Proposer optimal DA matching

Bloody Micro! by Impatient Researcher

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Question

Consider a two-sided matching problem. To make it concrete, let's consider a marriage problem where there is a group of male proposing to a group of female.

Show that:

1. Male-proposing DA results in a male-optimal outcome
2. Male-proposing DA results in a female-pessimal outcome

Answer

Part 1

Our strategy is this - we first define what an optimality means for the male group and then we prove this by contradiction.

If a (stable) matching is male-optimal, then by definition there **cannot** exist another (stable) matching where a man is matched to another woman for whom he strictly prefers. This is a bit mouthful so let's summarise this criterion of optimality into: "no man is rejected by an achievable woman".[1]

[1] Succinctly: w is achievable for some m if there $\exists$ a stable μ such that $\mu(m) = w$

This notion of optimality is important in terms of how we structure the contradiction statement.

1. Suppose not - the matching is not male-optimal, then by definition, there must exist at least one instance where a man is rejected by an achievable woman.
2. Consider the first man being rejected. Denote this man as m and the woman rejecting him as w.[2]
3. At this very step, w rejects m, she must be keeping some other man, call him m'.[3]
4. But supposition, this is the first step of the DA where a man is rejected by an achievable woman, then it must be the case that m' has not been rejected by any woman. Also, m' must first propose to the one he prefers the most[4] - this implies that $w \succ_{m'} \mu'(m')$.

[2] Our definition of achievable, again, is that there exists a stable matching μ' where $\mu'(m) = w$

[3] Note this man may not be the ultimate match. And also as $\mu'(m) = w$ by supposition, $\mu'(m') \neq w$ as we cannot have w matched to both m and m' simultaneously. Convention has it that we have $\mu'(m') = w'$

[4] This is how male-proposing DA works.

5. Then by the fact that w rejects m for m', it must be the case that $m' \succ_w \mu'(w)$. Thus m' and w form a blocking pair, implying that $\mu'(.)$ is not stable. $\square$

Part 2

1. Suppose not - the matching is not female-pessimal, then there must exist a worse stable matching outcome, $\mu'(.)$, for a at least one female.

2. Pick one unfortunate woman who by supposition can have an even worse match. Denote her as w, then we have $\mu'(w) = m'$ and $m \succ_w m'$.

3. We have shown previously that the male-proposing DA outcome is male-optimal, then we know for sure $w \succ_m w'$

4. Again under $\mu'(.)$, m and w can form a blocking pair, contradicting the claim that $\mu'(.)$ is stable. $\square$