Article

A new approximation for the perimeter of the ellipse Supplementary Material

Ellipse Approximation Collated by Sykora

Keplerian Equations

$$
\begin{aligned}
k_1 &= 2\pi\sqrt{ab}, \\
k_2 &= 2\pi\frac{(a+b)^2}{(\sqrt{a}+\sqrt{b})^2}, \\
k_3 &= \pi(a+b), \\
k_4 &= \pi(a+b)\left(\frac{3-\sqrt{1-h}}{2}\right), \\
k_5 &= \pi\sqrt{2(a^2+b^2)}, \\
k_6 &= 2\pi\left(\frac{2(a+b)^2-(\sqrt{a}-\sqrt{b})^4}{(\sqrt{a}+\sqrt{b})^2+2\sqrt{2(a+b)}\sqrt[4]{ab}}\right), \\
k_7 &= \frac{\pi}{2}\sqrt{6(a^2+b^2)+4ab}, \\
k_8 &= 2\pi\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3}, \\
k_9 &= \pi(a+b)\left(1+\frac{h}{8}\right)^2, \\
k_{10} &= \pi\left(3(a+b)-\sqrt{(a+3b)(3a+b)}\right), \\
k_{11} &= \frac{\pi}{4}\left(6+\frac{1}{2}\frac{(a-b)^2}{(a+b)^2}\right), \\
k_{12} &= \pi(a+b)\left(1+\frac{3h}{10+\sqrt{4-3h}}\right)
\end{aligned}
$$

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Keplerian Padè Equations

$$
\begin{aligned}
k_a &= \pi(a+b)\frac{16+3h}{16-h}, \\
k_b &= \pi(a+b)\frac{64+16h}{64-h^2}, \\
k_c &= \pi(a+b)\frac{64-3h^2}{64-16h}, \\
k_d &= \pi(a+b)\frac{256-48h-21h^2}{256-112h+3h^2}, \\
k_e &= \pi(a+b)\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2} \\
k_f &= m_p
\end{aligned}
$$

Optimized & Exact Extremes (No Crossing) Equations

$$
\begin{aligned}
o_2 &= \pi\sqrt{2(a^2+b^2) - \frac{(a-b)^2}{2.458338}},\\
e_1 &= \frac{\pi(a-b)}{\arctan\left(\frac{a-b}{a+b}\right)},\\
e_2 &= 4(a+b) - \frac{(8-2\pi)ab}{0.410117(a+b) + (1-2\times 0.410117)\left(\frac{\sqrt{(a+74b)(74a+b)}}{1+74}\right)},\\
e_3 &= 2\sqrt{\pi^2 ab + 4(a-b)^2},\\
e_4 &= 4\left(\frac{b^2}{a}\arctan\left(\frac{a}{b}\right) + \frac{a^2}{b}\arctan\left(\frac{b}{a}\right)\right),\\
e_5 &= \frac{4\pi ab + (a-b)^2}{a+b},\\
e_6 &= 4(a^s+b^s)^{\frac{1}{s}}, \quad s = \frac{\log 2}{\log\frac{\pi}{2}},
\end{aligned}
$$

Exact Extremes (No Crossing) Combined Padè Equations

$$
\begin{aligned}
e_{ad1} &= \frac{\pi}{4}\left(\frac{81}{64}\right) - 1, \quad e_{ad2} = \frac{\pi}{4}\left(\frac{19}{15}\right) - 1, \quad e_{ap} = \frac{e_{ad1}}{e_{ad1}-e_{ad2}},\\
e_a &= \pi(a+b)\left(e_{ap}\frac{16+3h}{16-h} + (1-e_{ap})\left(1+\frac{h}{8}\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{bd1} &= \frac{\pi}{4}\left(\frac{80}{63}\right) - 1, \quad e_{bd2} = \frac{\pi}{4}\left(\frac{61}{48}\right) - 1, \quad e_{bp} = \frac{e_{bd1}}{e_{bd1}-e_{bd2}},\\
e_b &= \pi(a+b)\left(e_{bp}\frac{64-3h^2}{64-16h} + (1-e_{bp})\frac{64+16h}{64-h^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{cd1} &= \frac{\pi}{4}\left(\frac{61}{48}\right) - 1, \quad e_{cd2} = \frac{\pi}{4}\left(\frac{187}{147}\right) - 1, \quad e_{cp} = \frac{e_{cd1}}{e_{cd1}-e_{cd2}},\\
e_c &= \pi(a+b)\left(e_{cp}\frac{256-48h-21h^2}{256-112h+3h^2} + (1-e_{cp})\frac{64-3h^2}{64-16h}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{dd1} &= \frac{\pi}{4}\left(\frac{187}{147}\right) - 1, \quad e_{dd2} = \frac{\pi}{4}\left(\frac{1573}{1236}\right) - 1, \quad e_{dp} = \frac{e_{dd1}}{e_{dd1}-e_{dd2}},\\
e_d &= \pi(a+b)\left(e_{dp}\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2} + (1-e_{dp})\frac{256-48h-21h^2}{256-112h+3h^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{ed1} &= \frac{\pi}{4}\left(\frac{1573}{1236}\right) - 1, \quad e_{ed2} = \frac{\pi}{4}\left(\frac{47707}{37479}\right) - 1, \quad e_{ep} = \frac{e_{ed1}}{e_{ed1}-e_{ed2}},\\
e_e &= \pi(a+b)\left(e_{ep}\frac{135168-85760h-5568h^2+3867h^3}{135168-119552h+22208h^2-345h^3} + (1-e_{ep})\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2}\right.
\end{aligned}
$$

Exact Extremes and Crossing Equations

$$
\begin{aligned}
c_1 &= \pi\sqrt{2(a^2+b^2)}\left(\frac{\sin(c_{1t})}{c_{1t}}\right), \quad c_{1t} = \frac{\pi}{4}\left(\frac{a-b}{b}\right),\\
c_2 &= 4a + 2(\pi-2)a\left(\frac{b}{a}\right)^{1.456},\\
c_3 &= 4\left(\frac{\pi ab + (a-b)^2}{a+b}\right) - \frac{89}{146}\left(\frac{b\sqrt{a} - a\sqrt{b}}{a+b}\right)^2,\\
c_4 &= 4(a+b) - \frac{2(4-\pi)ab}{\left(\left(\frac{a^{c_{4s}}+b^{c_{4s}}}{2}\right)^{1/c_{4s}}\right)}, \quad c_{4s} = 0.825056176207,\\
c_5 &= 4\left(\frac{\pi ab + (a-b)^2}{a+b}\right) - \frac{1}{2}\left(\frac{ab}{a+b}\right)\left(\frac{(a-b)^2}{\pi ab + (a+b)^2}\right),\\
c_6 &= \pi(a+b)\left(1 + \frac{3h}{10+\sqrt{4-3h}} + \left(\frac{4}{\pi} - \frac{14}{11}\right)h^{12}\right),\\
c_{7p} &= 3.982901, \quad c_{7q} = 66.71674, \quad c_{7r} = 56.2007, \quad c_{7s} = 18.31287, \quad c_{7t} = 23.39728,\\
c_7 &= 4(a+b) - \left(\frac{ab}{a+b}\right)\left(\frac{c_{7p}(a+b)^2 + c_{7q}ab + c_{7r}\left(\frac{ab}{a+b}\right)^2}{(a+b)^2 + c_{7s}ab + c_{7t}\left(\frac{ab}{a+b}\right)^2}\right)
\end{aligned}
$$

Algebraic Equations

$$a_1 = \frac{4(a^2 + b^2 + (\pi - 2)ab)}{a + b}$$

$$a_2 = \frac{4(a^3 + b^3 + a_{2t1}ab(a+b))}{a^2 + b^2 + 2a_{2s1}ab}, \quad a_{2t1} = 2.49808365277126, \quad a_{2s1} = 1.22694921875000,$$

$$a_{3t1} = 6.16881239339582, \quad a_{3t2} = 4.06617730084445, \quad a_{3s1} = 6.15241658169936,$$
$$a_3 = \frac{4(a^4 + b^4 + a_{3t1}ab(a^2+b^2) + 2a_{3t2}a^2b^2)}{a^3 + b^3 + a_{3s1}ab(a+b)},$$

$$a_{4t1} = 13.02487942169925, \quad a_{4t2} = 28.56997512074272, \quad a_{4s1} = 13.01750519704827, \quad a_{4s2} = 13.09$$
$$a_4 = \frac{4(a^5 + b^5 + a_{4t1}ab(a^3+b^3) + a_{4t2}a^2b^2(a+b))}{a^4 + b^4 + a_{4s1}ab(a^2+b^2) + 2a_{4s2}a^2b^2},$$

$$a_{5t1} = 27.301243680755, \quad a_{5t2} = 113.302483206429, \quad a_{5t3} = 76.50091476282086,$$
$$a_{5s1} = 27.297854333670, \quad a_{5s2} = 110.551872985869,$$
$$a_5 = \frac{4(a^6 + b^6 + a_{5t1}ab(a^4+b^4) + a_{5t2}a^2b^2(a^2+b^2) + 2a_{5t3}a^3b^3)}{a^5 + b^5 + a_{5s1}ab(a^3+b^3) + a_{5s2}a^2b^2(a+b)},$$

$$a_{6t1} = 51.447782789130, \quad a_{6t2} = 385.327854851892, \quad a_{6t3} = 790.4535392309255,$$
$$a_{6s1} = 51.445996674310, \quad a_{6s2} = 382.256974433855, \quad a_{6s3} = 347.212007887276,$$
$$a_6 = \frac{4(a^7 + b^7 + a_{6t1}ab(a^5+b^5) + a_{6t2}a^2b^2(a^3+b^3) + a_{6t3}a^3b^3(a+b))}{a^6 + b^6 + a_{6s1}ab(a^4+b^4) + a_{6s2}a^2b^2(a^2+b^2) + 2a_{6s3}a^3b^3},$$

$$a_{7t1} = 93.49235523473, \quad a_{7t2} = 1262.73239571330, \quad a_{7t3} = 4296.45229646421, \quad a_{7t4} = 2903.73561$$
$$a_{7s1} = 93.49135794687, \quad a_{7s2} = 1259.36473022183, \quad a_{7s3} = 4093.96201082922,$$
$$a_7 = \frac{4(a^8 + b^8 + a_{7t1}ab(a^6+b^6) + a_{7t2}a^2b^2(a^4+b^4) + a_{7t3}a^3b^3(a^2+b^2) + 2a_{7t4}a^4b^4)}{a^7 + b^7 + a_{7s1}ab(a^5+b^5) + a_{7s2}a^2b^2(a^3+b^3) + a_{7s3}a^3b^3(a+b)}.$$

Class S Equations

$$
\begin{aligned}
s_{0q} &= 4.16102118885517, \quad s_{0u} = \frac{4-\pi}{4-2\sqrt{2+s_{0q}}}, \quad s_{0p} = 1 - s_{0u}, \\
s_0 &= s_{0p}(a+b) + s_{0u}\sqrt{a^2+b^2+s_{0q}ab},
\end{aligned}
$$

$$
\begin{aligned}
s_{1q} &= 92.28480788617108, \quad s_{1v} = 0.04522028227769, \quad s_{1p2} = 0.99983439391729, \quad s_{1u} = 1 + s_{1v} \\
s_{1p1} &= \frac{\pi}{2}(2 + s_{1v}\sqrt{2+s_{1q}}) - (2s_{1p2} + 2s_{1u}\sqrt{2+s_{1q}}), \\
s_1 &= \frac{s_{1p2}(a^2+b^2) + s_{1p1}ab + s_{1u}(a+b)\sqrt{a^2+b^2+s_{1q}ab}}{(a+b) + s_{1v}\sqrt{a^2+b^2+s_{1q}ab}},
\end{aligned}
$$

$$
\begin{aligned}
s_{2q1} &= 13.66022044408346, \quad s_{2q2} = 37.30921886231118, \quad s_{2v} = -1.03788930003090, \quad s_{2t} = 5.519 \\
s_{2p2} &= 1.24957869093182, \quad s_{2u} = 1 + s_{2v} - s_{2p2}, \quad s_{2p1} = \frac{\pi}{4}(2 + s_{2t} + s_{2v}\sqrt{2+2s_{2q1}+s_{2q2}}) - (s_{2} \\
s_{2R} &= \sqrt{(a^4+b^4) + s_{2q1}ab(a^2+b^2) + s_{2q2}a^2b^2}, \\
s_2 &= \frac{s_{2p2}(a^3+b^3) + s_{2p1}ab(a+b) + s_{2u}(a+b)s_{2R}}{(a^2+b^2) + s_{2t}ab + s_{2v}s_{2R}},
\end{aligned}
$$

$$
\begin{aligned}
s_{1rq} &= 74.01745125408363, \quad s_{1rv} = 0.05027328304233, \quad s_{1ru} = s_{1rv}, \\
s_{1rp1} &= \frac{\pi}{2}(2 + s_{1rv}\sqrt{2+s_{1rq}}) - (2 + 2s_{1ru}\sqrt{2+s_{1rq}}), \\
s_{1r} &= \frac{(a^2+b^2) + s_{1rp1}ab + s_{1ru}(a+b)\sqrt{a^2+b^2+s_{1rq}ab}}{(a+b) + s_{1rv}\sqrt{a^2+b^2+s_{1rq}ab}},
\end{aligned}
$$

$$
\begin{aligned}
s_{ac1} &= \pi - 3, \quad s_{ac2} = \pi, \quad s_{ac3} = 0.5, \quad s_{ac4} = \frac{1+\pi}{2}, \quad s_{ac5} = 4, \\
s_{ak} &= 1 - \frac{s_{ac1}ab}{(a^2+b^2) + s_{ac2}\sqrt{s_{ac3}(ab)^2 + ab\sqrt{ab(s_{ac4}(a^2+b^2) + s_{ac5}ab)}}}, \\
s_a &= 4\frac{\pi ab + s_{ak}(a-b)^2}{a+b},
\end{aligned}
$$

$$
\begin{aligned}
s_{aoc1} &= 0.14220038049945, \quad s_{aoc2} = 3.30596250119242, \quad s_{aoc3} = 0.00135657637724, \\
s_{aoc4} &= 2.00637978782056, \quad s_{aoc5} = 5.3933761426286, \\
s_{aok} &= 1 - \frac{s_{aoc1}ab}{(a^2+b^2) + s_{aoc2}\sqrt{s_{aoc3}(ab)^2 + ab\sqrt{ab(s_{aoc4}(a^2+b^2) + s_{aoc5}ab)}}}, \\
s_{ao} &= 4\frac{\pi ab + s_{aok}(a-b)^2}{a+b},
\end{aligned}
$$

$$
\begin{aligned}
s_{ard1} &= 0.14220096377128, \quad s_{ard2} = 3.93490847789660, \quad s_{ard3} = 2.691437204515743, \\
s_{ark} &= 1 - \frac{s_{ard1}ab}{(a^2+b^2) + s_{ard2}\sqrt{\sqrt{(ab)^3(a^2+b^2+s_{ard3}ab)}}}, \\
s_{ar} &= 4\frac{\pi ab + s_{ark}(a-b)^2}{a+b}.
\end{aligned}
$$

We note for the functions s_0, s_1, s_2, we are unable to obtain accurate approximations at high precision as shown in Supplementary Fig. 1. The error shown is around 0.75, while the error observed in other approximations in the prior figures is around 1×10^{-9}, suggesting that either the function may be defined incorrectly or that when evaluated at high precision, the results are negatively impacted.

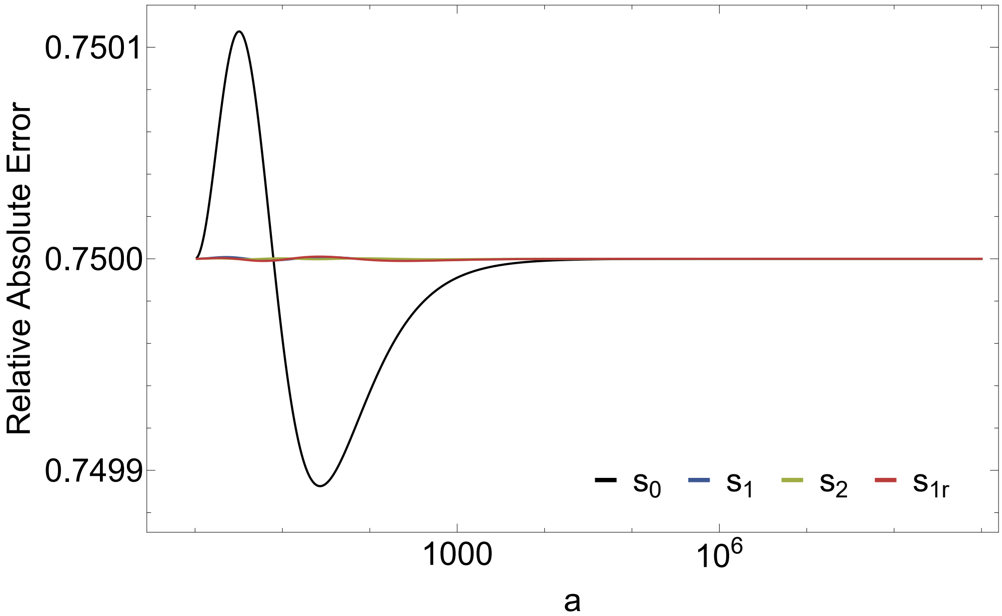

Figure 1. Log absolute relative error vs log a plot for S-class functions s_0, s_1 and s_2 when evaluated over the log space between a=1.05 and $a = 1 \times 10^9$.

Mathematica Software

A GitHub repository is available of the following code https://github.com/impvau/ellipse.