MECH 309: Assignment 1, Question 4

Cagri Arslan

January 21, 2026

Consider

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Basis for $\mathcal{N}(A)$.
Let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$. Solving $Ax = 0$ gives

$$\begin{cases} x_1 + 2x_3 + 3x_4 = 0, \\ x_2 + 4x_3 + 5x_4 = 0. \end{cases}$$

Let $x_3 = s$ and $x_4 = t$. Then $x_1 = -2s - 3t$ and $x_2 = -4s - 5t$, so

$$x = s \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

A basis is

$$\boxed{\mathcal{N}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

(b) Basis for $\mathcal{R}(A)$ in terms of the columns of A.
Let the columns of A be

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \quad a_4 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}.$$

The pivot columns are the first two, hence a basis for $\mathcal{R}(A)$ is

$$\boxed{\mathcal{R}(A) = \operatorname{span}\{a_1, a_2\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.}$$

(Indeed $a_3 = 2a_1 + 4a_2$ and $a_4 = 3a_1 + 5a_2$.)

(c) Nullity of A.
From part (a), $\dim(\mathcal{N}(A)) = 2$, so

$$\boxed{\operatorname{nullity}(A) = 2.}$$

(d) Rank of A.
From part (b), $\dim(\mathcal{R}(A)) = 2$ (two pivot columns), so

$$\boxed{\operatorname{rank}(A) = 2.}$$

(e) Verify Rank–Nullity.
Since $A : \mathbb{R}^4 \to \mathbb{R}^3$, the Rank–Nullity theorem states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = 4.$$

Using parts (c) and (d): $2 + 2 = 4$, so it holds.

(f) Basis for $\mathcal{N}(A^T)$.
Compute

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}.$$

Let $y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$. Solving $A^T y = 0$ gives

$$y_1 = 0, \qquad y_2 = 0,$$

and y_3 is free. Therefore

$$\boxed{\mathcal{N}(A^T) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.}$$

(g) Basis for $\mathcal{R}(A^T)$ in terms of the columns of A^T.
The columns of A^T are

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus a basis is

$$\boxed{\mathcal{R}(A^T) = \operatorname{span}\{c_1, c_2\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 5 \end{bmatrix} \right\}.}$$

(h) Verify: any $x \in \mathcal{N}(A)$ and any $q \in \mathcal{R}(A^T)$ are orthogonal.
If $q \in \mathcal{R}(A^T)$, then $q = A^T y$ for some y. If $x \in \mathcal{N}(A)$, then $Ax = 0$. Hence

$$x \cdot q = x^T q = x^T (A^T y) = (Ax)^T y = 0^T y = 0,$$

so x and q are orthogonal.

(i) Verify: any $y \in \mathcal{N}(A^T)$ and any $z \in \mathcal{R}(A)$ are orthogonal.
If $z \in \mathcal{R}(A)$, then $z = Ax$ for some x. If $y \in \mathcal{N}(A^T)$, then $A^T y = 0$. Hence

$$y \cdot z = y^T z = y^T (Ax) = (A^T y)^T x = 0^T x = 0,$$

so y and z are orthogonal.

(j) Strang Diagram.
See attached.

(k) Solve $Ax = b_j$ for $j = 1, 2, 3$.
Let

$$b_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad b_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}.$$

Writing $Ax = b$ as equations:

$$x_1 + 2x_3 + 3x_4 = b^{(1)}, \qquad x_2 + 4x_3 + 5x_4 = b^{(2)}, \qquad 0 = b^{(3)}.$$

So a solution exists if and only if the third component of b is 0.

k.1. $b_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$:

Existence: Yes, since $b_1^{(3)} = 0$.
Uniqueness: Not unique, because $\text{nullity}(A) = 2$ (there are free variables).
One solution: take $x_3 = 0,\ x_4 = 0$. Then $x_1 = 1,\ x_2 = -2$, so

$$\boxed{x = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}}.$$

k.2. $b_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$:

Existence: No, since the third equation forces $0 = b_2^{(3)}$, i.e. $0 = 2$, which is impossible.

$$\boxed{\text{No solution (inconsistent due to the zero third row of } A\text{).}}$$

k.3. $b_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$:

Existence: Yes, since $b_3^{(3)} = 0$.
Uniqueness: Not unique, because $\text{nullity}(A) = 2$.
One solution: take $x_3 = 0,\ x_4 = 0$. Then $x_1 = 3,\ x_2 = 5$, so

$$\boxed{x = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 0 \end{bmatrix}}.$$