Expected payment in standard auctions

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Question

Derive the expected payment in the following two standard auctions[1] under the assumption of Independent Private Valuation (IPV):

1. First price auction
2. Second price auction

[1] Standard auctions are simply auctions where the highest bidder gets the item.

Answer

Expected payment is the product of two things:

1. How much I have to pay in the event I win
2. The probability I win

   The details of the auction design is secondary.

First price auction

1. Derive the optimal bidding strategy yourself. I will directly state the answer here[2]:

$$\beta^I(\theta) = E\left[\max\left(Y_1, r\right)|Y_1 \le \theta\right]$$

2. The probability of winning is simply: $[F(\theta)]^{(N-1)}$ where F is the CDF of everyone's valuation. To ease notation, the literature often denote $G(.) \equiv [F(.)]^{(N-1)}$.
3. So we have $M^I(\theta) = G(\theta) \times E\left[\max\left(Y_1, r\right)|Y_1 \le \theta\right]$
4. Notice $E\left[\max\left(Y_1, r\right)|Y_1 \le \theta\right] = \frac{1}{G(\theta)} \int_0^\theta max(y,r)g(y)dy$, which can be simplified much further when breaking the limits of integration into two intervals and evaluate max(.) in each of them[3]. Ultimately, you will get:

$$E\left[\max\left(Y_1, r\right)|Y_1 \le \theta\right] = \frac{1}{G(\theta)}\left[rG(r) + \int_r^\theta yg(y)dy\right]$$

5. The expected payment is therefore[4]:

$$M^I(\theta) = G(\theta) \times \frac{1}{G(\theta)}\left[rG(r) + \int_r^\theta yg(y)dy\right] = rG(r) + \int_r^\theta yg(y)dy$$

[2] Y_1 is the second highest order statistic which is essentially the second highest valuation while acknowledging that it is a random variable. r is the reserve price, if any.

[3] $\frac{1}{G(\theta)}\left[\int_0^r max(y,r)g(y)dy + \int_r^\theta max(y,r)g(y)dy\right]$

[4] The second equality comes from Integration By Parts.

Second price auction

1. Argue that bidding one's true valuation is a weakly dominant strategy.
2. Thus in a symmetric equilibrium, everyone's truth telling and your winning probability is unsurprisingly $G(\theta)$.
3. And in the event you win, you pay the second highest bid which is the second highest valuation[5] in the equilibrium:

$$E\left[\max\left(Y_1, r\right) | Y_1 \leq \theta\right]$$

4. Therefore we have again:

$$M^{II}(\theta) = \left[rG(r) + \int_r^\theta yg(y)dy\right]$$

[5] Remember when the individual submits a bid, he/she only knows his/her valuation and not others. Thus he/she has to guess what the second highest valuation might be based on the only piece of information he/she has, which is the CDF.

Remark on the second price auction

In the second price setting, there is an alternative interpretation:

1. When the $(n-1)$ highest order statistic is **smaller** than r, then the "second highest" price in the model is effectively the reserve price (Assuming of course θ is as high as r), and that's the price you have to pay when win.
2. But if the $(n-1)$ highest order statistic is **higher** than r, then it's just business as usual, i.e. $(n-1)$ highest order statistic is the effective second highest price.