Statistical Finance (STAT W4290)

Assignment 6

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1

An bivariate Archimedean copula with a strict generator has the form

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v))$$

Now the Frank copula has the generator ϕ as

$$\phi(u) = -\log \frac{e^{(-\theta u)} - 1}{e^{(-\theta)} - 1}$$

Then, the inverse generator ϕ^{-1} is

$$\phi^{-1}(y) = -\frac{1}{\theta} \log \left[(e^{(-y)})(e^{(-\theta)} - 1) + 1 \right]$$

Then, the variate Frank copula is

$$\begin{aligned}
C(u,v) &= -\frac{1}{\theta}\log\left[(e^{(-\phi(u)-\phi(v))})(e^{(-\theta)}-1)+1\right] \\
&= -\frac{1}{\theta}\log\left[(e^{(\log\frac{e^{(-\theta u)}-1}{e^{(-\theta)}-1}+\log\frac{e^{(-\theta v)}-1}{e^{(-\theta)}-1})})(e^{(-\theta)}-1)+1\right] \\
&= -\frac{1}{\theta}\log\left[(e^{(\log\frac{(e^{(-\theta u)}-1)(e^{(-\theta v)}-1)}{(e^{(-\theta)}-1)^2})})(e^{(-\theta)}-1)+1\right] \\
&= -\frac{1}{\theta}\log\left[\frac{(e^{(-\theta u)}-1)(e^{(-\theta v)}-1)}{(e^{(-\theta)}-1)^2}(e^{(-\theta)}-1)+1\right] \\
&= -\frac{1}{\theta}\log\left[\frac{(e^{(-\theta u)}-1)(e^{(-\theta v)}-1)}{(e^{(-\theta)}-1)}+1\right]
\end{aligned}$$

2

To prove that $C(u,v) \leq C^+(u,v)$, recall that by definition of a bivariate copula, $C(u,1) = u$ and $C(1,v) = v$. The bivariate copula is also 2 increasing. Hence,

$$\begin{aligned}
C(u,v) &\leq C(u,1) = u \\
C(u,v) &\leq C(1,v) = v
\end{aligned}$$

Then, $C(u,v) \leq \min(u,v)$.

Now given that copulas are 2-increasing,

$$C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$$

Set $u_1 = u$, $u_2 = 1$, $v_1 = v$, $v_2 = 1$, we get

$$\begin{aligned}
1 - u - v + C(u,v) &\geq 0 \\
C(u,v) &\geq u + v - 1
\end{aligned}$$

Similarly, set $u_1 = 0$, $u_2 = u$, $v_1 = 0$, $v_2 = v$, by the grounded property, we get

$$
\begin{aligned}
C(u,v) - 0 - 0 + 0 &\geq 0 \\
C(u,v) &\geq 0
\end{aligned}
$$

Hence, $C(u,v) \geq \max(u+v-1, 0)$. Combining that with the previous result,

$$C^-(u,v) \leq C(u,v) \leq C^+(u,v)$$

3

3.1

Probability of both defaulting is

$$
\begin{aligned}
P(T_A \leq 1 \cap T_B \leq 1) &= C(P(T_A \leq 1), P(T_B \leq 1)) \\
&= C(1 - e^{-\lambda_A}, 1 - e^{-\lambda_B}) \\
&= e^{\left(-((-\log u)^\alpha + (-\log v)^\alpha)^{\frac{1}{\alpha}}\right)} \\
&= 0.00235139
\end{aligned}
$$

where u and v are $1 - e^{-\lambda_A}$ and $1 - e^{-\lambda_B}$ respectively.

```
lambdaa = 0.01
lambdab = 0.02
u = 1-exp(-lambdaa)
v = 1-exp(-lambdab)
alpha = 2

gumbel = function(u, v, alpha) {
  return(exp(-((-log(u))^alpha + (-log(v))^alpha)^(1/alpha))
    )
}

gumbel(u, v, alpha)
```

3.2

Probability of at least 1 defaulting is

$$\begin{aligned}
&P(T_A \leq 1) + P(T_B \leq 1) + P(T_A \leq 1 \cap T_B \leq 1) \\
=&P(T_A \leq 1) + P(T_B \leq 1) + C(P(T_A \leq 1), P(T_B \leq 1)) \\
=&0.0274001
\end{aligned}$$

```
default_p = function(lambda, t) {
  return(1 - exp(-lambda * t))
}

default_p(0.01, 1) + default_p(0.02, 1) - gumbel(u, v, alpha
  )
```

4

When $\alpha = 2$, fair value is $1,000,000 * P(T_A \leq 1 \cap T_B \leq 1) = 2351.39$

When $\alpha = 1$, fair value is 197.0265.

```
> 1000000*gumbel(u, v, 2)
[1] 2351.39
> 1000000*gumbel(u, v, 1)
[1] 197.0265
```

This makes sense, since higher α means that the assets are more correlated, hence if one goes under, the other is more likely to default too.

5

Kendall's τ is defined as

$$\begin{aligned}
\rho_\tau &= P((X - X^*)(Y - Y^*) > 0) - P((X - X^*)(Y - Y^*) < 0) \\
&= E(\text{sign}((X - X^*)(Y - Y^*)))
\end{aligned}$$

5.1

If $Y' = \frac{1}{Y}$, since both X and Y are always positive,

$$\begin{aligned}
\rho_\tau &= E\left(\operatorname{sign}\left((X - X^*)\left(\frac{1}{Y} - \frac{1}{Y^*}\right)\right)\right) \\
&= E\left(\operatorname{sign}\left((X - X^*)\left(\frac{Y * - Y}{YY*}\right)\right)\right) \\
&= E\left(\operatorname{sign}\left((X - X^*)\left(Y * - Y\right)\right)\right) && = -0.55
\end{aligned}$$

5.2

If $X' = \frac{1}{X}$,

$$\begin{aligned}
\rho_\tau &= E\left(\operatorname{sign}\left(\left(\frac{1}{X} - \frac{1}{X^*}\right)\left(\frac{1}{Y} - \frac{1}{Y^*}\right)\right)\right) \\
&= E\left(\operatorname{sign}\left((X^* - X)\left(Y * - Y\right)\right)\right) && = -(-0.55) = 0.55
\end{aligned}$$

6

Intuitively, it is because Spearman ρ involves transforming the random variable by its CDF, which is the same as computing the rank of the variable in a finite sample. The Pearson correlation does not contain this transformation. Similarly, in calculating Kendall's τ, we are only concerned with the relative ranks of the variables. Hence, a monotonically increasing formation (which $y = x^2$ is for the range 0 to 1) will not change the Kendall's τ for a variable. Hence, for both Spearman and Kendall, we are effectively calculating the rank correlation of X with itself, resulting in 1.

Let Pearson correlation coefficient be ρ_p. Then,

$$\begin{aligned}
\rho_p &= \frac{\operatorname{cov}(X, Y)}{\sigma_x \sigma_Y} \\
&= \frac{E(XY) - E(X)E(Y)}{\sqrt{E(X^2) - E(X)^2}\sqrt{E(Y^2) - E(Y)^2}} \\
&= \frac{E(X^3) - E(X)E(X^2)}{\sqrt{E(X^2) - E(X)^2}\sqrt{E(X^4) - E(X^2)^2}}
\end{aligned}$$

Now,

$$
\begin{aligned}
E(X) &= 0.5 \\
E(X^2) &= \frac{1}{3} \\
E(X^3) &= \frac{1}{4} \\
E(X^4) &= \frac{1}{5}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\rho_p &= \frac{E(X^3) - E(X)E(X^2)}{\sqrt{E(X^2) - E(X)^2}\sqrt{E(X^4) - E(X^2)^2}} \\
&= \frac{0.25 - 0.5 * \frac{1}{3}}{\sqrt{\frac{1}{3} - 0.25}\sqrt{0.2 - \frac{1}{9}}} \\
&= 0.9682458
\end{aligned}
$$

Whereas for Spearman ρ and Kendall's τ, the square function does not change the rank for a variable between 0 and 1.

7

The bivariate Frank copula is

$$
C(u, v) = -\frac{1}{\theta} \log \left[\frac{(e^{(-\theta u)} - 1)(e^{(-\theta v)} - 1)}{(e^{(-\theta)} - 1)} + 1 \right]
$$

As $\theta \to \infty$, $e^{(-\theta)} \to 0$, $e^{(-\theta u)} \to 0$ and $e^{(-\theta v)} \to 0$. Hence, we get nowhere, and have to use L'hospital.

However, we can simplify the equation first.

$$
\begin{aligned}
C(u,v) &= -\frac{1}{\theta}\log\left[\frac{(e^{(-\theta u)}-1)(e^{(-\theta v)}-1)}{(e^{(-\theta)}-1)}+1\right] \\
&= -\frac{1}{\theta}\log\left[\frac{e^{(-\theta u-\theta v)}-e^{(-\theta u)}-e^{(-\theta v)}+1+e^{(-\theta)}-1)}{(e^{(-\theta)}-1)}\right] \\
&=
\end{aligned}
$$

Now assume $u<v$. Then, $u<v<u+v$. Then,

$$e^{-\theta u} > e^{-\theta v} > e^{-\theta u-\theta v}$$

Since $u<v\leq 1$

$$e^{-\theta u} > e^{-\theta}$$

Then,

$$e^{(-\theta u-\theta v)}-e^{(-\theta u)}-e^{(-\theta v)}+1+e^{(-\theta)}-1) \sim e^{(-\theta u)}$$

as $\theta \to \infty$.

Then,

$$C(u,v) \sim -\frac{1}{\theta}\log\left[\frac{e^{(-\theta u)}}{e^{(-\theta)}-1}\right]$$

Taking limits naively, one still gets indeterminate forms. Apply L'hospital's rule. $\frac{d}{d\theta}\theta = 1$.

$$
\begin{aligned}
\frac{d}{d\theta}\log\left[\frac{e^{(-\theta u)}}{e^{(-\theta)}-1}\right] &= \left(\frac{e^{(-\theta)}-1}{e^{(-\theta u)}}\right)\left(\frac{-ue^{(-\theta u)}}{e^{(-\theta)}-1}+\frac{e^{(-\theta u)}e^{(-\theta)}}{(e^{(-\theta)}-1)^2}\right) \\
&= -u+e^{(-\theta)}
\end{aligned}
$$

As $\theta\to\infty$, $-u+e^{(-\theta)}\to -u$

Then, by L'hospital's rule,

$$\lim_{\theta \to \infty} C(u, v) = u$$

The same applies when $v < u$, then $\lim_{\theta \to \infty} C(u, v) = v$. Hence,

$$\lim_{\theta \to \infty} C(u, v) = \min(u, v)$$