STAT 105, Fall 2015
Section B
Homework #8, Solutions

1. Chapter 5, Section at end of Chapter, Exercise 35

a) The inspector is going to pass over 5 flaws in the chip, and each flaw has a .8 probability of detection.

If we consider each flaw a trial, and detection of a flaw a success, it is reasonable to consider Y to have a binomial distribution with $n=5$ and $p=.80$

Given $X=5$, the probability that $Y=3$ can be found by using the binomial pdf with $p=.8$ and $n=5$:

$$P(Y=3/X=5) = \frac{5!}{(5-3)!\,3!}(.8)^3(1-.8)^{5-3} = \frac{5!}{2!\,3!}(.8)^3(.2)^2$$

$$= \frac{5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{(2\cdot 1)(\cancel{3\cdot 2\cdot 1})}(.512)(.04)$$

$$= .2048$$

b)
$$f_Y(0) = \sum_{\text{all } x} f(x,0) = \sum_{x=0}^{\infty} f(x,0) = \sum_{x=0}^{\infty} f_{Y|X}(0|x)\cdot f_X(x)$$

$$= \sum_{x=0}^{\infty}\left(\frac{x!}{(x-0)!\,0!}(.8)^0(.2)^x\right)\left(\frac{e^{-3}\cdot 3^x}{x!}\right)$$

$$= \sum_{x=0}^{\infty}\frac{e^{-3}\cdot(.6)^x}{x!}$$

$$= e^{-3}\cdot\sum_{x=0}^{\infty}\frac{(.6)^x}{x!}$$

$$= e^{-3}\cdot e^{.6}$$

$$= e^{-2.4}$$

c)
$$f_Y(y) = \sum_{\text{all } x} f(x,y) = \sum_{x\ge y} f(x,y) \quad \text{(since \# flaws} \ge \text{\# flaws detected)}$$

$$= \sum_{x\ge y}\left(\frac{x!}{(x-y)!\,y!}(.8)^y(1-.8)^{x-y}\right)\left(\frac{e^{-3}\cdot 3^x}{x!}\right) \quad \text{(since } f(x,y) = f_{X|Y}(y|x)\text{)}$$

$$= \left(\frac{(.8)^y(.2)^{-y}e^{-3}}{y!}\right)\cdot\sum_{x\ge y}\frac{(.2)^x\cdot 3^x}{(x-y)!} \quad \text{(since } y \text{ is same in each term)}$$

c), continued

So

$$f_Y(y) = \frac{4^y e^{-3}}{y!} \sum_{x \geq y} \frac{(.6)^x}{(x-y)!}$$

$$= \frac{4^y e^{-3}}{y!} \left(\frac{(.6)^y}{(y-y)!} + \frac{(.6)^{y+1}}{((y+1)-y)!} + \frac{(.6)^{y+2}}{((y+2)-y)!} + \dots \right) \quad \text{(by writing out the sum)}$$

$$= \frac{4^y e^{-3}}{y!} \left((.6)^y \left(\frac{1}{0!} + \frac{(.6)^1}{1!} + \frac{(.6)^2}{2!} + \dots \right) \right) \quad \text{(factoring out } (.6)^y\text{)}$$

$$= \frac{(2.4)^y e^{-3}}{y!} \left(\frac{(.6)^0}{0!} + \frac{(.6)^1}{1!} + \frac{(.6)^2}{2!} + \dots \right)$$

$$= \frac{(2.4)^y e^{-3}}{y!} \left(\sum_{x=0}^{\infty} \frac{(.6)^x}{x!} \right)$$

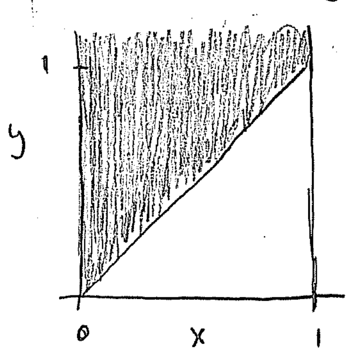
$$= \frac{(2.4)^y e^{-3}}{y!} \left(e^{.6} \right)$$

$$= \frac{(2.4)^y e^{-2.4}}{y!}$$

The marginal distribution for Y is a poisson with mean $\lambda = 2.4$.

2. Chapter 5, Exercise 37

a) Consider the following region:



The shaded area represents where $f(x,y) = e^{x-y} > 0$, the points for which $0 \leq x \leq 1$ and $x \leq y$

If $y = .3$, then $f(x,y)$ is only positive for $x \leq .3$.

However, if $y \geq 1$, $f(x,y) > 0$ for any $0 \leq x \leq 1$.

So:

$$P(Y \leq 1.5) = P(0 \leq Y < 1) + P(1 \leq Y \leq 1.5)$$

$$= \int_0^1 \int_0^y e^{x-y}\,dx\,dy + \int_1^{1.5} \int_0^1 e^{x-y}\,dx\,dy$$

$$= \int_0^1 e^{-y} \left(e^x \Big|_0^y \right) dy + \int_1^{1.5} e^{-y} \left[e^x \Big|_0^1 \right] dy$$

a), continued

$$= \int_0^1 e^{-y}(e^{y} - e^{0})\,dy + \int_1^{1.5} e^{-y}(e^{1} - e^{0})\,dy$$

$$= \int_0^1 1 - e^{-y}\,dy + \int_1^{1.5} e^{1-y} - e^{-y}\,dy$$

$$= (y + e^{-y})\Big|_0^1 + (-e^{1-y} + e^{-y})\Big|_1^{1.5}$$

$$= (1 + e^{-1}) - (0 + e^{-0}) + (-e^{-.5} + e^{-1.5}) - (-e^{0} + e^{-1})$$

$$= \cancel{1} + \cancel{e^{-1}} - \cancel{1} - e^{-.5} + e^{-1.5} + 1 - \cancel{e^{-1}}$$

$$= 1 - e^{-.5} + e^{-1.5}$$

b)

If $y \le 0$, $f(x,y) = 0$ and so $f_Y(y) = 0$.

If $0 < y < 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y e^{x-y}\,dy = e^{-y}\left[e^{x}\Big|_0^y\right] = 1 - e^{-y}$$

If $y \ge 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 e^{x-y}\,dy = e^{-y}\left[e^{x}\Big|_0^1\right] = e^{1-y} - e^{-y}$$

So

$$f_Y(y) = \begin{cases} 0 & y \le 0 \\ 1 - e^{-y} & 0 < y \le 1 \\ e^{1-y} - e^{-y} & 1 < y \end{cases}$$

If $x < 0$ or $x > 1$, $f(x,y) = 0$. So $f_X(x) = 0$ for $x > 1$ or $x < 0$

if $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{\infty} e^{x-y}\,dy = e^{x}\left[-e^{-y}\Big|_x^{\infty}\right] = e^{x}(-0 + e^{-x}) = 1$$

So

$$f_X(x) = \begin{cases} 1 & 0 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

c) They are not independent: $f(.2, .1) = 0$ but $f_X(.2) \cdot f_Y(.1) = 1 \cdot (1 - e^{-.2}) \neq 0$

So $f(x,y) \neq f_X(x) \cdot f_Y(y)$ for all pairs (x,y).

So they are not independent.

d) $f_{Y|X}(y|.25) = \dfrac{f(.25, y)}{f_X(.25)}$

$= \dfrac{e^{.25-y}}{1}$ if $y \geq .25$ and 0 otherwise

$= e^{.25-y}$ if $y \geq .25$ and 0 otherwise

$$f_{Y|X}(y|.25) = \begin{cases} e^{.25-y} & y \geq .25 \\ 0 & \text{otherwise.} \end{cases}$$

The general definition of expected value says $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$.

In this case, we are using the density function $f_{Y|X}(y|.25)$. This gives

$$E(Y|X=.25) = \int_{.25}^{\infty} y \cdot e^{.25-y} dy \qquad \left(\begin{array}{l} u = y \\ dv = e^{.25-y} \end{array} \Rightarrow \begin{array}{l} du = dy \\ v = -e^{.25-y} \end{array} \right.$$

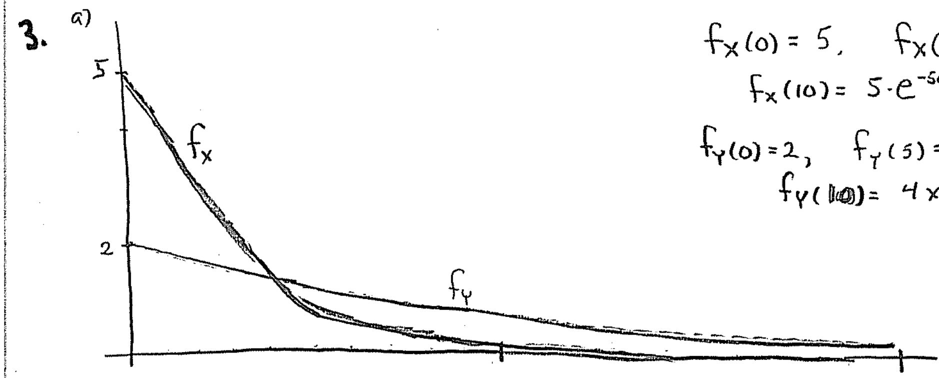
$$= \left[-y e^{.25-y} \Big|_{.25}^{\infty} \right] + \int_{.25}^{\infty} e^{.25-y} dy \qquad \left(\text{since } \lim_{y \to \infty} y e^{.25-y} = 0 \right)$$

$$= .25 + \left(-e^{.25-y} \Big|_{.25}^{\infty} \right)$$

$$= .25 + 1$$

$$= 1.25$$

3. a)



$f_X(0) = 5$, $f_X(5) = 5 \cdot e^{-25} \approx 6 \cdot \times 10^{-11}$

$f_X(10) = 5 \cdot e^{-50} \approx 0$

$f_Y(0) = 2$, $f_Y(5) = 2 \cdot e^{-10} \approx 9 \times 10^{-5}$

$f_Y(10) = 4 \times 10^{-9}$

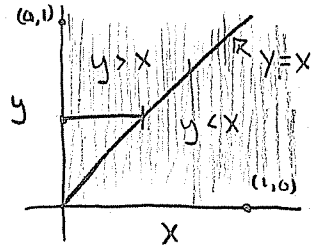
b) $P(X \geq 3) = \int_3^{\infty} 5e^{-5x} dx = -e^{-5x} \Big|_3^{\infty} = e^{-15}$

c) $P(Y \geq 3) = \int_3^{\infty} 2e^{-2y} dy = -e^{-2y} \Big|_3^{\infty} = e^{-9}$

d) $f(x, y) = f_X(x) \cdot f_Y(y) = 5e^{-5x} \cdot 2e^{-2y} = 10 e^{-5x-2y}$ if $x > 0$, $y > 0$

$= 0$ if otherwise.

e)



$$P(U=1) = P(Y>X) = \int_0^\infty \int_0^y 10e^{-5x-2y}\,dx\,dy$$

$$= \int_0^\infty 10e^{-2y}\left(-\tfrac{1}{5}e^{-5x}\Big|_0^y\right)dy$$

$$= \int_0^\infty -2e^{-2y}\left(e^{-5y}-1\right)dy$$

$$= \int_0^\infty -2e^{-7y} + 2e^{-2y}\,dy$$

$$= \left(\tfrac{2}{7}e^{-7y} - e^{-2y}\right)\Big|_0^\infty$$

$$= -\tfrac{2}{7} + 1$$

$$= \boxed{\tfrac{5}{7}}$$

4. Since Z_i are standard normal, $E(Z_i)=0$ and $Var(Z_i)=1$.

a)

$$E(X) = 3\cdot E(Z_i) + 5 = 3\cdot 0 + 5 = 5$$

$$Var(X) = (3)^2\, Var(Z_i) = 3^2\cdot 1 = 9$$

b)

$$Y = Z_1 - Z_2 = 1\cdot Z_1 + (-1)\cdot Z_2$$

$$E(Y) = 1\cdot E(Z_1) + (-1)\cdot E(Z_2) = 1\cdot 0 + (-1)\cdot 0 = 0$$

$$Var(Y) = (1)^2\, Var(Z_1) + (-1)^2\, Var(Z_2) = 1\cdot 1 + 1\cdot 1 = 2$$

c) $U = Z_1 - Z_1$

Notice: no matter what value Z_1 takes, $Z_1 - Z_1 = 0$.
This is the same as saying that $U = 0$, a constant.

$$E(U) = E(0) = 0$$

$$Var(U) = Var(0) = 0.$$

d)

$$W = \sum_{i=1}^n \frac{i}{n}\left(Z_i + \frac{i}{n}\right) = \sum_{i=1}^n \frac{i}{n}Z_i + \frac{i^2}{n^2}$$

$$E(W) = \sum_{i=1}^n \frac{i}{n}E(Z_i) + \frac{i^2}{n^2} = \sum_{i=1}^n \frac{i}{n}\cdot 0 + \frac{i^2}{n^2} = \sum_{i=1}^n \frac{i^2}{n^2} = \frac{1}{n^2}\sum_{i=1}^n i^2 = \frac{1}{n^2}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6\cdot n}$$

$$Var(W) = \sum_{i=1}^n \left(\frac{i}{n}\right)^2 Var(Z_i) = \sum_{i=1}^n \frac{i^2}{n^2}\cdot 1 = \sum_{i=1}^n \frac{i^2}{n^2} = \frac{(n+1)(2n+1)}{6\cdot n}$$