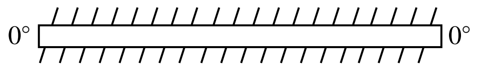
Chapter 2

Boundary Value Problems

2.1 BVP 1(A): Diffusion of Heat in a Thin Bar, Ends Maintained at 0°

Thin rod of length L, ends maintained at 0°, initial temperature $f(x)$, no heat generation, or absorption.



$$\psi(0,t) = 0, \quad \psi(L,t) = 0, \quad \psi(x,0) = f(x)$$

Start with the Diffusion Equation:

$$\frac{1}{\alpha^2}\frac{\partial \psi}{\partial t} - \nabla^2 \psi = \frac{Q}{K} \qquad Q = 0 \textit{ bc. no heat generation or absorption}$$

* *"Thin" implies temperature is independent of y and z* $\left(\frac{\partial \psi}{\partial y} = 0\right)$.

As there is no heat generation, the diffusion equation becomes:

$$\frac{1}{\alpha^2}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}, \quad 0 < x < L,\ t > 0 \quad \longrightarrow \quad \textit{Heat equation with Dirichlet B.C.s (MATH 264)}$$

We have

B.C.s: $\psi(0,t) = 0, \quad \psi(L,t) = 0$

And

I.C.: $\psi(x,0) = f(x)$

Recall solving separable PDEs in MATH 264, look for solutions of the form:

$u(x,t) = T(t)X(x)$, *Plug into PDE:*

$\frac{1}{\alpha^2}\frac{dT}{dt} = T \cdot \frac{d^2X}{dx^2}$, *Move all X terms to one side, all T terms to other side.*

$\left.\frac{1}{\alpha^2 T} \cdot \frac{dT}{dt} = \frac{1}{X} \cdot \frac{d^2X}{dx^2} = h(x,t)\right\}$ *h = constant (λ) because it is independent of x and t.*

Thus, we must solve the 2 ODEs:

$$\frac{1}{\alpha^2 T}\frac{dT}{dt} = \lambda, \qquad \text{Trivially, } \frac{\partial}{\partial x}\Big[\tfrac{1}{\alpha^2 T}\tfrac{dT}{dt}\Big] = 0,$$

$$\frac{1}{X}\frac{d^2X}{dx} = \lambda, \qquad \text{Trivially, } \frac{\partial}{\partial t}\Big[\tfrac{1}{X}\tfrac{d^2X}{dx}\Big] = 0.$$

$\frac{d^2X}{dx^2} + X\lambda,\ X(0) = 0,\ X(L) = 0.$ *λ can be 0, +ve, or -ve. (MATH 264)*

Case 1: $\lambda = 0$:

$$\frac{d^2X}{dx^2} = 0 \quad \Rightarrow \quad \Longrightarrow X(x) = A + Bx \quad \therefore B = 0 \quad \therefore AL = 0 \quad \Rightarrow \quad A = 0 \quad \}\textit{Trivial solution}$$

X(0)=0 X(L)=0

Case 2: λ + ve, ie $\lambda = \mu^2$

$$\frac{d^2X}{dx^2} + \mu^2 X = 0$$

We can express this as <u>either</u> $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$ or $X(x) = K_1\cosh(\mu x) + K_2\sinh(\mu x)$

* *These are mathematically equivivalent and interchangeable. Here, we choose the hyperbolics since B.C. $X(0) = 0$ kills the cosh term (cleaner).*

$X(x) = K_1\cosh(\mu x) + K_2\sinh(\mu x)$. $X(0) = 0$ $\quad {*}\cosh(0) = 1$, ${*}\sinh(0) = 0$ $\quad \therefore K_1 = 0$

$X(L) = 0 \quad \Rightarrow \quad K_2\sinh(\mu L) = 0 \quad \Rightarrow \quad K_2 = 0$ } Trivial solution, sinh is never nonzero outside $x = 0$... but sin is.

Case 3: λ - ve, i.e., $\lambda = -\mu^2$

$$\frac{d^2X}{dx^2} = -\mu^2 X \quad \Rightarrow \quad X(x) = A\cos(\mu x) + B\sin(\mu x) \quad \therefore A = 0 \quad \therefore 0 = B\sin(\mu L) \quad B \neq 0$$

Want $\sin(\mu L) = 0$

$\therefore \mu L = n\pi, \quad n \in \pm1, \pm2, \ldots$

$$\mu = \frac{n\pi}{L}, \quad \lambda = -\mu^2$$

$\therefore \lambda = -\dfrac{n^2\pi^2}{L^2} \quad \Rightarrow \quad X_n(x) = B_n \sin\left(\dfrac{n\pi x}{L}\right), n \in \mathbb{N}(1, 2, 3, \ldots)$

Now we consider:

$$\frac{1}{\alpha^2}\frac{dT}{dt} = \lambda = -\frac{n^2\pi^2}{L^2} \text{ with } \lambda = \frac{-n^2\pi^2}{L^2}$$

$\therefore \dfrac{dT}{dt} = \underbrace{-\dfrac{\alpha^2 n^2\pi^2}{L^2}}_{\text{constant}}\}\text{recall, } y' = y \quad \Rightarrow \quad y = Ce^x$

$\therefore T_n(t) = C_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}+}$

Combine $\psi_n(x,t) = X_n(x)T_n(t)$:

$$\psi_n = D_n \sin\left(\frac{n\pi x}{L}\right) \underbrace{e^{-\frac{\alpha^2 n^2\pi^2}{L^2}t}}_{=1 \text{ when } t=0}, \ n \in \mathbb{N}. \quad \textit{We must now find } D_n \textit{ using I.C. } \psi(x,0) = f(x)$$

$$D_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Can't resolve this unless $f(x)$ is a multiple of $\sin\left(\frac{n\pi x}{L}\right)$, which is a very rare edge case.

So we write the general:

$$\underbrace{\left[\frac{1}{\alpha^2}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right]}_{\mathcal{L}}(\psi(x,t)) = 0 \quad \therefore \mathcal{L}\psi = 0.$$

We want the nullspace of this (linear) operator $\mathcal{L}$.

By superposition, if $\psi_1, \psi_2, \ldots$ are solutions, so is their sum:

$$\therefore \psi(x,t) = \sum_{n=1}^{N} \psi_n(x,t)$$

$$f(x) = \sum_{n=1}^{N} D_n \sin\left(\frac{n\pi x}{L}\right) = \psi(x,0).$$

A solution for this exists for all piecewise smooth functions ($f(x)$, $f'(x)$ piecewise continuous, may have jump discontinuities).

For $f(x) = e^x = \sum\limits_{n=0}^{N} \frac{x^n}{n!}$ requires $N \to \infty$, so we must do $N \to \infty$ for all piecewise smooth $f(x)$.

$$\therefore f(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \quad \textit{We integrate both (Fourier Sine Series):}$$

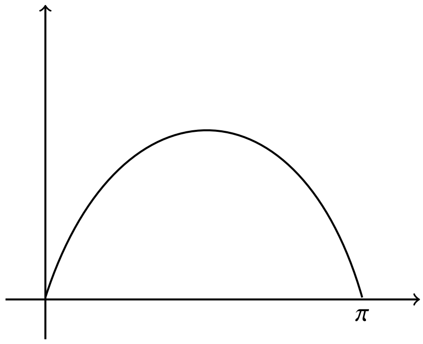
$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = \sum_{n=1}^{\infty} D_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

The RHS integral vanishes whenever $n \neq k$, and is equal to $\frac{L}{2}$ when $n = k$.

Interchanging n and k gives:

$$\boxed{D_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx, \text{ sub into } \psi(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}}$$

Example: $L = \pi,\ f(x) = x(\pi - x) = \psi(x,0)$



$$\psi(x,t) = \sum_{n=1}^{\infty} D_n \sin(nx) e^{-\alpha^2 n^2 t}$$

$$\psi(x,0) = X(\pi - x) = \frac{8}{\pi}\left[\frac{\sin(x)}{1^3} + \frac{\sin(3x)}{3^3} + \frac{\sin(5x)}{5^3} + \dots\right]$$

Now, incorporate $T(t)$:

$$\psi(x,t) = \frac{8}{\pi}\left[\frac{\sin(x)}{1^3}e^{-\alpha^2 t} + \frac{\sin(3x)}{3^3}e^{-9\alpha^2 t} + \frac{\sin(5x)}{5^3}e^{-25\alpha^2 t} + \dots\right] \quad (\textit{Converges rapidly})$$

2.2 BVP 1(B): Diffusion of Heat in a Thin Bar, Ends Insulated

$$\frac{1}{\alpha^2}\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2}, \qquad 0 < x < L, \quad t > 0$$

$$\left.\begin{array}{l} \text{B.C.s: } \psi_x(0,t) = 0, \quad \psi_x(L,t) = 0 \\ \text{I.C.: } \psi(x,0) = f(x) \end{array}\right\} \textit{Heat equation with Neumann B.Cs (while 1A is Dirichelet)}$$

Using the same procedure, we get:

$$\psi_n(x,t) = \underbrace{D_n \cos}_{\text{unlike sin for BVP 1A}}\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \qquad n = \underbrace{0}_{\text{start at zero because } \cos(0) \neq 0}, 1, 2, \dots$$

Once again, the I.C. $\psi(x,0)$ rarely gives a clean solution.

So we once again write the PDE as:

$$\left.\left[\frac{1}{\alpha^2}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right]\psi(x,t) = 0\right\} \quad \Rightarrow \quad \mathcal{L}\psi = 0 \qquad \textit{Want nullspace of } \mathcal{L}$$

$$\psi(x,t) = \sum_{n=0}^{\infty} \psi_n(x,t) = \sum_{n=0}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

We extract the $n = 0$ *term and deal with it separately (In B.V.P 1A irrelevant because* $\sin(0) = 0$*)*

$$\begin{aligned} \therefore \quad \psi(x,t) &= D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}\right) \quad \textit{Apply I.C.} \\ \psi(x,0) &= D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \quad \textit{Use Fourier Cosine series} \end{aligned}$$

$$\int_0^L \cos\left(\frac{k\pi x}{L}\right) f(x) dx = \underbrace{D_0 \int_0^L \cos\left(\frac{k\pi x}{L}\right) dx}_{0} + \sum_{n=1}^{N} D_n \underbrace{\int_0^L \cos\left(\frac{k\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx}_{\substack{\text{Like BVP 1A. this is } = 0 \\ \text{if } n \neq k \text{ and } = \frac{L}{2} \text{ if } n = k \\ \text{we use the Kronecker Delta} \\ \delta_{kn} = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}}} = \frac{L}{2} D_k,$$

interchange k and n.

$$\boxed{D_n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx} \tag{2.1}$$

In addition, for the $k = 0$ case:

$$\int_0^L (1) f(x) dx = \underbrace{D_0 \int_0^L (1) dx}_{L} + \sum_{n=1}^{\infty} D_n \underbrace{\int_0^1 (1) \cos\left(\frac{n\pi x}{L}\right) dx}_{0} = D_0 \cdot L \tag{2.2}$$

$$\therefore \quad \boxed{D_0 = \frac{1}{L} \int_0^L f(x)\, dx}$$

Sub in (2.1) and (2.2) into

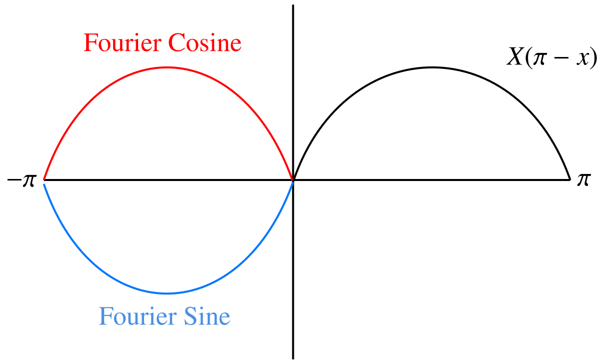
$$\boxed{\psi(x,t) = D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}}$$

for the solution.

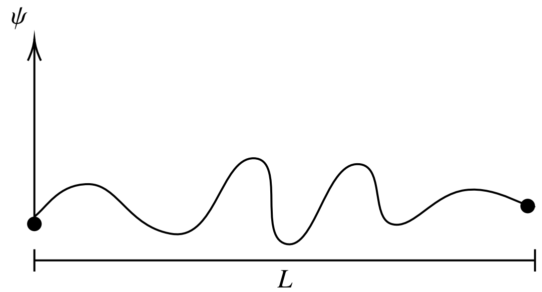
Example: $L = \pi,\ f(x) = x(\pi - x)$

$$\begin{aligned} \psi(x,t) &= D_0 + \sum_{n=1}^{\infty} D_n \cos(nx) e^{-\alpha^2 n^2 t} \\ \psi(x,0) &= x(\pi - x) = D_0 + \sum_{n=1}^{\infty} D_n \cos(nx) = \frac{\pi^2}{6} - \left[\frac{\cos(2x)}{1^2} + \frac{\cos(4x)}{2^2} + \frac{\cos(6x)}{3^2} + \dots\right. \\ \therefore \psi(x,t) &= \frac{\pi^2}{6} - \left[\frac{\cos(2x)}{1^2} e^{-4\alpha^2 +} + \frac{\cos(4x)}{2^2} e^{-16\alpha^2 +} + \frac{\cos(6x)}{3^2} e^{-36\alpha^2 +} + \dots \right. \quad \textit{(converges vapidly, but slower than 1A).} \end{aligned}$$

The Fourier Sine and Cosine series both extends the domain of $f(x)$, but in different ways.



2.3 BVP 2: Vibrating String



We now use the Wave Equation

$$a^2\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial t^2} = -F\Big\}\text{ Free vibrations. no external forces. } \therefore\ F = 0$$

$$\therefore \quad a^2\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2}$$

We now have the PDE:

$$\left.\begin{array}{l} a^2\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{\partial^2\psi}{\partial t^2} \quad \text{with B.C.s} \\ \psi(0,t) = 0,\ \psi(L,t) = 0,\ \text{and I.C.s} \\ \psi(x,0) = f(x),\ \psi_t(x,0) = g(x) \end{array}\right\} \textit{Wave Equation with Dirichlet B.C.s}$$

Physically, string with fixed/tied ends. Initial displacement of $f(x)$ and initial velocity of $g(x)$. Small vibrations.

Once again, assume $\psi(x,t) = X(x)T(t)$. Plug into PDE:

$$a^2T\frac{d^2X}{dx^2} = X\frac{d^2T}{dt^2}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{a^2T}\frac{d^2T}{dt^2} = \lambda \textit{ This separates to two second-order ODEs}$$

$$\frac{d^2X}{dx^2} = \lambda X \quad X(0) = 0, X(L) = 0$$

$$\left.\begin{array}{l}\text{Like in BVP 1A, } \lambda = -\dfrac{n^2\pi^2}{L^2} \\ X_n(x) = A_n \sin\left(\dfrac{n\pi x}{L}\right)\end{array}\right\} \quad n = 1, 2, \ldots \text{ because } \sin(0) = 0.$$

$$\frac{d^2T}{dt^2} = a^2\lambda T = -\underbrace{\frac{a^2n^2\pi^2}{L^2}}_{w_n^2}T, \quad w_n = \frac{an\pi}{L}$$

Allowed angular frequencies of vibration ($n = 1, 2, 3, \ldots$)

$\therefore \dfrac{d^2T}{dt^2} = -w_n^2 T$

General solution is:

$T_n(t) = B_n' \cos(w_n t) + C_n' \sin(w_n t)\}$ Both sin and cos terms exist because $IC \neq BC$, IC is not homogeneous.

$$\psi_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)[B_n \cos(W_n t) + C_n \sin(W_n t)] \begin{array}{l} B_n = B_n' \cdot A_n \\ C_n = C_n' \cdot A_n \end{array} \quad n = 1, 2, \ldots$$

Like before, we can rarely satisfy ICs with a single value of n. So we use a Fourier Series again

$$\therefore \underbrace{\left[a^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right]}_{\mathcal{L} \text{ (a linear operator)}}\psi(x,t) = 0 \quad \text{find nullspace of } \mathcal{L}.$$

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)[B_n \cos(\omega_n t) + C_n \sin(\omega_n t)]}$$

Apply position IC for B_n:

$\psi(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ *Using process in BVP 1*:

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

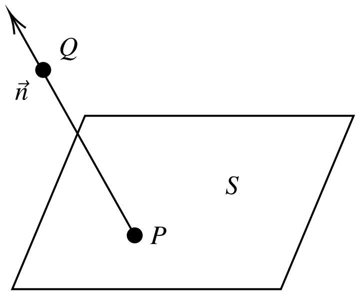
Apply velocity IC for C_n:

$$\psi_t(x,0) = \sum_{n=1}^{\infty} C_n w_n \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

$$\therefore \boxed{C_n w_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

Sub in B_n and C_n for the particular solution.

2.4 Newton's Law of Heating and Cooling



ψ_S = Temperature on surface S.

ψ_{med} = Temperature of external medium.

$$\left[\frac{\partial\psi}{\partial n}\right]_S = k\left[\psi_S - \psi_{med}\right]$$

Cooling.

$\psi_Q < \psi_P, \quad \therefore \left[\frac{\partial\psi}{\partial n}\right]_S < 0$ Point Q is colder than P, causing S to cool down.

Therefore, $\left[\frac{\partial\psi}{\partial n}\right]_S = -k[\psi_S - \psi_{med}]$

(*positive*), since external medium is colder in cooling

$$\therefore \left[\frac{\partial\psi}{\partial n} + k\psi\right]_S = k\psi_{med}$$

Heating.

$$\psi_Q > \psi_P, \quad \therefore \left[\frac{\partial\psi}{\partial n}\right]_S > 0$$

Therefore, $\left[\frac{\partial\psi}{\partial n}\right]_S = -k[\psi_S - \psi_{med}]$

(*negative*), since external medium is hotter in heating

$$\therefore \left[\frac{\partial\psi}{\partial n} + k\psi\right]_S = k\psi_{med}$$

2.5 BVP 3: Steady-State Temperature in Rectangular Regions

Recall Fourier Diffusion Equation:

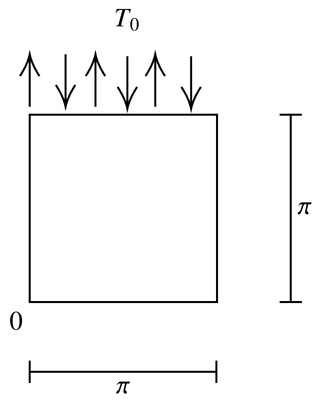
$$\frac{1}{\alpha^2}\frac{\partial\psi}{\partial t} - \nabla^2\psi = \frac{Q}{K} \qquad \textit{Steady-state, } \frac{\partial\psi}{\partial t} = 0$$

This reduces to Laplace's Equation if $Q = 0$

$$\therefore \nabla^2\psi = 0$$

For $2D$, $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$

Eg. Rectangular region insulated on 3 sides:



Assume $Q = 0$

Heats/Cools into a medium at T_0

B.C.s $\psi_x(0, y) = 0 \quad \psi_x(\pi, y) = 0$

$\psi_y(x, 0) = 0 \quad \left[\dfrac{\partial\psi}{\partial n} + k\psi\right]_{y=\pi} = \underset{(t)\ \text{constant}}{kT_0}$ (from Newton's law of Heating/Cooling.)

This is a well-posed Robin BVP.

Use separation of variables:

$\psi(x, y) = X(x)Y(y)$

$\dfrac{d^2X}{dx^2} \cdot Y + X \cdot \dfrac{d^2Y}{dy^2} = 0$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda.$$

$\dfrac{d^2X}{dx^2} = \lambda X \quad X'(0) = 0, \quad X'(\pi) = 0$ This is the same setup in BVP 1B (of which we have the solution of).

$X_n(x) = A_n \cos\left(\dfrac{n\pi x}{L}\right) = A_n \cos(nx)\Big|_{n=0}$ with $\lambda = \dfrac{-n^2\pi^2}{L^2} = -n^2$

In this case, because $L = \pi$

start at 0 because $\cos(0) \neq 0$

$\dfrac{d^2Y}{dy^2} = -\lambda Y \quad Y'(0) = 0 \cdot \left[\dfrac{d^2}{dy^2} - n^2\right] Y = 0$

The $(-)$ causes λ +ve (hyperbolic or exponential)

$$\begin{aligned}
Y_n(y) &= B_n \cosh(\mu y) + C_n \sinh(\mu y) \\
Y_n'(y) &= -\mu B_n \sinh(\mu y) + \mu C_n \cosh(\mu y) \quad \text{Applying } Y'(0) = 0 \\
0 &= -\mu B_n \sinh(0) + \mu C_n \cosh(0) \qquad \therefore C_n = 0 \quad (\text{for nonzero } \mu).
\end{aligned}$$

We can then combine:

$$\boxed{\psi_n(x,y) = D_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)}$$

Important: If the Neumann BCs were Dirichlet instead, simply replace cos *with* sin *and* cosh *with* sinh*. The physical interpretation would then become "3 sides kept at* $0°$ *(or a given constant)".*

Now we address the case where the 4th B.C. isn't nice (Fourier Series)

$\therefore Y_n(y) = B_n \cosh(\mu y)$ Extract $n = 0$ term

$$\cosh(0) = 1$$

$$\therefore Y_0(y) = B_0$$

NB: If the bottom edge is the non-constant term, y gets replaced with $\pi - y$ *on the hyperbolic.*

Likewise, $\cos(0) = 1$ $\therefore x_0(x) = A_0$ $\therefore \psi_0(x,y) = A_0 \cdot B_0 = D_0$

$$\psi(x,y) = \psi_0(x,y) + \sum_{n=1}^{\infty} \psi_n(x,y)$$

$\therefore D_0 + \sum_{n=1}^{\infty} D_n \cos(nx)\cosh(ny)$ NB: This is for the case where $L = \pi$. Otherwise keep the argument as $\frac{n\pi}{L}$.

Consider 4th BC: $\left[\frac{\partial\psi}{\partial n} + k\psi\right]_{y=r} = kT_0$

$$\psi_y(x,\pi) + k\psi(x,\pi) = kT_0$$

$$\sum_{n=1}^{\infty} D_n n \sinh(n\pi)\cos(nx) + kD_0 + \sum_{n=1}^{\infty} D_n k \cosh(n\pi)\cos(nx) = kT_0$$

Pulled out $n = 0$ term.

Collect D_n terms:

$\sum_{n=1}^{\infty} D_n[n\sinh(n\pi) + k\cosh(n\pi)]\cos(nx) + \overbrace{kD_0}^{a_0} \cdot 1 = \overbrace{kT_0}^{f(x)}$ Fourier Cosine Series

LHS is the Fourier Cosine series of kT_0

$$\therefore kD_0 = \frac{1}{\pi}\int_0^{\pi} kT_0 dx = \frac{kT_0\psi}{\pi} = kT_0 \Rightarrow D_0 = T_0$$

$$D_n[n\sinh(n\pi) + k\cosh(n\pi)] = \frac{2}{\pi}\int_0^{\pi} kT_0\cos(nx)dx = \frac{2kT_0}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} = 0 \quad (\sin(\pi) = 0)$$

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\therefore D_n = 0, \quad n = 1, 2, \dots \quad D_0 = T_0 \quad \therefore \psi(x, y) = T_0$$

This makes physical sense. If the only non-insulated side is held at a source at a temperature T_0, the steady-state temperature of this region will naturally tend towards this T_0 as time passes. If the non-insulated side has a less predictable heat transfer profile (ie. with spatial dependence, one may simple plug in the given function instead of kT_0 to the Fourier series and solve for D_n, before inputting it into the general solution.

2.6 BVP 4: Steady-State Temperature in Circular Regions

Solve $\nabla^2\psi(r, \theta) = 0$ Use Laplacian in polar coordinates.

$$\nabla^2\psi(r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

Let $\psi(r, \theta) = R(r)M(\theta)$

$\frac{M}{r} \cdot \frac{d}{dr}\left[r\frac{dR}{dr}\right] + \frac{R}{r^2} \cdot \frac{d^2M}{d\theta^2} = 0$ Multiply both sides by $\frac{r^2}{RM}$

$$\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] + \frac{1}{M}\frac{d^2M}{d\theta^2} = 0$$

$$\therefore -\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = \frac{1}{M}\frac{d^2M}{d\theta^2} = \lambda$$

Note that M is periodic with period $2\pi(M(0) = M(2\pi))$. Because it is the angular coordinate.

Angular ODE:

$$\frac{d^2M}{d\theta^2} = \lambda M$$

Case 1: $\lambda = 0 \quad \Rightarrow \quad M = A + B\theta$

$B = 0$ because M must be periodic.

$$\therefore M_0(\theta) = A_0$$

Case 2: λ positive, let $\lambda = \mu^2$.

$$\frac{d^2M}{d\theta^2} = \mu^2 M, \quad M = \underbrace{A\cosh(\mu\theta)}_{\text{Not periodic}} + \underbrace{B\sinh(\mu\theta)}_{\text{Not periodic}}$$

$\therefore A = B = 0$ (Trivial solution).

Case 3: λ negative, let $\lambda = -\mu^2$

$\dfrac{d^2M}{d\theta^2} = -\mu^2 M, \quad M = A\cos(\mu\theta) + B\sin(\mu\theta).$ Need $M(2\pi) = \mu(0)$.

$$\begin{matrix} \cos(2\pi\mu) = \cos(0) \\ \sin(2\pi\mu) = \sin(0) \end{matrix} \Rightarrow 2\pi\mu = 2n\pi \quad \therefore \mu = n$$

Thus; $M_0(\theta) = A_0. \quad M_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta). \quad n = 1, 2, 3, \ldots$

Radial ODE:

$$-\frac{r}{d}\left[r\frac{dR}{dr}\right] = \lambda = \underset{\substack{\uparrow \\ \text{established that } \mu = n}}{-n^2}$$

$$-\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = \lambda = -n^2$$

But only in a full circle. Otherwise, $\lambda =$ ends up being something like $-4n^2$. Go through substitution if not full circle (Roth likes to ask these questions).

Expand the differentials:

$r^2\dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} - n^2R = 0$ We solve with the substitution $R = r^k$ (Euler ODE).

$$\therefore r^2 \cdot (k-1) \cdot k \cdot r^{k-2} + r \cdot k \cdot r^{k-1} - n^2 \cdot r^k = 0 \qquad \therefore \frac{dR}{dr} = kr^{k-1}$$

$$\left[(k-1)\cdot k + k - n^2\right] r^k = 0 \quad \Rightarrow \quad k^2 - n^2 = 0 \quad \therefore k = \pm n \quad \therefore \frac{d^2R}{dr^2} = (k-1)kr^{k-2}$$

Thus, $R_1 = r^n,\ R_2 = r^{-n}$

Therefore $R_n(r) = C_n r^n + \dfrac{D_n}{r^n} \quad n = 1, 2, 3, \ldots$

For $n = 0$, we have:

$$r\frac{d}{dr}\left[r\frac{dR}{dr}\right] = 0 \quad \therefore \frac{d}{dr}\Big[\underset{\substack{\uparrow \\ \text{Must be constant}}}{r\frac{dR}{dr}}\Big] = 0$$

Integrate both sides:

$\therefore r\dfrac{dR}{dr} = D_0$

$\therefore \dfrac{dR}{dr} = \dfrac{D_0}{r} \quad \Rightarrow \quad R_0(r) = C_0 + D_0\ln(r).$

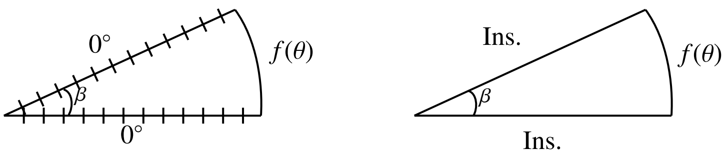
Now, the final solution to the PDE $\psi(r, \theta)$ is obtained by multiplying M by R.

$$\boxed{\psi(r,\theta) = E_0 + F_0\ln(r) + \sum_{n=1}^{\infty}\cos(n\theta)\left[E_n r^n + \frac{F_n}{r^n}\right] + \sum_{n=1}^{\infty}\sin(n\theta)\left[G_n r^n + \frac{H_n}{r^n}\right]}$$

Sectors

Two sectors possible:

a) b)



(Dirichlet) (Neumann)

For both, we separate regularly:

$$\frac{1}{M}\frac{d^2M}{d\theta^2} = -\frac{r}{R}\frac{d}{dr}\left[r\frac{dR}{dr}\right] = \lambda$$

For (a), angular ODE is:

$$\frac{d^2M}{d\theta^2} = \lambda M, \quad \begin{array}{l} M(0) = 0, \\ M(\beta) = 0, \end{array} \quad \therefore \quad M(\theta) = A_n \sin\left(\frac{n\pi\theta}{\beta}\right) \quad \text{(Like BVP 1A)}$$

* Ex 15
PSet 4

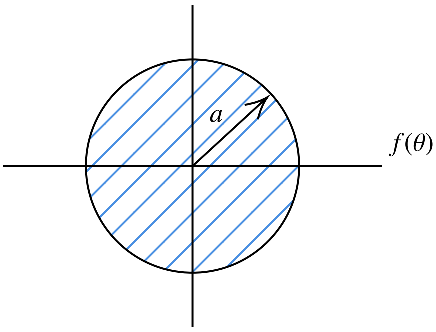
For (b), angular ODE is:

$$\frac{d^2M}{d\theta^2} = \lambda M, \quad \begin{array}{l} M'(0) = 0, \\ M'(\beta) = 0, \end{array} \quad \therefore \quad M(\theta) = A_n \cos\left(\frac{n\pi\theta}{\beta}\right) \quad \text{(Like BVP 1B)}$$

* Ex 9
Pset 4

You then multiply these by the general Radial ODE.

2.6.1 BVP 4a: Interior of a Disc (Long cylinder)



$$\begin{array}{ll} \nabla^2(r,\theta) = 0; & 0 \leqslant r \leqslant a \\ \psi(a,\theta) = f(\theta) & 0 \leqslant \theta \leqslant 2\pi \end{array}$$

We invoke the general solution:

$$\psi(r,\theta) = E_0 + F_0\ln(r) + \sum_{n=1}^{\infty}\cos(n\theta)\left[E_n r^n + \frac{F_n}{r^n}\right] + \sum_{n=1}^{\infty}\sin(n\theta)\left[G_n r^n + \frac{H_n}{r^n}\right]$$

The solution must be finite for $r = 0$. $\therefore F_0 = 0,\ F_n = 0,\ H_n = 0.$

$$\boxed{\psi_{\text{INT}}(r,\theta) = E_0 + \sum_{n=1}^{\infty} E_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} G_n r^n \sin(n\theta)}$$

Now we apply the IC to solve for the constants:

$$\psi_{\text{INT}}\,(a,\theta) = E_0 + \sum_{n=1}^{\infty} E_n a^n \cos(n\theta) + \sum_{n=1}^{\infty} G_n a^n \sin(n\theta) = f(\theta)$$

Fourier Series.

$$\int_0^{2\pi} \underbrace{E_0 \cos(k\theta)\,d\theta}_{=0} + \sum_{n=1}^{\infty} E_n a^n \int_0^{2\pi} \underbrace{\cos(k\theta)\cos(n\theta)\,d\theta}_{=\pi\cdot\delta_{n,k}} + \sum_{n=1}^{\infty} G_n a^n \int_0^{2\pi} \underbrace{\cos(k\theta)\sin(k\theta)\,d\theta}_{=0}$$
$$= \int_0^{2\pi} \cos(k\theta) f(\theta)\,d\theta$$

Only nonzero term is when $n = k$. So interchange $k = n$.

$$\pi E_n a^n = \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$\boxed{E_n = \frac{1}{a^n \pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta} \tag{1}$$

Doing the same thing with sin, we get:

$$\boxed{G_n = \frac{1}{a^n \pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta} \tag{2}$$

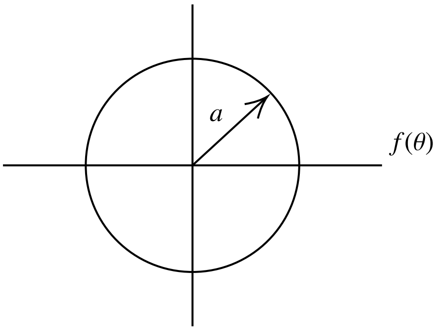
Finally, doing the same thing with 1:

$$\int_0^{2\pi} f(\theta) d\theta = \underbrace{E_0 \int_0^{2\pi} 1 d\theta}_{E_0\cdot 2\pi} + \sum_{n=1}^{\infty} E_n a^n \int_0^{2\pi} \underbrace{1\cdot\cos(n\pi)}_{0}\,d\theta + \sum_{n=1}^{\infty} G_n a^n \int_0^{2\pi} \underbrace{1\cdot\sin(n\theta)}_{0}\,d\theta$$

$$\boxed{\therefore E_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta} \tag{3}$$

Plug (1), (2), (3) for E_n, G_n, E_0 in the General Solution

2.6.2 BVP 4b: Exterior of a Disc (Long cylinder)



$$\nabla^2\psi(r,\theta) = 0 : \quad r > a$$
$$\psi(a,\theta) = f(x).$$

Once again, we invoke the general solution

$$\psi(r,\theta) = E_0 + F_0\ln(r) + \sum_{n=1}^{\infty}\cos(n\theta)\left[E_n r^n + \frac{F_n}{r^n}\right] + \sum_{n=1}^{\infty}\sin(n\theta)\left[G_n r^n + \frac{H_n}{r^n}\right]$$

This time, since $r > a$, the solution must be finite as $r \to \infty$

$$\therefore F_0 = 0,\ E_n = 0,\ G_n = 0$$

$$\boxed{\psi_{\text{EXT}}(r,\theta) = E_0 + \sum_{n=1}^{\infty}\frac{F_n}{r^n}\cos(n\theta) + \sum_{n=1}^{\infty}\frac{H_n}{r^n}\sin(n\theta)}$$

We apply the same trick as in BVP 4 a to get coefficients:

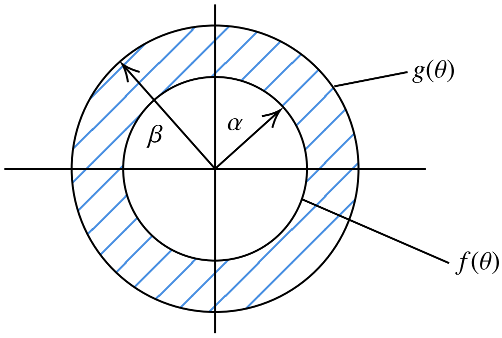
$$\boxed{E_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta}$$

$$\boxed{F_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}$$

$$\boxed{H_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}$$

Plug these into the General Solution

2.6.3 BVP 4c: Annular Regions



$$\nabla^2\psi(r,\theta) = 0, \quad \begin{matrix} \alpha < r < \beta \\ 0 < \theta < 2\pi \end{matrix} \quad \begin{matrix} \psi(\alpha,\theta) = f(\theta) \\ \psi(\beta,\theta) = g(\theta) \end{matrix}$$

Once again, we invoke the general formula.

$$\psi(r,\theta) = E_0 + F_0 \ln(r) + \sum_{n=1}^{\infty} \cos(n\theta)\left[E_n r^n + \frac{F_n}{r^n}\right] + \sum_{n=1}^{\infty} \sin(n\theta)\left[G_n r^n + \frac{H_n}{r^n}\right]$$

We can't kill any terms, though.

However, we can apply the same trick to create a system of equations

$$\psi(\alpha,\theta) = E_0 + F_0 \ln(\alpha) + \sum_{n=1}^{\infty} \cos(n\theta)\left[E_n \alpha^n + \frac{F_n}{\alpha^n}\right] + \sum_{n=1}^{\infty} \sin(\theta)\left[G_n \alpha^n + \frac{H_n}{\alpha^n}\right] = f(\theta)$$

$$\therefore E_0 + F_0 \ln(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta \qquad \bullet$$

$$\therefore E_n \alpha^n + \frac{F_n}{\alpha^n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta) d\theta \qquad \bullet$$

$$\therefore G_n \alpha^n + \frac{F_n}{\alpha^n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta) d\theta \qquad \bullet$$

$$\psi(\beta,\theta) = E_0 + F_0 \ln(\beta) + \sum_{n=1}^{\infty} \cos(n\theta)\left[E_n \beta^n + \frac{F_n}{\beta^n}\right] + \sum_{n=1}^{\infty} \sin(\theta)\left[G_n \beta^n + \frac{H_n}{\beta^n}\right] = g(\theta)$$

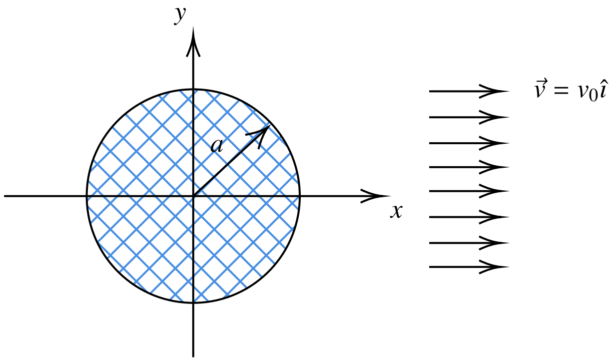
$$\therefore E_0 + F_0 \ln(B) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta \qquad \bullet$$

$$\therefore E_n B^n + \frac{F_n}{\beta^n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta) d\theta \qquad \bullet$$

$$\therefore G_n B^n + \frac{F_n}{\beta^n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta) d\theta \qquad \bullet$$

For every colour • • •, there are 2 equations and 2 unknowns. Thus all constant terms can be solved for.

2.6.4 BVP 4d: Flow Around a Long Circular Cylinder.



Assume that there is initially uniform glow parallel to the x-axis. Then the cylinder is inserted into the flow. The centre of the cylinder is at the origin.

Originally, $\vec{V} = V_0\hat{\imath} \quad \Rightarrow \quad \vec{\nabla}\psi = \frac{\partial\psi}{\partial x}\hat{\imath} + \frac{\partial\psi}{\partial y}\hat{\jmath}$

When the cylinder is inserted:

$\frac{\partial\psi}{\partial x} \to V_0$ and $\frac{\partial\psi}{\partial y} \to 0$

$$\therefore \psi \to V_0 x + c = V_0 \underbrace{r\cos(\theta)}_{x} + c$$

We invoke the general solution, extracting the $n = 1$ term from cos.

$$\psi(r,\theta) = E_0 + F_0\ln(r) + \cos(\theta)\left[E_1 r + \frac{F_R}{r}\right] + \sum_{n=2}^{\infty}\cos(n\theta)\left[E_n r^n + \frac{F_n}{r^n}\right] + \sum_{n=1}^{\infty}\sin(n\theta)\left[G_n r^n + \frac{H_n}{r^n}\right]$$

$$\psi(r,\theta) \longrightarrow C + V_0 r\cos(\theta)$$

Kill exponential growth terms for stability

$$\therefore \psi(r,\theta) = E_0 + V_0 r\cos(\theta) + \sum_{n=1}^{\infty}\frac{F_n}{r^n}\cos(n\theta) + \sum_{n=1}^{\infty}\frac{H_n}{r^n}\sin(n\theta)$$

Since the circumference of the cylinder is a physical boundary, fluid can't enter or leave the surface.

$$\therefore [V_n]_{r=a} = \left[\overrightarrow{\nabla\psi}\cdot\vec{n}\right]_{r=a} = \left[\frac{\partial\psi}{\partial r}\right]_{r=a} = 0$$

$$\frac{\partial\psi}{\partial r} = V_0\cos(\theta) - \sum_{n=1}^{\infty}\frac{nF_n}{r_{n+1}}\cos(n\theta) - \sum_{n=1}^{\infty}\frac{nF_n}{r^{n+1}}\sin(n\theta)$$

Since $\cos(\theta), \cos(2\theta), \ldots$ and $\sin(\theta), \sin(2\theta), \ldots$, are orthogonal, they are linearly independent in the complete Fourier Series.

$$0 = \left(V_0 - \frac{F_1}{a^2}\right)\cos(\theta) - \frac{2F_2}{a^3}\cos(2\theta) - \ldots -$$
$$- \frac{H_1}{a^2}\sin(\theta) - \frac{2H_2}{a^3}\sin(2\theta)$$

$H_n = 0$ (Otherwise the equation can't equal zero).

And, $V_0 - \dfrac{F_1}{a^2} = 0 \quad \Rightarrow \quad F_1 = V_0 a^2$. All other F_n terms must be 0 (For the same reason that $H_n = 0$).

$\therefore\ \psi(r,\theta) = E_0 + V_0 r\cos(\theta) + \dfrac{V_0 a^2}{r}\cos(\theta)$. E_0 is arbitrary, can be disregarded

$$\boxed{\therefore\ \psi(r,\theta) = V_0\cos(\theta) + \frac{V_0 a^2}{r}\cos(\theta)}$$

First term: Effect of pre-existing flow

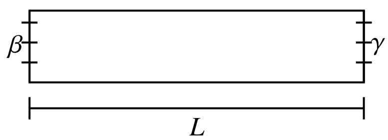
Second term: Effect of cylinder

Thus, $\vec{V} = \overrightarrow{\nabla\psi} = V_0\cos(\theta)\left[1 - \dfrac{a^2}{r^2}\right]\hat{\mu}_r - V_0\sin(\theta)\left[1 + \dfrac{a^2}{r^2}\right]\hat{\mu}_\theta$ and $|\vec{v}|_{r=a} = -2V_0\sin(\theta)\hat{\mu}_\theta$ (no radial component at the boundary)

2.7 BVP 5: Time-Indepedent Non-Homogenous Aspects

2.7.1 BVP 5a. Diffusion of Heat in a Thin Bar, Ends Maintained at β° and γ°

Identical to BVP 1a, but ends are at constant β° and γ° instead of 0°



Diffusion Equation: $\dfrac{1}{\alpha^2}\dfrac{\partial\psi}{\partial t} = \dfrac{\partial^2\psi}{\partial x^2}; \quad 0 < x < L,\ t > 0$

B.C.s: $\psi(0,t) = \beta, \quad \psi(L,t) = \gamma$

I.C.: $\psi(x,0) = f(x)$

Separation of variables won't work. Instead, we use a Slave Function of one variable to turn the B.C.s homogeneous. The motivation for this is that when subtracted from our initial equation $\psi(x,t)$, the Slave Function $\psi_S(x)$ will turn what's left $\psi_T(x)$ into a PDE with homogenous B.C.s, which can easily be solved.

We formalize the "Slave Function" as the "Steady-state" behaviour since it has no time dependency, while $\psi_T(x)$ models the transient, or time-dependent, behaviour of the PDE.

Let $\psi(x,t) = \psi_S(x) + \psi_T(x,t)$

$\psi_S(x)$ = Steady-state temperature.

Sub in this into the PDE:

$\psi_T(x,t)$ = Transient temperature.

$$0 + \frac{1}{\alpha^2}\cdot\frac{\partial\psi_T}{\partial t} = \frac{\partial^2\psi_S}{\partial x^2} + \frac{\partial^2\psi_T}{\partial x^2}$$

First we determine the slave function according to our BC/ICs.

For the steady-state part, $\dfrac{\partial^2\psi_S}{\partial x^2} = 0 \qquad \therefore \psi_S(x) = Ax + B$

$$\therefore \psi_S(0) = B = \beta \qquad \therefore \psi_S(x) = Ax + \beta$$

$$\psi_S(L) = AL + B = \gamma \qquad \therefore A = \frac{\gamma - \beta}{L}$$

Finally, $\boxed{\psi_S(x) = \left(\dfrac{\gamma-\beta}{L}\right)x + \beta}$

In essence, the slave function $\psi_S(x)$ "homogenizes" the boundary conditions for the transient function $\psi_T(x)$.

For the transient part, $\dfrac{1}{\alpha^2}\dfrac{\partial\psi_T}{\partial t} = \dfrac{\partial^2\psi_T}{\partial x^2}$

$$\left.\begin{array}{ll} \psi_T(0,t) = \psi(0,t) - \psi_S(0) = \beta - \beta = 0 & \text{Homogeneous.} \\ \psi_T(L,t) = \psi(L,t) - \psi_S(L) = \gamma - \gamma = 0 & \text{Homogeneous.} \end{array}\right\} \quad \text{B.C.'s}$$

$$\psi_T(x,0) = \psi(x,0) - \psi_S(x) = \underbrace{f(x) - \left[\frac{\gamma-\beta}{L}x + \beta\right]}_{F(x)} \quad \text{I.C.}$$

The problem is now like BVP 1a. However, keep in mind that the initial condition has changed (from $f(x)$ to $F(x)$ as defined in the line above).

Solve for $\psi_T(x,t)$ and add it to $\psi_S(x)$ for the final solution.

Thus, $\psi(x,t) = \psi_T(x,t) + \psi_S(x)$

$$\psi_T(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\int_0^L F(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\ \sin\left(\frac{n\pi x}{L}\right)e^{\frac{-\alpha^2n^2\pi^2}{L^2}t}$$

$$= \int_0^L G(x,t,\ \xi)F(\xi)d\xi\,, \quad G(x,t,\xi) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right)e^{\frac{-\alpha^2n^2\pi^2}{L^2}t} \quad \text{(Green's Function).}$$

2.7.2 BVP 5b: Heat Generation/Absorption in a Thin Bar

This is BVP 1, but with nonzero Q.

$$\frac{1}{\alpha^2}\cdot\frac{\partial\psi}{\partial t} - \nabla^2\psi = \frac{Q}{K}$$

Eg.

$$2\frac{\partial\psi}{\partial t} - \frac{\partial^2\psi}{\partial x^2} = -6x, \qquad \begin{array}{ll} \psi(0,t) = 3 & \psi(x,0) = x^3 + 2x + 3 \\ \psi(2,t) = 9 & \end{array}$$

Let $\psi(x,t) = \psi_S(x) + \Phi(x,t) \quad (\psi_T(x,t) = \Phi(x,t)) \quad$ Sub in:

$$0 + 2\frac{\partial\Phi}{\partial t} - \frac{d^2\psi_S}{dx^2} - \frac{\partial^2\Phi}{\partial x^2} = -6x$$

For the slave function ($\psi_S(x)$), it is in steady state, so $\dfrac{\partial \psi_S}{\partial t} = 0$

$$\therefore\ \frac{1}{\alpha^2}\cdot\frac{\partial \psi_S}{\partial t} - \nabla^2\psi_S = \frac{Q}{K} \quad \text{(which is 1 dimensional).}$$

$$-\frac{d^2\psi_S}{dx^2} = -6x \quad \Rightarrow \quad \psi_S(x) = x^3 + C_1x + C_2 \quad \text{Solve for } C_1, C_2 \text{ using B.C.'s}$$

$$\psi_S(0) = C_2 = 3. \quad \psi_S(x) = x^3 + C_1x + 3$$

$$\psi_5(2) = 8 + 2C_1x + 3 = 9, \quad \therefore C_1 = -1$$

$$\therefore\ \psi_S(x) = x^3 - x + 3$$

Now, address $\Phi(x,t)$:

$$\left.\begin{array}{rl} \text{B.C.'s:} & \Phi(0,t) = \psi(0,t) - \psi_S(0) = 3 - 3 = 0 \\ & \Phi(2,t) = \psi(2,t) - \psi_S(2) = 9 - 9 = 0 \\ \text{I.C:} & \Phi(x,0) = \left[x^3 + 2x + 3\right] - \left[x^3 - x + 3\right] = 3x \end{array}\right\}\text{BVP 1a}$$

$$\Phi(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) e^{\frac{-n^2\pi^2}{8}t}$$

$\Phi(x,0) = \sum\limits_{n=1}^{\infty} C_n \sin\left(\dfrac{n\pi x}{2}\right) = 3x$ Fourier Sine Series to resolve IC.

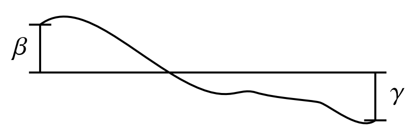
$C_n = \dfrac{2}{L}\displaystyle\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx = \frac{2}{2}\int_0^2 3x\sin\left(\frac{n\pi x}{2}\right)dx = \frac{-12(-1)^n}{n\pi}$ (Integrate by parts).

$$\therefore\ \psi(x,t) = \psi_S(x) + \Phi(x,t) = x^3 - x + 3 - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{2}\right)e^{\frac{-n^2\pi^2}{8}t}$$

Physically: $\psi(x,t)$ represents the temperature at a thin rod of length 3, insulated on the sides, initial temperature is $x^2 + 2x + 3$. (Q = rate of heat absorption/generation). Diffusivity $\left(\alpha^2\right) = \dfrac{1}{2}$. $\dfrac{Q(x)}{k} = -6x$ $\therefore Q(x) = -6xk$

2.7.3 BVP 5c: Vibrating String with Gravity

We have $a^2\dfrac{\partial^2\psi}{\partial x^2} - \dfrac{\partial^2\psi}{\partial t^2} = g$



Ends fixed at β and γ.

B.C.'s: $\psi(0,t) = \beta$, $\psi(L,t) = \gamma$.

I.C.'s: $\psi(x,0) = f(x)$, $\psi_t(x,0) = g(x)$.

Use Slave Function

Let $\psi(x,t) = \psi_S(x) + \Phi(x,t)$

Substitute into PDE:

$$a^2 \frac{d^2\psi_S}{dx^2} + \underbrace{a^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial t^2}}_{=0} = g$$

Note that we group our terms such that the transient component is homogeneous. In essence, we split up our inhomogeneous PDE into two: an inhomogeneous ODE $\psi_S(x)$ and a homogeneous PDE $\Phi(x,t)$, both of which are solvable.

$a^2 \dfrac{d^2\psi}{dx^2} = g \quad \psi_S(0) = \beta, \quad \psi_S(L) = \gamma$ Integrate twice

$\boxed{\psi_S(x) = -\dfrac{gx}{2a^2}(L-x) + \left(\dfrac{\gamma-\beta}{L}\right)x + \beta}$ Can be considered the static/equilibrium deflection of the string.

Once again, we are left with:

$$a^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial t^2} = 0$$

$$\left.\begin{array}{ll} \text{B.C.'s :} & \Phi(0,t) = \psi(0,t) - \psi_S(0) = \beta - \beta = 0 \\ & \Phi(L,t) = \psi(0,t) - \psi_S(0) = \gamma - \gamma = 0 \\ \text{I.C.'s :} & \Phi(x,0) = \psi(x,0) - \psi_S(x) = f(x) - \left\{\dfrac{-gx}{2a^2}(L-x) + \left(\dfrac{\gamma-\beta}{L}\right)x + \beta\right\} = F(x) \\ & \Phi_t(x,0) = \psi_t(x,0) = g(x) = G(x) \end{array}\right\} \quad \text{BVP 2}$$

As before, sum up $\psi_S(x)$ and $\Phi(x,t)$ to obtain $\psi(x,t)$.

2.8 BVP 6: Time-Dependent Non-Homogenous Aspects

2.8.1 BVP 6a: Generalized Diffusion, ends at $0°$

General Diffusion Equation: $\dfrac{1}{\alpha^2}\dfrac{\partial\psi}{\partial t} - \dfrac{\partial^2\psi}{\partial x^2} = \dfrac{Q(x,t)}{k} = h(x,t)$

In	BVP 1:	$Q = 0$
	BVP 5b:	$Q = f(x)$
	BVP 6a:	$Q = f(x,t)$

For simplicity, take $\alpha^2 = 1,\ L = \pi$ (but this can easily be generalized).

Assume ends are maintained at 0°.

$$\therefore \frac{\partial\psi}{\partial t} - \frac{\partial^2\psi}{\partial x^2} = h(x,t); \quad 0 < x < \pi, \quad t > 0$$

B.C.s: $\psi(0,t) = 0, \quad \psi(L,t) = 0$

I.C.: $\psi(x,0) = f(x)$

We know the solution for the homogeneous one:

$\psi_{\text{HOM}}(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) e^{-n^2 t}$ (Complimentary solution).

We can solve with "Variation of Parameters".

$$\text{Let } \boxed{\psi(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin(nx)}\ *$$

(Spatial part satisfies the B.C.'s, so the $e^{-n^2 t}$ gets incorporated into $C_n(t)$.)

Physically, $e^{-n^2 t}$ represented the decrease with time from an initial temperature of $f(x)$ to O (because there is no heat generation or absorption). However, this term is pointless with nonzero heat generation/absorption.

Sub * into the PDE:

$\sum_{n=1}^{\infty}\left[\frac{dC_n}{dt} + n^2 C_n\right]\sin(nx) = h(x,t)$ Apply the Fourier Sine trick (all 0 unless $n = k$).

$$\begin{aligned}
\sum_{n=1}^{\infty}\left[\frac{dC_n}{dt} + n^2 C_n\right]\int_0^{\pi} \underbrace{\sin(kx)\sin(nx)\,dx}_{\delta_{k,x}\cdot\frac{\pi}{2}} &= \int_0^{\pi} \sin(kx)h(x,t)\,dx \\
\frac{\pi}{2}\left[\frac{dC_k}{dt} + k^2 C_k\right] &= \int_0^{\pi} \sin(kx)h(x,t)\,dx \quad (n = k) \\
\frac{dC_n}{dt} + n^2 C_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)h(x,t)\,dx &= B_n(t)
\end{aligned}$$

This is a first-order linear ODE (we solve it by multiplying each side with an integrating factor $\mu(t)$):

$$\begin{aligned}
&\frac{dC_n}{dt} + \underbrace{n^2 C_n}_{p(t)} = \underbrace{B_n(t)}_{g(t)} \qquad \mu(t) = e^{\int p(t)\,dt} = e^{n^2 t} \\
&\underbrace{\frac{dC_n}{dt}\, e^{n^2 t} + n^2 C_n\, e^{n^2 t}}_{\text{reverse prod. rule}} = B_n(t)\, e^{n^2 t}
\end{aligned}$$

$$\frac{d}{dt}\left[C_n e^{n^2 t}\right] = B_n(t) e^{n^2 t}$$

$$\int_0^t \frac{d}{d\tau}\left[C_n e^{n^2 \tau}\right] d\tau = \int_0^t B_n(\tau) e^{n^2 \tau} d\tau$$

$$e^{-n^2 t}\left[C_n(t) e^{n^2 t} - C_n(0)\right] = \int_0^t B_n(\tau) e^{n^2 \tau} d\tau$$

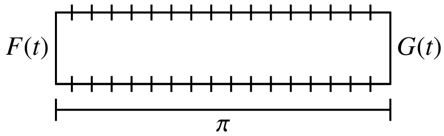
$$\boxed{C_n(t) = C_n(0) e^{-n^2 t} + e^{-n^2 t}\int_0^t B_n(\tau) e^{n^2 \tau} d\tau}$$

From the IC: $\psi(x,0) = \sum_{n=1}^{\infty} C_n(0)\sin(nx) = f(x)$

$$\therefore \boxed{C_n(0) = \frac{2}{\pi}\int_0^{\pi} f(\xi)\sin(n\xi)d\xi}$$

Plug $C_n(t)$ and $C_n(0)$ into the (boxed) general solution. τ and ξ are just dummy variables.

2.8.2 BVP 6b. Generalized Diffusion. Dirichlet, B.C. function of time



$$\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial t^2} = h(x,t) \qquad \begin{array}{l} \psi(0,t) = F(t) \\ \psi(\pi,t) = G(t) \end{array} \qquad \psi(x,0) = f(t)$$

Now we use a Slave Function of 2 variables. (since the B.C.s are both nonhomogenous and a function of time). Here, the slave function once again represents the long-term behaviour of the system, but this isn't time-indepdent; it just means that the transient components decays relative to the non-constant slave function.

$$\psi(x,t) = \psi_S(x,t) + \Phi(x,t)$$

In BVP 5a, we had $\psi(0,t) = \beta,\ \psi(L,t) = \gamma$ with $\psi_S(x) = Ax + B$.

With the time-variant component, we have:

$$\psi_S(x,t) = A(t)x + B(t)$$

Thus. $\psi_S(0,t) = B(t)$

$\therefore\ \psi_S(x,t) = A(t)x + F(t)$ Now, we incorporate the other B.C.

$\psi_S(\pi,t) = A(t)\pi + F(t) = G(t)$

or generally, L

$$\therefore\ \boxed{\psi_S(x,t) = \left[\frac{G(t) - F(t)}{\pi}\right] x + F(t)}$$

Sub in $\psi(x,t) = \psi_S(x,t) + \Phi(x,t)$ into PDE $\dfrac{\partial \psi}{\partial t} - \dfrac{\partial^2 \psi}{\partial x^2} = h(x,t)$.

$$\therefore\ \left\{\left[\frac{G'(t) - F'(t)}{\pi}\right] x + F'(t) - 0\right\} + \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} + 0 = h(x,t)$$

Moving everything to the RHS, we have:

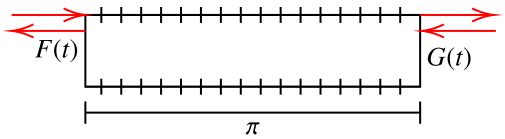
$$\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = h(x,t) - \left\{\left[\frac{G'(t) - F'(t)}{\pi}\right] x + F'(t)\right\},$$

$$\begin{aligned} \text{B.C.'s:} \quad & \Phi(0,t) = \psi(0,t) - \psi_S(0,t) = F(t) - F(t) = 0, \\ & \Phi(\pi,t) = \psi(\pi,t) - \psi_S(\pi,t) = G(t) - G(t) = 0, \\ \text{I.C.:} \quad & \Phi(x,0) = \psi(x,0) - \psi_s(x,0) \\ & \qquad = f(x) - \left\{\left[\frac{G(0) - F(0)}{\pi}\right] x + F(0)\right\} \\ & \qquad = \mathcal{F}(x) \end{aligned}$$

As we can see, the PDE of $\Phi(x,t)$ has reduced to an inhomogeneous, time dependent PDE with homogenous B.C.s. This is simply BVP 6a, and we can solve for $\Phi(x,t)$ as we did BVP 6a.

Finally, don't forget to add the slave function to your solution $\psi(x,t) = \Phi(x,t) + \psi_s(x,t)$

2.8.3 BVP 6c. Generalized Diffusion. Neumann, B.C. function of time



$$\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial x^2} = h(x,t) \qquad \begin{array}{l} \psi_x(0,t) = F(t) \quad \psi(x,0) = f(t) \\ \psi_x(\pi,t) = G(t) \end{array}$$

Rate of heat inflow (outflow) at ends represented by $F(x), G(x)$.

Same process, assume the general solution is equal to the sum of a transient function and slave function.

$$\psi(x,t) = \psi_S(x,t) + \Phi(x,t)$$

$\therefore \dfrac{\partial \psi_S}{\partial x} = A(t)$. This slave function can't satisfy two derivative-level (Neumann) B.C.s, so we go "up a power".

$\therefore \psi_S(x,t) = A(t)x^2 + B(t)x$

$\therefore \dfrac{\partial \psi_S}{\partial x} = 2A(t)x + B(t)$ Apply B.C.'s:

$$\begin{aligned} \left.\frac{\partial \psi_S}{\partial x}\right|_{x=0} &= B(t) = F(t) \\ \left.\frac{\partial \psi_S}{\partial x}\right|_{x=\pi} &= 2\pi A(t) + F(t) = G(t) \qquad \therefore A(t) = \frac{G(t) - F(t)}{2\pi} \end{aligned}$$

$$\therefore \boxed{\psi_S(x,t) = \left[\frac{G(t) - F(t)}{2\pi}\right] x^2 + F(t)x}$$

Sub in $\psi(x,t) = \left[\dfrac{G(t) - F(t)}{2\pi}\right] x^2 + F(t)x + \Phi(x,t)$ into the PDE:

$$\begin{aligned} &\left\{\left[\frac{G'(t) - F'(t)}{2\pi}\right] x^2 + F'(t)x - \left[\frac{G(t) - F(t)}{\pi}\right]\right\} + \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = h(x,t) \\ &\frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial x^2} = h(x,t) - \left\{\left[\frac{G'(t) - F'(t)}{2\pi}\right] x^2 + F'(t)x - \left[\frac{G(t) - F(t)}{\pi}\right]\right\} = \mathcal{H}(x,t) \end{aligned}$$

All that happened was that the RHS of the transient function $\Phi(x,t)$ got modified. In turn, its B.C.s will become homogenous, allowing us to solve for it. We did the same thing in BVP 6b.

Apply B.C.'s on $\Phi(x,t)$:

$$\begin{aligned} \Phi_x(0,t) &= \psi_x(0,t) - \frac{\partial \psi_S}{\partial x}(0,t) = F(t) - F(t) = 0 \\ \Phi_x(\pi,t) &= \psi_x(\pi,t) - \frac{\partial \psi_S}{\partial x}(\pi,t) = G(t) - G(t) = 0 \end{aligned}$$

I.C: $\Phi(x,0) = \psi(x,0) - \psi_S(x,0) = f(x) - \left\{\left[\frac{G(0)-F(0)}{2\pi}\right]x^2 + F(0)x\right\} = \mathcal{F}(x,t)$

Once again. this reduces to BVP 6a. However, with Neumann B.C.s instead of Dirichlet.

With the same reasoning in BVP 6a, we let:

$$\Phi(x,t) = \sum_{n=0}^{\infty} C_n(t)\cos(nx) = C_0(t) + \sum_{n=1}^{\infty} C_n(t)\cos(nx)$$

Sub into PDE:

$\left[\frac{dC_0}{dt}\right]\cdot 1 + \sum_{n=1}^{\infty}\left[\frac{dC_n}{dt} + n^2C_n\right]\cos(nx) = \mathcal{H}(x,t)$ Apply Fourier Cosine trick

$$\therefore \frac{dC_0}{dt} = \frac{1}{\pi}\int_0^{\pi}\mathcal{H}(x,t)\,dx = B_0(t)$$

$$\therefore \boxed{C_0(t) = C_0(0) + \int_0^t B_0(\tau)\,d\tau}$$

$$\therefore \frac{dC_n}{dt} + n^2C_n = \frac{2}{\pi}\int_0^{\pi}\mathcal{H}(x,t)\cos(nx)\,dx = B_n(t)$$

Like in BVP 6a, apply an integrating factor of e^{-n^2t} to get:

$$\boxed{C_n(t) = C_n(0)e^{-n^2t} + e^{-n^2t}\int_0^t e^{n^2\tau}B_n(\tau)\,d\tau}$$

Finally, since

$$\Phi(x,0) = C_0(0) + \sum_{n=1}^{\infty} C_n(0)\cos(nx) = \mathcal{F}(x) \quad \text{(Fourier Cosine Series),}$$

we have

$$\boxed{C_0(0) = \frac{1}{\pi}\int_0^{\pi}\mathcal{F}(\xi)\,d\xi}$$

$$\boxed{C_n(0) = \frac{2}{\pi}\int_0^{\pi}\mathcal{F}(\xi)\cos(n\xi)\,d\xi}$$

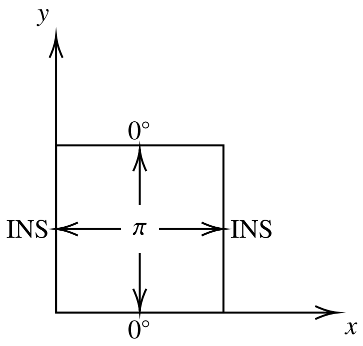
Plug and solve for $\Phi(x,t)$. Note that the use of τ and ξ as variables is just to prevent ambiguity with the t on the bounds of the integrals. Once all computations are done, all these functions are functions of time, t.

2.9 BVP 7: 3-Variable Diffusion

Analagous to BVP 3, but not steady state $\left(\frac{\partial\psi}{\partial t} \neq 0\right)$

So the temperature is $\psi(x,y,t)$ at any point $P(x,y)$

Assume $\alpha^2 = 1,\ L = \pi$ for simplicity.



$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \qquad \begin{matrix} 0 < x < \pi \\ 0 < y < \pi \end{matrix} \qquad \underbrace{\begin{matrix} \psi_x(0, y, t) = 0 \\ \psi(\pi, y, t) = 0 \end{matrix}}_{\text{2 x Neumann}} \qquad \underbrace{\begin{matrix} \psi(x, 0, t) = 0 \\ \psi(x, \pi, t) = 0 \end{matrix}}_{\text{2 x Dirichlet}} \qquad \psi(x, y, 0) = f(x, y)$$

Separation of Variables:

$$\begin{aligned} \psi(x, y, t) &= X(x)Y(y)T(t) = 0 \\ X'(0) = 0 &\quad Y(0) = 0 \\ X'(\pi) = 0 &\quad Y(\pi) = 0 \end{aligned}$$

$$\left\{ XY\frac{dT}{dt} = \frac{d^2X}{dx^2}YT + X\frac{d^2Y}{dy^2}T \right\} \cdot \frac{1}{XYT}$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

$$\frac{1}{T}\frac{dT}{dt} - \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{ constant. } = \lambda$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda, \quad X'(0) = 0,\ X'(\pi) = 0 \quad \text{BVP 1b}$$

$$\therefore X(x) = A_n \cos\left(\frac{n\pi x}{\pi}\right) = A_n \cos(nx), \quad n = 0, 1, \ldots \qquad \lambda = \frac{n^2\pi^2}{\pi^2} = n^2$$

Now we are left with:

$$\frac{1}{T} \cdot \frac{dT}{dt} - \frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda = -n^2$$

$$\therefore \frac{1}{T} \cdot \frac{dT}{dt} + n^2 = \frac{1}{Y}\frac{d^2Y}{dy^2} = \text{ constant } = \underset{\substack{\uparrow \\ \text{Different from } \lambda}}{\mu}$$

$$\frac{d^2Y}{dy^2} = Y\mu, Y(0) = 0, Y(\pi) = 0 \quad \text{BVP 1a}$$

$$Y(y) = B_k \sin(ky), k = 1, 2, 3 \ldots \quad \mu = \frac{k^2\pi^2}{\underset{\substack{\uparrow \\ \text{“H”}}}{\pi^2}} = k'^2$$

Finally, $\frac{dT}{dt} = T\left(-n^2 - \mu\right)$

$$\therefore T(t) = C_{nk} e^{-\left(n^2+k^2\right)t}$$

So the complete solution is:

$$\boxed{\psi_{nk}(x, y, t) = D_{nk} \cos(nx) \sin(ky) e^{-\left(n^2+k^2\right)t}}$$

cos because x B.C.'s are Neumann
sin because y B.C.'s are Dirichlet.

Same Fourier thing as BVP 1, but multivariable:

$$\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right] \psi(x, y, t) = 0$$

$\mathcal{L}\psi = 0$. Want nullspace of this linear operator ($\mathcal{L}$ is a linear operator, so we can use superposition)

$$\therefore \psi(x, y, t) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} D_{nk} \cos(nx) \sin(ky) e^{-\left(n^2+k^2\right)t}$$

Apply IC, extract $n = 0$ term:

$$\psi(x, y, 0) = \sum_{k=1}^{\infty} D_{0k} \sin(ky) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} D_{nk} \cos(nx) \sin(ky) = f(x, y)$$

We do the Fourier Sine Trick: (multiply by $\sin(ly)$, take double integral).

$$\begin{aligned}
&\sum_{k=1}^{\infty} D_{0k} \int_0^{\pi} \underbrace{(1)\, dx}_{\pi} \int_0^{\pi} \underbrace{\sin(ly) \sin(ky)}_{\delta_{l,k} \cdot \frac{\pi}{2}}\, dy + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} D_{nk} \int_0^{\pi} \underbrace{(1) \cos(nx)\, dx}_{0} \int_0^{\pi} \underbrace{\sin(ly) \sin(ky)\, dy}_{\delta_{l,k} \cdot \frac{\pi}{2}} \\
&= \int_0^{\pi} \int_0^{\pi} \sin(ly) f(x, y)\, dx\, dy \quad \text{only nonzero when } l = k, \\
&\frac{\pi^2}{2} D_{0l} = \int_0^{\pi} \int_0^{\pi} \sin(ly) f(x, y)\, dx\, dy.
\end{aligned}$$

$$\therefore \boxed{D_{0k} = \frac{2}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin(ly) f(x, y)\, dx\, dy}$$

Repeat with $\cos(mx)\sin(ly)$ for D_{nk} and sub in D_{0k} and D_{nk} into the general solution.

$$\begin{aligned}
&\sum_{k=1}^{\infty} D_{0k} \int_0^{\pi} \sin(ly) \sin(ky)\, dy \int_0^{\pi} \underbrace{(1) \cos(mx)\, dx}_{0} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} D_{nk} \int_0^{\pi} \underbrace{\cos(mx) \cos(nx)\, dx}_{\frac{\pi}{2}\, \delta_{m,n}} \int_0^{\pi} \underbrace{\sin(ly) \sin(ky)\, dy}_{\frac{\pi}{2}\, \delta_{l,k}} \\
&= \int_0^{\pi} \int_0^{\pi} \cos(mx) \sin(ly) f(x, y)\, dx\, dy.
\end{aligned}$$

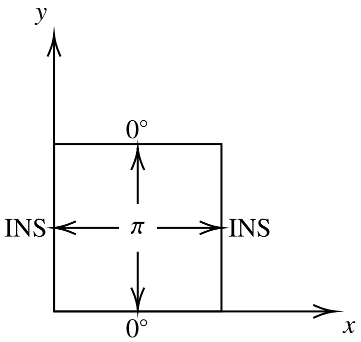
$$\therefore \boxed{D_{nk} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} \cos(nx) \sin(ky) f(x, y)\, dx\, dy}$$

2.10 BVP 8: Poisson's Equation

$$\nabla^2\psi = -F \quad \text{(Poisson's Equation)}$$

F can represent:

ψ = Velocity Potential, F = Rate of Fluid Generation.
ψ = Steady State Temperature, F = Rate of Heat Generation.
ψ = Displacement, F = External Force/ mass.
ψ = Electric Potential, F = Charge Density.
Eg. Steady-State Temp Distribution:



$$\nabla^2\psi = -F(x,y) \qquad \underbrace{\begin{matrix}\psi_x(0,y,t) = 0 \\ \psi_x(\pi,y,t) = 0\end{matrix}}_{\text{2 x Neumann}} \qquad \underbrace{\begin{matrix}\psi(x,0,t) = 0 \\ \psi(x,\pi,t) = 0\end{matrix}}_{\text{2 x Dirichlet}}$$

Unlike BVP 7, PDE is not homogeneous, but $\frac{\partial\psi}{\partial t} = 0$

Using the same principle from BVP 7:

$$\boxed{\psi(x,y) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} C_{nk}\, \underbrace{\cos(nx)}_{\text{Newman}}\, \underbrace{\sin(ky)}_{\text{Dirichlet}}}$$

This satisfies all 4 B.C.'s. Just need the PDE now.

$\nabla^2\psi(x,y) = -F(x,y)$. Plug $\psi(x,y)$ into PDE.

$$\therefore \nabla^2\psi(x,y) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} C_{nk}\left[-n^2 - k^2\right]\cos(nx)\sin(ky) = -F(x,y)$$

$$\therefore \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} C_{nk}\left[n^2 + k^2\right]\cos(nx)\sin(ky) = F(x,y) \tag{1}$$

Like in BVP 7, extract the $n = 0$ term.

$$\sum_{k=1}^{\infty} D_{0k}\sin(ky) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} D_{nk}\cos(nx)\sin(ky) = f(x,y) \quad \text{(From BVP 7)} \tag{2}$$

Equating (1) and (2) (since we know the solution to (1))

$$C_{0k}\cdot k^2 = D_{0k} = \frac{2}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(ky)\,dx\,dy$$

$$C_{nk}\left[n^2+k^2\right]=D_{nk}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi \cos(nx)\sin(ky)\,dx\,dy$$

$$\boxed{C_{0k}=\frac{2}{\pi^2k^2}\int_0^\pi\int_0^\pi \sin(ky)F(x,y)\,dx\,dy}$$

$$\boxed{C_{nk}=\frac{4}{\pi^2\left(n^2+k^2\right)}\int_0^\pi\int_0^\pi \cos(nx)\sin(ky)F(x,y)\,dx\,dy}$$

Sub into general solution.

What if B.C.'s are not homogeneous?

Let

$$\psi(x,y)=\overset{\text{Nonhomogeneous B.C. / Homogeneous PDE. (BVP 3)}}{\psi_H(x,y)}+\overset{\text{Homogeneous B.C. / Nonhomogeneous PDE. (what we just got)}}{\psi_P(x,y)}.$$

$$\nabla^2\psi(x,y)=-F(x,y)$$

We solve the two cases separately and then sum up the results. Note that if multiple B.C.s are not homogenous, they have to be addressed separately. That is, when getting $\psi_H(x,y)$, we sequentially consider all but one B.C. to be homogenous and solve, before adding up all results.

See question 4 on page 273 of the coursepack for an example. That being said, this is a pretty tough question and it is hard to think of this on the spot without knowing what to do *a priori*.