Project 1

1 The Physical Problem

The parachuter experiences the following forces :

- Gravity, F_g
- Drag, F_d
- Buouyancy, F_b

Newton's second law states that the sum of forces acting on the body equals the mass of the body times acceleration :

$$ma = \sum F_{ext}$$

or in the case of the parachuter

$$m\frac{dv}{dt} = F_g + F_d + F_b$$

where

$$F_g = mg, \quad F_d = -\frac{1}{2}C_D\rho|v|v, \text{ and } \; F_b = -\rho g V.$$

1.1 Numerical Scheme

In general we can write the ODE as following :

$$\dot{v} = -a(t)|v|v + b(t) \tag{1.a}$$
$$v(0) = v_0 \tag{1.b}$$

for any a(t) and b(t), and start velocity v_0. For our physical problem a = $\frac{C_D\rho}{2m}$ and b = g - $\frac{\rho g V}{m}$.Using the approximation $v'(t_n) \approx \frac{v^{n+1}-v^n}{t^{n+1}-t^n}$, the numerical scheme then takes the following form :

$$
\begin{aligned}
v^{n+1} &= \frac{v^n + b^{n+\frac{1}{2}}\Delta t}{1+\Delta t a^{n+\frac{1}{2}}|v^n|} \\
a^{n+\frac{1}{2}} &= \frac{a^{n+1}+a^n}{2} \\
b^{n+\frac{1}{2}} &= \frac{b^{n+1}+b^n}{2}
\end{aligned}
\tag{2}
$$

where $a^n = a(t_n)$ and $b^n = b(t_n)$ are function values evaluated at the mesh points.

We have managed to simplify the scheme by using geometric averaging for the term $|v|v$. By doing so we have linearized the nonlinear ODE (1.a). Finally we employed the Crank Nicolson scheme by evaluating the node points at $n+\frac{1}{2}$. Hence forth we will refer to b as the source term when we perform our tests.

2 Verification & Debugging

After we have implemented the solver, we should *always* test it for simple cases where we can expect machine precision. A good exercise is to check for a constant solution. To accomplish this, we have to construct the right ODE problem. We can specify any a(t), then we can fit the source term to the problem :

For $v_e = 4$[1],

$$
\begin{aligned}
0 &= -16a(t) + b(t) \\
&\Downarrow \\
b(t) &= 16a(t)
\end{aligned}
$$

Inserting this expression for b(t) into the scheme

$$
v^{n+1} = \frac{v^n + 16\Delta t a^{n+\frac{1}{2}}}{1+\Delta t a^{n+\frac{1}{2}}|v^n|}.
$$

If we now let $v^n = 4$, then the scheme becomes

$$
\begin{aligned}
v^{n+1} &= 4\frac{1+4\Delta t a^{n+\frac{1}{2}}}{1+4\Delta t a^{n+\frac{1}{2}}} \\
&= 4.
\end{aligned}
$$

We realize that it may be a good idea to design a general solver which can handle any a(t) and b(t). That is the reason why we have chosen to do an averaging of the function values.

In the exercise we are required to demonstrate that a linear funciton of **t** does not fullfill the discrete equation (because of the geometric mean used for the quadratic drag

[1] To satisfy this solution, the initial condition $v_e(0)$ must equal 4.

term). Let us then observe what happens if we try the solution $v_e = ct$. The discrete solution becomes $v^n = c\Delta tn$. This implies that we should expect $v^{n+1} = c\Delta t(n+1)$. Assume that the source term, b, is zero. Then the scheme (2) can be written as following

$$v^{n+1} = \frac{c\Delta tn}{1 + \Delta t a^{n+\frac{1}{2}}|c\Delta tn|}.$$

This is obviously not $c\Delta t(n+1)$ [1] . We can fix this problem by choosing b(t) such that it fits the scheme. As we did for the linear case, if we perform the same calculations as above, we end up with the desired result :

$$\begin{aligned}
c &= -a(t)ct|ct| + b(t) \\
&\Downarrow \text{Rearranging for b(t)} \\
b(t) &= c + a(t)ct|ct| \\
&\Downarrow \text{The discrete version, after using geometric averaging} \\
b^{n+\frac{1}{2}} &= c + a^{n+\frac{1}{2}} c\Delta t(n+1)|c\Delta tn| \\
&\Downarrow \text{Inserting this expression in our scheme} \\
v^{n+1} &= \frac{c\Delta t(n+1) + \Delta t a^{n+\frac{1}{2}} c\Delta t(n+1)|c\Delta tn|}{1 + \Delta t a^{n+\frac{1}{2}}|c\Delta tn|} \\
&= c\Delta t(n+1)
\end{aligned}$$

which is what we wanted.

[1] Note that c is simply a constant.