A Project In INF5620

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Abstract

For the mandatory project in INF5620, we (the authors) have decided to achieve the following :

- make a Navier-Stokes solver,
- make a RANS solver with the two-equation k-ε model.

These solvers will be written within the framework of the automated finite element PDE environment known as FEniCS; where a variational formulation must be given along with the boundary conditions in order to solve a boundary value problem. Finally some tests will be performed to find correct convergence rates (hopefully!).

Contents

1 Introduction

In this project we are going to start with the Navier-Stokes equations. We will formulate the variational formulation, thereupon discretize this form and hopefully shortly discuss the stratagies that can be employed for certain terms in the discretization; leading to different schemes for solving the Navier-Stokes equations.

We have decided to implement the Chorin scheme, first introduced by Alexandre Chorin [7] [1] and independently by Roger Temam [8][2]. The common terminology for such schemes are projection methods. This will be discussed more thoroughly later in the discretization section for the Navier-Stokes equations. The implementation will be written in FEniCS. After this we shall perform similar steps for the RANS equations using the k-ε model.

2 The Navier-Stokes Equations

The incompressible Navier-Stokes equations satisfying mass conservation are given as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla \cdot \left[\nu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)\right] + \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ is the fluids velocity, $\mathbf{f}$ external forces working upon the fluid, p pressure, ρ constant density of the fluid, and ν viscosity. Note that we have used a form which may seem unsual to new comers to Navier-Stokes equations, however this form is mathematically equivalent to using $\nabla^2\mathbf{u}$. Using the equation of continuity, (2), lead to the following calculations : $\nabla \cdot \nabla \mathbf{u}^T = \nabla\nabla \cdot \mathbf{u} = 0$.

For RANS we will have an additional "turbulent" viscosity which we shall model using the eddy viscosity model (this will be discussed later). The Navier-Stokes equations (1) can also be expressed with the Cauchy stress tensor, which for a Newtonian fluid is defined as

$$\sigma_{ij} = 2\nu S_{ij} - \frac{p}{\rho}\delta_{ij}, \tag{3}$$

Here, δ_{ij} is the delta function for the d-dimensional vector field $\mathbf{u}$ and S_{ij} is the symmetric gradient

$$S = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \tag{4}$$

Inserting equation (3) in (1) yields the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot \sigma + \mathbf{f} \tag{5}$$

Later, when we derive the weak formulation, expressing the viscosity term as we have, will bear fruits, as we will employ the so-called pseudo-traction or a do-nothing boundary condition in zeroing out the boundary terms in the linear form as was done by Liu [11].

2.1 Variational Formulation

We consider the time-dependent incompressible Navier-Stokes equations (1)-(2) on the time interval [0,T], and the domain $\Omega \subset \mathbb{R}^d$(d = 2,3) bounded by a sufficiently regular boundary $\partial\Omega$. Let $(\mathbf{u}, p) \in$ VxQ, where V = $[\mathrm{H}^1(\Omega)]^d$ and Q= L^2. These spaces are defined as following : a function u is in H^1 if all its weak derivative of order up to and including 1 are square integrable over Ω. Hence we require that

$$\int_\Omega u^2 + (\nabla u)^2 dx < \infty.$$

Similarly, a function q is in the L^2 space if it is square integrable over Ω

$$\int_\Omega q^2 dx < \infty.$$

[1] References vary slightly as to which articles of Chorin and Temam first introduced the original "projection method". Following articles support the citations : [6] and [9].

[2] See footnote 1.

The common denomination for these spaces is Sobolev spaces; vector space of functions equipped with a norm that is a combination of L^p-norms of the function itself as well as its derivatives up to a given order [3]. For the sake of simplicity we will only consider velocity functionals which exist in the trial space $\mathcal{H}_0^1$ defined as

$$\mathcal{H}_0^1 = \{\mathbf{u} \in H^1(\Omega) \mid \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega_D\}$$

Hence we only consider in this project boundary-value problems where we have no-slip conditions on the solid boundaries, i.e the walls for our physical cases. Further, we use the subspace of solenoidal functionals

$$\mathcal{J}_0^1 = \{\mathbf{u} \in \mathcal{H}_0^1 \mid \nabla \cdot \mathbf{u} = \mathbf{0} \text{ in } \Omega\}$$

We will use the short notation (., .) for the inner products between the test functions and trial functions. Note that for the diffusion term, after performing integration by parts, we will use the Frobenius inner product, denoted $\mathbf{A} : \mathbf{B}$, is the component-wise inner product of two matrices as though they are vectors [4]. The weak formulation for the Navier-Stokes equations (1) takes the form :

$$(\partial_t \mathbf{u}, \mathbf{v}) + \nu \overbrace{(\nabla \mathbf{u}, \nabla \mathbf{v})}^{\text{Frobenius inner product}} = \frac{1}{\rho}(p, \nabla \mathbf{v}) - (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V, \tag{6}$$

$$(\nabla \cdot \mathbf{u}, q) = 0 \qquad \forall q \in Q. \tag{7}$$

The derivations for this form have been included in the appendix.

2.2 Discretization

We intend to discretize the Navier-Stokes equations first in time and project the tentative solution onto the space of the solenoidal vector fields(i.e incompressible vector fields), in order to satisfy the incompressibility condition. Such methods are often called fractional-step projection or fractional-step methods [4], and are generally classified as projection methods [6] (first of such methods proposed was the Chorin scheme). In general the methods for solving the time-dependent Navier-Stokes equations can be grouped into two main classes : fractional-step methods and nonfractional-step methods. The most interesting feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure, making it very efficient for large scale numerical simulations[6].

The naive approach to discretizing the Navier-Stokes equations (1) would be to attempt a forward Euler discretization for the time derivative term, using the previous pressure value.

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n \cdot \nabla \mathbf{u}^n) = -\frac{1}{\rho}\nabla p + \nabla \cdot \nu(\nabla \mathbf{u}^n + \nabla(\mathbf{u}^T)^n) + \mathbf{f}^n$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t(\mathbf{u}^n \cdot \nabla \mathbf{u}^n) - \frac{\Delta t}{\rho}\nabla p^n + \Delta t \nabla \cdot \nu(\nabla \mathbf{u}^n + (\nabla \mathbf{u}^T)^n) + \Delta t \mathbf{f}^n$$

However, this leads quickly to a scheme which does not necessarily fullfill our requirement on functionals in the solenoidal space $\mathcal{J}_0^1$. Further more we end up with an equation where we get no updates for the pressure. Before we continue any further, let us state that the discrete solution $(\mathbf{u}^n, p^n)$ exists in the discrete trial spaces $V^h \text{x}\, Q^h \subset V \text{x} Q$. Hence we need a solution which satisfies the incompressibility. Now instead of considering the pressure in the previous time step, we use the current pressure value, and the new velocity will also satisfy the incompressibility condition :

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \Delta t \mathbf{u}^n \cdot \nabla \mathbf{u}^n - \Delta t \frac{1}{\rho}\nabla p^{n+1} + \Delta t \nabla \cdot \nu(\nabla \mathbf{u}^n + (\nabla \mathbf{u}^T)^n) + \Delta t \mathbf{f}^n, \tag{8}$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \tag{9}$$

Here is the proposed algorithm : Calculate the tentative velocity, $\mathbf{u}^*$:

$$\mathbf{u}^* = \mathbf{u}^n - \Delta t \mathbf{u}^n \cdot \nabla \mathbf{u}^n - \Delta t \frac{\beta}{\rho}\nabla p^n + \Delta t \nabla \cdot \nu(\nabla \mathbf{u}^n + (\nabla \mathbf{u}^T)^n) + \Delta t \mathbf{f}^n \tag{10}$$

[3] http://en.wikipedia.org/wiki/Sobolev_space

[4] http://en.wikipedia.org/wiki/Matrix_multiplication#Frobenius_product

We seek a correction $\delta\mathbf{u}$ such that $\mathbf{u^{n+1}}$ fullfills

$$\nabla \cdot \mathbf{u^{n+1}} = 0. \tag{11}$$

Let $\mathbf{u}^{n+1} = \mathbf{u}^* + \delta\mathbf{u}$, then subtract (8) with (10)

$$\delta\mathbf{u} = \mathbf{u}^{n+1} - \mathbf{u}^n = -\frac{\Delta t}{\rho}\nabla\underbrace{\left(p^{n+1} - \beta p^n\right)}_{\phi^n}. \tag{12}$$

For the Chorin scheme, the potential ϕ^n is defined as p^{n+1}, i.e for $\beta = 0$. However, according to Navier-Stokes experts [10], using a nonzero β values results in a better scheme. Striving towards flexibility, we will include such a variable seeking further modularity in our solver.

Taking the divergence of the velocity correction term (12), and using the imposed divergence condition (11) :

$$\nabla \cdot \mathbf{u}^* = \frac{\Delta t}{\rho}\nabla^2\phi^n \tag{13}$$

Summing up the procedure, the steps become

1. Solve (10).
2. Insert the tentative velocity (10) in (13) and solve for ϕ^n.
3. Calculate the updated velocity $\mathbf{u^{n+1}} = \mathbf{u}^* + \delta\mathbf{u}$.
4. Calculate the updated pressure $p^{n+1} = \beta p^n + \phi^n$.
5. Repeat these steps untill simulation ends.

In order to execute this scheme in FEniCS we need to specify the variational form. Let $\mathbf{u}^* \in V^h$ and $p^{n+1} \in Q^h$. The variational form for (10) becomes (again using the short notation as previously)

$$(\mathbf{u}^*, \mathbf{v}) = (\mathbf{u}^n, \mathbf{v}) - (\Delta t\mathbf{u}^n \cdot \nabla\mathbf{u}^n, \mathbf{v}) + (\Delta t\frac{\beta}{\rho}p^n, \nabla \cdot \mathbf{v}) - (\Delta t\nu(\nabla\mathbf{u}^n + (\nabla\mathbf{u}^T)^n), \nabla\mathbf{v}) + (\Delta t\mathbf{f}^n, \mathbf{v}) \tag{14}$$

Rannacher [9] interestingly notes that Chorin's projection method can be interpreted as a pressure stabilization (Petrov-Galerkin) method. In general, pseudo compressibility methods are often used to overcome difficulties caused by the incompressibility constaint. This implies adding additional terms in the continuity equation :

- $\nabla \cdot \mathbf{u} = \epsilon\nabla$ p. Pressure stabilization.
- $\nabla \cdot \mathbf{u} = -\epsilon$ p. Penalty method.
- $\nabla \cdot \mathbf{u} = -\epsilon\frac{\partial p}{\partial t}$. Artificial compressibilty.
- $\nabla \cdot \mathbf{u} = \epsilon\nabla^2$ p. Petrov-Galerkin.

Often in litterature this is referred to as circumventing the celebrated Babuška-Brezzi condition based on the article of Hughes [5].

3 The Reynolds Averaged Navier-Stokes Equations

The RANS equations are given as

$$\frac{\partial\mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{U} - \nabla \cdot \overline{\mathbf{uu}}, \tag{15}$$

$$\nabla \cdot \mathbf{U} = \mathbf{0}. \tag{16}$$

3.1 The Variational Formulation

Let $(\mathbf{U}, P) \in$ VxQ, where V = $[\mathrm{H}^1(\Omega)]^d$ and Q= L^2. The weak formulation then becomes

$$(\partial_t \mathbf{U}, \mathbf{v}) + ([\nu + \nu_T]\nabla\mathbf{U}, \nabla\mathbf{v}) = \frac{1}{\rho}(P, \nabla\mathbf{v}) - (\mathbf{U}\cdot\nabla\mathbf{U}, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \quad \forall\mathbf{v} \in V, \tag{17}$$

$$(\nabla\cdot\mathbf{U}, q) = 0 \qquad \forall q \in Q. \tag{18}$$

Here **v** and q are test functions.

An explanation for the turbulent viscosity, ν_T, entering the equation (17) is as follows. Assuming that the Boussinesq approximation is valid, then the Reynolds stresses can be expressed as

$$\overline{u_i u_j} = -\nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij}. \tag{19}$$

Inserting (19) into the RANS equations (15) results in the weak formulation we have listed as equation (17). RANS models of this type are referred to as eddy viscosity models.

3.2 Discretization

(Unfinished....must complete NS solver first....our original plan was to base our k-epsilon scheme upon the article by Kristian Valen-Senstad et al. [3].)

4 The k-ε Model

The k-ε model is the most widely used turbulence model. Another two-equation model widely used is the k-ω model, which according to Wilcox[1] is the better of the two, as it handles wall bounded flow much better, avoiding the use of wallfunctions. However, we believe that both models handle similarly. To avoid the use of these so-called wallfunctions, we will only consider low-Reynolds number flow and the corresponding standard k- ε model based on Launder & Sharma [2] that is integrated all the way up to solid walls where regular Dirichlet boundary conditions apply [3]. The model is summarized by the following equations :

$$\partial_t k + \mathbf{U}\cdot\nabla k = \nabla\cdot\left[\left(\nu + \frac{\nu_T}{\sigma_k}\right)\nabla k\right] + P - \varepsilon - 2\nu||\nabla\sqrt{k}||^2, \tag{20}$$

$$\partial_t \varepsilon + \mathbf{U}\cdot\nabla\varepsilon = \nabla\cdot\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\nabla\varepsilon\right] + \frac{\varepsilon}{k}(C_{\varepsilon_1}P - C_{\varepsilon_2}\varepsilon) + 2\nu\nu_T||\nabla^2\mathbf{U}||^2, \tag{21}$$

$$P = \overline{u_i u_j}\frac{\partial U_i}{\partial x_j}, \tag{22}$$

$$\nu_T = C_\mu k^2 \varepsilon. \tag{23}$$

The Reynolds stress tensor is given as equation (19). The closure coefficients are $\sigma_k = 1$, $\sigma_\varepsilon = 1.3$, $C_{\varepsilon_1} = 1.44$, $C_{\varepsilon_2} = 1.92$ and $C_\mu = 0.09$. The last term in both equation (20) and equation (21) have been taken from [3].

5 Mesh Generation

The following meshes have been created using a tool named Gmsh; a 3D finite element grid generator with a build-in CAD engine and post-processor. The specification to geometry and mesh generation can either be done interactively using the graphical user interface or in ASCII text files using Gmsh's own scripting language. FEniCS allows users to create mesh through the Dolfin library :

>> mesh = dolfin.UnitSquareMesh(5,5) #creates a unit square mesh with 5x5 cells (24)

It is also possible to load pre-generated meshes as .xml files :

>> myMesh = dolfin.mesh("test.xml") (25)

Gmsh generates files of extension .geo and .msh. The .geo files contain the Gmsh script which is used by Gmsh to generate the mesh, whilst the .msh needs to be converted to a .xml file in order to use this mesh in FEniCS. This can be accomplished by using the following command :

>> python dolfin-convert test.msh test.xml (26)

The python file dolfin-convert.py can easily be found through the internets, a conversion script made by Anders Logg. Here are some meshes that we have created using the Gmsh :

5.1 Channel with cylinder in the interior

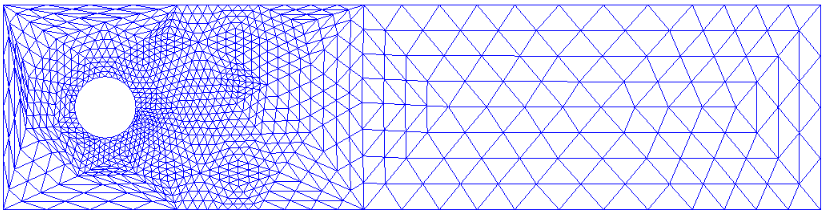

Figure 1.

The most relevant feature of the flow, at moderate values of the Reynolds number is the onset of a time-periodic regime characterized by alternate vortex shedding, known as the von Kármán vortex street.

In low turbulence, tall buildings can produce a Kármán street so long as the structure is uniform along its height. In urban areas where there are many other tall nearby structures the turbulence produced by these prevents the formation of coherent vortices. Periodic crosswind forces set up by vortices along object's sides can be highly undesirable, and hence it is important for engineers to account for the possible effects of vortex shedding when designing a wide range of structure. [5]

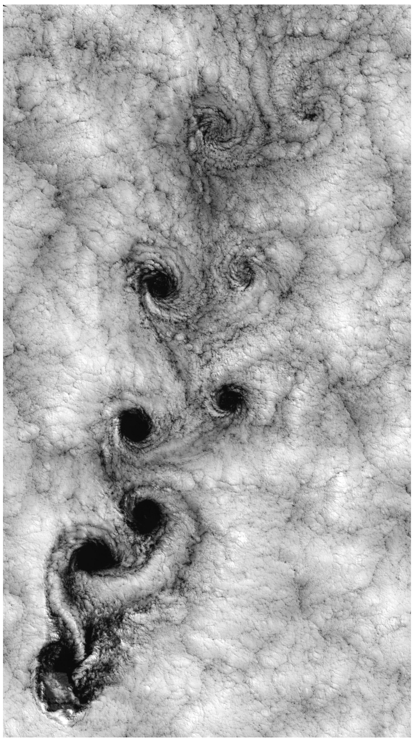

Figure 2 : Kármán vortex street caused by wind flowing around the Juan Fernández Islands off the Chilean coast. [6]

5.2 Channel with high resolution at moving boundary

In fluid dynamics, Couette flow is the laminar flow of a viscous fluid in the space between two parallel plates, one of which is moving relative to the other. The mesh has been made

[5] http://en.wikipedia.org/wiki/Kármán_vortex_street

[6] Taken from the wikipedia article.

to take into account the movement of the top plate, while the bottom plate remains fixed. For such flows the steady solution is given as

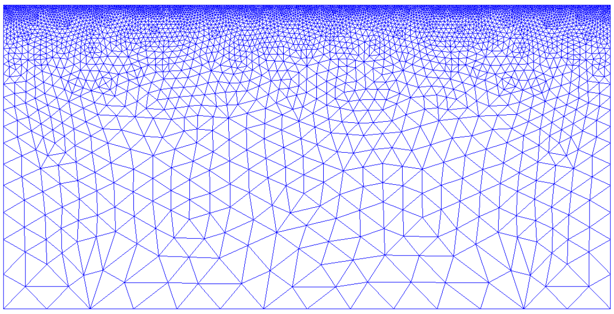

Figure 3.

$$u(y) = \frac{U}{2}\left(1 + \frac{y}{h}\right) \tag{27}$$

where the length of the width is 2h and the velocity of the top boundary is given as a constant U. For such flows, a coarse mesh is sufficient. However, we intend to simulate the unsteady problem, i.e the top boundary velocity starts at zero is suddenly increased to U. There exist an analytical solution for such a flow as well. This comes in handy when we want to see how well our solver approximates the solution by looking at the error between the exact solution and the discrete solution. The unsteady Couette flow solution is [12]

$$u(y, t) = U\left(1 - \operatorname{erf}\left(\frac{1}{2\sqrt{\nu t}}\right)\right), \tag{28}$$

where erf is the error function, defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t.$$

5.3 Backward facing step

Separation and reattachment of turbulent flows occur in many practical engineering applications. In these situations, the flow experiences an adverse pressure gradient, i.e. the pressure increases in the direction of the flow, which causes the boundary layer to separate from the solid surface. The flow subsequently reattaches downstream forming a recirculation bubble. Among the flow geometries used for the studies of separated flows, the most frequently selected is the backward-facing step.[13]

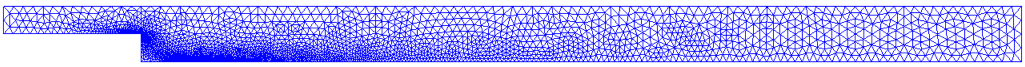

Figure 4.

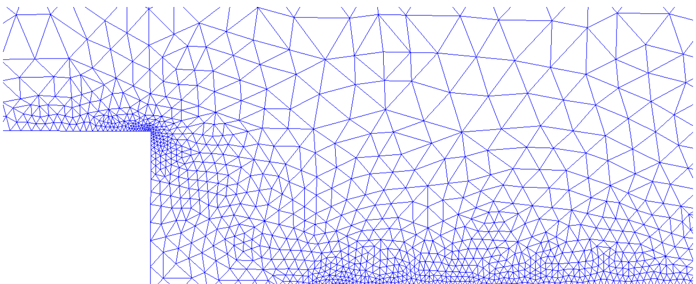

Figure 5 : A close up of the corner.

A Derivation of the Navier-Stokes Weak Formulation

In this section, we will derive the weak formulation listed as equations (6) - (7). For the sake of the clarity, we list the incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla\cdot\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] + \mathbf{f}, \tag{29}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{30}$$

The unsteady incompressible Navier-Stokes equations (29)-(30) are defined on the time interval [0,T], and the domain $\Omega \subset \mathbb{R}^d$ (d = 2,3) bounded by a sufficiently regular boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ consists of the Dirichlet and Neumann boundary. The trial funcitons are $(\mathbf{u}, p) \in$ VxQ, where V = $[\mathrm{H}^1(\Omega)]^d$ and Q= L^2. Define test functions $\mathbf{v}$,q $\in$ VxQ, and multiply (29) with $\mathbf{v}$, and (30) with q, and integrate over the domain Ω

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = -\int_\Omega \frac{1}{\rho}\nabla p\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \nabla\cdot\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right]\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}, \tag{31}$$

$$\int_\Omega (\nabla\cdot\mathbf{u})q\ \ \mathrm{d}\mathbf{x} = 0. \tag{32}$$

Next we perform integration by parts on the diffusion and the pressure gradient term :

$$-\int_\Omega \frac{1}{\rho}\nabla p\cdot\mathbf{v}d\mathbf{x} = \frac{1}{\rho}\int_\Omega p\nabla\cdot\mathbf{v}d\mathbf{x} - \frac{1}{\rho}\int_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n}ds \tag{33}$$

$$\int_\Omega \nabla\cdot\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right]\cdot\mathbf{v}d\mathbf{x} = -\int_\Omega\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] : \nabla\mathbf{v}d\mathbf{x} + \nu\int_{\partial\Omega}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{v}\cdot\mathbf{n}ds \tag{34}$$

where ds is a surface measure and $\mathbf{n}$ the normal vector; normal to the surface $\partial\Omega$. Inserting (33)-(34) in (31) :

$$\begin{aligned}\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \frac{1}{\rho}\int_\Omega p\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} = \\ -\frac{1}{\rho}\int_{\partial\Omega} p\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s + \nu\int_{\partial\Omega}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{35}$$

Collecting the boundary terms from (35) :

$$\int_{\partial\Omega}\mathbf{v}\cdot\underbrace{\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) - \frac{1}{\rho}pI\right]}_{=\sigma(\text{see eq.(3)})}\cdot\mathbf{n}\,\mathrm{d}s = \int_{\partial\Omega}\sigma\cdot\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}s. \tag{36}$$

Setting this to zero on the inflow/outflow boundaries ($\partial\Omega\backslash\Gamma_N$) is called pseudo-traction or a do-nothing boundary condition. It is often used on outlets where we are simply interested in letting the fluid escape with as little interference of boundary conditions as possible. It is called do-noting because you dont have to do anything to enforce it. [7]

We finally end up with the problem, find $(\mathbf{u}, p) \in$ VxQ such that

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega\left[\nu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] : \nabla\mathbf{v}\,\mathrm{d}\mathbf{x} = \frac{1}{\rho}\int_\Omega p\nabla\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x},$$

$$\int_\Omega (\nabla\cdot\mathbf{u})q\ \ \mathrm{d}\mathbf{x} = 0,$$

which is the result we have listed as equation (6) and (7).

B Derivations leading to the k-ε model

For convenience, we will first start by defining some well-worn symbols and notations that are familiar for the purpose of the subject.

The symbol $\tilde{u}_i$ will be used for the instantaneous velocity vector :

$$\tilde{u}_i = \begin{bmatrix} \tilde{u}(\boldsymbol{x},t) \\ \tilde{v}(\boldsymbol{x},t) \\ \tilde{w}(\boldsymbol{x},t) \end{bmatrix} \tag{37}$$

where $\boldsymbol{x} = \boldsymbol{x}(x,y,z)$ in cartecian coordinates or $\boldsymbol{x} = \boldsymbol{x}(r,\theta,z)$ in polar coordinates depending on the practical purpose. Further we will use u_i for fluctuating velocities, and U_i for the mean velocities. Same reasoning for the instantaneous, fluctuating and the mean pressure, respectively $\tilde{p}_i$, p_i and P_i. The instantaneous velocities can be decomposed as following:

$$\tilde{u}_i(\boldsymbol{x},t) = U_i(\boldsymbol{x},t) + u_i(\boldsymbol{x},t) \tag{38}$$

Where $U_i \equiv \overline{\tilde{u}}$, i.e averaged mean instantaneous velocity. Similarly with the instantaneous pressure where $P_i \equiv \overline{\tilde{p}}_i$ is the averaged mean instantaneous pressure.

[7] http://www.uio.no/studier/emner/matnat/math/MEK4300/v13/undervisningsmateriale/lecturenotes.pdf

B.1 Derivations of the Reynolds Averaged Navier Stokes equations

Starting from the instantaneous momentum equation i.e. The Navier-Stokes equation, and the continuity equations for an incompressible fluid:

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\tilde{p} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} \tag{39}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{40}$$

First we will assure that the continuity equation is fulfilled. By inserting the decomposed intantaneous velocities eq (38) in the continuity equation (40), and then taking the average, we get

$$\begin{aligned}
\overline{\frac{\partial (U_i + u_i)}{\partial x_i}} &= \frac{\partial \overline{(U_i + u_i)}}{\partial x_i} \\
&= \frac{\partial (\overline{U}_i + \overline{u}_i)}{\partial x_i} \\
&= \frac{\partial (U_i + 0)}{\partial x_i} \\
\frac{\partial (U_i)}{\partial x_i} &= 0
\end{aligned} \tag{41}$$

Which mean that the continuity equation is satisfied. No let us insert equation (38) in the Navier-Stokes equation (39), and then taking the average.

$$\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial (P+p)}{\partial x_i} + \nu \frac{\partial^2 (U_i + u_i)}{\partial x_j \partial x_j} \tag{42}$$

$$\overline{\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial (P+p)}{\partial x_i} + \nu \frac{\partial^2 (U_i + u_i)}{\partial x_j \partial x_j}}$$

$$\overline{\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j}} = \overline{-\frac{1}{\rho}\frac{\partial (P+p)}{\partial x_i} + \nu \frac{\partial^2 (U_i + u_i)}{\partial x_j \partial x_j}}$$

$$\overline{\frac{\partial (U_i + u_i)}{\partial t}} + \overline{(U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j}} = -\frac{1}{\rho}\overline{\frac{\partial (P+p)}{\partial x_i}} + \nu \overline{\frac{\partial^2 (U_i + u_i)}{\partial x_j \partial x_j}}$$

$$\frac{\partial}{\partial t}(\overline{U_i} = U_i + \overline{u_i} = 0) + \overline{(U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}(\overline{P} = P + \overline{p} = 0) + \nu \frac{\partial^2 (\overline{U_i} + \overline{u_i})}{\partial x_j \partial x_j} \tag{43}$$

Before handling the average of the convective term, let us employ the continuity equation to simplify this expression:

$$(U_j + u_j)\frac{\partial (U_i + u_i)}{\partial x_j} = \frac{\partial (U_i + u_i)(U_j + u_j)}{\partial x_j} \tag{44}$$

Using the rule: $\overline{U_i u_i} = \overline{U_i}\,\overline{u_i} = 0$. Averaging (43)

$$\begin{aligned}
\frac{\partial}{\partial x_j}\overline{(U_i + u_i)(U_j + u_j)} &= \frac{\partial}{\partial x_j}(U_i U_j + u_i u_j) \\
&= \frac{\partial}{\partial x_j}(U_i U_j) + \frac{\partial}{\partial x_j}(\overline{u_i u_J})
\end{aligned}$$

Now we can finally insert this expression into equation (42), and after rearraging some terms we obtain the Reynolds average Navier-Stokes equation.

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(\overline{u_i u_J}) \tag{45}$$

The last term on the right hand side of the equation is the Reynolds $\frac{\partial}{\partial x_j}(\overline{u_i u_J})$ is the Reynolds stress tensor that needs some more cumbersome processing which we will come back to, but first we will derive the fluctuating momentum equation.

B.2 The Fluctuating Momentum Equation

By substracting RANS equations (45) from the averaged Navier-Stokes equation (42), and rearranging some terms, we can show that the flutuating momentum equation becomes:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}(u_i u_j - (\overline{u_i u_J})) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{46}$$

In order to show that the fluctuating velocity field is divergence free, we can employ the definition of continuity equation for the instantaneous velocity field, (38), and the averaged velocity field, (40). Subtracting (38) from (40)

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial \tilde{u}_i}{\partial x_i} - \frac{\partial U_i}{\partial x_i} = 0 \tag{47}$$

Notice that since the partial derivative is a linear operator, we can rearrange the terms instead by using the definition (42) instead.

B.3 The derivations of the Reynolds Stress Transport Equations

In order to derive the Reynolds stress transport equation, we need to perform the following steps :

1. Multiply equation (46) with u_k and take the average of the resulting equation (47).
2. Interchange the indices i and k, and add the resulting interchanged equation (48) to the averaged equation from step 1.

Step 1
Mutiplying equation (46) with u_k:

$$u_k\left[\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}+u_j\frac{\partial U_i}{\partial x_j}+\frac{\partial}{\partial x_j}(u_iu_j-(\overline{u_iu_j})\right]=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

Take the average:

$$\overline{u_k\frac{\partial u_i}{\partial t}}+\overline{u_kU_j\frac{\partial u_i}{\partial x_j}}+\overline{u_ku_j\frac{\partial U_i}{\partial x_j}}+\overline{u_k\frac{\partial}{\partial x_j}(u_iu_j-(\overline{u_iu_j})}=-\frac{1}{\rho}\overline{u_k\frac{\partial p}{\partial x_i}}+\nu\overline{u_k\frac{\partial^2 u_i}{\partial x_j\partial x_j}}$$

We will show that this term $(\overline{u_iu_j})$ on the left hand side will fall out, because $\overline{u_k}=0$, then

$$\overline{u_k\overline{u_iu_j}}=\overline{u_k}\,\overline{u_iu_j}=0$$

Further, the U_j is interchangable so that it can be extracted out, and we get

$$\overline{u_k\frac{\partial u_i}{\partial t}}+U_j\overline{\frac{\partial}{\partial x_j}(u_iu_k)}=-\frac{1}{\rho}\overline{u_k\frac{\partial p}{\partial x_i}}+\nu\overline{u_k\frac{\partial^2 u_i}{\partial x_j\partial x_j}}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_j}-\overline{u_ku_j\frac{\partial u_i}{\partial x_j}} \tag{48}$$

The last term on right hand side follows by the continuity equation.

Step 2
For equation (48) interchange indices i and k (as they are both free indices) :

$$\overline{u_i\frac{\partial u_k}{\partial t}}+U_j\overline{\frac{\partial}{\partial x_j}(u_ku_i)}=-\frac{1}{\rho}\overline{u_i\frac{\partial p}{\partial x_k}}+\nu\overline{u_i\frac{\partial^2 u_k}{\partial x_j\partial x_j}}-\overline{u_iu_j}\frac{\partial U_k}{\partial x_j}-\overline{u_iu_j\frac{\partial u_k}{\partial x_j}} \tag{49}$$

Adding(48) to (47), and noticing that $\frac{\partial(u_iu_k)}{\partial t}=u_k\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_k}{\partial t}$:

$$\begin{aligned}\frac{\partial}{\partial t}\overline{(u_iu_k)}+U_j\overline{\frac{\partial}{\partial x_j}(u_iu_k)} &= -\frac{1}{\rho}\left(\overline{u_i\frac{\partial p}{\partial x_k}}+\overline{u_k\frac{\partial p}{\partial x_i}}\right)+\nu\left(\overline{u_i\frac{\partial^2 u_k}{\partial x_j\partial x_j}}+\overline{u_k\frac{\partial^2 u_i}{\partial x_k\partial x_k}}\right)\\ &\quad -\overline{u_iu_j}\frac{\partial U_k}{\partial x_j}-\overline{u_ku_j}\frac{\partial U_i}{\partial x_j}-\overline{u_iu_j\frac{\partial u_k}{\partial x_j}}-\overline{u_ku_j\frac{\partial u_i}{\partial x_j}}\end{aligned} \tag{50}$$

We can further simplify this expression. Let us rst start with the term with kinematic viscosity(the Laplacian terms). In order to simplify this expression, we need to use the product rule, and perform some manipulations with the new expressions :

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_j}(u_k)\right) &= \underbrace{\frac{\partial(u_i)}{\partial x_j}\frac{\partial(u_k)}{\partial x_j}}+u_i\frac{\partial^2 u_k}{\partial x_j\partial x_j}\\ &\Downarrow\quad \frac{\partial(u_k)}{\partial x_j}\frac{\partial(u_i)}{\partial x_j}\\ u_i\frac{\partial^2 u_k}{\partial x_j\partial x_j} &= \frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_j}(u_k)\right)-\frac{\partial(u_i)}{\partial x_j}\frac{\partial(u_k)}{\partial x_j}\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{\partial^2}{\partial x_j\partial x_j}(u_iu_k) &= \frac{\partial}{\partial x_j}\left(u_k\frac{\partial}{\partial x_j}(u_i)\right)+\frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_j}(u_k)\right)\\ &\Downarrow\\ \frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_j}(u_k)\right) &= \frac{\partial^2}{\partial x_j\partial x_j}(u_iu_k)-\frac{\partial}{\partial x_j}\left(u_k\frac{\partial}{\partial x_j}(u_i)\right)\end{aligned} \tag{52}$$

Insert (52) into (51) :

$$\begin{aligned}u_i\frac{\partial^2 u_k}{\partial x_j\partial x_j} &= \frac{\partial^2}{\partial x_j\partial x_j}(u_iu_k)-\frac{\partial}{\partial x_j}\left(u_k\frac{\partial}{\partial x_j}(u_i)\right)-\frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_j}(u_k)\right)\\ &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}(u_iu_k)-u_k\frac{\partial}{\partial x_j}(u_i)\right)-\frac{\partial}{\partial x_j}u_i\frac{\partial}{\partial x_j}(u_k)\end{aligned} \tag{53}$$

Performing similar calculations for $u_k \frac{\partial^2 u_i}{\partial x_j \partial x_j}$(or simply interchanging the i and k indices in equation (50)) :

$$u_k \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\left(u_k \frac{\partial}{\partial x_j}(u_i)\right) - \frac{\partial}{\partial x_j} u_k \frac{\partial}{\partial x_j}(u_i) \tag{54}$$

If we now add (52) and (53) we get

$$\begin{aligned} u_i \frac{\partial^2 u_k}{\partial x_j \partial x_j} + u_k \frac{\partial^2 u_i}{\partial x_j \partial x_j} &= \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}(u_i u_k) - u_k \frac{\partial}{\partial x_j}(u_i)\right) + \frac{\partial}{\partial x_j}\left(u_k \frac{\partial}{\partial x_j}(u_i)\right) - 2\frac{\partial}{\partial x_j} u_i \frac{\partial}{\partial x_j}(u_k) \\ &= \frac{\partial^2}{\partial x_j \partial x_j}(u_i u_k) - 2\frac{\partial}{\partial x_j} u_i \frac{\partial}{\partial x_j}(u_k) \end{aligned} \tag{55}$$

Finally due to continuity(as we have already shown that the fluctuating velocity field is divergence free, this assumption is thus valid), the last two terms in (50) can be simplified; again using the product rule :

$$u_i u_j \frac{\partial u_k}{\partial x_j} + u_k u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial (u_i u_k u_j)}{\partial x_j} \tag{56}$$

Inserting equation (55) and (56) in equation (54) we get the Reynolds stress transport equation.

$$\begin{aligned} \frac{\partial}{\partial t}\overline{(u_i u_k)} + U_j \overline{\frac{\partial}{\partial x_j}(u_i u_k)} &= -\frac{1}{\rho}\left(\overline{u_i \frac{\partial p}{\partial x_k}} + \overline{u_k \frac{\partial p}{\partial x_i}}\right) - 2\nu \overline{\left(\frac{\partial}{\partial x_j} u_i \frac{\partial}{\partial x_j}(u_k)\right)} \\ & \quad -\overline{u_i u_j}\frac{\partial U_k}{\partial x_j} - \overline{u_k u_j}\frac{\partial U_i}{\partial x_j} - \frac{\partial \overline{(u_i u_k u_j)}}{\partial x_j} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\overline{(u_i u_k)} \end{aligned} \tag{57}$$

B.4 Derivation of the $k - \epsilon$ model

In the previous subsection we derived the Reynolds stress transport equation (57) :

$$\begin{aligned} \partial_t(\overline{u_i u_k}) + U_j \partial_j \overline{u_i u_k} = &-\frac{1}{\rho}(\overline{u_i \partial_k p} + \overline{u_k \partial_i p}) \\ &-2\nu(\overline{\partial_j u_i \partial_j u_k}) \\ &-\overline{u_i u_j}\partial_j U_k - \overline{u_k u_j}\partial_j U_i \\ &-\partial_j \overline{(u_i u_k u_j)} \\ &+\nu \partial_j \partial_j \overline{u_i u_k}. \end{aligned}$$

Here we have used the short notation for the partial derivatives ∂_i. Multiplying this equation by half and setting i=k, we get the following equation (after rearranging the equation) :

$$\begin{aligned} \partial_t k + U_j \partial_j k = &\overbrace{\tau_{ij}\partial_j U_i}^{\text{production}} - \overbrace{\epsilon}^{\text{dissipation}} \\ &+ \partial_j \left[\nu \partial_j k - \underbrace{\frac{1}{2}\overline{u_i u_i u_j}}_{\text{turbulent transport}} - \overbrace{\frac{1}{\rho}\overline{p u_j}}^{\text{pressure diffusion}}\right] \end{aligned}$$

For the pressure term, we used that $\partial_j u_j = 0$. This equation is known as the transport equation for turbulence kinetic energy. ϵ is the dissipation per unit mass and is defined by the following correlation :

$$\epsilon = \nu \overline{\partial_j u_i \partial_j u_i}$$

τ_{ij} is the Reynolds-stress tensor $-\overline{u_i u_j}$, hence the specific(or per unit mass) turbulence kinetic energy (or simply the turbulence kintic energy) is half the trace of the Reynolds-stress tensor.

Reynolds-stess tensor : RANS turbulence models must predict the Reynolds-stresses, which prevents closure for the averaged momentum equations. The most common way of predicting these stresses is to assume that the Boussinesq approximation is valid. This implies that the Reynolds-stess tensor is given by

$$\tau_{ij} = 2\nu_T S_{ij} - \frac{2}{3}k\delta_{ij}$$

where S_{ij} is the mean strain rate of strain tensor :

$$S_{ij} = \partial_j U_i + \partial_i U_j$$

The RANS turbulence models of this type are referred to as eddy viscosity models. Assuming that Boussinesq approxmimation is valid, it follows that production P is given as

$$P = \left(2\nu_T S_{ij} - \frac{2}{3}k\right)\partial_j U_i \tag{58}$$

Gradient Transport Law : The "traditional" or standard approximation made to represent turbulent transport of scalar quantities in a turbulent flow is given on the form

$$\frac{1}{2}\overline{u_i u_i u_j} + \frac{1}{\rho}\overline{p u_j} = -\frac{\nu_T}{\sigma_k}\partial_j k$$

where σ_k is a closure coeffient and may be considered a turbulent Prandtl number. Inserting the last two equations into the transport equation for turbulence kinetic equation we finally end up with equation (20).

In order to derive the dissipation rate ϵ we need to perform the following calculations

1. Differentiate the Navier-Stokes equations by ∂_j, giving an equation for $\partial_j \tilde{u}_i$
2. Multiply this equation by $2\nu\partial_j u_i$
3. Take the average of the resulting equation.

Although this procedure has been listed, we will not attempt at deriving the equation for dissipation rate. In [14, page 28-29], a detailed physical interpretation is provided for all the terms.

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