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Final round 2021

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Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

1. Let (m, n) be a pair of positive integers. Julia has carefully planted m rows of n dandelions in an $m \times n$ array in her back garden. Now, Jana and Viviane decide to play a game with a lawnmower they just found. Taking alternating turns and starting with Jana, they can mow down all the dandelions in a straight horizontal or vertical line (and they must mow down at least one dandelion!). The winner is the player who mows down the final dandelion. Determine all pairs (m, n) for which Jana has a winning strategy.

Answer: Jana has a winning strategy if and only if $m + n$ is odd or if one of m, n is equal to 1.

Solution (Paul):

First we make two general observations about the problem. Firstly, since the game always finishes in a finite number of moves there is always a winner and consequently a player who has a winning strategy at the start. Secondly, when a player mows down a row or a column, it does not matter which row or column is chosen (as long as it is still a row respectively a column). The remaining dandelions are affected by further moves in the exact same manner no matter which rows or columns were previously mowed down.

Due to the second observation, each move results in a situation which can be viewed as the start of a game, but this time the other player begins and either m or n is decreased by 1. Therefore we can deduce which player has a winning strategy by considering the smaller array. This suggests an inductive approach to the problem.

If $m = 1$ or $n = 1$, then Jana can clearly win the game with a single move. Now we prove the answer above using induction over $m + n$, so suppose that $m + n$ is odd and $m, n > 1$. The case $m + n = 1$ is impossible and $m + n = 3$ has already been taken care of. If $m + n \geq 5$, then one of m, n is at least 3 and it is possible for Jana to mow down a line such that at least 2 rows and 2 columns are left. Then Viviane cannot win in her next move and afterwards the sum of the number of rows and columns will be odd again. By induction, we know that Jana will have a winning strategy for the remainder of the game.

If $m + n$ is even and $m, n > 1$, then Jana's first move will result in a configuration for which Viviane has a winning strategy.

Marking scheme

- 1P: Dealing with the case $m = 1$ or $n = 1$.
- 1P: Stating the rest of the correct answer.
- 1P: Noting that it does not matter which row or column is chosen in a move.
- 1P: Noting that after a move we get a new game with m or n decreased by 1 and opposite starting player.
- 2P: Proving that Jana has a winning strategy if $m + n$ is odd and $m, n > 1$.
- 1P: Proving that Jana does not have a winning strategy if $m + n$ is even and $m, n > 1$

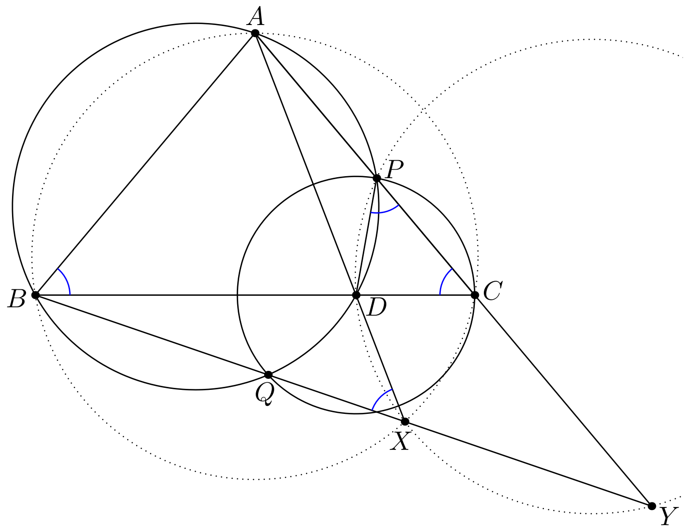
2. Let ABC be an acute triangle with $AB = AC$ and let D be a point on the side BC. The circle with centre D passing through C intersects the circumcircle of ABD in P and Q, where Q is the point closer to B. The line BQ intersects AD in X and AC in Y. Prove that $PDXY$ is cyclic.

Solution: We first claim that $P \in AC$. Indeed, let P' be the second intersection between the circle centered at D and AC. Then

$$\angle ABD = \angle ABC = \angle ACB = \angle P'CD = 180^\circ - \angle AP'D.$$

so that $ABDP'$ is cyclic. This implies that $P = P'$, in particular $P \in AC$. As $DP = DQ$, the arcs DP and DQ subtend angles of same measure on the circle $(APDQB)$, so that $\angle QBD = \angle PAD$. Hence, ΔDBX and ΔDAC are similar (alternatively $ABXC$ is cyclic), implying that $\angle DXB = \angle DCA$. This concludes the problem, as we wanted to prove that

$$\angle DPY = \angle DPC = \angle DCP = \angle DCA = \angle DXB = 180^\circ - \angle DXY.$$



Marking scheme

- 1P: Claim that P lies on AC.
- 2P: Prove that P lies on AC.
- 2P: Prove that $\angle QBD = \angle PAD$.
- 2P: Conclude. (Proving that $ABXC$ cyclic is enough to end the problem gives these two points for example)

3. Find all finite sets S of positive integers with at least two elements, such that if $m > n$ are two elements of S, then

$$\frac{n^2}{m-n}$$

is also an element of S.

Answer: $S = \{s, 2s\}$, for a positive integer s.

Solution(Arnaud):

Trying to apply number theoretical methods to deduce something from the fact that $m - n$ divides n^2 does not seem to lead anywhere. Instead, we will try to find the extreme values that the quotient $n^2/(m-n)$ can achieve. This will give us some interesting bounds on the elements of S.

(a) First, since S contains at least two elements, we take $l > s$ to be the largest and smallest elements of S. Since $s^2/(l-s)$ also belongs to S we must have

$$\frac{s^2}{l-s} \geq s \quad \Longrightarrow \quad 2s \geq l.$$

(b) Now, consider $k < l$ the second largest element of S. The number $k^2/(l-k)$ belongs to S. We claim that $k^2/(l-k) \neq l$, which implies

$$\frac{k^2}{l-k} \leq k \quad \Longrightarrow \quad 2k \leq l.$$

We prove the previous claim in the following lemma.

Lemma 1 *The equation $k^2/(l-k) = l$ has no positive integer solutions $l > k$.*

Proof. Assume such a solution exists, then it also solves

$$l^2 = k^2 + kl.$$

Since the equation is homogeneous (of degree 2), we can assume that $\gcd(k, l) = 1$. Since we must have $l \mid k^2$, we deduce that $l = 1$. This is a contradiction since k is a positive integer with $k < l$. $\square$

Combining the two relations above, and using that $s \leq k$, we get

$$l \leq 2s \leq 2k \leq l.$$

Therefore $k = s$ and S contains exactly two elements. Moreover, the inequalities above also implies $l = 2s$. So $S = \{s, 2s\}$ as desired.

Any such set satisfies the desired property because

$$\frac{s^2}{2s-s} = s \in S.$$

Marking scheme

Case $|S| = 2$ (3P)

- 1P: Stating the correct answer and checking that it indeed works.
- 2P: Statement and proof of the lemma.

Reduction to $|S| = 2$ (4P)

- 1P: Considering the smallest and largest elements of S.
- 1P: $l \leq 2k_1$ for some $k_1 \neq l$ in S (e.g. $k_1 = s$).
- 2P: Conclude.

-1P for logical flaws.

4. The real numbers a, b, c, d are positive and satisfy $(a+c)(b+d) = ac+bd$. Find the minimum of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

Answer: 8.

Solution(Raphael):

After trying out some things one might suspect that the minimum is reached when $a = c$ and $b = d$. So we'll try to use an inequality with equality case $a = c$ and $b = d$. Going for the easiest such inequality one comes across:

$$ac + bd = (a+c)(b+d) \geq 2\sqrt{ac} \cdot 2\sqrt{bd} \Leftrightarrow \sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}} \geq 4$$

Where we just used AM-GM on the terms $(a+c)$ and $(b+d)$ and then divided everything by $\sqrt{abcd}$, to get the second inequality.

Now if $a = c$ and $b = d$ then $\frac{a}{b} = \frac{c}{d}$ and $\frac{b}{c} = \frac{d}{a}$. So another inequality to try is:

$$\frac{a}{b} + \frac{c}{d} + \frac{b}{c} + \frac{d}{a} \geq 2\sqrt{\frac{ac}{bd}} + 2\sqrt{\frac{bd}{ac}}$$

which is once again AM-GM. So combining these two inequalities one gets:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2 \cdot \left(\sqrt{\frac{ac}{bd}} + \sqrt{\frac{bd}{ac}}\right) \geq 2 \cdot 4 = 8$$

with equality only when $\frac{a}{b} = \frac{c}{d}$ and $\frac{b}{c} = \frac{d}{a}$.

Now we still have to show that numbers $a = c$, $b = d$ exist which fulfil the condition $ac + bd = (a+c)(b+d)$ Plugging in $c = a$ and $d = b$ we get:

$$a^2 + b^2 = 4ab \Leftrightarrow \left(\frac{a}{b}\right)^2 - 4\frac{a}{b} + 1 = 0$$

which has the solutions $\frac{a}{b} = 2 \pm \sqrt{3}$. so using $a = c = (2+\sqrt{3}) \cdot b = (2+\sqrt{3}) \cdot d$ fulfils the condition.

Marking scheme

- 1P: Claiming $a = c \neq b = d$ is the equality case/reaches the minimum.
- 2P: Getting the first inequality (applying AM-GM on the condition).
- 2P: Getting the second inequality (applying AM-GM on the expression).
- 1P: Proving that the expression is indeed greater/equal to 8.
- 1P: Showing that the minimum is achieved.

5. For which integers $n \geq 2$ can we arrange the numbers $1, 2, \ldots, n$ in a row, such that for all integers $1 \leq k \leq n$ the sum of the first k numbers in the row is divisible by k?

Answer: This is only possible for $n = 3$.

Solution (Valentin): If we let $k = n$ we find that

$$n \,\Big|\, \frac{n(n+1)}{2}$$

which implies that n has to be odd. For $n = 3$ the arrangement $1, 3, 2$ satisfies the condition. From now on assume that $a_1, a_2, \ldots, a_n$ is a valid arrangement for an odd $n \geq 5$. If we now let $k = n - 1$ we find that

$$n - 1 \,\Big|\, \frac{n(n+1)}{2} - a_n \implies a_n \equiv n \cdot \frac{n+1}{2} \equiv \frac{n+1}{2} \mod n - 1.$$

But $1 \leq a_n \leq n$ and $n \geq 5$ implies that $a_n = \frac{n+1}{2}$ since $\frac{n+1}{2} + n - 1$ would already exceed n. Now we let $k = n - 2$ to find

$$n - 2 \,\Big|\, \frac{n(n+1)}{2} - \frac{n+1}{2} - a_{n-1} \implies a_{n-1} \equiv \frac{n^2 - 1}{2} = (n-1) \cdot \frac{n+1}{2} \equiv \frac{n+1}{2} \mod n - 2.$$

But similarly to before since $1 \leq a_{n-1} \leq n$ and $n \geq 5$ this implies that $a_{n-1} = \frac{n+1}{2}$ as well, a contradiction. We conclude that $n = 3$ is the only value for which this is possible.

Marking Scheme

- 1P: Proving that n must be odd.
- 1P: Finding a valid arrangement for $n = 3$, namely (1 3 2) or (3 1 2).
- 2P: Proving that the last number in the row must be $(n + 1)/2$.
- 2P: Proving that the second to last number in the row must be $(n + 1)/2$ as well.
- 1P: Finishing the proof.

6. Let $\mathbb{N}$ be the set of positive integers. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that for every $n \in \mathbb{N}$

$$f(n) - n < 2021 \quad \text{and} \quad \underbrace{f(f(\cdots f(f}_{f(n)}(n))\cdots)) = n.$$

Prove that $f(n) = n$ for infinitely many $n \in \mathbb{N}$.

Solution (Valentin):

We first prove that f is surjective. This follows directly from the second condition since for all $a \in \mathbb{N}$ we have

$$b = \underbrace{f(f(\cdots f(f}_{f(a)-1}(a))\cdots)) \implies f(b) = a.$$

Now pick any prime $p > 2021$ and n with $f(n) = p$. Now apply f to the second condition to get

$$p = f(n) = \underbrace{f(f(\cdots f(f}_{p+1}(n))\cdots)) = \underbrace{f(f(\cdots f(f}_{p}(p))\cdots)).$$

Instead setting $n = p$ in the second condition we find that

$$p = \underbrace{f(f(\cdots f(f}_{f(p)}(p))\cdots))$$

as well. Now let d_p be the smallest positive integer such that $p = \underbrace{f(f(\cdots f(f}_{d_p}(p))\cdots))$.

It follows that $d_p \mid p, f(p)$ since otherwise we would obtain a smaller value of d_p by repeatedly substituting this expression in the two equations above. Now if $d_p = 1$ we have $f(p) = p$ and if $d_p = p$ we have $f(p) - p < 2021 < p$ using the first condition and we find $f(p) = p$ as well. We conclude that every prime $p > 2021$ satisfies $f(p) = p$ and since there is an infinite number of such primes, this solves the problem.

Marking Scheme

- 1P: Prove (claim) that f is surjective.
- 2P: Proving $n = \underbrace{f(f(\cdots f(f}_{n}(n))\cdots))$ for all $n \in \mathbb{N}$ (or any set containing ∞ primes).
- 2P: Considering $d_n = \min\left\{k \in \mathbb{N} \mid n = \underbrace{f(f(\cdots f(f}_{k}(n))\cdots))\right\}$.
- 2P: Finishing the proof.

7. Let $m \geq n$ be positive integers. Frieder is given mn posters of Linus with different integer dimensions $k \times l$ with $1 \leq k \leq m$ and $1 \leq l \leq n$. He must put them all up one by one on his bedroom wall without rotating them. Every time he puts up a poster, he can either put it on an empty spot on the wall, or on a spot where it entirely covers a single visible poster and does not overlap any other visible poster. Determine the minimal area of the wall that will be covered by posters.
Remark: a wall is a structure often made of bricks and concrete (or cardboard if you live in Winterthur) commonly found in houses.

Answer: $m\frac{n(n-1)}{2}$

Solution (Tanish): We introduce the following definitions[1]:

- A *chain* is a sequence of posters, each of which covers the previous one completely.
- An *antichain* is a group of posters, none of which can be placed over any of the others because it would not cover any of the others completely.
- The *minimal* and *maximal* elements in a chain are the first (smallest) and last (largest) elements in that chain, respectively.
- The *degree* of a poster is the sum of its length and width.

With these definitions we can note two facts already: the total area of the wall covered is the total area of the maximal elements of all chains (this is trivial), and all elements of an antichain must belong to different chains (this is because the relationship "can be placed over" is transitive: if Poster A can be placed over Poster B and Poster B can be placed over Poster C, then Poster A can be placed over poster C).

Firstly, note that the n posters with dimension $m \times 1, m-1 \times 2, \ldots, m-n-1 \times n$ form an antichain (more generally, you can say all posters of a given degree x form an antichain) and so we must have at least n different chains. We therefore have at least n maximal posters of degree $\geq m+1$. Analogously we have at least $n+1-k$ maximal posters of degree $\geq m+k$ for any $k \in \{1, 2, \ldots, n\}$ (using the same argument as before with the antichain of posters of degree $m+k$).

With this information, let us try and find a lower bound on the total area of these maximal posters (we don't know if satisfying these facts will be enough, but we can use it to give a bound and then try to attain this bound with a construction). Suppose there are no maximal elements of some degree $m+k$; the pigeonhole principle tells us there must therefore be two or more maximal posters of some degree $m+l$, $l > k$. However for any poster of degree $m+l$ there is always a poster of degree $m+k$ that fits inside it, and so changing one of the maximal posters of degree $m+l$ for such a poster of degree $m+k$ will strictly reduce the total area. In other words, a set of maximal posters with minimal combined area would have at least one element of degree $m+k, \forall k \in \{1, 2, \ldots, n\}$. The smallest poster of degree $m+k$ is the poster $m \times k$ (once the perimeter is fixed the area is minimised by maximising the difference between height and width) and so a lower bound on the total area of the posters would be:

$$(m + 2m + \cdots + nm) = m\frac{n(n+1)}{2}.$$

However, this bound can easily be attained: one construction would be to create chains by placing all the posters $x \times y, 1 \leq x \leq m$ on top of each other. This gives a chain for each $y \in \{1, 2, \ldots, n\}$.

[1]Funnily enough, these are not randomly chosen words, but the same definitions you will see when you study partially ordered sets later on.

Marking scheme Contestants are awarded the maximum of the sum of the points they obtain from the additive section and the highest-scoring item they obtain from the non-additive section.
Upper bound (additive):

- +1P: Stating the answer $m\frac{n(n+1)}{2}$ or equivalent.
- +1P: Any correct construction for this value. If the contestant nests the posters in the "wrong coordinate" they should not be awarded points.

Upper bound (non-additive):

- 1P: Introducing the notion of degrees or an equivalent idea and claiming we want to put each poster over a poster whose degree is 1 lower.
- 1P: Correct answer for case $m = n$.

Lower bound (additive):

- +1P: Any antichain of size n with the claim that all these posters must be in different chains.
- +1P: The generalisation of the previous claim with a good choice of antichain: the $n+1-k$ posters of degree $m+k$ are in different chains.
- +1P: Any claim that minimal area would occur with at least one maximal poster of degree $m+k$ for $k \in \{1, 2, \ldots, n\}$
- +1P: Proof of the aforementioned claim.
- +1P: Concluding that a lower bound is $m\frac{n(n+1)}{2}$.

Lower bound (non-additive):

- 1P: Any reference to the fact that the posters form a poset.
- 1P: Any drawing that reflects the partial order, e.g. a Hasse diagram (or something that looks like one).
- 2P: Any flawed attempt to create chains starting at highest degree and working down (for example, putting the poster of degree $m+n$ in a chain, stating only one of the two posters of degree $m+n-1$ can go into this chain and so the other poster must maximise a new chain, and continuing downwards. This argument fails because you assume one of the two posters of degree $m+n-1$ must go into the chain with the poster of degree $m+n$ but you ignore the possibility that neither does (and you should prove that this would not be optimal for a perfect solution).
- 3P: Correct proof of bound for case $m = n$.

8. Let ABC be a triangle with $AB = AC$ and $\angle BAC = 20°$. Let D be the point on the side AB such that $\angle BCD = 70°$. Let E be the point on the side AC such that $\angle CBE = 60°$. Determine the value of the angle $\angle CDE$.

Answer: $\angle CDE = 20°$.

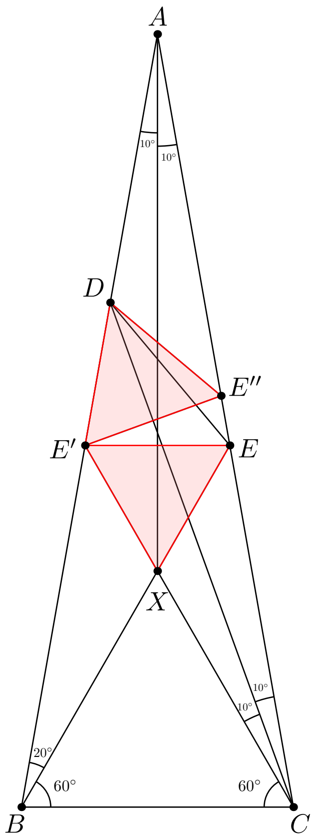
Solution 1: Define E' on the side AB such that $\angle BCE' = 60°$, and X the intersection of BE and CE'. Straightforward angle chasing gives that $\Delta E'EX$ is equilateral, as it has angles of $60°$. Moreover, $\Delta AE'C$ is isosceles at E', as $\angle E'AC = 20° = \angle E'CA$. Consider the reflection preserving this triangle, sending E' to itself, and exchanging A and C. Then D and X are also exchanged, as $\angle E'CD = 10° = \angle E'AX$, in particular $E'D = E'X$. As $\Delta E'EX$ is equilateral, we can now deduce that $E'D = E'X = E'E$, so that $\Delta DE'E$ is isosceles at E'. The rest is angle chasing:

$$\angle E'DE = \frac{180° - \angle DE'E}{2} = \frac{180° - 80°}{2} = 50°$$

and

$$\angle CDE = \angle E'DE - \angle E'DC = 50° - 30° = 20°.$$

Solution 2: Define E' on the side AB such that $\angle BCE' = 60°$, and E'' on the side AC such that $\angle BDE'' = 60°$. Straightforward angle chasing yields $\angle E'DC = 30° = \angle E''DC$ and $\angle E'CD = 10° = \angle E''CD$, proving that CD is the perpendicular bisector of $E'E''$. As $DE' = DE''$ and $\angle E'DE'' = 60°$, $\Delta DE'E''$ is equilateral so that $\angle DE'E'' = 60°$ and $\angle E''E'E = 20°$. This implies that $\angle E''EE' = 80° = \angle EE''E'$ so that $\Delta E''EE'$ is isosceles at E'. Looking at side lengths, this implies that $E'D = E'E'' = E'E$, so we also have that $\Delta DE'E$ is isosceles at E'. We can then conclude like in Solution 1.



Remark : This drawing has elements from Solutions 1 and 2 : the two constructed equilateral triangles are highlighted.

Marking scheme - Solution 1

- 0P: Claiming that $\angle CDE = 20^\circ$.
- 1P: Introducing E', the point on the side AB such that $\angle E'CB = 60^\circ$.
- 1P: Showing that $\Delta E'EX$ is equilateral.
- 2P: Showing that $E'D = E'X$.
- 3P: Conclude.

Marking scheme - Solution 2

- 0P: Claiming that $\angle CDE = 20^\circ$.
- 1P: Introducing E', the point on the side AB such that $\angle E'CB = 60^\circ$ and E''.
- 2P: Showing that $\Delta E'DE''$ is equilateral.
- 1P: Showing that $E'E'' = E'E$.
- 3P: Conclude.

$0 \leq d \leq c \leq b \leq a \leq 1 \implies$
$(a+b+c+d)^2(a+2b+3c+4d) \leq (a+b+c+d)^3$