Statistical Finance (STAT W4290)

Assignment 5

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1

$$S\ N(0.001, 0.015)$$

$$\begin{aligned} P(S < 0.001 - 0.015) &= P\left(\frac{S - 0.001}{0.015} < -1\right) \\ &= 0.1586553 \end{aligned}$$

2

2.1

$$R\ N(0.001, 0.015) = 0.015Z + 0.001$$

$$\begin{aligned} P(R < r) &= 0.1 \\ P(0.015Z + 0.001 < r) &= 0.1 \\ P(Z < \frac{r - 0.001}{0.015}) &= 0.1 \end{aligned}$$

$$
\begin{aligned}
VaR(0.1) &= -S * q(0.1) \\
&= -1000000 * q(0.1) \\
&= -1000000 * (q_{norm}(0.1) * 0.015 + 0.001) \\
&= -1000000 * -0.01822327 \\
&= 18223.27
\end{aligned}
$$

2.2

$$R\ N(0.1, 0.2)$$

$$
\begin{aligned}
P(R < r) &= 0.05 \\
P(0.2Z + 0.1 < r) &= 0.05 \\
P(Z < \frac{r - 0.1}{0.2}) &= 0.05
\end{aligned}
$$

$$
\begin{aligned}
VaR(0.05) &= -S * q(0.05) \\
&= -1000 * q(0.05) \\
&= -1000 * (q_{norm}(0.05) * 0.2 + 0.1) \\
&= 228.9707
\end{aligned}
$$

3

3.1

$$R\ t(df = 2, 0.001, 0.015)$$

$$
\begin{aligned}
P(R < r) &= 0.1 \\
P(0.015T + 0.001 < r) &= 0.1 \\
P(T < \frac{r - 0.001}{0.015}) &= 0.1
\end{aligned}
$$

$$
\begin{aligned}
VaR(0.1) &= -S * q(0.1) \\
&= -1000 * q(0.1) \\
&= -1000 * (q_{t,df=2}(0.1) * 0.015 + 0.001) \\
&= 27.28427
\end{aligned}
$$

3.2

$$R\ t(df = 5, 0.001, 0.015)$$

$$
\begin{aligned}
VaR(0.1) &= -S * q(0.1) \\
&= -1000 * q(0.1) \\
&= -1000 * (q_{t,df=5}(0.1) * 0.015 + 0.001) \\
&= 21.13826
\end{aligned}
$$

As expected, larger df leads to slimmer tails.

4

Compose a portfolio of stock A and B. Let the portfolio be S. Then $S\ N(500 * 0.01 + 1000 * 0.005, \sigma_s) = N(10, \sigma_s)$

4.1

Given independence, $\sigma_s = \sqrt{500^2 * \sigma_A^2 + 1000^2 * \sigma_B^2} = 26.92582$

$$
\begin{aligned}
VaR(0.05) &= -q(0.05) \\
&= -(q_{norm}(0.05) * sigma_s + 10) \\
&= 34.28904
\end{aligned}
$$

4.2

Given $\rho = 0.3$, $\sigma_s = \sqrt{500^2 * \sigma_A^2 + 1000^2 * \sigma_B^2 + 500 * 1000 * 2 * \rho * \sigma_A * \sigma_B} = 29.5804$

$$\begin{aligned} VaR(0.05) &= -q(0.05) \\ &= -(q_{norm}(0.05) * sigma_s + 10) \\ &= 38.65543 \end{aligned}$$

4.3

Given $\rho = -0.3$, $\sigma_s = \sqrt{500^2 * \sigma_A^2 + 1000^2 * \sigma_B^2 + 500 * 1000 * 2 * \rho * \sigma_A * \sigma_B} = 23.97916$

$$\begin{aligned} VaR(0.05) &= -q(0.05) \\ &= -(q_{norm}(0.05) * sigma_s + 10) \\ &= 29.4422 \end{aligned}$$

5

First, recognize that Z is not a variable, but determined by the function. Since

$$\int_{-\infty}^{\infty} \frac{1}{Z} \frac{|x+1|}{(x^2+1)^2} = 1$$

Then

$$Z = \int_{-\infty}^{\infty} \frac{|x+1|}{(x^2+1)^2} = 1.785398$$

```
> crazy_eqn = function (x) {
+   abs(x+1)/((x^2+1)^2)
+ }
> integrate(f=crazy_eqn, lower=-Inf,upper=+Inf)
1.785398 with absolute error < 6e-09
```

Now because my integration-fu is too bad, let me solve for this numerically. We want $VaR(0.05) = F(0.95)$ where F is the CDF ($\int f(x)\,dx$) of L. Hence

$$F(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{Z} \frac{|x+1|}{(x^2+1)^2} dx$$

Unfortunately, `R` does not have a function for numerical computation of quantiles for arbitrary distribution functions. However, we can build one.

Finding α above is equivalent to the following optimization problem:

$$\alpha^* = \arg\min_{\alpha} (F(\alpha) - 0.05)^2$$

Then all we need is an optimization routine to find α that minimizes the squared error $F(\alpha) - q)^2$. In `R` there exists the library `nlminb` for numerical optimization. The functions are below:

```
CDF = function (x, dist) {
  integrate(f = dist, lower = -Inf, upper = x)$value
}
objective = function (x, quantile, dist) {
  (CDF(x, dist) - quantile)^2
}
find_quantile = function (dist, quantile) {
  result = nlminb(start = 0, objective = objective, quantile
      = quantile, dist = dist)$par
  return (result)
}

crazy_eqn = function (x) {
  abs(x+1)/((x^2+1)^2)
}

z = integrate(crazy_eqn, lower=-Inf, upper=Inf)

crazy_eqn_to_integrate = function(x) {
  z$value * (abs(x+1)/((x^2+1)^2))
}

alpha = find_quantile(dist = crazy_eqn_to_integrate,
    quantile = 0.95)

integrate(crazy_eqn_to_integrate, lower=-Inf, upper=alpha)
```

Solving for α, we get $\alpha = 0.03161596$.

I also didn't realize there was a hint for this question on Courseworks. Please make an announcement next time please? `:)` I did solve the first portion of the integration and got a really funky equation with arctan, and thought I had to inverse that by hand (given the nature of this assignment).

6

6.1

$$\begin{aligned} P(L > VaR(\alpha)) &= P(\sigma Z + \mu > VaR(\alpha)) \\ &= P\left(Z > \frac{VaR(\alpha) - \mu}{\sigma}\right) \end{aligned}$$

Then $VaR(\alpha) = -(\sigma Q(\alpha) + \mu) = \sigma Q(1-\alpha) + \mu = \sigma Q(z_\alpha) + \mu$ where Q is the standard normal quantile function. Following the definition of expected shortfall,

$$\begin{aligned} ES(\alpha) &= -\frac{\int_0^\alpha VaR(x)\,dx}{\alpha} \\ &= -\frac{\int_0^\alpha \sigma Q(1-x) + \mu\,dx}{\alpha} \\ &= -\frac{\mu\alpha + \int_0^\alpha \sigma Q(1-x)\,dx}{\alpha} \\ &= -\frac{\mu\alpha + \int_1^{1-\alpha} \sigma Q(z)\,dz}{\alpha} \\ &= \mu + \frac{\sigma\phi(z_\alpha)}{\alpha} \end{aligned}$$

Sorry I didn't show the last step. I know $Q(z) = zf(z)$ where f is the density function, and that that is integrable using by parts, but I don't have enough time to solve this.

6.2

Plugging the numbers into the formula,

$$
\begin{aligned}
ES(0.05) &= S * \left(-\mu + \frac{\sigma \phi(z_\alpha)}{\alpha} \right) \\
&= 100000 * \left(0.04 + \frac{0.18\phi(Q(0.05))}{0.95} \right) \\
&= 33128.83
\end{aligned}
$$