Fourier, filtering, smoothing, and noise

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(with thanks to David Forsyth)

Plan for today

- we have talked about some theoretic of 2D DSP
- today we will see examples with real images
- we will:
  - talk about the discrete-space Fourier transform
  - sampling in 2D
  - visualize concepts such as frequency spectra and aliasing
  - see that very important vision operations are really just filtering, with some post processing
  - introduce some important filters that are widely used in practice
  - talk about the impact of noise and how to deal with it

The Discrete-Space Fourier Transform

- is, once again, a straightforward extension of the 1D Discrete-Time Fourier Transform

$$X(\omega_1,\omega_2)=\sum_{n_1}\sum_{n_2}x[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}$$

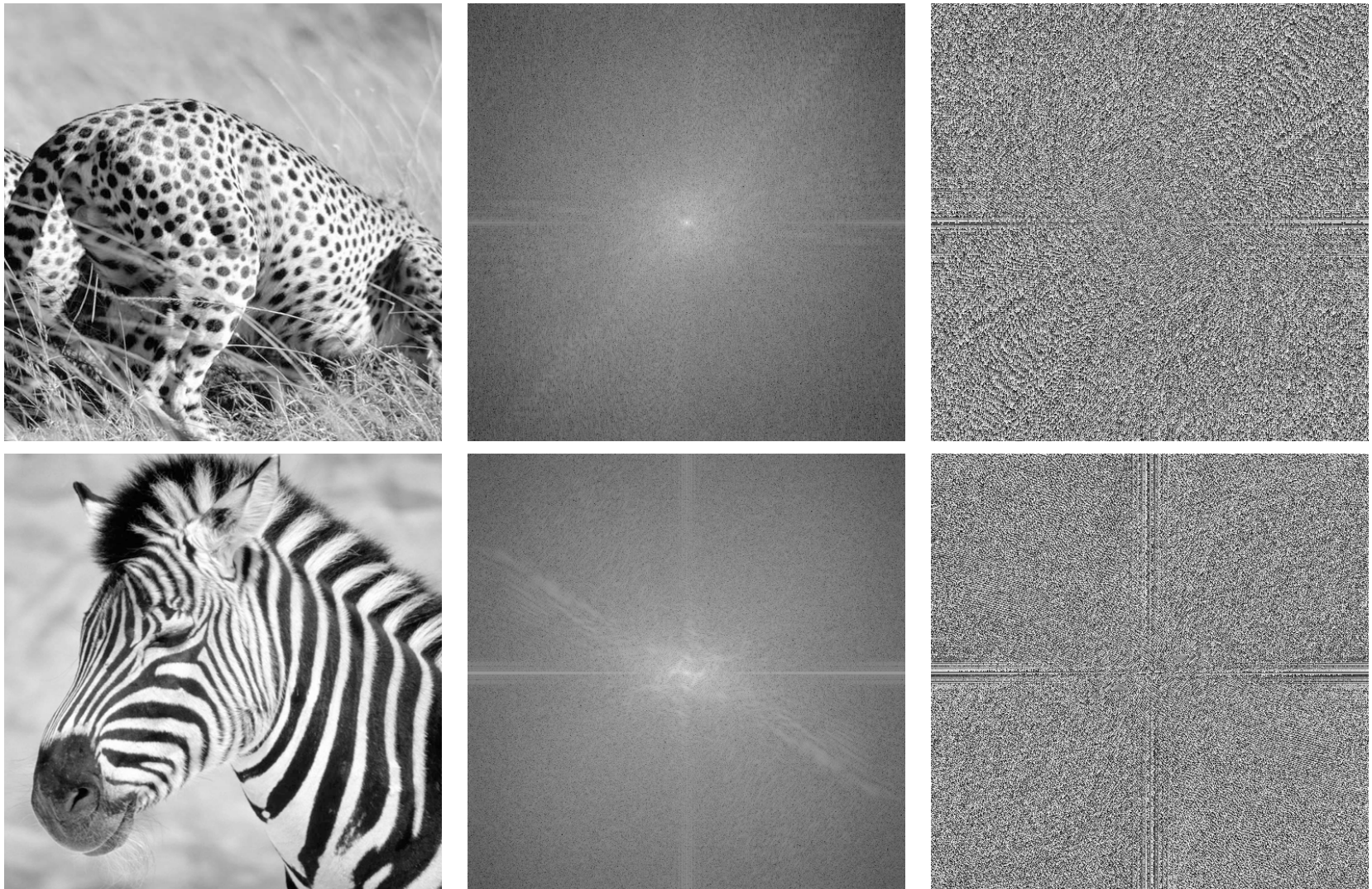
$$x[n_1,n_2]=\frac{1}{(2\pi)^2}\iint X(\omega_1,\omega_2)e^{j\omega_1 n_1}e^{j\omega_2 n_2}d\omega_1 d\omega_2$$

- properties:
  - basically the same as in 1D (see table in Lim, page 25)
  - only novelty is separability (homework)

$$x[n_1,n_2]=x_1[n_1]x_2[n_2]\leftrightarrow X(\omega_1,\omega_2)=X_1(\omega_1)X_2(\omega_2)$$

Image spectrum

- two images, the magnitude, and phase of their FTs



Phase and Magnitude

- curious fact
  - all natural images have about the same magnitude transform
  - monotonically decaying with frequency

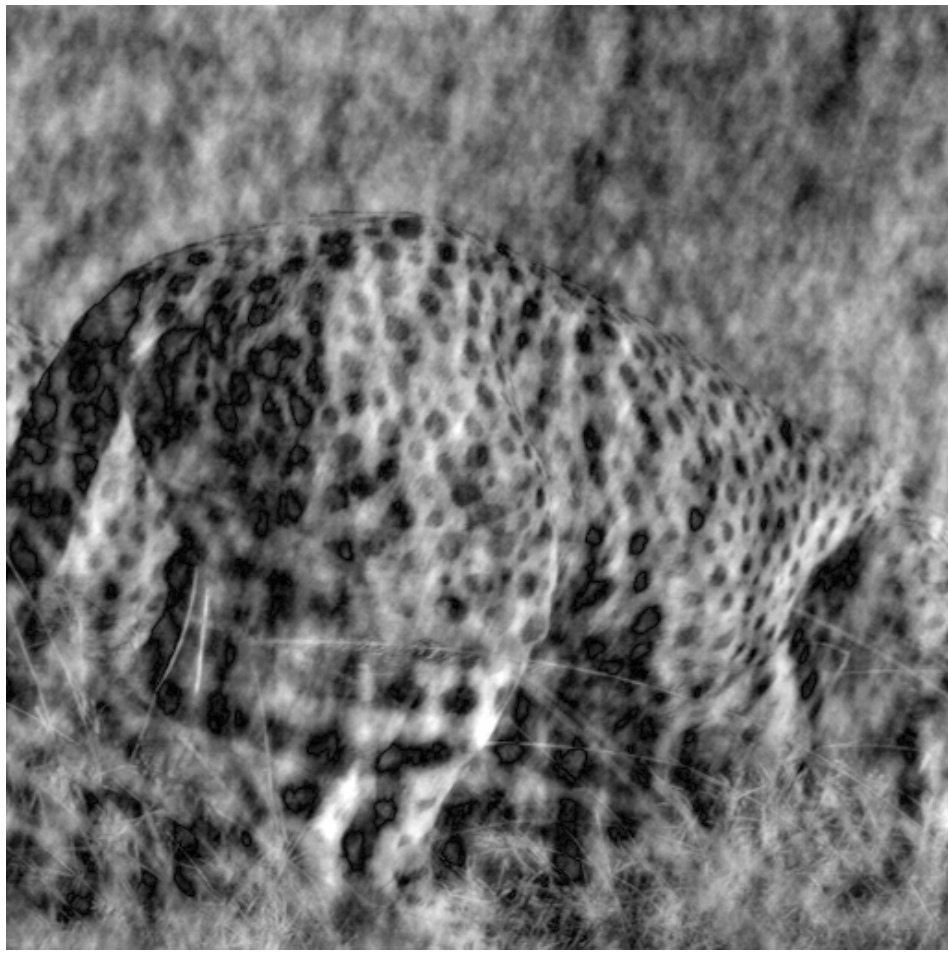
$$X(\omega_1, \omega_2) \propto \frac{1}{\omega_1^2 + \omega_2^2}$$

- hence, phase seems to matter, but magnitude largely doesn't
- we have seen this a little bit in the first problem set
- took two pictures, swap the phase transforms, compute the inverse
- here is another example

The importance of phase

Reconstruction with zebra phase, cheetah magnitude

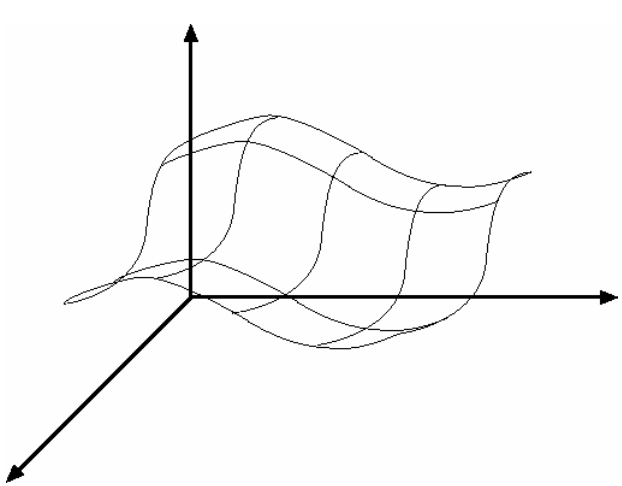
Reconstruction with cheetah phase, zebra magnitude

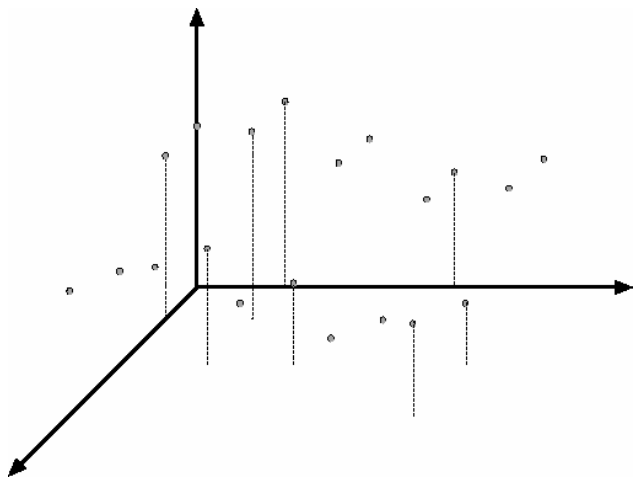


Sampling in 2D

- consider an analog signal $x_c(t_1,t_2)$ and let its analog Fourier transform be $X_c(\Omega_1,\Omega_2)$
  - we use capital Ω to emphasize that this is analog frequency
- sample with period (T_1,T_2) to obtain a discrete-space signal

$$x[n_1,n_2] = x_c(t_1,t_2)\Big|_{t_1=n_1T_1;t_2=n_2T_2}$$





Sampling in 2D

- relationship between the Discrete-Space FT of $x[n_1,n_2]$ and the FT of $x_c(t_1,t_2)$ is simple extension of 1D result

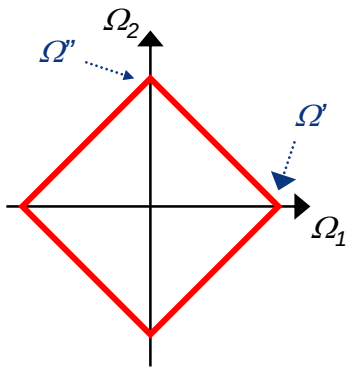
$$X(\omega_1,\omega_2)=\frac{1}{T_1T_2}\sum_{r_1=-\infty}^{\infty}\sum_{r_2=-\infty}^{\infty}X_c\left(\frac{\omega_1-2\pi r_1}{T_1},\frac{\omega_1-2\pi r_1}{T_1}\right)$$

DSFT of $x[n_1,n_2]$
"discrete spectrum"

FT of $x_c(\omega_1,\omega_2)$
"analog spectrum"

- Discrete Space spectrum is sum of replicas of analog spectrum
  - in the "base replica" the analog frequency Ω_1 (Ω_2) is mapped into the digital frequency Ω_1T_1 (Ω_2T_2)
  - discrete spectrum has periodicity $(2\pi,2\pi)$

For example



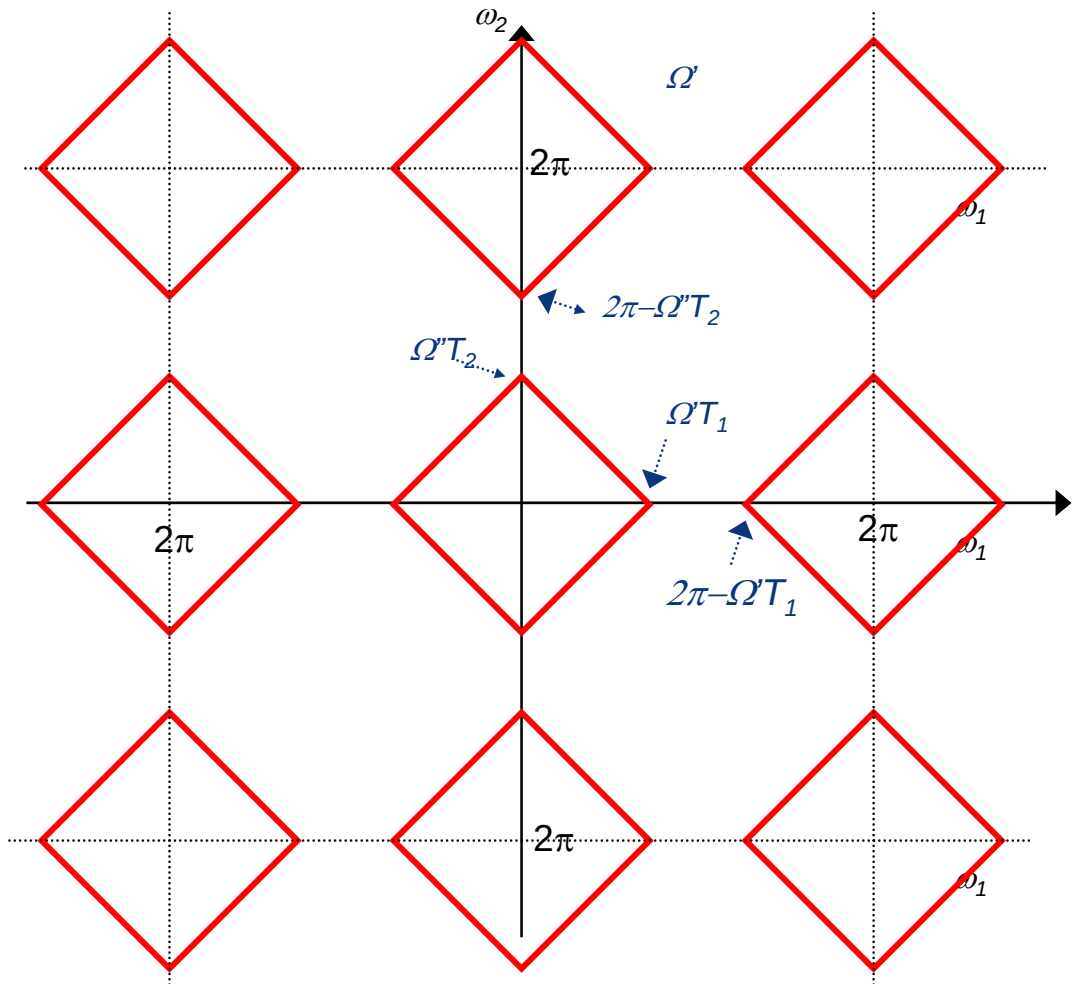
$$\Omega' \rightarrow \alpha = \Omega' T_1$$

$$\Omega'' \rightarrow \beta = \Omega'' T_2$$

- no aliasing if

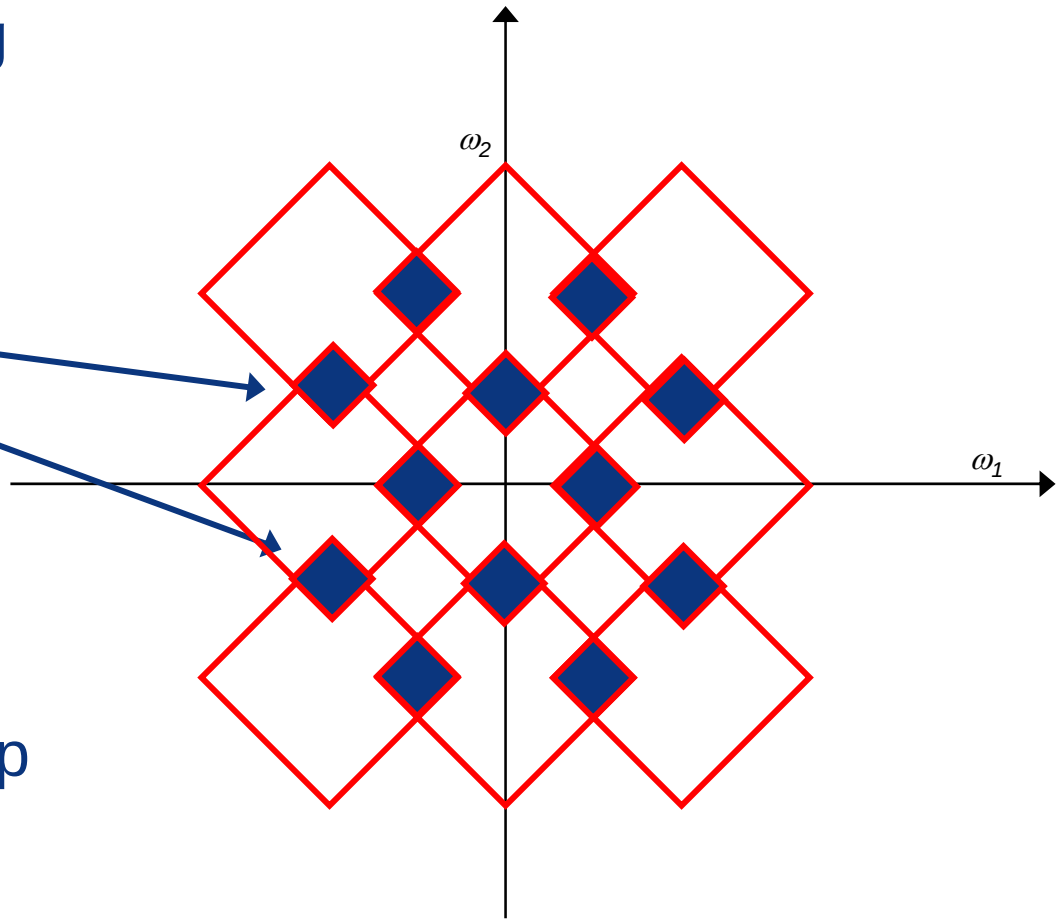
$$\begin{cases} \Omega' T_1 \le 2\pi - \Omega' T_1 \\ \Omega'' T_2 \le 2\pi - \Omega' T_2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} T_1 \le \pi / \Omega' \\ T_2 \le \pi / \Omega'' \end{cases}$$



Aliasing

- the frequency $(\Omega'/\pi,\Omega''/\pi)$ is the critical sampling frequency
- below it we have aliasing
- this is just like the 1D case, but now there are more possibilities for overlap



Reconstruction

- if there is no aliasing we can recover the signal in a way similar to the 1D case

$$y_c(t_1,t_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}x[n_1,n_2]\frac{\sin\frac{\pi}{T_1}(t_1-n_1T_1)}{\frac{\pi}{T_1}(t_1-n_1T_1)}\frac{\sin\frac{\pi}{T_2}(t_2-n_2T_2)}{\frac{\pi}{T_2}(t_2-n_2T_2)}$$

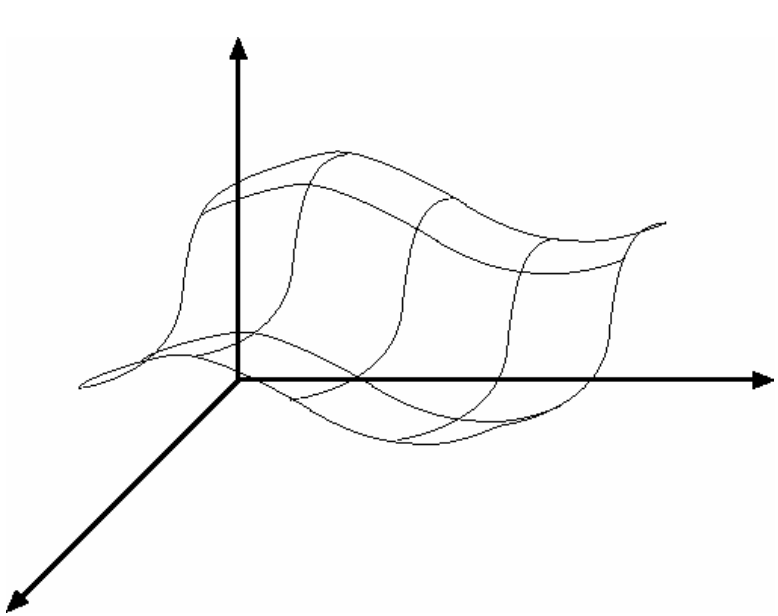
- note: in 2D there are many more possibilities than in 1D
  - e.g. the sampling grid does not have to be rectangular, e.g. hexagonal sampling when $T_2 = T_1/sqrt(3)$ and

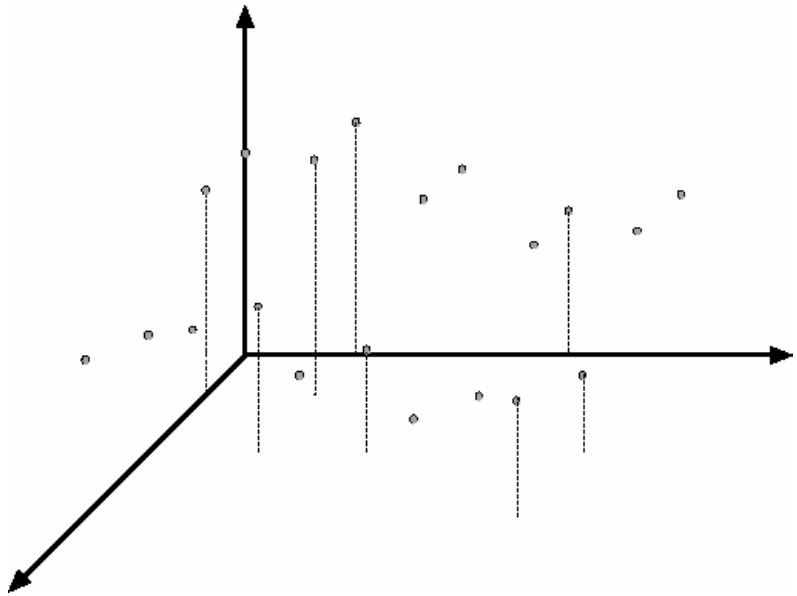
$$x[n_1,n_2]=\begin{cases}x_c(t_1,t_2)\big|_{t_1=n_1T_1;t_2=n_2T_2} & n_1,n_2 \text{ both even or odd}\\ 0 & \text{otherwise}\end{cases}$$

  - in practice, however, one usually adopts the rectangular grid

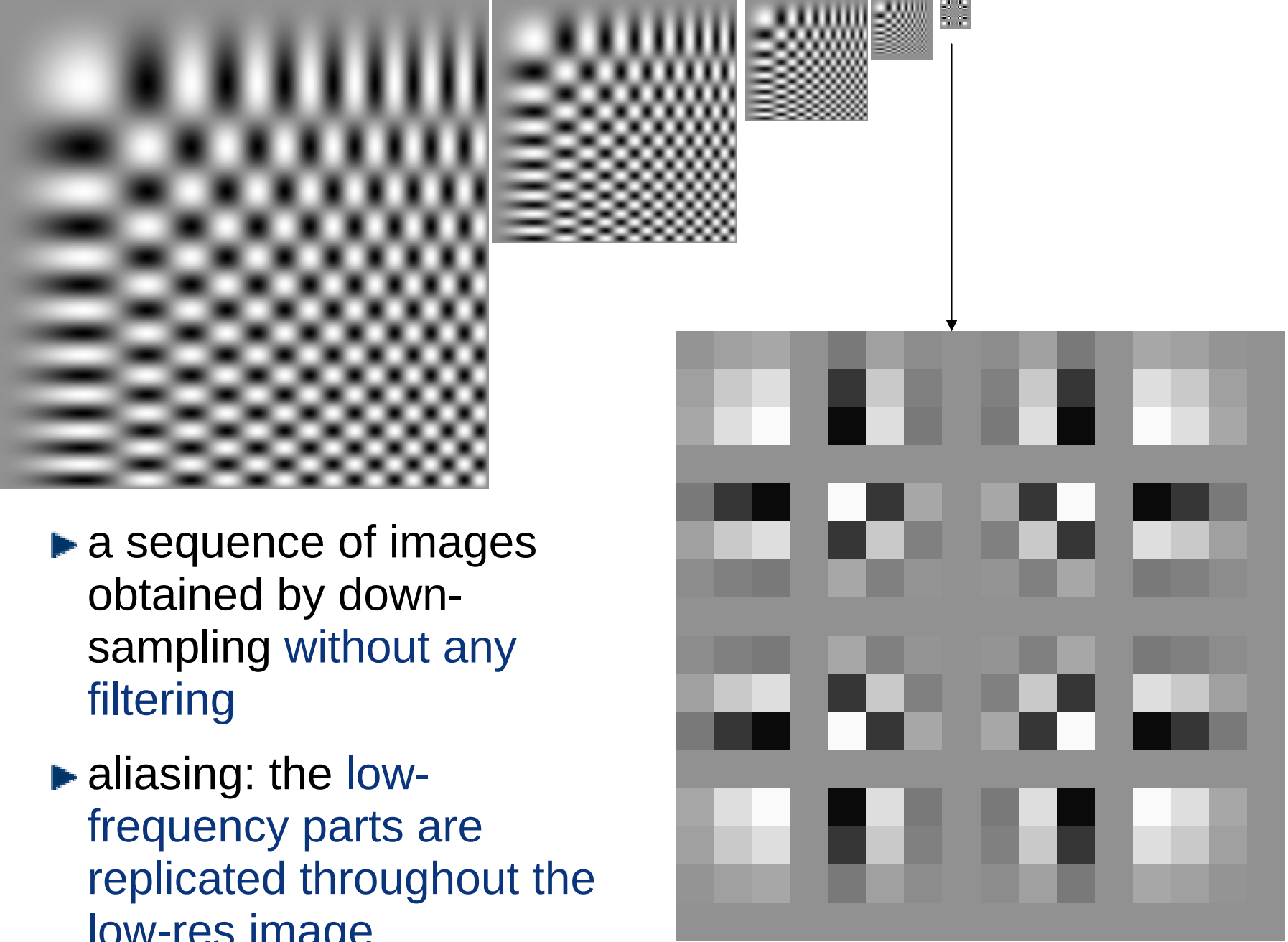
Sampling and aliasing in 2D

- in summary, sampling is not very different from the 1D case



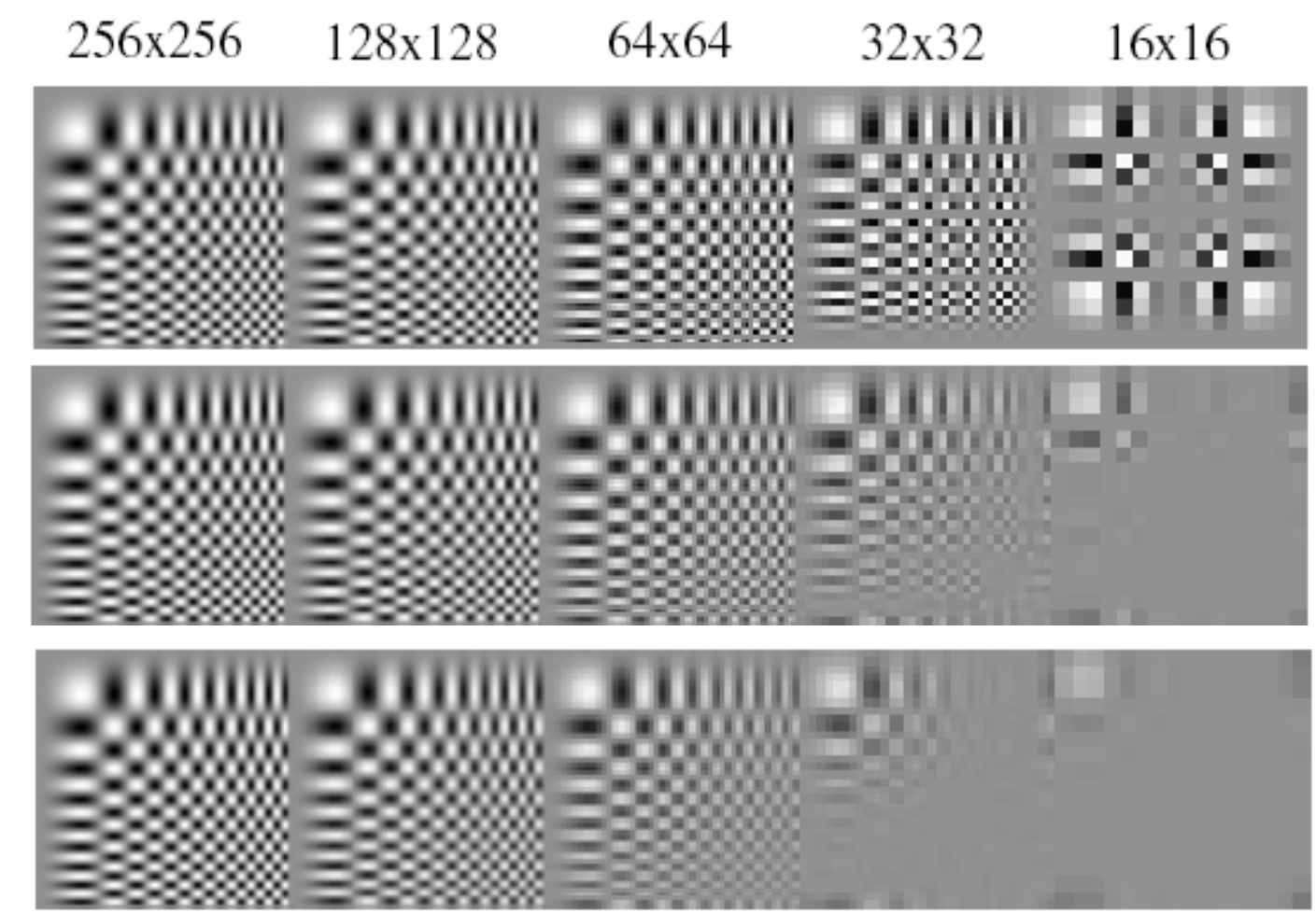


- but aliasing is a lot more fun to look at in images
- it shows up in video too (the wagon wheel effect)



- a sequence of images obtained by down-sampling without any filtering
- aliasing: the low-frequency parts are replicated throughout the low-res image

The role of smoothing



- too little leads to aliasing
- too much leads to loss of information

Linear Filtering

- smoothing is implemented with linear filters
- given an image $x(n_1,n_2)$, filtering is the process of convolving it with a kernel $h(n_1,n_2)$

$$y(n_1,n_2) = \sum_{k_1 k_2} x(k_1,k_2)h(n_1-k_1,n_2-k_2)$$

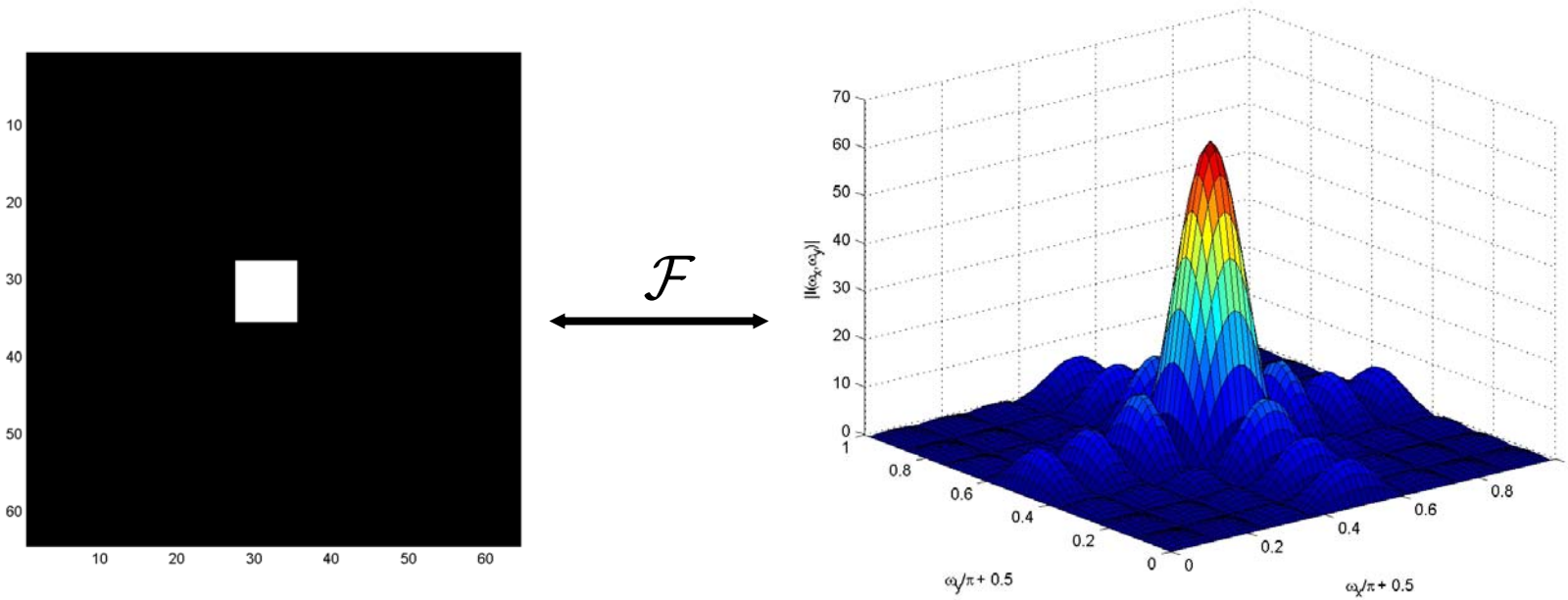
- some very common operations in image processing are nothing but filtering, e.g.
  - smoothing an image by low-pass filtering
  - contrast enhancement by high pass filtering
  - finding image derivatives
  - noise reduction

Popular filters

- box function

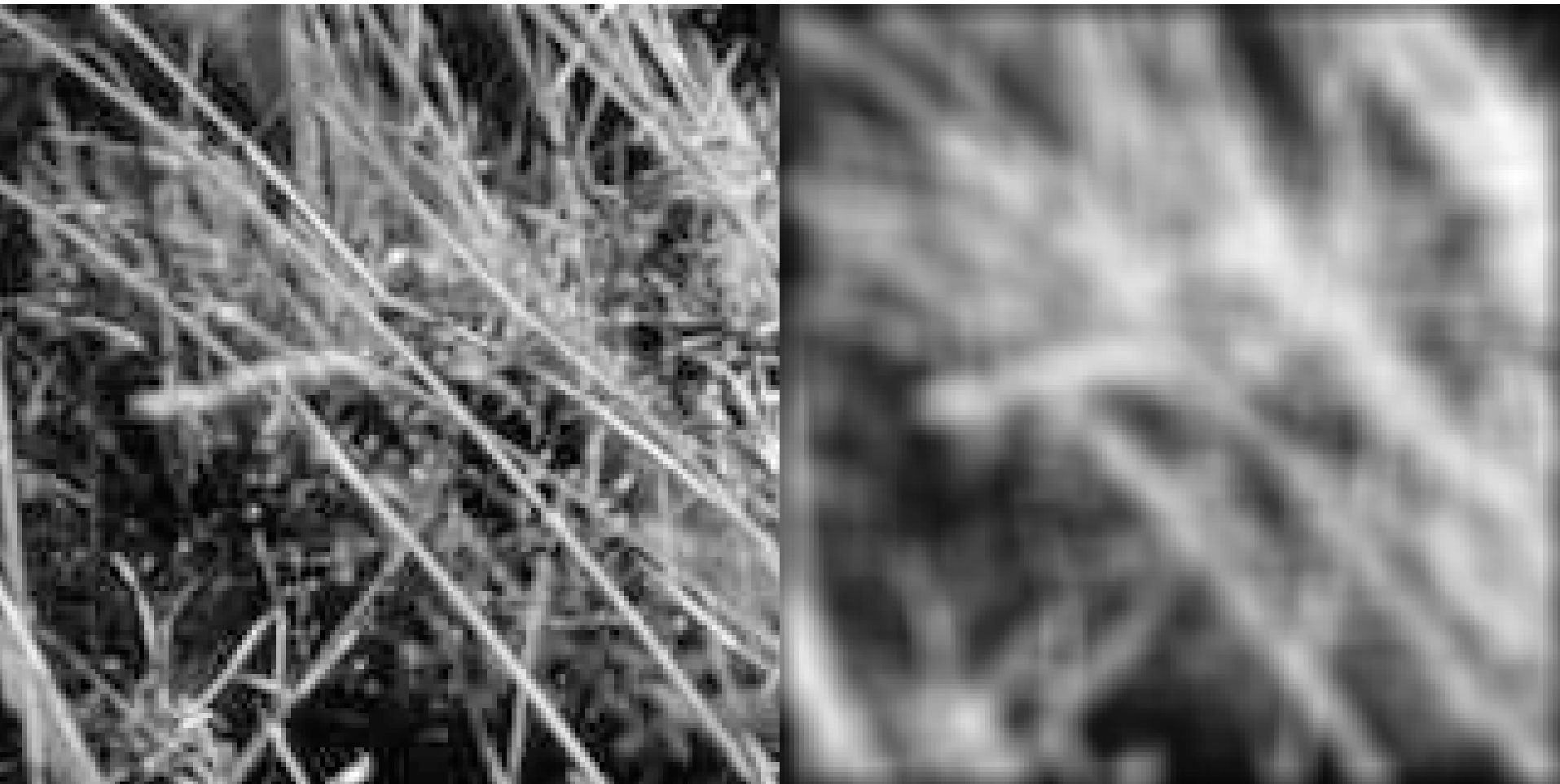
$$R_{N_1 x N_2}(n_1, n_2) = \begin{cases} 1, & 0 \le n_1 \le N_1 - 1, 0 \le n_2 \le N_2 - 1 \\ 0 & otherwise \end{cases}$$

- Fourier transform of a box is the sinc, low-pass filter



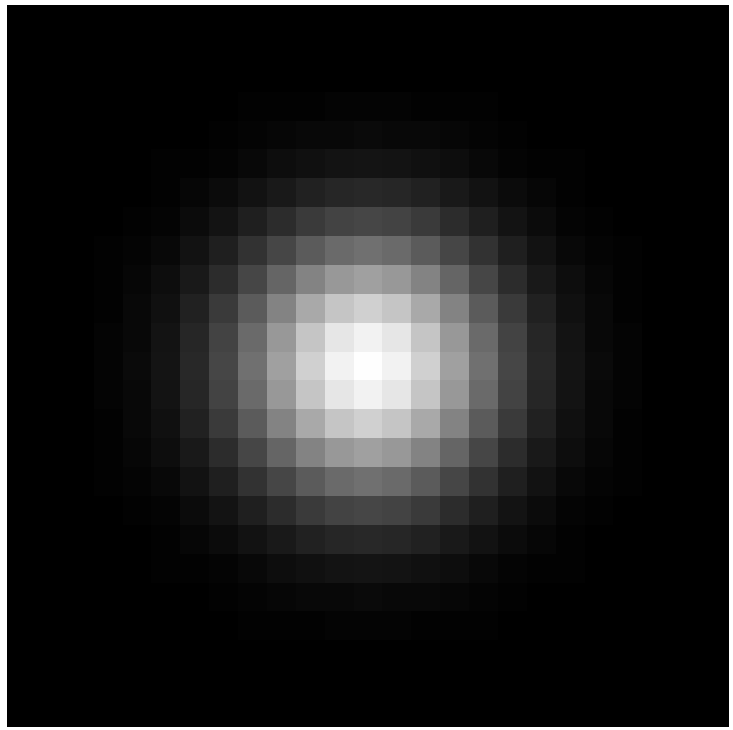
- side-lobes produce artifacts, smoothed image does not look like the result of defocusing

Example: Smoothing by Averaging



Camera defocusing

- if you point an out-of-focus camera at a very small white light (e.g. a light-bulb) at night, you get something like this
- the light can be thought of as an impulse
- this must be the impulse response
- well approximated by a Gaussian
- more natural filter for image blur than the box



$$h(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

The Gaussian

- the discrete space version is

$$h(n_1, n_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_1^2 + n_2^2}{2\sigma^2} \right)$$

- obviously separable

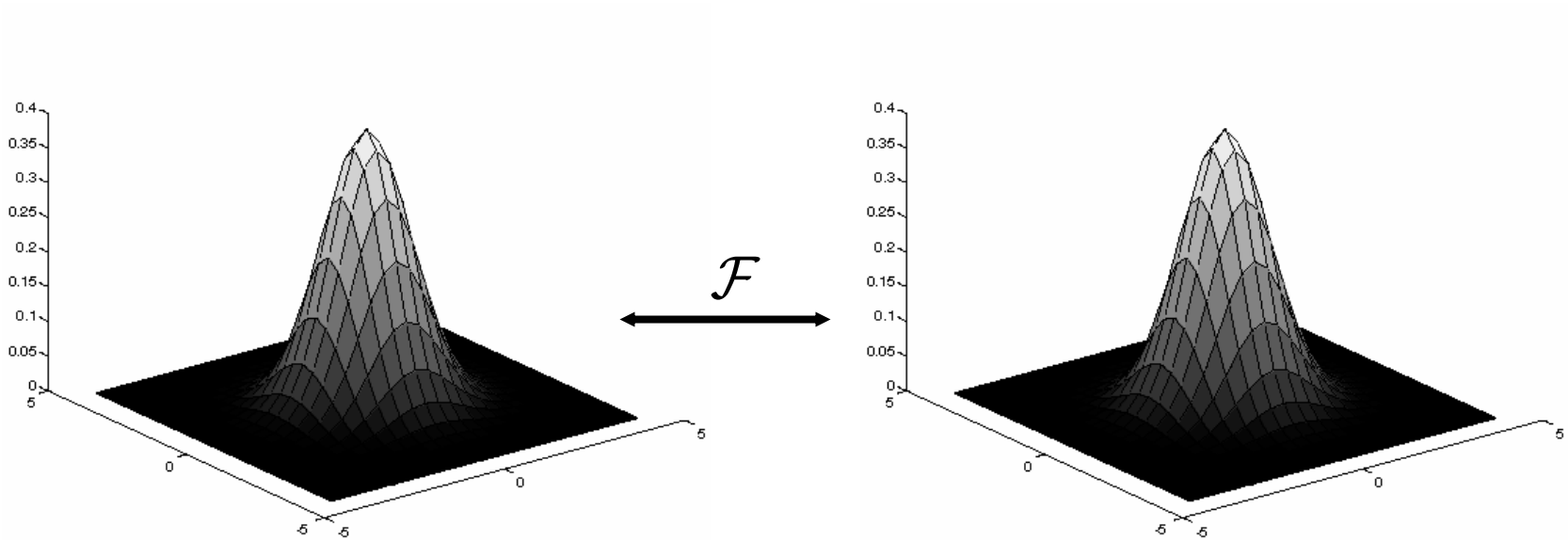
$$h(n_1, n_2) = \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n_1^2}{2\sigma^2}}}_{h(n_1)} \times \underbrace{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n_2^2}{2\sigma^2}}}_{h(n_2)}$$

- $h(n_1,n_2)$ has Fourier transform

$$H(\varpi_1, \varpi_2) = \exp\left(-\frac{\sigma^2(\varpi_1^2 + \varpi_2^2)}{2} \right)$$

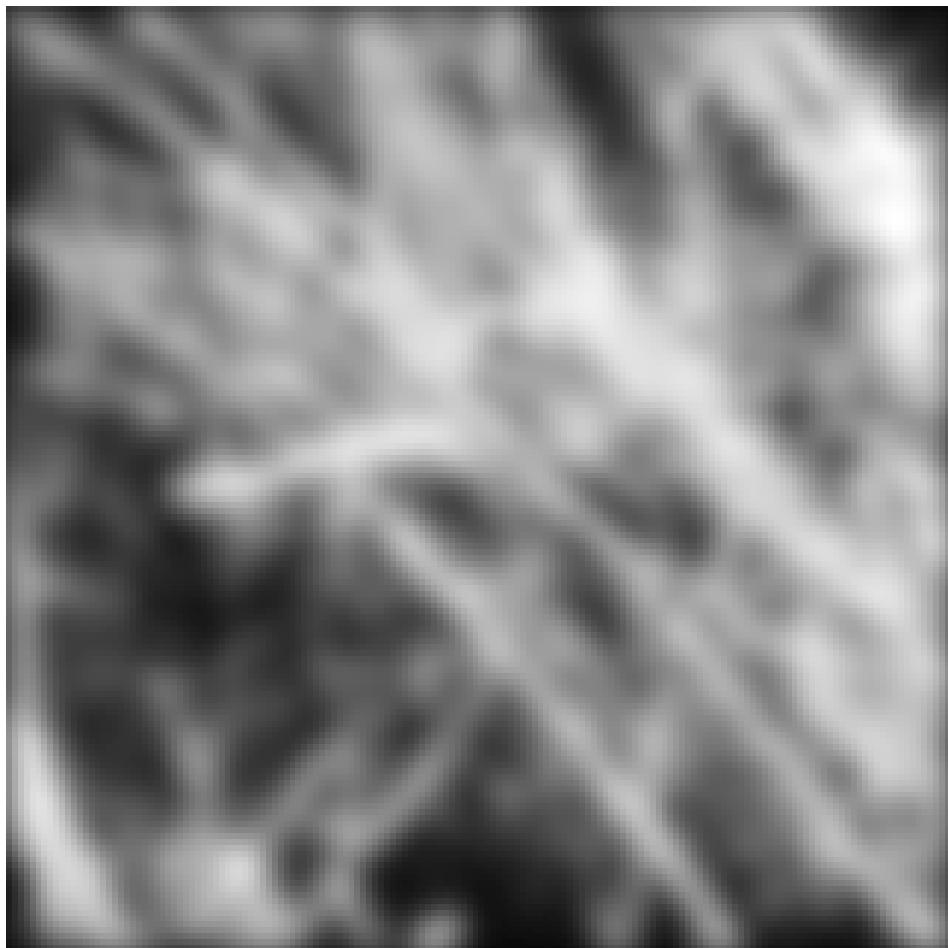
The Gaussian filter

- the Fourier transform of a Gaussian is a Gaussian
  $(\sigma_x,\sigma_y) \propto (1/\sigma_{w1},1/\sigma_{w2})$

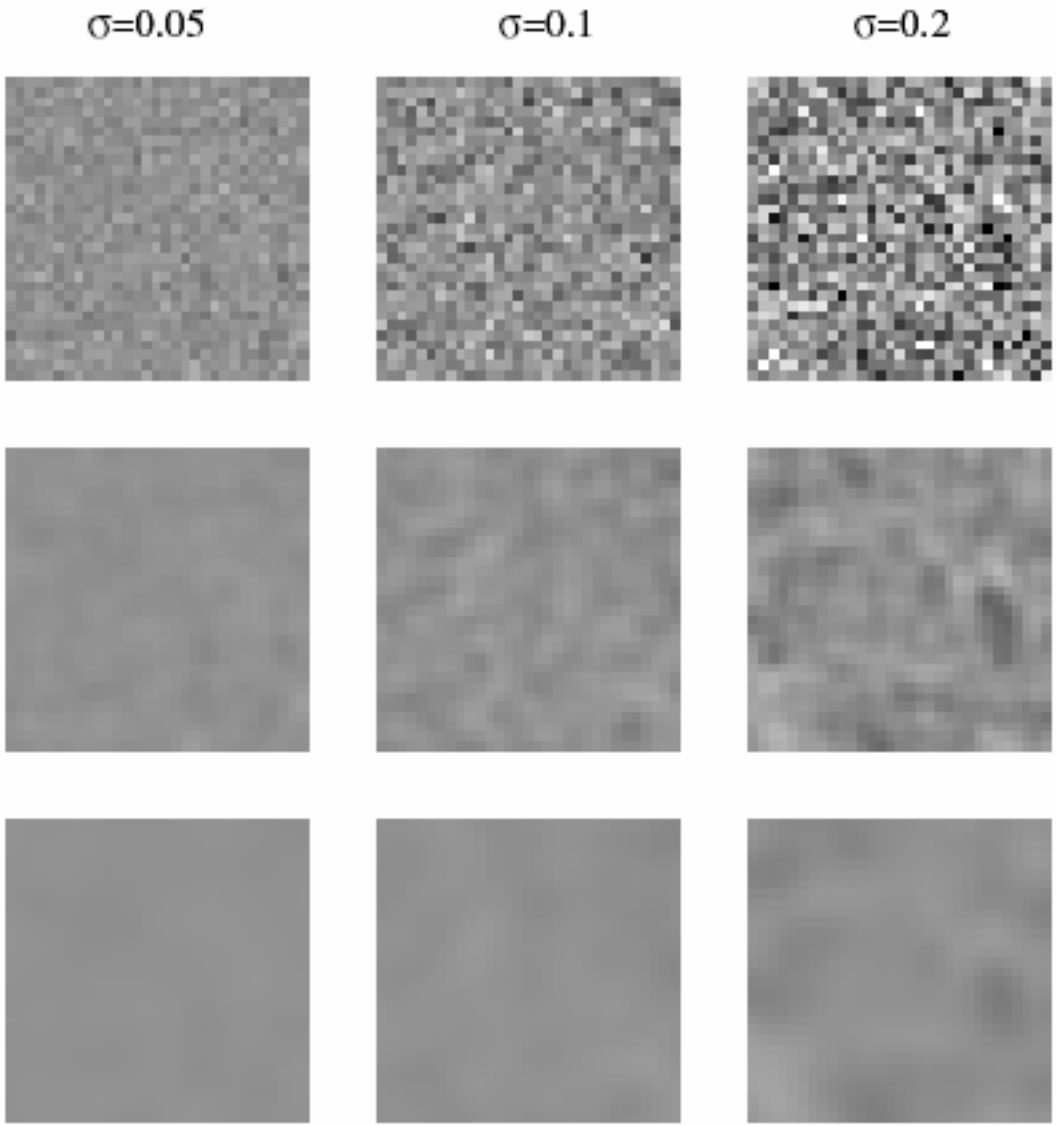


- note that there are no annoying side-lobes

Smoothing with a Gaussian



Role of the variance

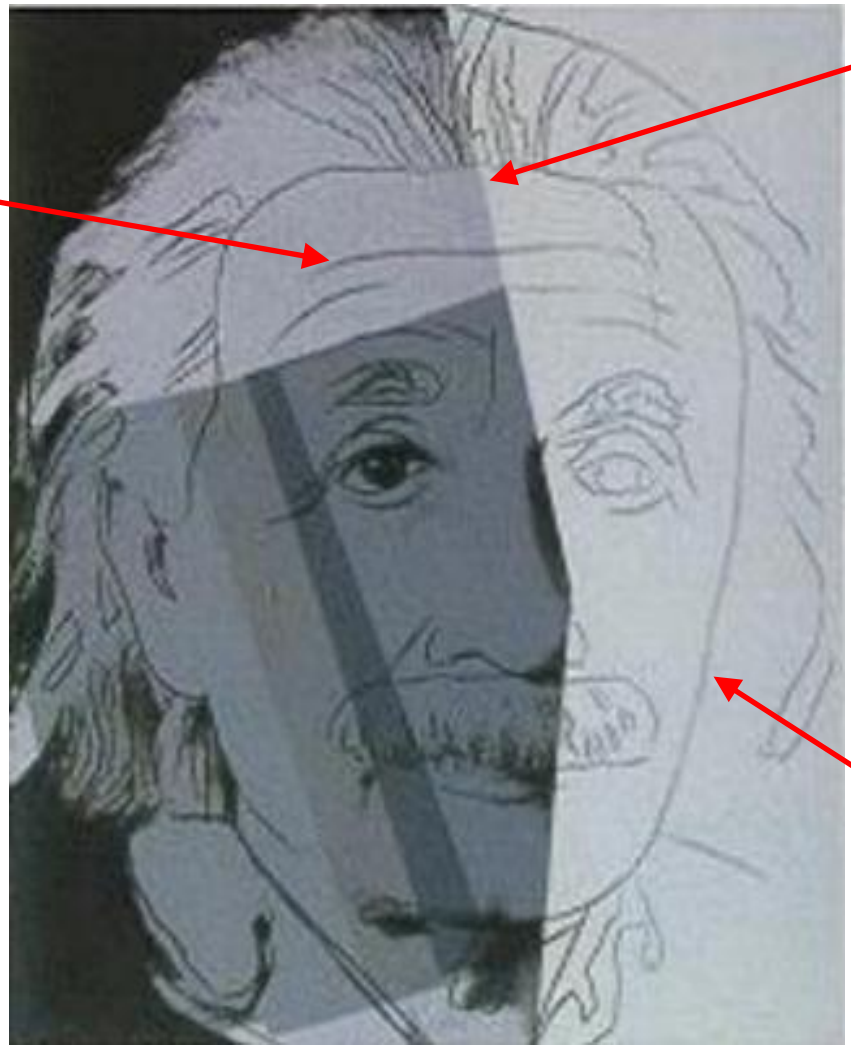


- the variance controls the amount of smoothing
- each column shows different realizations of an image of gaussian noise
- each row shows smoothing with gaussians of different σ

Gradients and edges

- for image understanding, one of the problems is that there is too much information in an image
- just smoothing is not good enough
- how to detect important (most informative) image points?
- note that derivatives are large at points of great change
  - changes in reflectance (e.g. checkerboard pattern)
  - change in object (an object boundary is different from background)
  - change in illumination (the boundary of a shadow)
- these are usually called **edge points**
- detecting them could be useful for various problems
  - segmentation: we want to know what are object boundaries
  - recognition: cartoons are easy to recognize and terribly efficient to transmit

The importance of edges



Gradients

- for a 2D function, *f(x,y)* the gradient at a point (x_0,y_0)

$$\nabla f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0)\right)^T$$

$$= \left(f_x(x_0,y_0),\ f_y(x_0,y_0)\right)^T$$

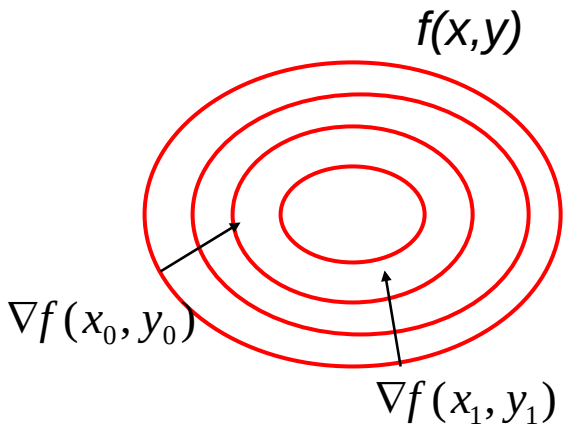
  is the direction of greatest increase at that point

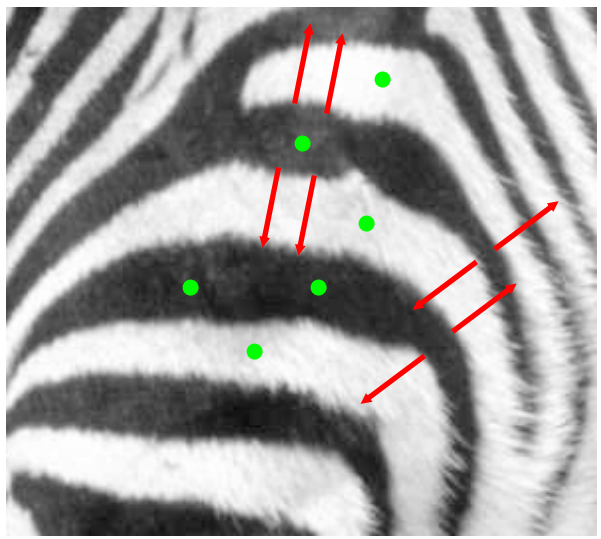
- the gradient magnitude

$$\left\|\nabla f(x_0,y_0)\right\|^2 = \left(\frac{\partial f}{\partial x}(x_0,y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^2$$

  measures the rate of change

- it is large at edges!





– large gradient magnitude

– small gradient magnitude

Derivatives and convolution

- recall that a derivative is defined as

$$\frac{\partial f(x)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- linear and shift invariant, so must be the result of a convolution.
- we could approximate as

$$\frac{\partial f(n)}{\partial n} = \frac{f(n+1) - f(n)}{1} = f(n+1) - f(n) = f * h(n)$$

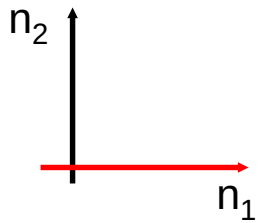
- where the derivative kernel is

$$h(n) = \delta(n+1) - \delta(n)$$

Finite difference kernels

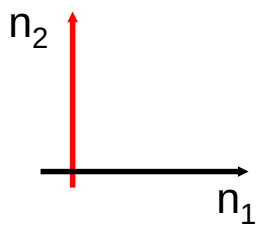
- in two dimensions we have various possible kernels
- e.g. , $N_1=2$, $N_2=3$, derivative along n_1, *(line $n_2=k$)* (horizontal)

$$\begin{matrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{matrix}$$



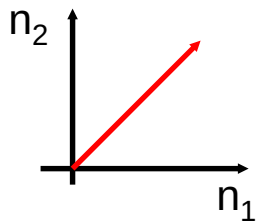
- derivative along n_2, *(line $n_1=k$)* (vertical)

$$\begin{matrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} -1 & -1 & -1 \\ 1 & 1 & 1 \end{matrix}$$



- derivative along line $n_1=n_2$ (diagonal)

$$\begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$$



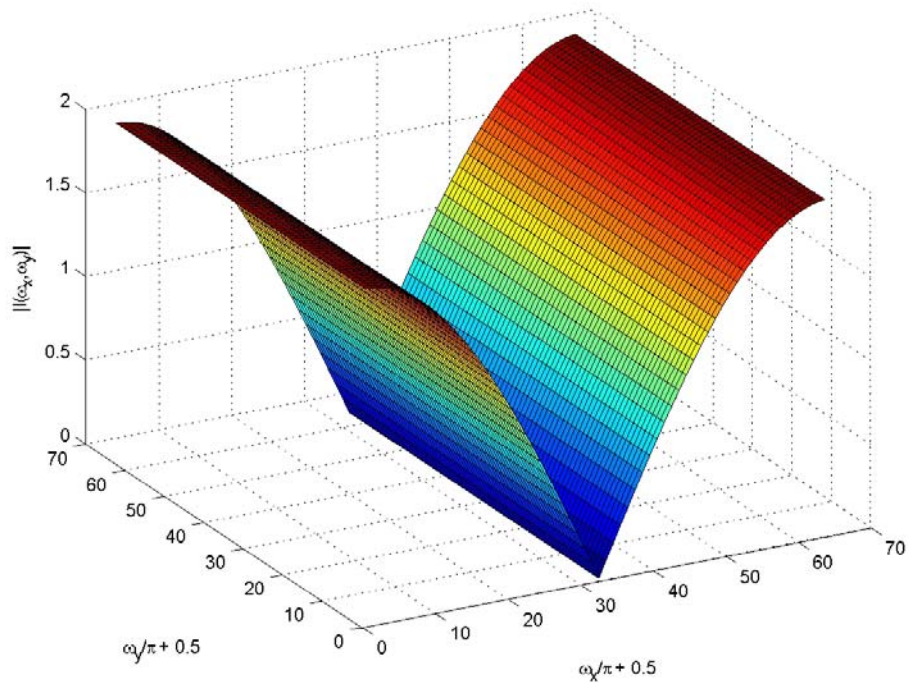
Finite difference kernels

- note that, when

$$h(n_1,n_2) = \begin{matrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{matrix}$$

- we have

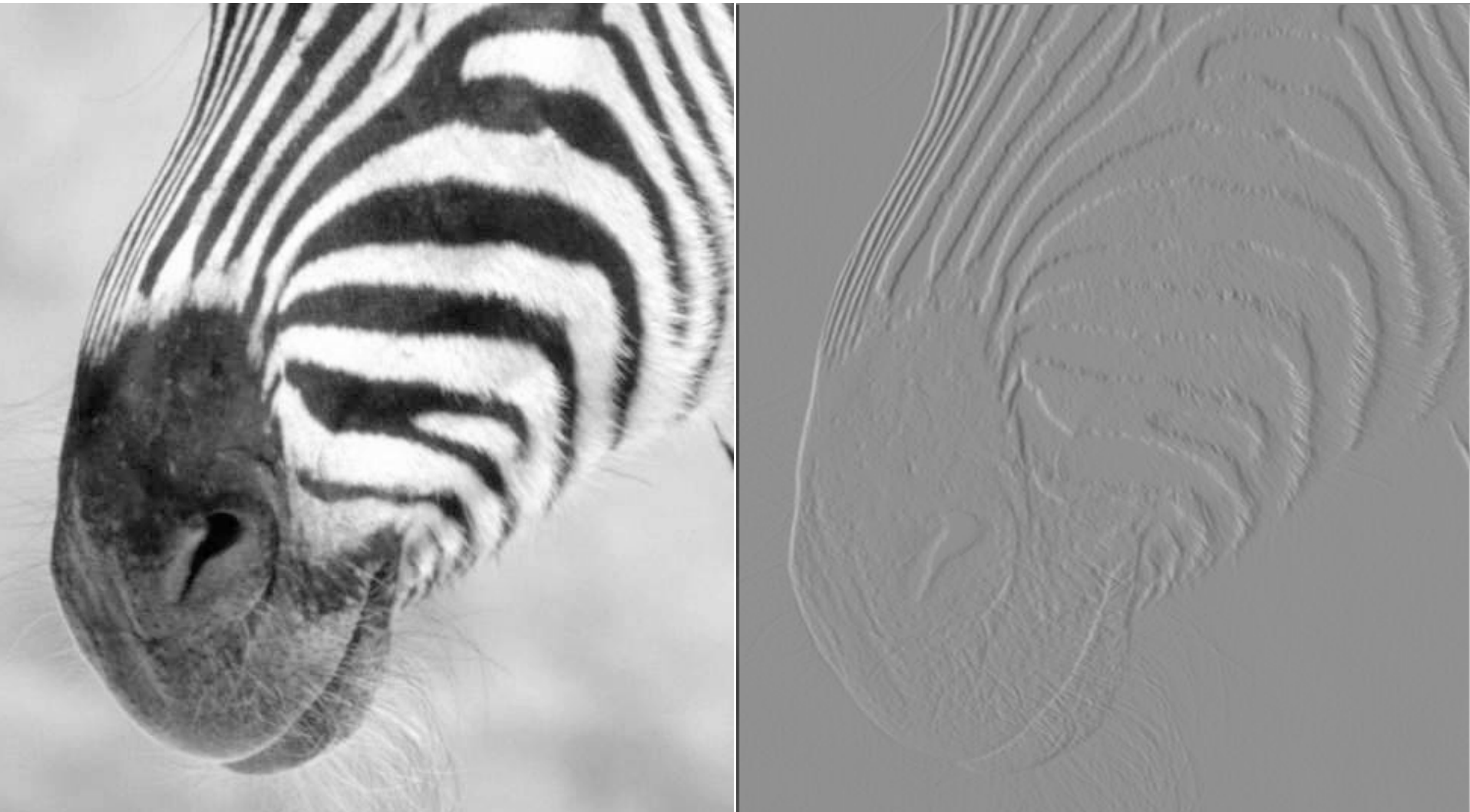
$$H(\varpi_1,\varpi_2) = e^{j\varpi_1} - 1$$
$$= \left(e^{j\frac{\varpi_1}{2}} - e^{-j\frac{\varpi_1}{2}} \right) e^{j\frac{\varpi_1}{2}}$$
$$= 2\sin\left(\frac{\varpi_1}{2}\right) e^{j\frac{\varpi_1}{2}}$$



- derivative is a high-pass filter
- hw: check that this holds for all others
- intuitive, because a derivative is a measure of the rate of change of a function

Finite differences

- Q: which one do we have here? (gray=0,white=+,dark=-)



Finite differences and noise

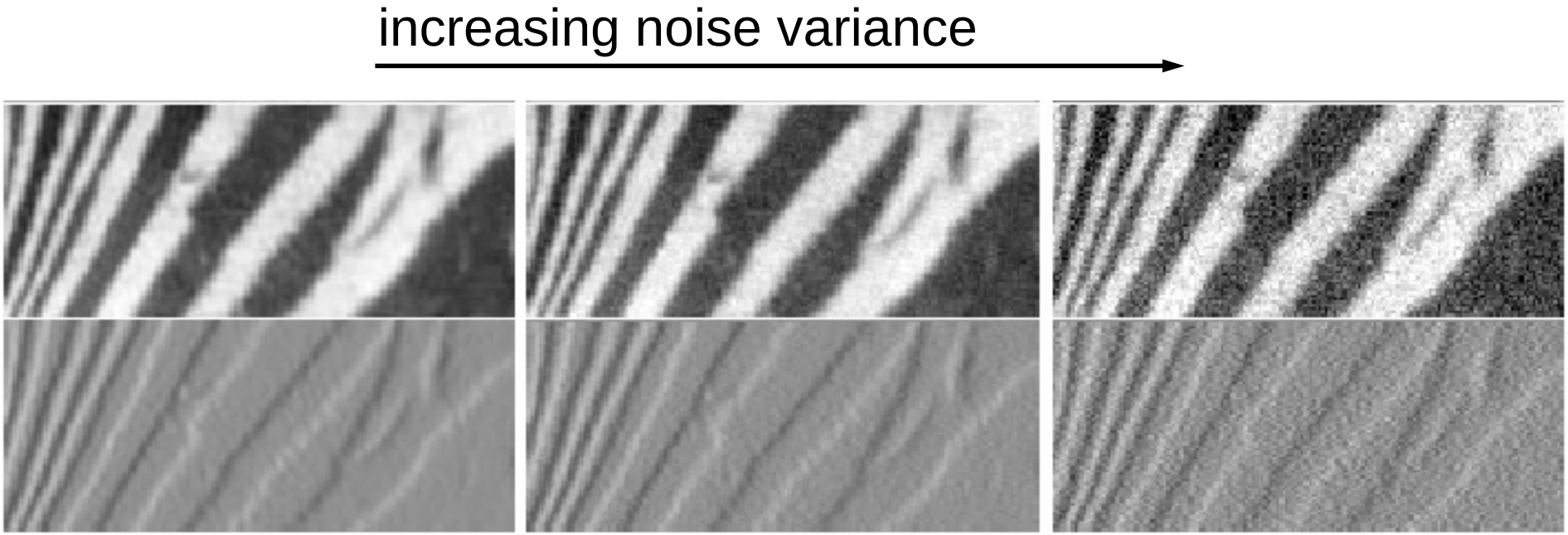
- because they perform high-pass filtering, finite difference filters respond strongly to noise
- generally, the larger the noise the stronger the response
- for noisy images it is usually best to apply some smoothing before computing derivatives
- what do mean by noise?
- we only consider the simplest model
  - independent stationary additive Gaussian noise
  - the noise value at each pixel is given by an independent draw from the same normal probability distribution

$$y(n_1, n_2) = x(n_1, n_2) + \varepsilon(n_1, n_2), \qquad \varepsilon \sim N(0, \sigma^2)$$

sigma=1

sigma=16

Finite differences responding to noise



- note that as the noise variance increases the estimates of the image derivative are also very noisy
- Q: would a larger filter do better?

The response of a linear filter to noise

- suppose we apply filter $h(n_1,n_2)$ to

$$y(n_1,n_2) = x(n_1,n_2) + \varepsilon(n_1,n_2), \qquad \varepsilon \sim N(0,\sigma^2)$$

- by linearity

$$h * y = \underbrace{h * x}_{\text{output signal}} + \underbrace{h * \varepsilon}_{\text{output noise}}$$

- consider the noise term

$$\nu(n_1,n_2) = \sum_{k_1 k_2} \varepsilon(k_1,k_2) h(n_1-k_1, n_2-k_2)$$

- by linearity of expectation

$$E[\nu(n_1,n_2)] = \sum_{k_1 k_2} E[\varepsilon(k_1,k_2)] h(n_1-k_1, n_2-k_2) = 0$$

The response of a linear filter to noise

- and since the $\varepsilon(n_1,n_2)$ are independent

$$\mathrm{var}\left[\nu(n_1,n_2)\right]=\sum_{k_1k_2}\mathrm{var}\left[\varepsilon(k_1,k_2)\right]h^2(n_1-k_1,n_2-k_2)$$

$$=\sigma^2\sum_{k_1k_2}h^2(n_1-k_1,n_2-k_2)$$

- output error has
  - zero mean
  - variance that increases with the size of the filter and the input variance
- increasing the filter size would definitely not help!

Smoothing reduces noise

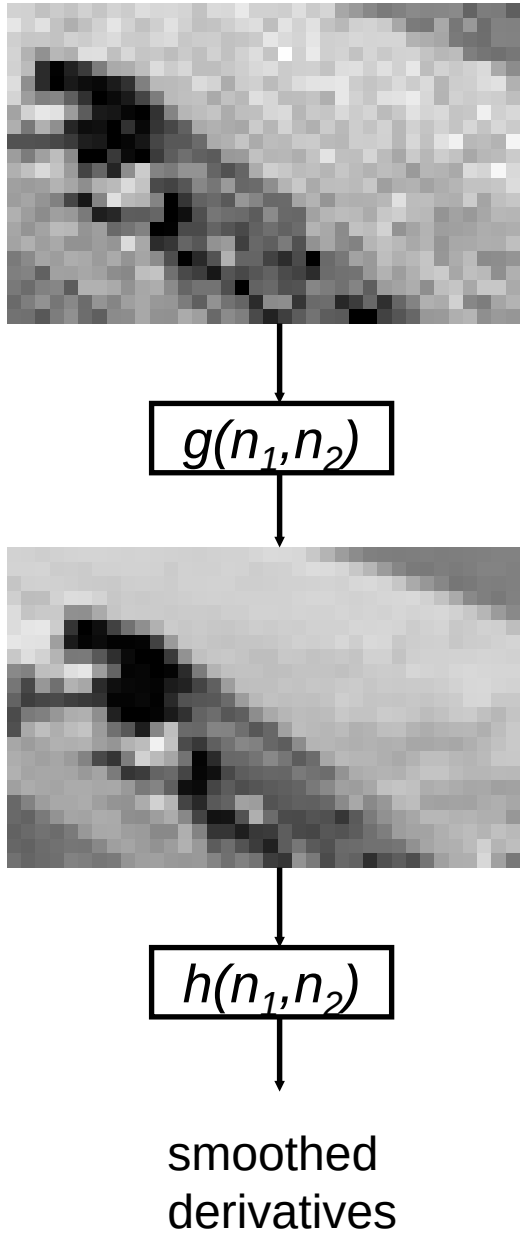
- noise has a lot of high-frequencies
- strategy:
  1. start by low-pass filtering, to suppress noise
  2. compute derivative on smoothed image
- i.e. for a smoothing filter $g(n_1,n_2)$ compute

$$h * (g * x)$$

- note that, by associativity of convolution, this is equal to

$$(h * g) * x$$

- i.e. filter the image with the filter $h*g$



The derivative of a Gaussian

- let's consider, for example,

$$h(n_1, n_2) = \delta(n_1 + 1, n_2) - \delta(n_1, n_2)$$

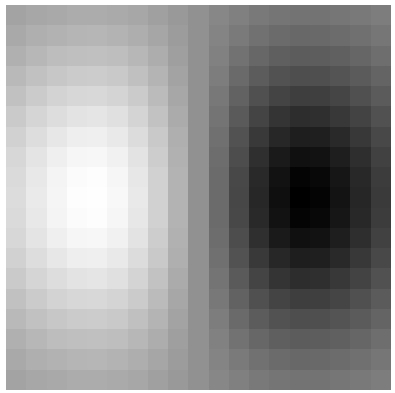
- in which case

$$h * g(n_1, n_2) = g(n_1 + 1, n_2) - g(n_1, n_2)$$

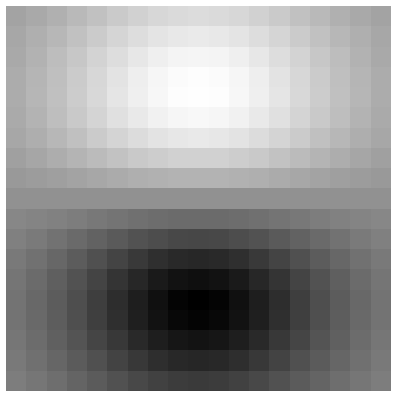
  is a difference of two Gaussians

- this is the derivative of a Gaussian (DoG) filter
- for other definitions of *h* we have a similar result

DoG along n_1

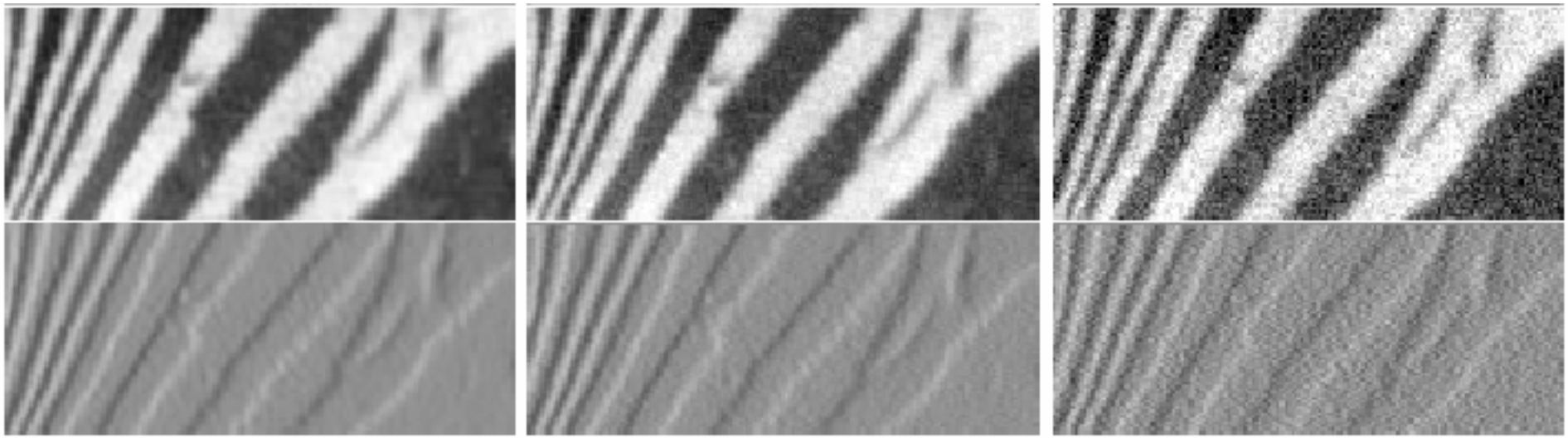


DoG along n_2

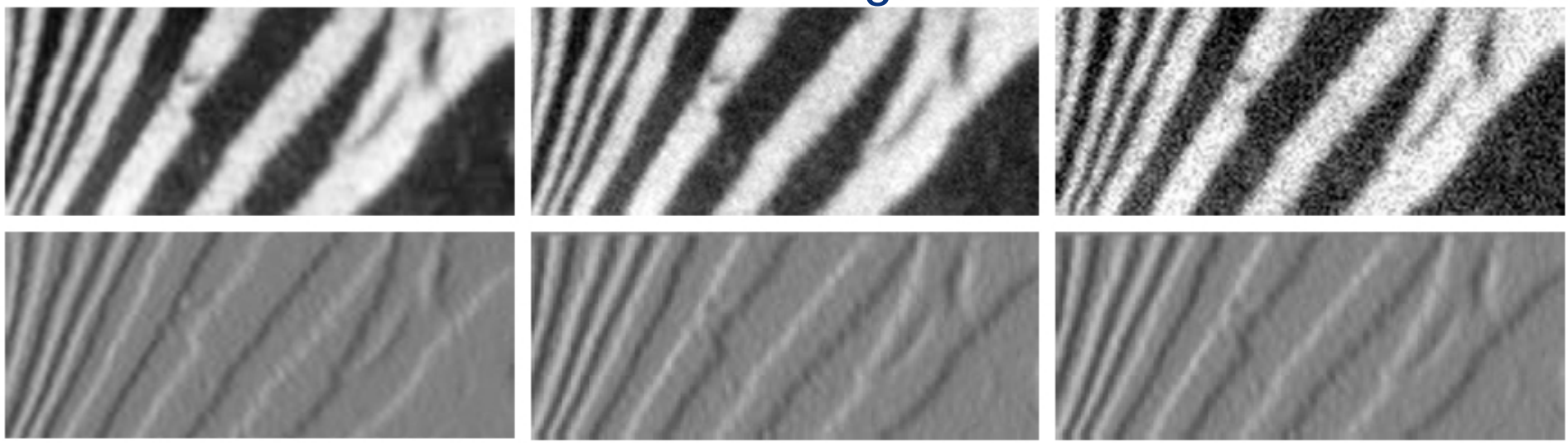


Smoothed derivatives

no smoothing



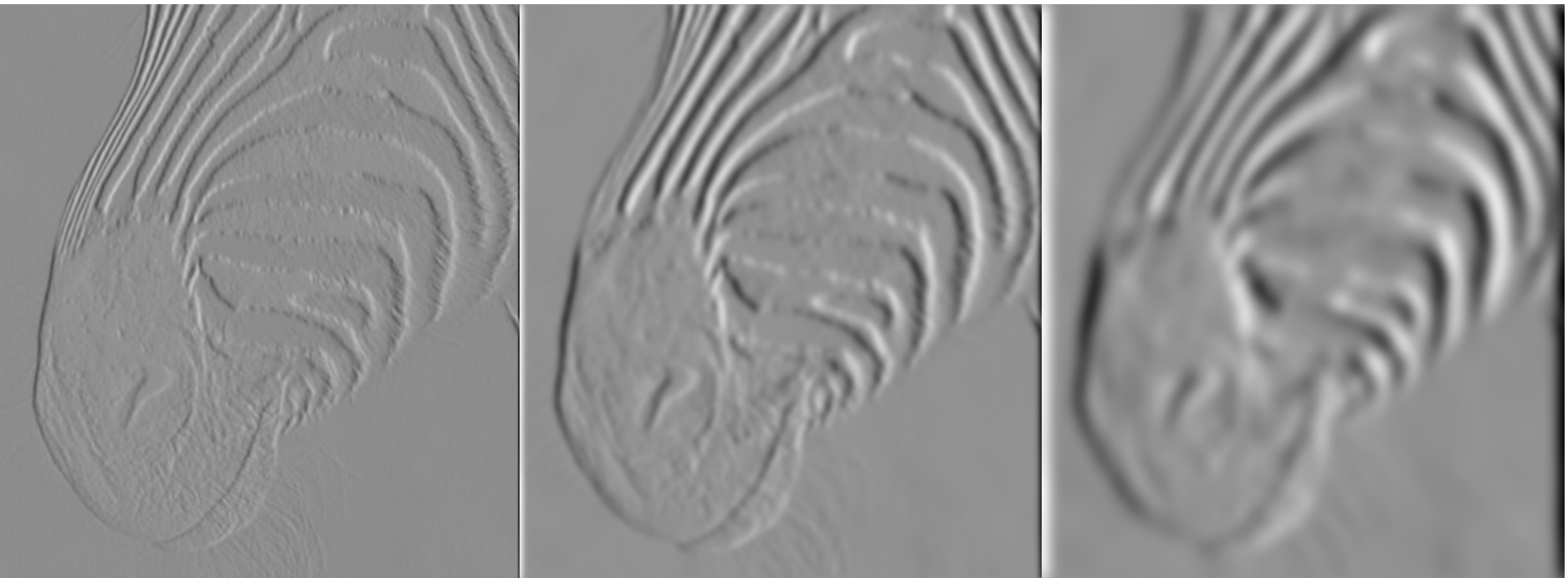
with smoothing



Choosing the right scale

- the scale of the smoothing filter affects
  - derivative estimates,
  - the semantics of the derivative image
- trade-off between noise and ability to detect detail

1 pixel 3 pixels 7 pixels




Any questions?

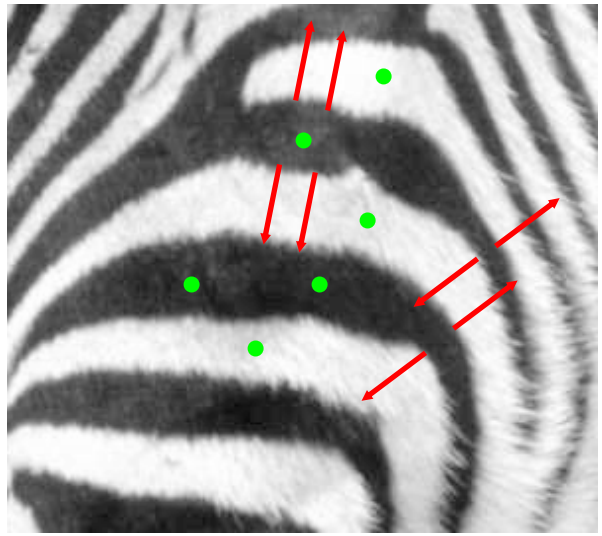
Gradients and edges

- by now we have a good idea of how to differentiate images
- remember that edges are points of large gradient magnitude
- edge detection strategy

1. determine magnitude of image gradient

$$\|\nabla f(x_0, y_0)\|^2 = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2$$

2. mark points where gradient magnitude is particularly large wrt neighbours (ideally, curves of such points)



– large gradient magnitude

– small gradient magnitude

Looks easy but

- three major issues (to discuss next class):
  - 1) gradient magnitude at different scales is different (see below); which should we choose?
  - 2) gradient magnitude is large along thick trail; what are the significant points?
  - 3) how do we link the relevant points up into curves?