Announcements

- Exam I: Over Now!
- R: Learn by watching?
  HW 4 will have R elements.

STAT 430: Lecture 13

Joint Distributions

SAS

```
proc reg data=d;
y;hrs;!
```

~~What is a Function of a Random Variable?~~

~~A Function of Functions?~~

~~A Function of Variables?~~

Joint Distributions

General

Two or More Random Variables

Random variables are functions from a sample space to the real numbers:

$$X : \Omega \to \mathbb{R}$$

the way we describe the likelihood that a random variable takes a certain value is through the random variables distribution, which we can describe using it's CDF:

$$F_X(x) = P(X \leq x)$$

discrete: $p_X(x) = P(X=x)$

continuous: $f_X(x) \neq P(X=x)$

when we have more than two random variables at work, for instance X and Y we can talk about they're behavior separately:

$$X : \Omega \to \mathbb{R} Y : \Omega \to \mathbb{R}$$

or **jointly**]

Joint Distributions

General

Two or More Random Variables

If we are more interested in how they behavior simultaneously (or **jointly**), we are talking about a pair of outcomes - this means we are now talking about a **point** in $\mathbb{R} \times \mathbb{R}$.

$$(X, Y) : \Omega \times \Omega \to \mathbb{R} \times \mathbb{R}$$

And we can describe the simultaneous behavior of the two random variables using a **joint** distribution, which can be defined by (for instance) the **joint cumulative distribution function**:

$$F_{XY}(x, y) = P(\underline{X \le x}, \underline{Y \le y})$$

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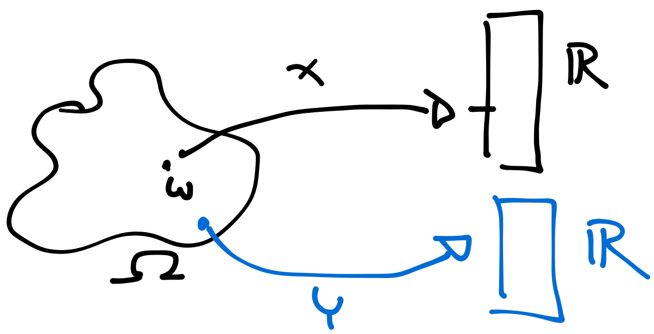

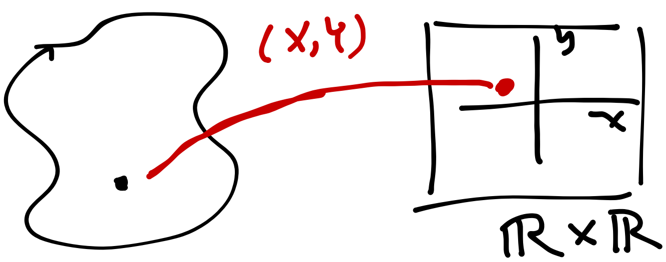

Joint Distributions

General

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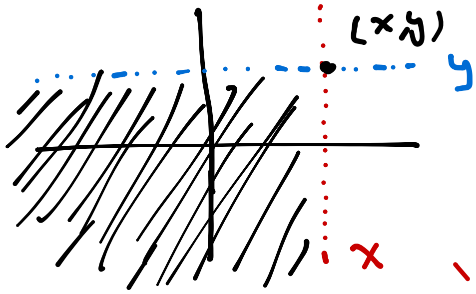
Joint Distributions

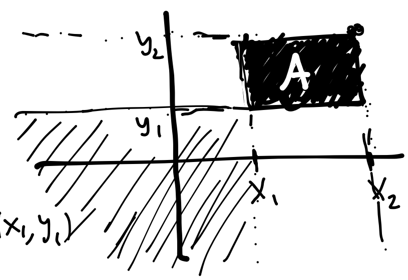
Illustration: Joint CDF and Rectangles

for continuous RV:

General $F_{XY}(x,y)$



$P((x,y)\in A)$ if A is a rectangle in $\mathbb{R}\times\mathbb{R}$



$$P((x,y)\in A) = F_{XY}(x_2,y_2) - F_{XY}(x_1,y_2) - F_{XY}(x_2,y_1) + F_{XY}(x_1,y_1)$$

too much — left of rectangle — below — correction for subtracting twice

Illustration: Joint CDF and Rectangles

Joint Distributions and Discrete Random Variables

Ideas and Notation

Basics

Discrete Case

Joint Discrete Random Variables

For two discrete random variables we have a **joint probability mass function**:

$$p(x, y) = P(X = x, Y = y)$$

You may recall from previous chapters that this can be written as:

$$P(X=x, Y=y) = P(X=x \mid Y=y) \cdot P(Y=y)$$
$$= \underbrace{P(Y=y \mid X=x)}_{\text{dependency?}} \cdot P(X=x)$$

the question of dependency and how to deal with.

$$p(x, y) = P(X = x, Y = y) = P(X = x|Y = y)P(Y = y)$$

or

Discrete Case

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Basics

Random Variables

Example: Flip a coin

Suppose I flip a coin three times and let X be the number of heads and Y be the number of tails

Sample space:

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$P(X=2) = 3/8$
$P(Y=2) = 3/8$

$\Big\}$ marginal distributions for X and Y

Joint

$$P(X=2, Y=2) = 0$$

probability table

X \ Y	0	1	2	3	X
0	0	0	0	1/8	1/8
1	0	0	3/8	0	3/8
2	0	3/8	0	0	3/8
3	1/8	0	0	0	1/8
	1/8	3/8	3/8	1/8	1

Basics

Discrete Case

Example: Multinomial Distributions

A Special discrete distribution

Scenario: In a binomial experiment, there are n independent iterations of an experiment and the probability of success is p. That is, we have two possible outcomes. But what if we have three possible outcomes?

Suppose that we have exactly three outcomes: Outcome A, Outcome B and Outcome C with probability of success p_A , p_B , and p_C

Think

$$p_A + p_B + p_C = 1$$
$$k_A + k_B + k_C = n$$

X: # of trials with outcome A

Y: # of trials with outcome B

Z: # of trials with outcome C.

$$P(X=x, Y=y, Z=z) = P(\underbrace{X=x}_{\text{binomial?}}, \underline{Y=y, Z=n-x-y})$$

$$= \underbrace{P(Y=y, Z=n-x-y \mid \underbrace{X=x}_{\text{take out "x" experiments.}})}_{\text{binomial for Y.}} \cdot P(X=x)$$

n − x experiments

Y = # of success

probability of success?

$$\frac{p_B}{p_B+p_C}, \quad 1-\frac{p_B}{p_B+p_C} = \frac{p_C}{p_B+p_C}$$

p_A = prob. of outcome A

p_B = prob of outcome B

p_C = prob of outcome C.

$$= P(X=x) \cdot P(\text{"}Y\text{"} = \text{"}y\text{"})$$

$$= \binom{n}{x}(p_A)^x (p_B+p_C)^{n-x} \cdot \binom{n-x}{y} \left(\frac{p_B}{p_B+p_C}\right)^y \left(\frac{p_C}{p_B+p_C}\right)^{n-x-y}$$

$$P(X=x, Y=y, Z=z) = \frac{n!}{x!\,\cancel{(n-x)!}}(p_A)^x (p_B+p_C)^{n-x} \cdot \frac{\cancel{(n-x)!}}{(n-x-y)!\,y!}\left(\frac{p_B}{p_B+p_C}\right)^y \left(\frac{p_C}{p_B+p_C}\right)^{n-x-y}$$

$$= \frac{n!}{x!\,y!\,(n-x-y)!}\cdot(p_A)^x \cdot \cancel{\frac{(p_B+p_C)^{n-x}}{(p_B+p_C)^y (p_B+p_C)^{n-x-y}}}(p_B)^y \cdot (p_C)^{-x-y}$$

$$= \frac{n!}{x!\,y!\,(n-x-y)!}\,(p_A)^x \cdot (p_B)^y (p_C)^{n-x-y}$$

For a multinomial experiment, with 3 outcomes

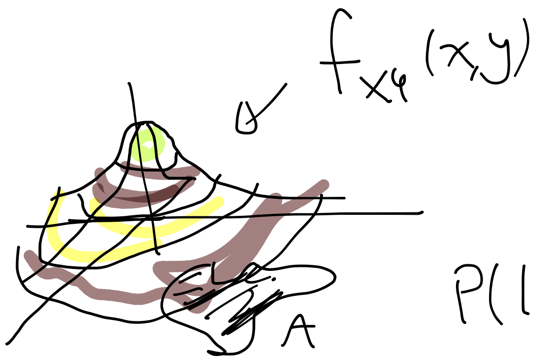
$$p(k_1,k_2,k_3) = \begin{cases} \dfrac{n!}{k_1!\,k_2!\,k_3!}(p_1)^{k_1}(p_2)^{k_2}\cdot(p_3)^{k_3} & \text{if } k_1+k_2+k_3=n \\ 0 & \text{otherwise} \end{cases}$$

where $p_1+p_2+p_3=1$ are the probabilities of outcome 1, 2, 3 on any given experiment

Illustration:

With continous random variables, we now have two dimensions at play:

2D: X, Y



Volume under the curve?

$$P((X,Y) \in A) = \iint_A f_{XY}(x,y)\,dx\,dy$$

Basics

Discrete Case

Continuous Case

Joint Density Functions

The joint cdf and pdf are connected in a similar way as the single variable case:

marginal

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$$

and

marginal

$$F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(y)dy$$

But jointly:

$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int\int_A f_{XY}(x,y)dxdy$$

where $A = \{ (a,b) \in \mathbb{R}^2 \mid a \leq x, b \leq y \}$

Joint Density Functions

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$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(x)dx$$

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$$F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(y)dy$$

But jointly:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y) = \int\int_{A} f_{XY}(x, y)dxdy$$

Basics

Discrete Case

Continuous Case

Example: Joint Exponential

Suppose you have two pieces of equipment with two different failure times...

Suppose that the failure time for piece 1 is on average $\lambda = 5$ and for piece 2 is on average $\lambda = 10$.

If we let T_1 = time to failure of piece 1

T_2 = time to failure of piece 2.

*Assume independent

and both are modeled exponentially

What is the probability that piece 2 fails before piece 1?