Homework 3

2.8 Binomial Expansion: for any $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k} = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \dots + \binom{n}{n-1} a^{n-1} b^1 + \binom{n}{n} a^n b^0$$

which means that for $X \sim \text{binom}(n, p)$

$$P(X=0) + P(X=1) + \dots + P(X=n) = \binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} + \dots + \binom{n}{n} p^n (1-p)^0$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (p + (1-p))^n \quad \text{by binomial expansion}$$

$$= (1)^n$$

$$= 1$$

So for any n and p, the sum of the binomial probabilities is 1.

2.11

So we want to find the integer k where $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ is maximized. Since the derivative is usually used to find the maximum of continuous functions (and since $(n-k)!$ is hard to work with) the book suggests using the ratio.

$$\frac{P(X=k+1)}{P(X=k)} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-(k+1)}}{\binom{n}{k} p^k (1-p)^{n-k}}$$

$$= \frac{k!\,(n-k)!}{(k+1)!\,(n-k-1)!} \frac{p}{1-p}$$

$$= \frac{n-k}{k+1} \frac{p}{1-p}$$

if $\frac{n-k}{k+1} \frac{p}{1-p} \geq 1$ then $\frac{P(X=k+1)}{P(X=1)} \geq 1$ and $P(X=k+1) \geq P(X=k)$

so $n-k \geq \frac{1-p}{p}(k+1) \Rightarrow n-k \geq \left(\frac{1-p}{p}\right)k + \left(\frac{1-p}{p}\right)$

$$\Rightarrow n - \frac{1-p}{p} \geq \left(\frac{1-p}{p}\right)k + k$$

$$\Rightarrow p \cdot n - 1 + p \geq (1-p)k + pk$$

$$\Rightarrow p(n+1) - 1 \geq k$$

continued on next page

So if $k \leq (n+1)p - 1$ then $P(X=k+1) \geq P(X=k)$
this means that the largest value of k for which
$P(X=k+1) \geq P(X=k)$ is the largest integer less than $(n+1)\cdot p - 1$
notice: $k+1$ in this case would be the mode.

so the largest integer less than $(n+1)p$ is the mode.

2.19 A geometric random variable is described by the pmf

$$P(X=k) = (1-p)^{k-1} \cdot p$$

so

$$\begin{aligned}
P(X \leq k) &= 1 - P(X \geq k+1) \\
&= 1 - \sum_{i=k+1}^{\infty} P(X=i) \\
&= 1 - \sum_{i=k+1}^{\infty} p(1-p)^i \\
&= 1 - \sum_{i=1}^{\infty} p(1-p)^{k+i} \\
&= 1 - (1-p)^k \sum_{i=j}^{\infty} p(1-p)^i \\
&= 1 - (1-p)^k \cdot (1) \qquad \text{since } \sum_{k=1}^{\infty} p(1-p)^k = 1 \\
&= 1 - (1-p)^k
\end{aligned}$$

2.21

$$\begin{aligned}
P(X > n+k-1 \mid X > n-1) &= \frac{P(X > n+k-1 \text{ and } X > n-1)}{P(X > n-1)} \\
&= \frac{P(X > n+k-1)}{P(X > n-1)} \\
&= \sum_{i=n+k}^{\infty} P(X=i) \Big/ \sum_{i=n}^{\infty} P(X=i) \\
&= \sum_{i=n+k}^{\infty} p(1-p)^i \Big/ \sum_{i=n}^{\infty} p(1-p)^i \\
&= \sum_{i=n}^{\infty} p(1-p)^{i+k} \Big/ \sum_{i=1}^{\infty} p(1-p)^i \\
&= (1-p)^k \cdot \sum_{i=n}^{\infty} p(1-p)^i \Big/ \sum_{i=1}^{\infty} p(1-p)^i \\
&= (1-p)^k \\
&= P(X > k)
\end{aligned}$$

2.29

For $X \sim \text{poisson}(\lambda)$, $P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$

so $p_0 = P(X=0) = \frac{\lambda^0}{0!}e^{-\lambda} = e^{-\lambda}$

and $p_k = \frac{\lambda^k}{k!}e^{-\lambda} = \frac{\lambda \cdot \lambda^{k-1}}{k(k-1)!}e^{-\lambda} = \frac{\lambda}{k}\left[\frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda}\right] = \frac{\lambda}{k} \cdot p_{k-1}$

2.33

Again, we want to find the values for k where $P(X=k+1)/P(X=k) \geq 1$

From problem 29, we know $P(X=k+1)/P(X=k) = \frac{\lambda}{k+1}$

This means that if $\frac{\lambda}{k+1} \geq 1$ the probability that $X = k+1$ is larger than the probability that $X = k$

So $\frac{\lambda}{k+1} \geq 1 \Rightarrow \lambda \geq k+1$

So our function is increasing or constant for $\lambda \geq k+1$ which means the mode must occur at the largest integer less than or equal to λ.

Note: if λ is an integer then there is some k so that

$$\lambda = k+1 \Rightarrow \frac{\lambda}{k+1} = 1 \Rightarrow P(X=k+1) = P(X=k)$$

in that case both k and $k+1$ would be "the mode"

It's a little sloppy, but (as this problem shows) statisticians don't make much distinction between "a mode" and "the mode" in the general sense.

2.36 notice that if $U \sim \text{Uniform}(0,1)$ then $0 \leq U \leq 1$ which means $0 \leq n \cdot U \leq n$. This means that $0 \leq [nU] \leq n$ and the only way $[nU] = n$ is if $U = 1$

(cont)

So suppose $k \in \{0, 1, 2, \ldots n-1\}$

then $P(X=k) = P([nU]=k) = P(k-1 \geq nU \leq k)$

$= P(k \leq nU < k+1)$

$= P(\frac{k}{n} \leq U \leq \frac{k+1}{n})$

$= \frac{1}{n}$ (by uniform probability)

So for $k=0,1,2,\ldots,n-1$ $P(X=k) = 1/n$

but what about $P(X=n)$?

Since U is uniform $P(U=1)=0$ and thus $P(X=n)=0$ as well.

38. i) f, g are both pdf.

then $f(x) \geq 0$ for all x and $g(x) \geq 0$ for all x and thus

since $0 \leq \alpha \leq 1$ then $\alpha \cdot f(x) \geq 0$ and $(1-\alpha)g(x) \geq 0$ for all x

thus $\alpha \cdot f(x) + (1-\alpha)f(x) \geq 0$

ii) Since $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\int_{-\infty}^{\infty} g(x)dx = 1$ then

$$\int_{-\infty}^{\infty} \alpha \cdot f(x) + (1-\alpha)g(x)\, dx = \alpha \int_{-\infty}^{\infty} f(x)dx + (1-\alpha)\int_{-\infty}^{\infty} g(x)dx$$

$$= \alpha(1) + (1-\alpha)(1)$$

$$= \alpha + (1-\alpha)$$

$$= 1$$

(3) piecewise continuous.

It is enough to say that the sum of two piecewise continuous functions will also be piecewise continuous. (This is covered in the prerequisites)

2.45

If $X \sim$ exponential (λ) then $P(X \leq t) = \int_0^t \lambda e^{-\lambda x} dx$

so $P(X \leq t) = \int_0^t \lambda e^{-\lambda x} dx = -e^{-\lambda x} \Big|_0^t = 1 - e^{-\lambda t}$

and $P(X > t) = 1 - P(X \leq t) = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$

a) $P(X \leq 10) = 1 - e^{-(.1)(10)} = 1 - e^{-1}$

b) $P(5 \leq X \leq 15) = P(X \leq 15) - P(X \leq 5)$

$= 1 - e^{-(15 \cdot (.1))} - 1 + e^{-(5)(.1)}$

$= e^{-0.5} - e^{-1.5}$

c) Need $P(X > t) = 0.1$

so $P(X > t) = e^{-t/10} = \frac{1}{10} \Rightarrow -t/10 = \ln\left(\frac{1}{10}\right) \Rightarrow t = -10 \cdot \ln\left(\frac{1}{10}\right)$

so $t = 10 \cdot \ln(10)$