n-1}x^{n-1} \cdots + a_0$ and let $Q(x) = (x + \frac{a_{n-1}}{n})^n$.
Claim: For any ε, we can find a constant $C_\varepsilon > 0$, such that $|Q(x - \varepsilon)| < |P(x)| < |Q(x + \varepsilon)|$ for all $|x| > C_\varepsilon$.
Indeed, for n odd, we have

$$(x + \frac{a_{n-1}}{n} + \varepsilon)^n > P(x) \Leftrightarrow n\varepsilon \cdot x^{n-1} + \sum_{i=0}^{n-2} \binom{n}{i} \left(\frac{a_{n-1}}{n} + \varepsilon\right)^{n-i} \cdot x^i - a_i \cdot x^i > 0,$$

which is true for all $|x| > \frac{S}{n\varepsilon}$, where $S_+ = \sum_{i=0}^{n-2} \left|\binom{n}{i} \frac{a_{n-1}+n\varepsilon}{n}\right| + |a_i|$. Similarly we have $P(x) > (x + \frac{a_{n-1}}{n} - \varepsilon)^n$ for all $x > \frac{S_-}{n\varepsilon}$, where $S_- := \sum_{i=0}^{n-2} |a_i| + \left|\binom{n}{i} \frac{a_{n-1}-n\varepsilon}{n}\right|$. Picking $C_\varepsilon = \max(S_+, S_-)$ finishes the case n odd, as $P(x)$ and $Q(x \pm \varepsilon)$ have are positive for $x > C_\varepsilon$.
In the case n even, we can still pick $C_\varepsilon = \max(S_+, S_-)$. The only difference to n odd, is that we have to swap all inequality signs for $x < 0$. But since $P(x)$ and $Q(x - \varepsilon)$ are both negative, when $x > C_\varepsilon$, the claim still holds.

We now distinguish between two cases:
1) $n \mid a_{n-1}$. Picking $\varepsilon = 1$, we see that the only way that $P(c)$ for c sufficiently large is an integer n-th power, is if $P(c) = Q(c)$. Since this now must hold for infinitely c, we have $P \equiv Q$, which is the same form as the claimed solution set.
2) $n \nmid a_{n-1}$. Picking $\varepsilon = \min\left(\lceil\frac{a_{n-1}}{n}\rceil - \frac{a_{n-1}}{n}, \frac{a_{n-1}}{n} - \lfloor\frac{a_{n-1}}{n}\rfloor\right)$, instantly gives us that $P(c)$ can't be an integer n-th power for c sufficiently large.

Now at last it's easy to check that $P(x) \equiv 1 = (x - d)^0$ is the only constant polynomial that works, finishing our proof.

Marking Scheme

(a) 1 point: Claiming the solution to be the set of polynomials of the form $P(x) = (x - d)^n$.

(b) 1 point: Showing that for every set $b_1, \ldots, b_k \in \mathbb{Z}$ there exists an integer c, such that $P(c) = \prod_{i=1}^{k} P(b_i)$.

(c) 1 point: Claiming that the polynomial lies between $(x - c_1)^n$ and $(x + c_2)^n$.

(d) 2 points: proving that the claim holds and that ε can be chosen sufficiently small.

(e) 1 point: Case n divides a_{n-1}.

(f) 1 point: Case n doesn't divide a_{n-1}.

IMO Selection 2024

Second Exam – Solutions

Bern
May 5, 2024

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

4. Let $a_1, \dots, a_{2^{2024}}$ be a sequence of pairwise distinct positive integers. Define

$$S_n = \frac{1}{1+a_1} + \frac{a_1}{(1+a_1)(1+a_2)} + \dots + \frac{a_1 a_2 \cdots a_{n-1}}{(1+a_1)(1+a_2)\cdots(1+a_n)}.$$

Determine how many sequences $a_1, \dots, a_{2^{2024}}$ exist, such that $S_{2^i} = \frac{2^i}{2^i+1}$ for all $0 \leq i \leq 2024$.

Solution

By induction one can show that $S_n = 1 - \frac{a_1 a_2 \cdots a_n}{(1+a_1)(1+a_2)\dots(1+a_n)}$. Indeed $S_1 = \frac{1}{1+a_1} = 1 - \frac{a_1}{1+a_1}$ and

$$\begin{aligned}
S_n =& S_{n-1} + \frac{a_1 \cdots a_{n-1}}{(1+a_1)\cdots(1+a_n)} \\
=& 1 - \frac{a_1 \cdots a_{n-1}}{(1+a_1)\cdots(1+a_{n-1})} + \frac{a_1 \cdots a_{n-1}}{(1+a_1)\cdots(1+a_n)} \\
=& 1 - \frac{a_1 \cdots a_n}{(1+a_1)\dots(1+a_n)}.
\end{aligned}$$

Let $T_n = \frac{a_1 \cdots a_n}{(1+a_1)\dots(1+a_n)}$. We want that $T_{2^i} = \frac{1}{2^i+1}$ for all $i \leq 2024$. Since $\frac{x}{1+x}$ is an increasing function in x, we have that $T_n \geq \frac{1 \cdot 2 \cdots n}{(1+1)\cdot(2+1)\cdots(n+1)} = \frac{1}{n+1}$, with equality if and only if $a_1, \dots, a_n$ is a permutation of $1, \dots, n$.

So the condition is fulfilled by exactly the sequences, such that $a_{2^{i-1}+1}, \dots, a_{2^i}$ is a permutation of $2^{i-1}+1, \dots, 2^i$, for all $1 \leq i \leq 2024$ and $a_1 = 1$. Since there are $\prod_{i=1}^{2024} 2^{i-1}!$ such sequences, the answer is $2! \cdot 4! \cdots 2^{2023}!$.

Marking Scheme

For a full solution 7 points will be awarded. Else the following marking scheme for partials applies, where (a) and (b) are non-additive.

(a) 1 point: Proving that for $a_i = i$, we have $S_{2^i} = \frac{2^i}{1+2^i}$. The point is also awarded if this result is directly implied by their result.

(b) 1 point: Proving that $S_n = 1 - \frac{a_1 \cdots a_n}{(1+a_1)\cdots(1+a_n)}$.

(c) 3 points: Proving that $S_n = \frac{n}{n+1}$ if and only if $a_1, \ldots, a_n$ is a permutation of $1, \ldots, n$.

   i. 1 point: Claiming the statement.

   ii. 1 point: Proof that $\max(a_1, \ldots, a_{2^i}) \leq 2^i$. (only if direction)

   iii. If point (b) was awarded, +1 point for claiming $S_n \geq \frac{n}{n+1}$.

   iv. 1 point:

(d) 1 point: Correct classification of all possible sequences.

(e) 1 point: Stating the correct answer of $2! \cdot 4! \cdots 2^{2023}!$.

In (c) only i and ii are additive.

5. Let $n \geq 4$ be an integer and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sequences of positive integers such that the $n+1$ products

$$\begin{gathered} a_1 a_2 \cdots a_{n-1} a_n, \\ b_1 a_2 \cdots a_{n-1} a_n, \\ b_1 b_2 \cdots a_{n-1} a_n, \\ \vdots \\ b_1 b_2 \cdots b_{n-1} a_n, \\ b_1 b_2 \cdots b_{n-1} b_n, \end{gathered}$$

taken in this order, form a strictly increasing arithmetic progression. Determine the smallest possible common difference of this arithmetic progression in terms of n.

Remark: An arithmetic progression is a sequence of the form $a, a+r, a+2r, \ldots, a+kr$ where a, r and k are integers and r is called the common difference.

Solution

The answer is $n!$.

We first give a construction that achieves this value: Take $a_k = k$ and $b_k = k+1$ for all $1 \leq k \leq n$. The k-th of the $n+1$ numbers then becomes

$$b_1 b_2 \cdots b_{k-2} b_{k-1} \cdot a_k a_{k+1} \cdots a_n = 2 \cdot 3 \cdots (k-1)k \cdot k(k+1) \cdots n = k \cdot n!$$

and thus they form an arithmetic progression with $n!$ as the common difference.

Let us now prove that we cannot do any better than $n!$. Set $A = a_1 a_2 \cdots a_n$ and let d denote the common difference. Taking the ratio of two consecutive terms of the arithmetic progression, we find that

$$\frac{b_k}{a_k} = \frac{A + kd}{A + (k-1)d}.$$

Thus, we must have

$$A + (k-1)d \mid a_k \cdot \gcd\big(A + kd, A + (k-1)d\big) = a_k \cdot \gcd(A, d),$$

by comparing denominators and using the properties of the greatest common divisor. But now, note that

$$a_k \geq \frac{A + (k-1)d}{\gcd(A, d)} = \frac{A}{\gcd(A, d)} + (k-1)\frac{d}{\gcd(A, d)} \geq 1 + (k-1) = k$$

and using that $b_1 > a_1$, we conclude

$$d = (A + d) - A = (a_1 - b_1)a_2 a_3 \cdots a_n \geq 2 \cdot 3 \cdots n = n!$$

Marking Scheme

- 2P: Finding the construction for $n!$
- 5P: Proving that $n!$ is best possible
  - 1P: Claiming that $n!$ is best possible
  - 1P: Considering the ratios b_k/a_k
  - 1P: Proving that $A + (k-1)d \leq a_k \cdot \gcd(A, d)$, or an equivalent statement
  - 1P: Proving that $a_k \geq k$

In case of no points from the above, one point can be awarded for the observation that $b_i > a_i$ for all $i = 1, ..., n$.

6. Let $n \geq 2$ be an integer. Kaloyan has a $1 \times n^2$ strip of unit squares, where the i-th square is labelled with i for all $1 \leq i \leq n^2$. He cuts the strip into several pieces, each piece consisting of a number of consecutive unit squares. He then places the pieces, without rotation or reflection, on an $n \times n$ square such that the square is covered entirely and the unit square in the i-th row and j-th column contains a number congruent to $i + j$ modulo n.

Determine the smallest number of pieces for which this is possible.

Solution

The minimal number of pieces is $2n - 1$.

There are several possible constructions; we give two of them. One option is to have $n - 1$ pieces of length n with pieces of length 1 between them and two pieces of length 1 at the end; another option is to have a piece of length n and alternate between pieces of length i and $n - i$, with i increasing from 1 to $n - 1$.

Now, let us work modulo n. We may assume that the strip is instead a closed loop, with the first and last unit square adjacent, as we can assume the first cut (the demarcation between two separate pieces that touch) is between n and 1, since the validity of a set of cuts is shift-invariant. We may also assume the numbers are congruent to $i + j - 1$ modulo n, as by shifting rows this is equivalent to the original problem and allows us to say that row i starts with the number i. We want to show that the total number of cuts is at least $2n-1$. We describe three approaches:

Method 1: Create a directed graph G with n vertices of the form $(i, i + 1)$ with i varying from 1 to n. We direct an edge from $(i, i + 1) \to (j, j + 1)$ if the row starting with $i + 1$ and ending with i has a cut inside it between $j, j + 1$. G is connected: assume this was not the case and consider two disjoint vertex sets A and $B := V \setminus A$. Considering all the pieces used to cover the rows corresponding to A, there must be a piece not used for these rows adjacent to one that is used. If these two pieces are separated at $(k, k + 1)$ then either $(k + 1, k)$ is a row in A and a cut in a row in B or a row in B and a cut in a row in A. In both cases, we get an arrow from $(k, k + 1)$ to or from a vertex in the opposite set.

This means we have at least $n - 1$ cuts inside rows, as each cut yields an edge. Add this to the n cuts provided by the ends of the rows and we have at least $2n - 1$ cuts.

Method 2: Create a multigraph G with n vertices corresponding to the rows. We draw an edge between two rows if they use two pieces that were adjacent in the strip (the first and last piece are also considered adjacent!), and colour this edge in colour i if the two pieces are separated at $(i, i + 1)$. It is easy to see that the edges of a given colour induce a disjoint union of cycles, with the total number of edges of that colour being the number of rows with a cut between $(i, i+1)$ where the pieces on either side of the cut were not adjacent on the initial strip.

This graph is Eulerian: if we consider the pieces in sequential order from the original strip, then every time we have two adjacent pieces in different rows, we get a new edge, in such a manner that we visit each edge exactly once and two consecutive edges thus obtained are always adjacent (with one corresponding to entering a row and the other to leaving it). In particular, it is connected. Furthermore, if we consider the partition of the graph into disjoint cycles that is induced by the colours, we can then remove one edge from each cycle and still have a connected graph (if two cycles had a common vertex before, they still do now, and we can travel around every cycle without using the missing edge). So we have at least $(n-1+k)$ edges, which give us $(n-1+k)$ cuts. Now, for each colour i not present in the graph, none of these cuts is between $(i, i+1)$. But at least one such cut exists, because of the endpoints of the row starting with $i+1$ and ending with i. This gives us at least $(n-k)$ extra cuts, for a total of $2n-1$.

Method 3: Consider a valid set of pieces. We will describe an algorithm to move the cuts around that does not affect the validity of the set. Our algorithm will decrease the number of rows with no internal cuts until only one is remaining, at which point every other row will have at least two pieces, allowing us to conclude. The algorithm proceeds as follows:

- If there are at least two rows without internal cuts, pick a row i with no internal cuts. Otherwise, terminate.
- The row starts with $i+1$ and end with i. Consider the left neighbour (James) of the single piece constituting the whole row, which we say starts with j and ends with i, and appears in row k. Introduce a cut between $(k-1, k)$ in row i.
- We can now swap the piece starting with $i+1$ and ending with $k-1$ we just created in row i (Jim), and the pieces starting with $i+1$ and ending with $k-1$ in row k (essentially, all the pieces to the right of the aforementioned left neighbour within row k).
- Finally, this means that James and Jim are adjacent in both the strip and the square, so we can remove the cut between them. Baptize this new piece John. If John is of length n, then repeat from step two onwards on John. Otherwise, go back to step one.

Note that every time we return to step two, we move one piece leftwards on the strip, changing the pieces by moving a single cut as we go along. In other words, this algorithm looks at a piece of length x to the left of a piece of length n, shifts the cut between the two, and returns to the second step if the piece of length x is now of length n - without modifying the other pieces. If this continues indefinitely then eventually we will hit another piece of length n on the left - meaning two pieces of length n are adjacent to each other, contradiction. So we must eventually return to step one, at which point the number of rows without internal cuts has decreased by 1. So the algorithm always terminates after revisiting step one enough times.

Method 4: Consider the following alternative setup: instead of the full square and the full strip, we have a strip from 1 to kn, and and $k \leq n-1$ rows of length n where *none of the rows begin with* 1 and they all begin with different numbers. We will prove by induction that we need at least $2k$ pieces for this. This is clear for $k=1$. Now, assume it holds true up to $k \leq n-2$ and let us consider $k+1$. Firstly, if every row has at least 2 pieces then we are done, so we may assume there is a piece of length n somewhere. Now, some notation: we call a "bad" cut any cut on the strip between $i, i+1$ where none of the rows we are currently working with start with $i+1$ and end with i. All bad cuts are internal in the rows, i.e. they cannot be at the endpoints of a row.

Take the piece of length n (say it covers $j+1$ to j) out and glue the two remaining pieces of the strip together (if the full row was not at either end) to get a valid partition for the case with k rows. Now, we want to show that this has at least $2k+1$ pieces. We know this partition has a bad cut between $j, j+1$. Now take the union of all bad cuts between $j, j+1$ and move them all simultaneously to the right. We can do this as if you imagine what is happening topologically on the strip, we have a bunch of pieces ending at j and a bunch starting with $j+1$ and they are paired off as neighbours. By moving the cuts rightwards, we are elongating the left-neighbours by 1 and shrinking the right-neighbours by 1, but this doesn't change the validity of the partition since the pieces still fit together. Eventually, we will shrink a piece to size 0 and remove it, and again have reduced the number of pieces by one whilst retaining a valid partition for k rows, meaning again we had at least $2k+1$ pieces prior.

Now in the square case for the original problem we can assume WLOG that the last n unit squares are a single piece (valid partitions are shift-invariant, and we need at least one piece of size n). Now we remove this piece to get a strip from 1 to $n(n-1)$, and we know the remaining $n-1$ rows, none of which start or end with 1, need at least $2n-2$ pieces.

Marking Scheme

For the upper bound, the following points are available.

(a) 1P: Any explicit construction and/or small case that clearly generalises to all n.

For the lower bound, the following points are available for each of the three solutions. Contestants should obtain the maximal number of points from any one of the three schemes.
Method 1:

(a) 1P: Considering the placements of the internal cuts in a row as a function of the row number. Any kind of map that completely describes all the internal cuts in terms of which row they are in and which two values they separate will be rewarded.

(b) +1P: Any claim equivalent to "the graph is connected" in the graph contextualisation described in the solution.

(c) +3P: Proving the above claim

(d) +1P: Concluding that there are at least $n - 1$ internal cuts. This point can be awarded even if the contestant does not prove the claim, as long as they recognise that connectivity allows them to finish.

Other unproved claims about the map structure implying the existence of $n - 1$ cuts may be rewarded points based on how difficult they are to prove.
Method 2:

(a) 1P: Considering the relation on rows induced by pieces which are adjacent on the strip but not in the square. Any kind of map that completely describes all such links will be rewarded.

(b) +1P: Further differentiating different relationships based on which two values the pieces met at (i.e., colouring the edges in the solution) **and** noting that each colour induces a disjoint union of cycles/each row has degree 2 or 0 in the induced graph.

(c) +3P: Proving the graph has at least $(n - 1 + k)$ edges, where k is the number of colours.

(d) +1P: Noting that every colour not present in the graph gives an extra cut, for a total of $2n - 1$ cuts.

Method 3:

(a) 1P: Proving that, via moving the pieces around and merging cuts, we can introduce a cut in a row with no cuts whilst removing a cut elsewhere.

(b) +2P: Doing this in a manner that will eventually remove a cut in a row that did not previously have exactly one cut **and** claiming this is the case **and** explaining why this would allow one to conclude.

(c) +3P: Proving that the proposed manner actually works.

If the student does not find a manner described in the second item above but still claims that one exists and explains why it would allow them to conclude, they will earn one partial point that is additive only with the first point above.
Method 4:

(a) 2P: The inductive hypothesis that if you have a strip from 1 to nk and k rows *not starting with 1*, you need $2k$ pieces, along with the base case. **This point is necessary to obtain all subsequent points, as the observations that follow are not useful unless we are within this specific inductive framwork.**

(b) +1P: Considering cuts not matching start and end values of the rows we are *currently* working with within the induction.

(c) +2P: Proving there is at least one such cut and moving all the cuts of the same value in one direction until you remove a piece, and explaining well that this does not change the validity of the partition.

(d) +1P: Concluding by stating in the original problem we must have a piece of length n, and the remaining pieces can be rearranged to give a strip starting with a value not at the start of any of the remaining rows.

IMO Selection 2024

Third Exam – Solutions

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

Bern
May 18, 2024

Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

7. Let $m, n \geq 2$ be integers. On each unit square of a $m \times n$ grid there is a coin. Initially all coins show heads. Jérôme repeatedly performs the following operation. First, he selects a 2×2 square within the grid and then does one of:

- Flipping all coins in the chosen 2×2 square except the top-right one.
- Flipping all coins in the chosen 2×2 square except the bottom-left one.

Determine all pairs (m, n) for which, at some point, Jérôme can make all coins show tails at the same time.

Solution

The answer is all pairs (m, n) where at least one of m and n is a multiple of 3.

Firstly, in order to prove that other values do not work, consider a coloration of the grid with each of the top-right-to-bottom-left diagonals alternatingly marked with 0s, 1s and 2s. Ergo, fill a diagonal with 0s, the above it with 1s, then 2s, then 0s, and so forth. It is easy to see that any operation will flip three coins from differently coloured diagonals. It is therefore clear that the sum of the squares on which the coin is a head is invariant modulo 3.

If each of m and n is congruent to 1 or 2 modulo 3, then since any 3×1 or 1×3 rectangle contains three different numbers, the total sum modulo 3 is just the sum of the bottom-right 1×2, 2×1 or 1×1 rectangle, so it suffices to make sure that these sums are not divisible by 3. We can ensure this by picking the bottom-right corner to be a 0 in the first two cases, or a 1 in the third case, and filling in the rest of the grid in the manner described above.

Finally, it remains to be shown that it is actually possible when at least one of m or n is a multiple of 3. WLOG, assume m is a multiple of 3, as the other case can be obtained by reflecting the moves over the diagonal. We can cut the rectangle into $3 \times n$ strips and handle each strip separately. It suffices to show that 3×2 and 3×3 are both possible, as any $n \geq 4$ can be decomposed into a sum of 2s and 3s. The former is obvious; for the latter, we can play the following five moves:

- Top-left 2×2 without top-right.
- Bottom-right 2×2 without top-right.
- Bottom-left 2×2 without top-right.
- Bottom-left 2×2 without bottom-left.
- Top-right 2×2 without bottom-left.

This finishes the proof.

Marking Scheme

For the colouring, the following points are available.

(a) 2P: Providing the colouring described in the solution.

(b) +1P: Completing the proof that any rectangle whose area is not divisible by 3 works (with particular attention on placing the values so that the total sum is not 0 mod 3).

If the contestant just states that the sum reduces to that in the bottom $i \times j$ rectangle but does not specify that the values can actually be placed so that we do not get 0, they will not be awarded the second item. For the construction, the following points are available. The positive points are not additive.

(a) 1P: Claiming the answer

(b) 1P: Any construction that works for all $3n \times 2m$ or $2n \times 3n$ and can be adapted to work for all cases using a small odd case (e.g 3×3.

(c) 2P: Any construction that works for all $3 \times m$ or $n \times 3$.

(d) 3P: Any construction that works for all $3n \times m$ or $n \times 3m$.

(e) -1P: Any construction that works for all but a number of cases for which it is easy to fix.

Contestants proving only one of the two without explicitly stating that the other case is covered by symmetry will not be penalized.

8. Determine all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$x\left(f(x)+f(y)\right) \geq f(y)\left(f(f(x))+y\right)$$

for all $x, y \in \mathbb{R}_{>0}$.

Solution

The answer is the family of functions $f(x) = C/x$ where $C \in \mathbb{R}_{>0}$.

Let f be a function satisfying the given condition, which we will denote by $P(x, y)$. Moreover, we will write f^n to denote the n-fold application of f.

Using $P(x, x)$ we deduce

$$2xf(x) \geq f(x)f(f(x)) + xf(x) \implies x \geq f^2(x)$$

and using $P(f(y), y)$ we get

$$f(y)\left(f^2(y) + f(y)\right) \geq f(y)(f^3(y) + y) \implies f^2(y) + f(y) \geq f^3(y) + y.$$

Rearranging and repeatedly replacing x by $f(x)$ leads to the chain of inequalities

$$f^{2n-2}(x) - f^{2n}(x) \geq f^{2n-3}(x) - f^{2n-1}(x) \geq f^{2n-4}(x) - f^{2n-2}(x) \geq \cdots \geq x - f^2(x).$$

Now if $x - f^2(x) = c > 0$, then summing every second term in the above chain gives

$$x - f^{2n}(x) \geq nc,$$

which can't be true for $n > x/c$. Thus, since $x - f^2(x) \geq 0$,

$$f(f(x)) = x.$$

With this the inequality, $P(x, f(y))$ reduces to

$$x(f(x) + y) \geq y(x + f(y)).$$

By swapping x and y, we deduce that equality must hold, and thus

$$xf(x) + xy = yx + yf(y) \implies xf(x) = yf(y).$$

Setting $y = 1$ gives us $f(x) = C/x$, where $C = f(1) > 0$ is a constant. Indeed, the inequality

$$x\left(\frac{C}{x} + \frac{C}{y}\right) \geq \frac{C}{y}(x + y)$$

holds for every $C \in \mathbb{R}_{>0}$, and we conclude that every function f of this form works.

Marking Scheme

(a) 1P: Showing that any function $f(x) = C/x$ with any $C \in \mathbb{R}_{>0}$ works.

(b) 1P: Proving $f(x) - f^3(x) \geq x - f^2(x)$ for all $x \in \mathbb{R}_{>0}$, written in this form.

(c) 1P: Proving $f^{n-2}(x) - f^n(x) \geq x - f^2(x)$ for all $x \in \mathbb{R}_{>0}$.

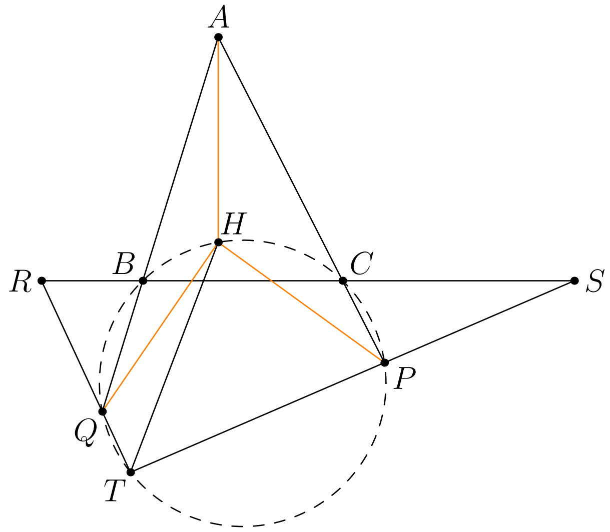
(d) 1P: Showing that $x - f^{2n}(x) \geq n(x - f^2(x))$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$.

(e) 2P: Proving that $f(f(x)) = x$ for all $x \in \mathbb{R}_{>0}$.

(f) 1P: Finishing from the fact that $f(f(x)) = x$ for all $x \in \mathbb{R}_{>0}$.

9. Let ABC be an acute triangle with orthocenter H, satisfying $AC > AB > BC$. The perpendicular bisectors of AC and AB intersect line BC at R and S respectively. Let P and Q be points on lines AC and AB respectively, both different from A, such that $AB = BP$ and $AC = CQ$. Prove that the distances from point H to lines SP and RQ are equal.

Solution



As usual we let $\angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$.

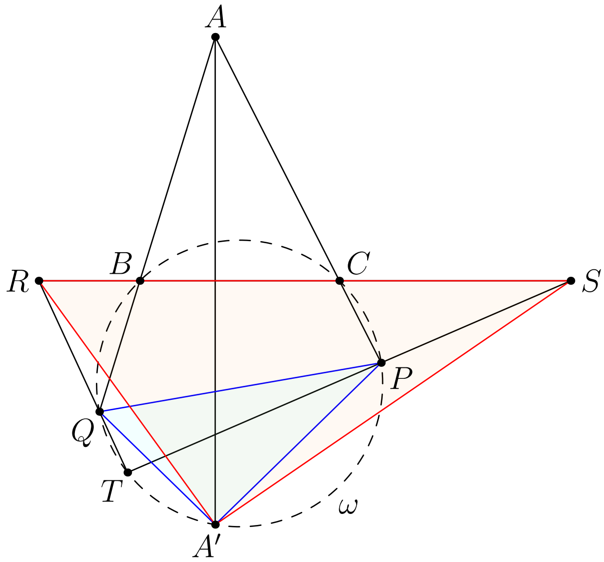
We introduce the intersection T of lines SP and RQ. Clearly, the problem boils down to proving that TH bisects the angle $\angle RTS$.

By angle chasing, one can notice that H is the circumcenter of triangle APQ. Indeed, the fact that $CA = CQ$ and $CH \perp AQ$ means that CH is the perpendicular bisector of AQ and thus $HA = HQ$. Similarly, $HA = HP$. In particular, we get that $HP = HQ$, which means that it's sufficient to prove that $HPQT$ is a cyclic quadrilateral. In fact, as

$$\angle CPB = \angle CQB = \alpha = 180° - \angle BHC$$

(by angle chasing), B, C, H, P, Q all lie on a circle ω and it suffices to show that $T \in \omega$. We propose three appproaches to prove this claim.

- **Spiral similarity**



Let A' be the reflection of A across line BC. Notice that $\angle CA'B = \alpha$ so $A' \in \omega$. Therefore, we can easily the compute that

$$\angle QPA' = \angle QCA' = \angle BCA' - \angle BCQ = \gamma - (180^\circ - 2\alpha - \gamma) = 2\alpha + 2\gamma - 180^\circ = 180^\circ - 2\beta,$$

and by symmetry

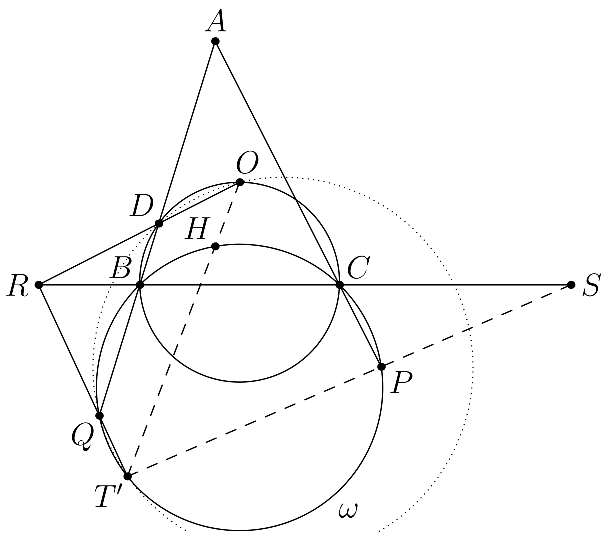
$$180^\circ - 2\beta = \angle ASR = \angle RSA'.$$

Similarly

$$\angle PQA' = \angle SRA',$$

meaning that triangles $\triangle A'PQ$ and $\triangle A'SR$ are spiral similar. Therefore, triangles $\triangle A'PS$ and $\triangle A'PR$ are also spiral similar and, by angle chase, $T = SP \cap RQ$ lies on the circumcircle of $A'PQ$, that is ω.

- **Power of a point**



Let O be the circumcenter of triangle ABC, and D be the intersection of OR with AB. As $OR \perp AC$ by construction we easily get that $ODBC$ is cyclic by angle chasing. Therefore, we can compute the power of point R to get

$$RD \cdot RO = RB \cdot RC = RQ \cdot RT',$$

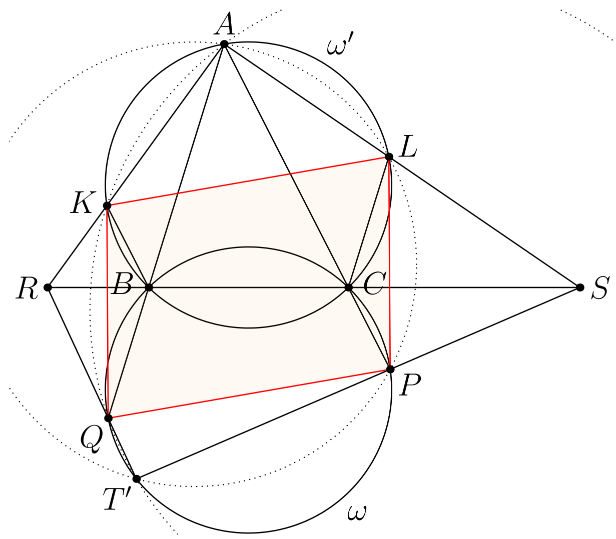
where T' is the second intersection of ω with RQ. In other words, $DOT'Q$ is cyclic. But, one can notice by angle chase that

$$\angle ODA = 90^\circ - \alpha = \angle HPQ = \angle HTQ,$$

which, combined with the above result, directly implies that O, H, T' are collinear, or in other words the second intersection T' of OH with ω lies on RQ. Similarly, T' must lie on SP, meaning that $T = T' \in \omega$.

Note that one can get to a similar solution by inverting at R with radius $\sqrt{RB \cdot RC}$, as $TROC$ is cyclic.

- **Radical axis**



Here, we introduce $K \neq A$, resp. $L \neq A$ the second intersection of the circumcircle ω' of ABC with AR, resp. AS. Clearly, as RAC and SAB are isosceles, $AKBC$ and $ALBC$ are isosceles trapezoids, yielding that $KB \parallel PC$ and $LC \parallel QB$. Now, by considering the symmetry centered at the midpoint of BC, we know that B is sent to C and ω' is sent to ω (as it's well known that $(ABC) = \omega'$ and $(HBC) = \omega$ symmetrical across line BC. Moreover, by the above parallelisms, K is sent to P and L is sent to Q, meaning that $KLPQ$ is parallelogram.
We denote the second intersection of the circles (AKQ) and (ALP) by T'. A quick angle chase shows us that

$$\angle PT'Q = \angle PT'A + \angle AT'Q = \angle PLS + \angle RKQ = \angle RAS = 180^\circ - 2\alpha$$

as $PL \parallel KQ$. We deduce that $T' \in \omega$, since $\angle QHP = 2\alpha$.
Now we conclude easily as R is the radical center of circles ω, ω' and (AKQ), and so QT' passes through R. Similarly, PT' passes through S and therefore $T = T' \in \omega$.

Marking Scheme

(Non-additive)

Part 1 : Preparing the ground (2P)

- 1P : Proving that B, C, H, P, Q all lie on a circle ω
- 1P : Claiming that T lies on ω (or any other equivalent claim)
- 1P : Proving that $HP = HQ$
- 2P : Proving that it's sufficient to prove that T lies on ω

Part 2 : Proving that $T \in \omega$ (5P)

- 1P : Introducing A'
- 2P : Claiming that H, O and T are collinear
- 4P : Proving that H, O and T are collinear
- 1P : Introducing D and proving that $ODBC$ is cyclic (or any other equivalent statement)
- 1P : Introducing K and L
- 3P : Proving $KQ \parallel PL$
- 2P : Obtaining useful equalities by computing the power of point R or S (approaches 2, 3)

IMO Selection 2024

Fourth Exam – Solutions

Bern

May 19, 2024

Duration: 4.5 hours
Difficulty: The problems are ordered by difficulty.
Points: Each problem is worth 7 points.

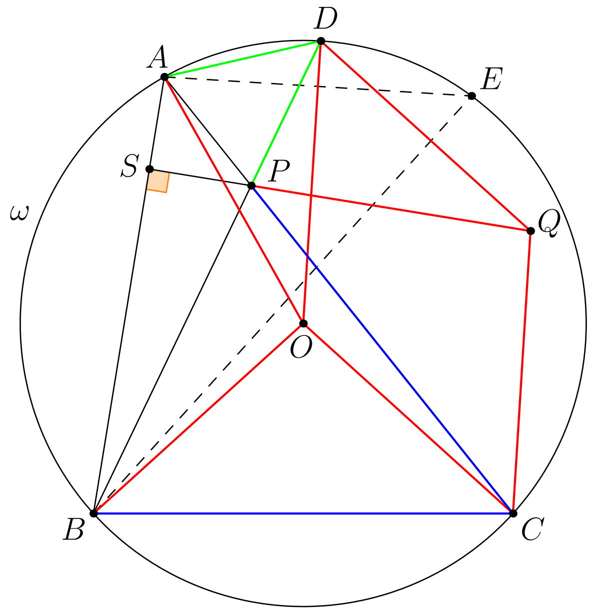
Preliminary Remark

A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a full solution for (minor) flaws. Partial marks are attributed according to the marking schemes. In the case of multiple marking schemes for the same problem, the score is the maximum among all the marking schemes.

10. Let ABC be a triangle with $AC > BC$. Let ω be the circumcircle of triangle ABC and let r be the radius of ω. Let point P lie on the segment AC such that $BC = CP$ and let S be the foot of the perpendicular from P to line AB. Let the line BP intersect ω again at $D \neq B$. Let Q lie on line SP such that $PQ = r$ and such that S, P and Q lie on the line in that order. Finally, let the line perpendicular to CQ from A intersect the line perpendicular to DQ from B at E.

Prove that E lies on ω.

Solution



Let α, β and γ be the angles of triangle ABC and O its circumcircle. Then

$$\angle CPQ = \angle APS = 90^\circ - \alpha = \angle CBO$$

so by the SAS criterion, we have $\triangle BOC \cong \triangle PQC$. In particular, $QC = r$.

As $\triangle ADP \sim \triangle PCB$, we also have that $AD = DP$. From

$$\angle QPD = \angle SPB = 90^\circ - \left(\beta + \frac{\gamma}{2} - 90^\circ\right) = \alpha + \frac{\gamma}{2} = \angle OAD$$

we can also use the SAS criterion to obtain $\triangle AOD \cong \triangle PQD$. In particular, $DQ = r$.

From these lengths, we have that $DOCQ$ is a rhombus, in particular $DO \parallel QC$ and $CO \parallel DQ$. We are now ready to prove that E lies on ω, for example

$$\angle CAE = \angle OAE - \angle OAC = (90^\circ + \alpha - \beta) - (90^\circ - \beta) = \alpha = \angle CBE,$$

finishing the proof.

Marking Scheme

Points from each part are non-additive: only a single item can be chosen for each part.

Part A: Q is the circumcentre of DPC (4P)

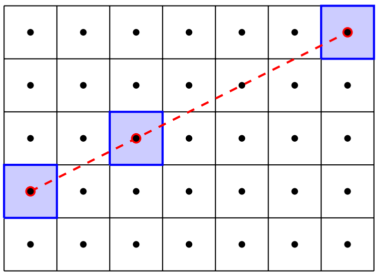
- 1P : Introducing O and proving that $\angle CBO = \angle CPQ$ or $\angle OAD = \angle QPD$
- 2P : Proving that $QC = r$
- 2P : Proving that $DQ = r$

Part B: Finishing (3P)

- 1P: Proving that $DO \parallel QC$ or $CO \parallel QD$
- 2P: Computing one of the angles needed to prove that E lies on ω: for example $\angle CAE$ or $\angle CBE$.

11. Let $m, n \geq 3$ be integers. Nemo is given an $m \times n$ grid of unit squares with one chip on every unit square initially. They can repeatedly carry out the following operation: first, they pick any three distinct collinear unit squares and then they move one chip from each of the outer two squares onto the middle square. They may only do this operation if the outer two squares are not empty, but the middle square is allowed to be empty.

As a function of (m, n), either determine the maximum number of operations Nemo can make before they cannot continue anymore, or prove that they can carry out an arbitrarily large number of operations.



An example of three squares with collinear centres

Solution

We number the squares with coordinates in such a manner that if both m, n are odd then $(0,0)$ is the central square; if m is even and n is odd then $(0,0)$ and $(1,0)$ are the central squares and if m, n are even then $(0,0), (1,0), (0,1)$ and $(1,0)$ are the central squares.

Consider the two following functions:

$$f_0(x) = \begin{cases} 2^x - 2 \text{ if } x \geq 1; \\ 2^{1-x} - 2 \text{ otherwise.} \end{cases}$$

$$f_1(x) = \begin{cases} 0 \text{ if } x = 0; \\ 3 \cdot 2^{|x|-1} - 2 \text{ otherwise.} \end{cases}$$

Now, assign to the square (x, y) the value $f_i(x) + f_j(y)$ where $i \equiv m$ and $j \equiv n \pmod 2$.

Firstly, we claim that the total value of chips (with a chip having the value of the square it is currently on) is a monovariant that decreases by at least 2 each move. Consider a legal move: we are moving two chips from the squares $(x + ka, y + kb)$ and $(x + la, x + lb)$ into the square (x, y) for integers x, y, l, k, a, b with k, l nonzero and having different signs. The total change in the monovariant is therefore

$$2f_i(x) - f_i(x + ka) - f_i(x + \ell a) + 2f_j(y) - f_j(y + kb) - f_j(y + \ell b).$$

It suffices to show that the term $2f_i(x) - f_i(x + ka) - f_i(x + \ell a)$ is at most -2, as this would be true for the second term by symmetry. Now ka and ℓa are just nonzero integers with different signs, so we may as well assume $a = 1$ for this part and say, WLOG, k has the same sign as x. We also note that the value of the function f_i increases the further away you get from 0 (technically 0.5 for f_0) so, for a fixed x, the expression

$$2f(x) - f(x + k) - f(x + l)$$

is maximised when $x + k$ and $x + l$ are as close as possible to 0 (or 0.5). The latter can just be equal to 0; the former is at least one further away. It is easy to check that for both functions $f(0) = 0$ and

$$2f(x) - f(x + \operatorname{sgn}(x)) = -2,$$

as the functions are actually just doubling and adding 2 every time you go a step away from 0 – almost as if this was done on purpose. So the claim is proven, and furthermore, we must be making moves entirely within a line or column for the monovariant to decrease by exactly 2.

Now we prove that a sequence is possible where we decrease by exactly 2 at each step. It suffices to do the one-dimensional case and to show that all the chips in a row can be brought into the central column(s), as we can do this (making sure to have the same number of chips in the right and left column if there are two central columns) and then apply the same moves vertically several times to bring the chips in the central column(s) into the central square(s).

We use induction here to give a construction. We have two inductive hypotheses:

- A sequence is possible from the starting point where we have a single chip on the square k to the left of and the square k to the right of the central square(s), and $k - 1$ chips on the two central squares/$2k - 1$ chips on the central square in the even and odd case respectively.
- A sequence is possible from the starting point where we have one chip on all the squares from the square k to the left of up until the square k to the right of the central square(s).

Both these hypotheses are obvious in the base case $k = 1$ for even and odd respectively, which just correspond to the sequence of one move for $m = 3$ and the sequence of two moves for $m = 4$. Suppose both hypotheses hold true up to k, and now consider $k + 1$.

We initially have one chip on all the squares from the square $k + 1$ to the left of up until the square $k + 1$ to the right of the central square(s), which is the first hypothesis for $k + 1$. Now, we apply the first assumption for k on the inner $2k + 1$ or $2k + 2$ squares to bring them all to the central column(s), reducing the monovariant by only 2 at a time. We have now arrived at the second hypothesis for $k + 1$.

We then make two moves: first, we pick the square $k + 1$ to the left of the central square(s), the square k to the left of the central square(s), and the left central square (just the central square if there is only one). This move results in 2 chips on the square k to the left of the central squares. We then do the same move mirrored on the other side. We now also have 2 chips on the square k to the right of the left squares. We also have $k - 1$ chips on the two central squares/$2k - 1$ chips on the central square in the even and odd case respectively - so this is the second assumption for k, with an extra chip on the outer squares.

We can now apply the second assumption twice to bring these two chips to the center with moves only reducing the monovariant by only 2 at a time - in the second run, we will have two extra chips in the center but this does not affect anything and we can ignore them if we want. We therefore have proved by induction that a sequence of moves reducing the monovariant by the minimal amount each time is possible.

Thus we have shown that a sequence of moves exists that reduces the monovariant by the minimal amount each time, and that concludes with the minimal value of the monovariant, namely 0. It follows that this sequence of moves must be optimal and it suffices to calculate the number of moves. We simply need to sum all the values in all the squares and divide by 2. Since each square is the sum of two functions, we can just sum for the first function and then the second function and add them together. Noting that

$$\sum_{i=-k+1}^{k} f_0(i) = 4(2^k - 1) - 4k$$
$$\sum_{i=-k}^{k} f_1(i) = 6(2^k - 1) - 4k$$

and defining

$$g(x) = \begin{cases} 2(2^{\frac{x}{2}} - 1) - x \text{ if } x \text{ even;} \\ 3(2^{\frac{x-1}{2}} - 1) - x + 1 \text{ if } x \text{ odd.} \end{cases}$$

and using that two functions only depend on one coordinate each, it follows that the maximal number of moves for (m, n) is $ng(m) + mg(n)$.

Marking Scheme

Contestants that show (d) below without shown (a), (b) and (c) will only receive 1 point for (d).

(a) 1P: Creating any monovariant equal to a sumproduct of the chips that reduces by a minimal constant factor for the "best" move(s) (i.e., those that are entirely contained in a row/column and involve a central square, and two adjacent squares on either side of it). A diagram that clearly generalises suffices.

(b) +2P: Proving the above.

(c) +1P: Claiming that one can reduce by this constant factor at each step and describing the optimal move.

(d) +2P: Proving the existence of an optimal sequence of moves in the one dimensional case.

(e) +1P: Concluding.

Contestants that obtain none of the above points may obtain the following (non-additive) partials:

(a) 1P: Claiming the answer is equal to $ng(m) + mg(n)$, where $g(x)$ is the answer for the one-dimensional case.

Minor errors, such as flaws in either the odd or even case, will face a deduction of 1 point. If the contestant only does one of the two parity cases completely, they will receive 3P, with up to two additional points if the method could be adapted to work in the other case, depending to what extent they make progress. Contestants that fail to prove or consider the third point above but get everything else will be awarded 5P. Computational errors for the final answer are not penalised.

12. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$\underbrace{f(f(\cdots f}_{bf(a)}(a+1)\cdots)) = (a+1)f(b)$$

holds for all $a, b \in \mathbb{N}$.

Solution

The only solution is $f(n) = n+1$ for all $n \in \mathbb{N}$.

Let $f : \mathbb{N} \to \mathbb{N}$ be a function with the desired property, which we abbreviate by $P(a,b)$. Moreover, we will denote by f^n the n-fold application of f.

We say that f has a *cycle* if there are integers $a_1, a_2, \ldots, a_n \in \mathbb{N}$ with $f(a_i) = a_{i+1}$ for $i = 1, \ldots, n-1$ and $f(a_n) = a_1$.

Lemma 1: The function f has no cycles.

Assume that $a_1, a_2, \ldots, a_n$ forms a cycle of f and set $M = \max(a_1, a_2, \ldots, a_n)$. Since $f(1) = 1$ together with $P(1,1)$ imply $f(2) = 2$, assume WLOG that $M > 1$. Now the LHS of $P(M-1, b)$ is at most M, but the RHS is larger than $Mf(b)$. Since this must hold for all b, the only case is $f \equiv 1$ which is clearly not a solution.

Lemma 2: The function f is injective.

If $f(x) = f(y)$ for $x \neq y$, we compare $P(1,x)$ with $P(1,y)$ and find

$$f^{xf(1)}(2) = 2f(x) = 2f(y) = f^{yf(1)}(2).$$

But since $xf(1) \neq yf(1)$, this implies the existence of a cycle, a contradiction.

Lemma 3: We have $f(x) > 1$ for all $x \in \mathbb{N}$.

Assume that $f(x) = 1$ for some $x \in \mathbb{N}$. But now $P(a,x)$ tells us that

$$f^{xf(a)}(a+1) = a+1,$$

which implies that f has a cycle, a contradiction.

Lemma 4: We have $f^{yf(f(x)-1)}(x) = f^{xf(f(y)-1)}(y)$ for all $x, y \in \mathbb{N}$.

From $P(f(x)-1, y)$ (note that $f(x) - 1 \in \mathbb{N}$ by lemma 3), we find that

$$f(x)f(y) = f^{yf(f(x)-1)}(f(x)) = f^{yf(f(x)-1)+1}(x).$$

Since the LHS of the above equation is symmetric, we get

$$f^{yf(f(x)-1)+1}(x) = f^{xf(f(y)-1)+1}(y)$$

and thus also

$$f^{yf(f(x)-1)}(x) = f^{xf(f(y)-1)}(y)$$

by injectivity.

Lemma 5: All integers $x > 1$ are in the image of f.

If we set $y = 1$ in Lemma 4, we find that

$$f^{f(f(x)-1)}(x) = f^{xf(f(1)-1)}(1).$$

Using the injectivity of f, we can split into three cases:

- If $f(f(x)-1) > xf(f(1)-1)$, then $f^n(x) = 1$ for some $n \in \mathbb{N}$.
- If $f(f(x)-1) < xf(f(1)-1)$, then $f^n(1) = x$ for some $n \in \mathbb{N}$.
- If $f(f(x)-1) = xf(f(1)-1)$, then $x = 1$.

The first case contradicts lemma 3. The second case implies that x is in the image of f.

Lemma 6: We have $f^n(x)f(f(x)-1) = xf(f^{n+1}(x)-1) + n$ for all $x, n \in \mathbb{N}$.

If we set $y = f^n(x)$ in Lemma 4, we get

$$f^{f^n(x)f(f(x)-1)}(x) = f^{xf(f(f^n(x))-1)}(f^n(x)) = f^{xf(f^{n+1}(x)-1)+n}(x)$$

which implies that

$$f^n(x)f(f(x)-1) = xf(f^{n+1}(x)-1) + n,$$

since otherwise f would have a cycle.

Finish: Note that setting $n = 1$ in lemma 6 yields

$$f(x)f(f(x)-1) = xf(f^2(x)) - 1) + 1.$$

This equation implies that $\gcd(x, f(f(x)-1)) = 1$. Using this on lemma 6, we find that

$$x \mid f^n(x) \iff x \mid n.$$

But using $P(x, 1)$, we now have

$$x + 1 \mid (x+1)f(1) = f^{f(x)}(x+1)$$

and thus $x + 1 \mid f(x)$ by the remark above (with $x \to x+1$ and $n \to f(x)$).

This in turn now shows that $f(x) \geq x + 1$ for all $x \in \mathbb{N}$, but because the image of f contains every integers $x > 1$, we must have $f(x) = x + 1$ for all $x \in \mathbb{N}$ by a simple counting argument. It is easy to check that f satisfies P.

Marking Scheme

(a) +1P (non-additive) Claiming and checking that $f(n) = n + 1$ works

(b) +1P: Showing that f has no cycle

(c) +1P: Showing that f is injective

(d) +1P: Proving that 1 is not in the image of f

(e) +1P: Proving that $f^{yf(f(x)-1)}(x) = f^{xf(f(y)-1)}(y)$ for all $x, y \in \mathbb{N}$

(f) +1P: Deducing that all integers $n > 1$ are in the image of f

(g) +1P: Deducing that $f^n(x)f(f(x) - 1) = xf(f^{n+1}(x) - 1) + n$ for all $x, n \in \mathbb{N}$

(h) +1P: Concluding

Point (b) is still awarded if a contestant gives a proof by contradiction in (c) or (d) where they construct a cycle and use only the existence of this cycle to obtain a contradiction but don't explicitly state that no cycle exists.

Point (e) will still be awarded if a contestant does not show (d) but shows that (e) holds for all $x, y \in \mathbb{N}$ such that $f(x) > 1$ and $f(y) > 1$.

At most 6P will be awarded to a contestant that does not check that $f(n) = n + 1$ works.