Announcements

- Exam I is next Tuesday, October 3rd
- Solutions are up on the course page

STAT 430: Lecture 12

- Practice Exam is in progress.

Functions of a Random Variable

Note: Check slides.

Course page: imouzon.github.io/stat430

What is a Function of a Random Variable?

A Function of Functions?

A Function of Variables?

Basics

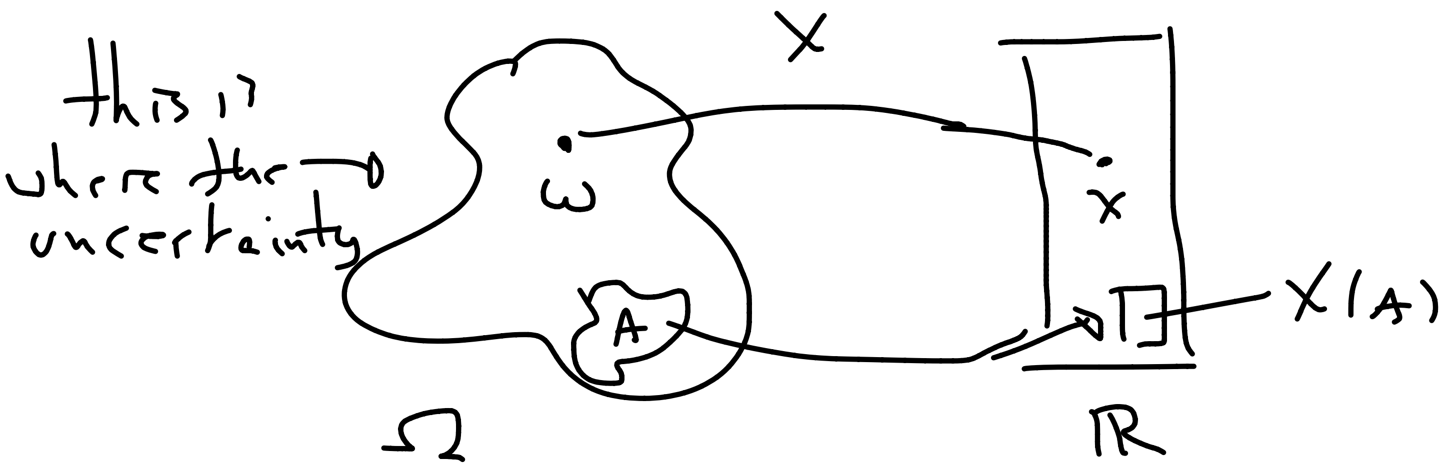
Random Variables Are Already Functions

Random Variables

When we talk about random variables, we are talking about a function that takes observations from the samples space, Ω, to the reals, $\mathbb{R}$.

$$X : \Omega \to \mathbb{R}$$

illustration:



Basics

Random Variables

Functions of Random Variables

So when we define a function that takes values of the random variable to some other real value, the ultimate result is a composite function that takes values from Ω to the reals:

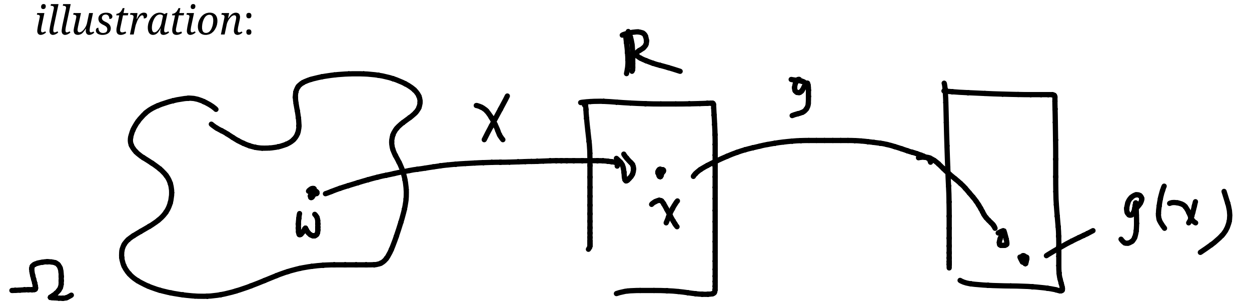
$$X : \Omega \to \mathbb{R}$$

$$g : \mathbb{R} \to \mathbb{R}$$

which means

$$g \circ X : \Omega \to \mathbb{R}$$

illustration:



Basics

Random Variables

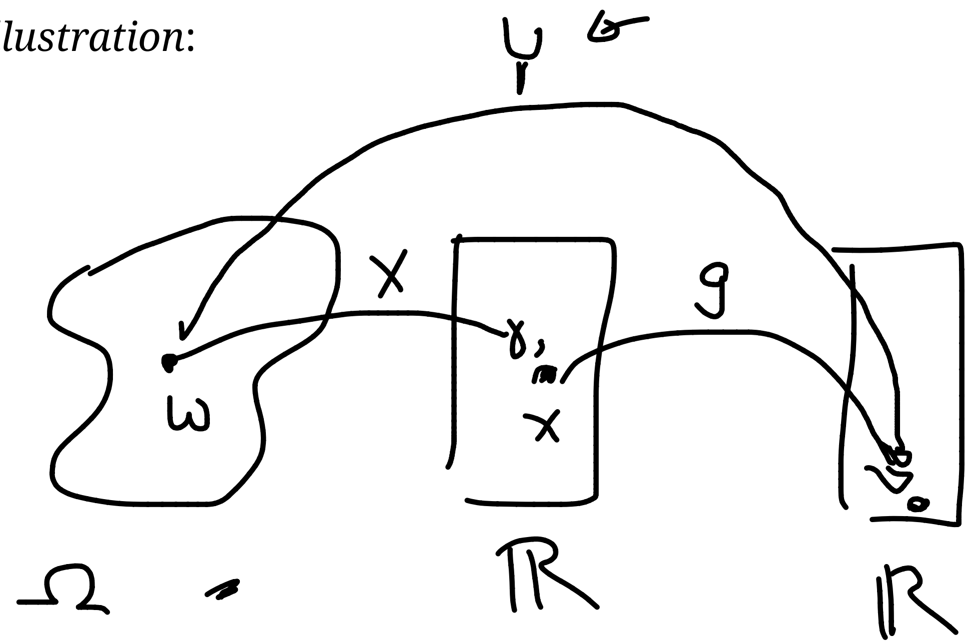
Functions of Random Variables

This means that $g \circ X$ (aka, $g(X)$) is also a random variable

So "functions of random variables are random variables" - essentially, by taking a function of a random variable we can define *new* random variables.

For instance if we say X is a random variable, $g(X)$ is some nice function, and $\boxed{Y = (g \cdot X)(\omega)}$ then we can write $Y : \Omega \to \mathbb{R}$.

illustration:



New Random Variables With New Distributions

Example 1

If $\underline{U} \sim U(0, 1)$ and

$$h(U) = \mathbb{I}_{\{u \in \mathbb{R} : u > 0.2\}}(U)$$

$\mathbb{I}_{(\{u>0.2\})}(u)$

$\mathbb{I}_{(u>0.2)}(u)$

if $u = 0.1$ then $h(u) = 0$

where

$$\mathbb{I}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

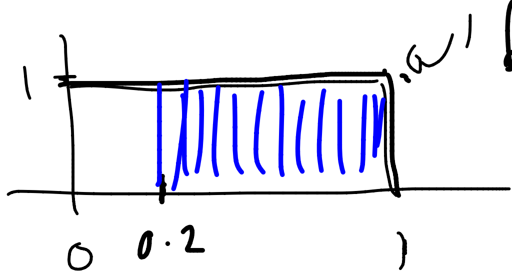
1 if TRUE

0 if False

is an indicator function

then $h(U)$ will be a Bernoulli random variable with $p = 0.8$.

pdf of a $U(0,1)$ random var.



w/ $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$P(h(u) = 1) = P(u > 0.2) = \int_{0.2}^{1} 1 \, du = x \Big|_{0.2}^{1} = 0.8$$

$$p(h(u) = 0) = P(u \le 0.2) = 0.2$$

$$\begin{cases} h(u) = 1 \text{ with probability } 0.8 \\ \quad\;\; = 0 \text{ with probability } 0.2 \end{cases}$$

$$h(u) \sim \text{Bernoulli}(0.8)$$

Finding the Distribution of the (g(X))

So if X is random variable then $Y = g(X)$ is a random variable. But that means we might want to think about the distribution of Y. How can we figure that out?

Let's start by thinking about the **cumulative distribution function** of X, $F_X(x) = P(X \leq x)$ and with a random variable Y created by the simple function $g(X) = a + b \cdot X$.

Start: Find $P(Y \leq y)$ then we have the CDF of Y – that means we know its distribution

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$
$$= P(a + bX \leq y)$$
$$= P\left(X \leq \frac{y-a}{b}\right)$$
$$= F_X\left(\frac{y-a}{b}\right)$$

Additionally, since $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ then $\frac{d}{dx} F_X(x) = f_X(x)$

then $\frac{d}{dy} F_Y(y) = f_Y(y) \Rightarrow \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-a}{b}\right)$

$$= f_X\left(\frac{y-a}{b}\right)\left|\frac{1}{b}\right|$$

Finding the Distribution of the (g(X))

Since $Y = g(X)$ is a random variable, it will also have a cdf which we might call $F_Y(y) = P(Y \leq y)$. Then we can write:

$$
\begin{aligned}
\underbrace{F_Y(y)}_{\text{cdf of } Y} &= P(Y \leq y) \\
&= P(aX + b \leq y) \\
&= P\left(X \leq \frac{y-b}{a}\right) \\
&= F_X\left(\frac{y-b}{a}\right) \quad \text{cdf of } X
\end{aligned}
$$

and since $\underline{\frac{d}{dx}F_X(x)} = f_X(x)$ we can write

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy}F_Y(y) \\
&= \frac{d}{dy}F_X\left(\frac{y-b}{a}\right) \\
&= f_X\left(\frac{y-b}{a}\right)\frac{1}{a}
\end{aligned}
$$

we had take the derivative of "this inside part"

Finding the Distribution of the (g(X))

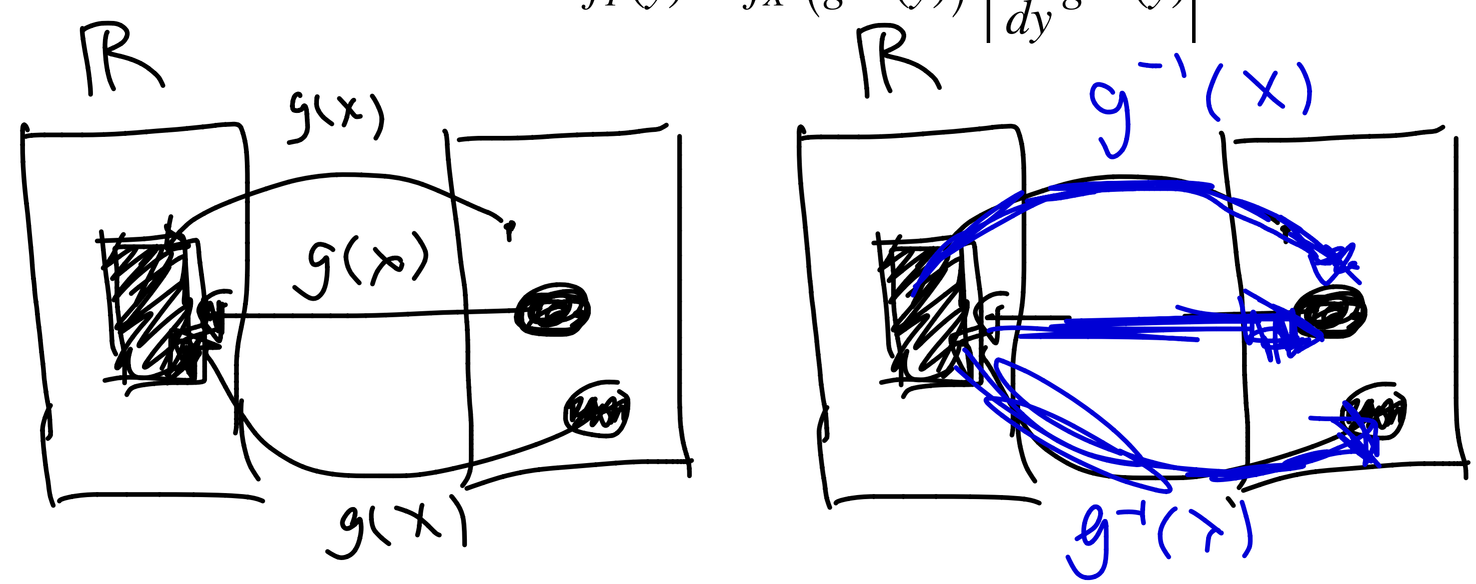
"strictly"

This process doesn't require much: in fact, the key step was taking the inverse of $Y = g(X)$. We can write this process generally as:

Proposition B

Let X be a continuous random variable and let with density $f_X(x)$ and let $Y = g(X)$ where g is differentiable and strictly monotonic wherever $f(x) > 0$. Then Y has the density

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$$



strictly increasing: if $x > y$ $f(x) > f(y)$

strictly decreasing: if $x > y$, $f(x) < f(y)$

strictly monotonic: f is either strictly inc or strictly dec

Basics

Random Variables

New Distributions

New Random Variables With New Distributions

Example 2

If $Z \sim N(0,1)$ (i.e., Z is a standard normal random variable) then for any a and any $b > 0$ we can say that $g(Z) = a + bX$ will be $N(a, b^2)$.

$Y = g(Z) = a + bZ$

$$F_Y(y) = P(Y \le y) = P(a + bZ \le y)$$
$$= P\left(Z \le \frac{y-a}{b}\right)$$
$$= F_Z\left(\frac{y-a}{b}\right)$$
$$= \Phi\left(\frac{y-a}{b}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_Z\left(\frac{y-a}{b}\right)$$
$$= f_Z\left(\frac{y-a}{b}\right)\frac{1}{b}$$
$$= \frac{1}{\sqrt{2\pi \cdot 1}} e^{-\frac{1}{2\cdot 1}\left(\left(\frac{y-a}{b}\right)-0\right)^2} \cdot \frac{1}{b}$$
$$= \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2}\cdot\frac{(y-a)^2}{b^2}}$$
$$= \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2b^2}(y-a)^2}$$

Last Step: $Z \in (-\infty, \infty) \Rightarrow a + bZ \in (-\infty, \infty)$
$\Rightarrow y \in (-\infty, \infty)$

the pdf of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{1}{2b^2}(y-a)^2}, \quad -\infty \le y \le \infty$$

Which is the pdf of a $N(a, b^2)$ R.V.

Therefore: Y is $N(a, b^2)$

Basics

Random Variables

New Distributions

possible values of Y?

$0 \le y \le 1$

New Random Variables With New Distributions

Example 3

Suppose that X is any random variable. What is the distribution of $Y = F_x(x)$?

in this case, the "g" is F_X,
the cumulative density function of X.

$$F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$$

$$F_Y(y) = P(Y \le y) = P(F_X(x) \le y) = y$$

Basics

Random Variables

New Distributions

New Random Variables With New Distributions

Example 4

If $X \sim \exp(\lambda)$ and c is any positive constant, then what is the distribution of $Y = c \cdot X$?

$$f_Y(y) = f_X(c \cdot x) \cdot |c|$$

$$= \lambda e^{-\lambda \cdot cy} \cdot |c|$$

$$= c \cdot \lambda \cdot e^{-c\lambda x}$$

$$Y \sim \exp(c\lambda)$$

Basics

Random Variables

New Distributions

New Random Variables With New Distributions

Example 5

If $Z \sim N(0, 1)$ (i.e., Z is a standard normal random variable) then what is the distribution of $Y = Z^2$?

Basics

New Random Variables With New Distributions

Random Variables

New Distributions

Example 6

If $X \sim \exp(\lambda)$ and c is any positive constant, then what is the distribution of $Y = c \cdot \ln(X)$?

after class additions

first: since $c > 0$ then $c \cdot \ln(x)$ is strictly increasing so we can use "Proposition B"

since $g(x) = c \cdot \ln(x) \Rightarrow y = c \cdot \ln(x)$

$\Rightarrow x = c \ln(y)$

$\Rightarrow e^{x/c} = y$

so $g^{-1}(x) = e^{x/c}$, $0 \le x \le y$

also, since $0 \le x < \infty$, $-\infty \le c \cdot \ln(x) \le \infty$