STAT 105, Fall 2015
Section B
Homework #9, Solutions

1. a) Since $n = 40$ is greater than 25, we can use normal approximation.

From table B-3, we get $Z_{1-\frac{.10}{2}} = Z_{.95} = 1.645$ (since $P(Z \le 1.645) = .95$)

So a 95% confidence interval for the mean is

$$\bar{X} \pm Z_{1-\frac{\alpha}{2}} \cdot \sqrt{\frac{\sigma^2}{n}} = 193.8 \pm 1.645\sqrt{\frac{80}{40}} = (191.4736, 196.1264)$$

With 90% confidence we can say the average time to build this house is between 191.47 and 196.13 days.

b) In this case, we want 95% confidences and $Z_{1-\frac{.05}{2}} = Z_{.975} = 1.96$ (from table B-3)

So $$\bar{X} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma^2}{n}} = 193.8 \pm 1.96 \cdot \sqrt{\frac{80}{40}} = (191.0281, 196.5719)$$

with 95% confidence we can say the average time required to build the house is between 191.03 and 196.57 days.

c) For a 99% confidence interval, we need $\alpha = .01 \Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{.995}$

Since $P(Z \le 2.575) = .995$ (from table B-3), $Z_{.995} = 2.575$

So $$\bar{X} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma^2}{n}} = 193.8 \pm 2.575\sqrt{\frac{80}{40}} = (190.1584, 191.4416)$$

with 99% confidence, we can say that the average time required to build a house is between 190.16 days and 191.44 days.

d) The widths of the intervals are as follows:

confidence	interval width
90%	4.65 days
95%	5.54 days
99%	7.28 days

However, the change in days, when talking about a process that is taking 193.8 days on average with our sample, may not be important.

The 99% confidence interval is less than 3 days wider than the 90%, but the difference in accuracy is 1/100 chance of being wrong vs. 1/10 chance of being wrong.

The widths do not seem practically important. The 99% would be preferred.

2\. a) Since X_i are each from the same conditions, we can say.

$E(X_i) = \mu_X$, $Var(X_i) = \sigma_X^2 = (1.5)^2 \cdot \frac{krads^2}{s^4}$

So $E(\bar{X}) = \frac{1}{25}E(X_1) + \frac{1}{25}E(X_2) + \dots + \frac{1}{25}E(X_{25})$

$= \frac{1}{25}\mu_X + \frac{1}{25}\mu_X + \dots + \frac{1}{25}\mu_X$

$= 25\left(\frac{1}{25}\mu_X\right)$

$= \mu_X$

b) $Var(\bar{X}) = \left(\frac{1}{25}\right)^2 \cdot Var(X_1) + \left(\frac{1}{25}\right)^2 Var(X_2) + \dots + \left(\frac{1}{25}\right)^2 Var(X_{25})$

$= \left(\frac{1}{25}\right)^2 \sigma_X^2 + \left(\frac{1}{25}\right)^2 \sigma_X^2 + \dots + \left(\frac{1}{25}\right)^2 \sigma_X^2$

$= 25\left(\frac{1}{25}\right)^2 \sigma_X^2$

$= \left(\frac{1}{25}\right)\sigma_X^2 = \frac{\left(1.5\ \frac{krads}{s^2}\right)^2}{25}$

c) Since $n \geq 25$, $\bar{X}$ will have a normal distribution.

Specifically, $\bar{X}$ will be a normal random verible with mean μ_X and variance $\sigma_X^2/25 = (1.5)^2/25$.

d) $E(\bar{Y}) = \frac{1}{25}E(Y_1) + \dots + \frac{1}{25}E(Y_{25})$

$= \frac{1}{25}\mu_Y + \dots + \frac{1}{25}\mu_Y$

$= 25\left(\frac{1}{25}\mu_Y\right)$

$= \mu_Y$

e) $Var(\bar{Y}) = \left(\frac{1}{25}\right)^2 Var(Y_1) + \dots + \left(\frac{1}{25}\right)^2 Var(Y_{25})$

$= \left(\frac{1}{25}\right)^2 \cdot \sigma_Y^2 + \dots + \left(\frac{1}{25}\right)^2 \cdot \sigma_Y^2$

$= 25\left(\frac{1}{25}\right)^2 \sigma_Y^2$

$= \frac{\sigma_Y^2}{25} = \frac{(1.5)^2}{25}$ (in $krads^2/s^4$)

f) As with $\bar{X}$, since $n \geq 25$ we can say that (by the central limit theorem) $\bar{Y}$ is normally distributed with mean μ_Y and variance $\frac{\sigma_Y^2}{25} = \frac{(1.5)^2}{25}$

g) $\bar{D} = \bar{X} - \bar{Y}$

Since $\bar{X}$ and $\bar{Y}$ are just two random variables,

$$E(\bar{D}) = E(\bar{X}) - E(\bar{Y}) = \mu_X - \mu_Y$$

h) Again, $\bar{X}$ and $\bar{Y}$ are just two random variables.

We can also say that $\bar{X}$ and $\bar{Y}$ are independent, since the experiments were performed consistently and without any obvious systematic errors.

So $\bar{D} = \bar{X} - \bar{Y} = 1 \cdot \bar{X} + (-1) \cdot \bar{Y}$ and

$$\begin{aligned} Var(\bar{D}) &= (1)^2 \, Var(\bar{X}) + (-1)^2 \, Var(\bar{Y}) \\ &= 1 \cdot \frac{\sigma_X^2}{25} + 1 \cdot \frac{\sigma_Y^2}{25} \\ &= \frac{(1.5)^2}{25} + \frac{(1.5)^2}{25} \\ &= \frac{2(1.5)^2}{25} \end{aligned}$$

i) If we knew that $E(\bar{D}) > 0$ than $\mu_X - \mu_Y > 0$ and $\mu_X > \mu_Y$

Since we are looking for the helmet with the smaller mean rotation, this would imply that Prototype Y is better.

j) For 95% C.I. with $n \geq 25$, use $Z_{1-\frac{.05}{2}} = Z_{.975} = 1.96$

$$\begin{aligned} \bar{X} \pm Z_{.975}\sqrt{\frac{\sigma_X^2}{n}} &= 13.84 \pm 1.96\sqrt{\frac{(1.5)^2}{25}} \\ &= 13.84 \pm 1.96\left(\frac{1.5}{5}\right) \\ &= (13.252,\ 14.428) \end{aligned}$$

k) Again, $Z_{1-\frac{.05}{2}} = Z_{.975} = 1.96$

$$\begin{aligned} \bar{Y} \pm Z_{.975}\sqrt{\frac{\sigma_Y^2}{n}} &= 15.08 \pm 1.96\sqrt{\frac{(1.5)^2}{25}} \\ &= 15.08 \pm 1.96\left(\frac{1.5}{25}\right) \\ &= (14.492,\ 15.668) \end{aligned}$$

l) $\bar{D}$ took the value $\bar{X} - \bar{Y} = 13.84 - 15.08 = -1.24$

m) Since we know $\bar{D}$ follows a normal distribution, then we can construct a 95% confidence interval for $\mu_X - \mu_Y$:

$$\bar{D} \pm Z_{1-\frac{.05}{2}} \sqrt{Var(\bar{D})}$$

Note: for the confidence interval for $\bar{X}$ we used
$$\bar{X} \pm Z_{1-\frac{.05}{2}} \sqrt{Var(\bar{X})}$$
this $\bar{D}$ is no different

Which in our case is

$$-1.24 \pm 1.96 \sqrt{\frac{2(1.5)^2}{25}} = -1.24 \pm 1.96 \frac{\sqrt{2}(1.5)}{5}$$
$$= -1.24 \pm 0.8316$$
$$= (-2.072, -0.408)$$

This means we are 95% confident that $\mu_X - \mu_Y$ is somewhere between -2.072 and -0.408. This interval only includes negative values.

In other words, we can be 95% confident that $\mu_X - \mu_Y < 0$!
this means that we are 95% confident that $\mu_X < \mu_Y$.

There is evidence that $\mu_X < \mu_Y$ then, suggesting prototype X is preferable to prototype Y.