

Show **all** of your work on this assignment and answer each question fully in the given context.

### 1. Sets

One common operation involving sets that we briefly mentioned in class is the Cartesian product (" $\times$ "), which, for any two sets  $A$  and  $B$ , is defined by:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Also consider a new definition: set difference (" $-$ ") which for any two sets  $A$  and  $B$  we will define as:

$$A - B = \{a \in A : a \notin B\}$$

Using this definition, as well as the other operations discussed in class, prove or disprove the following:

- (a)  $A - B = A \cap B^c$
- (b)  $(A - B) \cup (A - B^c) = A$
- (c) If  $A \times B = B \times A$  then  $A = B$
- (d)  $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2. Show that for any sample space  $\Omega$  and any subset  $A$  of  $\Omega$  the set

$$\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$$

follows the three properties needed to define a probability on  $\mathcal{F}$  - that is that (1) the empty set is in  $\mathcal{F}$ , (2) for any set in  $\mathcal{F}$ , the complement is also in  $\mathcal{F}$ , and that (3) a countable union of sets from  $\mathcal{F}$  is still a set in  $\mathcal{F}$

### 3. Functions

Define the function  $F : \{[a, b] : a, b \in \mathbb{R}, 0 \leq a \leq b\} \rightarrow \mathbb{R}$  as

$$F([a, b]) = \int_a^b 3e^{-3x} dx$$

- (a) Show that  $F$  is bounded (i.e., there are two values in the range that will contain any value of the function).
- (b) Show that if  $A, B$  are two intervals in the domain for which  $A \subset B$  then  $F(A) \leq F(B)$ .
- (c) Show that for any interval  $A = [a, b]$  in the domain,

$$F([a, b]) = 1 - F([0, a]) - \lim_{n \rightarrow \infty} F[b, n]$$

### 4. Measurements

Consider the Snow Volume Measurement Tool discussed in class - we know before hand that it can measure volumes over any rectangular area of any field imaginable.

- (a) What is the smallest measurement that this tool could return? (for instance, suppose you try to measure the volume of snow over a rectangular field in a park near your home today - what would the tool report?)

Imagine a rectangular field with width 10 meters and length 10 meters in which the snow is uniformly 1 meter deep - meaning the snow is spread out equally over every point in the field

- (b) If you measure the entire field, what is the total volume of snow?
- (c) Suppose that I went around the field and chose 10 points and removed all the snow over the over those points. Can this field still be measured with the tool you have? Why or why not? (provide a clear mathematical argument for your conclusion)
- (d) Now suppose that I went around the field and chose  $n$  points and removed all the snow over the over those points. Can this field still be measured with the tool you have? Why or why not? (provide a clear mathematical argument for your conclusion)