# A new approximation for the perimeter of the ellipse
# Supplementary Material

## Ellipse Approximation Collated by Sykora

### Keplerian Equations

$$k_1 = 2\pi\sqrt{ab},$$
$$k_2 = 2\pi\frac{(a+b)^2}{(\sqrt{a}+\sqrt{b})^2},$$
$$k_3 = \pi(a+b),$$
$$k_4 = \pi(a+b)\left(\frac{3-\sqrt{1-h}}{2}\right),$$
$$k_5 = \pi\sqrt{2(a^2+b^2)},$$
$$k_6 = 2\pi\left(\frac{2(a+b)^2-(\sqrt{a}-\sqrt{b})^4}{(\sqrt{a}+\sqrt{b})^2+2\sqrt{2(a+b)\sqrt[4]{ab}}}\right),$$
$$k_7 = \frac{\pi}{2}\sqrt{6(a^2+b^2)+4ab},$$
$$k_8 = 2\pi\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3},$$
$$k_9 = \pi(a+b)\left(1+\frac{h}{8}\right)^2,$$
$$k_{10} = \pi\left(3(a+b)-\sqrt{(a+3b)(3a+b)}\right),$$
$$k_{11} = \frac{\pi}{4}\left(6+\frac{1}{2}\frac{(a-b)^2}{(a+b)^2}\right),$$
$$k_{12} = \pi(a+b)\left(1+\frac{3h}{10+\sqrt{4-3h}}\right)$$

### Keplerian Padè Equations

$$k_a = \pi(a+b)\frac{16+3h}{16-h},$$
$$k_b = \pi(a+b)\frac{64+16h}{64-h^2},$$
$$k_c = \pi(a+b)\frac{64-3h^2}{64-16h},$$
$$k_d = \pi(a+b)\frac{256-48h-21h^2}{256-112h+3h^2},$$
$$k_e = \pi(a+b)\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2}$$
$$k_f = m_p$$

### Optimized & Exact Extremes (No Crossing) Equations

$$
\begin{aligned}
o_2 &= \pi\sqrt{2(a^2+b^2)-\frac{(a-b)^2}{2.458338}},\\
e_1 &= \frac{\pi(a-b)}{\arctan\left(\frac{a-b}{a+b}\right)},\\
e_2 &= 4(a+b)-\frac{(8-2\pi)ab}{0.410117(a+b)+(1-2\times 0.410117)\left(\frac{\sqrt{(a+74b)(74a+b)}}{1+74}\right)},\\
e_3 &= 2\sqrt{\pi^2 ab+4(a-b)^2},\\
e_4 &= 4\left(\frac{b^2}{a}\arctan\left(\frac{a}{b}\right)+\frac{a^2}{b}\arctan\left(\frac{b}{a}\right)\right),\\
e_5 &= \frac{4\pi ab+(a-b)^2}{a+b},\\
e_6 &= 4\left(a^s+b^s\right)^{\frac{1}{s}}, \quad s=\frac{\log 2}{\log\frac{\pi}{2}},
\end{aligned}
$$

### Exact Extremes (No Crossing) Combined Padè Equations

$$
\begin{aligned}
e_{ad1} &= \frac{\pi}{4}\left(\frac{81}{64}\right)-1, \quad e_{ad2}=\frac{\pi}{4}\left(\frac{19}{15}\right)-1, \quad e_{ap}=\frac{e_{ad1}}{e_{ad1}-e_{ad2}},\\
e_a &= \pi(a+b)\left(e_{ap}\frac{16+3h}{16-h}+(1-e_{ap})\left(1+\frac{h}{8}\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{bd1} &= \frac{\pi}{4}\left(\frac{80}{63}\right)-1, \quad e_{bd2}=\frac{\pi}{4}\left(\frac{61}{48}\right)-1, \quad e_{bp}=\frac{e_{bd1}}{e_{bd1}-e_{bd2}},\\
e_b &= \pi(a+b)\left(e_{bp}\frac{64-3h^2}{64-16h}+(1-e_{bp})\frac{64+16h}{64-h^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{cd1} &= \frac{\pi}{4}\left(\frac{61}{48}\right)-1, \quad e_{cd2}=\frac{\pi}{4}\left(\frac{187}{147}\right)-1, \quad e_{cp}=\frac{e_{cd1}}{e_{cd1}-e_{cd2}},\\
e_c &= \pi(a+b)\left(e_{cp}\frac{256-48h-21h^2}{256-112h+3h^2}+(1-e_{cp})\frac{64-3h^2}{64-16h}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{dd1} &= \frac{\pi}{4}\left(\frac{187}{147}\right)-1, \quad e_{dd2}=\frac{\pi}{4}\left(\frac{1573}{1236}\right)-1, \quad e_{dp}=\frac{e_{dd1}}{e_{dd1}-e_{dd2}},\\
e_d &= \pi(a+b)\left(e_{dp}\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2}+(1-e_{dp})\frac{256-48h-21h^2}{256-112h+3h^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
e_{ed1} &= \frac{\pi}{4}\left(\frac{1573}{1236}\right)-1, \quad e_{ed2}=\frac{\pi}{4}\left(\frac{47707}{37479}\right)-1, \quad e_{ep}=\frac{e_{ed1}}{e_{ed1}-e_{ed2}},\\
e_e &= \pi(a+b)\left(e_{ep}\frac{135168-85760h-5568h^2+3867h^3}{135168-119552h+22208h^2-345h^3}+(1-e_{ep})\frac{3072-1280h-252h^2+33h^3}{3072-2048h+212h^2}\right),
\end{aligned}
$$

**Exact Extremes and Crossing Equations**

$$c_1 = \pi\sqrt{2\left(a^2+b^2\right)}\left(\frac{\sin(c_{1t})}{c_{1t}}\right), \quad c_{1t} = \frac{\pi}{4}\left(\frac{a-b}{b}\right),$$

$$c_2 = 4a + 2(\pi-2)a\left(\frac{b}{a}\right)^{1.456},$$

$$c_3 = 4\left(\frac{\pi ab + (a-b)^2}{a+b}\right) - \frac{89}{146}\left(\frac{b\sqrt{a} - a\sqrt{b}}{a+b}\right)^2,$$

$$c_4 = 4(a+b) - \frac{2(4-\pi)ab}{\left(\left(\frac{a^{c_{4s}}+b^{c_{4s}}}{2}\right)^{1/c_{4s}}\right)}, \quad c_{4s} = 0.825056176207,$$

$$c_5 = 4\left(\frac{\pi ab + (a-b)^2}{a+b}\right) - \frac{1}{2}\left(\frac{ab}{a+b}\right)\left(\frac{(a-b)^2}{\pi ab + (a+b)^2}\right),$$

$$c_6 = \pi(a+b)\left(1 + \frac{3h}{10+\sqrt{4-3h}} + \left(\frac{4}{\pi} - \frac{14}{11}\right)h^{12}\right),$$

$$c_{7p} = 3.982901, \quad c_{7q} = 66.71674, \quad c_{7r} = 56.2007, \quad c_{7s} = 18.31287, \quad c_{7t} = 23.39728,$$

$$c_7 = 4(a+b) - \left(\frac{ab}{a+b}\right)\left(\frac{c_{7p}(a+b)^2 + c_{7q}ab + c_{7r}\left(\frac{ab}{a+b}\right)^2}{(a+b)^2 + c_{7s}ab + c_{7t}\left(\frac{ab}{a+b}\right)^2}\right)$$

### Algebraic Equations

$$a_1 = \frac{4(a^2+b^2+(\pi-2)ab)}{a+b}$$

$$a_2 = \frac{4(a^3+b^3+a_{2t1}ab(a+b))}{a^2+b^2+2a_{2s1}ab}, \quad a_{2t1} = 2.49808365277126, \quad a_{2s1} = 1.22694921875000,$$

$$a_{3t1} = 6.16881239339582, \quad a_{3t2} = 4.06617730084445, \quad a_{3s1} = 6.15241658169936,$$
$$a_3 = \frac{4(a^4+b^4+a_{3t1}ab(a^2+b^2)+2a_{3t2}a^2b^2)}{a^3+b^3+a_{3s1}ab(a+b)},$$

$$a_{4t1} = 13.02487942169925, \quad a_{4t2} = 28.56997512074272, \quad a_{4s1} = 13.01750519704827, \quad a_{4s2} = 13.09922140579137,$$
$$a_4 = \frac{4(a^5+b^5+a_{4t1}ab(a^3+b^3)+a_{4t2}a^2b^2(a+b))}{a^4+b^4+a_{4s1}ab(a^2+b^2)+2a_{4s2}a^2b^2},$$

$$a_{5t1} = 27.301243680755, \quad a_{5t2} = 113.302483206429, \quad a_{5t3} = 76.50091476282086,$$
$$a_{5s1} = 27.297854333670, \quad a_{5s2} = 110.551872985869,$$
$$a_5 = \frac{4(a^6+b^6+a_{5t1}ab(a^4+b^4)+a_{5t2}a^2b^2(a^2+b^2)+2a_{5t3}a^3b^3)}{a^5+b^5+a_{5s1}ab(a^3+b^3)+a_{5s2}a^2b^2(a+b)},$$

$$a_{6t1} = 51.447782789130, \quad a_{6t2} = 385.327854851892, \quad a_{6t3} = 790.4535392309255,$$
$$a_{6s1} = 51.445996674310, \quad a_{6s2} = 382.256974433855, \quad a_{6s3} = 347.212007887276,$$
$$a_6 = \frac{4(a^7+b^7+a_{6t1}ab(a^5+b^5)+a_{6t2}a^2b^2(a^3+b^3)+a_{6t3}a^3b^3(a+b))}{a^6+b^6+a_{6s1}ab(a^4+b^4)+a_{6s2}a^2b^2(a^2+b^2)+2a_{6s3}a^3b^3},$$

$$a_{7t1} = 93.49235523473, \quad a_{7t2} = 1262.73239571330, \quad a_{7t3} = 4296.45229646421, \quad a_{7t4} = 2903.735611540449,$$
$$a_{7s1} = 93.49135794687, \quad a_{7s2} = 1259.36473022183, \quad a_{7s3} = 4093.96201082922,$$
$$a_7 = \frac{4(a^8+b^8+a_{7t1}ab(a^6+b^6)+a_{7t2}a^2b^2(a^4+b^4)+a_{7t3}a^3b^3(a^2+b^2)+2a_{7t4}a^4b^4)}{a^7+b^7+a_{7s1}ab(a^5+b^5)+a_{7s2}a^2b^2(a^3+b^3)+a_{7s3}a^3b^3(a+b)}.$$

**Class S Equations**

$$s_{0q}=4.16102118885517,\quad s_{0u}=\frac{4-\pi}{4-2\sqrt{2+s_{0q}}},\quad s_{0p}=1-s_{0u},$$
$$s_0=s_{0p}(a+b)+s_{0u}\sqrt{a^2+b^2+s_{0q}ab},$$

$$s_{1q}=92.28480788617108,\quad s_{1v}=0.04522028227769,\quad s_{1p2}=0.99983439391729,\quad s_{1u}=1+s_{1v}-s_{1p2},$$
$$s_{1p1}=\frac{\pi}{2}(2+s_{1v}\sqrt{2+s_{1q}})-(2s_{1p2}+2s_{1u}\sqrt{2+s_{1q}}),$$
$$s_1=\frac{s_{1p2}(a^2+b^2)+s_{1p1}ab+s_{1u}(a+b)\sqrt{a^2+b^2+s_{1q}ab}}{(a+b)+s_{1v}\sqrt{a^2+b^2+s_{1q}ab}},$$

$$s_{2q1}=13.66022044408346,\quad s_{2q2}=37.30921886231118,\quad s_{2v}=-1.03788930003090,\quad s_{2t}=5.51954143485218,$$
$$s_{2p2}=1.24957869093182,\quad s_{2u}=1+s_{2v}-s_{2p2},\quad s_{2p1}=\frac{\pi}{4}(2+s_{2t}+s_{2v}\sqrt{2+2s_{2q1}+s_{2q2}})-(s_{2p2}+s_{2u}\sqrt{2+2s_{2q1}+s_{2q2}}),$$
$$s_{2R}=\sqrt{(a^4+b^4)+s_{2q1}ab(a^2+b^2)+s_{2q2}a^2b^2},$$
$$s_2=\frac{s_{2p2}(a^3+b^3)+s_{2p1}ab(a+b)+s_{2u}(a+b)s_{2R}}{(a^2+b^2)+s_{2t}ab+s_{2v}s_{2R}},$$

$$s_{1rq}=74.01745125408363,\quad s_{1rv}=0.05027328304233,\quad s_{1ru}=s_{1rv},$$
$$s_{1rp1}=\frac{\pi}{2}(2+s_{1rv}\sqrt{2+s_{1rq}})-(2+2s_{1ru}\sqrt{2+s_{1rq}}),$$
$$s_{1r}=\frac{(a^2+b^2)+s_{1rp1}ab+s_{1ru}(a+b)\sqrt{a^2+b^2+s_{1rq}ab}}{(a+b)+s_{1rv}\sqrt{a^2+b^2+s_{1rq}ab}},$$

$$s_{ac1}=\pi-3,\quad s_{ac2}=\pi,\quad s_{ac3}=0.5,\quad s_{ac4}=\frac{1+\pi}{2},\quad s_{ac5}=4,$$
$$s_{ak}=1-\frac{s_{ac1}ab}{(a^2+b^2)+s_{ac2}\sqrt{s_{ac3}(ab)^2+ab\sqrt{ab\,(s_{ac4}(a^2+b^2)+s_{ac5}ab)}}},$$
$$s_a=4\frac{\pi ab+s_{ak}(a-b)^2}{a+b},$$

$$s_{aoc1}=0.14220038049945,\quad s_{aoc2}=3.30596250119242,\quad s_{aoc3}=0.00135657637724,$$
$$s_{aoc4}=2.00637978782056,\quad s_{aoc5}=5.3933761426286,$$
$$s_{aok}=1-\frac{s_{aoc1}ab}{(a^2+b^2)+s_{aoc2}\sqrt{s_{aoc3}(ab)^2+ab\sqrt{ab\,(s_{aoc4}(a^2+b^2)+s_{aoc5}ab)}}},$$
$$s_{ao}=4\frac{\pi ab+s_{aok}(a-b)^2}{a+b},$$

$$s_{ard1}=0.14220096377128,\quad s_{ard2}=3.93490847789660,\quad s_{ard3}=2.691437204515743,$$
$$s_{ark}=1-\frac{s_{ard1}ab}{(a^2+b^2)+s_{ard2}\sqrt{\sqrt{(ab)^3(a^2+b^2+s_{ard3}ab)}}},$$
$$s_{ar}=4\frac{\pi ab+s_{ark}(a-b)^2}{a+b}.$$

We note for the functions $s_0$, $s_1$, $s_2$, we are unable to obtain accurate approximations at high precision as shown in Fig. 1. The error shown is around 0.75, while the error observed in other approximations in the prior figures is around $1 \times 10^{-9}$, suggesting that either the function may be defined incorrectly or that when evaluated at high precision, the results are negatively impacted.

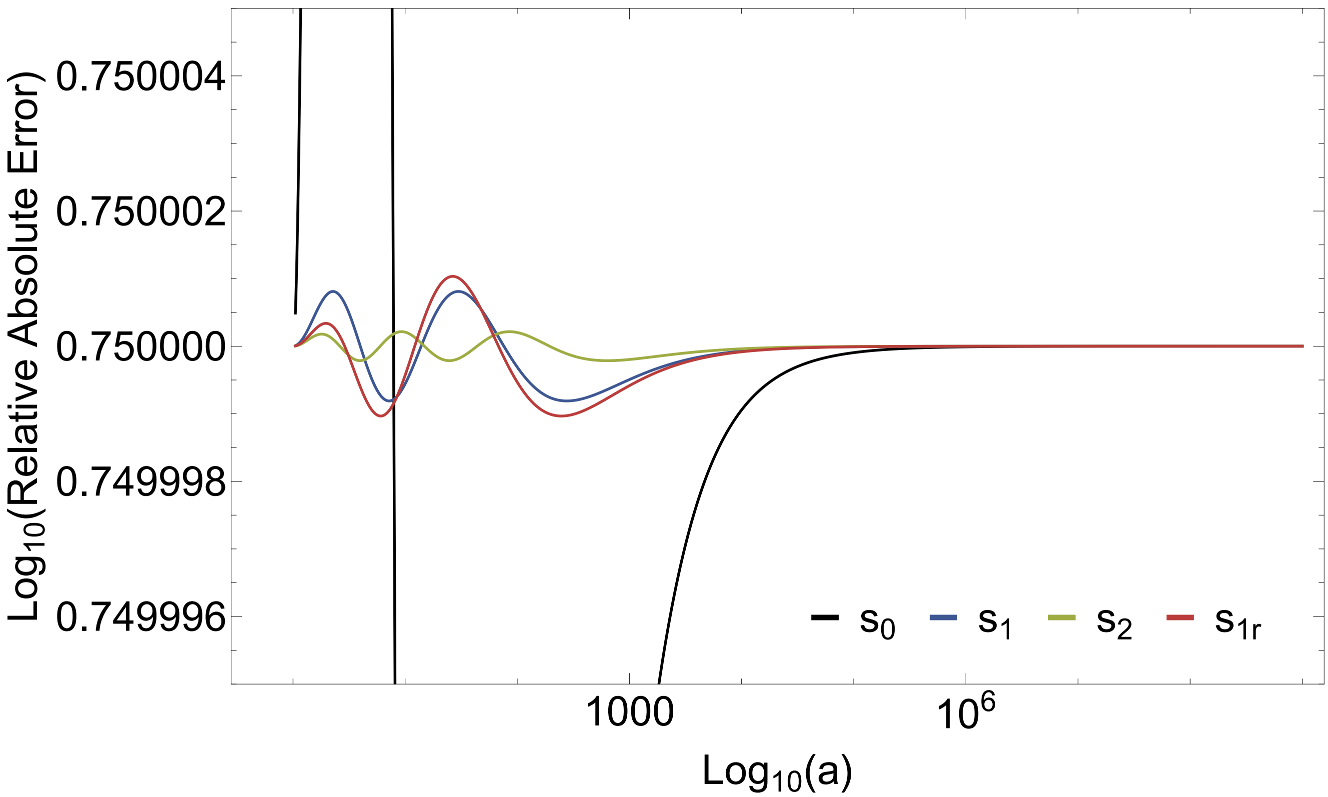


**Figure 1.** Log absolute relative error vs log a plot for S-class functions $s_0$, $s_1$ and $s_2$ when evaluated over the log space between $a$=1.05 and $a = 1 \times 10^9$.

### Mathematica Software

A GitHub repository is available at

#### ellipse.nb

**Listing 1.** Entry point for the functionality that re-generates all data files, tables and plots

```
(*Set the desired precision*)
prec = 200;
x = "a";
n = 10000;

SetDirectory[NotebookDirectory[]];

Needs["EllipseFuncs`"];
Needs["EllipseData`"];
Needs["Plots`"];

(* Evaluate points initial training points *)
results = GetEllipseFuncs[DataTrain[]];

(* Output results to CSV file *)
(*selectedKeys={"yd","y","a","h","h1","h2","hd","h1d","h2d","Mu","MR",\
"MRu","Mp", "M6"}; *)
selectedKeys = {"ynp", "h"};
selection = results[All, selectedKeys];
formatNumber[num_] :=
  ToString[
   NumberForm[N[num, 200], {200, 200}, ExponentFunction -> (Null &),
    NumberPadding -> {"", "0"}]];
data = results /.
   assoc_Association :> (formatNumber[assoc[#]] & /@ selectedKeys);
data = Prepend[data, selectedKeys];
Export["ellipse_hp_yd.csv", data, "CSV", "TextDelimiters" -> ""]

(*Check the precision of the result*)
(*
actualPrecision = Precision[results[[All,"y"]]];
Print["Precision of the result: ",actualPrecision];
*)

(*Extract and transform for the table *)
selectedKeys = {x, "MR Err", "Mu Err", "MRu Err", "Mp Err", "MMC Err"};
selection =
  Transpose[SetPrecision[results[[All, #]] & /@ selectedKeys, prec]];

formatNumber[num_] :=
  Module[{absNum = Abs[num]},
   If[absNum > 1 && absNum < 1000000000000000000000000000000000000,
    ToString[num],
    ToString[
     ScientificForm[N[num], NumberFormat -> (Row[{#1, "E", #3}] &)]]]];

formattedTable =
  MapIndexed[
```

```
   Function[{value, index},
    If[NumericQ[value],
     If[index[[2]] == 1,(*For the first column'a'*)
      NumberForm[N[value, prec], {Infinity, 2},
       ExponentFunction -> (Null &)],(*Avoid scientific notation*)
      formatNumber[
       value] (*Apply existing logic for other columns*)],
     value (*Handle non-numeric gracefully*)]], selection, {2}];

formattedTable = Prepend[formattedTable, selectedKeys];

Grid[formattedTable, Frame -> All, Alignment -> Left, Dividers -> All]

(* Evaluate points with a sampling assuming log10 plot *)
dataPairs =
  Table[{a, 1}, {a,
    Table[10^((Log10[1.05]*(n - i) + Log10[1000000000]*(i - 1))/(n -
          1)), {i, n}]}];
Grid[dataPairs, Frame -> All];
results = GetEllipseFuncs[dataPairs];

(* Currently must run this after changing the range in Plots.m to the \
commented version *)
(*
selectedKeys = {x, "MMC Err","MR Err","MRu Err" ,"Mu Err", "Mp Err" \
};
createAndSavePlotSR[ "plot_err_m6_vs_intro_80.pdf", selectedKeys, \
results, x, "Linear", "Linear"];
*)

selectedKeys = {x, "MMC Err", "MR Err", "MRu Err", "Mu Err",
   "Mp Err" };
createAndSavePlotSR[ "plot_err_m6_vs_intro.pdf", selectedKeys,
  results, x, "Log", "Log"];

selectedKeys = {x, "MMC Err", "Sa Err", "Sao Err", "Sar Err"};
pltAlg =
 createAndSavePlotSR[ "plot_err_m6_vs_s.pdf", selectedKeys, results,
  x, "Log", "Log",
  ToString[Subscript["m", "MC"], StandardForm] <> " vs S-Class" ]


selectedKeys = {x, "MMC Err", "K1 Err", "K2 Err", "K3 Err", "K4 Err",
   "K5 Err", "K6 Err", "K7 Err", "K8 Err", "K9 Err", "K10 Err",
   "K11 Err", "K12 Err"};
pltKelp =
  createAndSavePlotSR[ "plot_err_m6_vs_kelp.pdf", selectedKeys,
   results, x, "Log", "Log",
   Row[{Style["a) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Keplarian"}]];

selectedKeys = {x, "MMC Err", "Ka Err", "Kb Err", "Kc Err", "Kd Err",
   "Ke Err", "Mp Err"};
pltPade =
```

```
  createAndSavePlotSR[ "plot_err_m6_vs_kelp_pade.pdf",  selectedKeys,
   results, x, "Log", "Log",
   Row[{Style["b) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Keplerian Pad "}]];

selectedKeys = {x, "MMC Err", "E1 Err", "E2 Err", "E3 Err", "E4 Err",
   "E5 Err", "E6 Err", "O2 Err"};
pltExtr1 =
  createAndSavePlotSR[ "plot_err_m6_vs_extreme_nox.pdf",
   selectedKeys, results, x, "Log", "Log",
   Row[{Style["c) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Exact Extermes No-Crossing"}]];

selectedKeys = {x, "MMC Err", "Ea Err", "Eb Err", "Ec Err", "Ed Err",
   "Ee Err"};
pltExtr2 =
  createAndSavePlotSR[ "plot_err_m6_vs_extreme_nox_pade.pdf",
   selectedKeys, results, x, "Log", "Log",
   Row[{Style["d) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Exact Extermes Pad "}]];

selectedKeys = {x, "MMC Err", "C1 Err", "C2 Err", "C3 Err", "C4 Err",
   "C5 Err", "C6 Err", "C7 Err"};
pltExtr3 =
  createAndSavePlotSR[ "plot_err_m6_vs_extreme_x.pdf",  selectedKeys,
   results, x, "Log", "Log",
   Row[{Style["e) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Exact Extermes Crossing"}]];

selectedKeys = {x, "MMC Err", "A1 Err", "A2 Err", "A3 Err", "A4 Err",
   "A5 Err", "A6 Err", "A7 Err"};
pltAlg =
  createAndSavePlotSR[ "plot_err_m6_vs_algebra.pdf",  selectedKeys,
   results, x, "Log", "Log",
   Row[{Style["f) ", Bold],
     ToString[Subscript["m", "MC"], StandardForm] <>
      " vs Algebraic"}]];

plots = GraphicsGrid[{{pltKelp, pltPade}, {pltExtr1,
     pltExtr2}, {pltExtr3, pltAlg}}];

Export["plot_compare.pdf", plots, ImageSize -> 1750];
```

**data.m**

**Listing 2.** Generates the training data points for use in the main notebook

```
Remove[DataTrain];
Remove[DataAll];

BeginPackage["EllipseData`"];

(* Export all symbols that will be defined in subpackages *)
DataTrain::usage = "DataTrain[] generates data used to train on.";
DataAll::usage = "DataAll[minZ, maxZ, n] generates data used to plot and evaluate over a larg

Begin["`Private`"];

(* Function to generate and return sorted combinations *)
DataTrain[] := Module[
  {additionalCombinations, aRangeCombinations, combinations},

  (* Define additional specific combinations *)
  additionalCombinations = {
    (*{1,1},*){105/100,1},{115/100,1},{125/100,1},{135/100,1},{145/100,1},{155/100,1},{165/100
  };

  (* Generate a range of combinations for a = 2 to 30 *)
  aRangeCombinations = Table[{a, 1}, {a, 2, 30}];

  (* Join and sort combinations by increasing a *)
  combinations = Join[additionalCombinations, aRangeCombinations];
  SortBy[combinations, First]
];

(* Function to generate combinations based on a transformation formula *)
DataAll[minZ_, maxZ_, n_Integer] := Module[
  {zValues, aValues, combinations},
  zValues = Subdivide[minZ, maxZ, n - 1]; (* Uniform z values from minZ to maxZ *)
  aValues = #/(1 - #) & /@ zValues;       (* Calculate corresponding a values using the trans
  combinations = Table[{aValues[[i]], zValues[[i]]}, {i, 1, n}]; (* Pair each a value with it
  combinations
];

End[];
EndPackage[];
```

**plot.m**

**Listing 3.** Generate plots according to scientific reports functionality

```
Remove[createAndSavePlotSR]
BeginPackage["Plots`"];

createAndSavePlotSR::usage =
  "createAndSavePlotSR[filename, selectedKeys, results, x, scalingX, scalingY, plotTitle] cre

Begin["`Private`"];

createAndSavePlotSR[filename_String, selectedKeys_List, results_, x_, scalingX_:"Linear", sc
  {
    errorTypes, colors, plotData, legendLabels, plot, validData
  },

  errorTypes = DeleteCases[selectedKeys, x];

  colors = Table[If[i == 1, Black,
    ColorData["DarkRainbow"][(i - 2)/(Length[errorTypes] - 2)]], {i, Length[errorTypes]}];

  plotData = Table[Tooltip[
    Table[{results[[idx, x]], results[[idx, errorType]]}, {idx, Length[results]}],
    Style[Subscript[StringTake[errorType, 1],
      StringTake[errorType, {2, First[Flatten[{StringPosition[errorType, " "], Length[errorTy
  ], {errorType, errorTypes}];

  legendLabels = Style[Subscript[ToLowerCase[StringTake[#, 1]],
    StringTake[#, {2, First[Flatten[{StringPosition[#, " "], Length[#] + 1}]] - 1}]], FontSiz

  plot = ListPlot[
    plotData,
    ScalingFunctions -> {scalingX, scalingY},
    Axes -> False,
    Frame -> True,
    FrameLabel -> {{Style[ToString[Subscript["Log","10"],StandardForm] <> "(Relative Absolute
    (*FrameLabel -> {{Style["Relative Absolute Error", 30, FontFamily -> "Helvetica"], None},
    (*PlotLabel -> If[plotTitle != "", Style[plotTitle, 30, FontFamily -> "Helvetica"], None]
(* New line for title *)*)
    PlotLabel -> If[plotTitle =!= Null, Style[plotTitle, 30, FontFamily -> "Helvetica"], None
    LabelStyle -> {FontSize -> 30, FontFamily -> "Helvetica"},
    Joined -> True,
    (*PlotRange -> {Automatic, {0.750005,0.749995}},*)
    (*PlotRange -> {{0,80}, All},*)
    PlotRange -> {Automatic, Automatic},
    Ticks -> {Automatic, Automatic},
    ImageSize -> {969, 603},
    GridLines -> None,
    PlotStyle -> (Directive[Thick, #] & /@ colors),
    PlotLegends -> Placed[LineLegend[
      (Directive[Thick, AbsoluteThickness[4], #] & /@ colors),
      (Style[#, 30] & /@ legendLabels),
      LegendLayout -> "Row"],
      (*{Right, Top},*)
      {Scaled[{0.95, 0.05}], {Right, Bottom}},
```

```
        LegendFunction -> (Framed[#, FrameMargins -> 10, Background -> White] &)]
    ];

    Export[filename, plot];

    plot

  ];

End[];

EndPackage[];
```

**funcs.m**

**Listing 4.** Define a series of functions that are evaluated on the training data or plotting data

```
Remove[GetEllipseFuncs];

BeginPackage["EllipseFuncs`"];

GetEllipseFuncs::usage =
  "GetEllipseFuncs[data_List] calculates various ellipse approximations \
expressions based on a list of inputs {a, b}.";

Begin["`Private`"];

(* Set precision for all calculations *)
prec = 200;

(* Redefine GetEllipseFuncs to accept a list of {a, b} pairs *)
GetEllipseFuncs[data_List] := Module[
   {results, pi = SetPrecision[Pi, prec]},
   results = Table[
     Module[
       {a = SetPrecision[pair[[1]], prec], b = SetPrecision[pair[[2]], prec], f = <||>},

       (* Direct calculations with consistent precision *)
       f["a"] = a;
       f["z"] = SetPrecision[a/(a+1),prec];
       f["b"] = b;
       f["y"] = SetPrecision[4*EllipticE[1 - b^2/a^2]*a, prec];
       f["ynp"] = SetPrecision[f["y"]/(pi*(a+b)), prec];
       f["h"] = SetPrecision[((a - b)/(a + b))^2, prec];
       f["k"] = SetPrecision[Sqrt[1 - Min[a, b]^2/Max[a, b]^2], prec];



       (* Calculations depending on previously computed values, all with explicit precision s
       f["MR"] = SetPrecision[pi*(a + b)*(1 + (3*f["h"])/(10 + Sqrt[4 - 3*f["h"]])), prec];
       f["Mu"] = SetPrecision[pi*(a + b)*(1 + (44/pi - 11)*f["h"]/(10 + Sqrt[4 - 3*f["h"]])),
       f["h1"] = SetPrecision[(((f["MR"] - f["Mu"])^2)/((f["MR"] + f["Mu"])^2)), prec];
       f["MRu"] = SetPrecision[f["MR"] + 9938*f["Mu"]*f["h"]^7*f["h1"], prec];
       f["Mp"] = SetPrecision[pi*(a + b)*(135168 - 85760*f["h"] - 5568*f["h"]^2 + 3867*f["h"]
       f["h2"] = SetPrecision[(((f["Mp"] - f["Mu"])^2)/((f["Mp"] + f["Mu"])^2)), prec];
       f["M1"] = SetPrecision[f["Mu"] - ((((-125494663)/(a + 59)) + 2392263)*f["h1"]), prec];
       f["M2"] = SetPrecision[f["Mp"]*(9686.95*(f["h1"] - f["h2"])) + f["Mp"], prec];
       f["M3"] = SetPrecision[f["Mp"]*(-581184*f["h1"]/(39 + a)) + f["Mu"], prec];
       f["M4"] = SetPrecision[f["Mp"]*Power[f["Mu"]/f["Mp"],Power[(f["h1"]/615)*f["h1"],f["h1
       f["M5"] = SetPrecision[f["Mp"]*Power[f["Mu"]/f["Mp"],Power[22,((-13)/(f["h2"]*a))*f["h
       (* Old one with less effective coefficient
       f["M6"] = SetPrecision[f["Mp"]*Power[f["Mu"]/f["Mp"],Power[f["h1"]*f["h1"]/435,(f["h1"
       *)
       f["M6"] = SetPrecision[f["Mp"]*Power[f["Mu"]/f["Mp"],Power[f["h1"]*f["h1"]/615,(f["h1"

       f["hd"] = SetPrecision[ 1/f["h"], prec];
                 f["h1d"] = SetPrecision[ 1/f["h1"], prec];
                 f["h2d"] = SetPrecision[ 1/f["h2"], prec];
```

```
(* Equations from Sykora *)
(* Keplerian *)
f["K1"] = SetPrecision[2*pi*Sqrt[a*b],prec];
f["K2"] = SetPrecision[2*pi*Power[(a+b),2]/Power[(Sqrt[a]+Sqrt[b]),2],prec];
f["K3"] = SetPrecision[pi*(a+b),prec];
f["K4"] = SetPrecision[pi*(a+b)*((3-Sqrt[1-f["h"]])/2),prec];
f["K5"] = SetPrecision[pi*Sqrt[2*(Power[a,2]+Power[b,2])],prec];
f["K6"] = SetPrecision[2*Pi*((2*(a + b)^2 - (Sqrt[a] - Sqrt[b])^4) / ((Sqrt[a] + Sqrt[
f["K7"] = SetPrecision[(pi/2)*Sqrt[6*(Power[a,2]+Power[b,2])+4*a*b],prec];
f["K8"] = SetPrecision[(2*pi)*Power[(Power[a,3/2]+Power[b,3/2])/2,2/3],prec];
f["K9"] = SetPrecision[pi*(a+b)*Power[1+f["h"]/8,2],prec];
f["K10"] = SetPrecision[pi*(3*(a+b)-Sqrt[(a+3*b)*(3*a+b)]),prec];
f["K11"] = SetPrecision[(pi/4)*(6+(1/2)*(Power[a-b,2])/(Power[a+b,2])),prec];
f["K12"] = f["MR"];
(* Pade *)
f["Ka"] = SetPrecision[ pi*(a+b)*((16+3*f["h"])/(16-f["h"])), prec];
f["Kb"] = SetPrecision[ pi*(a+b)*((64+16*f["h"])/(64-Power[f["h"],2])), prec];
f["Kc"] = SetPrecision[ pi*(a+b)*((64-3*Power[f["h"],2])/(64-16*f["h"])), prec];
f["Kd"] = SetPrecision[ pi*(a+b)*((256-48*f["h"]-21*Power[f["h"],2])/(256-112*f["h"]+3
f["Ke"] = SetPrecision[ pi*(a+b)*((3072-1280*f["h"]-252*Power[f["h"],2]+33*Power[f["h"]
(*Optimized Peano*)
f["O2"] = SetPrecision[ pi*Sqrt[2*(Power[a,2]+Power[b,2])-Power[a-b,2]/2.458338], prec
(*Extremes*)
f["E1"] = SetPrecision[pi*(a-b)/ArcTan[(a-b)/(a+b)],prec];
f["E2"] = SetPrecision[4*(a+b)-((8-2*pi)*a*b)/(0.410117*(a+b)+(1-2*0.410117)*(Sqrt[(a+
f["E3"] = SetPrecision[2*Sqrt[Power[pi,2]*a*b+4*Power[a-b,2]],prec];
f["E4"] = SetPrecision[4*((Power[b,2]/a)*ArcTan[a/b]+(Power[a,2]/b)*ArcTan[b/a]),prec]
f["E5"] = SetPrecision[(4*pi*a*b+Power[a-b,2])/(a+b), prec];
f["s"] = SetPrecision[Log[2]/Log[pi/2],prec];
f["E6"] = SetPrecision[4*Power[Power[a,f["s"]]+Power[b,f["s"]],1/f["s"]], prec];
(*Extreme Pades*)
f["Ead1"] = SetPrecision[(pi/4)*(81/64)-1,prec];
f["Ead2"] = SetPrecision[(pi/4)*(19/15)-1,prec];
f["Eap"]=f["Ead1"]/(f["Ead1"]-f["Ead2"]);
f["Ea"] = SetPrecision[pi*(a+b)*(f["Eap"]*(16+3*f["h"])/(16-f["h"])+(1-f["Eap"])*Power

f["Ebd1"] = SetPrecision[(pi/4)*(80/63)-1,prec];
f["Ebd2"] = SetPrecision[(pi/4)*(61/48)-1,prec];
f["Ebp"]=f["Ebd1"]/(f["Ebd1"]-f["Ebd2"]);
f["Eb"] = SetPrecision[pi*(a+b)*(f["Ebp"]*(64-3*Power[f["h"],2])/(64-16*f["h"])+(1-f["

f["Ecd1"] = SetPrecision[(pi/4)*(61/48)-1,prec];
f["Ecd2"] = SetPrecision[(pi/4)*(187/147)-1,prec];
f["Ecp"]=f["Ecd1"]/(f["Ecd1"]-f["Ecd2"]);
f["Ec"] = SetPrecision[pi*(a+b)*(f["Ecp"]*(256-48*f["h"]-21*Power[f["h"],2])/(256-112*

f["Edd1"] = SetPrecision[(pi/4)*(187/147)-1,prec];
f["Edd2"] = SetPrecision[(pi/4)*(1573/1236)-1,prec];
f["Edp"]=f["Edd1"]/(f["Edd1"]-f["Edd2"]);
f["Ed"] = SetPrecision[pi*(a+b)*(f["Edp"]*(3072-1280*f["h"]-252*Power[f["h"],2]+33*Pow

f["Eed1"] = SetPrecision[(pi/4)*(1573/1236)-1,prec];
f["Eed2"] = SetPrecision[(pi/4)*(47707/37479)-1,prec];
f["Eep"]=f["Eed1"]/(f["Eed1"]-f["Eed2"]);
```

```
f["Ee"] = SetPrecision[pi*(a + b)*((f["Eep"])*(135168 - 85760*f["h"] - 5568*f["h"]^2 +

f["C1t"] = SetPrecision[(pi/4)*((a-b)/b),prec];
f["C1"] = SetPrecision[pi*Sqrt[2*(Power[a,2]+Power[b,2])]*(Sin[f["C1t"]]/f["C1t"]),pre

f["C2"] = SetPrecision[4*a+2*(pi-2)*a*Power[b/a,1.456],prec];
f["C3"] = SetPrecision[4*((pi*a*b+Power[a-b,2])/(a+b))-(89/146)*Power[(b*Sqrt[a]-a*Sqr

f["C4s"]=0.825056176207;
f["C4"] = SetPrecision[4*(a+b)-(2*(4-pi)*a*b)/Power[(Power[a,f["C4s"]]+Power[b,f["C4s"

f["C5"] = SetPrecision[4*((pi*a*b+Power[a-b,2])/(a+b))-(1/2)*(a*b/(a+b))*(Power[a-b,2]
f["C6"] = SetPrecision[pi*(a + b)*(1 + (3*f["h"])/(10 + Sqrt[4 - 3*f["h"]])+(4/pi-14/1

f["C7p"] = 3.982901;
f["C7q"] = 66.71674;
f["C7r"] = 56.2007;
f["C7s"] = 18.31287;
f["C7t"] = 23.39728;
f["C7"] = SetPrecision[4*(a+b)-((a*b)/(a+b))*((f["C7p"]*Power[a+b,2]+f["C7q"]*a*b+f["C

f["A1"] = SetPrecision[4*(Power[a,2]+Power[b,2]+(pi-2)*a*b)/(a+b),prec];

    f["A2t1"] = SetPrecision[2.49808365277126, prec];
    f["A2s1"] = SetPrecision[1.22694921875000, prec];
f["A2"] = SetPrecision[(4*(a^3 + b^3 + f["A2t1"]*a*b*(a+b))/(a^2 + b^2 + 2*f["A2s1"]*a

         f["A3t1"] = SetPrecision[6.16881239339582, prec];
         f["A3t2"] = SetPrecision[4.06617730084445, prec];
         f["A3s1"] = SetPrecision[6.15241658169936, prec];
         f["A3"] = SetPrecision[(4*(a^4 + b^4 + f["A3t1"]*a*b*(a^2 + b^2)+2*f["A3t2"]*

         f["A4t1"] = SetPrecision[13.02487942169925, prec];
         f["A4t2"] = SetPrecision[28.56997512074272, prec];
         f["A4s1"] = SetPrecision[13.01750519704827, prec];
         f["A4s2"] = SetPrecision[13.09922140579137, prec];
         f["A4"] = SetPrecision[(4*(a^5 + b^5 + f["A4t1"]*a*b*(a^3 + b^3) + f["A4t2"]*

         f["A5t1"] = SetPrecision[27.301243680755, prec];
         f["A5t2"] = SetPrecision[113.302483206429, prec];
         f["A5t3"] = SetPrecision[76.50091476282086, prec];
         f["A5s1"] = SetPrecision[27.297854333670, prec];
         f["A5s2"] = SetPrecision[110.551872985869, prec];
         f["A5"] = SetPrecision[(4*(a^6 + b^6 + f["A5t1"]*a*b*(a^4 + b^4) + f["A5t2"]*

         f["A6t1"] = SetPrecision[51.447782789130, prec];
         f["A6t2"] = SetPrecision[385.327854851892, prec];
         f["A6t3"] = SetPrecision[790.4535392309255, prec];
         f["A6s1"] = SetPrecision[51.445996674310, prec];
         f["A6s2"] = SetPrecision[382.256974433855, prec];
         f["A6s3"] = SetPrecision[347.212007887276, prec];
         f["A6"] = SetPrecision[(4*(a^7 + b^7 + f["A6t1"]*a*b*(a^5 + b^5) + f["A6t2"]*
                 (a^6 + b^6 + f["A6s1"]*a*b*(a^4 + b^4) + f["A6s2"]*a^2*b^2*(a^2 + b^2

  f["A7t1"] = SetPrecision[93.49235523473, prec];
```

```
        f["A7t2"] = SetPrecision[1262.73239571330, prec];
        f["A7t3"] = SetPrecision[4296.45229646421, prec];
        f["A7t4"] = SetPrecision[2903.735611540449, prec];
        f["A7s1"] = SetPrecision[93.49135794687, prec];
        f["A7s2"] = SetPrecision[1259.36473022183, prec];
        f["A7s3"] = SetPrecision[4093.96201082922, prec];
        f["A7"] = SetPrecision[(4*(a^8 + b^8 + f["A7t1"]*a*b*(a^6 + b^6) + f["A7t2"]*
                (a^7 + b^7 + f["A7s1"]*a*b*(a^5 + b^5) + f["A7s2"]*a^2*b^2*(a^3 + b^3

f["S0q"] = SetPrecision[4.16102118885517, prec];
        f["S0u"] = SetPrecision[(4 - pi)/(4 - 2*Sqrt[2+f["S0q"]]),prec];
        f["S0p"] = SetPrecision[1 - f["S0u"], prec];
        f["S0"] = SetPrecision[f["S0p"]*(a + b) + f["S0u"]*Sqrt[a^2 + b^2 + f["S0q"]*

        f["S1q"] = SetPrecision[92.28480788617108, prec];
        f["S1v"] = SetPrecision[0.04522028227769, prec];
        f["S1p2"] = SetPrecision[0.99983439391729, prec];
        f["S1u"] = SetPrecision[1 + f["S1v"] - f["S1p2"], prec];
        f["S1p1"] = SetPrecision[(pi/2)*(2 + f["S1v"]*Sqrt[2+ f["S1q"]]) - (2*f["S1p2

        (* Function S1 *)
        f["S1"] = SetPrecision[(f["S1p2"]*(a^2 + b^2) + f["S1p1"]*a*b + f["S1u"]*(a +
                              ((a + b) + f["S1v"]*Sqrt[a^2 + b^2 + f["S1q"]*a*b]),

        (* Constants for S2 *)
        f["S2q1"] = SetPrecision[13.66022044408346, prec];
        f["S2q2"] = SetPrecision[37.30921886231118, prec];
        f["S2v"] = SetPrecision[-1.03788930003090, prec];
        f["S2t"] = SetPrecision[5.51954143485218, prec];
        f["S2p2"] = SetPrecision[1.24957869093182, prec];
        f["S2u"] = SetPrecision[1 + f["S2v"] - f["S2p2"], prec];
        f["S2p1"] = SetPrecision[(pi/4)*(2 + f["S2t"] + f["S2v"]*Sqrt[2 + 2*f["S2q1"]
                                 (f["S2p2"] + f["S2u"]*Sqrt[2 + 2*f["S2q1"] + f["S2q
        f["S2R"] = SetPrecision[Sqrt[(a^4 + b^4) + f["S2q1"]*a*b*(a^2 + b^2) + f["S2q
        f["S2"] = SetPrecision[(f["S2p2"]*(a^3 + b^3) + f["S2p1"]*a*b*(a + b) + f["S2
                              ((a^2 + b^2) + f["S2t"]*a*b + f["S2v"]*f["S2R"]), pre

        f["S1rq"] = SetPrecision[74.01745125408363, prec];
        f["S1rv"] = SetPrecision[0.05027328304233, prec];
        f["S1ru"] = f["S1rv"];
        f["S1rp1"] = SetPrecision[(pi/2)*(2 + f["S1rv"]*Sqrt[2 + f["S1rq"]]) - (2 + 2
        f["S1r"] = SetPrecision[((a^2 + b^2) + f["S1rp1"]*a*b + f["S1ru"]*(a + b)*Sqr
                                ((a + b) + f["S1rv"]*Sqrt[a^2 + b^2 + f["S1rq"]*a*b]

        f["Sac1"] = SetPrecision[Pi - 3, prec];
        f["Sac2"] = SetPrecision[Pi, prec];
        f["Sac3"] = SetPrecision[0.5, prec];
        f["Sac4"] = SetPrecision[(1 + pi)/2, prec];
        f["Sac5"] = SetPrecision[4, prec];
        f["Sak"] = SetPrecision[
          1 - (f["Sac1"]*a*b)/(
            (a^2 + b^2) + f["Sac2"]*Sqrt[f["Sac3"]*(a*b)^2 + a*b*Sqrt[a*b*(f["Sac4"]*
          ),
          prec
        ];
```

```
(* Define the function S_a using constants from the association *)
f["Sa"] = SetPrecision[
  4 * ((pi*a*b + f["Sak"]*(a - b)^2)/(a + b)),
  prec
];

f["Saoc1"] = SetPrecision[0.14220038049945, prec];
f["Saoc2"] = SetPrecision[3.30596250119242, prec];
f["Saoc3"] = SetPrecision[0.00135657637724, prec];
f["Saoc4"] = SetPrecision[2.00637978782056, prec];
f["Saoc5"] = SetPrecision[5.3933761426286, prec];

f["Saok"] = SetPrecision[
  1 - (f["Saoc1"]*a*b)/(
    (a^2 + b^2) + f["Saoc2"]*Sqrt[f["Saoc3"]*(a*b)^2 + a*b*Sqrt[a*b*(f["Saoc4
  ),
  prec
];

(* Define the function S_a using constants from the association *)
f["Sao"] = SetPrecision[
  4 * ((pi*a*b + f["Saok"]*(a - b)^2)/(a + b)),
  prec
];

f["Sard1"] = 0.14220096377128;
f["Sard2"] = 3.93490847789660;
f["Sard3"] = 2.691437204515743;
f["Sark"] = SetPrecision[1-(f["Sard1"]*a*b)/((Power[a,2]+Power[b,2])+f["Sard2
f["Sar"] = SetPrecision[4*(pi*a*b+f["Sark"]*Power[a-b,2])/(a+b),prec];

f["Mn"] = SetPrecision[ Power[f["MR"]/f["Mp"],148/a]*f["Mu"]*Power[f["Mp"]/f[


f["Mcfr"] = SetPrecision[(-0.0005410347761165108l9*f["h"]+1.00000000884005358
f["Mn1"] = SetPrecision[ Power[a,f["h1"]/Sqrt[a]]*f["Sar"], prec];
f["Mn2"] = SetPrecision[ Sqrt[f["M6"]*f["Sa"]], prec];
f["Mn3"] = SetPrecision[ f["Sa"]/Power[f["Sa"]/f["M6"],0.360886], prec];
f["Mn4"] = SetPrecision[ f["M6"]*Power[f["Mu"],Power[f["h2"],f["h2"]/f["h1"]]
f["Mn6"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],120]/(a*a))), prec];
f["Mn7"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],91]/(a*Sqrt[a*39*Sqrt[a]]))
f["Mn8"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],20]/(a*a*(1+(f["MR"]/(-3674
f["Mn9"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],21]/(a*a*(1+(f["Mu"]/(-10))
f["Mn10"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],14]/(a*a*(1+(-32))))), pre

(* Probably the best candidate *)
f["MMC"] = SetPrecision[ f["M6"]*(1+(Power[f["h"],18]/(a*a*(1+(-1*a))))), pre
f["err"] = SetPrecision[(f["y"]-f["MMC"])*10^15,prec];
f["Mn12"] = SetPrecision[ f["M6"]*(1+((f["h"]^2/(a^2+267246))*(1+(19*(-1+f["h
f["MA1"] = SetPrecision[ 6.44122277800000 + (f["a"] - 1.05000000000000)/(0.262
f["MA2"] = SetPrecision[ 6.44122277811400412 + (f["a"] - 1.05000000000000004)
0.2966645029406122506134554808539764968940370083140355400664283604262481061060557855417 65
    1.75000000000000000)/(-127.2634922841017189469050084996877417278583242979752552380270 57
        5.)/(-0.0466370014221251685451647264815880055798528387638653761111766889442802278139
```

```
            500.)/(-1.13505515211833401538618234037833353863978462154908997972035430460449195
            36361.628558908774755148948001494216629448130019937942743268302509857392244377718
, prec];
              f["MA3"] = SetPrecision[6.44122277811400412 + (f["a"] - 1.05000000000000004)/(
,prec];

f["MA4"] = SetPrecision[6.44122277811400412 + (f["MMC"] - 6.44122277811400412)/(0.999999999999

,prec];

(*
f["MA5"] = SetPrecision[ pi*(a+b)*(1.0000000000000000000000000000000000000000000000000000000
),prec];
*)
f["MA5"] = SetPrecision[pi*(a+b)*(1.00000000000000000000000000000000000000000000000000000000
,prec];


(*f["err"] = SetPrecision[(f["err"] - Min[f["err"]]) / (Max[f["err"]] - Min[f["err"]]),prec];
f["yd"] = SetPrecision[f["err"],prec];*)
(*f["yd"] = SetPrecision[f["y"] / f["maxY"], prec];*)
(*f["yd"] = SetPrecision[ (f["y"]/f["MMC"]), prec];*)

      keys = {
      "MR", "Mu", "MRu", "Mp",
      "M1", "M2", "M3", "M4", "M5", "M6", "M7", "M8", "M9",
      "K1", "K2", "K3", "K4", "K5", "K6", "K7", "K8", "K9", "K10", "K11", "K12",
      "Ka","Kb","Kc","Kd","Ke",
      "E1","E2","E3","E4","E5","E6", "O2",
      "Ea","Eb","Ec","Ed","Ee",
      "C1","C2","C3","C4","C5","C6","C7",
      "A1","A2","A3","A4","A5","A6","A7",
      "S0","S1","S2","S1r","Sa","Sao","Sar",
      "Mn", "Mcfr", "Mn1", "Mn2", "Mn3", "Mn4", "Mn6", "Mn7", "Mn8", "Mn9", "Mn10", "Mn11",
      };

      calculateErrors = Function[key,
        f[key <> " Err"] = SetPrecision[Abs[f["y"] - f[key]]/f["y"], prec]
      ];
      Map[calculateErrors, keys];

      f
    ],
    {pair, data}
  ];
  results
];

End[];
EndPackage[];
```