Statistical Finance (STAT W4290)

Assignment 1

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Question 1

(a)

Let investment amount at t be P_t. Set $P_0 = 1000$.

Given that $\log(1 + R_{0,1}) \sim N(0.001, 0.015^2)$,

$$
\begin{aligned}
P(P_1 < 990) &= P\left(\frac{P_1}{P_0} < \frac{990}{P_0}\right) \\
&= P\left(1 + R_{0,1} < \frac{990}{1000}\right) \\
&= P\left(\log(1 + R_{0,1}) < \log\frac{990}{1000}\right) \\
&= 0.2306557
\end{aligned}
$$

```
> pnorm(log(990/1000), mean=0.001, sd=0.015)
[1] 0.2306557
```

(b)

$$
\begin{aligned}
P(P_5 < 990) &= P\left(\frac{P_5}{P_0} < \frac{990}{1000}\right) \\
&= P\left(1 + R_{0,5} < \frac{990}{1000}\right) \\
&= P\left((1+R_{0,1})(1+R_{1,2})(1+R_{2,3})(1+R_{3,4})(1+R_{4,5}) < \frac{990}{1000}\right) \\
&= P\left(\log\left((1+R_{0,1})(1+R_{1,2})(1+R_{2,3})(1+R_{3,4})(1+R_{4,5})\right) < \log\frac{990}{1000}\right) \\
&= P\left(\log\left(1+R_{0,1}\right) ... + \log\left(1+R_{4,5}\right) < \log\frac{990}{1000}\right)
\end{aligned}
$$

Now, given that daily log returns are independent,

$$\log\left(1+R_{0,1}\right) ... + \log\left(1+R_{4,5}\right) \sim N(0.005, 5 * 0.015^2)$$

Then,

$$P\left(\log\left(1+R_{0,1}\right) ... + \log\left(1+R_{4,5}\right) < \log\frac{990}{1000}\right) = 0.3268189$$

```
> pnorm(log(990/1000), mean=0.005, sd=sqrt(5) * 0.015)
[1] 0.3268189
```

Question 4

(a)

$$
\begin{aligned}
R_2 &= \frac{P_2 + D_2}{P_1} - 1 \\
&= \frac{54 + 0.2}{52} - 1 \\
&= 0.04230769
\end{aligned}
$$

(b)

$$
\begin{aligned}
R_4(3) &= \frac{P_4 + D_4}{P_3} * \frac{P_3 + D_3}{P_2} * \frac{P_2 + D_2}{P_1} - 1 \\
&= \frac{59 + 0.25}{53} * \frac{53 + 0.2}{54} * \frac{54 + 0.2}{52} - 1 && = 0.1479588
\end{aligned}
$$

(c)

$$
\begin{aligned}
r_3 &= \log 1 + R_3 \\
&= \log \frac{P_3 + D_3}{P_2} \\
&= \log \frac{53 + 0.2}{54} \\
&= -0.01492565
\end{aligned}
$$

Trivial `R` code omitted.

Question 6

(a)

$$
\begin{aligned}
X_k &= X_0 \exp\left(r_1 + ... + r_k\right) \\
\log X_k - \log X_0 &= r_1 + ... + r_k
\end{aligned}
$$

Then,

$$
\log X_k - \log X_0 \sim N(k\mu, k\sigma^2)
$$

$$
\begin{aligned}
P(X_2 > 1.3X_0) &= P\left(\frac{X_2}{X_0} > 1.3\right) \\
&= P\left(\log X_2 - \log X_0 > \log 1.3\right)
\end{aligned}
$$

Given that $\log X_2 - \log X_0 \sim N(2\mu, 2\sigma^2)$

$$P\left(\log X_2 - \log X_0 > \log 1.3\right) = 1 - P\left(\log X_2 - \log X_0 < \log 1.3\right)$$

(b)

Recall that suppose X is a random variable with PDF $f_X(x)$ and $Y = g(X)$ for g is a strictly increasing function. Since g is strictly increasing, it has an inverse, which we denote by h. Then Y is also a random variable and its CDF is:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le h(y) = F_X(h(y))$$

Differentiating, we find the PDF of Y

$$f_Y(y) = f_X(h(y))h'(y)$$

In this case, $r \sim N(\mu, \sigma^2)$

$$f_R(r) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right)$$

Since $Y = X_1 = X_0 \exp R = g(R)$,

$$h(Y) = R = \log \frac{X_1}{X_0} = \log X_1 - \log X_0$$

$$h'(Y) = \frac{\delta R}{\delta X_1} = \frac{1}{X_1}$$

Then,

$$f_Y(y) = f_{X_1}(x) = \frac{1}{x}\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\log x - \log X_0 - \mu)^2}{2\sigma^2}\right)$$

(c)

$$X_k = X_0 \exp r_1 + ... + r_k = X_0 \exp R$$

Then, $R \sim N(k\mu, k\sigma^2)$

Then,

$$f_{X_k}(x) = \frac{1}{x}\frac{1}{\sqrt{2\pi k}\sigma}\exp\left(-\frac{(\log x - \log X_0 - k\mu)^2}{2k\sigma^2}\right)$$

Thus transformation from R to X_k is monotonic. Then, we find the 0.9 quantile of $R \sim N(k\mu, k\sigma^2)$. Denote this value with $r_{0.9}$. Then, the 0.9 quantile of X_k is simply $X_0 \exp r_{0.9}$

(d)

$$X_k^2 = X_0^2 \exp\left(2(r_1 + ... + r_k)\right) = X_0^2 \prod_{i=1}^{k} \exp\left(2r_i\right)$$

Then, the expectation $E(X_k^2)$ can be found by (since each r_i is iid)

$$\begin{aligned}
E(X_k^2) &= X_0^2 \int_{-\infty}^{\infty} \left(\prod_{i=1}^{k} \exp\left(2r_i\right) p(r_i) dr_i\right) \\
&= X_0^2 \left(\int_{-\infty}^{\infty} \exp\left(2r\right) \frac{1}{\sqrt{2\pi}\sigma} \exp -\frac{(r-\mu)^2}{2\sigma^2} dr\right)^k \\
&= X_0^2 \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(2r - \frac{(r-\mu)^2}{2\sigma^2}\right) dr\right)^k \\
&= X_0^2 \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right) r - \frac{r^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) dr\right)^k \\
&= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right) r - \frac{r^2}{2\sigma^2}\right) dr\right)^k
\end{aligned}$$

Working on the inner exponent,

$$\begin{aligned}
\left(2 + \frac{2\mu}{2\sigma^2}\right) r - \frac{r^2}{2\sigma^2} &= -\frac{1}{2\sigma^2}\left(r^2 - 2\sigma^2\left(2 + \frac{\mu}{\sigma^2}\right) r\right) \\
&= -\frac{1}{2\sigma^2}\left(r^2 - 2(2\sigma^2 + \mu) r\right) \\
&= -\frac{1}{2\sigma^2}\left(r^2 - 2(2\sigma^2 + \mu) r + (2\sigma^2 + \mu)^2 - (2\sigma^2 + \mu)^2\right) \\
&= -\frac{1}{2\sigma^2}\left(\left(r - (2\sigma^2 + \mu)\right)^2 - (2\sigma^2 + \mu)^2\right)
\end{aligned}$$

Continuing where we left off earlier,

$$\begin{aligned}
E(X_k^2) &= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right) r - \frac{r^2}{2\sigma^2}\right) dr \right)^k \\
&= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp \frac{(2\sigma^2 + \mu)^2}{2\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(r - (2\sigma^2 + \mu))^2\right) dr \right)^k \\
&= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp \frac{(4\sigma^4 + 4\sigma^2\mu)^2}{2\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2} r^2\right) dr \right)^k
\end{aligned}$$

Recall that

$$\int_{-\infty}^{\infty} \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

Then,

$$\begin{aligned}
E(X_k^2) &= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \sqrt{2\pi}\sigma \exp\left(2\mu + 2\sigma^2\right) \right)^k \\
&= X_0^2 \exp\left(2k\mu + 2k\sigma^2\right)
\end{aligned}$$

(e)

$$Var(X_k) = E(X_k^2) - E(X_k)^2$$

(I'm running late for the assignment and had no time to typeset this). Using basically the same stuff as (d) except replace X_k^2 with X_k

$$E(X_k) = X_0 \exp\left(\frac{k\sigma^2}{2} + k\mu\right)$$

Then,

$$\begin{aligned}
Var(X_k) &= X_0^2 \exp\left(2k\mu + 2k\sigma^2\right) - X_0^2 \exp\left(\frac{2k\sigma^2}{2} + 2k\mu\right) \\
&= X_0^2 \left(\exp\left(2k\mu + 2k\sigma^2\right) - \exp\left(\frac{2k\sigma^2}{2} + 2k\mu\right) \right)
\end{aligned}$$

Question 7

Given that $\log{(1+R_t)} \sim N(0.0002, 0.03^2)$, $\log{(1+R_{0,20})} \sim N(20*0.0002, 20*0.03^2)$

$$
\begin{aligned}
P(P_{20} > 100) &= P\left(\frac{P_{20}}{P_0} > \frac{100}{P_0}\right) \\
&= P\left(1 + R_{0,20} > \frac{100}{97}\right) \\
&= P\left(\log{(1+R_1)} + \log{(1+R_2)} + ... + \log{(1+R_{20})} > \log\frac{100}{97}\right) \\
&= 1 - P\left(\log{(1+R_1)} + \log{(1+R_2)} + ... + \log{(1+R_{20})} < \log\frac{100}{97}\right) \\
&= 0.4218295
\end{aligned}
$$

```
> 1 - pnorm(log(100/97), mean=20*0.0002, sd=sqrt(20) * 0.03)
[1] 0.4218295
```