Chapter 1

Partial Differential Equations in Engineering

1.1 Fundamental Lemma of ODEs

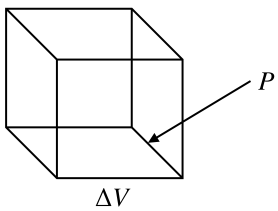
If $\iiint\limits_V f\,d\tau = 0$ and
1. f is continuous
2. V is arbitrary.

Then $f = 0$ (take as fact).

Proof by contradiction:

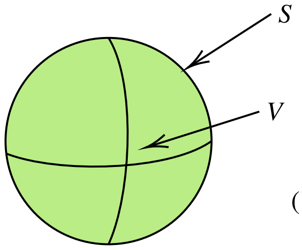
Assume $f \neq 0$ at a point P. Because f is continuous, $f \neq 0$ in a volume V surrounding P. (Assume $f > 0$ instead of $f < 0$).

Thus, $\iiint\limits_{\Delta v} f\,d\tau > 0$ (contradiction).



1.2 Fluid Flow

Let S be a closed surface bounding volume V:



(V is arbitrary)

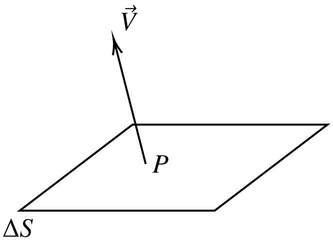
Rate of change of mass in V is $\dfrac{dM}{dt}$

$$\underset{\underset{3}{\uparrow}}{\frac{dM}{dt}} = \underset{\underset{1}{\uparrow}}{\text{Rate of entry/(exit)}} + \underset{\underset{2}{\uparrow}}{\text{Rate of generation/(absorption)}}$$

$M = \iiint_V \delta d\tau$ where δ is density.

$\therefore \dfrac{dM}{dt} = \dfrac{d}{dt}\displaystyle\iiint\limits_V \delta d\tau = \iiint\limits_V \dfrac{\partial \delta}{\partial t} d\tau$. (Leibniz integral rule: can swap derivative and integral).

(1) $-\displaystyle\oiint\limits_S \int \underset{\substack{\uparrow \\ \text{Velocity} \\ \text{vector}}}{\vec{v}} \cdot \underset{\substack{\uparrow \\ \text{unit normal} \\ \text{vector}}}{\vec{n}} dS = -\oiint\limits_S \delta\vec{v}\cdot\vec{n}\underset{\substack{\uparrow \\ \text{Surface element}}}{dS}$ (hand wavey - but $m' = \delta$).



(2) Let $Q(\vec{r}, t)$ be the rate at which fluid is generated/(absorbed). t is time, $\vec{r}$ is the position vector. Unit of Q is (ie.) $g/cc/s$.

$\therefore$ Net generation/(absorption) is:

(2) $= \iiint_V Q(\vec{r}, t) d\tau$.

Recall definition of Del operator $\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$.

By the Divergence Theorem: $\displaystyle\oiint\limits_S \vec{F}\cdot d\vec{S} = \iiint\limits_V \vec{\nabla}\cdot\vec{F} dv$

$\therefore (1) = -\displaystyle\iiint\limits_\tau \vec{\nabla}\cdot\delta\vec{v} d\tau$ integration in (1) and (2) done over same volume. (arbitrary)

$$\iiint_\tau \left[\frac{\partial \delta}{\partial t} + \vec{\nabla}\cdot(\delta\vec{v}) - Q\right]\delta\tau = \frac{dM}{dt} = 0$$

$\therefore$ By Fundamental Lemma: $\dfrac{\partial\delta}{\partial t} + \vec{\nabla}\cdot(\delta\vec{v}) - Q = 0$ "Law of Conservation of Mass"

This can be further simplified if special conditions are met.

If incompressible (δ constant, $\dfrac{\partial \delta}{\partial t} = 0$): $0 + \delta\vec{\nabla}\cdot\vec{v} = Q$. $\quad \therefore \vec{\nabla}\cdot\vec{v} = Q/\delta$

If also irrotational (no swirling $\rightarrow \vec{\nabla}\times\vec{v} = \overrightarrow{0}$)

$\therefore \vec{\nabla} = \vec{\nabla}\psi\left(\dfrac{\partial\psi}{\partial x}, \dfrac{\partial\psi}{\partial y}, \dfrac{\partial\psi}{\partial z}\right)$. (curl = 0)

* See the water flowing down a hill analogy. ψ = "scalar potential".

$\therefore \nabla^2\psi = Q/\delta.$ }"Poisson's Equation".

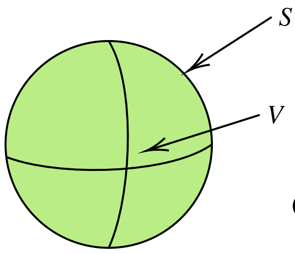
If also $Q = 0$ (no fluid generated/consumed).

$$\therefore \nabla^2\psi = 0\Big\} \text{ "Laplace's Equation" (with } \vec{v} = \vec{\nabla}\psi)$$

1.3 Diffusion of Heat

$\psi(\vec{r}, t)$ is temperature at direction vector $\vec{r}$ at time t.

Let S be a closed surface bounding volume V:



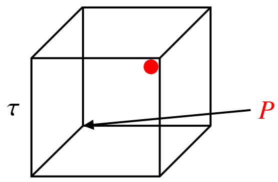
Rate of change of heat = Rate of entry/(exit) + Rate of generation/(absorption.).

Like for fluid flow:

$$(3) = (1) + (2)$$

Solving for (3).

The amount of heat in a small mass element ($\Delta m = \delta\Delta\tau$) is: $\approx [c\psi(\vec{r}, t)\delta]_p \Delta\tau$

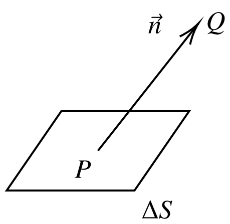


$\therefore H = \iiint\limits_\tau c\delta\psi d\tau$ differentiate both sides and apply Leibniz' Rule.

$$\therefore \frac{dH}{dt} = \iiint\limits_\tau \frac{\partial}{\partial t}[c\delta\psi]d\tau = (3)$$

Solving for (1).

Heat flows from hot to cold (entropy).



I.e. heat will leave the surface if $\psi(P) > \psi(Q)$ (along normal)

Rate at which heat leaves ΔS is:

$$\approx -\left[K\frac{\partial\psi}{\partial n}\right]_p \Delta S$$

(K = conductivity).

This means that net heat exit if $\frac{\partial\psi}{\partial n} < 0$ ($\psi(P) > \psi(Q)$). Net rate of heat entry is $\oiint K\frac{\partial\psi}{\partial n}dS = (1)$

If (1) is positive, net heat in.

Solving for (2):

Simply, if $Q(\vec{r},t)$ is rate of heat generation/(absorption) as (i.e.) cal/c.c./sec, then:

Net heat generation/(absorption) is

$$(2) = \iiint_V Q(\vec{r},t)d\tau$$

Before combining, use Divergence Theorem on (1)

$\oiint K\frac{\partial\psi}{\partial n}dS = \oiint K\nabla\vec{\psi}\cdot\vec{n}dS = \iiint_V \vec{\nabla}(K\nabla\vec{\psi})d\tau$ Triple Integral, can combine.

$$\iiint_\tau \frac{\partial}{\partial t}[c\delta\psi]d\tau = \iiint_V \vec{\nabla}(K\vec{\nabla}\psi)d\tau + \iiint_V Q(\vec{r},t)d\tau$$

Move everything to same side:

$$\underbrace{\iiint_\tau}_{\text{Arbitrary}} \underbrace{\left[\frac{\partial}{\partial t}[c\delta\psi] - \vec{\nabla}(K\vec{\nabla}\psi) - Q(\vec{r},t)\right]}_{\text{Assume continuous}} d\tau = 0.$$

By Fundamental Lemma: $\frac{\partial}{\partial t}[c\delta\psi] - \vec{\nabla}(K\vec{\nabla}\psi) - Q(\vec{r},t) = 0.$

If c, K, δ all constant: $c\delta\frac{\partial\psi}{\partial t} - K\nabla^2\psi - Q(\vec{r},t) = 0$

$\therefore \left.\frac{1}{\alpha^2}\frac{\partial\psi}{\partial t} - \nabla^2\psi = \frac{Q}{K}\right\}$ "Fourier Diffusion Equation" ($\alpha^2 = \frac{K}{\delta c}$, "diffusivity")

If $Q = 0 : \frac{1}{\alpha^2}\frac{\partial\psi}{\partial t} - \nabla^2\psi = 0 = \left[\frac{1}{\alpha^2}\frac{\partial}{\partial t} - \nabla^2\right]\psi = 0$ If in steady-state, temperature is time-invariant. $\left(\frac{\partial}{\partial t} = 0\right)$.

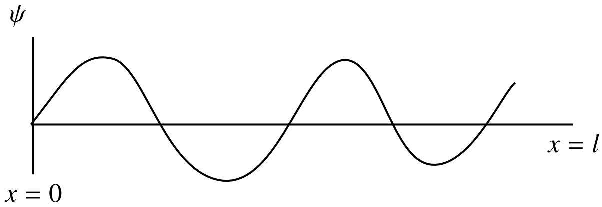
If steady-state:

$\left.\nabla^2\psi = -Q/k\right\}$ "Poisson's Equation"

If also $Q = 0$:

$\left.\nabla^2\psi = 0\right\}$ "Laplace's Equation"

1.4 Vibrating String



Displacement ψ is a function of time and position

$$a^2\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2}{\partial t^2} = -F$$

$$F = \frac{\text{Force}}{\text{Mass}}, \quad a = \sqrt{\frac{T}{\delta}}$$

T = Tension
δ = Linear Density

General solution to the PDE is: $\psi = \underbrace{F(x-at)}_{\text{moving right}} + \underbrace{G(x+at)}_{\text{moving left}}$ "with speed a"

* Derivation and solution in coursepack.

For an "infinite" string:

$$\psi(x,t) = \frac{f(x-at)+f(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)d\xi, \text{ I.C. } \begin{array}{l}\psi(x,0) = f(x)\\ \psi(t,0) = g(x)\end{array}$$

e.g $f(x) = \sin(x) \quad g(x) = xe^{-x^2}$

$$\psi(x,t) = \frac{\sin(x-at)+\sin(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \xi xe^{-\xi^2}d\xi$$
$$= \sin(x)\cos(at) - \frac{1}{4a}\left[e^{-\xi^2}\right]_{x-at}^{x+at}$$

1.5 Vibrating Membrane

$F = ma$ and external forces $\Rightarrow \oiint$

Tension forces $\Rightarrow \oint$

For static deflections: $\dfrac{\partial^2\psi}{\partial t^2} = 0$

strings (1D): $a\dfrac{\partial^2\psi}{\partial x^2} = -F$

membranes (2D): $\left.\nabla^2\psi = -F/a^2\right\}$ "Poisson's Equation"

1.6 Three Fundamental Equations

There are three significant PDEs that govern many engineering systems

1. Poisson's Equation

   $\left.\nabla^2\psi = -F\right\}$ if 0, Laplace's (special case of Poisson's).

2. Diffusion Equation

$$\nabla^2 \psi - \frac{1}{\alpha^2}\frac{\partial \psi}{\partial t} = -F$$

3. Wave Equation

$$a^2 \nabla^2 \psi - \frac{\partial^2 \psi}{\partial t^2} = -F$$

Many (but not all) PDEs are governed by one of these 3 equations.

1.7 Solving PDEs

For the homogeneous Wave equation:

$a^2 \dfrac{\partial^2 \psi}{\partial x^2} - \dfrac{\partial^2 \psi}{\partial t^2} = 0,$ we have a general solution.

$$\psi(x,t) = F(x - at) + G(x + at)$$

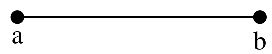
$$\text{Eg.} \quad \begin{aligned} f(z) &= z^2 = (x+iy)^2 \\ &= x^2 + 2ixy - y^2 \\ &= x^2 - y^2 + i(2xy) = u(x,y) + iv(x,y) \end{aligned}$$

$$\left.\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned}\right\} \quad \begin{aligned} \nabla^2 u &= 2 - 2 = 0 \\ \nabla^2 v &= 0 \end{aligned}$$

The two functions generates two harmonics. This is common. For a well-posed problem, we need a unique solution

1. Dirichlet B.C.

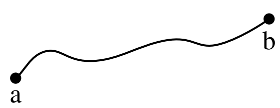
Value of u is constant with time on boundary.



$$\begin{cases} u(a,t) = 0 \\ u(b,t) = 0 \end{cases} \quad \forall t.$$

2. Neumann B.C.

Spatial derivative of u is constant with time on boundary.



$$\begin{cases} u_x(a,t) = 0 \\ u_x(b,t) = 0 \end{cases} \quad \forall t$$

3. Robin B.C.

A linear combination of Dirichelet and Neumann.

$$Au(a,t) + Bu_x(a,t) = c, \quad \forall t$$

1.8 Uniqueness Theorems

"For Poisson's Equation (or Laplace's), the solution of a Dirichlet problem is unique, and Neumann problem is unique to an additive constant."

Proof: Let ψ_1 and ψ_2 be two solutions to the same problem, i.e:

$\nabla^2\psi_1 = -F$ and $\nabla^2\psi_2 = -F\}$ Goal: Prove $\psi_1 = \psi_2$ for Dirichlet and $\psi_1 - \psi_2 = c$ for Neumann

either $[\Psi_1]_s = [\psi_2]_s$ (Dirichlet)

or $\left[\dfrac{\partial\psi_1}{\partial n}\right]_s = \left[\dfrac{\partial\psi_2}{\partial n}\right]_s$ (Neumann)

Let $u = \psi_1 - \psi_2$, then $\nabla^2 u = \nabla^2\psi_1 - \nabla^2\psi_2 = -F - (-F) = 0$ on τ.

Either $[u]_s = 0$ (Dirichlet) or.

$\left[\dfrac{\partial u}{\partial n}\right]_S = 0$ (Neumann)

for this to hold true.

Consider $\oiint_s u\dfrac{\partial u}{\partial n}dS = 0$, this is equal to $\oiint_s u\overrightarrow{\nabla u}\cdot\vec{n}dS$ (Def. of Del operator).

Applying Divergence Theorem:

$0 = \iiint_\tau \vec{\nabla}\cdot[u\overrightarrow{\nabla u}]d\tau$ Apply vector identity (Product Rule) from Tutorial 1 .

$$0 = \iiint_\tau \underbrace{\overrightarrow{\nabla u}\cdot\overrightarrow{\nabla u}}_{\|\overrightarrow{\nabla u}\|^2}d\tau + \iiint_\tau u\underbrace{\nabla^2 u}_{=0}d\tau$$

However, we may not automatically conclude $\|\overrightarrow{\nabla u}\|^2 = 0$ by the Fundamental Lemma since τ is not arbitrary. Why is it not arbitrary? We have restricted our analysis to Dirichlet or Neumann B.C.s, rather than *any* boundary conditions.

Instead, we use the limit of a sum argument for integrals because $\|\overrightarrow{\nabla u}\|^2 \geqslant 0$.

And $\iiint_\tau \underbrace{\|\overrightarrow{\nabla u}\|^2}_{\geq 0} d\tau = 0$

$\therefore \|\overrightarrow{\nabla u}\|^2 = 0$ in τ

$\therefore \overrightarrow{\nabla u} = 0 \quad\Rightarrow\quad u = \psi_1 - \psi_2 = c$ in τ.

This proves the Neumann part.

For Dirichlet, we have $[u]_s = 0$ and $[u]_\tau = c$

Since τ can be infinitesimally close to s, we can't continuously go from 0 on S to c on τ, if $c \neq 0 \therefore c = 0 \Rightarrow \psi_1 = \psi_2$, which proves the Dirichlet case.

Note on arbitrary domains:

Consider $\int_\tau xdx$. This is clearly not always zero. However, if domain $\tau = x \in [-1, 1]$, then $\int_{-1}^{1} xdx = 0$. This doesn't mean x is zero, since the domain is not arbitrary. This is the definition of "arbitrary τ" in the Fundamental Lemma. It can only be applied if the domain is arbitrary (and thus integral = 0 regardless of bounds).

“For Poisson’s Equation (or Laplace’s), the solution of a Robin problem is unique.

Proof: Let ψ_1 and ψ_2 be two solutions to the same problem, i.e:

$\nabla^2\psi_1 = -F$ and $\nabla^2\psi_2 = -F$ in τ.

$\dfrac{\partial\psi_1}{\partial n} + h\psi_1 = \dfrac{\partial\psi_2}{\partial n} + h\psi_2$ on s. the boundary of τ, with h positive.

Let $u = \psi_1 - \psi_2$, then $\nabla^2 u = 0$ in τ.

$$\frac{\partial}{\partial n}(\psi_1 - \psi_2) + h(\psi_1 - \psi_2) = 0 \text{ or } \frac{\partial u}{\partial n} + hu = 0 \text{ on } S.$$

Now, $\oiint\limits_s u\dfrac{\partial u}{\partial n}dS = -\oiint\limits_s hu^2 dS \quad \left(\frac{\partial u}{\partial n} = -hu\right)$

$$= \oiint_s u\overrightarrow{\nabla u}\cdot\vec{n}\;dS \Rightarrow \iiint_\tau \vec{\nabla}\cdot(u\overrightarrow{\nabla u})d\tau \text{ (by the Divergence Theorem)}$$

$$\therefore -\oiint_S h\,u^2\,dS = \iiint_\tau \nabla\cdot(u\,\nabla u)\,d\tau = \iiint_\tau \underbrace{\nabla u\cdot\nabla u}_{\|\nabla u\|^2\geqslant 0}\,d\tau. + \cancel{\iiint_\tau u\,\nabla^2 u\,d\tau} \text{ (Poisson problem: } \nabla^2 u = 0)$$

Only way for L.H.S. = R.H.S. is if both are 0.

The gist of this proof is that the L.H.S. $\leqslant 0$ and the R.H.S. is $\geqslant 0$. The only way for this to be possible is if L.H.S = R.H.S. = 0.

$\iiint\limits_\tau \|\overrightarrow{\nabla u}\|^2 d\tau = 0 \Rightarrow u = \text{constant in } \tau$.

$$\oiint_S \underbrace{hu^2}_{\geqslant 0}\,dS = 0 \Rightarrow u = 0 \text{ on } S.$$

Hence by continuity, $u = \psi_1 - \psi_2 = 0$, thus $\psi_1 = \psi_2$.

Theorem 3: For the diffusion equation, solutions of Dirichlet and Neumann problems are unique if $\psi(\vec{r}, 0)$ is specified.

Theorem 4: For the wave equation, solutions of Dirichlet and Neumann problems are unique if $\psi(\vec{r}, 0)$ and $\psi_+(\vec{r}, 0)$ is specified.

(Proofs not needed).