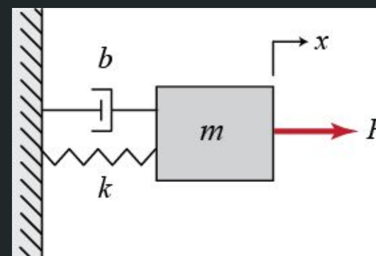
Dynamic Systems Analysis

Renan G. Maidana
renan.maidana@acad.pucrs.br

Porto Alegre, 2018

Previously...

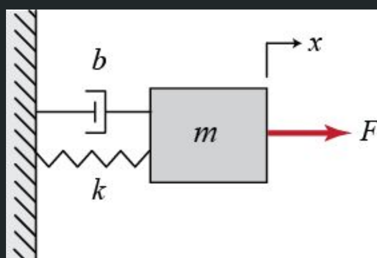
- **Dynamic Systems Modelling**
  - Differential Equations
  - Transfer Functions
  - Laplace and Z Transforms
  - Discretization
  - Identification
  - State-Space
  - Examples and Exercises



$$m\ddot{x} + b\dot{x} + kx = u$$

Previously...

- Challenge: **Spring-Mass-Damper system identification**



$$m\ddot{x} + b\dot{x} + kx = 0$$

1) **Find the system's transfer function, G(s)**
2) **Discretize G(s)**
3) **Use your discretized model to approximate G(s) through identification**
    a) Tip: Use matlab, octave, python...
4) **Define the previously unknown mass, damping coefficient (b) and spring constant (k)**

Today!

- **Dynamic Systems Analysis**
  - Transfer Function Poles and Zeros
  - Stability Criteria
  - Transient Response
  - Steady-State Error

- **Digital Control Systems Design**
  - Compensator Design
  - PID Control
  - PID Tuning
  - Anti-Windup

Transfer Function Poles and Zeros

- In order to analyze a dynamic system, we explore the concept of a transfer function's **poles** and **zeros**

$$G(s) = \frac{s^n + b_0 s^{n-1} + ... + b_{n-1} s + b_n}{s^n + a_0 s^{n-1} + ... + a_{n-1} s + a_n}$$

Zeros

Poles

- The arrangement of poles and zeros define the **natural response** of a transfer function

- The **total response** is the sum of the **forced response** (reaction to a given input) and the **natural response** (autonomous behavior of a TF)

Transfer Function Poles and Zeros

- The **poles** are the values of ***s*** that:
  - Make the result of the transfer function to become infinite
  - Are denominator roots common to the numerator

$$G(s) = \frac{s+\alpha}{s(s+\beta)}, \quad G(s) = \infty \text{ if } s = \{0, -\beta\}$$

- A transfer function becomes infinite when the denominator is zero
- Thus, the TF's denominator roots are the poles
- The second definition is because, although the effect of a pole can be nullified by a zero, it still contributes to the system's overall order, and thus is still considered a pole

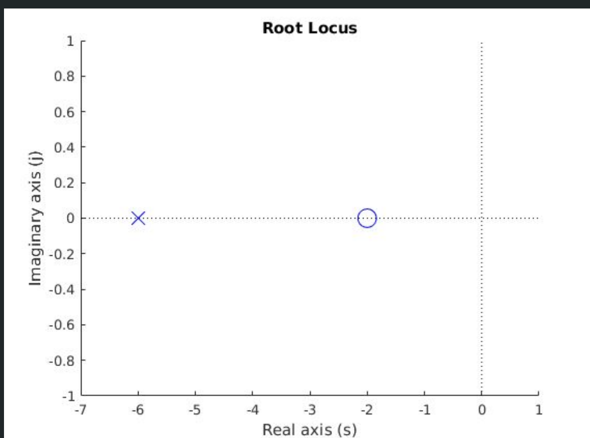
Transfer Function Poles and Zeros

- The **zeros** are values of ***s*** that:
  - Make the result of the transfer function to become zero
  - Are numerator roots common to the denominator

$$G(s) = \frac{s+\alpha}{s(s+\beta)}, \quad G(s) = 0 \text{ if } s = -\alpha$$

Transfer Function Poles and Zeros

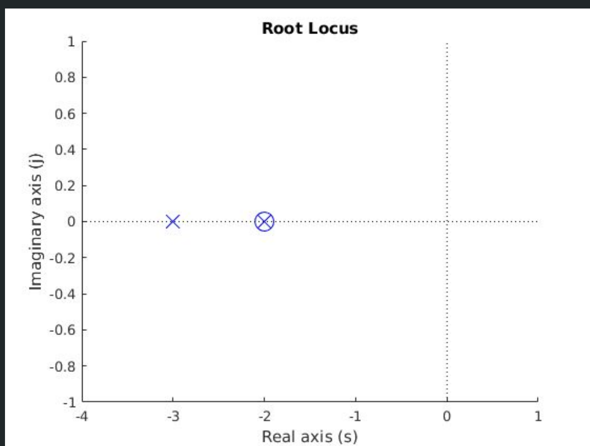
- We can represent the poles and zeros graphically in a **root locus plot**



$$G(s) = \frac{s+2}{s+6}$$

Transfer Function Poles and Zeros

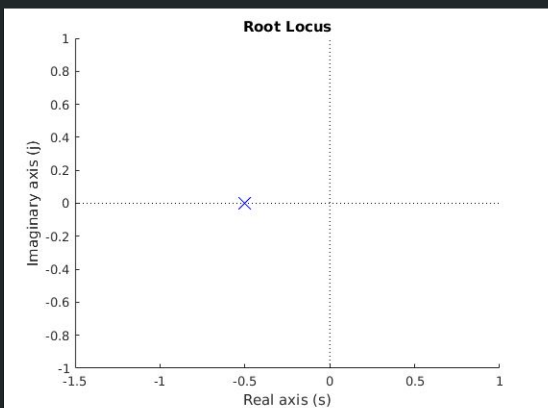
- We can represent the poles and zeros graphically in a **root locus plot**

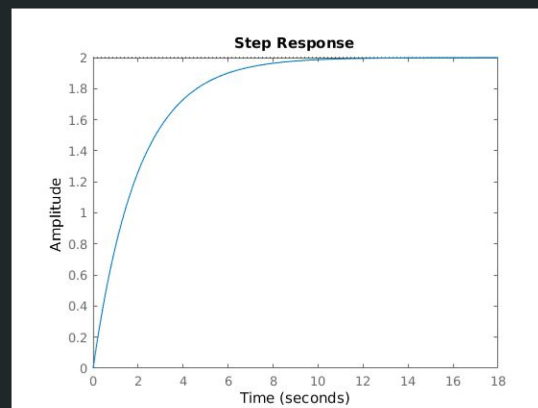


$$G(s) = \frac{s+2}{(s+2)(s+3)}$$

Transfer Function Poles and Zeros

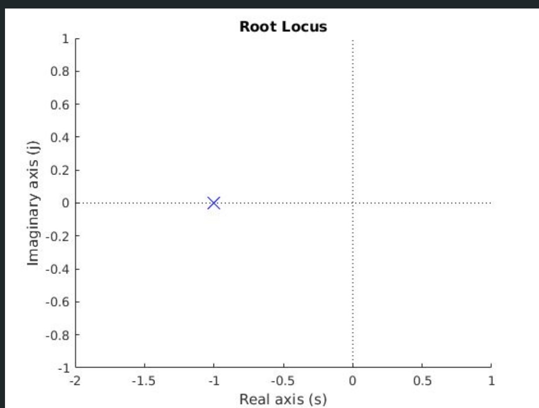
- Effect of pole location in the system response

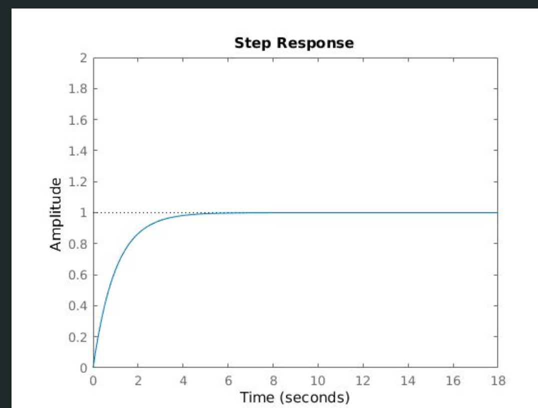




Transfer Function Poles and Zeros

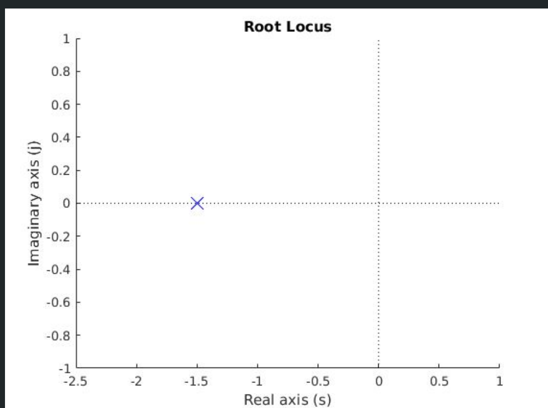
- Effect of pole location in the system response

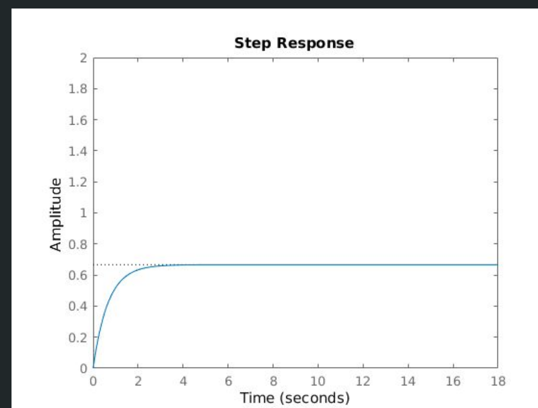




Transfer Function Poles and Zeros

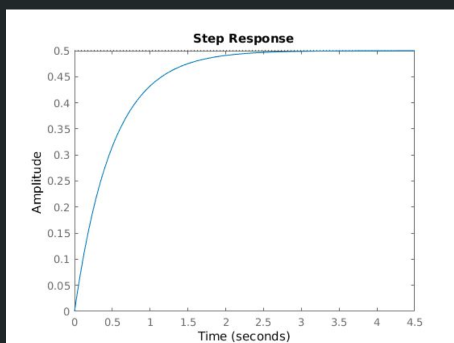
- Effect of pole location in the system response

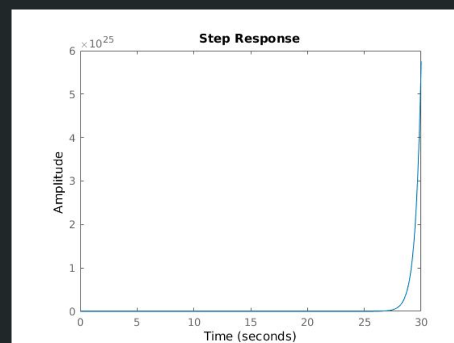




Stability Criteria

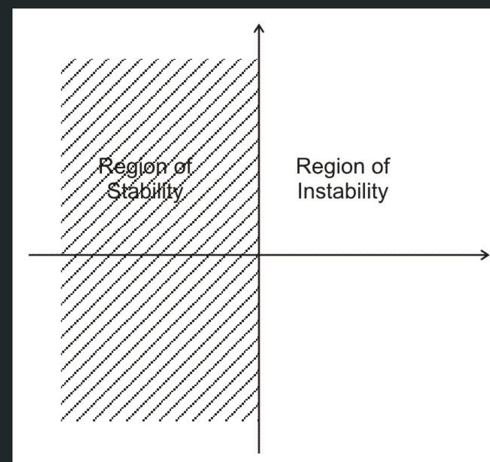
- The concept of stability defines if a dynamic system has **runaway behavior**, given a certain input
    - Formally, a system is **stable** if it produces a **bounded output** given a **bounded input**





Stability Criteria

- A very simple way to determine if a system is stable is to analyze its **transfer function poles**
  - The **Routh-Hurwitz** criterion states that the system is stable if the poles' real components are **negative**



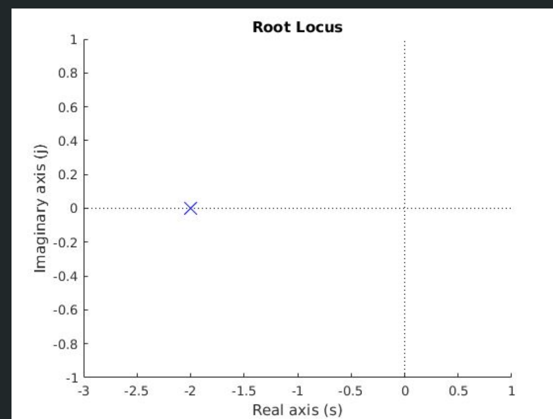
- In other words, if (s < 0) -> stable

Stability Criteria

- Example 1:

$$G(s) = \frac{1}{s+1}$$

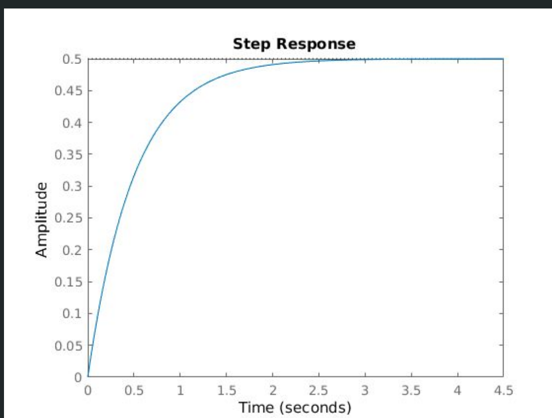
- This system is stable, because there is only one pole at -1

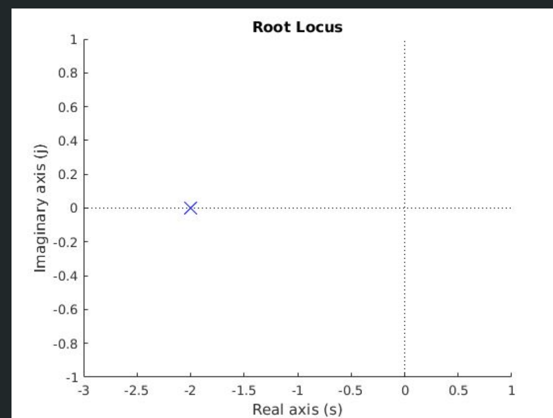


- In other words, if (s < 0) -> stable

Stability Criteria

- Example 1:





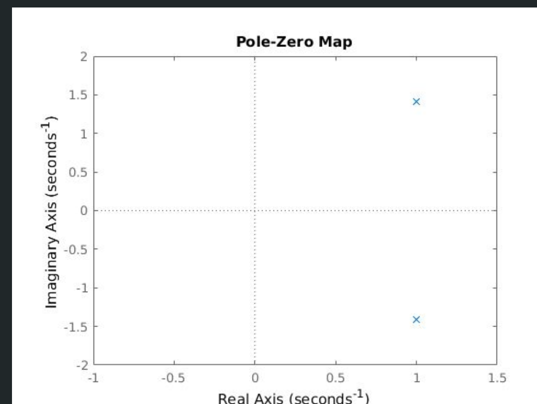
- In other words, if (s < 0) -> stable

Stability Criteria

- Example 2:

$$G(s) = \frac{1}{s^2 - 2s + 3}$$

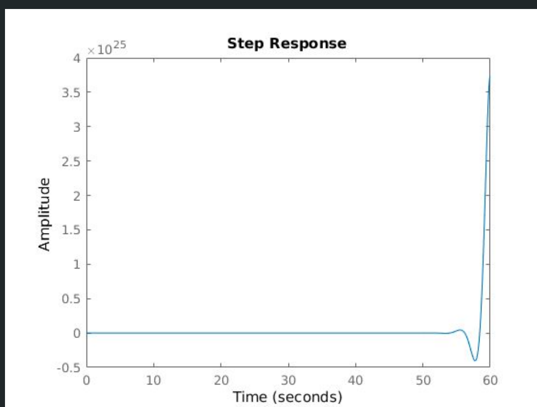
- This system is unstable, because there are two positive poles at 1 + 1.5j

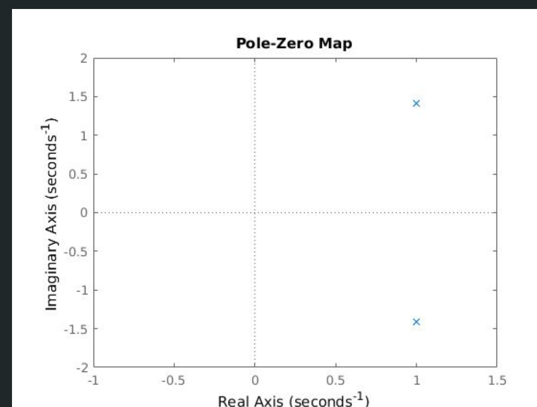


- In other words, if (s > 0) -> unstable

Stability Criteria

- Example 2:

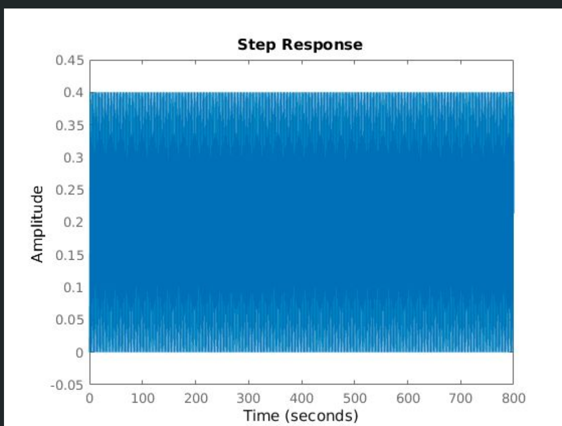


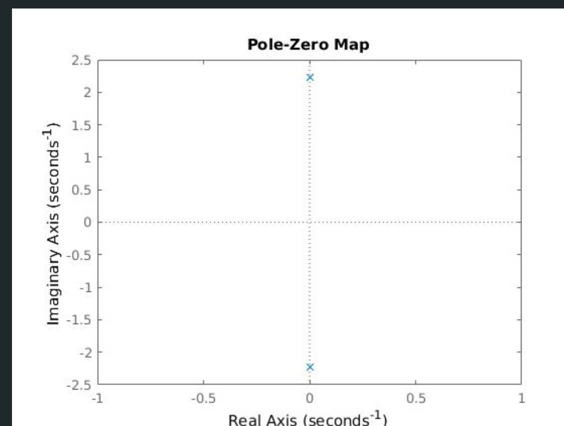


- In other words, if (s > 0) -> unstable

Stability Criteria

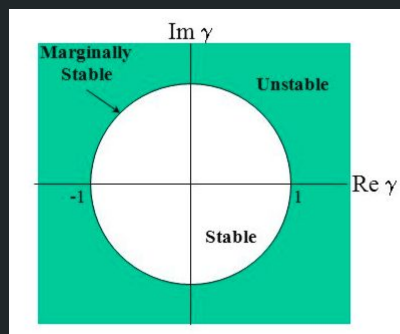
- A special case is **marginal stability**, when there are complex poles at the **imaginary axis** (the real part of the poles is 0)





Stability Criteria

- In discrete-time, a system is:
  - **Stable** if the discrete poles stay within a circle of radius 1
  - **Unstable** if the poles are outside of the circle
  - **Marginally stable** if there are complex poles on the circle's boundary (real part = 1)



- In other words, if (z < 1) -> stable

Transient Response

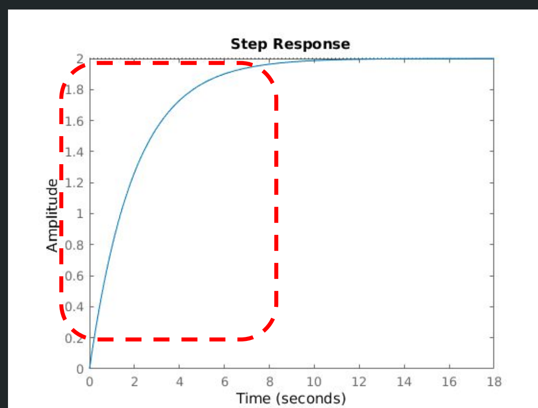
- Given a dynamic system, there are three main aspects to consider when designing a controller:
  - Stability
  - **Transient Response**
  - **Steady-State Response (Error)**

- Apart from these, it is also worth considering:
  - Actuator Limitations
  - Systemic Effects
  - Disturbance Effects

- The three main aspects are known as the **Analysis and Controller Project Objectives**
- Examples of systemic effects could be transport delay (e.g., the time delay between actuation and reaction/sensing), or systemic errors in sensing (e.g., constant bias in the sensor)
- Disturbances can be any un-modelled effects in the system response (compensating these effects is called *Robust Control*)

Transient Response

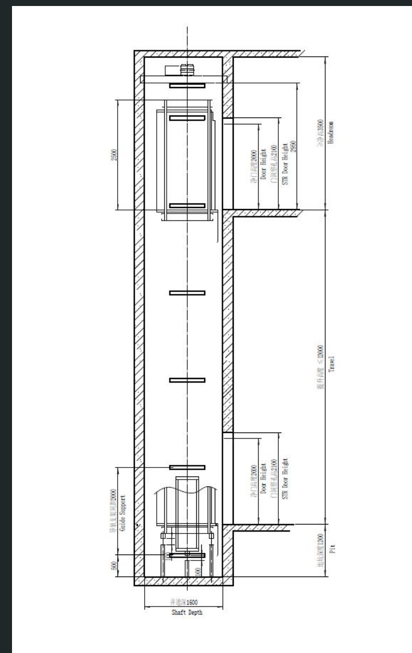
- The **Transient Response** is the immediate reaction of a system to an input
  - In other words, it is **how fast a system reacts**



- We manipulate the transient response to control how a system behaves before reaching a steady-state response

Transient Response

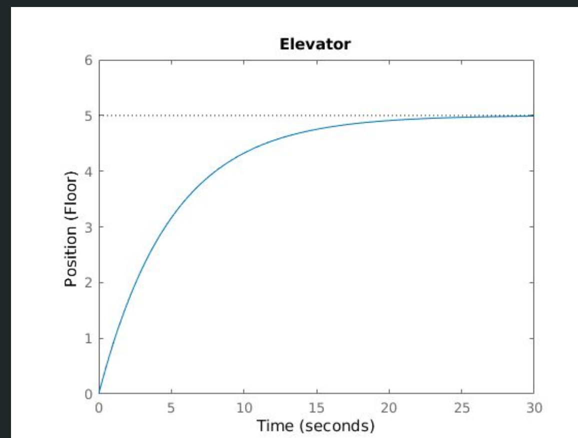
- Why should be concerned with that?
  - Consider the example of an elevator



Transient Response

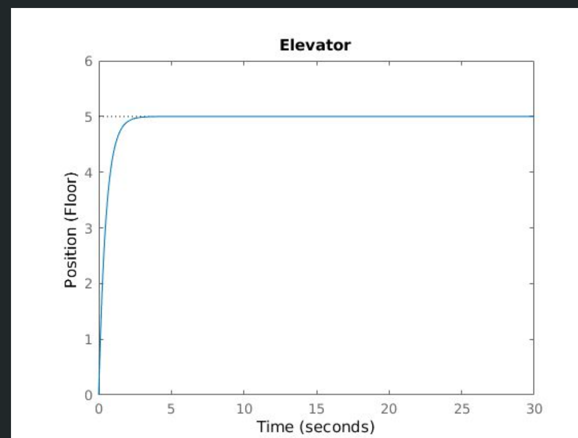
- We can change the elevator’s position by controlling the elevator’s speed

- If the elevator takes 30 seconds to rise 5 floors, it is way too slow!



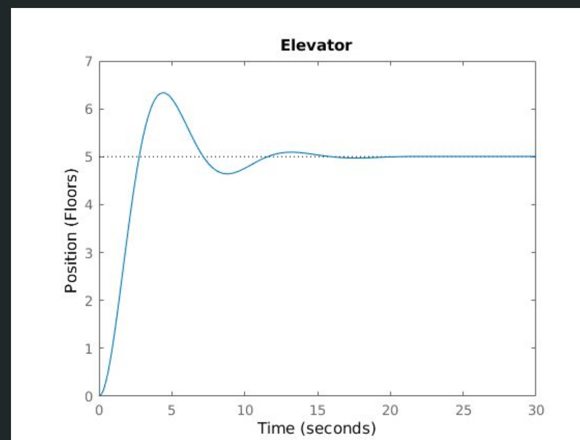
Transient Response

- If the elevator takes 2 seconds to rise 5 floors, it is way too fast!



Transient Response

- The elevator now takes 5 seconds to rise 5 floors (ok!), but oscillates around the 5th floor (what!?!)



Transient Response

- The transient response can be analyzed with **three metrics**:
  - **Time Constant**
  - **Rise Time**
  - **Settling Time**

- The **Time Constant (τ)** is the time it takes for a system's step response to reach **63% of its final value**

Transient Response

- The **Time Constant (τ)** is the time it takes for a system's step response to reach **63% of its final value**

$$Y(s) = U(s) \times G(s) = \frac{1}{s} \times \frac{a}{s+a}$$

$$\therefore^{\mathcal{L}^{-1}} \; y(t) = 1 - e^{-at}$$

$$y\left(\frac{1}{a}\right) = 1 - e^{-1} = 1 - 0.37 = 0.63$$

- Case study: First-order System

Transient Response

- The **Time Constant (τ)** is the time it takes for a system's step response to reach **63% of its final value**

$$Y(s) = U(s) \times G(s) = \frac{1}{s} \times \frac{a}{s+a}$$

$$\therefore^{\mathcal{L}^{-1}} \; y(t) = 1 - e^{-at}$$

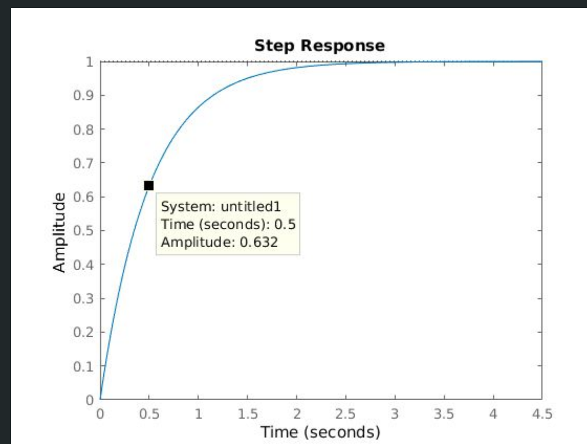
$$y\left(\overset{\tau}{\boxed{\frac{1}{a}}}\right) = 1 - e^{-1} = 1 - 0.37 = 0.63$$

- We want to know e^{-1} because it is the exponential decay rate of a system

Transient Response

- Example:

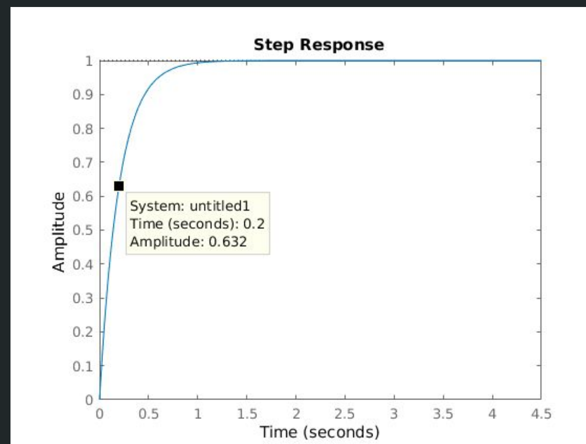
$$G(s) = \frac{\boxed{2}^{\,a}}{s+\boxed{2}} \therefore \tau = \frac{1}{2} = 0.5$$



Transient Response

- Example:

$$G(s) = \frac{5}{s+5} \therefore \tau = \frac{1}{5} = 0.2$$



Transient Response

- The **Rise Time (Tr)** is the time for the system response to go from **10% to 90% of its final value**

$$T_r = \frac{ln\left(\frac{0.9}{0.1}\right)}{a}$$

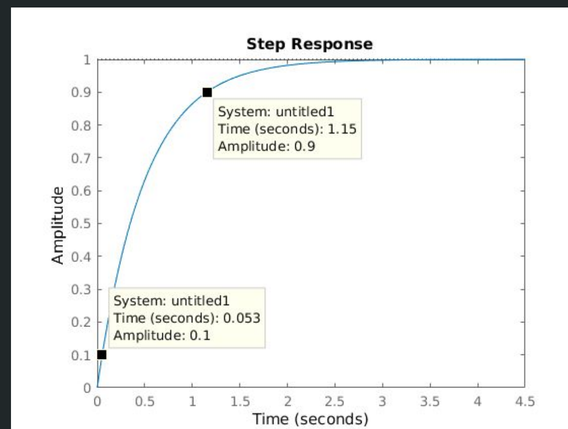
- Tr is the time difference between y(t) = 0.1 and y(t) = 0.9
- Isolate t in the system response: t(y(t)) = ln(1-y(t))/(-a)
- Tr = t(0.9)-t(0.1)
- The percentages are arbitrary

Transient Response

- Example:

$$G(s) = \frac{2}{s+2}$$

$$T_r = \frac{ln\left(\frac{0.9}{0.1}\right)}{2} = 1.0986\ s$$



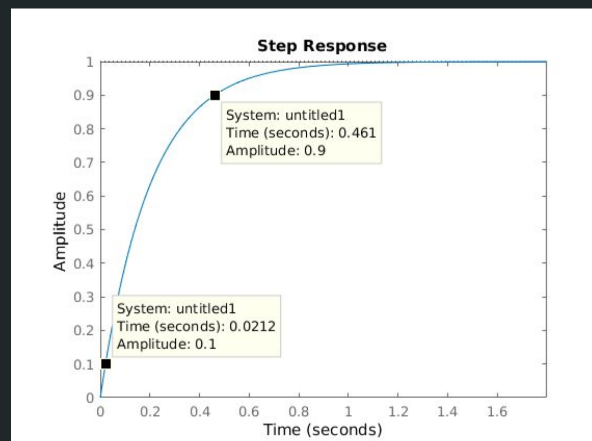
- We can check the rising time graphically also: Tr = 1.15 - 0.053 = 1.097 s

Transient Response

- Example:

$$G(s) = \frac{5}{s+5}$$

$$T_r = \frac{ln\left(\frac{0.9}{0.1}\right)}{5} = 0.439\ s$$



- We can check the rising time graphically also: Tr = 0.461 - 0.0212 = 0.4398 s

Transient Response

- Finally, the **Settling Time (Ts)** is the time for the system response to reach **98% of its final value**

$$T_s = -\frac{ln(0.02)}{a}$$

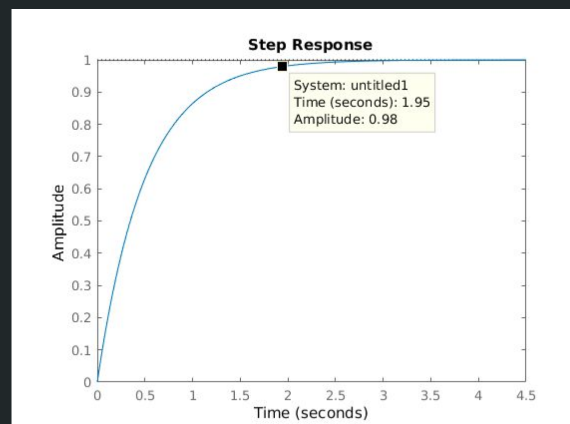
- Ts is the time for y(t) = 0.98
- Isolate t in the system response: t(y(t)) = ln(1-y(t))/(-a)
- Ts = ln(1-0.98)/(-a)
- This percentage is arbitrary (we say the system is in steady-state when it reaches 2% error)

Transient Response

- Example:

$$G(s) = \frac{2}{s+2}$$

$$T_s = -\frac{ln(0.02)}{2} = 1.956\ s$$



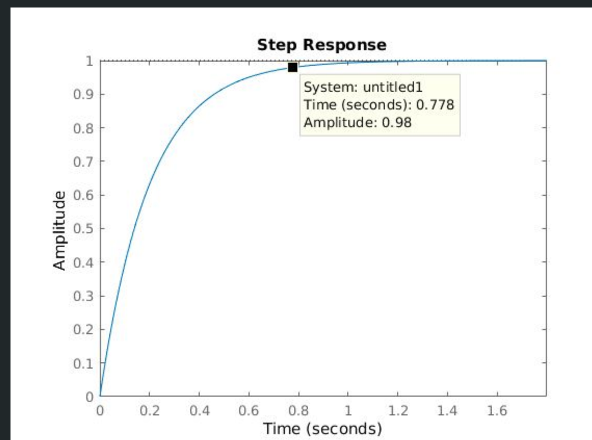
- We can check the rising time graphically also: Ts = 1.95 s

Transient Response

- Example:

$$G(s) = \frac{5}{s+5}$$

$$T_s = -\frac{ln(0.02)}{5} = 0.782\ s$$



- We can check the rising time graphically also: Ts = 0.778 s

Transient Response

- For second and higher order systems, the mathematical definitions for time constant, rise time and settling time become complex

- Thus, it is more advantageous to analyse these systems **graphically**

- They are not a function of the time constant, and thus must be deduced with the inverse Laplace transform

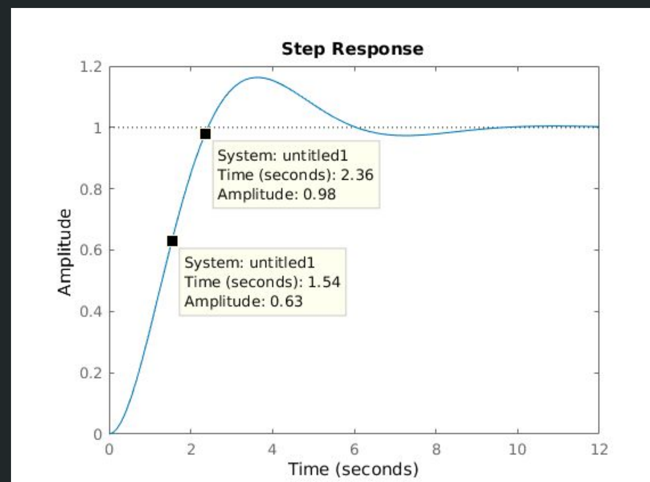
Transient Response

- Example:

$$G(s) = \frac{1}{s^2 + s + 1}$$

Ts = 2.36 s

τ = 1.54 s **(not equal to 1/a!)**

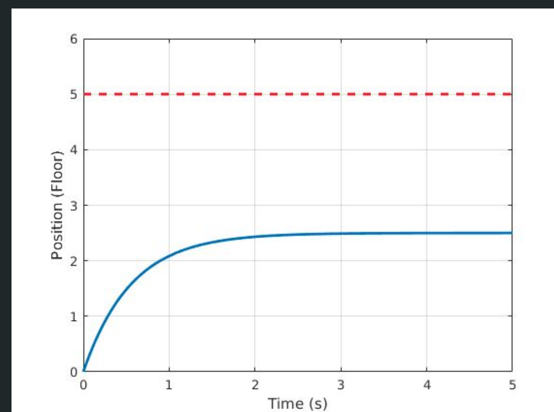


- What can we do with transient response analysis?
  - We can develop closed-loop control systems which have desired rise and settling times, according to a controller C

Steady-State Response

- If Transient Response if the analysis of system behavior in time, **Steady-State Response** is the analysis of system behavior in amplitude

- Considering the elevator example, an elevator is useless if it **rises in an adequate time but never reaches the desired floor**



Steady-State Response

- The steady-state response is the state of the system **after it has settled**

- Thus, we can define a system's steady-state response as the **error w.r.t to the desired state (steady-state error)**

Steady-State Response

- The state of an open-loop system after it has settled is:

$$y(\infty) = \lim_{s \to 0} \{U(s)G(s)\}$$

- Thus, the error can be expressed as:

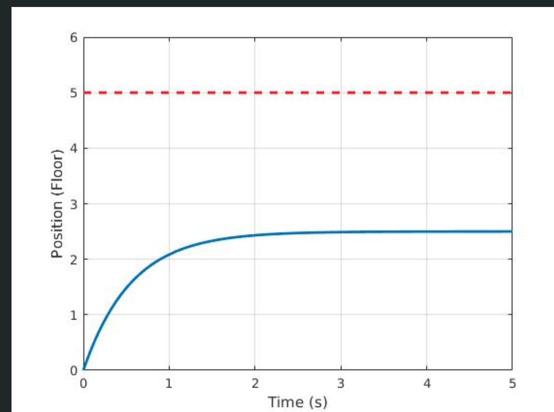
$$e_{ss} = \text{desired} - \lim_{s \to 0} \{U(s)G(s)\}$$

- When the frequency tends to 0, time tends to infinity

Steady-State Response

- Example:

$$U(s) = 5 \quad ; \quad G(s) = \frac{5}{s+10}$$



$$e_{ss} = 5 - \lim_{s \to 0} \left(\frac{25}{s+10} \right) = 5 - 2.5 = 2.5$$

Steady-State Response

- In general way, the error for closed-loop systems is defined as:

$$e_{ss} = \lim_{s \to 0} \left(\frac{sR(s)}{1 + C(s)G(s)} \right)$$

- We can define **types of error relative to the type of input**

- A closed-loop feedback assumes E(s) = R(s) - Y(s)

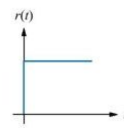
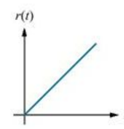
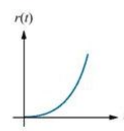
Steady-State Response

- We can thus define **types of error relative to the type of input**

$$\lim_{s \to 0} \left(\frac{1}{1 + C(s)G(s)} \right)$$

$$\lim_{s \to 0} \left(\frac{s^{-1}}{1 + C(s)G(s)} \right)$$

$$\lim_{s \to 0} \left(\frac{s^{-2}}{1 + C(s)G(s)} \right)$$

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

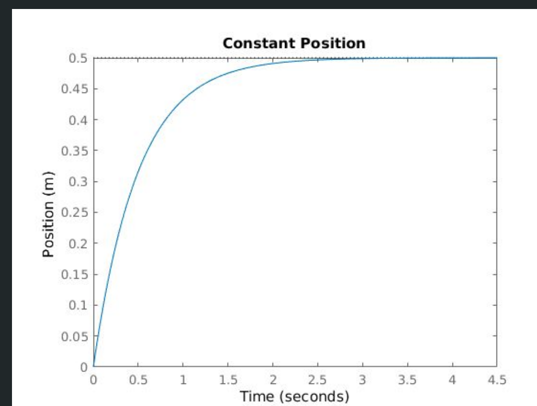
Steady-State Response

- Example: Mobile Robot position

$$C(s) = 1 \qquad G(s) = \frac{1}{s+1}$$

$$R(s) = \frac{1}{s}$$

$$e_{ss} = \frac{1}{1 + \frac{1}{0+1}} = 0.5$$



- E_ss comes from the first equation in slide 44, substituting R, C and G

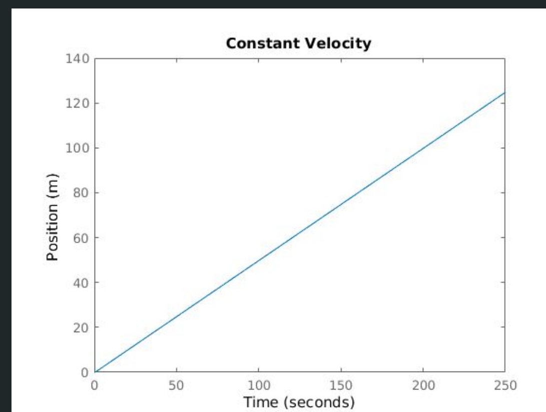
Steady-State Response

- Example: Mobile Robot position

$$C(s) = 1 \quad G(s) = \frac{1}{s+1}$$

$$R(s) = \frac{1}{s^2}$$

$$e_{ss} = \frac{1}{0 \times (1 + \frac{1}{0+1})} = \infty$$



- E_ss comes from the first equation in slide 44, substituting R, C and G

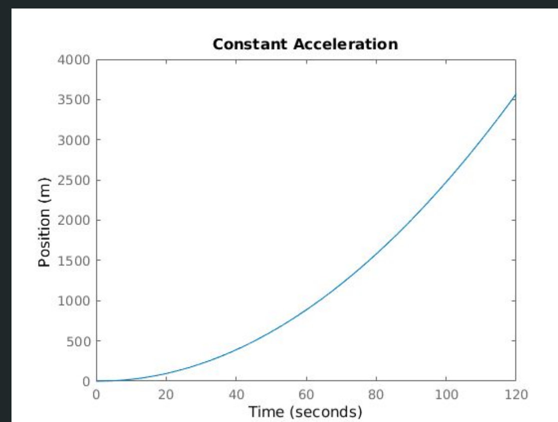
Steady-State Response

- Example: Mobile Robot position

$$C(s) = 1 \quad G(s) = \frac{1}{s+1}$$

$$R(s) = \frac{1}{s^3}$$

$$e_{ss} = \frac{1}{0 \times (1 + \frac{1}{0+1})} = \infty$$



- E_ss comes from the first equation in slide 44, substituting R, C and G

Conclusion

- With these analysis tools, we can design controllers which:
  - Stabilize an unstable system (or un-stabilize a stable one…)
  - Alter the Rise and Settling Time
  - Dampen a system’s oscillatory behavior
  - Decrease the steady-state error to a known input

To be continued!