Inner Product Spaces

For a vector space V, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following four postulates (applicable to real vector spaces):

1. $\langle x, y \rangle = \langle y, x \rangle$ (Symmetry)
2. $\langle cx, y \rangle = c\langle x, y \rangle$ (Linearity in the first argument)
3. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$ (Additivity)
4. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$ (Positivity)

An inner product space is not unique. A vector space may have any amount of inner product spaces as long as the four postulates are met.

Complex Vector Spaces

For complex vector spaces, the first postulate changes to:

$$\langle x, y \rangle = \overline{\langle y, x \rangle},$$

ensuring that the inner product remains real-valued when evaluated. The other postulates remain similar, with linearity applying only in the first argument to maintain consistency.

Examples

- For $\mathbb{R}^2$ or $\mathbb{R}^3$, an example of an inner product is:

  $$\langle x, y \rangle = \|x\| \|y\| \cos(\theta),$$

  where θ is the angle between vectors x and y. Another example is:

  $$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

  The dot product is an inner product in $\mathbb{R}^n$.

- For integrable functions $f(x)$ and $g(x)$ over an interval $[a, b]$, the inner product is defined as:

  $$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

Integral Example

Consider $f(x) = e^{3x}(\cos(3x) + i \sin(3x))$ and $g(x) = e^{-x}(2\cos(5x) - i \sin(5x))$. The inner product over a suitable interval is:

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$

The norm squared:

$$\langle f, f \rangle = \int_a^b f(x)^2 \, dx,$$

is always positive, and zero only when $f(x) = 0$, satisfying the fourth postulate.