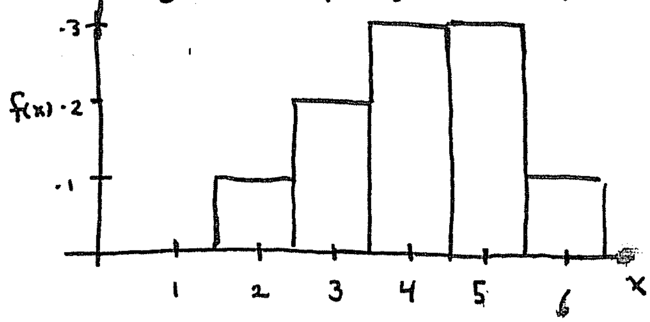
STAT 105, FALL 2015
Section B
Homework #7 Solutions

1.) 5.1.1 (pg. 243)

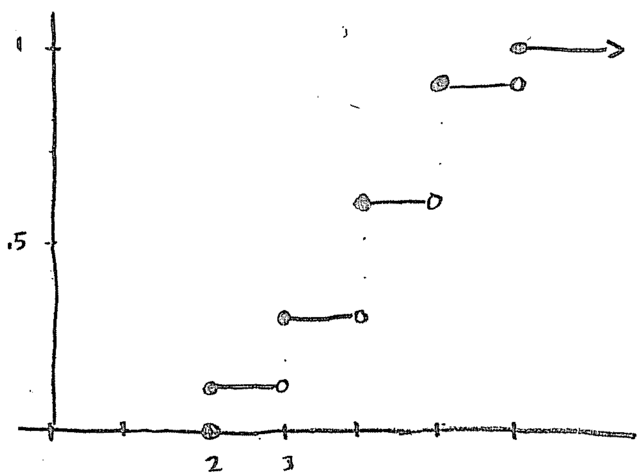
a) The histogram should center the boxes of the specified values with heights corresponding the the probability.



The cumulative probability function, $F(x)$, is just $P(X \le x)$

X	1	2	3	4	5	6
$F(x)$	0	.1	.3	.6	.9	1

← it jumps at the values that have $f(x) > 0$.



Note: $F(2.5) = f(2) = .1$
$F(2.9) = f(2) = .1$
$F(2.999) = f(2) = .1$
$F(3) = .3$
So there are "●—" "—○" at the jumps

b) The mean of X is the expected value: $E(X) = \sum_{\text{all } x} x \cdot f(x)$

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = 2 \cdot f(2) + 3 \cdot f(3) + 4 \cdot f(4) + 5 \cdot f(5) + 6\, f(6)$$

$$= 2(.1) + 3(.2) + 4(.3) + 5(.3) + 6(.1)$$

$$= .2 + .6 + 1.2 + 1.5 + .6$$

$$= 4.1$$

(Variance and standard deviation on next page)

In order to get the standard deviation, we need to get the variance.

$$Var(X) = \sum_{all\,X} (X - E(X))^2 f(x)$$

$$= (2-4.1)^2 \cdot f(2) + (3-4.1)^2 \cdot f(3) + (4-4.1)^2 f(4) + (5-4.1)^2 \cdot f(5) + (6-4.1)^2 \cdot f(6)$$

$$= (-2.1)^2 \cdot (.1) + (-1.1)^2 \cdot (.2) + (.1)^2 (.3) + (.9)^2 \cdot (.3) + (1.9)^2 \cdot (.1)$$

$$= 1.29$$

So $\sigma^2 = Var(X) = 1.29 \Rightarrow \sigma = \sqrt{1.29} = 1.135782$

Rounding, we get standard deviation of 1.136.

2.) 5.1.2 (pg. 243)

a) Since X is the number of people correctly identifying the artificial soda, X could be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 - so we need to find f(x) for each of those.

If there is no difference in taste, then subjects may still guess the artificial sweetened soda - its just going to be by chance though. Since there are 3 choices, the chance they pick the the artificial sweetener is 1/3 (assuming, as we just said no difference).

Lets call a subject selecting the artificial soda a "success." There are 10 subjects, $p = \frac{1}{3}$ is the probability of success and the people are doing the test independantly of each other.

So we can think of X as having a binomial distribution with $n = 10$ and $p = \frac{1}{3}$. (i.e., $X \sim$ binomial$(10, \frac{1}{3})$).

For each $X = 0, 1, 2, \ldots, 10$ $\quad f(x) = \frac{10!}{(10-X)!\,X!} \left(\frac{1}{3}\right)^X \left(1-\frac{1}{3}\right)^{10-X}$ so

X	0	1	2	3	4
f(x)	$\left(\frac{2}{3}\right)^{10}$	$10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^9$	$\frac{10\cdot9}{2}\left(\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right)^8$	$\frac{10\cdot9\cdot8}{3\cdot2\cdot1}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7$	$\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^6$

X	5	6	7	8	9	10
f(x)	$\frac{10\cdot9\cdot8\cdot7\cdot6}{5\cdot4\cdot3\cdot2\cdot1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5$	$\frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^4$	$\frac{10\cdot9\cdot8}{3\cdot2\cdot1}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^3$	$\frac{10\cdot9}{2\cdot1}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2$	$\frac{10}{1}\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right)$	$\left(\frac{1}{3}\right)^{10}$

b) If 7 of the 10 correctly identify the artificial sweetener then we can say $X=7$.

If the subjects can't <u>really</u> tell the difference, then

$$P(X=7) = f(7) = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^3$$

$$= \frac{126\cdot 2^3}{3^{10}} \approx 0.016$$

So there is about a 1.6% chance that could have happened if they can't really tell a difference. This is so unlikely that we should probably believe they can taste a difference (meaning P(identify artificial) = .6, or .8 is more reasonable than P(identify artificial) = 1/3.

3.) 5.1.5 (244)

W = # of the 8 specimens we detect the crack in. so $f(w) = \frac{8!}{(8-w)!\cdot w!}p^w(1-p)^{n-w}$

We want to consider W to be binomial with $n=8$ and $p=.20$

a) $$P[W=3] = f(3) = \frac{8!}{5!\cdot 3!}(.2)^3(.8)^5 = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1}(.2)^3(.8)^5 = 56(.2)^3(.8)^5$$

$$\approx .1468$$

b) $$P(W\leq 2) = f(0)+f(1)+f(2)$$

$$= \frac{8!}{8!\cdot 0!}(.2)^0(.8)^8 + \frac{8!}{7!\,1!}(.2)^1(.8)^7 + \frac{8!}{6!\,2!}(.2)^2(.8)^6$$

$$\approx 0.1678 + 0.3355 + 0.2936$$

$$\approx .7969$$

c) $$EW = n\cdot p = 8(.2) = \frac{16}{10} = 1.6$$

d) $$Var(W) = n\cdot p\cdot(1-p) = 8(.2)(1-.2) = 8(.2)(.8) = 1.28$$

4.) X can be 2, 3, 4, ..., 12 ; as illustrated below.

Red Die	Blue Die 1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

the entries inside the table are the values of X that would result.

for instance, if we roll Red = 2 and Blue = 5 we get a 7.

There are 36 results here - and each one is equally likely.
So this means that each spot has a 1/36 chance of occurring.

Notice $P(X=2) = P(\text{Red} = 1 \text{ and Blue} = 1) = 1/36$

<u>BUT</u>

$$P(X=3) = P(\text{Red}=1 \text{ and Blue}=2) + P(\text{Red}=2 \text{ and Blue}=1) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$$

<u>So</u>

X	f(x)
1	0
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36
13	0

b) So, as we just saw, $P(X=7) = 6/36 = 1/6$.

If we call rolling a 7 a success and let $Y =$ # of successes on 4 rolls, then Y is binomial with probability of success $p = \frac{1}{6}$ and $n = 4$

So $P(Y=3) = \frac{4!}{(4-3)!3!}\left(\frac{1}{6}\right)^3\left(1-\frac{1}{6}\right)^{4-3}$

$= (4)\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^1$

$\approx .0154$

b) Now we have 6 rolls - we can consider $V =$ # times we roll a 7 in 6 tries and V is binomial with $n=6$ and $p = 1/6$

$P(V=3) = \frac{6!}{(6-3)!3!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{6-3}$

$= \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3$

$= 20\left(\frac{1}{216}\right)\left(\frac{125}{216}\right)$

$= \frac{(20)(125)}{46656}$ $\qquad \frac{2^2\cdot 5^4}{2^6\cdot 3^6} = \frac{5^4}{2^4\cdot 3^6}$

$= \frac{625}{11\,664}$

$\approx .0535$

c) $T =$ first time 7 is rolled $\Rightarrow$ T is geometric with $p = 1/6$

$P(T \le 5) = P(T=1) + P(T=2) + P(T=3) + P(T=4) + P(T=5)$

$= \left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)^{1-1} + \left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)^{2-1} + \left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)^{3-1} + \left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)^{4-1} + \left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)^{5-1}$

$= \frac{1}{6} + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^4$

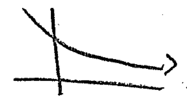
$\approx .598$

d) $P(T \ge 2) = 1 - P(T=1) = 1 - \frac{1}{6} = 5/6$ $\qquad$ or $P(T \ge 1) = 1$ since $T \ge 1$ covers all the values that T can actually take - so $P(T \ge 2)$ is only missing $P(T=1)$

e) "should I expect" always means you should think about expected value.

$E(T) = \frac{1}{p} = \frac{1}{1/6} = 6$

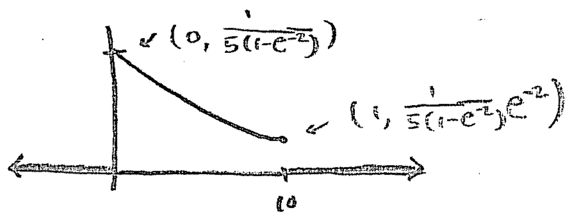
So we should expect to roll the die 6 times to get a 7.

5.) a) e^{-x} has a shape like this: 

So this function just transforms that.

$$f(0) = \frac{1}{5(1-e^{-2})} e^{-0/5} = \frac{1}{5(1-e^{-2})} \cdot 1 = \frac{1}{5(1-e^{-2})}$$

$$f(10) = \frac{1}{5(1-e^{-2})} e^{-10/5} = \frac{1}{5(1-e^{-2})} e^{-2}$$



b) We know $f(w) \geq 0$ if $w > 10$ or $w < 0$ (since any of those kinds of w will give us $f(w) = 0$ and $0 \geq 0$ is true.)

So consider $0 \leq w \leq 10$.

$$f(w) = \frac{1}{5(1-e^{-2})} e^{-w/5}.$$

- $e^{-w/2} > 0$ no matter what
- $1 - e^{-2} > 0$ so $\frac{1}{5(1-e^{-2})} > 0$ no matter what.

so $f(w) > 0$ no matter what.

Now,

$$\int_{-\infty}^{\infty} f(w)\,dw = \int_0^{10} \frac{1}{5(1-e^{-2})} e^{-\frac{w}{5}}\,dw = \frac{1}{5(1-e^{-2})}(-5)e^{-\frac{w}{5}}\Big|_0^{10}$$

$$= \left[-\frac{1}{(1-e^{-2})} e^{-\frac{10}{2}}\right] - \left[-\frac{1}{(1-e^{-2})} e^{-\frac{0}{2}}\right]$$

$$= \left[-\frac{e^{-2}}{(1-e^{-2})}\right] - \left[-\frac{1}{(1-e^{-2})}\cdot(1)\right]$$

$$= -\frac{e^{-2}}{(1-e^{-2})} + \frac{1}{(1-e^{-2})}$$

$$= \frac{-e^{-2}+1}{(1-e^{-2})}$$

$$= \frac{1-e^{-2}}{1-e^{-2}}$$

$= 1$. so $f(w)$ is a valid p.d.f.

For each of these problems, we are looking for

$$P[a \le W \le b] = \int_a^b \frac{1}{5(1-e^{-2})} e^{-\frac{w}{5}} dw = \frac{e^{-a/5} - e^{-b/5}}{(1-e^{-2})}, \text{ for } a, b \text{ in } [0, 10].$$

Note: $P[W \le w] = P[0 \le W \le w]$ in this case.

c) $$P[W \le 2] = \frac{e^{-0/5} - e^{-2/5}}{(1-e^{-2})} = \frac{1-e^{-2/5}}{1-e^{-2}}$$

d) $$P[2 \le W \le 5] = \frac{e^{-2/5} - e^{-5/5}}{1-e^{-2}} = \frac{e^{-2/5} - e^{-1}}{1-e^{-2}}$$

e) $$P[5 \le W \le 10] = \frac{e^{-5/5} - e^{-10/5}}{1-e^{-2}} = \frac{e^{-1} - e^{-2}}{1-e^{-2}}$$

f) $$P[2 \le W \le 10] = \frac{e^{-2/5} - e^{-10/2}}{1-e^{-2}} = \frac{e^{-2/5} - e^{-2}}{1-e^{-2}}$$