41. There are $7+8+9=24$ socks

So there are 24 ways to choose the first sock and 23 ways to choose the second

That gives $24 \cdot 23$ ways to select the two socks

Now suppose we just want the pairs.

black pairs: $7 \cdot 6$

blue pairs: $8 \cdot 7$

green pairs: $9 \cdot 8$

$$P(\text{"socks match"}) = \frac{\#\text{ matching pairs}}{\#\text{ pairs total}} = \frac{7\cdot 6 + 8\cdot 7 + 9\cdot 8}{24\cdot 23}$$

b) $$\frac{\#\text{ black pairs}}{\#\text{ total pairs}} = \frac{8\cdot 7}{24\cdot 23}$$

43.

$$\underbrace{\underline{10}\ \underline{9}\ \underline{8}}_{\text{Job 1}}\ \underbrace{\underline{7}\ \underline{6}\ \underline{5}}_{\text{Job 2}}\ \underbrace{\underline{4}\ \underline{3}\ \underline{2}\ \underline{1}}_{\text{Job 3}} \Rightarrow 10! \text{ unique orderings}$$

but since being "in job 2" doesn't depend on <u>order</u> we would overcount the same groupings.

For any specific grouping, there are 3! ways to order job 1, 3! ways to order job 2, 4! ways to order job 3

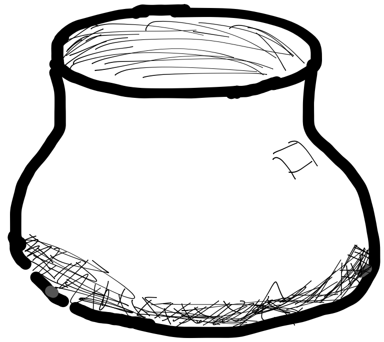
So total ways to group into 3 jobs: $\frac{10!}{3!\,3!\,4!}$

44. Same idea as above:

$\frac{12!}{4!\,4!\,4!}$ if people are assigned individually

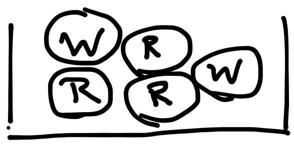
<u>If</u> people are assigned in pairs we are only assigning 6 pairs, and each game can take 2 $\rightarrow \frac{6!}{2!\,2!\,2!}$

46. "Urns" are common parts of elementary probability problems. They work as a sort of "black box" and usually are full of colored balls or lettered tiles.



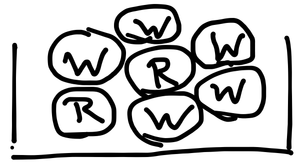
Outside of probability the only time I've ever heard a container called an urn is when as a container of human remains. I have no idea how urns came to have such a traditional place in these kinds of probability questions.

URN A

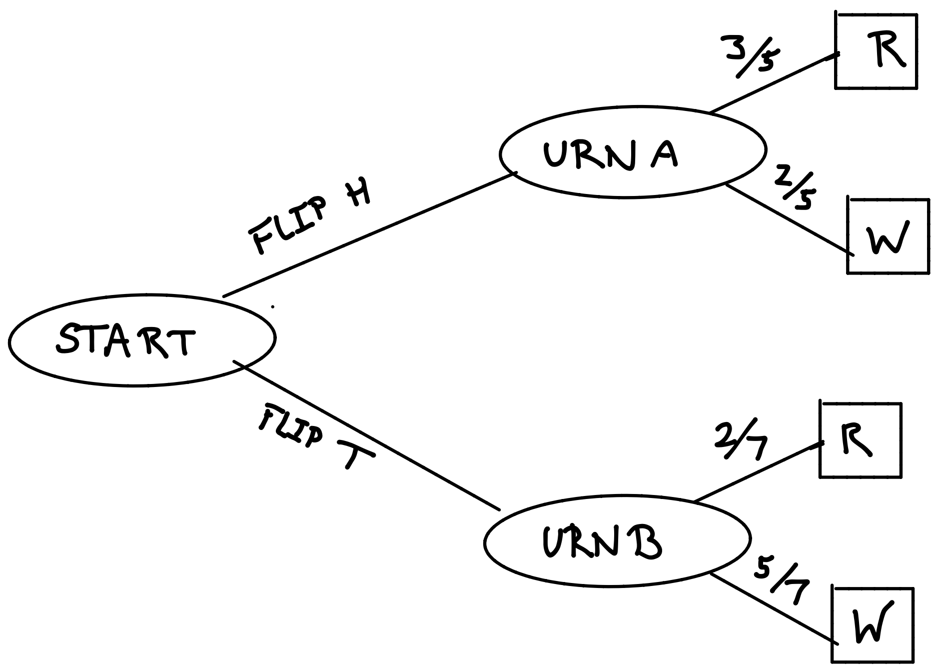


3 Red, 2 White
choose if coin flip is H

URN B



2 Red, 5 White
Choose if coin flip is T



a) $$P(R) = P((R\cap A)\cup(R\cap B))$$
$$= P(R\cap A) + P(R\cap B)$$
$$= P(R|A)\cdot P(A) + P(R|B)\cdot P(B)$$
$$= (3/5)(1/2) + (2/7)(1/2)$$
$$= 3/10 + \frac{2}{14}$$
$$= 21/70 + 10/70$$
$$= 31/70$$

so the probability of drawing a red ball is $31/70$

b) $$P(A|R) = \frac{P(A\cap R)}{P(R)} = \frac{21/70}{31/70} = 21/31$$

So if a red ball is drawn, then the probability that we flipped heads is $21/31$

67.

In this case, consider the foll. **TRUE FACTS**

- $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$
- $(A \cap B) \cap (A \cap B^c) = \emptyset$, $(A \cap B) \cap (A^c \cap B) = \emptyset$, $(A \cap B^c) \cap (A^c \cap B) = \emptyset$
- If $A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
- $A = (A \cap B) \cup (A \cap B^c)$, $B = (A \cap B) \cup (A^c \cap B)$

<u>Then</u>

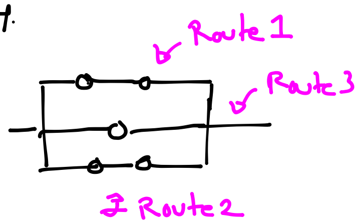
$$
\begin{aligned}
P(A \cup B) &= P\big((A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)\big) \\
&= P(A \cap B) + P(A^c \cap B) + P(A \cap B^c) \\
&= P(A \cap B) + \big[P(B) - P(A \cap B)\big] + \big[P(A) - P(A \cap B)\big] \\
&= P(A) + P(B) - 2P(A \cap B) + P(A \cap B)
\end{aligned}
$$

So $P(A \cup B) = P(A) + P(B) - P(A \cap C)$

Since $P(A \cap B) = P(A) \cdot P(B)$ for independent A, B then

$P(A \cap B) = P(A) + P(B) - P(A) \cdot P(B)$

74.



$P(\text{"system works"}) = 1 - P(\text{"system fails"})$

Let R_i = route i does not fail
R_i^c = route i fails

$$
\begin{aligned}
P(\text{"system fails"}) &= P(R_1^c \cap R_2^c \cap R_3^c) \\
&= P(R_1^c) \cdot P(R_2^c) \cdot P(R_3^c)
\end{aligned}
$$

In R_1, let U_1 = unit 1 does not fail and U_2 = unit 2 does not fail

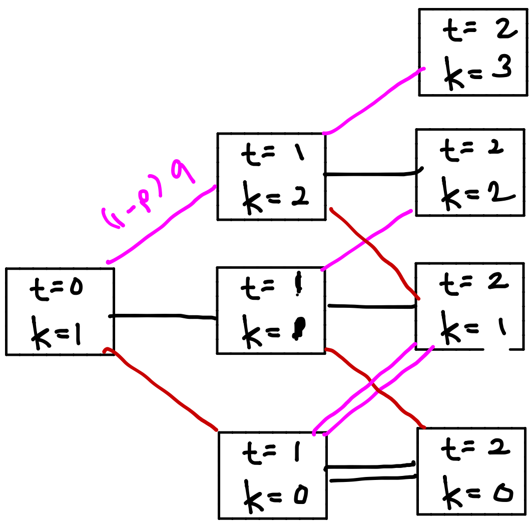
Then $P(R_1^c) = P(U_1^c \cup U_2^c) = P(U_1^c) + P(U_2^c) \quad P(U_1^c) \cdot P(U_2^c)$

$$
\begin{aligned}
&= p + p - p \cdot p \\
&= 2p - p^2
\end{aligned}
$$

Similarly, $P(R_2^c) = p$ and $P(R_3^c) = 2p - p^2$ so

$P(\text{"system does not fail"}) = 1 - (2p - p^2) \cdot p \cdot (2p - p^2) = 1 - p^3(2-p)^2$

76. Let $k = \#$ people in the queue. Then we can draw a tree diagram for this problem using



Blue path: we gain a user, we lose no one i.e., the probability of going from k at time t to $k+1$ at time $t+1$ or $P(k \to k+1)$

$$P(k \to k+1) = P(\text{gain user and lose no one}) = P(\text{gain user}) \cdot P(\text{lose no one}) = (1-p) \cdot q$$

Red path: we lose a person, gain no one

i.e. $P(k \to k-1) = P(\text{"gain no one" and "lose someone"}) = (1-q)p$

Black Path: We take the black path if we don't take blue or red

$$P(k \to k) = 1 - P(k \to k+1) - P(k \to k-1)$$
$$= 1 - p(1-q) - (1-p)q$$
$$= 1 - p + p \cdot q - q + p \cdot q$$
$$= 1 - p - q + 2pq$$

* Special Case *

If $k = 0$, then we cannot lose anyone.

So $P(0 \to 1) = P(\text{"gain user" and "lose no one"})$
$= P(\text{"gain user"}) \cdot P(\text{"lose no one"})$
$= q$

So double black has probability $1-q$ and double blue has prob q

The probability of following a certain 2-step path can be found by multiplying the single steps probabilities

k at time t=2	path	prob
0	$1 \to 0,\ 0 \to 0$	$(1-q)p(1-q)$
	$1 \to 1,\ 1 \to 0$	$(1-p-q+2pq)$
1	$1 \to 1,\ 1 \to 1$	$(1-p-q+2pq)^2$
	$1 \to 0,\ 0 \to 1$	$(1-q)pq$
	$1 \to 2,\ 2 \to 1$	$(1-p) \cdot q(1-q) \cdot p$
2	$1 \to 2,\ 2 \to 2$	$(1-p)q(1-p-q+2pq)$
	$1 \to 1,\ 1 \to 2$	$(1-p-q+2pq)(1-p)q$
3	$1 \to 2,\ 2 \to 3$	$(1-p)^2 q^2$

So $P(k=0) = p(1-q)^2 + (1-p-q+2pq)$

$P(k=1) = (1-p-q+2pq)^2 + (1-q)pq + (1-p)^2 qp$

$P(k=2) = 2(1-p)q(1-p-q+2pq)$

$P(k=3) = (1-p)^2 q^2$