

# Dynamic Systems Modelling

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Renan G. Maidana

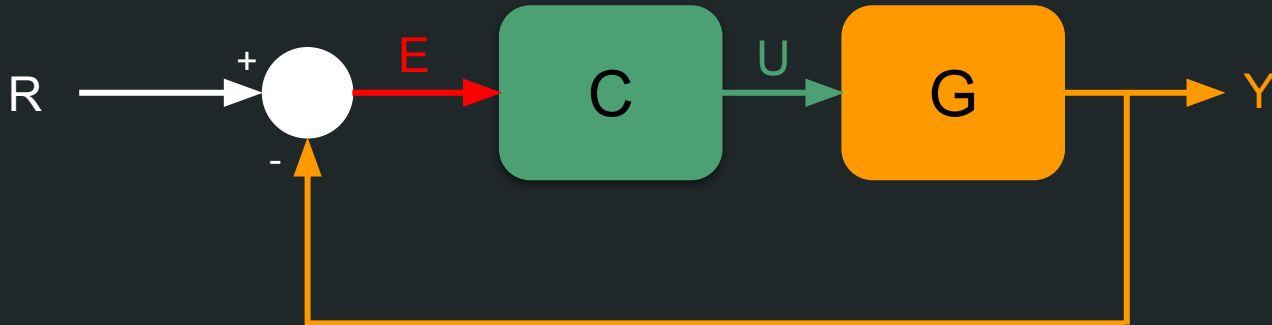
Porto Alegre, 2018

# Previously...

- What is feedback?

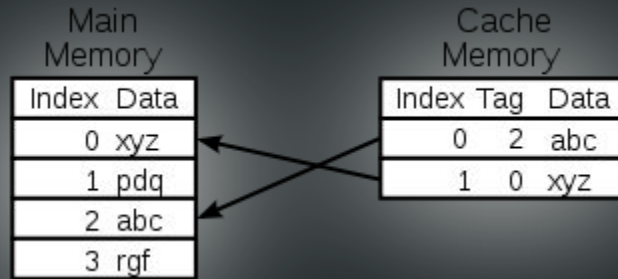


- What is control?



# Previously...

- Practical examples
- Advantages/Disadvantages
- Applications (general and in Robotics)
- Challenge: Cache Hit Ratio

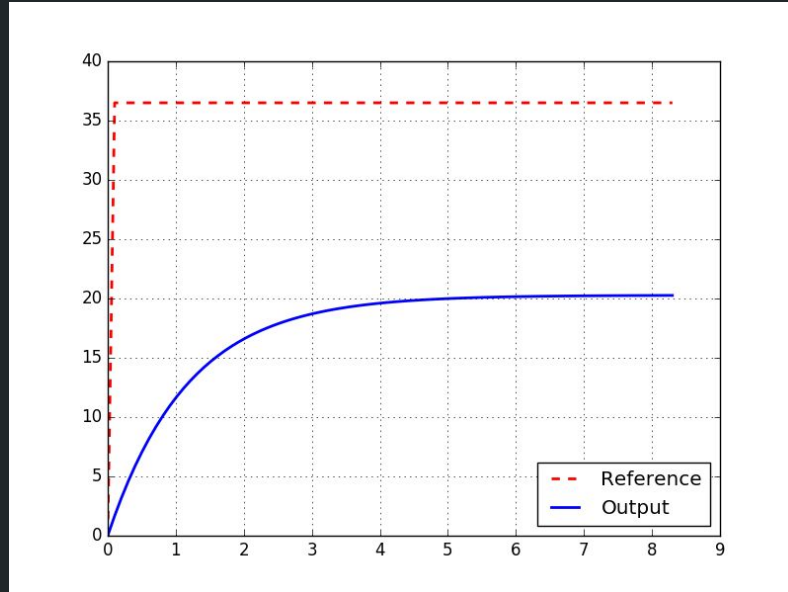


# Today!

- **Dynamic Systems Modelling**
  - Introduction
  - Differential Equations
  - Transfer Functions
  - Laplace and Z Transforms
  - Discretization
  - Systems Identification
  - State-spaces
  - Examples and Exercises

# Introduction to Dynamic Systems Modelling

- A **dynamic model** is a mathematical representation of a system in which the response to a given input is **not immediate**



# Introduction to Dynamic Systems Modelling

- In control theory, a model represents **the behavior of a dynamic system between its input and output**



# Introduction to Dynamic Systems Modelling

- In control theory, a model represents **the behavior of a dynamic system between its input and output**



- In order to design and/or analyze control systems, the controlled process model (G) must be known

# Introduction to Dynamic Systems Modelling

- How can we obtain these models?
  - **Differential Equations**
  - **Systems Identification**
  - Markov Chain/HMM
  - Artificial Neural Networks
  - Graphs
  - Petri nets
  - ...



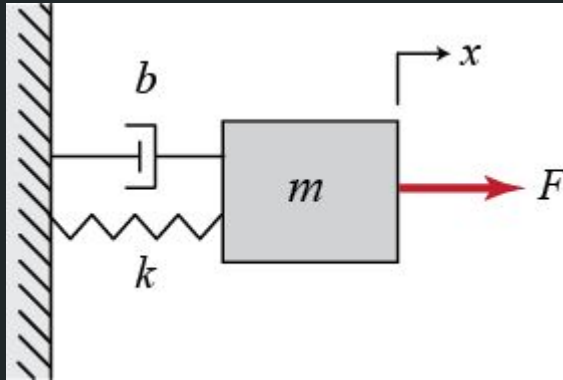
# Introduction to Dynamic Systems Modelling

- **Differential equations** are perhaps the most popular way of modelling dynamic systems
- A differential equation typically represents the relationship between **rates of change** of the various elements in a system
  - In dynamic systems, an ordinary differential equation (ODE) is used to model the rate of change of the output w.r.t the input

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = u(t)$$

# Differential Equations

- Example: **Spring-Mass-Damper system**

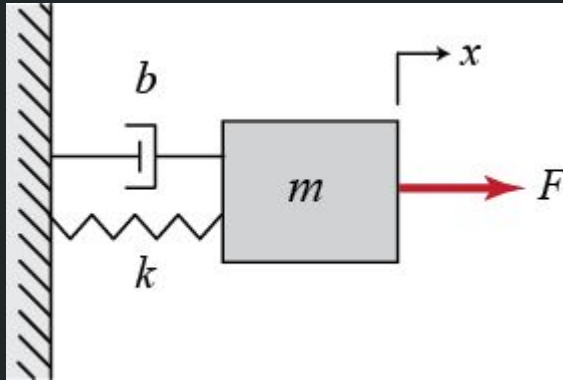


- $m$  = Object mass (kg)
- $x$  = Object displacement (m)
- $b$  = Damping ratio
- $k$  = Spring constant (N/m)

$$m\ddot{x} + b\dot{x} + kx = 0$$

# Differential Equations

- Example: **Spring-Mass-Damper system**



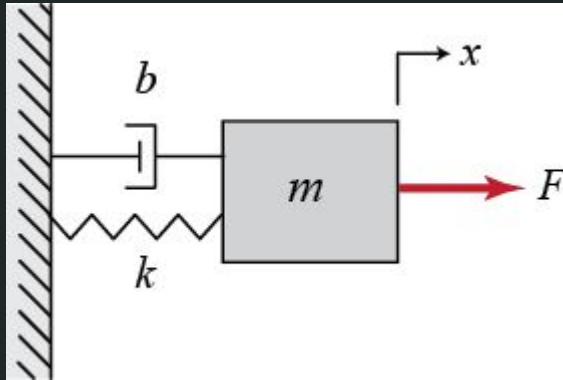
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$$m\ddot{x} + b\dot{x} + kx = 0$$

It is the sum of the equilibrated forces acting upon the system

# Differential Equations

- Example: **Spring-Mass-Damper system**

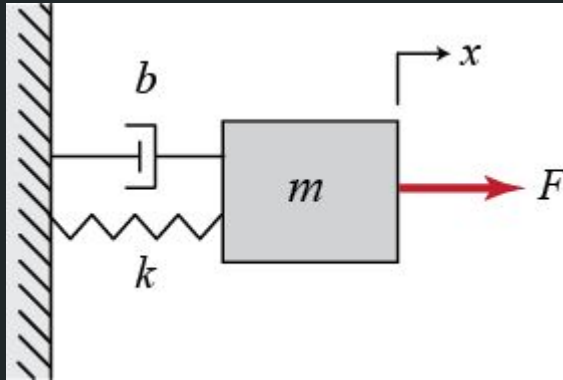


- The SMD system as modelled here is known as an **autonomous system**, because it is unaffected by external influences

$$m\ddot{x} + b\dot{x} + kx = 0$$

# Differential Equations

- Example: **Spring-Mass-Damper system**



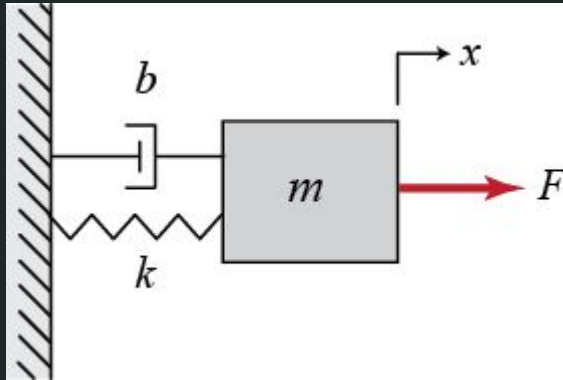
- The SMD system as modelled here is known as an **autonomous system**, because it is unaffected by external influences
- In control theory, it is useful to model external disturbances or controlled forces

$$m\ddot{x} + b\dot{x} + kx = u$$

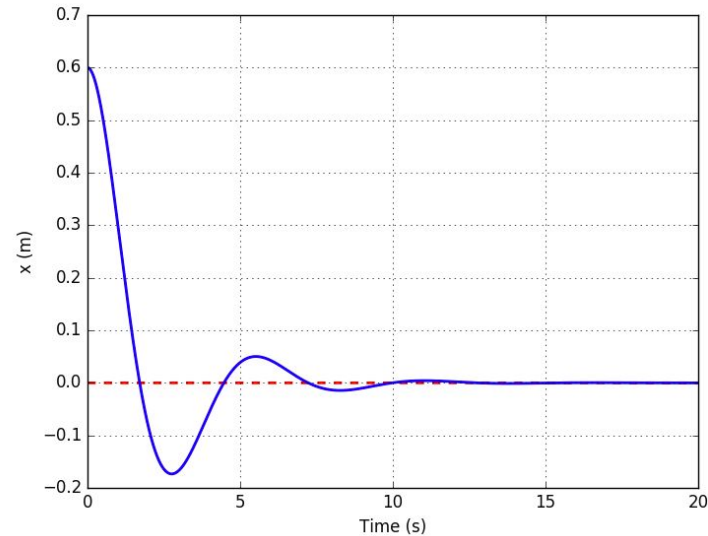
- This is known as a **forced** or **controlled** differential equation

# Differential Equations

- Example: **Spring-Mass-Damper system**



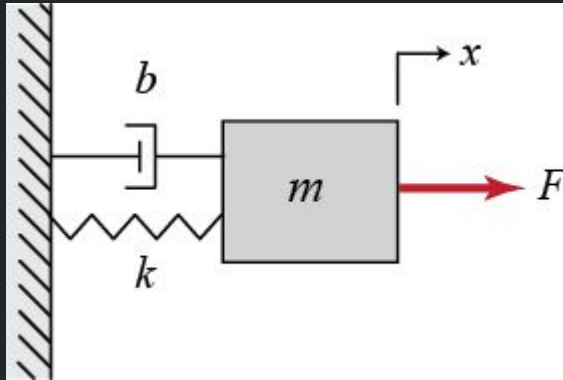
$$m\ddot{x} + b\dot{x} + kx = 0$$



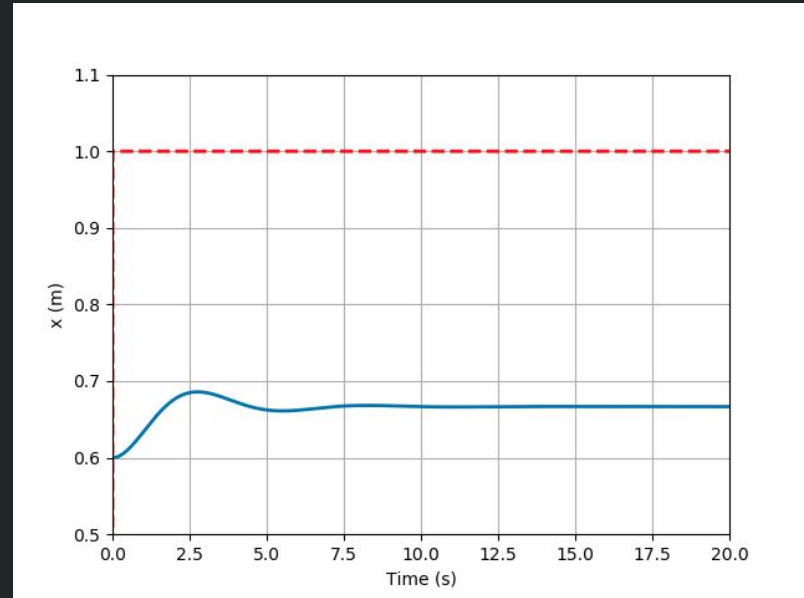
$m = 1$  kg;  $b = 1$ ;  $k = 1.5$  N/m;  
Initial conditions:  $x = 0.6$  m

# Differential Equations

- Example: **Spring-Mass-Damper system**



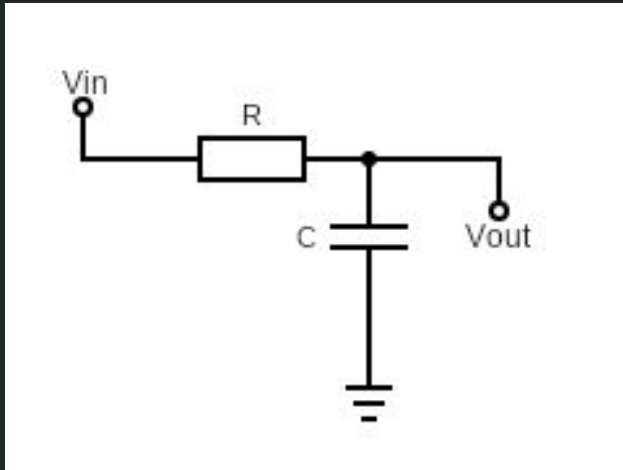
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$m = 1$  kg;  $b = 1$ ;  $k = 1.5$  N/m;  
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# Differential Equations

- Practice: **RC Low-pass filter**

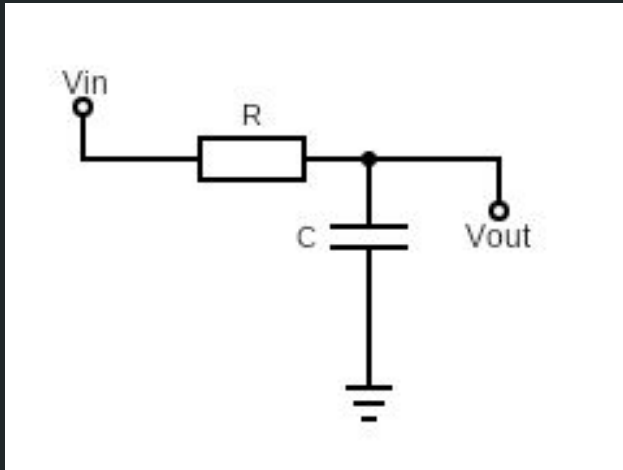


- How can we model this system?
- Tip: Use Kirchoff's first law (node current)



# Differential Equations

- Practice: **RC Low-pass filter**



$$\frac{V_i - V_o}{R} = C\dot{V}_o$$

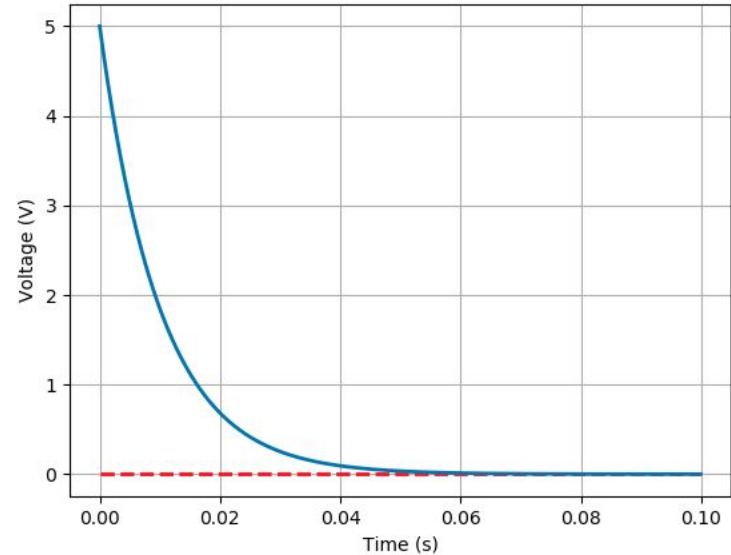
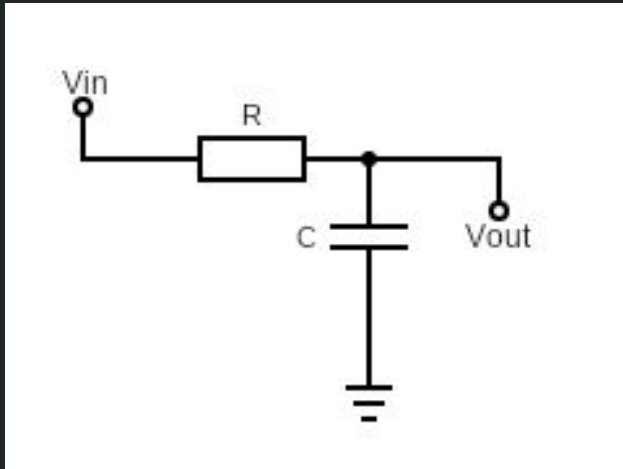
$$V_i - V_o = RC \times \dot{V}_o$$

$$RC \times \dot{V}_o + V_o = V_i$$

**Forced Differential Equation**

# Differential Equations

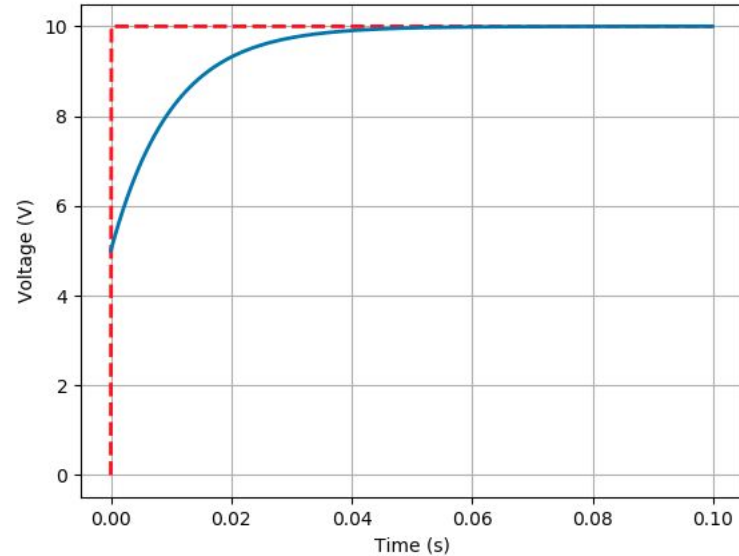
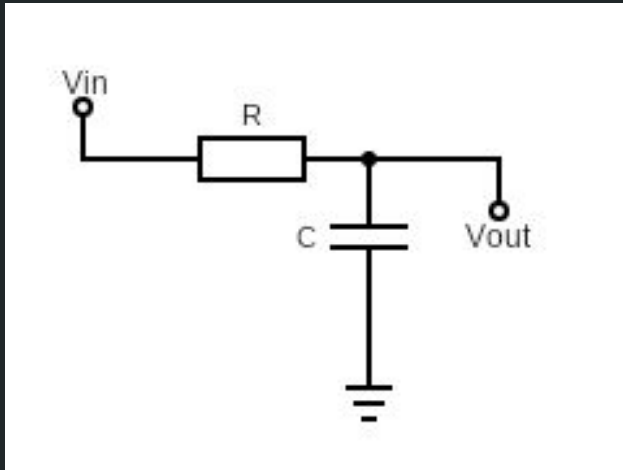
- Practice: **RC Low-pass filter**



$R = 10 \text{ k}\Omega$ ;  $C = 1 \text{ }\mu\text{F}$ ;  
Initial conditions:  $V_{out} = V_C = 5 \text{ V}$

# Differential Equations

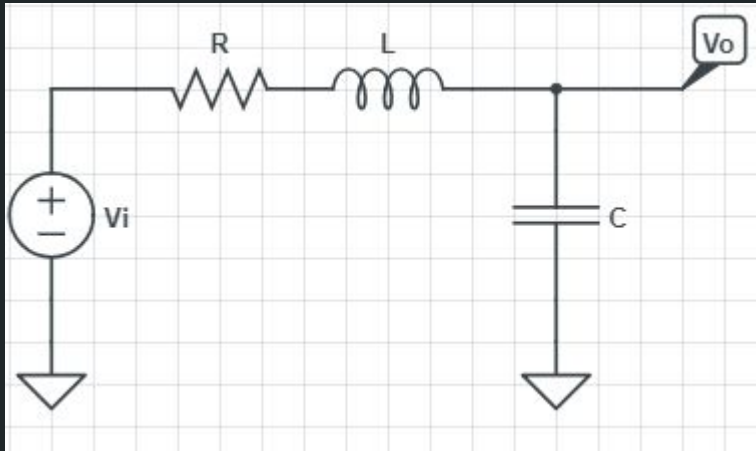
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# Differential Equations

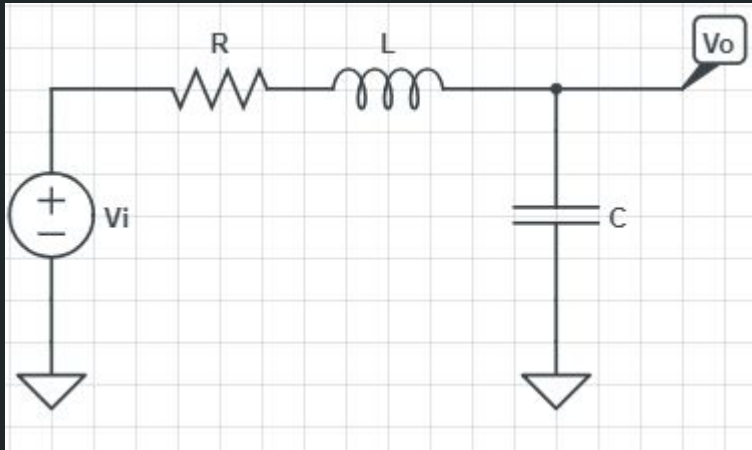
- Practice: **RLC low-pass filter**



- What are the autonomous and forced equations?

# Differential Equations

- Practice: **RLC low-pass filter**

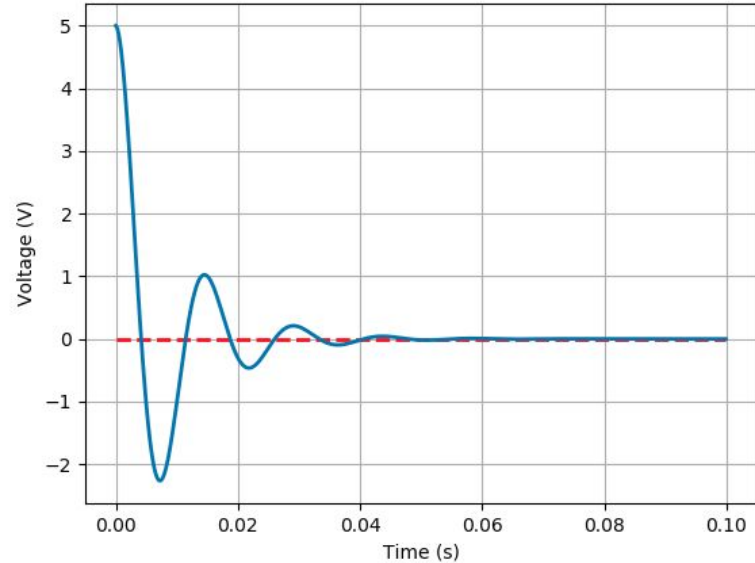
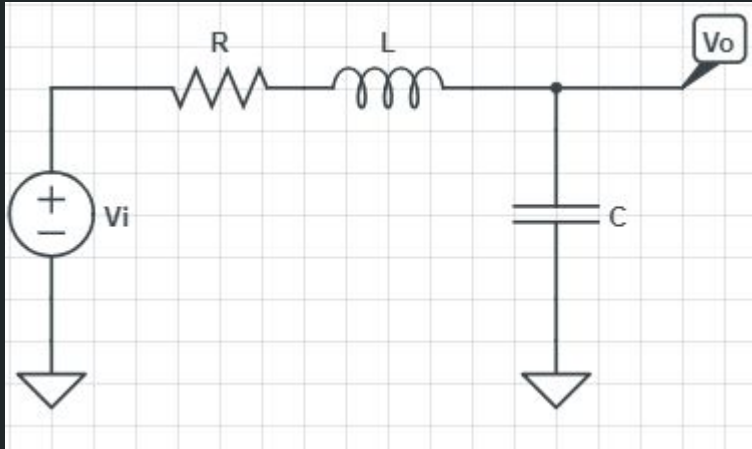


$$LC\ddot{V}_o + RC\dot{V}_o + V_o = V_i$$

- Second-order forced differential equation**
  - When  $V_i = 0$ , it becomes autonomous

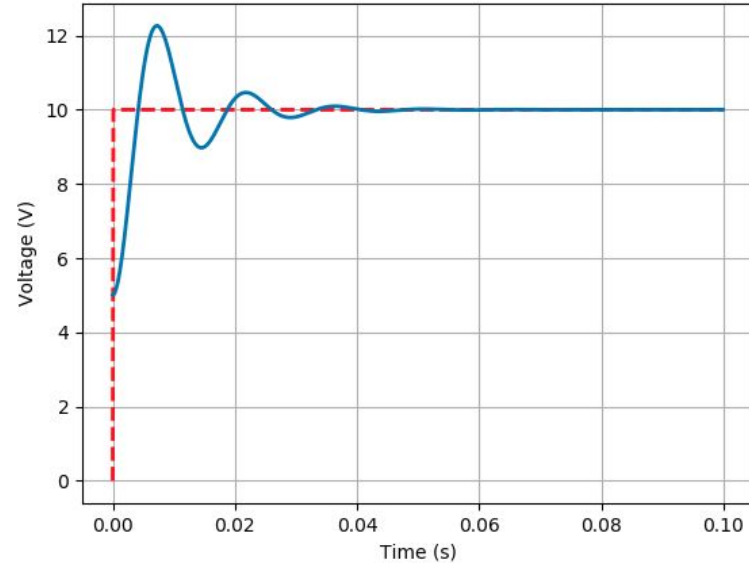
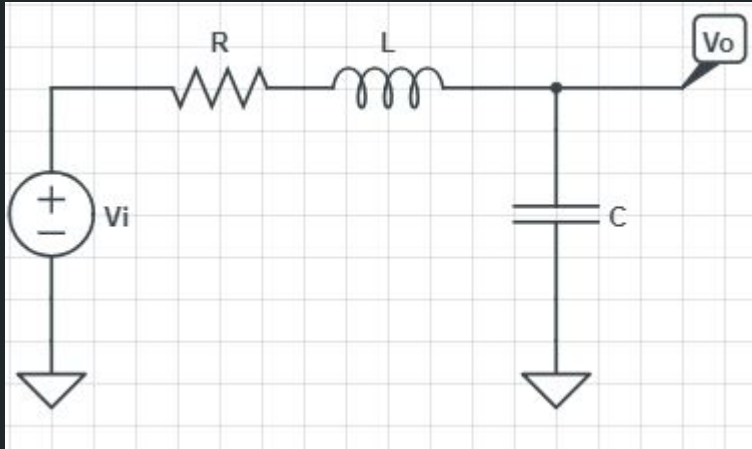
# Differential Equations

- Practice: **RLC** low-pass filter



# Differential Equations

- Practice: **RLC** low-pass filter



# Differential Equations

- The solution to an ordinary differential equation is the **combination of independent variables** (e.g, time) which satisfies the ODE's condition

$$RC\dot{V}_o + V_o = V_i \quad \longrightarrow \quad RCdV_o + V_o dt = V_i dt$$

$$\begin{array}{c} \text{[Integral on all terms]} \\ V_o(t + RC) = V_i t \end{array} \quad \longrightarrow \quad V_o(t) = \frac{V_i \times t}{t + RC} + C_1$$

**General Solution**



# Differential Equations

- Given any **initial conditions**, an ODE may have countless valid **particular solutions**

$$\text{If } V_o(0) = 5 \text{ V} \quad \therefore \quad 5 = \frac{V_i \times 0}{0 + RC} + C_1 \quad \therefore \quad C_1 = 5$$

$$V_o(t) = \frac{V_i \times t}{t + RC} + 5$$

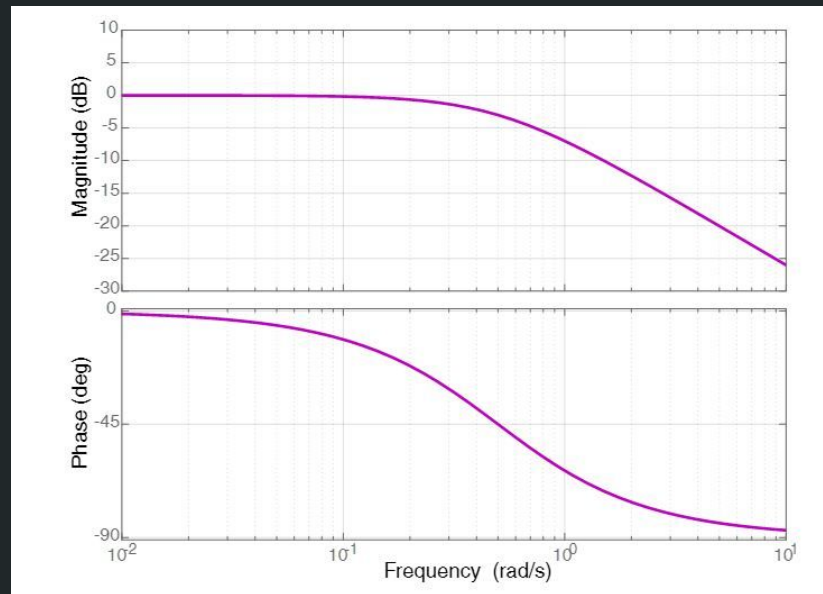
**Particular Solution**

# Transfer Functions

- A disadvantage of modelling dynamic systems with differential equations is **the need to know the system's initial conditions, which may not be constant**
- Instead of using a time-domain model, we can look at the system from a **frequency domain** perspective
- A **Transfer Function** (TF) represents the input/output dynamics of a system through a generalized complex frequency  $\mathbf{s}$  ( $= j\omega$ ), rather than the time  $\mathbf{t}$

# Transfer Functions

- Typically, TFs are used in Electrical Engineering to model complex behaviors in circuits, and to analyze their frequency response
- With transfer functions, we can:
  - Represent voltage/current instant gain
  - Analyze a system's gain/phase margin
  - Analyze a system's stability and steady-state response
  - Etc...



# Laplace and Z Transforms

- From a continuous-time differential equation, we can obtain a **continuous-time transfer function** by applying **Laplace Transforms**

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s)$$

# Laplace and Z Transforms

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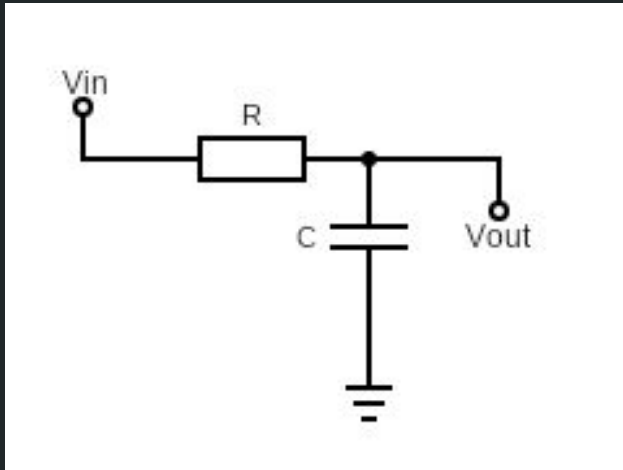
$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \qquad \mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s)$$

- From a discrete-time differential equation, we can obtain a **discrete-time transfer function** by applying **Z Transforms**

$$\mathcal{Z}\{f[n]\} = \sum_{n=0}^{\infty} f[n]z^{-n}$$

# Transfer Func. (cont.)

- Continuous-time example: **RC Low-pass filter**



$$RC\dot{V}_o + V_o = V_i$$

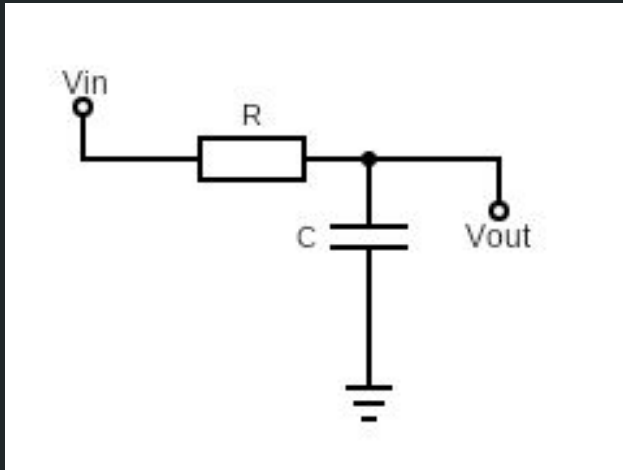
$$RC\dot{y} + y = u$$

$$RC\mathcal{L}\{\dot{y}\} + \mathcal{L}\{y\} = \mathcal{L}\{u\}$$

$$Y(s)(RCs + 1) = U(s) \quad \therefore \quad \frac{Y(s)}{U(s)} = \frac{1/RC}{s + 1/RC}$$

# Transfer Func. (cont.)

- Continuous-time example: **RC Low-pass filter**



$$RC\dot{V}_o + V_o = V_i$$

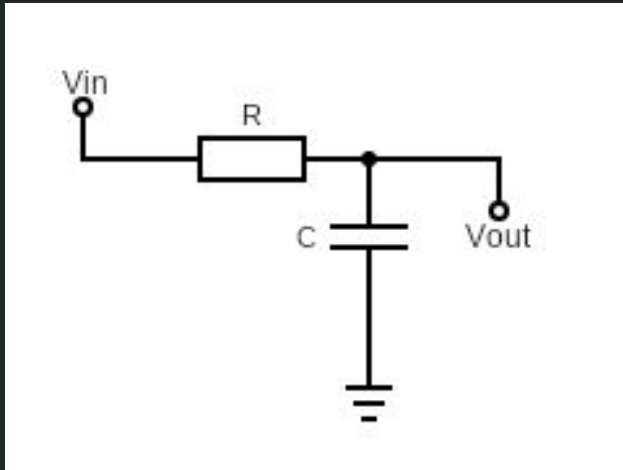
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# Transfer Func. (cont.)

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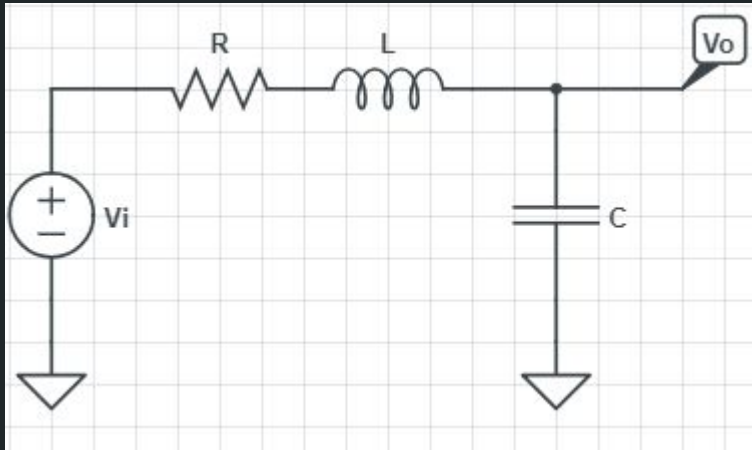


- What about discrete-time?



# Transfer Func. (cont.)

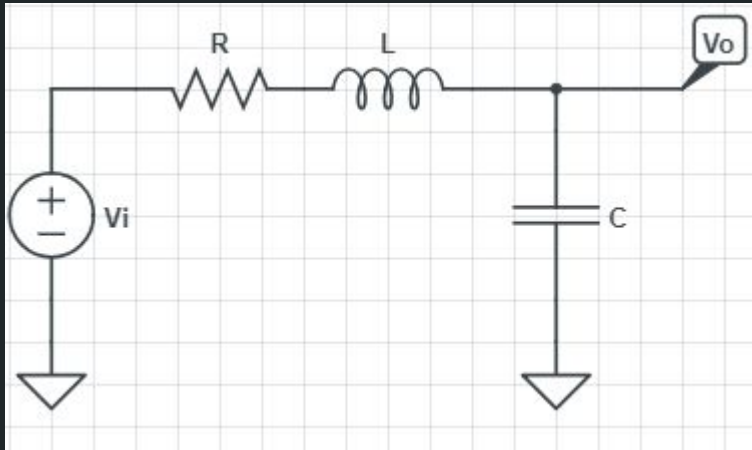
- Practice: **RLC** low-pass filter



$$LC\ddot{V}_o + RC\dot{V}_o + V_o = V_i$$

## Transfer Func. (cont.)

- Practice: **RLC low-pass filter**

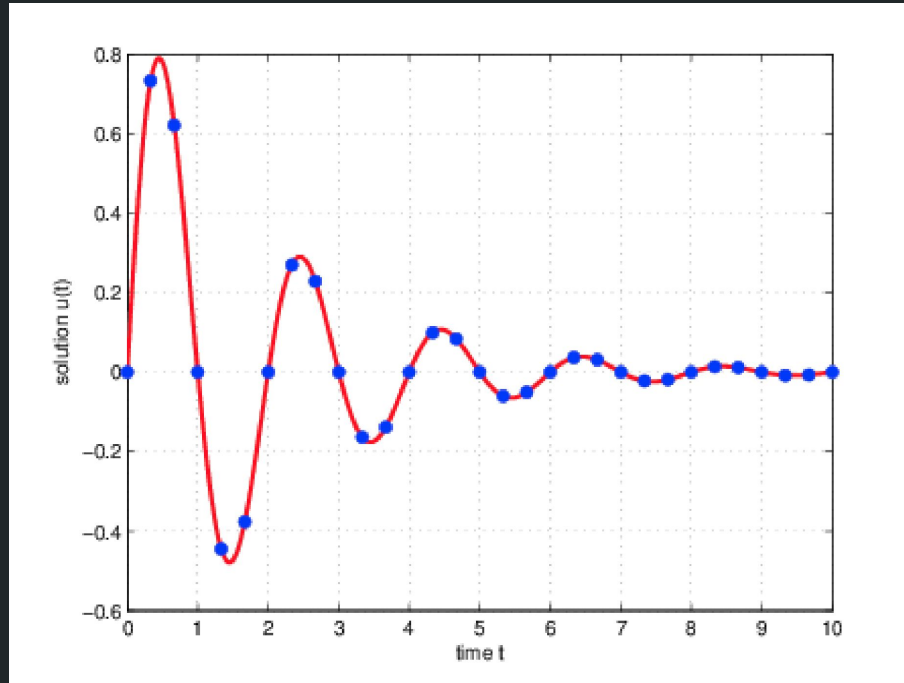


$$LC\ddot{V}_o + RC\dot{V}_o + V_o = V_i$$

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

# Discretization

- To obtain a system's discrete-time TF, we can **discretize** the modelled ODE



# Discretization

- A discrete-time system can be represented by a **difference equation**, a discrete approximation of a differential equation

$$\dot{y} + \frac{y}{RC} = \frac{u}{RC} \qquad \dot{y} \approx \frac{y[k] - y[k - 1]}{T}$$

# Discretization

- A discrete-time system can be represented by a **difference equation**, a discrete approximation of a differential equation

$$\dot{y} + \frac{y}{RC} = \frac{u}{RC} \qquad \dot{y} \approx \frac{y[k] - y[k-1]}{T}$$

$$y[k] = \frac{1}{1 + \frac{T}{RC}} \left( \frac{T}{RC} u[k] + y[k-1] \right)$$

# Discretization

- Finally, the Z-transform gives us the discrete-time transfer function

$$y[k] = \frac{1}{1 + \frac{T}{RC}} \left( \frac{T}{RC} u[k] + y[k - 1] \right)$$

$$\frac{Y(z)}{U(z)} = G(z) = \frac{\frac{\alpha}{1+\alpha} z}{z - \frac{1}{1+\alpha}}, \text{ where } \alpha = \frac{T}{RC}$$

# Discretization

Euler-Backward

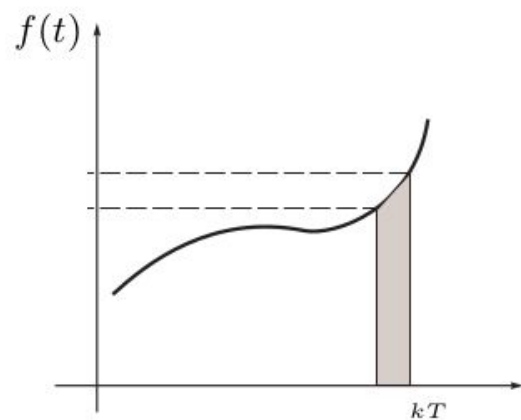
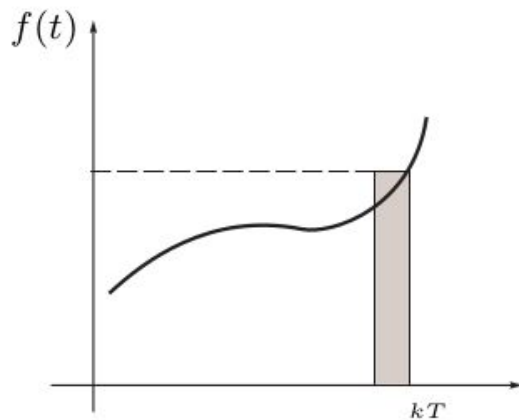
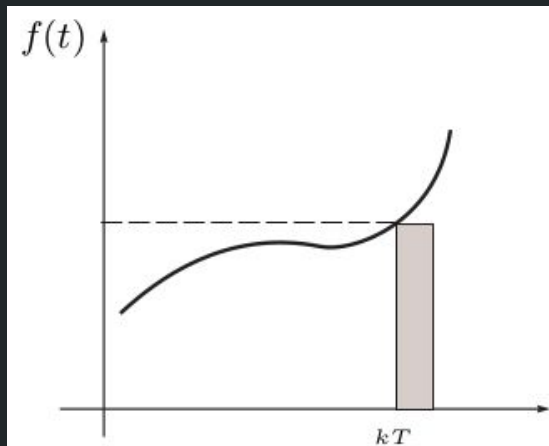
$$s \approx \frac{z - 1}{Tz}$$

Euler-Forward

$$s \approx \frac{z - 1}{T}$$

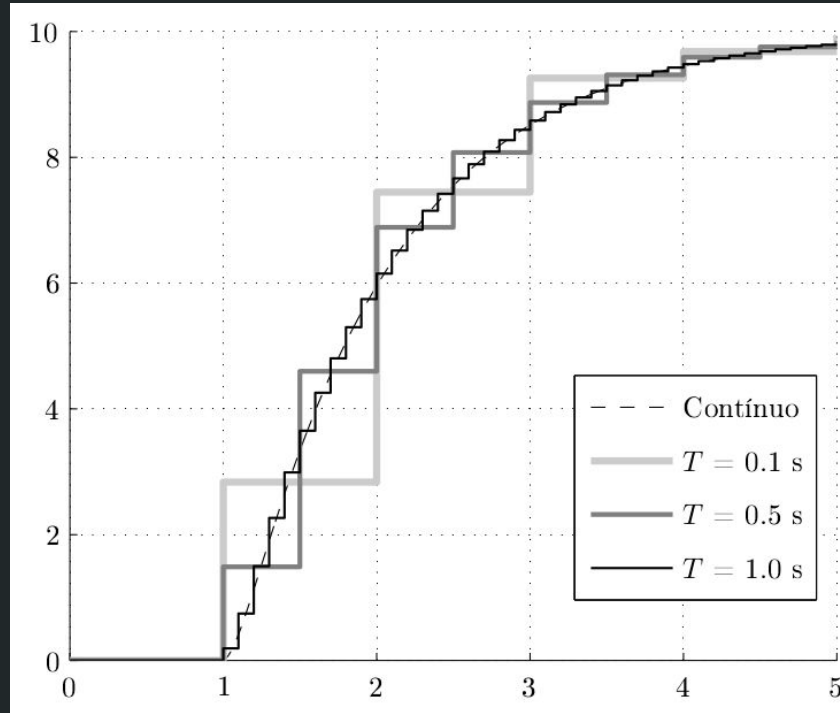
Tustin

$$s \approx \frac{2(z - 1)}{T(z + 1)}$$



# Discretization

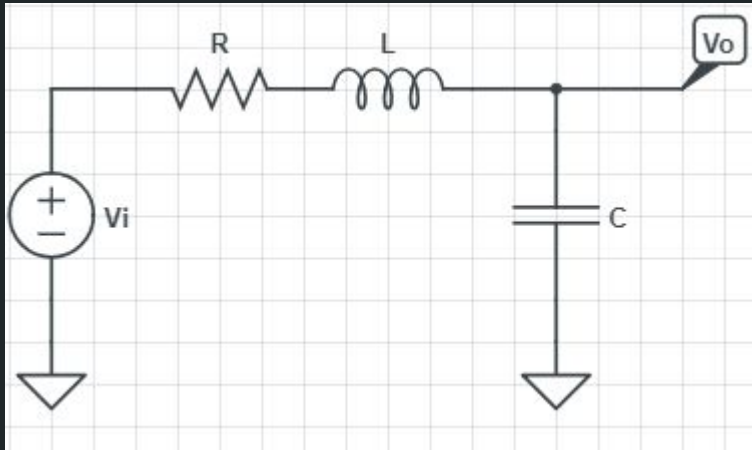
- Effects of sampling time in discretization





# Discretization

- Practice: **Discretization of the RLC low-pass filter**



$$LC\ddot{V}_o + RC\dot{V}_o + V_o = V_i$$

# Discretization

- Practice: **Discretization of the RLC low-pass filter**

$$LC\ddot{V}_o + RC\dot{V}_o + V_o = V_i$$

$$\frac{Y(z)}{U(z)} = \frac{\frac{T^2}{\alpha} z^2}{z^2 - \frac{2LC+TRC}{\alpha} z + \frac{1}{\alpha}}, \text{ where } \alpha = T^2 + TRC + LC$$

# Systems Identification

- For complex systems, modelling transfer functions through differential equations can be challenging
- If we stimulate a system with a known input and record its output, it is possible to **approximate the transfer function through some regression**
  - Normal equation

# Systems Identification

- To do this, we must first “guess” an approximation with the same (unknown) order of our system

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

# Systems Identification

- Then we discretize it

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$y[k] = \frac{T^2 w_n^2}{T^2 w_n^2 + 2T\xi w_n + 1} u[k] + \frac{2(1 + T\xi w_n)}{T^2 w_n^2 + 2T\xi w_n + 1} y[k-1] - \frac{1}{T^2 w_n^2 + 2T\xi w_n + 1} y[k-2]$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k-1] + \theta_3 y[k-2]$$

# Systems Identification

- Then we discretize it

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$y[k] = \left[ \frac{T^2 w_n^2}{T^2 w_n^2 + 2T\xi w_n + 1} \right] u[k] + \left[ \frac{2(1 + T\xi w_n)}{T^2 w_n^2 + 2T\xi w_n + 1} \right] y[k-1] - \left[ \frac{1}{T^2 w_n^2 + 2T\xi w_n + 1} \right] y[k-2]$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k-1] + \theta_3 y[k-2]$$

# Systems Identification

- Then we represent it as a matrix multiplication

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[n] \end{bmatrix} = \begin{bmatrix} u[0] & y[-1] & y[-2] \\ u[1] & y[0] & y[-1] \\ u[2] & y[1] & y[0] \\ \vdots & \vdots & \vdots \\ u[n] & y[n-1] & y[n-2] \end{bmatrix} \times \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$Y = \psi\theta$$

# Systems Identification

- Then we solve for  $\theta$

$$Y = \psi\theta \qquad \theta = (\psi^T\psi)^{-1}\psi^TY$$



# Systems Identification

- Then we solve for  $\theta$

$$Y = \psi\theta$$



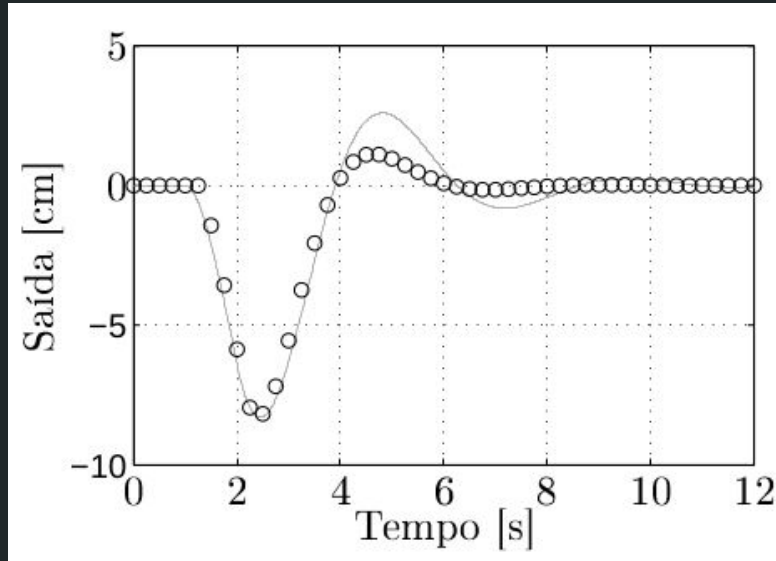
Regressors'  
Matrix

$$\theta = (\psi^T \psi)^{-1} \psi^T Y$$

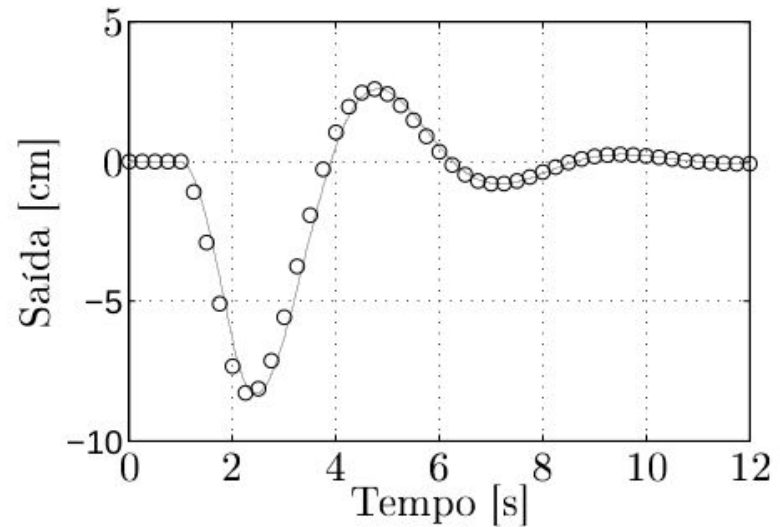
**Normal Equation**

# Systems Identification

- Effect of the approximation order on the output



$$\theta_1 y[k] + \theta_2 y[k-1] + \theta_3 u[k-1]$$



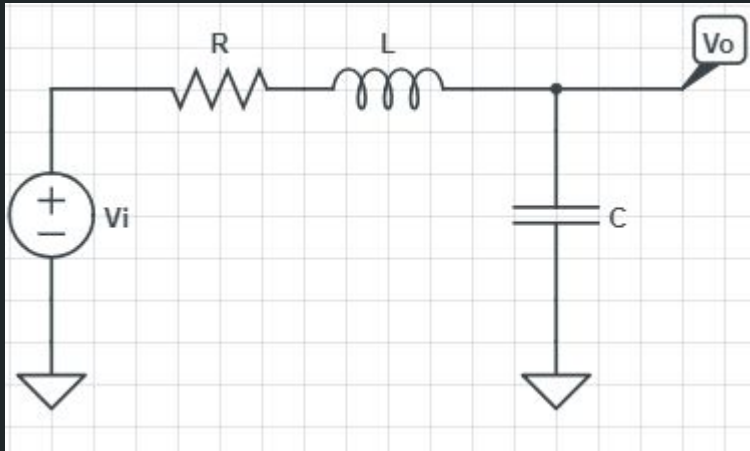
$$\theta_1 y[k] + \theta_2 y[k-1] + \theta_3 u[k] + \theta_4 u[k-1]$$

# Systems Identification

- **Beware!** While higher-orders provide better approximations, modelling them becomes increasingly complex
- What you are approximating in higher orders may not really be your system, but something else with a similar behavior

# Systems Identification

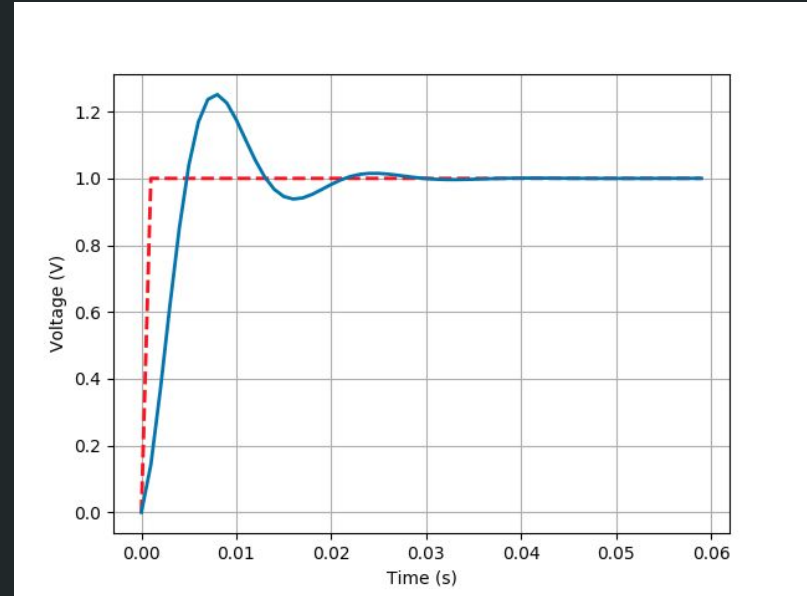
- Example: Identifying the RLC filter model



$R = ?$

$L = ?$

$C = ?$



Step input ( $u = 1\text{V } \forall t$ )

$T = 0.001\text{ s}$

# Systems Identification

- Example: **Identifying the RLC filter model**

$$\frac{Y(s)}{U(s)} = \frac{a}{s + b}$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1]$$

**First-order**

# Systems Identification

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$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1]$$

**First-order**

$$\theta = \begin{bmatrix} 0.2477 \\ 0.7612 \end{bmatrix}$$

# Systems Identification

- Example: **Identifying the RLC filter model**

$$\frac{Y(s)}{U(s)} = \frac{a}{s + b}$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1]$$

**First-order**

$$\theta = \begin{bmatrix} 0.2477 \\ 0.7612 \end{bmatrix}$$

**Second-order**

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1] + \theta_3 y[k - 2]$$

# Systems Identification

- Example: **Identifying the RLC filter model**

$$\frac{Y(s)}{U(s)} = \frac{a}{s + b}$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1]$$

**First-order**

$$\theta = \begin{bmatrix} 0.2477 \\ 0.7612 \end{bmatrix}$$

$$\theta = \begin{bmatrix} 0.1428 \\ 1.5714 \\ -0.7143 \end{bmatrix}$$

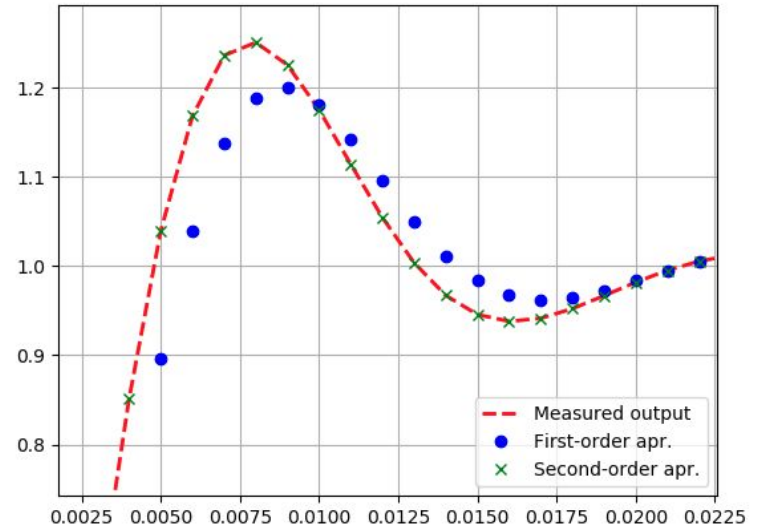
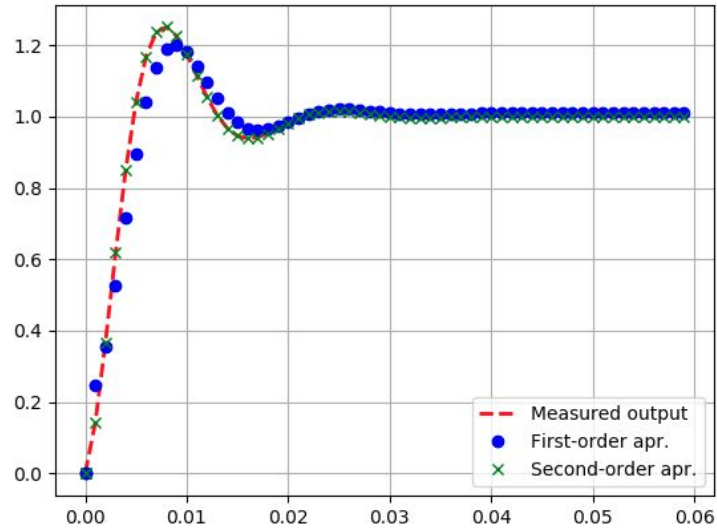
**Second-order**

$$\frac{Y(s)}{U(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2}$$

$$y[k] = \theta_1 u[k] + \theta_2 y[k - 1] + \theta_3 y[k - 2]$$

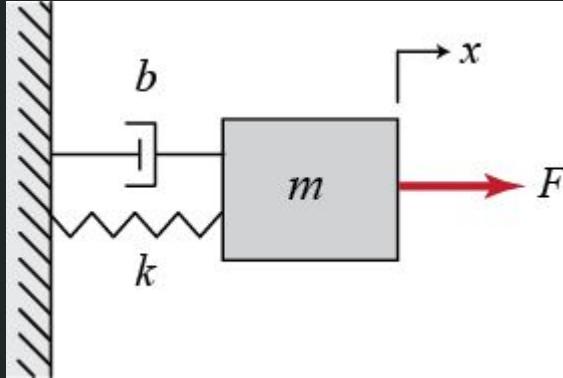


# Systems Identification



# Systems Identification

- Challenge: **Spring-Mass-Damper system**

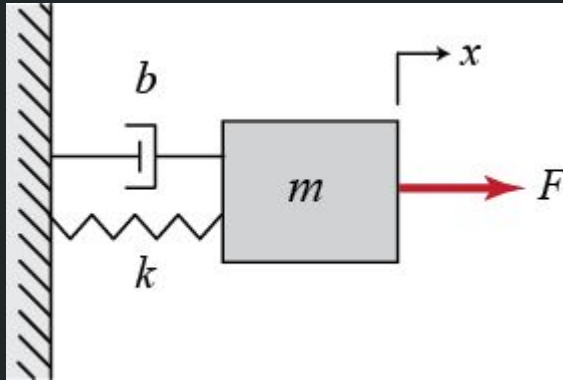


- 1) Find the system's transfer function,  $G(s)$
- 2) Discretize  $G(s)$
- 3) Use your discretized model to approximate  $G(s)$  through identification
  - a) Tip: Use matlab, octave, python...
- 4) Define the previously unknown mass, damping coefficient ( $b$ ) and spring constant ( $k$ )

$$m\ddot{x} + b\dot{x} + kx = 0$$

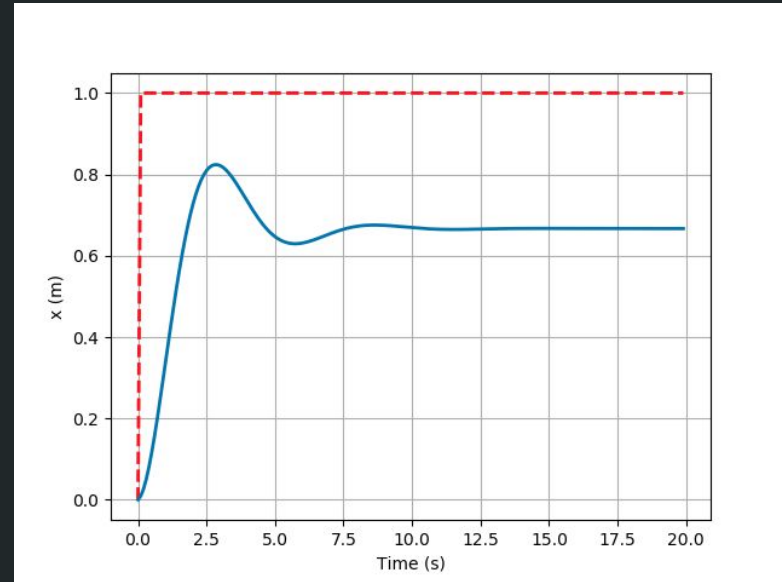
# Systems Identification

- Challenge: **Spring-Mass-Damper system**



$m = ?; b = ?; k = ?$

Sampling time  $T = 0.1$  s



- Data available in Github!**  
([https://github.com/imr-pucrs/didactic\\_resources](https://github.com/imr-pucrs/didactic_resources))

# State-Spaces

- In modern control theory, we typically use **state-spaces** to represent dynamic systems
  - Robust, optimal and predictive control;
  - Kalman Filters;
  - Etc;
- In a simple interpretation, they are **a set of equations which describe how a system changes in time**, relative to the **states** of the input and output

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

# State-Spaces

- Example: **Spring-Mass-Damper system**

$$m\ddot{y} + b\dot{y} + ky = u$$

# State-Spaces

- Example: **Spring-Mass-Damper system**

$$m\ddot{y} + b\dot{y} + ky = u$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overset{\text{A}}{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overset{\text{B}}{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}} u$$

$$y = \underset{\text{C}}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du$$

# State-Spaces

- To convert from transfer functions to state-spaces, we can use the **canonical forms**

Controllable  
Form

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \dots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

# State-Spaces

- To convert from transfer functions to state-spaces, we can use the **canonical forms**

Observable  
Form

$$\begin{aligned}A_{obs} &= A_{cont}^T \\ B_{obs} &= C_{cont}^T \\ C_{obs} &= B_{cont}^T \\ D_{obs} &= D_{cont}\end{aligned}$$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$



# State-Spaces

- Example: **RLC Low-pass filter**

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

# State-Spaces

- Example: **RLC Low-pass filter**

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad \text{Controllable}$$

$$y = \begin{bmatrix} \frac{1}{LC} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# State-Spaces

- Example: **RLC Low-pass filter**

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{LC} \\ 1 & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{LC} \\ 0 \end{bmatrix} u \quad \text{Observable}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# State-Spaces

- Practice: **RC Low-pass filter**

$$\frac{Y(s)}{U(s)} = \frac{1/RC}{s + 1/RC}$$

$$RC\dot{y} + y = u$$

- What are the canonical controllable and observable forms?
- How do they compare to the state-space modelled from the differential equation?