

Statistical Finance (STAT W4290)

Assignment 1

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Question 1

(a)

Let investment amount at t be P_t . Set $P_0 = 1000$.

Given that $\log(1 + R_{0,1}) \sim N(0.001, 0.015^2)$,

$$\begin{aligned} P(P_1 < 990) &= P\left(\frac{P_1}{P_0} < \frac{990}{P_0}\right) \\ &= P\left(1 + R_{0,1} < \frac{990}{1000}\right) \\ &= P\left(\log(1 + R_{0,1}) < \log \frac{990}{1000}\right) \\ &= 0.2306557 \end{aligned}$$

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1 > pnorm(log(990/1000), mean=0.001, sd=0.015)
2 [1] 0.2306557
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(b)

$$\begin{aligned}P(P_5 < 990) &= P\left(\frac{P_5}{P_0} < \frac{990}{1000}\right) \\&= P\left(1 + R_{0,5} < \frac{990}{1000}\right) \\&= P\left((1 + R_{0,1})(1 + R_{1,2})(1 + R_{2,3})(1 + R_{3,4})(1 + R_{4,5}) < \frac{990}{1000}\right) \\&= P\left(\log((1 + R_{0,1})(1 + R_{1,2})(1 + R_{2,3})(1 + R_{3,4})(1 + R_{4,5})) < \log \frac{990}{1000}\right) \\&= P\left(\log(1 + R_{0,1}) \dots + \log(1 + R_{4,5}) < \log \frac{990}{1000}\right)\end{aligned}$$

Now, given that daily log returns are independent,

$$\log(1 + R_{0,1}) \dots + \log(1 + R_{4,5}) \sim N(0.005, 5 * 0.015^2)$$

Then,

$$P\left(\log(1 + R_{0,1}) \dots + \log(1 + R_{4,5}) < \log \frac{990}{1000}\right) = 0.3268189$$

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1 > pnorm(log(990/1000), mean=0.005, sd=sqrt(5) * 0.015)
2 [1] 0.3268189
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Question 4

(a)

$$\begin{aligned}R_2 &= \frac{P_2 + D_2}{P_1} - 1 \\&= \frac{54 + 0.2}{52} - 1 \\&= 0.04230769\end{aligned}$$

(b)

$$\begin{aligned} R_4(3) &= \frac{P_4 + D_4}{P_3} * \frac{P_3 + D_3}{P_2} * \frac{P_2 + D_2}{P_1} - 1 \\ &= \frac{59 + 0.25}{53} * \frac{53 + 0.2}{54} * \frac{54 + 0.2}{52} - 1 = 0.1479588 \end{aligned}$$

(c)

$$\begin{aligned} r_3 &= \log 1 + R_3 \\ &= \log \frac{P_3 + D_3}{P_2} \\ &= \log \frac{53 + 0.2}{54} \\ &= -0.01492565 \end{aligned}$$

Trivial R code omitted.

Question 6

(a)

$$\begin{aligned} X_k &= X_0 \exp(r_1 + \dots + r_k) \\ \log X_k - \log X_0 &= r_1 + \dots + r_k \end{aligned}$$

Then,

$$\log X_k - \log X_0 \sim N(k\mu, k\sigma^2)$$

$$\begin{aligned} P(X_2 > 1.3X_0) &= P\left(\frac{X_2}{X_0} > 1.3\right) \\ &= P(\log X_2 - \log X_0 > \log 1.3) \end{aligned}$$

Given that $\log X_2 - \log X_0 \sim N(2\mu, 2\sigma^2)$

$$P(\log X_2 - \log X_0 > \log 1.3) = 1 - P(\log X_2 - \log X_0 < \log 1.3)$$

(b)

Recall that suppose X is a random variable with PDF $f_X(x)$ and $Y = g(X)$ for g is a strictly increasing function. Since g is strictly increasing, it has an inverse, which we denote by h . Then Y is also a random variable and its CDF is:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq h(y)) = F_X(h(y))$$

Differentiating, we find the PDF of Y

$$f_Y(y) = f_X(h(y))h'(y)$$

In this case, $r \sim N(\mu, \sigma^2)$

$$f_R(r) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(r - \mu)^2}{2\sigma^2}\right)$$

Since $Y = X_1 = X_0 \exp R = g(R)$,

$$h(Y) = R = \log \frac{X_1}{X_0} = \log X_1 - \log X_0$$

$$h'(Y) = \frac{\delta R}{\delta X_1} = \frac{1}{X_1}$$

Then,

$$f_Y(y) = f_{X_1}(x) = \frac{1}{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\log x - \log X_0 - \mu)^2}{2\sigma^2}\right)$$

(c)

$$X_k = X_0 \exp r_1 + \dots + r_k = X_0 \exp R$$

Then, $R \sim N(k\mu, k\sigma^2)$

Then,

$$f_{X_k}(x) = \frac{1}{x} \frac{1}{\sqrt{2\pi k}\sigma} \exp\left(-\frac{(\log x - \log X_0 - k\mu)^2}{2k\sigma^2}\right)$$

Thus transformation from R to X_k is monotonic. Then, we find the 0.9 quantile of $R \sim N(k\mu, k\sigma^2)$. Denote this value with $r_{0.9}$. Then, the 0.9 quantile of X_k is simply $X_0 \exp r_{0.9}$

(d)

$$X_k^2 = X_0^2 \exp(2(r_1 + \dots + r_k)) = X_0^2 \prod_{i=1}^k \exp(2r_i)$$

Then, the expectation $E(X_k^2)$ can be found by (since each r_i is iid)

$$\begin{aligned} E(X_k^2) &= X_0^2 \int_{-\infty}^{\infty} \left(\prod_{i=1}^k \exp(2r_i) p(r_i) dr_i \right) \\ &= X_0^2 \left(\int_{-\infty}^{\infty} \exp(2r) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) dr \right)^k \\ &= X_0^2 \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(2r - \frac{(r-\mu)^2}{2\sigma^2}\right) dr \right)^k \\ &= X_0^2 \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right)r - \frac{r^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) dr \right)^k \\ &= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right)r - \frac{r^2}{2\sigma^2}\right) dr \right)^k \end{aligned}$$

Working on the inner exponent,

$$\begin{aligned} \left(2 + \frac{2\mu}{2\sigma^2}\right)r - \frac{r^2}{2\sigma^2} &= -\frac{1}{2\sigma^2} \left(r^2 - 2\sigma^2 \left(2 + \frac{\mu}{\sigma^2}\right)r\right) \\ &= -\frac{1}{2\sigma^2} (r^2 - 2(2\sigma^2 + \mu)r) \\ &= -\frac{1}{2\sigma^2} (r^2 - 2(2\sigma^2 + \mu)r + (2\sigma^2 + \mu)^2 - (2\sigma^2 + \mu)^2) \\ &= -\frac{1}{2\sigma^2} \left((r - (2\sigma^2 + \mu))^2 - (2\sigma^2 + \mu)^2\right) \end{aligned}$$

Continuing where we left off earlier,

$$\begin{aligned}
E(X_k^2) &= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \int_{-\infty}^{\infty} \exp\left(\left(2 + \frac{2\mu}{2\sigma^2}\right)r - \frac{r^2}{2\sigma^2}\right) dr \right)^k \\
&= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\frac{(2\sigma^2 + \mu)^2}{2\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}(r - (2\sigma^2 + \mu))^2\right) dr \right)^k \\
&= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \exp\frac{(4\sigma^4 + 4\sigma^2\mu)^2}{2\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}r^2\right) dr \right)^k
\end{aligned}$$

Recall that

$$\int_{-\infty}^{\infty} \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

Then,

$$\begin{aligned}
E(X_k^2) &= X_0^2 \left(\frac{1}{\sqrt{2\pi}\sigma} \sqrt{2\pi}\sigma \exp(2\mu + 2\sigma^2) \right)^k \\
&= X_0^2 \exp(2k\mu + 2k\sigma^2)
\end{aligned}$$

(e)

$$\text{Var}(X_k) = E(X_k^2) - E(X_k)^2$$

(I'm running late for the assignment and had no time to typeset this). Using basically the same stuff as (d) except replace X_k^2 with X_k

$$E(X_k) = X_0 \exp\left(\frac{k\sigma^2}{2} + k\mu\right)$$

Then,

$$\begin{aligned}
\text{Var}(X_k) &= X_0^2 \exp(2k\mu + 2k\sigma^2) - X_0^2 \exp\left(\frac{2k\sigma^2}{2} + 2k\mu\right) \\
&= X_0^2 \left(\exp(2k\mu + 2k\sigma^2) - \exp\left(\frac{2k\sigma^2}{2} + 2k\mu\right) \right)
\end{aligned}$$

Question 7

Given that $\log(1 + R_t) \sim N(0.0002, 0.03^2)$, $\log(1 + R_{0,20}) \sim N(20 * 0.0002, 20 * 0.03^2)$

$$\begin{aligned} P(P_{20} > 100) &= P\left(\frac{P_{20}}{P_0} > \frac{100}{P_0}\right) \\ &= P\left(1 + R_{0,20} > \frac{100}{97}\right) \\ &= P\left(\log(1 + R_1) + \log(1 + R_2) + \dots + \log(1 + R_{20}) > \log \frac{100}{97}\right) \\ &= 1 - P\left(\log(1 + R_1) + \log(1 + R_2) + \dots + \log(1 + R_{20}) < \log \frac{100}{97}\right) \\ &= 0.4218295 \end{aligned}$$

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1 > 1 - pnorm(log(100/97), mean=20*0.0002, sd=sqrt(20) * 0.03)
2 [1] 0.4218295
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