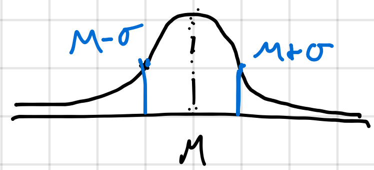
Chapter 2:

*Normal Random Variables

- $X \sim N(\mu, \sigma^2)$
- μ is the mean of the random variable
- σ^2 is the variance

Note: $E(X) = \mu$
$Var(X) = \sigma^2$

graph of pdf



- centered over μ
- σ tells us where the inflection points are located

pdf

$$\phi(x) = f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

This function is represented by ϕ.

CDF

$$\Phi(x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt$$

(no closed form)

Question: How do we actually figure out what the probability that a normal random variable is less than x?

Answer: Paper and pencil? Also a table.

otherwise

Use a computer:

in R: 'qnorm' is a function that reports the values of the CDF.

Standard Normal Distribution

Z is a standard normal RV means $Z \sim N(0,1)$

$\Phi(z)$ is the CDF, $\phi(z)$ is the pdf

Use: If Z is a standard normal

then $X = \sigma \cdot Z + \mu$ is a $N(\mu, \sigma^2)$

Our table (B-2) describes cumulative probabilities for standard normal RVs only.

so, for values of z, we can find $\Phi(z)$

$$\Phi(z) = P(Z \le z) = P(\sigma \cdot Z \le \sigma \cdot z) \quad (\text{if } \sigma > 0)$$

$$= P(\sigma \cdot Z + \mu \le \sigma \cdot z + \mu) \quad (\text{for any } \mu)$$

But $\sigma \cdot Z + \mu$ would be a normal RV with mean μ and variance σ^2

We can also go the other way:

if $X \sim N(\mu, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim N(0,1)$ i.e. $Z = \frac{X-\mu}{\sigma}$

for any x, $P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

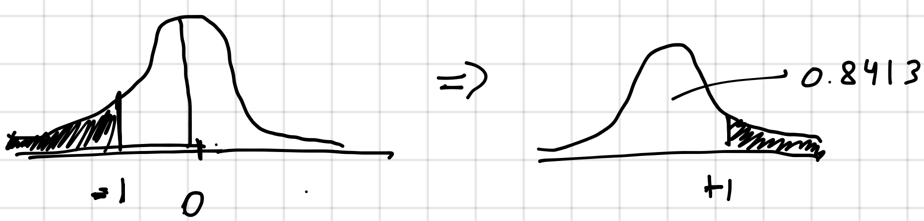
Ex $X \sim N(3,4)$ then $P(X \le 7) = P\left(\frac{X-3}{2} \le \frac{7-3}{2}\right)$
$= P(Z \le 2)$
$= .9772$

Ex $X \sim N(3,4)$, what is the probability that $X \le 1$

since $X \sim N(3,4)$ then $\frac{X-3}{2}$ is standard normal

$$P(X \le 1) = P\left(\frac{X-3}{2} \le \frac{1-3}{2}\right) = P(Z \le -1)$$

This value is not listed on our table, but notice



Key: the normal distribution is symmetric around the mean

Next: If we take the graph on the right, we need
$P(Z \ge 1)$ but our table is based on $P(Z \le 1) = 0.8413$
But since $P(Z \ge 1) = 1 - P(Z \le 1) = 1 - 0.8413$
$= .1587$

The Gamma Distribution

$X \sim \text{Gamma}(\alpha, \lambda)$ if the pdf of X is

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \qquad x \ge 0$$

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \qquad x \geq 0$$

where

$$\Gamma(t) = \int_0^\infty u^{t-1} \cdot e^{-u} \, du$$

pdf

important $\Gamma(k) = (k-1)!$ if k is an integer

- Connection to exponential

if $k=1$, $\lambda = 1$

$$f(x) = \frac{(1)^1}{\Gamma(1)} x^{1-1} \cdot e^{-1 \cdot x} = e^{-x}$$

since $\Gamma(1) = 0! = 1$

ie. $X \sim$ exponential (1)

if only is that $X \sim \text{Gamma}(\alpha = 1, \lambda)$

$$f(x) = \frac{\lambda^1}{\Gamma(1)} \cdot x^{1-1} \cdot e^{-\lambda x} = \lambda \cdot e^{-\lambda x},$$

since $\Gamma(1) = 1$

Q: What does this mean in practical terms?

A: We use exponential distributions to describe failure times and arrival times.

If α is an integer, we can use the gamma distribution to describe the total time to the $\alpha^{\underline{th}}$ failure.

- α is called "the shape parameter"
  it controls the center of our random variable
- λ is called "the scale parameter" - think sharpness

Beta Distribution

If X is a Beta(a,b) then the pdf of X is:

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 \le x \le 1$$

if $a=1$, $b=1$ then $\Gamma(a+b) = \Gamma(2) = 1! = 1$

$\Gamma(a) = 0! = 1$

$\Gamma(b) = 0! = 1$

(because if k is a intergen $\Gamma(k) = (k-1)!$

$$f(x) = \frac{1}{1\cdot 1} x^{1-1}(1-x)^{1-1} = 1$$

i.e., X is uniform on $[0,1]$