MECH 309: Assignment 2, Question 2

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Question 2(a)

Consider the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is square and full rank. Let $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ be the true solution, and let $\hat{\mathbf{x}}$ be a numerical approximation. Define the output error

$$\delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x},$$

and the residual

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}.$$

We show that $\|\mathbf{r}\| = 0 \implies \|\delta\mathbf{x}\| = 0$.

Proof

If $\|\mathbf{r}\| = 0$, then (by positive definiteness of the norm) $\mathbf{r} = \mathbf{0}$. Using $\mathbf{b} = \mathbf{A}\mathbf{x}$,

$$\mathbf{0} = \mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) = -\mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}) = -\mathbf{A}\,\delta\mathbf{x}.$$

Thus $\mathbf{A}\,\delta\mathbf{x} = \mathbf{0}$, i.e. $\delta\mathbf{x} \in \ker(\mathbf{A})$. Since $\mathbf{A}$ is full rank (hence invertible), $\ker(\mathbf{A}) = \{\mathbf{0}\}$, so $\delta\mathbf{x} = \mathbf{0}$ and therefore $\|\delta\mathbf{x}\| = 0$. $\square$

Question 2(b)

Does multiplying both $\mathbf{A}$ and $\mathbf{b}$ by $\alpha \in \mathbb{R}$ change the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$?

Answer: No (for $\alpha \neq 0$).

Let $\mathbf{A}_\alpha = \alpha\mathbf{A}$ and $\mathbf{b}_\alpha = \alpha\mathbf{b}$. The scaled system is

$$\mathbf{A}_\alpha\mathbf{x} = \mathbf{b}_\alpha \quad \iff \quad (\alpha\mathbf{A})\mathbf{x} = \alpha\mathbf{b}.$$

If $\alpha \neq 0$, dividing both sides by α gives $\mathbf{A}\mathbf{x} = \mathbf{b}$, so the solution is unchanged:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

(If $\alpha = 0$, the system becomes $\mathbf{0}\mathbf{x} = \mathbf{0}$, which no longer has a unique solution.)

Question 2(c)

Does multiplying both $\mathbf{A}$ and $\mathbf{b}$ by $\alpha \in \mathbb{R}$ change the residual? In particular, does $\|\mathbf{r}\|$ change?

Answer: Yes (it scales by $|\alpha|$).

Define the scaled residual using the suggested notation:

$$\mathbf{r}_\alpha = \mathbf{b}_\alpha - \mathbf{A}_\alpha \hat{\mathbf{x}} = \alpha \mathbf{b} - \alpha \mathbf{A} \hat{\mathbf{x}} = \alpha(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = \alpha \mathbf{r}.$$

Taking norms and using absolute homogeneity of norms,

$$\|\mathbf{r}_\alpha\| = \|\alpha \mathbf{r}\| = |\alpha|\,\|\mathbf{r}\|.$$

Thus, scaling $\mathbf{A}$ and $\mathbf{b}$ by α generally changes the residual norm (unless $\alpha = 1$ or $\mathbf{r} = \mathbf{0}$).

Question 2(d)

Is a small $\|\mathbf{r}\|$ a good measure of solution accuracy? Equivalently, does small $\|\mathbf{r}\|$ imply $\hat{\mathbf{x}}$ is accurate?

Answer: No.

One reason is that $\|\mathbf{r}\|$ is *scale-dependent*. From part (c),

$$\|\mathbf{r}_\alpha\| = |\alpha|\,\|\mathbf{r}\|.$$

So by choosing $|\alpha|$ very small, the residual norm can be made arbitrarily small even though $\hat{\mathbf{x}}$ (and hence the error $\delta\mathbf{x}$) has not changed at all. Therefore, small $\|\mathbf{r}\|$ alone is not a reliable measure of accuracy.

A second reason (formalized in part (e)) is that even when $\|\mathbf{r}\|$ is small for a fixed system, an ill-conditioned matrix $\mathbf{A}$ can amplify that residual into a large solution error.

Question 2(e)

Show that

$$\frac{\|\delta\mathbf{x}\|}{\|\hat{\mathbf{x}}\|} \le \operatorname{cond}(\mathbf{A}) \frac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|},$$

where the matrix norm is induced by the vector norm, and $\operatorname{cond}(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$.

Proof

Using $\mathbf{b} = \mathbf{A}\mathbf{x}$,

$$\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) = -\mathbf{A}\,\delta\mathbf{x}.$$

Multiplying by $-\mathbf{A}^{-1}$ gives

$$\delta\mathbf{x} = -\mathbf{A}^{-1}\mathbf{r}.$$

Taking norms and using submultiplicativity of induced norms,

$$\|\delta\mathbf{x}\| = \|\mathbf{A}^{-1}\mathbf{r}\| \le \|\mathbf{A}^{-1}\|\,\|\mathbf{r}\|.$$

Divide by $\|\hat{\mathbf{x}}\|$ and multiply by $\|\mathbf{A}\|/\|\mathbf{A}\|$:

$$\frac{\|\delta\mathbf{x}\|}{\|\hat{\mathbf{x}}\|} \le \frac{\|\mathbf{A}^{-1}\|\,\|\mathbf{r}\|}{\|\hat{\mathbf{x}}\|} = \left(\|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|\right) \frac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|} = \operatorname{cond}(\mathbf{A}) \frac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|}.$$

$\square$

Question 2(f)

Does multiplying both $\mathbf{A}$ and $\mathbf{b}$ by $\alpha \in \mathbb{R}$ change the *relative residual* $\dfrac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|}$?

Answer: No (for $\alpha \neq 0$).

From part (c), $\mathbf{r}_\alpha = \alpha \mathbf{r}$, so $\|\mathbf{r}_\alpha\| = |\alpha|\,\|\mathbf{r}\|$. Also,

$$\|\mathbf{A}_\alpha\| = \|\alpha \mathbf{A}\| = |\alpha|\,\|\mathbf{A}\|.$$

Therefore,

$$\frac{\|\mathbf{r}_\alpha\|}{\|\mathbf{A}_\alpha\|\,\|\hat{\mathbf{x}}\|} = \frac{|\alpha|\,\|\mathbf{r}\|}{|\alpha|\,\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|} = \frac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|}.$$

(If $\alpha = 0$, then $\|\mathbf{A}_\alpha\| = 0$ and the expression is undefined.)

Question 2(g)

Is a small relative residual along with a well-conditioned problem a good measure of solution accuracy? Equivalently, does small relative residual and well-conditioned $\mathbf{A}$ imply $\hat{\mathbf{x}}$ is accurate?

Answer: Yes, in the sense guaranteed by the bound in (e).

From part (e),

$$\frac{\|\delta \mathbf{x}\|}{\|\hat{\mathbf{x}}\|} \leq \operatorname{cond}(\mathbf{A}) \quad \underbrace{\frac{\|\mathbf{r}\|}{\|\mathbf{A}\|\,\|\hat{\mathbf{x}}\|}}_{\text{relative residual}} \quad .$$

Thus, if $\mathbf{A}$ is well-conditioned (so $\operatorname{cond}(\mathbf{A})$ is not large) and the relative residual is small, then the right-hand side is small, forcing $\|\delta \mathbf{x}\|/\|\hat{\mathbf{x}}\|$ to be small. In that sense, a small relative residual together with a well-conditioned matrix implies $\hat{\mathbf{x}}$ is close to the true solution.