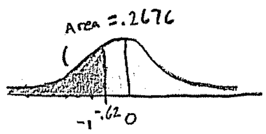
STAT 105, Fall 2015
Section B
Homework #7, solutions

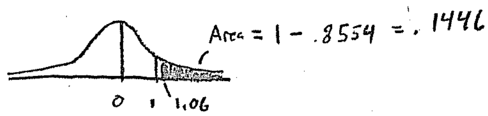
1. Ch. 5, Sec. 2, Ex. 2 (pg. 263)

a) From the standard normal cumulative probability table (Table B-3)

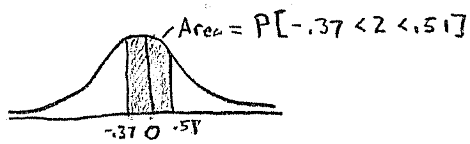
$P(Z \le -.62) = .2676$



b) $P[Z > 1.06] = 1 - P[Z \le 1.06] = 1 - .8554 = \boxed{.1446}$

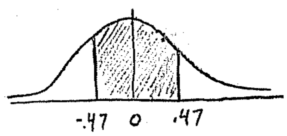


c) Thinking about the picture



This is just $P[Z < .51] - P[Z \le -.37] = .6950 - .3557 = \boxed{.3393}$

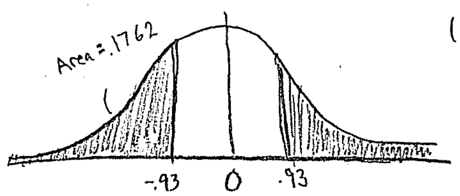
d) The sketch helps in this case: for $|Z| \le .47$, we have



(For instance $|-.4| \le .47$ is true, $|1.2| \le .47$ is NOT true)

So, we can find $P(Z \le .47) - P(Z \le -.47) = .6808 - .3192 = \boxed{.3616}$

e) Again, sketching the region helps:

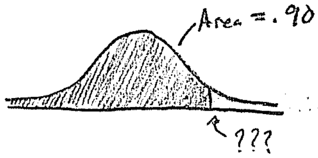


(We know which area to shade by picking test values, for instance $|-1.2| > .93$ is true, $|0.5| > .93$ is NOT.)

$P(Z \le -.93) = .1762$, and since the curve is symmetric, $P(Z > .93) = .1762$ also!

So $P(|Z| > .93) = .1762 + .1762 = \boxed{0.3524}$

g) Let's draw a picture: We know the area must be .90

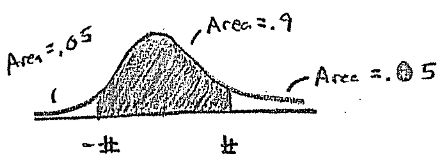


but we don't know what number does this – looking at the table though we see

$P(Z \le 1.28) = .8997$ and $P(Z \le 1.29) = .9015$

So if $P(Z \le \#) = .9$, # must be between 1.28 and 1.29 – so we can approximate $\boxed{\# = 1.285}$

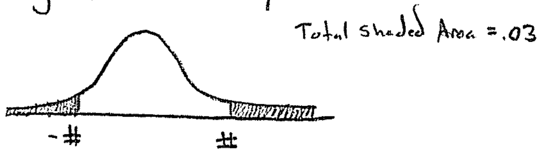
h) Same idea: We draw the picture, shading over values where $|Z| < \#$ would hold.



Since the shaded part has an area of .9, that leaves an area of .1 for both tails – and since the graph is symmetric, each tail has an area of .05

This means $P(Z \le \#) = .95$ – from table B-3, $P(Z \le 1.64) = .9495$ and $P(Z \le 1.65) = .9505$. So we can approximate $\boxed{\# = 1.645}$

i) Again, we draw a picture



Since the curve is symmetric, $P(Z < -\#) = .015$ and $P(Z > \#) = .015$

From the table, we find $P(Z \le -2.17) = .0150$

So $\boxed{\# = 2.17}$

2.) <u>Exercise 5.2.3</u>

The trick for each of these problems is to "convert" $X \sim N(43, (3.6)^2)$ to $Z \sim N(0,1)$ and then use the table B-3.

a) For $X \sim N(\mu, \sigma^2)$ $Z = \frac{X-\mu}{\sigma}$ follows a standard normal. This means

$$P(X \le 45.2) = P\left(\frac{X-43}{3.6} \le \frac{45.2-43}{3.6}\right) = P\left(Z \le \frac{2.2}{3.6}\right) = P(Z \le .611)$$

From B-3, $P(Z \le .61) = .7291$, so $\boxed{P(X \le 45.2) \approx .7291}$

b) $P(X \le 41.7) = P\left(\frac{X-43}{3.6} \le \frac{41.7-43}{3.6}\right) = P\left(Z \le \frac{-1.3}{3.6}\right) = P(Z \le .3611)$

From B-3, $P(Z \le -.36) = .3594$

So $P(X \le 41.7) = .3594$

c) $P(43.8 < X \le 47.0) = P\left(\frac{43.8-43}{3.6} < \frac{X-43}{3.6} \le \frac{47.0-43}{3.6}\right)$

$$= P(.8/36 < Z \le 4.0/3.6)$$
$$= P(.22\bar{2} < Z \le 1.1\bar{1})$$
$$\approx P(.22 < Z \le 1.11)$$
$$= P(Z \le 1.11) - P(Z \le 0.22)$$
$$= .8665 - .5871$$
$$= .2794$$

d) $|X - 43.0|$ is already in the form $X-\mu$ - so $\left|\frac{X-\mu}{\sigma}\right|$ is $|Z|$

$$P(|X-43.0| \le 2.6) = P\left(\left|\frac{X-43.0}{3.6}\right| \le \frac{2.0}{3.6}\right) = P(|Z| \le .55\bar{5})$$

As we saw in 1), $P(|Z| < .55\bar{5}) = P(-.55\bar{5} \le Z \le .55\bar{5})$

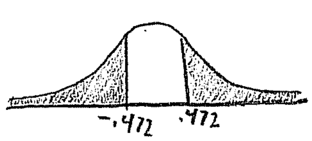
$$= P(Z \le .55\bar{5}) - P(Z \le -.55\bar{5})$$
$$\approx P(Z \le .55) - P(Z \le -.55)$$
$$= .7088 - .2912$$
$$= .4176$$

So $P(|X-43.0| \le 2.6) = .4176$

e) $P(|X-43.0| > 1.7) = P\left(\left|\frac{X-43.0}{3.6}\right| > \frac{1.7}{3.6}\right) = P(|Z| > 0.47\bar{2})$

Sketch

$$P(Z < -.47) + P(Z > .47) = .3192 + .3192$$
$$= .6384$$



So $P(|X-43.0| > 1.7) = .6384$

f) $P(X < \#) = .95$

$$P\left(\frac{X-43.0}{3.6} < \frac{\#-43.0}{3.6}\right) = .95 \Rightarrow P\left(Z \le \frac{\#-43.0}{3.6}\right) = .95$$

From B-3, we can say $P(Z \le 1.645) \approx .95$

So $\frac{\#-43.0}{3.6} = 1.645 \Rightarrow \# = 1.645(3.6) + 43.0$

$= 48.922$

So $P(X < 48.922) = .95$ and $\boxed{\# = 48.922}$

g) $P[X \ge \#] = P\left[\frac{X-43.0}{3.6} \ge \frac{\#-43.0}{3.6}\right] = P\left[Z \ge \frac{\#-43.0}{3.6}\right] = .30$

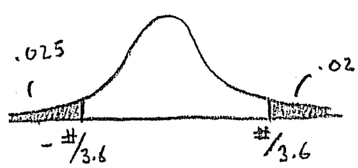
$P\left[Z < \frac{\#-43.0}{3.6}\right] = 1 - .30 = .7$

From Table B-3, $P[Z < .525] \approx .7$, so $\frac{\#-43.0}{3.6} = .525$

So $\# = .525(3.6) + 43.0 = 44.89$

So $P(X \ge 44.89) \approx .3$ and $\boxed{\# = 44.89}$

h) $P[|X-43.0| > \#] = P\left[\left|\frac{X-43.0}{3.6}\right| > \#/3.6\right] = P[|Z| > \#/3.6] = .05$



From table B-3,
$P(Z < -1.96) = .025$, $P(Z > 1.96) = .025$

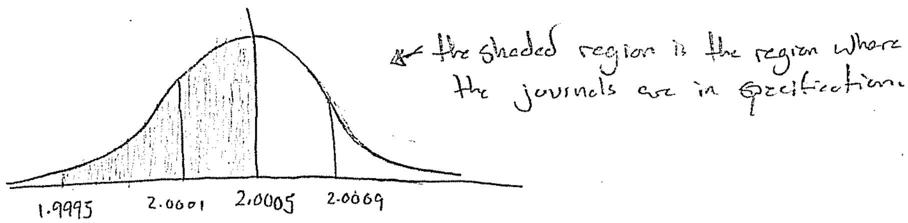
So $\frac{\#}{3.6} = 1.96 \Rightarrow \# = (1.96) \cdot 3.6 = 7.056$

Thus $P[|X-43.0| > 7.056] = .05$ and $\boxed{\# = 7.056}$

3. 5.2.4

Let X be the diameter of the bearing journals.

We can sketch the distribution of X:



a) In other words, what is $P[1.9995 \le X \le 2.0005]$

We can convert this to standard normal, $P\left[\frac{1.9995-2.0005}{.0004} \le \frac{X-2.0005}{.0004} \le \frac{2.0005-2.0005}{.0004}\right]$ ← Z

and use the table:

$$P\left[\frac{-.0010}{.0004} \le Z \le \frac{0.0000}{0.0004}\right] = P[-2.5 \le Z \le 0]$$
$$= P[Z \le 0] - P[Z \le -2.5]$$
$$= 0.5 - .0062$$
$$= .4938$$

So 49.38% of bearing journals are with the specs.

b) If we could shift the distribution curve so that the thicker part was more in the shaded area we would have a higher probability of the bearing journals between 1.9995 and 2.0005.

To do this, we might want to adjust the mean - we could adjust it so our process had a mean of 2.000 - in this case the fattest part of our curve is between (1.9995, 2.0005)

If we made that adjustment, so that $X \sim N(2.0000, .0004)$

$$P(1.9995 < X < 2.0005) = P\left(\frac{1.9995-2.0000}{.0004} < \frac{X-2.0000}{.0004} < \frac{2.0005-2.0000}{.0004}\right)$$
$$= P[-1.25 < Z < 1.25]$$
$$= P(Z < 1.25) - P(Z < -1.25)$$
$$= .8944 - .1056$$
$$= .7888$$

So moving the mean to 2.0000 means 78.88% of bearing journals will now be in specification.

c) In other words, we need to pick σ so that, for $X \sim N(2.0000, \sigma^2)$ makes $P(1.9995 < X < 2.0005) = .95$

$$P\left(\frac{1.9995 - 2.0000}{\sigma} < \frac{X - 2.0000}{\sigma} < \frac{2.0005 - 2.0000}{\sigma} \right) = .95$$

$$\Rightarrow P\left(\frac{-.0010}{\sigma} < Z < \frac{.0010}{\sigma} \right) = .95$$

From Table B-3, we have $P(-1.96 < Z < 1.96) = .95$

So $1.96 = \frac{0.0010}{\sigma} \Rightarrow \boxed{\sigma \approx 0.00051}$ $\quad \left(\sigma = \frac{0.0010}{1.96} \text{, literally} \right)$

.975

4.)

a) $P[X > 1] = 1 - P[X \le 1]$

$= 1 - F(1)$

$= 1 - \left(\frac{1}{8} + \frac{3}{16}\right)$

$= 1 - \left(\frac{5}{16}\right)$

$= {}^{11}/_{16}$

"exceeds" means "more than"

So $P(X > 1) = {}^{11}/_{16}$

b) $P[X \le 0.5] = F(0.5)$

$= \frac{0.5}{8} + \frac{3}{16}(.5)^2$

$= \frac{1}{16} + \frac{3}{16} \cdot \frac{1}{4}$

$= \frac{4}{64} + \frac{3}{64}$

$= {}^{7}/_{64}$

So $P(X \le 0.5) = {}^{7}/_{64}$

c) $P(1.0 \le X \le 1.5) = P(X \le 1.5) - P(X \le 1.0)$

$= F(1.5) - F(1.0)$

$= \left(\frac{1}{8}(1.5) + \frac{3}{16}(1.5)^2\right) - \left(\frac{1}{8}(1) + \frac{3}{16}(1)^2\right)$

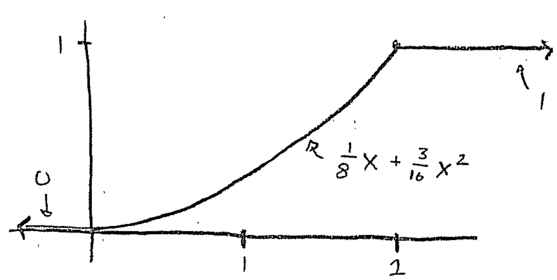
$= \left(\frac{3}{16} + \frac{3}{16} \cdot \frac{9}{4}\right) - \left(\frac{1}{8} + \frac{3}{16}\right)$

$= \left(\frac{12}{64} + \frac{27}{64}\right) - \left(\frac{8}{64} + \frac{12}{64}\right)$

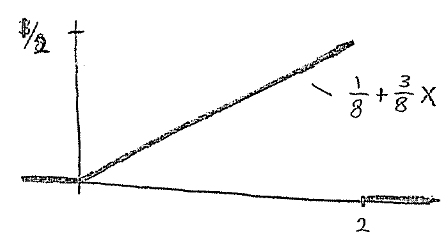
$= \frac{39}{64} - \frac{20}{64}$

$= {}^{19}/_{64}$

d)



f)



e) The derivative of a CDF is the PDF: $\frac{d}{dx}F(x) = f(x)$.

- For $X \ge 2$, $\frac{d}{dx}F(x) = 0$ since $F(x)$ is constant
- For $X < 0$, $\frac{d}{dx}F(x) = 0$, since $F(x)$ is constant.
- For $0 \le X \le 2$

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{8}X + \frac{3}{16}X^2\right) = \frac{1}{8} + \frac{6}{16}X = \frac{1}{8} + \frac{3}{8}X$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}X & 0 \le X \le 2 \\ 0 & \text{otherwise} \end{cases}$$

5.)

For any values of μ and σ^2.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

So

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{1}{\sigma^2}x^2 - \frac{2\mu}{\sigma^2}x + \frac{\mu^2}{\sigma^2}\right)} dx = 1$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}} dx = 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} \cdot e^{-\frac{\mu}{2\sigma^2}} dx = 1 \qquad (\text{Since } e^{a-b} = e^{a} \cdot e^{-b})$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mu}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x} dx = \sqrt{2\pi\sigma^2}\, e^{\mu/2\sigma^2}$$

So in the case of

$$\int_{-\infty}^{\infty} e^{3x-x^2} dx, \quad -\frac{1}{2\sigma^2} = -1 \text{ and } \frac{\mu}{\sigma^2} = 3 \Rightarrow \sigma^2 = \frac{1}{2}$$

$$\mu = 3\sigma^2 = 3/2$$

$$\text{So} \int_{-\infty}^{\infty} e^{3x-x^2} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2(\frac{1}{2})}x^2 + \left(\frac{3/2}{1/2}\right)x} dx = \sqrt{2\pi\left(\tfrac{1}{2}\right)}\, e^{\frac{3/2}{2(1/2)}}$$

$$= \sqrt{\pi}\, e^{3/2}$$

Thus

$$\boxed{\int_{-\infty}^{\infty} e^{3x-x^2} dx = \sqrt{\pi}\, e^{3/2}}$$