SMO - Final round 2018

First exam - 16 January 2018

Duration: 4 hours
Difficulty: The problems are orderd by difficulty.
Points: Each problem is worth 7 points.

1. The cells of an 8×8 chessboard are all coloured in white. A move consists in inverting the colours of a rectangle 1×3 horizontal or vertical (the white cells become black and conversely). Is it possible to colour all the cells of the chessboard in black in a finite number of moves ?

2. Let a, b and c be natural numbers. Determine the smallest value that the following expression can take:
$$\frac{a}{\gcd(a+b,a-c)} + \frac{b}{\gcd(b+c,b-a)} + \frac{c}{\gcd(c+a,c-b)}.$$
Remark: $\gcd(6,0) = 6$ *and* $\gcd(3,-6) = 3$.

3. Determine all natural integers n for which there is no triplet (a,b,c) of natural numbers such that:
$$n = \frac{a \cdot \operatorname{lcm}(b,c) + b \cdot \operatorname{lcm}(c,a) + c \cdot \operatorname{lcm}(a,b)}{\operatorname{lcm}(a,b,c)}.$$

4. Let D be a point inside an acute triangle ABC such that $\angle BAD = \angle DBC$ and $\angle DAC = \angle BCD$. Let P be a point on the circumcircle of triangle ADB. We assume that P lies outside of the triangle ABC. A line through P intersects the ray BA in X and the ray CA in Y such that $\angle XPB = \angle PDB$. Prove that BY and CX intersect on AD.

   Remark: For two points F and G, the ray FG is the set of points on the line FG that lies on the same side of F than G.

5. Prove that there exists no function $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for every $x, y \in \mathbb{R}_{>0}$
$$f(xf(x) + yf(y)) = xy.$$