CHAPTER 5

GAMES PLAYED ON NETWORKS

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5.1 Introduction

This chapter studies games played on networks. The games capture a wide variety of economic settings, including local public goods, peer effects, and technology adoption. Pairwise links can represent respectively, geographic proximity, peer relations, and industry ties. The collection of links form a network, and given the network, agents choose actions. Individual payoffs derive from own actions and the actions of linked parties. Because sets of links overlap, ultimately the entire network structure determines equilibrium outcomes.

The chapter develops a guide to study all these settings by nesting the games in a common framework. Start with local public goods, for example. Individuals choose some positive level of goods, which benefit themselves and their neighbors. Provision is individually costly. Players' actions are then strategic substitutes; when a player's neighbor provides more, he provides less. A peer effect game requires just two modifications. The first is a change in the sign of a parameter, so that players' actions are strategic complements rather than substitutes. The second is an upper bound on players' actions, since, for example, students can study no more than twenty-four hours in a day. Next consider a coordination game, as in technology adoption. Individuals' actions are complements; they want to choose the same action as their neighbors. Here there is one additional modification—restricting individuals to binary actions: "adopt technology A" or "adopt technology B." We show that in these games individuals have the same underlying incentives expressed in linear best replies, which are more or less constrained. The chapter systematically introduces the modifications—changes in parameter sign and/or constraints on agents' choices—and shows how they alter the analysis and affect outcomes.

The chapter has three overarching objectives. The first is to establish a common analytical framework to study this wide class of games. So doing, the chapter establishes new connections between games in the literature—particularly the connection between binary choice games, such as coordination and best-shot games, and games with continuous actions, such as public goods, peer effects, and oligopoly games. The second objective is to review and advance existing results by showing how they tie together within the common framework. The final objective is to outline directions for future research.

All the games considered in this chapter are simultaneous-move, complete information games.[1] The analysis thus employs classic solution concepts: Nash equilibrium and stable equilibrium, which is a Nash equilibrium robust to small changes in agents' actions.

The chapter studies how equilibrium outcomes relate to features of the network. In any strategic setting, researchers study the existence, uniqueness, and stability and possibly comparative statics of equilibria. In a network game, researchers strive to answer these questions in terms of the network: What features of the network determine the Nash and stable equilibrium set? How do individual network positions determine individual play? How do outcomes change when links are added to or subtracted from a network?

The chapter reviews known results and highlights open questions. Characterizations of equilibrium sets often involve conditions on the eigenvalues of the network matrix. It has been long known, for example, that in a continuous-action game with pure complementarities, such as peer effects, a contraction property based on the highest eigenvalue guarantees the existence and uniqueness of Nash equilibrium. Contraction ensures that the complementarities and network effects are sufficiently small so there is a convergence in the best replies.

In a continuous-action game with any substitutabilities, such as an oligopoly game, it is the lowest eigenvalue that appears in the equilibrium conditions. For example, a unique Nash equilibrium exists if the magnitude of the lowest eigenvalue is sufficiently small. While the highest eigenvalue is a positive number, the lowest eigenvalue is a negative number, and its magnitude captures the extent of substitutabilities in the overall network. When this condition does not hold, the computation of the Nash and stable equilibrium set is necessarily complex.

As for individual play, when network effects are sufficiently small so that there is a unique interior equilibrium, individual actions are proportional to players' Bonacich centralities. Little is known, however, about how individual play relates to network position when the network effects are larger and equilibria involve agents who are constrained in their best replies.

The chapter further engages a novel question concerning networks and equilibria: how is one agent's action affected by an exogenous shock to another agent? The

[1] We discuss network games of incomplete information in the Conclusion.

chapter develops a new notion: equilibrium *interdependence* of agents. When one firm's production cost is reduced, for example, it possibly affects equilibrium production of all firms—not just the firm's direct competitors. We study whether two players who are not directly connected are nonetheless *interdependent*, relating interdependence to changes in individual parameters. We show that path-connectedness is necessary but not sufficient for interdependence. A third player along the path could absorb the impact of one player on the other. We study comparative statics and how shocks to individuals propagate or do not propagate through a network.[2]

This consideration of individual heterogeneity opens new avenues for investigation. Individuals are characterized not only by their position in the network but also by individual costs and benefits of actions. This heterogeneity allows the study of interdependence and the connection to the empirical literature on social interaction, equilibria, and the reflection problem. This connection is a rich area for current and future research and is discussed below.

The chapter is organized as follows. Section 5.2 presents the basic simultaneous-move game on a network and constructs the common set of (linear) best replies. Section 5.3 gives examples of games in the literature that fit in the framework. Section 5.4 analyzes the simplest case—unconstrained actions—which serves as a foil for the constrained cases that follow. Sections 5.5 and 5.6 study the games where agents actions are continuous but constrained, first to be positive, then to be positive and below an upper bound. In Section 5.7, we study binary action games. Section 5.8 relates the theory of network games to empirical work on social interactions. Section 5.9 discusses related games and topics that fall outside our framework.

5.2 Games on a Network and (Modified) Linear Best Replies

This section introduces the basic game played on a network and identifies the key economic parameters. It then poses the mathematical system underlying the Nash and stable equilibria for the class of games covered in this chapter.

5.2.1 Players, Links, and Payoffs

There are n agents, and N denotes the set of all agents. Agents simultaneously choose actions; each agent i chooses an x_i in $X_i \subseteq \mathbb{R}$. Agents are embedded in a fixed network represented by an $n \times n$ matrix, or graph, $\mathbf{G}$, where $g_{ij} \in \mathbb{R}$ represents a link between

[2] The chapter by Daron Acemoglu, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, in this volume, also discusses how macroeconomic outcomes are induced by microeconomic shocks under network interactions.

agents i and j. Note that g_{ij} can be weighted and positive or negative. For most of the chapter, links are assumed to be undirected; i.e., $g_{ij} = g_{ji}$. In the games below, only the actions of i's *neighbors*—the agents to whom i is linked—enter an agent i's payoff π_i.

Each agent's payoff is a function of own action, x_i, others' actions, $\mathbf{x}_{-i}$, the network, and a global parameter $\delta \in [-1,1]$, called the "payoff impact parameter," which gives the sign and magnitude of the effect of players' actions on their neighbors: $\pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G})$. For a given $\mathbf{G}$ and δ, we will say a property holds *generically* (i.e., for *almost any* δ), if it holds for every $\delta \in [-1,1]$ except for possibly a finite number of values. For any square matrix $\mathbf{M}$, let $\lambda_{\min}(\mathbf{M})$ denote the lowest eigenvalue and let $\lambda_{\max}(\mathbf{M})$ denote the highest eigenvalue. Note these eigenvalues can always be written in terms of each other; that is, $\lambda_{\max}(-\mathbf{M}) = -\lambda_{\min}(\mathbf{M})$.

The signs of δ and g_{ij} determine the type of strategic interactions between i and j. As we will see, i's and j's actions are strategic complements when $\delta g_{ij} < 0$, and they are strategic substitutes when $\delta g_{ij} > 0$.[3] We say that a game has *pure complements* if $\delta < 0$ and $\forall i, j, g_{ij} \geq 0$ and *pure substitutes* if $\delta > 0$ and $\forall i, j, g_{ij} \geq 0$. The literature has paid much attention to these polar cases, at the risk of neglecting the analysis of the general case.[4] We pay careful attention to this issue in what follows.

5.2.2 Best Replies, Nash Equilibria, and Stable Equilibria

We consider pure-strategy Nash equilibria of these games.[5] Let

$$f_i(\mathbf{x}_{-i}; \delta, \mathbf{G}) = \arg\max_{x_i} \{\pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G})\} \tag{5.1}$$

denote agent i's best reply to other agents' actions. The following is then the system of best replies:

$$\begin{aligned} x_1 &= f_1(\mathbf{x}_{-1}; \delta, \mathbf{G}) \\ &\vdots \\ x_n &= f_n(\mathbf{x}_{-n}; \delta, \mathbf{G}). \end{aligned} \tag{5.2}$$

A Nash equilibrium is a vector $\mathbf{x} = (x_1, \ldots, x_n)$ that satisfies this system.

The chapter considers both the full set of Nash equilibria and the subset of Nash equilibria that are *stable*. Stable, here, refers to the criterion that an equilibrium is robust

[3] An agent i's action is a strategic complement (substitute) to j's action when i's best reply is increasing (decreasing) in j's action. See Bulow, Geanakoplos, and Klemperer (1985).

[4] For example, with pure complements the Perron-Frobenius Theorem and derivative results (e.g., the addition of a link increases the largest eigenvalue) apply. But these results do not apply if there are any substitutabilities.

[5] For many continuous action games in this class, the payoff functions are concave and no mixed strategy Nash equilibria exist.

to small changes in agents' actions. Appropriate stability notions necessarily differ for continuous and binary action games. For binary actions, we use a notion of stochastic stability based on asynchronous best-reply dynamics and payoffs (Blume 1993; Young 1998).

For continuous actions, we consider a classic definition of stability which is a continuous version of textbook Nash tâtonnement.[6] Starting with a Nash equilibrium **x**, and changing agents' actions by a little bit, we ask whether the best replies lead back to the original vector. Consider the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(\mathbf{x}_{-1};\delta,\mathbf{G}) - x_1 \\ &\vdots \\ \dot{x}_n &= f_n(\mathbf{x}_{-n};\delta,\mathbf{G}) - x_n. \end{aligned} \tag{5.3}$$

By construction, a vector **x** is a stationary state of this system if and only if it is a Nash equilibrium. We say a Nash equilibrium **x** is *asymptotically stable* when (5.3) converges to **x** following any small enough perturbation.[7]

5.2.3 Class of Games and Restrictions on the Strategy Space

We consider games whose payoffs are special cases of the following generalized payoff function:

$$\pi_i(x_i,\mathbf{x}_{-i};\delta,\mathbf{G}) = v_i\left(x_i - x_i^0 + \delta\sum_j g_{ij}x_j\right) + w_i(\mathbf{x}_{-i}) \tag{5.4}$$

where v_i is increasing on $(-\infty,0]$, decreasing on $[0,+\infty)$ and symmetric around 0, so that 0 is the unique maximum of v_i, and w_i can take any shape. The individual parameter x_i^0 represents agent i's optimal action absent social interactions ($\delta = 0$ and/or $g_{ij} = 0$). A higher x_i^0 would correspond, for example, to i's greater personal benefit from actions or lower private cost. As $|\delta|$ increases, the payoff externalities of agents' actions become globally stronger.

In the base case, actions can take any real value: for each agent i, $x_i \in X_i = \mathbb{R}$. With payoffs (5.4), best replies are linear in other agents' actions:

$$f_i(\mathbf{x}_{-i}) = x_i^0 - \delta\sum_j g_{ij}x_j. \tag{5.5}$$

[6] See, e.g., Fisher (1961).

[7] Formally, following Weibull (1995, Definition 6.5, p. 243), introduce $B(\mathbf{x},\varepsilon) = \{\mathbf{y} \in \mathbb{R}_+^n : ||\mathbf{y}-\mathbf{x}|| < \varepsilon\}$ and $\xi(t,\mathbf{y})$, the value at time t of the unique solution to the system of differential equations that starts at $\mathbf{y}$. An equilibrium $\mathbf{x}$ is *asymptotically stable* if $\forall\varepsilon > 0, \exists\eta > 0 : \forall\mathbf{y} \in B(\mathbf{x},\eta), \forall t \geq 0$, $\xi(t,\mathbf{y}) \in B(\mathbf{x},\varepsilon)$ and if $\exists\varepsilon > 0 : \forall\mathbf{y} \in B(\mathbf{x},\varepsilon)$, $\lim_{t\to\infty}\xi(t,\mathbf{y}) = \mathbf{x}$.

The agent essentially compares his autarkic optimum, x_i^0, to the weighted sum of his neighbors' actions, $\delta \sum_j g_{ij} x_j$; his best reply is the difference between the two.

While in principle, a player's action could be any real number, all games in the literature place restrictions on players' actions which represent different real-world situations. For example, for peer effects in a classroom, there are natural lower and upper bounds—a student can study no less than zero hours and no more than twenty-four hours in a day. For technology adoption, an individual is often restricted to two actions: "adopt" or "not adopt."

For each restriction on the action space, we determine the corresponding best reply. First, agents' actions are constrained to be non-negative, which corresponds to, for example, the production of goods or services. For each agent i, $x_i \in [0, \infty)$ and the corresponding best reply is

$$f_i(\mathbf{x}_{-i}) = \max\left(0, \left(x_i^0 - \delta \sum_j g_{ij} x_j\right)\right). \tag{5.6}$$

Second, agents' actions must not be below zero nor be above some finite upper bound L: for each agent i, $x_i \in X_i = [0, L]$ with $0 < L < \infty$. The corresponding best reply is

$$f_i(\mathbf{x}_{-i}) = \min\left(\max\left(x_i^0 - \delta \sum_j g_{ij} x_j, 0\right), L\right). \tag{5.7}$$

In both cases a player's best reply is to choose, as much as possible, the difference between x_i^0, and the weighted sum $\delta \sum_j g_{ij} x_j$.

Finally, agents must choose between two discrete values: $x_i \in X_i = \{a, b\}$ with $a \leq b$. Agent i's best reply can be written in terms of a threshold value $t_i = x_i^0 - \frac{1}{2}(a+b)$. If the weighted sum of neighbors' actions is above the threshold, i's best response is a; if the weighted sum is below the threshold, agent i's best response is b; if the sum is equal to the threshold, i is indifferent between a and b. We have:

$$\begin{aligned} f_i(\mathbf{x}_{-i}) &= a \text{ if } \delta \sum_j g_{ij} x_j > t_i; \\ f_i(\mathbf{x}_{-i}) &= b \text{ if } \delta \sum_j g_{ij} x_j < t_i; \\ f_i(\mathbf{x}_{-i}) &= \{a, b\} \text{ if } \delta \sum_j g_{ij} x_j = t_i. \end{aligned} \tag{5.8}$$

The best replies for the constrained actions, (5.6), (5.7), and (5.8), can all be obtained from (5.5). Let $\widehat{x}_i(\mathbf{x}_{-i}) \equiv x_i^0 - \delta \sum_j g_{ij} x_j$ denote the *unconstrained optimum*. When agents' choices are constrained, agent i's best reply is simply the value which is closest to $\widehat{x}_i(\mathbf{x}_{-i})$ within the restricted space.

5.2.4 Game Class

The chapter studies all games whose best replies have the above form. Since the best replies are equivalent, to analyze the equilibria for all games, we can consider one payoff function that satisfies the conditions of (5.4). We make extensive use of payoffs with the following quadratic form:

$$\Pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = -\frac{1}{2}\left(x_i - x_i^0 - \delta \sum_j g_{ij} x_j\right)^2 + w_i(\mathbf{x}_{-i}). \tag{5.9}$$

5.3 Examples of Games in the Literature

Many games in the economics and the network literature fall in this class, with different specifications of the action spaces, X_i, the link values g_{ij}, and the payoff impact parameter δ. All games in the literature involve some restriction on the strategy space.

5.3.1 Constrained Continuous Actions: Quadratic Payoffs and Benefit/Cost Payoffs

Quadratic payoffs are common and have been used to represent a variety of settings including peer effects, consumption externalities, and oligopoly. Players choose some action $x_i \in X_i = [0, \infty)$ and payoffs have a form such as

$$\pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = x_i^0 x_i - \frac{1}{2}x_i^2 - \delta \sum_j g_{ij} x_i x_j, \tag{5.10}$$

which is a special case of (5.9).[8] With positive links, $g_{ij} = g_{ji} \geq 0$, and negative payoff parameter, $\delta \leq 0$, these payoffs give a pure complements game, as in peer effects. For pure substitutes, $g_{ij} = g_{ji} \geq 0$, and $\delta \geq 0$, as in a Cournot game with n firms producing substitute products. The links $g_{ij} = g_{ji} = 1$ indicate firm i and firm j compete directly, and a function of δ is the overall extent of substitutability among goods. Quadratic payoffs have also been used to model settings with both substitutes and complements, as in crime games.[9]

[8] For a prominent example see Ballester, Calvó-Armengol, and Zenou (2006).

[9] The benefit from x_i is higher when the overall crime level is lower, capturing the possibility that criminals may compete for victims or territory. The cost of x_i is lower when i's friends engage in more crime, capturing the possibility of positive peer effects. See Bramoullé, Kranton and D'Amours (2014), which refines Calvó-Armengol and Zenou (2004) and Ballester, Calvó-Armengol, and Zenou (2010).

Another type of payoffs specifies the trade-off between the benefits from own and others' actions and individual costs, as in the private provision of local public goods.[10] Each agent chooses a level $x_i \in X_i = [0, \infty)$ and earns

$$\pi_i(x_i, \mathbf{x}_{-i}; \delta, \mathbf{G}) = b_i(x_i + \delta \sum_j g_{ij} x_j) - \kappa_i x_i,$$

where $b_i(\cdot)$ is strictly increasing and strictly concave, $\kappa_i > 0$ is i's marginal cost, and $b_i'(0) > \kappa_i > b_i'(+\infty)$ for all i. The links are positive, $g_{ij} = g_{ji} \geq 0$, and $\delta \geq 0$ so that agents' payoffs increase when their neighbors provide more public goods. Best replies correspond to (5.6) for x_i^0 such that $b_i'(x_i^0) = \kappa_i$. The substitutability of own and others' goods is scaled by δ.

5.3.2 Binary Actions: Coordination, Anti-Coordination, and Best-Shot Games

In many games in the literature, players have the choice between two actions such as "buy" vs. "not buy," "vote yes" vs. "vote no," and so on. For judicious choices of a, b, δ, and x_i^0, binary games played on a network have the form of best reply (5.8). Consider the payoffs of a symmetric 2×2 game with actions A and B:

	A	B
A	π_{AA}, π_{AA}	π_{AB}, π_{BA}
B	π_{BA}, π_{AB}	π_{BB}, π_{BB}

This matrix gives payoffs for a classic coordination game between two players when $\pi_{AA} > \pi_{BA}$ and $\pi_{BB} > \pi_{AB}$. On a network with all $g_{ij} \geq 0$, a player earns the sum of bilateral payoffs: $\pi_i(x_i, \mathbf{x}_{-i}) = \sum_j g_{ij} \pi(x_i, x_j)$.[11]

Agent i's best reply has a simple form. Define $p_B = (\pi_{AA} - \pi_{BA})/(\pi_{AA} + \pi_{BB} - \pi_{AB} - \pi_{BA})$, where $0 < p_B < 1$. Let $k_i = \sum_j g_{ij}$ be the weighted links of agent i's neighbors, and let $k_{iB} = \sum_j g_{ij}(B)$ be the weighted links of i's neighbors who play B. Then i strictly prefers to play B if and only if the weighted majority of his neighbors play B: $k_{iB} > p_B k_i$. To establish the correspondence with (5.8), assign numbers a and b to actions A and B, let $k_{iA} = \sum_j g_{ij}(A)$ be the weighted links of i's neighbors who play A, and note that $\sum_j g_{ij} x_j = a k_{iA} + b k_{iB} = (b-a) k_{iB} + a k_i$. The threshold $t_i = x_i^0 - \frac{1}{2}(a+b)$ is then constructed by setting $x_i^0 = \frac{1}{2}(a+b) + |\delta| k_i (a + (b-a) p_B)$ for any $\delta < 0$.

More generally, these games include any threshold game of complements as defined in Jackson (2008, p. 270). The "majority game" is thus easily recast as a coordination

[10] Bramoullé and Kranton (2007) study the private provision of local public goods. Further studies include Galeotti and Goyal (2010) and Allouch (2014).

[11] See Blume (1993, 1995), Ellison (1993), Morris (2000), Young (1998), Jackson and Watts (2002), Goyal and Vega-Redondo (2005).

game on a network. Agents earn payoffs when they choose the same action as the majority of their neighbors. Setting $\pi_{AA} = \pi_{BB} = 1$ and $\pi_{AB} = \pi_{BA} = 0$, gives $p_B = 1/2$, which corresponds to (5.8). For any $\delta < 0$ and for $a = -1$, $b = 1$, we have $x_i^0 = 0$.

For binary choice games with substitutes, such as anti-coordination games, agents want to differentiate from, rather than conform to, their neighbors. The Hawk-Dove game is one example, where in the payoff matrix above $\pi_{AA} < \pi_{BA}$ and $\pi_{BB} < \pi_{AB}$.[12] Agent i strictly prefers to play B if and only if a payoff-weighted *minority* of his neighbors plays B: $k_{iB} < p_B k_i$. A network anti-coordination game then has best responses of the form (5.8) by setting $x_i^0 = \frac{1}{2}(a+b) + \delta k_i(a+(b-a)p_b)$ for any $\delta > 0$.

In the "best-shot" game,[13] actions are also strategic substitutes and agents' actions can represent a discrete local public good. Each agent i chooses either 0 or 1, with $c \in (0,1)$ as the individual cost of taking action 1. Agents earn a benefit of 1 if any neighbor has played 1. The best reply is then $f_i = 1$ if $\sum_j g_{ij}x_j = 0$ and $f_i = 0$ if $\sum_j g_{ij}x_j > 0$. This game gives another particular case of (5.8) with $a = 0$, $b = 1$, $\delta = 1$ and $x_i^0 \in (\frac{1}{2}, \frac{3}{2})$.

5.4 Unconstrained Actions

When agents' actions are unconstrained, the games can be analyzed with relatively straightforward linear algebra. Even so, equilibrium behavior on networks gives rise to rich and complex patterns. This complexity is amplified by the introduction of constraints, studied in the next section.

5.4.1 Nash and Stable Equilibria

For unconstrained actions, a Nash equilibrium is simply a solution to the system of linear equations defined by the best replies (5.5). For $X_i = \mathbb{R}$, the system of best replies is, in matrix notation,

$$\mathbf{x}(\mathbf{I}+\delta\mathbf{G}) = \mathbf{x}^0.$$

Generically, there exists a unique Nash equilibrium. A unique equilibrium exists if $\det(\mathbf{I}+\delta\mathbf{G}) \neq 0$,[14] and then the equilibrium actions are determined by

$$\mathbf{x} = (\mathbf{I}+\delta\mathbf{G})^{-1}\mathbf{x}^0. \tag{5.11}$$

For convenience, we will label this unique unconstrained equilibrium vector $\mathbf{x}^*$. This argument also clearly holds for any directed network.

[12] For anti-coordination games played on networks see Bramoullé (2007), Bramoullé et al. (2004).

[13] See Hirshleifer (1983) and Pin and Boncinelli (2012).

[14] Note that $\det(\mathbf{I}+\delta\mathbf{G}) \neq 0$ for almost every δ. The invertibility of $(\mathbf{I}+\delta\mathbf{G})$ is a sufficient but not necessary condition for existence of a Nash equilibrium. Continua of equilibrium can exist when $\mathbf{I}+\delta\mathbf{G}$ is not invertible.

This equilibrium is asymptotically stable according to the standard conditions for the stability of a system of linear differential equations, which is here $|\lambda_{\min}(\delta\mathbf{G})| < 1$, which can also be written $|\lambda_{\max}(-\delta\mathbf{G})| < 1$. The stability condition imposes a joint restriction on the payoff impact, δ, and the network structure, which jointly give what we call the "network effects" of players' actions. The equilibrium is stable only when these network effects are small enough. When network effects are strong, the equilibrium is unstable. Bounds on actions, which also often represent real-world situations, are necessary for the existence of stable Nash equilibria.

5.4.2 Network Position

How do individual network positions affect individual actions? Ballester, Calvó-Armengol, and Zenou (2006) first establish the connection between equilibrium action and Bonacich centrality (Bonacich 1987).[15] In their model, individuals are homogenous but for their network position ($x_i^0 = x^0$ for all i). For a network $\mathbf{M}$ and a scalar q such that $(\mathbf{I} - q\mathbf{M})$ is invertible, $\mathbf{z}(q, \mathbf{M}) = (\mathbf{I} - q\mathbf{M})^{-1}\mathbf{M}\mathbf{1}$ is the vector of Bonacich centralities. It is easy to see that equilibrium actions $\mathbf{x}^*$ can be directly written in terms of centralities: $\mathbf{x}^* = x^0(\mathbf{I} + \delta\mathbf{G})^{-1}\mathbf{1} = x^0(\mathbf{1} - \delta\mathbf{z}(-\delta, \mathbf{G}))$.

These centralities have an interpretation in terms of paths in the network if $|q|\,\lambda_{\max}(\mathbf{M}) < 1$, so that $\det(\mathbf{I} - q\mathbf{M}) \neq 0$ and $\mathbf{z}(q, \mathbf{M}) = \sum_{k=0}^{+\infty} q^k \mathbf{M}^{k+1}\mathbf{1}$. For agent i, $z_i(q, \mathbf{M})$ is then equal to a weighted sum of the number of paths starting from i, where paths of lengths k are weighted by q^{k-1}. Agents' actions are increasing in centrality under pure complements and decreasing under pure substitutes.

Figure 5.1 below illustrates the unique equilibria—contrasting pure complements and pure substitutes—on a line with five agents when $\mathbf{x}^0 = \mathbf{1}$. For $\delta = -0.3$, agents' actions are strategic complements. The agent with the highest Bonacich centrality is in the middle of the line, as the scalar q is positive. This agent, then, has the highest level of play and agents' centralities and actions decrease moving away from the middle of the line. The outcome is quite different for $\delta = 0.3$, when agents' actions are strategic substitutes. The scalar q is now negative, giving positive weight to an agent's neighbors, but negative weight to the neighbors of neighbors. The agent in the middle of the line is not the most central agent. The intermediate agents are more central, as Bonacich centrality weights go up and down along network paths. Since the agents on the ends of the line have no other neighbors—and hence no further substitutes for their actions—their actions are highest, which leads to a lower level for the agents in the intermediate positions, which leads to a higher level for the agent in the middle.

The simple relationship between equilibrium actions and Bonacich centrality fails to hold when individuals are heterogeneous. Yet, Bonacich centrality still affords an intuitive comparative static. Let $\mathbf{x}^*(\mathbf{x}^0)$ be the unique equilibrium for a given

[15] The chapter by Yves Zenou in this volume provides further discussion of this connection, and how it relates to key-player policies.

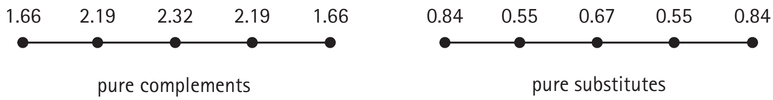

FIGURE 5.1 Bonacich centrality and equilibrium actions: complements vs. substitutes.

$\mathbf{x}^0 = (x_1^0, \ldots, x_1^0)$. Suppose each agent i's autarkic action, x_i^0, changes by the same amount s. An increase could represent, for instance, a policy intervention that lowers all agents' individual costs. Agents' Bonacich centralities give precisely the change in their equilibrium actions. It is readily evident that

$$\mathbf{x}^*(\mathbf{x}^0 + s\mathbf{1}) - \mathbf{x}^*(\mathbf{x}^0) = s(\mathbf{1} - \delta \mathbf{z}(-\delta, \mathbf{G})).$$

5.4.3 Interdependence

Equilibrium actions depend on how the network connects different agents. We say two agents i and j are *interdependent* if a (small, exogenous) change in agent j's autarkic action would lead to an adjustment in i's action. In $\mathbf{x}^*$, consider $\partial x_i^* / \partial x_j^0$. Intuitively, x_j^0 first affects x_j^* which then affects the action of j's neighbors and then the actions of their neighbors, and so on. Through the network, this change potentially impacts all agents. We have

$$\frac{\partial x_i^*}{\partial x_j^0} = \left[(\mathbf{I} + \delta \mathbf{G})^{-1}\right]_{ij} = \sum_{k=0}^{+\infty} (-\delta)^k \left[\mathbf{G}^k\right]_{ij},$$

where the second equality holds if $|\delta| \lambda_{\max}(\mathbf{G}) < 1$. The marginal impact of x_j^0 on x_i^* is equal to a weighted sum of the number of paths from i to j. When $\mathbf{G}$ is connected (i.e., there is a path from any agent i to any other agent j), $\partial x_i^* / \partial x_j^0 \neq 0$ for all i and j and almost every δ.[16]

The direction and magnitude of the interdependence depend on whether actions are complements or substitutes. Under pure complements ($\delta < 0$), all the terms in the infinite sum are non-negative. An increase in x_j^0 leads to an increase in x_j^*, which leads to an increase in the actions of j's neighbors, which leads to an increase in the actions of their neighbors, and so on.

Figure 5.2 illustrates the interdependence of agents in a network that represents connected communities. For $\delta = -\frac{1}{4}$, and $\mathbf{x}^0 = \mathbf{1}$, in the equilibrium $\mathbf{x}^*$ the agents with no connection to the other community play 5.33 and the two agents connecting the communities play 6.67. The figure shows the value of the partial derivatives for the play of each agent following a small positive impact to x^0 for the agent to the extreme left of the graph. This impact ultimately affects the play of all agents in the network.

[16] In the presence of substitutes, the positive and negative effects could cancel each other completely for specific values of δ.

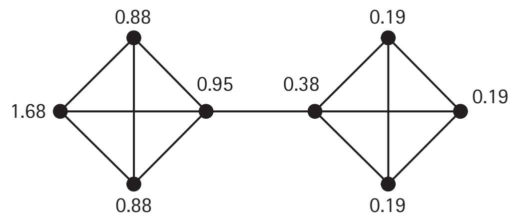

FIGURE 5.2 Measures of interdependence under complementarities.

The magnitude of the interdependence follows directly from the formula for $\partial x_i^*/\partial x_j^0$ above. With pure complements, agents are more (less) interdependent when they are connected by more and shorter (less and longer) paths.

The situation is more complex under pure substitutes ($\delta > 0$). In that case, the even terms in the infinite sum are non-negative, while the odd terms are non-positive. An increase in x_j^0 leads to an increase in x_j, which leads to a decrease in the actions of j's neighbors, which leads to an increase in the actions of their neighbors, and so on with alternating signs. Generically, the aggregate could be positive or negative.

While there are few results that relate the aggregate impacts under substitutes to the structure of the network, bipartite graphs serve as a benchmark. Bipartite graphs are the only networks where direct and indirect effects are all aligned. A graph is bipartite, by definition, when the agents can be partitioned in two sets U and V such that $g_{ij} = 0$ if $i, j \in U$ or $i, j \in V$. We can show that a graph is bipartite if and only if the length of all paths connecting two agents is exclusively even or odd. The length is even when the two agents belong to the same set (U or V) and odd when they belong to different sets. In a connected bipartite graph, $\partial x_i/\partial x_j^0 > 0$ if i and j belong to the same set, and $\partial x_i/\partial x_j^0 < 0$ if i and j belong to different sets.[17] It would be interesting to try to extend these results to more elaborate structures and interactions.

5.5 Tools for Constrained Actions: Potential Function

When players' actions are constrained, as in all economic games, the analysis of the equilibrium set is more complex. When $x_i^0 > 0$ for all players and $|\delta|$ is sufficiently small, the constrained and unconstrained equilibrium coincide.[18] All players choose actions in the interior of the action space, and the analysis of Section 5.4 applies. However, as

[17] See Appendix of Bramoullé, Kranton, and D'amours (2011).

[18] This is the case studied by Ballester, Calvó-Armengol, and Zenou (2006). In our notation, their sufficient condition for the existence of a unique interior equilibrium is $\delta < 1/(\overline{g} + \lambda_{\max}(\overline{g}\mathbf{C} - \mathbf{G}))$, where $\overline{g}$ is the value of the strongest substitute link in $\mathbf{G}$ (i.e. $\overline{g} = \max_{ij}(0, g_{ij})$) and $\mathbf{C}$ is the complete graph.

$|\delta|$ becomes larger, network effects become important, and some players will be driven to actions at the boundaries. The constraints then affect the analysis quite deeply.

To construct and analyze equilibria, we develop the following terminology. Agents who choose actions which are strictly positive and strictly lower than the upper bound are called *unconstrained*. Agents who play 0 or play L are called *constrained*. A constrained agent i is *strictly constrained* if i would remain constrained even with a small change in neighbors' actions. That is, i is strictly constrained if $x_i^0 - \delta \sum_j g_{ij} x_j > L$ when $x_i = L$ and $x_i^0 - \delta \sum_j g_{ij} x_j < 0$ if $x_i = 0$. For a network $\mathbf{G}$ and a subset of agents S, let $\mathbf{G}_S$ denote the subgraph that contains only links between the agents in S and $\mathbf{x}_S$ denote the actions of agents in S.

5.5.1 Potential Function

To analyze these games and solve for the Nash equilibria, we use the theory of potential games developed by Monderer and Shapley (1996).[19] A function $\varphi(x_i, \mathbf{x}_{-i})$ is a *potential function* for a game with payoffs $V_i(x_i, \mathbf{x}_{-i})$ if and only if for all x_i and x_i' and all $\mathbf{x}_{-i}$

$$\varphi(x_i, \mathbf{x}_{-i}) - \varphi(x_i', \mathbf{x}_{-i}) = V_i(x_i, \mathbf{x}_{-i}) - V_i(x_i', \mathbf{x}_{-i}) \qquad \text{for all } i.$$

A potential function mirrors each agent's payoff function. Changing actions from x_i to x_i' increases the potential by exactly the same amount as it increases agent i's payoffs. Not all payoff functions $V_i(x_i, \mathbf{x}_{-i})$ allow for a potential function. Monderer and Shapley (1996) show for (continuous, twice-differentiable) payoffs V_i, there exists a potential function if and only if $\frac{\partial^2 V_i(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial^2 V_j(\mathbf{x})}{\partial x_j \partial x_i}$ for all $i \neq j$. A key property is that the potential is preserved when restricting the domain. That is, for any possible choices x_i and x_i' in an agent's strategy space, the potential increases by exactly the same amount as agent i's payoffs.

5.5.2 Best Reply Equivalence and Potential for Quadratic Payoffs

Since all games in the class have the same best replies, we can analyze the equilibria for all games in the class by studying the equilibria for one game in the class. Any game with continuous actions and quadratic payoffs (5.9) has a potential function when $g_{ij} = g_{ji}$ since $\frac{\partial^2 \Pi_i}{\partial x_i \partial x_j} = -\delta g_{ij} = \frac{\partial^2 \Pi_j}{\partial x_j \partial x_i}$. We can then analyze the Nash equilibria for all the games

[19] Blume (1993) and Young (1998) introduced potential techniques to the study of discrete network games. Blume (1993) focuses on lattices, while Young (1998) looks at 2×2 coordination games played on networks. Bramoullé, Kranton, and D'amours (2014) first apply potential techniques to the study of network games with continuous actions.

in the class using the potential function for quadratic payoffs (5.10):

$$\varphi(x_i, \mathbf{x}_{-i}) = \sum_i (x_i^0 x_i - \frac{1}{2}x_i^2) - \frac{1}{2}\delta \sum_i \sum_j g_{ij} x_i x_j$$

or, in matrix form,

$$\varphi(\mathbf{x}) = (\mathbf{x}^0)^T \mathbf{x} - \frac{1}{2}\mathbf{x}^T(\mathbf{I} + \delta \mathbf{G})\mathbf{x}.$$

Since the potential property holds for constrained actions, we can analyze all games with best replies (5.6), (5.7), and (5.8) with this potential function.

5.6 Equilibria in Constrained Continuous Action Games

In this section we analyze the equilibria in games where agents' actions are continuous, but constrained. We begin with preliminary results relating the maxima of the potential function to the set of Nash and asymptotically stable equilibria.

5.6.1 The Potential Function, Nash Equilibria, and Asymptotically Stable Equilibria

For games with continuous actions, an action vector $\mathbf{x}$ is a Nash equilibrium if and only if it solves the first-order conditions for maximizing the potential on X (Bramoullé, Kranton, and D'Amours 2014). To see this, consider maximizing the potential function:

$$\max_{\mathbf{x}} \varphi(\mathbf{x}; \delta, \mathbf{G}) \text{ s.t. } x_i \in X_i \text{ for all } i, \tag{P}$$

where $X_i = \mathbb{R}$, $X_i = [0, \infty)$ or $X_i = [0, L]$. The first-order conditions for this problem mimic each agent i's individual best reply.[20] Each agent chooses his action in the game *as if* he wants to maximize the potential, given other agents' actions. Thus we have:

Lemma 1 *With continuous actions, a profile* $\mathbf{x}$ *is a Nash equilibrium if and only if* $\mathbf{x}$ *satisfies the first-order conditions of problem (P).*

The potential function can also be used to identify asymptotically stable equilibria. Mathematically, it plays the role of a Lyapunov function for the system of differential equations. Say that a local maximum is *strict* if it is the only maximum in some open neighborhood. Starting from an equilibrium which is a strict local maximum of the potential and modifying actions slightly, individual adjustments will lead back to the

[20] $\varphi(x_i, \mathbf{x}_{-i})$ is strictly concave in each x_i, so for any $\mathbf{x}_{-i}$ a single x_i satisfies the i^{th} Kuhn-Tucker condition.

equilibrium. Locally, there is no way to increase the potential, which reflects all agents' best replies. In contrast, if the equilibrium is not a strict local maximum, but, say, a saddle point, then modifying agents' actions slightly, there will be a direction in which the potential is increasing and individual reactions lead away from the equilibrium. Formally:

Lemma 2 *With continuous actions, a profile* $\mathbf{x}$ *is a stable Nash equilibrium if and only if* $\mathbf{x}$ *is a strict local maximum of* φ *over* X.

With these tools in hand, we will attack the analysis of the constrained continuous action games, where $X_i = [0, \infty)$ for all i or $X_i = [0, L]$ for all i.

5.6.2 Existence of Nash Equilibria

For games with a finite upper bound, existence of a Nash equilibrium is guaranteed. First, by standard results, for $X_i = [0, L]$ the strategy space is compact and convex. Since the best reply (5.7) is continuous, existence follows from Brouwer's fixed point theorem. This argument holds also for any directed network. Alternatively, since the potential function φ is continuous, it has a global maximum over X, and by Lemma 1 this maximum is a Nash equilibrium.

With no upper bound, a Nash equilibrium may fail to exist. When $X_i = [0, \infty)$, existence depends on whether actions are strategic substitutes or complements and on the extent of network effects. In a game of pure complements, if $|\delta|\lambda_{\max}(\mathbf{G}) < 1$, there exists a Nash equilibrium that is equivalent to the unconstrained equilibrium $\mathbf{x}^*$. For $|\delta|\lambda_{\max}(\mathbf{G}) > 1$, there is no Nash equilibrium with positive actions. Social interactions feed back into each other and diverge to infinity. With pure substitutes, on the other hand, existence is guaranteed. An agent will never choose an action that is greater than his autarkic optimum (i.e., $f_i(\mathbf{x}_{-i}) \leq x_i^0$). We can then assume without loss of generality that actions are bounded from above by $L = \max_i x_i^0$ and existence follows. All these arguments extend to directed networks. In general, with a mix of complements and substitutes, Bramoullé, Kranton, and D'amours (2014) show that if $|\lambda_{\min}(\delta\mathbf{G})| < 1$, a Nash equilibrium exists.[21]

The literature lacks existence results for larger payoff impacts. We conjecture that existence holds if the substitutes somehow dominate the complements in the strategic mix.

5.6.3 Unique versus Multiple Equilibria

Uniqueness is naturally related to the curvature of the potential function, as shown in Lemma 1. In particular, when φ is strictly concave, the first-order conditions of problem

[21] Note that $|\lambda_{\min}(\delta\mathbf{G})| = |\delta|\,\lambda_{\max}(\mathbf{G})$ under pure complements.

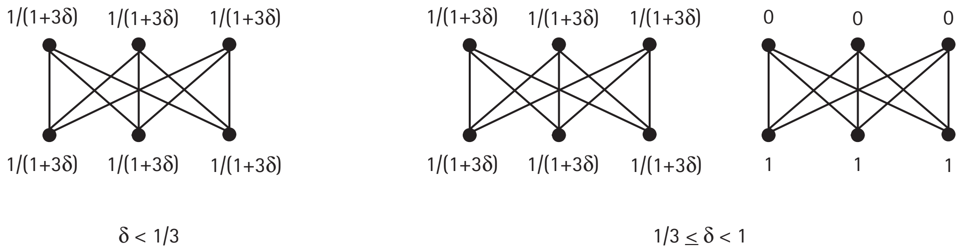

FIGURE 5.3 Unique vs. multiple equilibria in a regular graph.

(P) have at most one solution. Thus, for $X_i = [0, \infty)$, or $X_i = [0, L]$, there is a unique equilibrium when $|\lambda_{\min}(\delta \mathbf{G})| < 1$, since $\nabla^2 \varphi = -(\mathbf{I} + \delta \mathbf{G})$, and the potential is strictly concave when $\mathbf{I} + \delta \mathbf{G}$ is positive definite (Bramoullé, Kranton, and D'amours, 2014):

Proposition 1 *There is a unique Nash equilibrium if* $|\lambda_{\min}(\delta \mathbf{G})| < 1$.

Proposition 1 provides the best known condition valid for all continuous action games in the class. Researchers have derived stronger results for specific cases. Belhaj, Bramoullé, and Deroian (2014) show that the equilibrium is unique for any game with pure complements. For pure substitutes and homogeneous agents, Proposition 1's condition is necessary and sufficient for regular graphs (Bramoullé, Kranton, and D'amours 2014). When $\delta > 0$, Proposition 1's condition becomes $|\lambda_{\min}(\mathbf{G})| < 1/\delta$. The lowest eigenvalue—a negative number—gives a measure of overall substitutabilities in the network. When it is small in magnitude, the magnitude of the ups and downs in the network is smaller, and there is only one equilibrium.

Figure 5.3 illustrates in a complete bipartite graph for six agents. The lowest eigenvalue for this network is -3. Hence for $\delta < 1/3$, there is a unique Nash equilibrium where all agents play $1/(1+3\delta)$. For higher δ, there are three equilibria. One of these additional equilibria is illustrated, involving all agents on one side of the network playing action 0 and agents on the other side playing 1. The third equilibrium has the same pattern but with the play of the sides reversed.

The special case of local public goods with perfect substitutes and homogeneous agents ($\delta = 1$, $g_{ij} \in \{0, 1\}, x_i^0 = x^0$) yields precise structural results (Bramoullé and Kranton 2007). Every maximal independent set of the graph yields a Nash equilibrium. Agents inside the set choose x^0 and all agents outside the set choose 0.[22] In any connected graph, there are multiple equilibria and the number of equilibria can grow exponentially with the number of agents.

[22] An *independent set* of agents is a set such that no agent in the set is linked. A *maximal independent set* is an independent set that is not a subset of any other independent set. A maximal independent set has the property that all agents outside the set are linked to at least one agent in the set.

FIGURE 5.4 Maximal independent sets and equilibria in the star for $\delta = 1$.

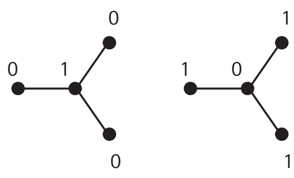

Figure 5.4 shows the equilibria in a star for $\delta = 1$. The agent in the center constitutes a maximal independent set, and the agents in the periphery constitute a maximal independent set.

Our general knowledge of how unique versus multiple equilibria depend on parameters and the network is still very fragmented. We conjecture that multiplicity tends to be higher when $|\delta|$ is greater, when there are more substitutes in the strategic mix, and when $|\lambda_{\min}(\mathbf{G})|$ is greater.

5.6.4 Stability and the Lowest Eigenvalue

In light of the large possible number of equilibria, stability is a natural refinement. From Lemma 2, we can show that a stable equilibrium exists for any $\mathbf{G}$ and almost any δ. Moreover, stability is then related to the local curvature of the potential.

Consider a Nash equilibrium $\mathbf{x}$ with unconstrained agents U and all other agents strictly constrained. Now perturb agents' actions slightly by adding a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ such that $\mathbf{x} + \varepsilon \in X$. When $|\lambda_{\min}(\delta \mathbf{G}_U)| < 1$ the potential function is strictly concave in $\mathbf{x}_U$, and the best replies converge back to $\mathbf{x}$. When $|\lambda_{\min}(\delta \mathbf{G}_U)| > 1$, the potential function is not concave in $\mathbf{x}_U$. Some small perturbation ε can lead to large changes in best replies, and the equilibrium is not stable.

Proposition 2 *A Nash equilibrium with unconstrained agents U and all other agents strictly constrained is stable if and only if* $|\lambda_{\min}(\delta \mathbf{G}_U)| < 1$.

The lowest eigenvalue is key to the set of Nash and stable equilibria. There are only a few results that relate this eigenvalue to a network's structure.[23] Intuitively, the lowest eigenvalue—a negative number—gives the extent of substitutabilities in the network. Overall, $|\lambda_{\min}(\mathbf{G})|$ tends to be larger when the network is more "two-sided," so that agents can be subdivided into two sets with few links within the sets but many links between them. Loosely speaking, an action then reverberates between the two sides. For n agents, the graph with the highest $|\lambda_{\min}(\mathbf{G})|$ is the complete bipartite network with as equal sides as possible (that is, agents are divided into as equal size sets as possible and all agents in each set are linked to all agents in the other set, with no links within the sets).

[23] See summary in Bramoullé, Kranton, and D'Amours (2014).

FIGURE 5.5 Stable equilibria and the lowest eigenvalue.

Figure 5.5 illustrates Proposition 2 in two graphs that highlight the importance of the lowest eigenvalue. Each network contains six agents, nine links, and three links per agent. The network on the left is a complete bipartite graph, with lowest eigenvalue of -3. The network on the right has lowest eigenvalue of -2. Consider in each network perturbing the play of agent 1 in the lower left corner. In the bipartite graph, this perturbation directly impacts the three agents on the other side of the network (agents 4, 5, and 6) who must adjust, leading to adjustments on the other side, and so on. In the prism graph in the right panel, the perturbation directly impacts three agents (2, 3, and 6), two of whom are linked to each other (2 and 3). The play of these two agents jointly adjusts to the perturbation, dampening its effect. In the bipartite graph, the interior Nash equilibrium is stable for $\delta \leq 1/3$; for the prism graph, the equilibrium is stable for greater impact parameters $\delta \leq 1/2$.

Overall, among the multiple Nash equilibria, stable equilibria tend to contain more constrained agents. If $|\lambda_{\min}(\delta \mathbf{G})| > 1$, any stable equilibrium involves at least one constrained agent. More generally, the number of constrained agents in stable equilibria tends to increase when $|\delta|$ and $|\lambda_{\min}(\mathbf{G})|$ increase. In addition, with pure substitutes stable equilibria involve the largest sets of constrained agents among all Nash equilibria.[24]

At this point, however, we know little about the selection power of stability. Exploratory simulations show that a large proportion of Nash equilibria is typically unstable. We do not yet know how this proportion depends on the structure of the netwok. The shape of stable equilibria and the selection power of stability deserves to be studied more systematically.

5.6.5 Individual Network Position and Equilibrium Action

The general knowledge of how individual network positions affect equilibrium actions is still spotty. While alignment with Bonacich centralities is preserved in some circumstances, this relationship does not hold generally.[25]

[24] See Proposition 6 in Bramoullé, Kranton, and D'amours (2014).
[25] See the example in Section IV.A. in Belhaj, Bramoullé, and Deroian (2014).

In the following discussion, we consider homogeneous agents ($x_i^0 = x^0$ for all i) and study how the possible nested structure of neighborhoods affects play. Whether an agent i plays more or less than agent j depends on whether actions are pure substitutes or complements. In particular, suppose that i's neighborhood is nested in j's, so that j's neighbors are a superset of i's neighbors: $\forall k \neq i,j$, $g_{ik} \leq g_{jk}$. Then under pure complements, $x_i \leq x_j$ in the unique equilibrium.[26] An agent with more neighbors plays a higher action. In contrast, under pure substitutes and if $\forall i,j, \delta g_{ij} < 1$, then $x_i \geq x_j$ in any Nash equilibrium. An agent with more neighbors now plays a lower action. The usefulness of these results, of course, depends on the structure of the network and the extent to which agents are nested.

Recent theoretical work has drawn attention to a specific class of networks—nested split graphs—where any two agents' neighborhoods are always nested.[27] On nested split graphs, action is then weakly increasing in an agent's degree (i.e., number of neighbors), or centrality, under pure complements and weakly decreasing in degree under pure substitutes. Future research could possibly extend these results to more complex structures and interactions.

5.6.6 Interdependence

When actions are unconstrained, all agents are interdependent but constraints on actions can break this pattern. Strictly constrained agents do not change their actions in response to small changes in neighbors' actions and hence break the chain reaction from a possibly distant exogenous shock. Depending on the agents' positions in the network, this "dam" effect can break interdependence and leave parts of the network without effects on other parts.

Consider first pure complements ($\delta < 0$). In $\mathbf{x}^*$ all direct and indirect effects are aligned, and $\partial x_i/\partial x_j^0 > 0$ for any i and j who are path-connected. With constraints on actions, the direction of alignment is unchanged,[28] but x_i may now be unaffected by x_j^0, and x_i is only weakly increasing in x_j^0 for any $|\delta|$. Since agents who reach the upper bound do not transmit positive shocks, the right derivative $(\partial x_i/\partial x_j^0)^+ > 0$ if and only if i and j are connected by a path of unconstrained agents.[29] In that case,

$$(\partial x_i/\partial x_j^0)^+ = \left[(\mathbf{I} + \delta \mathbf{G}_U)^{-1}\right]_{ij} = \sum_{k=0}^{+\infty} (-\delta)^k [\mathbf{G}_U^k]_{ij}, \tag{5.12}$$

[26] See Proposition 4 of Belhaj, Bramoullé, and Deroian (2014), which shows that this property holds in any Nash equilibrium for a broad class of network games with non-linear best-replies.

[27] Konig, Tessone, and Zenou (2104) and Baetz (2014) show that nested split graphs emerge as outcomes of natural network formation processes. Belhaj, Bervoets, and Deroian (2014) show that they solve network design problems. We refer to these papers for precise definitions and further discussions of these networks' properties.

[28] See Corollary 4 of Belhaj, Bramoullé, and Deroian (2014).

[29] See Proposition 5 in Belhaj, Bramoullé, and Deroian (2014). The left and right derivatives of x_i with respect to x_j^0 may differ. The argument extends to left-derivatives.

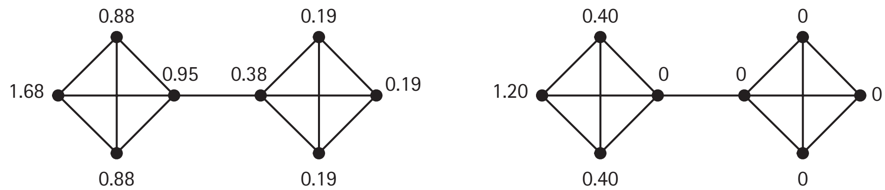

FIGURE 5.6 Measures of interdependence: Complements with unconstrained vs. constrained agents.

and the magnitude of the impact is equal to a weighted sum of the number of these interior paths. As $\mathbf{x}^0$ increases, more agents reach the upper bound and interdependence decreases on both margins. Fewer agents affect each other, and all the positive pairwise impacts have lower magnitude.

Figure 5.6 illustrates consequences of constraints to the interdependence of agents—in the communities graph under complementarities. The left panel contains the communities graph shown previously, with the partial derivatives of the impact of play from an impact to the agent on the far left in the unconstrained equilibrium $\mathbf{x}^*$. The right panel shows the magnitudes of the partial derivatives for interdependence in the equilibrium when agents are constrained to play in the range $[0, 6]$. The two agents connecting the communities are strictly constrained, playing 6, and all other agents play 5. A small change in x^0 of the agent to the far left has no impact on the agent in his community with links to the other community. This agent then blocks the impact from reaching further in the network, and the agents in the two communities are not interdependent.

Substitutes involve a number of complexitites, but the basic idea holds. Consider pure substitutes ($\delta > 0$) and a stable equilibrium $\mathbf{x}$ with unconstrained agents U and strictly constrained agents.[30] Now, a necessary condition for $(\partial x_i/\partial x_j^0)^+ \neq 0$ is that i and j are connected by a path of unconstrained agents, and this condition is generically sufficient. Equation (5.12) holds.

5.6.7 Network Comparative Statics—Adding Links

When studying network comparative statics, a natural first step is to look at the addition or strengthening of links. That is, consider the equilibria for a network $\mathbf{G}$ and compare them to the equilibria for $\mathbf{G}'$ where $\forall i, j$, $g_{ij} \leq g'_{ij}$. An increase in g_{ij} is in many ways similar to a simultaneous shock to x_i^0 and x_j^0. Hence we can use insights from the study of interdependence to determine the effect of the new or stronger link.

As with interdependence, the effect depends on substitutes vs. complements. Under pure complements, the action of every agent in $\mathbf{G}'$ is greater than or equal to the action

[30] These conditions ensure the existence of stable equilibrium with same set of unconstrained agents following a small enough increase in $\mathbf{x}^0$.

in **G**.[31] The action of an agent k is affected by g_{ij} if and only if there is a path of unconstrained agents connecting k with i or j. Under pure substitutes, the comparative statics are more complex. We do not know how to sign the effects at the individual level and equilibrium multiplicity further aggravates the issue.

The potential function, however, gives some traction on the problem, at least for aggregate outcomes. When $\delta > 0$, the potential is higher in **G** for any vector of actions: $\varphi(\mathbf{x}, \mathbf{G}') \leq \varphi(\mathbf{x}, \mathbf{G})\ \forall \mathbf{x}$. In addition, $\varphi(\mathbf{x}, \mathbf{G}) = \frac{1}{2}(\mathbf{x}^0)^T\mathbf{x}$ for any equilibrium $\mathbf{x}$. Thus the largest $\sum_i x_i^0 x_i$ in equilibrium decreases weakly following an expansion of the network.[32] In this sense, the direct and indirect negative impacts of the new link dominate the indirect positive effects.

5.7 Binary Action / Threshold Games

This section studies binary action games and shows how the common framework advanced in this chapter can make progress on questions of existence, uniqueness, and stability of equilibria. A full-fledged analysis is a ripe topic for future research. For ease of exposition in this section, we set the two actions to $a = -1$ and $b = 1$, so $X = \{-1, 1\}^n$.

5.7.1 Existence

While existence of a pure strategy equilibrium is not guaranteed a priori, we find existence follows naturally from the potential formulation in Section 5.5.1. Since the potential property is preserved on a constrained domain, the maxima of the potential function within the constrained space are Nash equilibria. Thus we can state the following new result:

Proposition 3 *In any network where $g_{ij} = g_{ji}$ and agents play a game with best replies (5.8), there exists a Nash equilibrium in pure strategies.*

The existence of a pure strategy Nash equilibrium does not extend to directed networks, except in the special case of pure complements. When agents always desire to take the same actions as their neighbors, existence of a pure strategy equilibrium is implied by standard results of the theory of supermodular games. However, for directed networks and a strategic mix there is no guarantee, as in Jackson's (2008, p. 271) "fashion" game where some agents desire to differentiate from neighbors, and there is no pure strategy equilibrium.

[31] See Corollary 4 of Belhaj, Bramoullé, and Deroian (2014).

[32] See Bramoullé, Kranton, and D'amours (2014). Theorem 2 in Ballester, Calvó-Armengol, and Zenou (2006) is a special case.

5.7.2 Unique versus Multiple Equilibria

For binary action games, strict concavity of the potential function does not guarantee a unique equilibrium. With the constraints on the action space, there can be more than one vector that maximizes a strictly concave potential subject to the constraints.

We can construct, however, a sufficient condition for a unique equilibrium that depends on the network structure. When each agent has a dominant strategy, there is a unique equilibrium, and whether each agent has a dominant strategy, in turn, depends on the impact parameter δ and each agent's degree. For pure complements ($\delta < 0$), an agent with $x_i^0 > 0$ strictly prefers to play 1 no matter what his neighbors choose if and only if $|\delta|k_i < x_i^0$. Similarly, playing -1 is strictly dominant for an agent with $x_i^0 < 0$ if and only if $x_i^0 < -|\delta|k_i$. An agent with $x_i^0 = 0$ is a priori indifferent and hence does not have a strictly dominant strategy. Similar conditions hold for pure substitutes. We thus have:

Proposition 4 *Consider pure substitutes or pure complements. A binary action game has a unique Nash equilibrium in dominant strategies if and only if*

$$|\delta| < \min_i \left(\frac{|x_i^0|}{k_i} \right).$$

Uniqueness in binary games then generally depends on the heterogeneity of agents and the distribution of idiosyncratic preferences.

This uniqueness condition bears a similarity to the conditions for continuous action games in that there is a unique equilibrium in binary action games when payoff impacts are small enough. With continuous actions, a small enough $|\delta|$ guarantees dampened adjustments to others' play. With binary actions, a small enough $|\delta|$ guarantees that idiosyncratic preferences dominate social interactions altogether, and agents never adjust to their neighbors.

While small payoff impacts guarantee unique equilibria, large impacts generally lead to multiple equilibria. Consider $|\delta| \geq \max_i \left(\frac{|x_i^0|}{k_i} \right)$. Under pure complements, all agents choosing -1 and all agents choosing 1 are both Nash equilibria. Social interactions completely swamp any idiosyncratic preferences, and full coordination occurs if and only if $|\delta| \geq \max_i \left(\frac{|x_i^0|}{k_i} \right)$. Under pure substitutes, equilibria involve agents playing different actions. But full anticoordination is impossible as soon as the network has a triangle; two connected agents then must play the same action. It is not possible for three agents to play different actions when only two actions are available.

For particular binary games with large payoff impacts, Nash equilibria have an intuitive graph-theoretic characterization. An action profile is a Nash equilibrium of the best-shot game if and only if the set of contributors is a maximal independent set of the graph (see Section 5.6). Each contributor is connected to agents who free-ride on his contribution.[33] This result implies that in the best shot game any connected network has multiple equilibria and, moreover, the number of equilibria may increase exponentially with n (Bramoullé and Kranton, 2007).

5.7.3 Stability

To refine the set of Nash equilibria, we can invoke a notion of stability. As noted above, asymptotic stability—defined by the system of differential equations (5.3)—does not apply to discrete action spaces. For binary choice games, we use asynchronous best-reply dynamics to study stability. For asynchronous best-reply dynamics subject to log-linear trembles, profiles that globally maximize the potential are the stochastically stable outcomes for all potential games (Blume 1993; Young 1998).[34] This stability notion depends on the specific payoffs of each game.[35]

In what follows, we provide a first discussion of stability in a binary game where agents have quadratic payoffs (5.10). Let $g_{ij} \in \{0, 1\}$. Simple computations show that the potential function is then

$$\varphi(\mathbf{x}) = \sum_{x_i=b} x_i^0 - \sum_{x_i=a} x_i^0 - \delta(n_{aa} + n_{bb} - n_{ab}) - \frac{1}{2}n$$

where n_{aa} is number links between a-players and similarly for n_{ab} and n_{bb}.

Consider pure complements ($\delta < 0$). Since $n_{aa} + n_{bb} + n_{ab} = |\mathbf{G}|$ (i.e., the number of links in the graph), the potential is

$$\varphi(\mathbf{x}) = \sum_{x_i=b} x_i^0 - \sum_{x_i=a} x_i^0 + 2|\delta|(n_{aa} + n_{bb}) - |\delta||\mathbf{G}| - \frac{1}{2}n.$$

This potential combines two forces. On one hand, φ is greater when individuals play the action for which they have some intrinsic preference: b for $x_i^0 > 0$ and a for $x_i^0 < 0$. On the other hand, φ is greater when there are more links between agents playing the same action. These two forces can be aligned. When all individuals intrinsically prefer the same action, full coordination on this action is the unique stable equilibrium. In general, however, a stable equilibrium can involve coordination on different actions in different parts of the network.

[33] The relation between Nash equilibria and maximal independent sets also appears when agents play games of anti-coordination on the network and one action has a much higher relative payoff than the other, see Bramoullé (2007).

[34] In the literature, researchers have analyzed stochastic stability in coordination games (Blume 1993; Young 1998, Jackson and Watts 2002) anti-coordination games (Bramoullé 2007) and the best-shot game (Boncinelli and Pin 2012).

[35] Hence, while all games with best replies (5.8) have the same Nash equilibria, the stochastically stable equilibrium sets could diverge.

Next consider strategic substitutes ($\delta > 0$). The potential is then

$$\varphi(\mathbf{x}) = \sum_{x_i=b} x_i^0 - \sum_{x_i=a} x_i^0 + 2\delta n_{ab} - \delta|\mathbf{G}| - \frac{1}{2}n.$$

Here φ is greater when there are more links between agents playing different actions. The two forces are aligned only in the special case when no link connects two agents who prefer the same action, in which case the network is bipartite. The profile where every agent plays his preferred action is then the unique stable equilibrium.

5.7.4 Interdependence

With binary actions, in equilibrium agents are not typically indifferent between the two actions, and small changes in individual parameters do not lead to a change in play. We then consider larger changes, and show changes in individual parameters only have impact on own play and play of others in critical configurations where what we call *switching cascades* can occur.

To illustrate, consider a pure complements game where $g_{ij} \in \{0, 1\}$ and $\delta = -1$. Consider one of the extremal equilibria—either an equilibrium where most agents play 1 or one where most agents play -1. Consider an initial change from a situation where $x_j^0 \ll 0$ and j plays -1 to the situation where $x_j^0 \gg 0$ and j plays 1. When does this change in j's preferences and action affect the play of an agent i? A clear necessary condition is a path of agents playing -1 connecting i to j in the initial equilibrium. As with bounded agents in Section 5.6, agents playing 1 cannot transmit positive shocks. If there is an agent playing 1 on all paths between i and j, x_i is unaffected by the change in x_j^0.

Unlike in Section 5.6, however, this condition is typically not sufficient. Agents playing -1 might not change their actions if some of their neighbors switch to 1. Changing actions depends on idiosyncratic preferences and on the number of switching neighbors. Overall, we observe that interdependence displays a non-monotonic pattern. When $\mathbf{x}_{-j}^0$ is low, agents have a strong preference for playing -1 and an increase on x_j^0 does not propagate. When $\mathbf{x}_{-j}^0$ is high, an increase in x_j^0 also has no impact because agents playing 1 block the transmission of shocks. The increase in x_j^0 eventually affects x_i only when $\mathbf{x}_{-j}^0$ takes some critical intermediate value. In future research, it would be interesting to understand more deeply how shocks propagate in binary action games.

5.8 Econometrics of Social Interactions

In this section, we connect the above theory to the empirical analysis of social interactions. The connections provide econometric models a precise game-theoretic microfoundation and set the stage for estimation of equilibria. Social scientists have long been trying to assess the importance of social interactions for outcomes as diverse as academic performance, welfare participation, smoking, obesity, and delinquent behavior.[36] In a typical regression, researchers try to estimate the impact of peers' outcomes $\sum_j g_{ij}x_j$ on individual outcome x_i. Because individual and peers' outcomes are determined at the same time, regressions define a set of simultaneous equations. This econometric system of equations is formally equivalent to the system of equations characterizing Nash equilibria with best replies (5.5), (5.6), (5.7), or (5.8).

Simultaneity raises two main econometric challenges: multiplicity and endogeneity. First, the econometric system may have multiple solutions.[37] Second, the variable $\sum_j g_{ij}x_j$ on the right hand side of the regressions is endogenous. To address these challenges, applied researchers must determine the *reduced form* of the system of simultaneous equations. That is, they must understand how outcomes $\mathbf{x}$ depend on parameters, observables, and unobservables. Formally, determining the reduced-form is equivalent to solving for the Nash equilibria of a network game.[38]

When the outcome is continuous and unbounded, a standard econometric model of peer effects is: $x_i = x_i^0 + \delta\sum_j g_{ij}x_j + \varepsilon_i$ where δ is a key parameter to be estimated, usually called the "endogenous peer effect," ε_i is an error term, and x_i^0 depends on individual and peers' covariates. We know from Section 5.4 that this system generically has a unique solution. The reduced-form is then given by $\mathbf{x} = (\mathbf{I} - \delta\mathbf{G})^{-1}(\mathbf{x}^0 + \varepsilon)$, and this equation provides the basis of many empirical analysis of social and spatial effects.[39]

However, most outcomes of interest (academic performance, etc.) are naturally bounded. These bounds are neglected in the previous approach and, in fact, in most studies of peer effects, which can yield biased estimates. A truncated version of the previous model,

$$x_i = \min(\max(0, x_i^0 + \delta\sum_j g_{ij}x_j + \varepsilon_i), L),$$

is a way to incorporate bounds on continuous actions. As for binary outcomes, a direct extension of classical discrete choice models is:

$$\hat{x}_i = x_i^0 + \delta\sum_j g_{ij}x_j + \varepsilon_i$$

with $x_i = 1$ if $\hat{x}_i \geq 0$ and $x_i = -1$ otherwise (Koreman and Soetevent 2007). These correspond to best reply (5.8).

[36] See the chapters by Vincent Boucher and Bernard Fortin, by Sinan Aral, by Emily Breza, and by Lori Beaman in this handbook.

[37] A related issue is that the system may not have any solution.

[38] In the literature, researchers have also considered games of incomplete information when agents do not know the outcomes and error terms of others (Brocke and Durlauf 2001; Lee, Li, and Lin 2014; Blume et al. 2014). Interestingly, the information structure has little impact on the analysis of continuous unbounded outcomes but deeply modifies the econometrics of binary actions.

[39] See, for example, Case (1991), Bramoullé, Djebbari, and Fortin (2009), Lee (2007), and Anselin's (2000) reviews.

The techniques and results presented in this chapter can then be combined with classical methods to estimate models with multiple equilibria. The researcher could, for instance, assume that all equilibria are equally likely (Koreman and Soetevent 2007); consider a flexible selection mechanism (Bajari, Hong and Ryan, 2010); build a likelihood from some evolutionary process (Nakajima 2007); or derive informative bounds from dominance relations (Tamer 2003).

5.9 Conclusion

This chapter presents a formal framework and technical tools to analyze a broad class of network games. These games all share the same underlying incentives—linear best replies with successive constraints on agents' choices. The chapter makes new connections between continuous action games and binary action games that share the same basic structure. We conclude here by reiterating future research directions and by connecting the analysis in this chapter to studies of games outside our class.

While much progress has been made on the network features that yield unique and stable equilibria, many interesting issues are still little understood. The following are some areas for future research: (i) existence of equilibria when network effects are large, (ii) the relationship between network structure and equilibrium multiplicity, (iii) comparative statics on the network in the presence of substitutes, (iv) interdependence in binary action games, and (v) the implications of our framework for the analysis of network games with discrete action spaces.

The research discussed below considers successively further departures from the assumptions that characterize the class of games covered by our framework.

Directed networks. In general, the analysis does not cover directed networks ($g_{ij} \neq g_{ji}$) since an exact potential function does not exist. The analysis does extend, however, in at least three cases. First, a game has a "weighted" potential when there are scalars α_i such that $\forall i,j, \alpha_i g_{ij} = \alpha_j g_{ji}$, and most results hold.[40] In particular, it is easy to extend the model to situations with undirected links and individual-specific δ_i. Second, when network effects are small, we can apply the theory of concave games developed by Rosen (1965) and obtain the following generalization of Proposition 1. For any network $\mathbf{G}$, the games with continuous actions have a unique Nash equilibrium if $|\lambda_{\min}(\delta(\mathbf{G}+\mathbf{G}^T)/2)| < 1$. Third, Belhaj and Deroian (2013) identify a balance condition under which the comparative statics on aggregate actions presented in Section 5.6 generalize to directed networks.

Non-linear best replies. In our framework, underlying best replies are linear and constraints add complexity. Researchers have started to analyze games with broader forms of non-linearities. Allouch (2014) studies the private provision of local public goods, combining the frameworks of Bergstrom, Blume and Varian (1986) and

[40] See Monderer and Shapley (1996) and Section VI.B in Bramoullé, Kranton and D'amours (2014).

Bramoullé and Kranton (2007). With pure substitutes ($\delta > 0$), Proposition 1 and the key role of the lowest eigenvalue extend to a wide class of non-linear best replies. The analysis does not consider, however, large network effects.

Researchers have also applied results from supermodular games to network games with pure complements. Belhaj and Deroian (2010) consider payoffs with indirect network effects and some specific symmetric networks. They show that action is aligned with Bonacich centrality in the highest and lowest equilibria, but not on intermediate equilibria. Belhaj, Bramoullé, and Deroian (2014) analyze network games with continuous, bounded actions, pure complements, and non-linear best replies. They derive a novel uniqueness condition and study interdependence.

Multidimensional strategies. In the games mentioned thus far, players choose a single number. In some contexts, players' actions are naturally multidimensional. A firm could choose both quality and quantity of a good, for example. Individuals can adopt different technologies to interact with different people. Bourlès and Bramoullé (2014) advance a model of altruism in networks, where players care about the utility of their neighbors and can transfer money to each other. An individual strategy specifies a profile of transfers. In recent research on conflict and networks, agents may also allocate different levels of resources to conflicts with different neighbors.[41] More generally, multidimensional strategies emerge when players can play different actions with different neighbors. Little research has been conducted to date on such games.

Incomplete information. The papers discussed so far analyze games of complete information, where payoffs and the network structure are common knowledge. In the games studied in this chapter, however, the best replies and convergence to a Nash equilibrium do not require that agents know the whole network or what all agents play. Agents respond simply to the play of their neighbors. In some contexts, however, these assumptions may be inappropriate. Agents may not have complete local information, or changes in the network may prevent convergence. Agents may then face residual uncertainty on others' connections when taking their actions. Actions would then depend on players' beliefs about their own and neighbors' positions and actions. Galeotti et al. (2010) study such network games of incomplete information.[42]

Network formation. In the games studied in this chapter, networks are fixed. In reality, networks evolve, possibly in a way that can be influenced by actions. Researchers have analyzed the joint determination of actions and links for many of the network games discussed in this chapter, as elaborated in Fernando Vega-Redondo's chapter in this volume. This work includes: coordination games (Jackson and Watts 2002; Goyal and Vega-Redondo 2005); anti-coordination games (Bramoullé et al. 2004); public goods in networks (Galeotti and Goyal 2010); and games with quadratic payoffs and continuous actions (Cabrales, Calvó-Armengol, and Zenou 2011). A broad conclusion of this literature is that endogenizing the network can lead to fewer possible outcomes, as equilibrium networks tend to have specific shapes. Thus far researchers generally

[41] See Franke and Ozturke (2009), Huremovic (2014), and Sanjeev Goyal's chapter in this volume.

[42] The literature on these games is reviewed in Jackson and Zenou (2014).

assume that the payoffs from the actions constitute the only incentive for network formation. In reality, people form friendships or other links for a variety of reasons, and there could be possible multiple costs and benefits from making and breaking links. Future research could engage these more challenging but potentially fruitful avenues.

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