Topics:

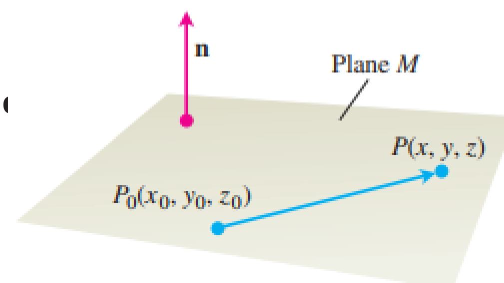
- **Vector Equations of Plane in Space,**
- **Distance from a Point to a Plane in Space**

Equation of a Plane in Space:

A **plane in space** is determined by knowing **a point on the plane** and its “**tilt**” or **orientation**. This “**tilt**” is defined by specifying a **vector** that is **perpendicular or normal to the plane**.

Suppose that a plane $\boldsymbol{M}$ passes through a point $\boldsymbol{P_o}\left(\boldsymbol{x_o, y_o, z_o}\right)$ and is **normal** to the non-zero vector $\vec{\boldsymbol{n}} = \boldsymbol{a}\,\hat{\boldsymbol{\imath}} + \boldsymbol{b}\hat{\boldsymbol{\jmath}} + \boldsymbol{c}\hat{\boldsymbol{k}}$, i.e., $\vec{\boldsymbol{n}}$ is a non-zero **normal orthogonal vector**. Then $\boldsymbol{M}$ is the set of all points $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ for which $\vec{\boldsymbol{u}} = \overrightarrow{\boldsymbol{P_o P}}$ is **orthogonal** to $\vec{\boldsymbol{n}}$. i.e., $\boldsymbol{P_0 P} \perp \vec{\boldsymbol{n}}$. Then

$$\vec{\boldsymbol{u}} = <\boldsymbol{x} - \boldsymbol{x_0}, \boldsymbol{y} - \boldsymbol{y_0}, \boldsymbol{z} - \boldsymbol{z_0}>$$



Since $\vec{\boldsymbol{n}}$ is **orthogonal to the plane** thus **it is orthogonal to**

vector on the plane. The vector $\vec{\boldsymbol{u}}$ lies on the plane.

Thus, the dot product

$$\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{u}} = \boldsymbol{0}$$

$$\left(\boldsymbol{a}\,\hat{\boldsymbol{\imath}} + \boldsymbol{b}\hat{\boldsymbol{\jmath}} + \boldsymbol{c}\hat{\boldsymbol{k}}\right).\left[(\boldsymbol{x} - \boldsymbol{x_0})\hat{\boldsymbol{\imath}} + (\boldsymbol{y} - \boldsymbol{y_0})\hat{\boldsymbol{\jmath}} + (\boldsymbol{z} - \boldsymbol{z_0})\hat{\boldsymbol{k}}\right] = \boldsymbol{0}$$

$$\boxed{\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x_0}) + \boldsymbol{B}(\boldsymbol{y} - \boldsymbol{y_0}) + \boldsymbol{C}(\boldsymbol{z} - \boldsymbol{z_0}) = \boldsymbol{0} \quad \text{----------------------------- (1)}}$$

This equation (1) is known as the equation of plane in space.

Remark:

Another form of the equation of plane.

$Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$

$Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$

$Ax + By + Cz = Ax_0 + By_0 + Cz_0$

$Ax + By + Cz = D$ where $D = Ax_0 + By_0 + Cz_0$

Example 1:

Find **an equation for the plane** passing through the point $\boldsymbol{P_0(-3, 0, 7)}$ and **perpendicular to the unit normal** vector $\boldsymbol{\vec{n} = 5\,\hat{\imath} + 2\hat{\jmath} - \hat{k}}$**.**

Solution: Since the given point is $\boldsymbol{P_0(x_0, y_0, z_0) = P_0(-3, 0, 7)}$

Tha is, $\boldsymbol{x_0 = -3}, \quad \boldsymbol{y_0 = 0}, \quad \boldsymbol{z_0 = 7}$

And the unit normal vector is $\boldsymbol{\vec{n} = a\,\hat{\imath} + b\hat{\jmath} + c\hat{k} = 5\,\hat{\imath} + 2\hat{\jmath} - \hat{k}}$

Tha is, $\boldsymbol{a = 5}, \quad \boldsymbol{b = 2}, \quad \boldsymbol{c = -1}$

Now, the equation of the plane in space is

$$a(x - x_0) + B(y - y_0) + C(\,z - z_0) = 0$$

Putting values in the above equation, we get

$$5(x - (-3)) + 2(y - 0) + (-1)(\,z - 7) = 0$$

$$5(x + 3) + 2y - z + 7 = 0$$

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z + 22 = 0$$

$$\boxed{\boldsymbol{5x + 2y - z = -22}}$$

Example 2: Find an **equation for the plane** passing through three points $\boldsymbol{A\,(0, 0, 1), B\,(2, 0, 0)}$ and $\boldsymbol{C\,(0, 3, 0)}$.

Solution: Given points are $\boldsymbol{A\,(0, 0, 1), B\,(2, 0, 0)}$ and $\boldsymbol{C\,(0, 3, 0)}$**.** Consider we fix a point $\boldsymbol{A}$**,** then

$$\boldsymbol{\overrightarrow{AB} = < 2 - 0, 0 - 0, 0 - 1 > = \, < 2, 0, -1 > = 2\,\hat{\imath} + 0\hat{\jmath} - \hat{k}}$$

$$\boldsymbol{\overrightarrow{AC} = < 0 - 0, 3 - 0, 0 - 1 > = < 0, 3, -1 > = 0\,\hat{\imath} + 3\hat{\jmath} - \hat{k}}$$

Now, a vector normal to the plane is:

$$\boldsymbol{\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}}$$

$$\boldsymbol{\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}}$$

$$\vec{n} = \hat{\imath}\begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}$$

$$\vec{n} = \hat{\imath}[(0-(-3))] - \hat{\jmath}[-2-0] + \hat{k}[6-0]$$

$$\vec{n} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$$

Now, the equation of the plane in space is

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$3(x-0) + 2(y-0) + 6(z-1) = 0$$

$$\boxed{3x + 2y + 6z = 6}$$

Practice questions:

Thomas Calculus Ex. 12.5: 21-26

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$

24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$

25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A

Distance from a Point to a Plane in Space

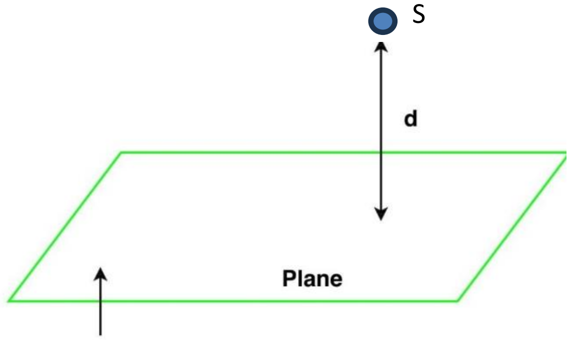


Figure 1

Let $\boldsymbol{P}$ be a point on the plane, and $\boldsymbol{S}$ be any point in the space, then $\vec{\boldsymbol{n}}$ is the unit normal vector to the point $\boldsymbol{P}$. Let the vector $\overrightarrow{\boldsymbol{PS}}$ be denoted by the vector $\vec{\boldsymbol{u}}$, i.e.

$$\overrightarrow{\boldsymbol{PS}} = \vec{\boldsymbol{u}}$$

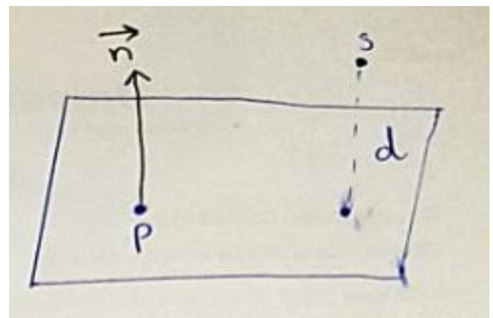


Figure 2

Now considering the shortest distance from the point $\boldsymbol{S}$ to the plane is perpendicular to the plane, then

$$\boldsymbol{d} = \frac{|\,\vec{\boldsymbol{u}}\;.\;\vec{\boldsymbol{n}}\,|}{|\vec{\boldsymbol{n}}|}$$

Proof:

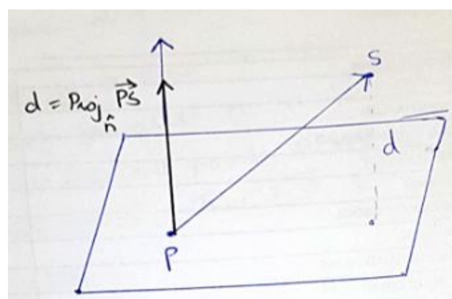


Figure 3

From the figure, we can write

$$\boldsymbol{cos}\,(\boldsymbol{\theta}) = \frac{\boldsymbol{Base}}{\boldsymbol{Hypotenuse}} = \frac{\boldsymbol{d}}{|\vec{\boldsymbol{u}}\,|}$$

$$\boldsymbol{cos}\,(\boldsymbol{\theta}) = \frac{\boldsymbol{d}}{|\vec{\boldsymbol{u}}\,|}$$

$$|\vec{\boldsymbol{u}}\,|\,\boldsymbol{cos}\,(\boldsymbol{\theta}) = \boldsymbol{d}$$

$$\boldsymbol{d} = |\vec{\boldsymbol{u}}\,|\,\boldsymbol{cos}\,(\boldsymbol{\theta})$$

$$\boldsymbol{d} = \frac{|\vec{\boldsymbol{u}}\,||\vec{\boldsymbol{n}}|\,\boldsymbol{cos}\,(\boldsymbol{\theta})}{|\vec{\boldsymbol{n}}|}$$

$$\boldsymbol{d} = \frac{|\vec{\boldsymbol{u}}\;.\vec{\boldsymbol{n}}|}{|\vec{\boldsymbol{n}}|}$$

Hence

$$\boldsymbol{d} = \frac{|\vec{\boldsymbol{u}}\ .\vec{\boldsymbol{n}}|}{|\vec{\boldsymbol{n}}|}$$

Example:

Find the distance from the point $\boldsymbol{S}\ (\mathbf{1}, \mathbf{1}, \mathbf{3})$ to the plane $\mathbf{3}\boldsymbol{x} + \mathbf{2}\boldsymbol{y} + \mathbf{6}\boldsymbol{z} = \mathbf{6}$.

Solution:

The corresponding unit normal vector $\vec{\boldsymbol{n}}$ to the plane is

$$\vec{\boldsymbol{n}} = \mathbf{3}\hat{\boldsymbol{\imath}} + \mathbf{2}\hat{\boldsymbol{\jmath}} + \mathbf{6}\hat{\boldsymbol{k}}$$

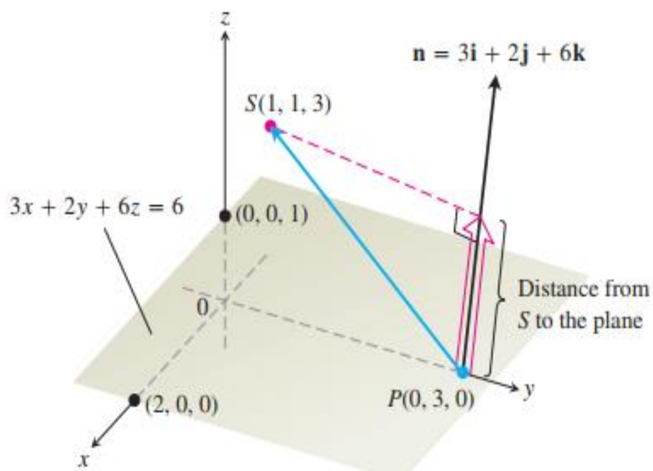
And the given point is $\boldsymbol{S} = (\mathbf{1}, \mathbf{1}, \mathbf{3})$. We have to find the point $\boldsymbol{P}$ on the plane. i.e.,

$$\boldsymbol{P} = ?$$

We find a point $\boldsymbol{P}$ in the plane and calculate the length of the vector projection of $\overrightarrow{\boldsymbol{PS}}$ onto a vector $\vec{\boldsymbol{n}}$ normal to the plane. The point on plane easiest to find from the plane's equation are the intercepts.

To find the point $\boldsymbol{P}$, we consider $\boldsymbol{y}$ –intercept.

That is, $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\mathbf{0}, \mathbf{3}, \mathbf{0})$ then



$$\overrightarrow{\boldsymbol{PS}} = \vec{\boldsymbol{u}} = (1-0)\hat{\boldsymbol{\imath}} + (1-3)\hat{\boldsymbol{\jmath}} + (3-0)\hat{\boldsymbol{k}}$$

$$= \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}$$

$$|\vec{\boldsymbol{n}}| = \sqrt{(\mathbf{3})^2 + (\mathbf{2})^2 + (\mathbf{6})^2} = \sqrt{\mathbf{9} + \mathbf{4} + \mathbf{36}} = \sqrt{\mathbf{49}} = \mathbf{7}$$

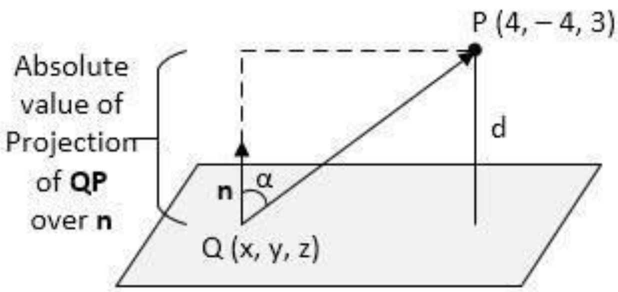
The distance from S to the plane is

$$d = \left(\frac{|\vec{u}.\vec{n}\ |}{|\vec{n}|}\right) = \left|(\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}).\left(\frac{3}{7}\ \hat{\boldsymbol{\imath}} + \frac{2}{7}\ \hat{\boldsymbol{\jmath}} + \frac{6}{7}\ \hat{\boldsymbol{k}}\right)\right|$$

$$d = \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \left|\frac{17}{7}\right| = \frac{17}{7}$$

Note: We can consider x-intercept or y-intercept or z-intercept to find the point P.

Example:



Suppose the equation of the Plane is 2x−2y+5z+8=0 and point P = (4, – 4, 3). We want to find the distance (shortest) from P to the plane.

SOLUTION: From the equation of plane we know the Normal vector **n** = <2, – 2, 5>.
Now, **QP** = <4 – x, – 4 – y, 3 – z>

Absolute value of Projection of **QP** on **n** = $\frac{\mathbf{QP} \bullet \mathbf{n}}{\|\mathbf{n}\|}$ This is d.

So, Distance d = $\frac{\mathbf{QP} \bullet \mathbf{n}}{\|\mathbf{n}\|} = \frac{\langle 4-x, -4-y, 3-z\rangle \bullet \langle 2, -2, 5\rangle}{\sqrt{2^2+(-2)^2+5^2}}$

We still have three unknowns. How was the distance found in the example that is demonstrated in the link that I provided?

Practice Questions:

Thomas Calculus 12 Edition: Ex. 12.5: 39-44, 45, 46

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $x + 2y + 2z = 13$

40. $(0, 0, 0)$, $3x + 2y + 6z = 6$

41. $(0, 1, 1)$, $4y + 3z = -12$

42. $(2, 2, 3)$, $2x + y + 2z = 4$

43. $(0, -1, 0)$, $2x + y + 2z = 4$

44. $(1, 0, -1)$, $-4x + y + z = 4$

45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

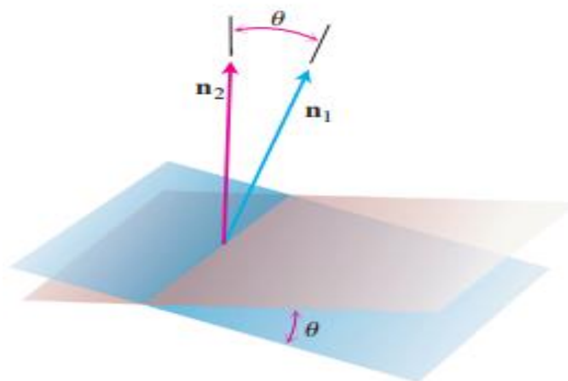
46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angle between Two Planes

The angle between two intersecting planes is defined

to be the angle between their normal vectors.

$$\boldsymbol{\theta = \cos^{-1}\left(\frac{\overrightarrow{n_1} \, . \, \overrightarrow{n_2}}{|\overrightarrow{n_1}| \, |\overrightarrow{n_2}|}\right)}$$



Example:

Find the angle between the planes $\boldsymbol{3x - 6y - 2z = 15}$ and $\boldsymbol{2x + y - 2z = 5}$.

Solution:

$$\boldsymbol{\overrightarrow{n_1} = 3\hat{\imath} - 6\hat{\jmath} - 2\hat{k}}$$

$$\boldsymbol{\overrightarrow{n_2} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}}$$

$$\boldsymbol{\overrightarrow{n_1} . \overrightarrow{n_2} = 6 - 6 + 4}$$

$$\boldsymbol{\overrightarrow{n_1} . \overrightarrow{n_2} = 4}$$

$$\boldsymbol{|\overrightarrow{n_1}| = \sqrt{(3)^2 + (-6)^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7}$$

$$\boldsymbol{|\overrightarrow{n_2}| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3}$$

Now

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{n_1} \, . \, \overrightarrow{n_2}}{|\overrightarrow{n_1}| \, |\overrightarrow{n_2}|}\right)$$

Putting values, we get

$$\theta = \cos^{-1}\left(\frac{4}{7 \times 3}\right)$$

$$\theta = \cos^{-1}\left(\frac{4}{21}\right)$$

$$\boxed{\boldsymbol{\theta = 79^{\circ}}}$$

<u>Practice Questions:</u>

Thomas Calculus 12 Edition Ex. 12.5: Q # 47-52

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, \quad 2x + y - 2z = 2$

48. $5x + y - z = 10, \quad x - 2y + 3z = -1$

Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, \quad 2x - 2y - z = 5$

50. $x + y + z = 1, \quad z = 0$ (the xy-plane)

51. $2x + 2y - z = 3, \quad x + 2y + z = 2$

52. $4y + 3z = -12, \quad 3x + 2y + 6z = 6$