When WARP guarantees rationality

Bloody Micro! by Impatient Researcher

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Question

Show that if observations of what an agent chooses satisfy WARP, and the budget set $\mathscr{B}$ includes all subsets of X up to 3 elements, then WARP guarantees rationality[1].

[1] It should be noted that rationality guarantees WARP but WARP **may not** imply rationality.

Refresher

Before diving into the formal definition, the idea is actually extremely simple. We want to see if the choice an individual makes is the same as what an individual with rational preferences would make.

Rationality: Given a choice structure $(\mathscr{B}, C(.))$, if $C(B) = C^*(B, \succcurlyeq), \forall B \in \mathscr{B}$, then the preference relation $\succcurlyeq$ rationalises $C(.)$ relative to $\mathscr{B}$.

Answer

1. Define the preference relation $\succcurlyeq$ as this[2]:

$$x \in C(x, y) \iff x \succcurlyeq y$$

[2] This is simply specifying the preference relation $\succcurlyeq$ (unobserved) with something we do observe $C(.)$. We do not assume $\succcurlyeq$ to be rational here.

2. WLOG[3], suppose $x \succcurlyeq y \succcurlyeq z$

[3] This is quite standard and it is innocuous in the sense that x, y, z are simply placeholders and are not assigned to any specific choice.

3. Then we know:
   (a) $x \in C(x, y) \because x \succcurlyeq y$
   (b) $y \in C(y, z) \because y \succcurlyeq z$

4. Given WARP, is our $\succcurlyeq$ defined in this fashion rational?
   (a) Completeness: yes because $C(.)$ is a choice function and by definition, it's never empty (you have got to choose something) and therefore $\succcurlyeq$ is complete.
   (b) Transitivity? Need more work to show.

5. To show transitivity, suppose $z \in C(x, y, z)$. Then $\because$ WARP[4]:
   (a) $y \in C(y, z)$ and $z \in C(x, y, z) \implies y \in C(x, y, z) \because$ Condition β
   (b) $x \in C(x, y)$ and $y \in C(x, y, z) \implies x \in C(x, y, z) \because$ Condition β
   (c) $x \in C(x, y, z) \implies x \in C(x, z) \because$ Condition α

[4] WARP $\iff$ Condition α and β

6. Then we have shown transitivity as:

$$x \succsim y \text{ and } y \succsim z \implies x \succsim z$$

7. We then need to show that $C(.) = C^*(., \succsim)$. To prove equivalence, we have to prove both ways:

   (a) $x \in C(B, \succsim) \implies x \in C(B)\ \forall B \in \mathscr{B}$

      i. For a given $B \in \mathscr{B}$
      ii. By the definition of $x \in C(B, \succsim)$, we know $x \succsim y,\ \forall y \in B$
      iii. As $C(.)$ is never empty[5], WLOG let y^* be some item such that $y^* \in C(B)$. And of course $y^* \in B$.
      iv. As $x \succsim y,\ \forall y \in B$, then we know that $x \succsim y^* \implies x \in C(\{x, y^*\})$
      v. By Condition β, we have $x \in C(\{x, y^*\})$ and $y^* \in C(B) \implies x \in C(B)$. We have come a full circle. $\square$

   (b) $x \in C(B) \implies x \in C(B, \succsim)\ \forall B \in \mathscr{B}$

      i. For a given $B \in \mathscr{B}$
      ii. $x \in C(B) \implies x \in C(x, z),\ \forall z \in B,\ \because$ Condition α
      iii. Given how our $\succsim$ is defined, then $x \in C(x, z),\ \forall z \in B \implies x \succsim z,\ \forall z \in B$
      iv. Which then of course implies that $x \in C(B, \succsim)\ \forall B \in \mathscr{B}$. $\square$

[5] Again it is a choice function, you have to got to choose something. It can be more than one item though.