Lec4

Moving average

- moving average system is shift invariant and linear
  - Convolution with impulse response defines a shift-invariant-linear system
  - Impulse response of moving average system is
    - filter = [[1,1,1],[1,1,1],[1,1,1]]/9
  - Such filters are defined by
    - Width and Height
    - Usually sum(filters)==1 and all entries > 0
      - Sigma

Statistical justification

- If observation is signal (deterministic) + random noise (probabilistic, iid)
- iid — independent identically distributed
    - That means if signal length is n, and noise length is also n, each element of noise is independently drawn from the same distribution
        - Or n times the same coin is independently flipped
- Then variance in noise is \sigma
- Variance in averaged noise \sigma/\sqrt(N) (#Samples averaged over)

4

In general, for any distribution of x_i that has a variance

$$\text{The sum } \sum_{i=0}^{N-1} x_i \text{ has variance } \sigma^2_{total} = \sum_{i=0}^{N-1} \sigma_i^2$$

if the x_i are independent.

$$\text{The average} \frac{1}{N} \sum_{i=0}^{N-1} x_i \text{ has variance } \sigma^2_{ave} = \frac{1}{N} \sum_{i=0}^{N-1} \sigma_i^2$$

which in terms of the question.

$$\sigma_{ave} = \frac{\sigma_i}{\sqrt{N}}$$

if the x_i have the same variance, or in other terms is iid (independent identically distributed)

Gaussian or not, this is true, and it doesn't depend on the Central Limit Theorem. It just has to be iid. Gaussianity is irrelevant.

When $E\{x_i\} \neq 0$ you need to be careful when calculating the standard deviation.

$$E\{x_i^2\} = \sigma^2 + E\{x_i\}^2$$

For sums of independent random variables, variances add if they have a variance. As an examp
a Cauchy distribution doesn't have a variance.

https://dsp.stackexchange.com/questions/48205/why-does-signal-averaging-reduces-noise-levels-by-more-than-sqrtn

Fourier (Spectrum) justification

- Frequency interpretation
  - Fourier spectrum — basis of exponential at different frequencies
- All real signals are band limited (by nature or by sampling process)
  - Bandlimited here means its low frequency — [0.. fs]
- Noise is not band limited and it will be great if its high frequency >fs
- FT(Signal + Noise) = FT(Signal) + FT (Noise)
- == Low Pass Signal + High Pass Signal
- Threshold in frequency to remove High Pass Noise
  - FT(Observed)
  - Take only low frequencies of FT(Observed)
  - Take IFT of that
- This is equivalent to
  - Average Filter convoluted with Observed (in space or time domain)
- REFER TO pages 15-22 of Filtering.pdf

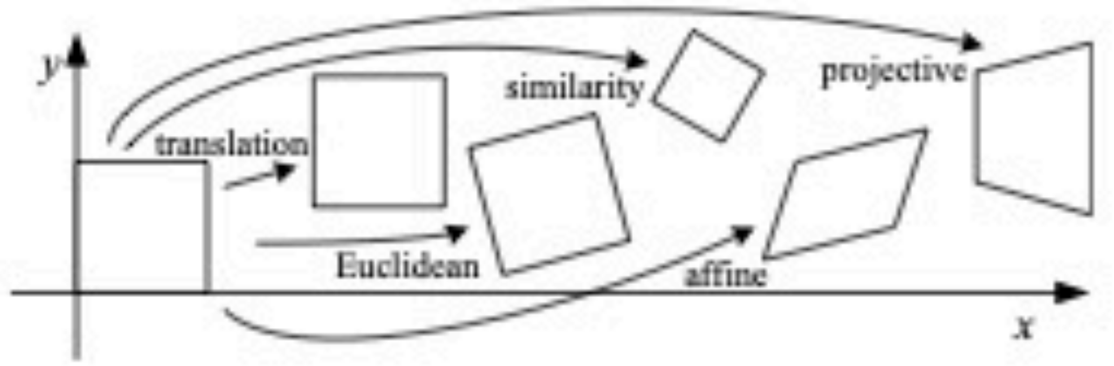


Figure 2.4 Basic set of 2D planar transformations.

Translation. 2D translations can be written as $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{t}$ or

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}} \tag{2.14}$$

where $\boldsymbol{I}$ is the (2×2) identity matrix or

$$\bar{\boldsymbol{x}}' = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \bar{\boldsymbol{x}} \tag{2.15}$$

where $\boldsymbol{0}$ is the zero vector. Using a 2×3 matrix results in a more compact notation, whereas using a full-rank 3×3 matrix (which can be obtained from the 2×3 matrix by appending a $[\boldsymbol{0}^T\ 1]$ row) makes it possible to chain transformations using matrix multiplication. Note that in any equation where an augmented vector such as $\bar{\boldsymbol{x}}$ appears on both sides, it can always be replaced with a full homogeneous vector $\tilde{\boldsymbol{x}}$.

Rotation + translation. This transformation is also known as *2D rigid body motion* or the *2D Euclidean transformation* (since Euclidean distances are preserved). It can be written as $\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$ or

$$\boldsymbol{x}' = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}} \tag{2.16}$$

where

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.17}$$

is an orthonormal rotation matrix with $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$ and $|\boldsymbol{R}| = 1$.

Scaled rotation. Also known as the *similarity transform*, this transformation can be expressed as $\boldsymbol{x}' = s\boldsymbol{R}\boldsymbol{x} + \boldsymbol{t}$ where s is an arbitrary scale factor. It can also be written as

$$\boldsymbol{x}' = \begin{bmatrix} s\boldsymbol{R} & \boldsymbol{t} \end{bmatrix} \bar{\boldsymbol{x}} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \end{bmatrix} \bar{\boldsymbol{x}}, \tag{2.18}$$

where we no longer require that $a^2 + b^2 = 1$. The similarity transform preserves angles between lines.

Affine. The affine transformation is written as $\boldsymbol{x}' = \boldsymbol{A}\bar{\boldsymbol{x}}$, where $\boldsymbol{A}$ is an arbitrary 2×3 matrix, i.e.,

$$\boldsymbol{x}' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{\boldsymbol{x}}. \tag{2.19}$$

Parallel lines remain parallel under affine transformations.

Projective. This transformation, also known as a *perspective transform* or *homography*, operates on homogeneous coordinates,

$$\tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}}, \tag{2.20}$$

where $\tilde{\boldsymbol{H}}$ is an arbitrary 3×3 matrix. Note that $\tilde{\boldsymbol{H}}$ is homogeneous, i.e., it is only defined up to a scale, and that two $\tilde{\boldsymbol{H}}$ matrices that differ only by scale are equivalent. The resulting homogeneous coordinate $\tilde{\boldsymbol{x}}'$ must be normalized in order to obtain an inhomogeneous result $\boldsymbol{x}$, i.e.,

$$x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \quad \text{and} \quad y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}}. \tag{2.21}$$

Perspective transformations preserve straight lines (i.e., they remain straight after the transformation).

Hierarchy of 2D transformations. The preceding set of transformations are illustrated in Figure 2.4 and summarized in Table 2.1. The easiest way to think of them is as a set of (potentially restricted) 3×3 matrices operating on 2D homogeneous coordinate vectors. Hartley and Zisserman (2004) contains a more detailed description of the hierarchy of 2D planar transformations.

The above transformations form a nested set of *groups*, i.e., they are closed under composition and have an inverse that is a member of the same group. (This will be important later when applying these transformations to images in Section 3.6.) Each (simpler) group is a subset of the more complex group below it.

Co-vectors. While the above transformations can be used to transform points in a 2D plane, can they also be used directly to transform a line equation? Consider the homogeneous equation $\tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0$. If we transform $\boldsymbol{x}' = \tilde{\boldsymbol{H}}\boldsymbol{x}$, we obtain

$$\tilde{\boldsymbol{l}}' \cdot \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{l}}'^T \tilde{\boldsymbol{H}}\tilde{\boldsymbol{x}} = (\tilde{\boldsymbol{H}}^T \tilde{\boldsymbol{l}}')^T \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}} \cdot \tilde{\boldsymbol{x}} = 0, \tag{2.22}$$

i.e., $\tilde{\boldsymbol{l}}' = \tilde{\boldsymbol{H}}^{-T}\tilde{\boldsymbol{l}}$. Thus, the action of a projective transformation on a *co-vector* such as a 2D line or 3D normal can be represented by the transposed inverse of the matrix, which is equivalent to the *adjoint* of $\tilde{\boldsymbol{H}}$, since projective transformation matrices are homogeneous. Jim

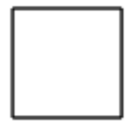
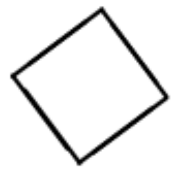
Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

Table 2.1 Hierarchy of 2D coordinate transformations. Each transformation also preserves the properties listed in the rows below it, i.e., similarity preserves not only angles but also parallelism and straight lines. The 2×3 matrices are extended with a third $[\mathbf{0}^T\ 1]$ row to form a full 3×3 matrix for homogeneous coordinate transformations.

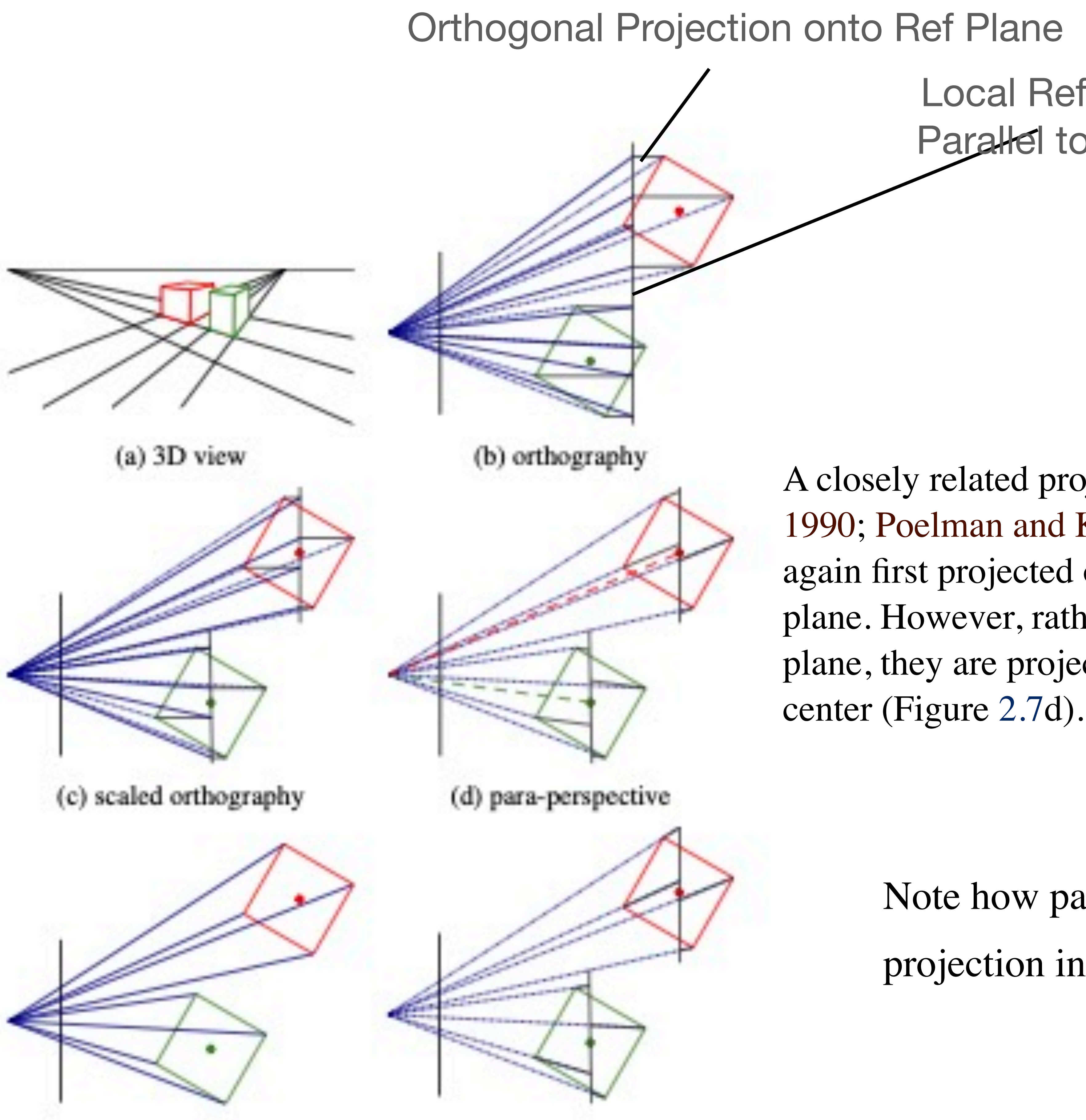


Figure 2.7 Commonly used projection models: (a) 3D view of world, (b) orthography, (c) scaled orthography, (d) para-perspective, (e) perspective, (f) object-centered. Each diagram shows a top-down view of the projection. Note how parallel lines on the ground plane and box sides remain parallel in the non-perspective projections.

A closely related projection model is *para-perspective* (Aloimonos 1990; Poelman and Kanade 1997). In this model, object points are again first projected onto a local reference parallel to the image plane. However, rather than being projected orthogonally to this plane, they are projected *parallel* to the line of sight to the object center (Figure 2.7d).

Note how parallel lines in 3D remain parallel after projection in Figure 2.7b–d.