An Euler tour of a strongly connected, directed graph $G=(V,E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once. Show that G has an Euler tour if and only if $indegree(v)=outdegree(v)$ for each vertex $v \in V$.

Proof:

Necessity Let $G(V,E)$ be a strongly connected directed graph. Thus G contains a cycle C. Let this cycle start and end at the vertex $u \in V$. Since each visit to an intermediate vertex v of Z contributes 1 to $indegree(v)$ and 1 to $outdegree(v)$ and since C traverses each edge exactly once, $indegree(v)=outdegree(v)$ holds true for every such vertex. Each intermediate visit to the initial vertex u contributes 1 to both $indegree(u)$ and $outdegree(u)$, and also the initial and final edges of C contribute 1 to $indegree(u)$ and $outdegree(v)$ respectively. So, $indegree(u)=outdegree(u)$ holds true for u as well. Thus, if G has an Euler tour, $indegree(v)=outdegree(v)$ for each vertex $v \in V$.

Sufficiency Let G_n be a strongly connected directed graph with n vertices, for which $indegree(v)=outdegree(v)$ for each vertex $v \in V$.

Base case: G_2 has a Euler tour. Let the vertices be u and v As we have a graph of two vertices with $indegree(u)=outdegree(u)$ and $indegree(v)=outdegree(v)$, we can travel from u to v and then back to u k times, resulting in an Euler tour

Hypothesis: Lets assume that there exists an Euler tour in any arbitrary graph G_n.

Inductive step: We have to prove that G_{n+1} has an Euler tour. Every vertex in G_n+1 has equal indegree and outdegree. So, for any vertex v, we can arbitrarily pair up all of v's in edges and out edges. Now we can remove vertex v from G_n+1 and for every pair of edges (u,v) and (v,w) we add an edge (u,w) in G_n. In this way, in-degree and out-degree of u and w remains the same. Thus, there is an Euler tour in G_n

(Inductive Hypothesis). Let the Eulerian cycle be $C=x_0,x_1,\ldots,u,w,\ldots,x_0$

Now we add the vertex v back to G_n and traverse edge (u,v) and then (v,w) in the place of (u,w) in C. The resulting path $P=x_0,x_1,\ldots,u,v,w,\ldots,x_0$ is also an Eulerian cycle in G_n+1. Thus, G_n+1 has an Euler tour.Thus G has an Euler tour if $indegree(v)=outdegree(v)$ for each vertex $v \in V$

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