MECH 309: Assignment 1, Question 1

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January 29, 2026

Question 1(a)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The induced matrix norm (induced by the vector norm $\|\cdot\|$) is

$$\|\mathbf{A}\| := \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

We show that

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|.$$

Proof

Take any $\mathbf{x} \neq \mathbf{0}$ and define

$$\mathbf{u} := \frac{\mathbf{x}}{\|\mathbf{x}\|}.$$

Then $\|\mathbf{u}\| = 1$ and $\mathbf{x} = \|\mathbf{x}\|\mathbf{u}$. Using homogeneity of the vector norm,

$$\|\mathbf{A}\mathbf{x}\| = \|\mathbf{A}(\|\mathbf{x}\|\mathbf{u})\| = \|\|\mathbf{x}\|(\mathbf{A}\mathbf{u})\| = \|\mathbf{x}\|\,\|\mathbf{A}\mathbf{u}\|.$$

Hence

$$\frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x}\|\,\|\mathbf{A}\mathbf{u}\|}{\|\mathbf{x}\|} = \|\mathbf{A}\mathbf{u}\| = \left\|\mathbf{A}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\right\|.$$

As $\mathbf{x}$ ranges over all nonzero vectors, $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|$ ranges over all unit vectors. Therefore,

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} = \max_{\|\mathbf{u}\|=1} \|\mathbf{A}\mathbf{u}\|.$$

Renaming the dummy variable $\mathbf{u}$ as $\mathbf{x}$ gives

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|.$$

$\square$

Question 1(b)

Assume now that $\mathbf{A}$ is nonsingular (invertible). We show that

$$\|\mathbf{A}^{-1}\| = \frac{1}{\min_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|}.$$

Proof

Start from the induced norm definition:

$$\|\mathbf{A}^{-1}\| = \max_{\mathbf{y}\neq\mathbf{0}} \frac{\|\mathbf{A}^{-1}\mathbf{y}\|}{\|\mathbf{y}\|}.$$

Since $\mathbf{A}$ is invertible, the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is bijective (equivalently, injective and surjective). In particular, injectivity implies

$$\ker(T) = \{\mathbf{0}\},$$

so $\mathbf{A}\mathbf{x} = \mathbf{0}$ holds if and only if $\mathbf{x} = \mathbf{0}$. Said differently: the *only* vector sent to $\mathbf{0}$ is $\mathbf{0}$ itself, and every nonzero $\mathbf{x}$ must be sent to a nonzero $\mathbf{y} = \mathbf{A}\mathbf{x}$. Therefore, restricting T to nonzero vectors gives a bijection

$$T : \{\mathbf{x}\neq\mathbf{0}\} \longrightarrow \{\mathbf{y}\neq\mathbf{0}\}.$$

Consequently, maximizing over all $\mathbf{y}\neq\mathbf{0}$ is equivalent to maximizing over all $\mathbf{x}\neq\mathbf{0}$ after defining $\mathbf{y} = \mathbf{A}\mathbf{x}$:

$$\|\mathbf{A}^{-1}\| = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}^{-1}(\mathbf{A}\mathbf{x})\|}{\|\mathbf{A}\mathbf{x}\|} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{x}\|}{\|\mathbf{A}\mathbf{x}\|}.$$

Now write $\mathbf{x} = \|\mathbf{x}\|\mathbf{u}$ where $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|$ and $\|\mathbf{u}\| = 1$. Then

$$\frac{\|\mathbf{x}\|}{\|\mathbf{A}\mathbf{x}\|} = \frac{\|\mathbf{x}\|}{\|\mathbf{A}(\|\mathbf{x}\|\mathbf{u})\|} = \frac{\|\mathbf{x}\|}{\|\|\mathbf{x}\|(\mathbf{A}\mathbf{u})\|} = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|\,\|\mathbf{A}\mathbf{u}\|} = \frac{1}{\|\mathbf{A}\mathbf{u}\|}.$$

Therefore,

$$\|\mathbf{A}^{-1}\| = \max_{\|\mathbf{u}\|=1} \frac{1}{\|\mathbf{A}\mathbf{u}\|}.$$

For a positive quantity, the reciprocal reverses inequalities, so the largest value of $1/\|\mathbf{A}\mathbf{u}\|$ occurs when $\|\mathbf{A}\mathbf{u}\|$ is minimized. Thus,

$$\max_{\|\mathbf{u}\|=1} \frac{1}{\|\mathbf{A}\mathbf{u}\|} = \frac{1}{\min_{\|\mathbf{u}\|=1}\|\mathbf{A}\mathbf{u}\|}.$$

Hence

$$\|\mathbf{A}^{-1}\| = \frac{1}{\min_{\|\mathbf{u}\|=1}\|\mathbf{A}\mathbf{u}\|}.$$

Renaming the dummy variable $\mathbf{u}$ as $\mathbf{x}$ yields the desired result:

$$\boxed{\|\mathbf{A}^{-1}\| = \frac{1}{\min_{\|\mathbf{x}\|=1}\|\mathbf{A}\mathbf{x}\|}.}$$

$\square$

Question 1(c)

- **Meaning of $\|\mathbf{A}\|$.** Using part (a),

$$\|\mathbf{A}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|.$$

Thus $\|\mathbf{A}\|$ is the *maximum stretch factor* of $\mathbf{A}$: among all unit vectors, it measures how much $\mathbf{A}$ can enlarge (stretch) the most "responsive" direction. In other words, there is some unit vector whose length is increased by a factor of $\|\mathbf{A}\|$, and no unit vector can be stretched by a larger factor than this.

- **Meaning of $\|\mathbf{A}^{-1}\|$.** From part (b),

$$\|\mathbf{A}^{-1}\| = \frac{1}{\min_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|}.$$

So $\|\mathbf{A}^{-1}\|$ grows like the reciprocal of the *minimum* stretch of $\mathbf{A}$. Equivalently, it reflects how strongly $\mathbf{A}$ can *compress* some unit vector: if $\|\mathbf{A}^{-1}\|$ is large, then the minimum value of $\|\mathbf{A}\mathbf{x}\|$ over unit vectors is small, meaning there is a particularly "accommodating" direction that $\mathbf{A}$ shrinks by a large factor.