Chapter 4

Linear Algebra

4.1 Linear Dependence and Independence

Let $\bar{V}_1, \bar{V}_2, \bar{V}_3, \ldots, \bar{V}_n$ be nonzero elements in vector space V and $C_1, C_2, C_3, \ldots, C_n \in \mathbb{R}$

If the only solution to $C_1\bar{V}_1 + C_2\bar{V}_2 + C_3\bar{V}_3 + \ldots + C_n\bar{V}_n = 0$ is the trivial solution ($C_1 = C_2 = \ldots = C_n = 0$), the set of $\bar{V}_1, \ldots, \bar{V}_n$ is linearly independent. Otherwise they are linearly dependent.

4.2 Wronskian

If $W[f_1, f_2, f_3, \ldots, f_n]$ ever $\neq 0$, the set is linearly independent. If $W[f_1, f_2, f_3, \ldots, f_n] = 0$ always and the set af functions are all analytic (can be expressed as a convergent power series on the domain of interest), the set is linearly dependent.

NB: $f_1, \ldots, f_n$ must be continuous and differentiable on $[a, b]$.

4.3 Dimension and Basis

Dimension: The dimension of vector space V is the cardinality of the basis of V. (number of basis vectors).

Corollary: Any set of vectors with cardinality > $\dim(V)$ is linearly dependent.

Corollary 2: The maximum number of linearly independent vectors is $\dim(V)$.

Basis: A set of linearly independent vectors that span V is a basis of V.

Corollary: All elements in V can be expressed as a linear combination of the elements of a basis of V. (Unique to the basis).

i.e. $\{\hat{\imath}, \hat{\jmath}\}$ is a basis in $\mathbb{R}^2$. So is any pair of non colinear vectors.
$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ is a basis in $\mathbb{R}^3$. So are any 3 non coplanar vectors.

4.4 Linear Operators

τ is an operator on V if $\tau\bar{x} \in V$ where $\bar{X} \in V$.

If for any $\bar{x} \in V, c \in \mathbb{R}$.

$$\tau(\bar{x}_1 + \bar{x}_2) = \tau(\bar{x}_1) + \tau(\bar{x}_2) \text{ and } \tau(c\bar{x}) = c\tau(\bar{x}).$$

τ is a linear operator and the image of $\tau : V \to V$ forms a subspace.

Eg. The operator $\tau = \dfrac{d}{dx}$ is a linear operator on the vector space V of polynomials of degree n or less.

NB: So is the operator $\tau = \int^x dt$, but its image is not confined in V

$$
\begin{aligned}
\tau &= \frac{d}{dx}, \tau : V \to V \\
\tau &= \int^x dx, \tau : V \to W,\ W = p_{n+1}(x),\ V = p_n(x)
\end{aligned}
$$

4.5 Null Spaces

The null space of a linear operator $\mathcal{L}$ on V is the set of elements $v \in V$ s.t. $\mathcal{L}v = \mathbb{O}$.

I.e. The null space of $\mathcal{L} = \dfrac{d}{dx}$ is $v = c,\ c \in \mathbb{R}$.

Obviously, since $\dfrac{d}{dx}c = \mathcal{L}c = 0$. The derivative of a constant is zero.

4.6 Gram-Schmidt Process

By definition, $\langle X \mid Y \rangle = |X||Y| \cos(X, Y)$

And, $\langle X \mid X \rangle = |X|^2, \quad \therefore |X| = \sqrt{\langle X \mid X \rangle}$

The distance between two elements (ie. functions) is:

$$\text{dist}(f, g) = |f - g|.$$

The Gram-Schmidt process turns a basis into an orthogonal basis.

Given a basis $\{v_1, v_2, v_3, \ldots, v_n\} \xrightarrow{G-S} \{u_1, u_2, u_3, \ldots, u_n\}$

Orthogonality is defined as $\langle u_n, u_k \rangle = 0$ if $n \neq k$. The Gram-Schmidt algorithm is as follows:

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \\
u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2
\end{aligned}
$$

Essentially it subtracts the non-orthogonal components of the vectors away. Note that the G-S process does not normalize the basis. That has to be done separately.

To normalize, $\varnothing_1 = \dfrac{u_1}{\sqrt{\langle u_1, u_1 \rangle}}$ for all u_n.

To get the "best approximation" of an element f outside a vector space V (ie. x^4 in $p_3(x)$):

$f \approx \sum_{k=1}^{n} a_k \phi_k \quad$ where $a_k = \langle \phi_k, f \rangle$

NOTE: $\varnothing$ must be an orthonormal basis, not just orthogonal.

I.e. Estimate τ^4 with $\varnothing = \left\{\frac{1}{\sqrt{2}},\ \sqrt{\frac{3}{2}}\tau,\ \sqrt{\frac{5}{8}}\left(3\tau^2 - 1\right),\ \sqrt{\frac{7}{8}}\left(5\tau^3 - 3\tau\right)\right\}$

$a_1 = \left\langle \frac{1}{\sqrt{2}}, \tau^4 \right\rangle$

$a_2 = \left\langle \sqrt{\frac{3}{2}}\tau, \tau^4 \right\rangle$, and so on for a_3, a_4.

4.7 Eigenvalues/Eigenfunctions of Differential Equations.

For the purposes of this course, consider a general solution like $\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2}{L^2}\cdot t}$ BVP 1-A

The eigenvalue will be $\frac{n^2\pi^2}{L^2}$, the eigenfunction will be $\sin(\sqrt{\lambda}x)e^{-\lambda t}$.

Mathematically, it is in the form:

$$\mathcal{L}[f(x)] = \lambda f(x)$$

Where $\mathcal{L}$ is a linear operator (i.e. derivative), λ is a scalar eigenvalue, $f(x)$ is the eigenfunction, maintaining shape under $\mathcal{L}$.

I.e. $\mathcal{L} = \frac{d^2}{dx^2}$

$$\therefore \frac{d^2 f(x)}{dx^2} = \lambda f(x)$$

As always, there are 3 cases ($\lambda > 0,\ \lambda = 0,\ \lambda < 0$)

The only oscillatory solution is $\underbrace{\lambda = -k^2}_{\text{eigenvalues}}$

$$\therefore f''(x) = -k^2 f(x)$$

$$f(x) = A \underbrace{\cos(kx)}_{\text{eigenfunction}} + B \underbrace{\sin(kx)}_{\text{eigenfunction}}.$$

The specific values of λ, A, B depend on the B.C.s.

I.e. with homogeneous Dirichlet conditions on $x \in [0, L]$ yields

$$k = \frac{n\pi}{L} \Rightarrow \lambda = -\frac{n^2\pi^2}{L^2}$$
$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

4.8 Hermitian Operators

A Hermitian operator $\mathcal{L}$ over an inner product is a linear operator with the property: $\langle X, \mathcal{L}[Y]\rangle = \langle \mathcal{L}[Y], X\rangle$

The inner product for functions is defined (with real values) by

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx,$$

where $w(x)$ is a given weight function.

If $\mathcal{L}$ is a Hermitian operator, then for all functions $f(x)$ and $g(x)$ in the domain of $\mathcal{L}$, the following equality holds:

$$\int_a^b f(x)\,\mathcal{L}[g](x)\,dx = \int_a^b \mathcal{L}[f](x)\,g(x)\,dx.$$

NB, this is a tough criterion, most operators are not Hermitian.

Hermitian operators are obviously not absolute. If the inner product is defined differently, a operator may cease to be Hermitian.

Hermitian operators have the following properties:

1. **The eigenvalues of a Hermitian operator are real.**
   Let $\mathcal{L}$ be a Hermitian operator on an inner product space and let f be an eigenvector of $\mathcal{L}$ with eigenvalue λ, i.e.,
   $$\mathcal{L}[f] = \lambda f.$$
   Then, using the properties of the inner product, we have
   $$\langle f, \mathcal{L}[f]\rangle = \langle f, \lambda f\rangle = \lambda\langle f, f\rangle.$$
   Since $\mathcal{L}$ is Hermitian,
   $$\langle f, \mathcal{L}[f]\rangle = \langle \mathcal{L}[f], f\rangle = \langle \lambda f, f\rangle = \overline{\lambda}\langle f, f\rangle.$$
   Equating the two expressions and noting that $\langle f, f\rangle \neq 0$ (since f is nonzero), we conclude
   $$\lambda = \overline{\lambda},$$
   which implies that λ is real, as only real numbers are equal to their complex conjugate.

2. **The eigenvectors of a Hermitian operator corresponding to different eigenvalues are orthogonal.**
   Let f and g be eigenvectors of the Hermitian operator $\mathcal{L}$ corresponding to distinct eigenvalues λ and μ respectively:
   $$\mathcal{L}[f] = \lambda f \quad \text{and} \quad \mathcal{L}[g] = \mu g.$$
   Consider the inner product $\langle f, \mathcal{L}[g]\rangle$. On one hand, using the eigenvalue equation for g,
   $$\langle f, \mathcal{L}[g]\rangle = \langle f, \mu g\rangle = \mu\langle f, g\rangle.$$
   On the other hand, using the Hermitian property of $\mathcal{L}$ and the eigenvalue equation for f,
   $$\langle f, \mathcal{L}[g]\rangle = \langle \mathcal{L}[f], g\rangle = \langle \lambda f, g\rangle = \lambda\langle f, g\rangle.$$
   Equating these two expressions gives
   $$\lambda\langle f, g\rangle = \mu\langle f, g\rangle.$$
   Since $\lambda \neq \mu$, it follows that
   $$\langle f, g\rangle = 0,$$
   meaning that f and g are orthogonal.

An operator being Hermitian may depend on the boundary conditions imposed on the functions. For example, consider the operator

$$\mathcal{L} = \frac{d^2}{dx^2}.$$

For $\mathcal{L}$ to be Hermitian, the boundary term that arises from integration by parts must vanish. To see this, let f and g be functions defined on $[0, L]$ and consider

$$\begin{aligned}
\langle \mathcal{L}[f], g\rangle &= \int_0^L \frac{d^2 f}{dx^2}\,g(x)\,dx \\
&= \left[f'(x)g(x) - f(x)g'(x)\right]_0^L + \int_0^L f(x)\frac{d^2 g}{dx^2}\,dx \quad \text{(by integration by parts)} \\
&= \text{B.T.} + \langle f, \mathcal{L}[g]\rangle,
\end{aligned}$$

where the boundary term (B.T.) is given by

$$\text{B.T.} = [f'(x)g(x) - f(x)g'(x)]_0^L .$$

Thus, for $\mathcal{L}$ to be Hermitian (i.e., for

$$\langle \mathcal{L}[f], g\rangle = \langle f, \mathcal{L}[g]\rangle \quad \text{for all } f, g),$$

the boundary term must vanish:

$$[f'(x)g(x) - f(x)g'(x)]_0^L = 0.$$

This condition depends on the specific boundary conditions imposed on the functions.

4.9 Parseval's Theorem

Optional. There is more to be covered but it gets pretty advanced and not too useful for people without a real analysis or advanced algebra background.

For $f(x)$ piecewise smooth on $(0, L)$:

$$f(x) = \sum_{n=1}^{\infty} b_n g_n$$

$$\langle f(x) \mid f(x)\rangle = \left\langle f(x) \mid \sum_{n=1}^{\infty} b_n g_n \right\rangle = \sum_{n=1}^{\infty} b_n \langle f(x) \mid g_n\rangle = \sum_{n=1}^{\infty} b_n \cdot \frac{L\overline{b_n}}{2} = \frac{L}{2}\sum_{n=1}^{\infty} |b_n|^2$$

If $\langle g_n \mid f(x)\rangle = \frac{L}{2} b_n$, then $\langle f(x) \mid g_n\rangle = \frac{L}{2}\overline{b_n}$

$$\therefore \frac{2}{L}\langle f(x) \mid f(x)\rangle = \sum_{n=1}^{\infty} |b_n|^2$$

$$\therefore \sum_{n=1}^{\infty} |b_n|^2 = \frac{2}{L}\int_0^L |f(x)|^2 dx \quad \text{Fourier transform is unitary.}$$