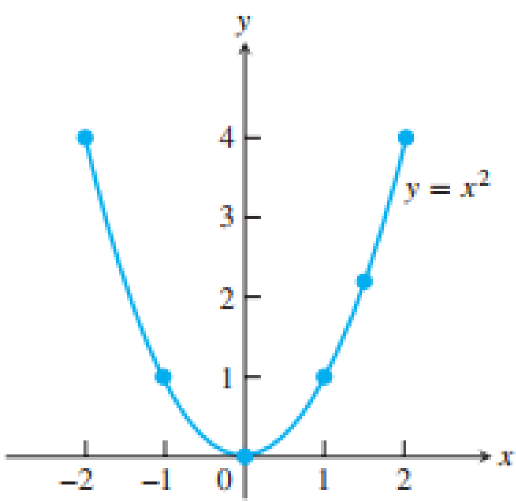
Topics:

- **Introduction to Multivariate Calculus (MVC)**
- **Introduction to vectors**
- **Dot product**
- **Cross product**
- **Projections**

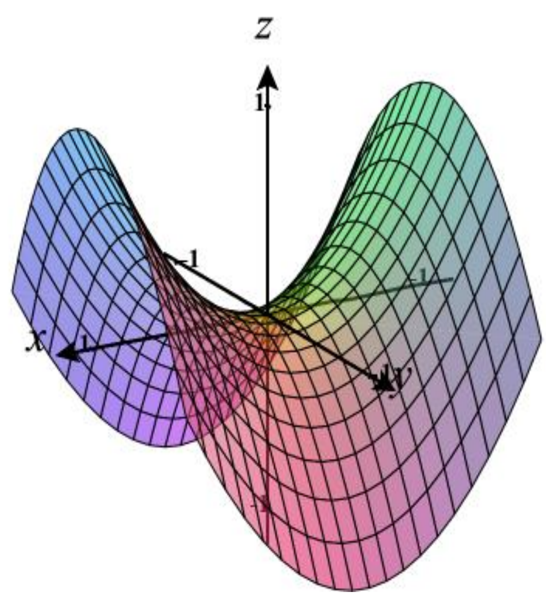
Introduction to Multivariable Calculus (MVC)

What do you understand from the word Multivariable Calculus?

Give some examples around you.



$$y = f(x) = x^2$$



$$f(x, y) = x^2 - y^2$$

Real-Life Examples:

Example 1:

Consider an airline's ticket price pattern. To avoid flying planes with many empty seats, it sells some tickets at full price and some at a discount. For a particular route, the airline's revenue, $\boldsymbol{R}$, earned in a given time period is determined by the number of full-price tickets, $\boldsymbol{x}$, and the number of discount tickets, $\boldsymbol{y}$, sold. We say that $\boldsymbol{R}$ is a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, and we write

$$\boldsymbol{R} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$$

This is just like the function notation of one-variable calculus. The variable $\boldsymbol{R}$ is the dependent variable and the variables x and y are the independent variables. The letter f stands for the ***function*** or ***rule*** that gives the value, or output, of $\boldsymbol{R}$ corresponding to given values of $\boldsymbol{x}$ and $\boldsymbol{y}$.

The revenue, $\boldsymbol{R}$, (in dollars) from a particular airline route is shown in table **1** as a function of the number of full-price tickets and of discount tickets sold.

Revenue from ticket sales as a function of x and y

		Number of full-price tickets, x			
		100	200	300	400
Number of discount tickets, y	200	75,000	110,000	145,000	180,000
	400	115,000	150,000	185,000	220,000
	600	155,000	190,000	225,000	260,000
	800	195,000	230,000	265,000	300,000
	1000	235,000	270,000	305,000	340,000

(Table 1)

Values of $\boldsymbol{x}$ are shown across the top, values of $\boldsymbol{y}$ are down the left side, and the corresponding values of $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ are in the table. For example, to find the value of $\boldsymbol{f}(\mathbf{300}, \mathbf{600})$, we look in the column corresponding to $x = 300$ at the row $\boldsymbol{y} = \mathbf{600}$, where we find the number $\mathbf{225,000}$. Thus,

$$\boldsymbol{f}(\mathbf{300}, \mathbf{600}) = \mathbf{225,000}$$

This means that the revenue from $\mathbf{300}$ full-price tickets and $\mathbf{600}$ discount tickets is $\$\mathbf{225,000}$.

Example 2:

Suppose you want to calculate your monthly payment on a five-year car loan; this depends on both the amount of money you borrow and the interest rate. These quantities can vary separately: the loan amount can change while the interest rate remains the same, or the interest rate can change while the loan amount remains the same. To calculate your monthly payment, you need to know both. If the monthly payment is $\$\boldsymbol{m}$, the loan amount is $\$\boldsymbol{L}$, and the interest rate is $\boldsymbol{r}$ %, then we express the fact that m is a function of $\boldsymbol{L}$ and $\boldsymbol{r}$ by writing

$$\boldsymbol{m} = \boldsymbol{f}\,(\boldsymbol{L}, \boldsymbol{r})$$

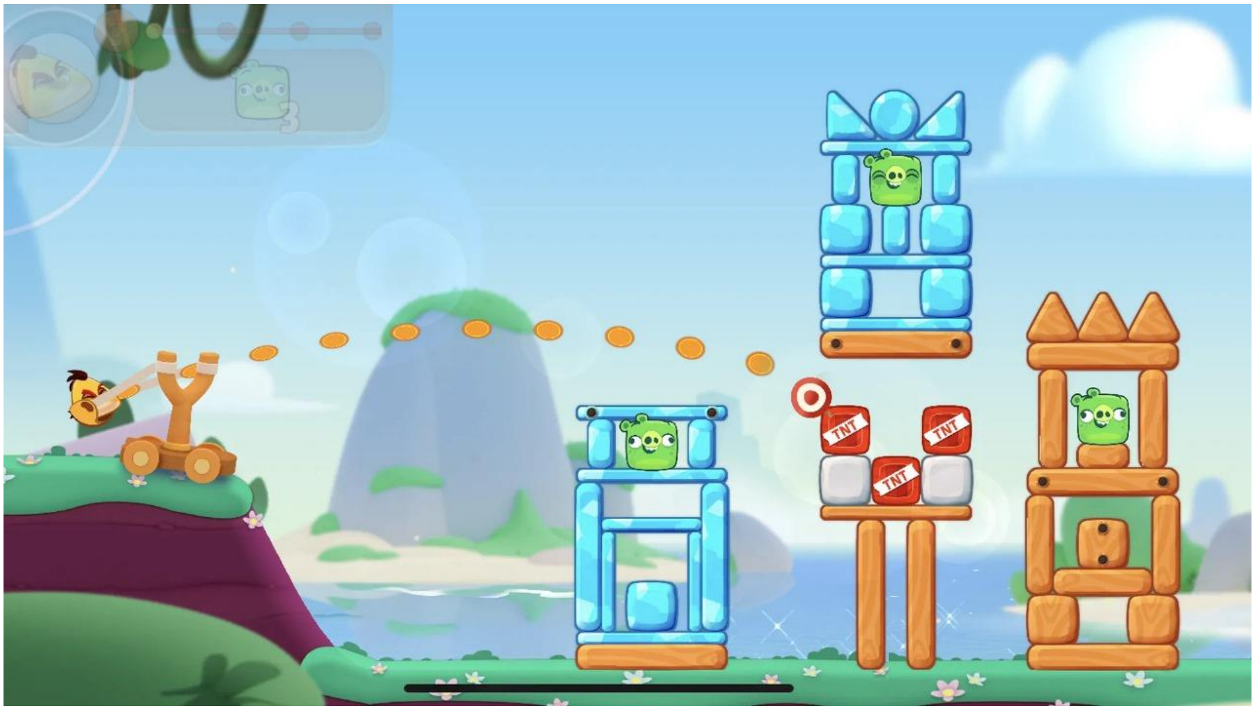
This is just like the function notation of one-variable calculus. The variable $\boldsymbol{m}$ is called the dependent variable, and the variables $\boldsymbol{L}$ and $\boldsymbol{r}$ are called the independent variables. The letter $\boldsymbol{f}$ stands for the *function* or rule that gives the value of $\boldsymbol{m}$ corresponding to given values of $\boldsymbol{L}$ and $\boldsymbol{r}$.

Definition: Multivariable Calculus

Multivariable Calculus is the **extension of calculus** in one variable to calculus with functions of several variables.

Single Variable Function (One independent variable)	**Multivariable Function (More than one independent variable)**
$f(x) = x^2$	$R = f(x, y) = x + y$
For $x = 1$, $\Longrightarrow f\,(1) = 1$ For $x = 2$, $\Longrightarrow f\,(2) = 4$	For $x = 1$, and $y = 2$, the function is $\Longrightarrow f\,(1{,}2) = 1 + 2 = 3$

Introduction to Vectors



In Games, vectors are used for determining **positions, directions and velocities**, etc.

- The **position vectors** indicate how far the object is.
- The **velocity** vector indicates how much time it will take or how much force we should give.
- The **direction vector** indicates in which way we should apply the force.

Just for Fun:

- **Zelda Game using vectors**
  In the link given below, you will see the practical coding of the game Zelda using vectors and their properties.
  https://www.youtube.com/watch?v=d42w8QyPI7I
- **Genshin Impact Gameplay**

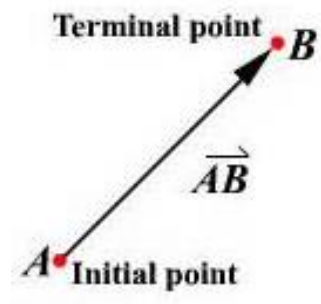
  https://www.youtube.com/watch?v=QsHWebfAwpY

- **Angry Birds Slingshot Stories**

  https://www.youtube.com/watch?app=desktop&v=b-GDMlWOPr0

Vector:

The physical quantities that have magnitude, SI unit and direction is called a vector.



Examples: Displacement, Force, Acceleration, Velocity

- A vector in plane/space is directed line segment. The directed line segment $\overrightarrow{AB}$ has initial point A and the terminal point B.

Examples:

- Let we have points $\boldsymbol{A}\ (\boldsymbol{x_1}, \boldsymbol{y_1})$ and $\boldsymbol{B}(\boldsymbol{x_2}, \boldsymbol{y_2})$ in plane, then

$$\overrightarrow{\boldsymbol{AB}} = <\ \boldsymbol{x_2} - \boldsymbol{x_1}, \quad \boldsymbol{y_2} - \boldsymbol{y_1} >$$

- Let we have points $\boldsymbol{A}\ (\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1})$ and $\boldsymbol{B}(\boldsymbol{x_2}, \boldsymbol{y_2}, \boldsymbol{z_2})$ in space, then

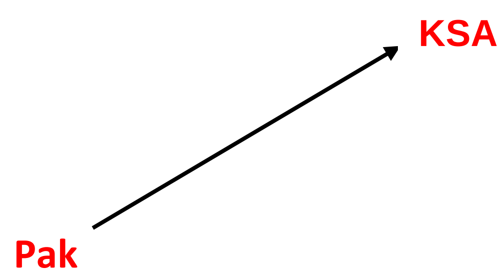
$$\overrightarrow{\boldsymbol{AB}} = <\ \boldsymbol{x_2} - \boldsymbol{x_1}, \quad \boldsymbol{y_2} - \boldsymbol{y_1}, \quad \boldsymbol{z_2} - \boldsymbol{z_1} >$$

Displacement vector

Suppose you are a pilot and you are planning a flight from **Pakistan (Pak) to Kingdom of Saudi Arabia (KSA)**.

You must know two things:

I. The distance to be travelled.
II. The direction to go. Both these quantities together specify the displacement or displacement vector between the two countries.

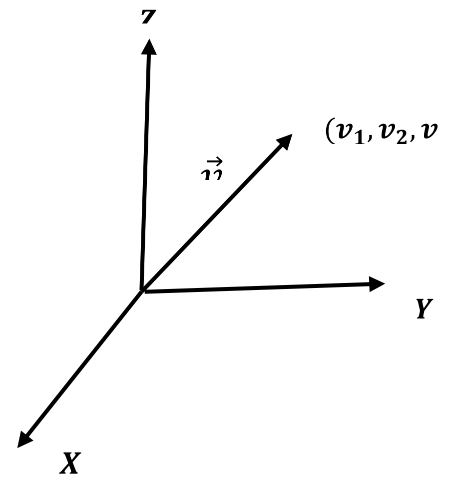


Note: The **magnitude (length/norm)** of the displacement vector is the distance between the points and is represented by the length of the arrow.

Component form and Standard form of a Vector

Let $\vec{\boldsymbol{v}}$ be a three-dimensional vector with initial point at the origin $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ and terminal point is $(\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3})$ then the component form of $\vec{\boldsymbol{v}}$ is

$$\vec{v} = < \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} >$$



and the standard form is

$$\vec{v} = v_1\, \hat{\boldsymbol{\imath}} + v_2\, \hat{\boldsymbol{\jmath}} + v_3\, \widehat{\boldsymbol{k}}$$

Magnitude or Length of a Vector

The length or magnitude of a vector

$$\vec{v} = v_1\, \hat{\boldsymbol{\imath}} + v_2 \hat{\boldsymbol{\jmath}} + v_3\, \widehat{\boldsymbol{k}}$$

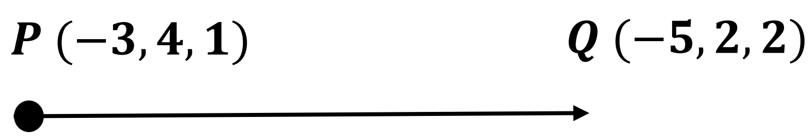
is

$$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$$

Note: It is a non-negative number.

Example 1: Component Form of a Vector and Length of a Vector

a) Find the component form of the vector with initial point $\boldsymbol{P}(-\mathbf{3}, \mathbf{4}, \mathbf{1})$ and terminal point $\boldsymbol{Q}(-\mathbf{5}, \mathbf{2}, \mathbf{2})$.
b) Find the length of the vector $\overrightarrow{\boldsymbol{PQ}}$.



Solution:

a) The initial point $\boldsymbol{P}$ has the coordinates $(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1}) = (-\mathbf{3}, \mathbf{4}, \mathbf{1})$. This implies that

$$\boldsymbol{x_1} = -\mathbf{3}, \qquad \boldsymbol{y_1} = \mathbf{4}, \qquad \boldsymbol{z_1} = \mathbf{1}$$

Similarly, the terminal point $\boldsymbol{Q}$ has the coordinates $(\boldsymbol{x_2}, \boldsymbol{y_2}, \boldsymbol{z_2}) = (-\mathbf{5}, \mathbf{2}, \mathbf{2})$. This implies that

$$x_2 = -5, \qquad y_2 = 2, \qquad z_2 = 2$$

The formula to find the vector $\overrightarrow{\boldsymbol{PQ}}$ is

$$\overrightarrow{\boldsymbol{PQ}} = < \boldsymbol{x_2 - x_1}, \quad \boldsymbol{y_2 - y_1}, \quad \boldsymbol{z_2 - z_1} >$$

Putting values in the formula, we get

$$\overrightarrow{\boldsymbol{PQ}} = < \boldsymbol{-5-(-3)}, \quad \boldsymbol{2-4}, \quad \boldsymbol{2-1} >$$

$$\overrightarrow{\boldsymbol{PQ}} = < \boldsymbol{-5+3}, \quad \boldsymbol{-2}, \quad \boldsymbol{1} >$$

$$\overrightarrow{\boldsymbol{PQ}} = < \boldsymbol{-2}, \quad \boldsymbol{-2}, \quad \boldsymbol{1} >$$

which is the required component form of vector $\overrightarrow{\boldsymbol{PQ}}$**.**

b) Since the vector $\overrightarrow{\boldsymbol{PQ}}$ is obtained in part **(a)**, that is

$$\vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{PQ}} = < \boldsymbol{-2}, \quad \boldsymbol{-2}, \quad \boldsymbol{1} >$$

or

$$\vec{\boldsymbol{v}} = < \boldsymbol{v_1, v_2, v_3} > \; = < \boldsymbol{-2}, \quad \boldsymbol{-2}, \quad \boldsymbol{1} >$$

The length or magnitude of vector $\vec{\boldsymbol{v}} = \overrightarrow{\boldsymbol{PQ}}$ is given by the formula

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Putting values, we get

$$|\vec{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2}$$

$$|\vec{v}| = \sqrt{4+4+1}$$

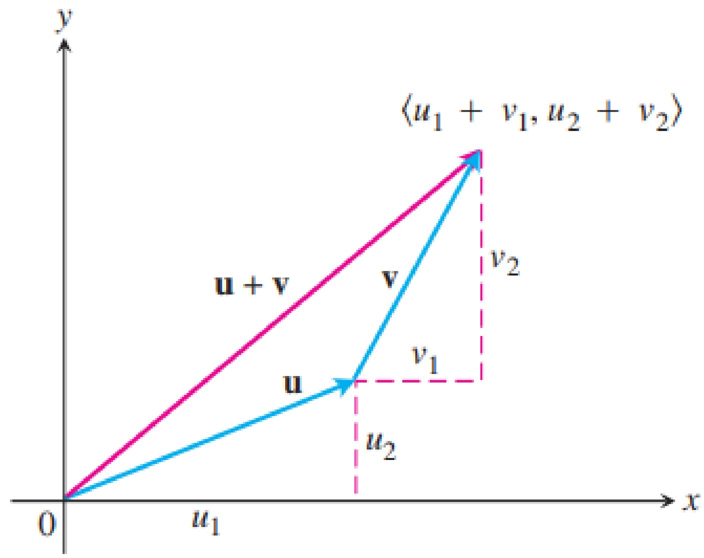
$$|\vec{v}| = \sqrt{9}$$

$$|\vec{\boldsymbol{v}}| = \boldsymbol{3}$$

So, the length of vector $\vec{v} = \overrightarrow{PQ}$ is **3**.

Vector Algebra

Two principal operations involving vectors are vector addition and scalar multiplication.



Vector Addition:

1) Let $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$

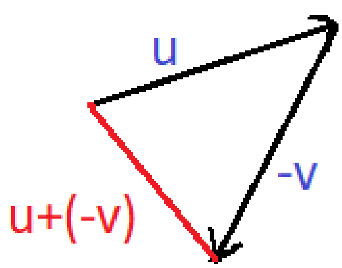
$\vec{u} + \vec{v} = =< u_1 + v_1, u_2 + v_2, u_3 + v_3 >$

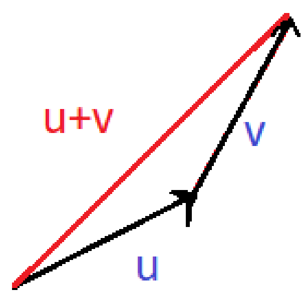
Example:

Let $\vec{u} =< u_1, u_2 >$ and $\vec{v} =< v_1, v_2 >$

Vectors can be added geometrically using head to tail rule.

2) **u** - **v** = **u** + (-**v**)





Question: Find $\vec{\boldsymbol{u}} - \vec{\boldsymbol{w}}$ when $\vec{\boldsymbol{u}} = 3\hat{\imath} - 4.5\hat{\jmath}$ and $\vec{\boldsymbol{w}} = -2.8\hat{\imath} + 9\hat{k}$. Also compute the magnitude of the resultant vector.

Scalar Multiplication of Vectors

A scalar is simply a real number, it can be positive, negative or zero.

Let $\vec{\boldsymbol{v}} =< \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} >$

then $\boldsymbol{k}\vec{\boldsymbol{v}} = \boldsymbol{k} < \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} > =< \boldsymbol{kv_1}, \boldsymbol{kv_2}, \boldsymbol{kv_3} >$ where $\boldsymbol{k}$ is any scalar.

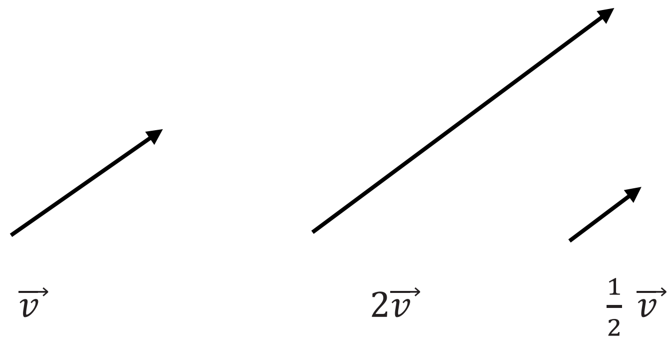
Note: If $\boldsymbol{k} > \boldsymbol{0}$ then $\boldsymbol{k}\vec{\boldsymbol{v}}$ has the same direction as $\vec{\boldsymbol{v}}$; if $\boldsymbol{k} < \boldsymbol{0}$ then the direction of $\boldsymbol{k}\vec{\boldsymbol{v}}$ is opposite to that of $\vec{\boldsymbol{v}}$.

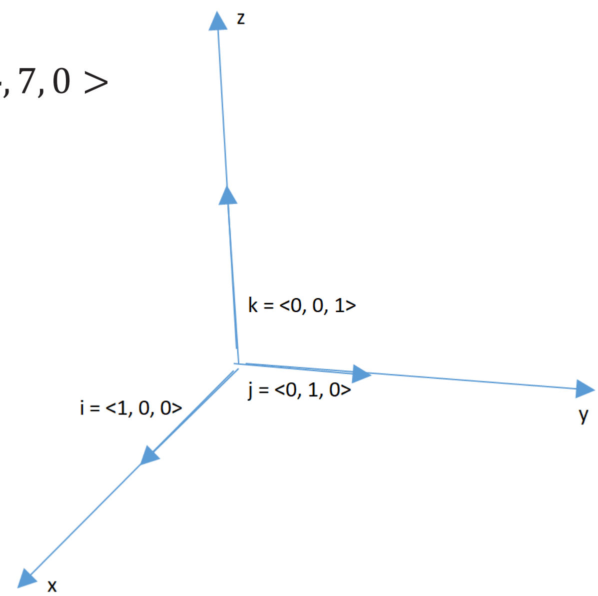
Examples:

If $\overrightarrow{v} =< \ 1,2,3 >$,

Then,

- $2\overrightarrow{v} =< \ 2,4,6 >$
- $\frac{1}{2}\overrightarrow{v} =< \ \frac{1}{2},\frac{2}{2},\frac{3}{2} >$





Question: Let $\overrightarrow{u} =< \ -1,3,1 >$, and $\overrightarrow{v} =< \ 4,7,0 >$

a. Find $2\overrightarrow{u}$ + $3\overrightarrow{v}$
b. $\overrightarrow{u} - \overrightarrow{v}$
c. $|\frac{1}{2}\overrightarrow{v}|$

<u>Unit Vector</u>

A vector $\overrightarrow{\boldsymbol{v}}$ of length $\mathbf{1}$ is called unit vector.

It is represented by $\widehat{\boldsymbol{v}}$.

- The standard unit vectors in $\boldsymbol{R}^3$ (space) are

$$\hat{\boldsymbol{\imath}} \ =< 1,0,0 >, \qquad \hat{\jmath} \ =< \ 0,1,0 >, \qquad \widehat{\boldsymbol{k}} \ =< 0,0,1 >$$

- Any vector $\overrightarrow{v} =< \ v_1, v_2, v_3 >$ can be written as a linear combination of the standard unit vectors as follows:

$$\overrightarrow{v} =< \ v_1, v_2, v_3 >= \ v_1 < \ 1,0,0 > +v_2 < \ 0,1,0 > +v_3 < \ 0,0,1 >$$

$$\overrightarrow{v} =< \ v_1, v_2, v_3 >= v_1\hat{\boldsymbol{\imath}} + v_2\,\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$$

Example:

$$\overrightarrow{v} =< \ 2,4,6 >= 2 < \ 1,0,0 > +4 < \ 0,1,0 > +6 < \ 0,0,1 >$$

$$=< \ 2,0,0 > +< \ 0,4,0 > +< \ 0,0,6 >$$

$$= \ < \ 2,4,6 >$$

$$\overrightarrow{v} \quad = 2\hat{\boldsymbol{\imath}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Remark:

If a vector is not unit vector, then we can make it unit vector by dividing it with magnitude of vector. i.e.,

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

Example:

If $\vec{v} =< 1, 2, 3 >$, then

$|\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \neq 1$

Vector $\vec{v}$ is not unit vector because its magnitude is not 1.

$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{<1,2,3>}{\sqrt{14}} = < \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} >$

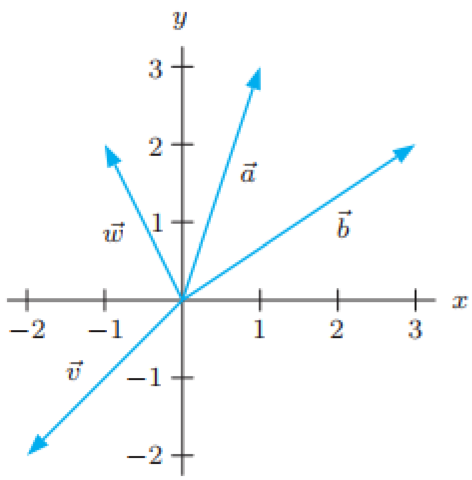
Now it is unit vector in the direction of vector $\vec{v}$.

<u>Practice Problems</u>

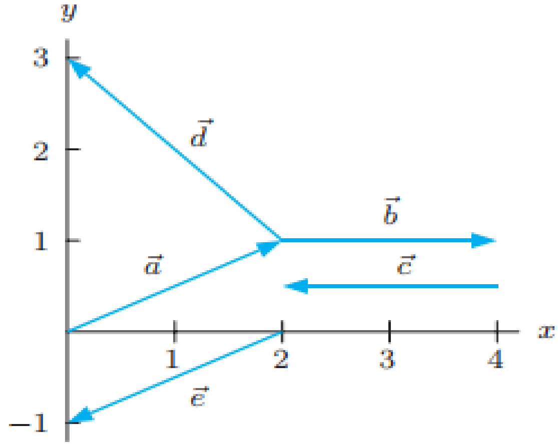
Question 1: Find the unit vector in the opposite direction to $\hat{\imath} - \hat{\jmath} + \hat{k}$.

Question 2: Resolve the vectors into components:

1.



2.



3. A vector starting at the point $Q = (4, 6)$ and ending at the point $P = (1, 2)$.

4. A vector starting at the point $P = (1, 2)$ and ending at the point $Q = (4, 6)$.

Thomas Calculus; 12th Edition

Ex. 12.2: 1-9,11,12,17-22

Dot Product of two vectors

We have seen how to add vectors. Can we multiply two vectors together?

Product of the vectors are of two types. A vector has both magnitude and direction, and based on this, the two products of vectors are,

- the dot product of two vectors, and
- the cross product of two vectors.

The dot product of two vectors is also referred to as scalar product, as the resultant value is a scalar quantity.

The cross product is called the vector product as the result is a vector, which is perpendicular to these two vectors.

The Scalar Product (Dot Product) → Produces Scalar

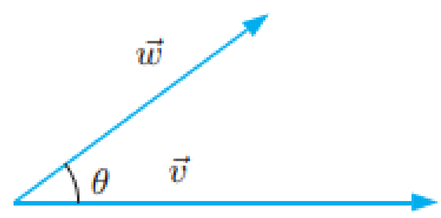
The Vector Product (Cross Product) → Produces Vector

Dot Product (Scalar Product)

Let $\vec{v} = v_1\,\hat{\boldsymbol{\imath}} + v_2\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$ **and** $\vec{w} = w_1\,\hat{\boldsymbol{\imath}} + w_2\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$

Then the dot product of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ can be defined in two ways:

- Dot Product of two vectors is equal to the product of the magnitudes of the two vectors, and the cosine of the angle between these two vectors.



Note: The dot product of two vectors is a scalar number, not a vector.

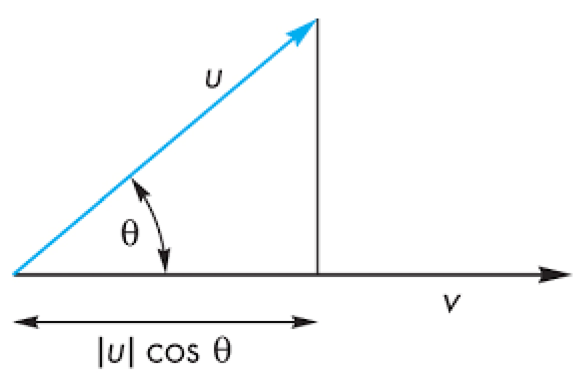
- **Geometric definition**

$$\vec{\boldsymbol{v}}.\vec{\boldsymbol{w}} = |\vec{\boldsymbol{v}}||\vec{\boldsymbol{w}}|\cos\boldsymbol{\theta}$$

where $\boldsymbol{\theta}$ is the angle between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ and $\mathbf{0} \le \boldsymbol{\theta} \le \boldsymbol{\pi}$.

- **Algebraic definition**

$$\vec{\boldsymbol{v}}.\vec{\boldsymbol{w}} = \boldsymbol{v_1 w_1} + \boldsymbol{v_2\,w_2} + \boldsymbol{v_3 w_3}$$



Example 1 Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$ both geometrically and algebraically.

Solution To use the geometric definition, see Figure 13.27. The angle between the vectors is $\pi/4$, or $45°$, and the lengths of the vectors are given by

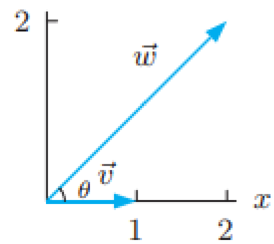
$$\|\vec{v}\| = 1 \quad \text{and} \quad \|\vec{w}\| = 2\sqrt{2}.$$

Thus,

$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 1 \cdot 2\sqrt{2} \cos\left(\frac{\pi}{4}\right) = 2.$$

Using the algebraic definition, we get the same result:

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + 0 \cdot 2 = 2.$$



Zero Vectors

The Vectors that have **0** magnitude are called **zero vectors**, denoted by

$$\vec{\mathbf{0}} = < \mathbf{0}, \mathbf{0}, \mathbf{0} >$$

The zero vector has **zero magnitude** and **no direction**. It is also called the **additive identity** of vectors.

Orthogonal Vectors:

Two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are orthogonal if they are perpendicular, i.e., they form a right angle with each other, or the dot product of two vectors is equal to zero.

Two non-zero vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ are perpendicular, or orthogonal, if and only if

$$\vec{v} \cdot \vec{w} = 0$$

For example:

$\hat{\imath} \cdot \hat{\jmath} = 0$, means that unit vector $\hat{\imath}$ and $\hat{\jmath}$ are orthogonal vectors

$\hat{\jmath} \cdot \hat{k} = 0$, means that unit vector $\hat{\jmath}$ and $\hat{k}$ are orthogonal vectors

$\hat{\imath} \cdot \hat{k} = 0$, means that unit vector $\hat{\imath}$ and $\hat{k}$ are orthogonal vectors

Remarks:

1) If we take the dot product of a vector with itself, then $\boldsymbol{\theta} = \mathbf{0}$, and

$$\mathbf{cos}\,\boldsymbol{\theta} = \mathbf{cos}(\mathbf{0}) = \mathbf{1}$$

For example:

$$\hat{\imath}.\hat{\imath} = \mathbf{1}, \qquad \hat{\jmath}.\hat{\jmath} = \mathbf{1}, \qquad \hat{k}.\hat{k} = \mathbf{1}$$

2) Magnitude and dot product are related as follows:

$$\vec{\boldsymbol{v}}.\vec{\boldsymbol{v}} = |\boldsymbol{v}|^{\mathbf{2}}$$

i.e., the dot product of the vector with itself is equal to the square of its magnitude.

Example 1: Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u}=\hat{\imath} + 3\,\hat{k}\ , \quad \vec{v}=\hat{\imath} + 3\hat{\jmath}\ , \quad \vec{w}=3\hat{\imath} + \hat{\jmath} - \hat{k}.$$

Solution: The geometric definition tells us that two vectors are perpendicular if and only if their dot product is zero. Since the vectors are given in components, we calculate dot products using the algebraic definition:

$$\vec{u} \cdot \vec{v} = (\hat{\imath} + 3\hat{\jmath} + 0\hat{k}) \cdot (\hat{\imath} + 0\hat{\jmath} + 3\,\hat{k}\,) = 1 \cdot 1 + 3 \cdot 0 + 0 \cdot 3 = 1 \neq 0$$

$$\vec{v} \cdot \vec{w} = (\hat{\imath} + 3\hat{\jmath} + 0\hat{k}) \cdot \left(3\hat{\imath} + \hat{\jmath} - \hat{k}\right) = 1 \cdot 3 + 3 \cdot 1 + 0 \cdot (-1) = 6 \neq 0$$

$$\vec{u} \cdot \vec{w} = \left(\hat{\imath} + 0\hat{\jmath} + 3\hat{k}\right) \cdot \left(3\hat{\imath} + \hat{\jmath} - \hat{k}\right) = 1 \cdot 3 + 0 \cdot 1 + 3 \cdot (-1) = 0$$

So, the only two vectors that are perpendicular (orthogonal) are $\vec{u}$ and $\vec{w}$ because

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}} = \mathbf{0}$$

Example 2: Compute the angle between the vectors $\vec{v}$ and $\vec{w}$.

$\vec{v} = \hat{\imath} + \sqrt{3}\,\hat{\jmath}$, $\quad \vec{w} = \sqrt{3}\,\hat{\imath} + \hat{\jmath} - \hat{k}$

Solution: Since, we know that

$$\vec{v}.\vec{w} = |\vec{v}|\,|\vec{w}|Cos\theta$$

By rearranging, we can write

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\,|\vec{w}|} - - - -(1)$$

First, we will find the dot product as

$\vec{v} \cdot \vec{w} = (\hat{\imath} + \sqrt{3}\,\hat{\jmath}) \cdot (\sqrt{3}\,\hat{\imath} + \hat{\jmath} - \hat{k}) = (1) \cdot (\sqrt{3}) + (\sqrt{3}) \cdot (1) + (0) \cdot (1)$

$\vec{v} \cdot \vec{w} = 2\sqrt{3}$

Now,

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

$$\cos(\theta) = \frac{2\sqrt{3}}{\sqrt{(1)^2 + \left(\sqrt{3}\right)^2 + (0)^2}\,\sqrt{\left(\sqrt{3}\right)^2 + (1)^2 + (-1)^2}}$$

$$\cos(\theta) = \frac{2\sqrt{3}}{\sqrt{1+3}\,\sqrt{3+1+1}} = \frac{2\sqrt{3}}{2\,\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{\sqrt{5}}\right) = 39.3^o$$

Practice Question: Compute the angle between the vectors $\vec{\boldsymbol{v}} = \mathbf{2}\,\hat{\boldsymbol{\imath}} + \mathbf{5}\hat{\boldsymbol{\jmath}} - \mathbf{9}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{w}} = \mathbf{7.31}\,\hat{\boldsymbol{\imath}} + \mathbf{8.64}\hat{\boldsymbol{\jmath}} + \mathbf{4.25}\hat{\boldsymbol{k}}$.

Food for Thought:

What would a dot product be between two orthogonal vectors?

Find any 2 vectors that are orthogonal.

Angle Between Two Vectors

The angle between two vectors is calculated as the cosine of the angle between the two vectors. The cosine of the angle between two vectors is equal to the dot product of the individual constituents of the two vectors, divided by the product of the magnitude of the two vectors.

Let $\vec{\boldsymbol{v}} = <\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}>$ and $\vec{\boldsymbol{w}} == <\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}>$**,** then the formula for the angle between the two vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{w}}$ is as follows.

$$\cos\theta = \frac{\vec{v}.\vec{w}}{|\vec{v}|\,|\vec{w}|}$$

$$\cos(\theta) = \frac{\vec{v}.\vec{w}}{\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}\,\sqrt{(w_1)^2 + (w_2)^2 + (w_3)^2}}$$

Finding the Angle of a Triangle

Example: Find the angle $\boldsymbol{\theta}$ in the triangle $\Delta\boldsymbol{ABC}$ determined by the vertices

$$\boldsymbol{A} = (\mathbf{0}, \mathbf{0}), \boldsymbol{B} = (\mathbf{3}, \mathbf{5}), \boldsymbol{C} = (\mathbf{5}, \mathbf{2})$$

Solution:

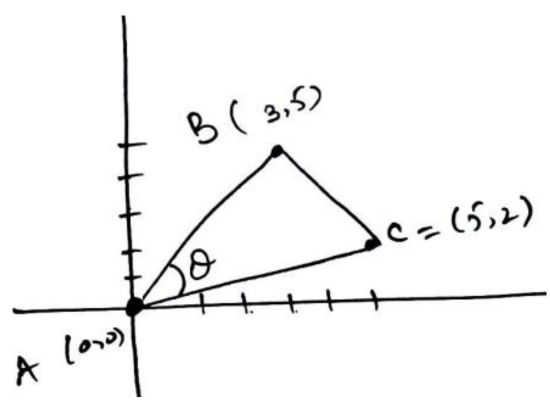
The angle θ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\overrightarrow{AB} =< 3-0, 5-0 >=< 3, 5 >= 3\hat{\imath} + 5\hat{\jmath}$

$\overrightarrow{AC} =< 5-0, 2-0 >=< 5, 2 >= 5\hat{\imath} + 2\hat{\jmath}$

$$Cos\theta = \frac{\overrightarrow{AB}\,.\,\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{15+10}{(\sqrt{3^2+5^2})\sqrt{5^2+2^2})}$$

$$= \frac{25}{(\sqrt{34})\sqrt{29})} = 35.9°$$



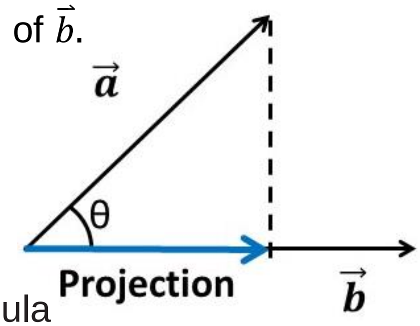
Vector Projection

The vector projection of one vector over another vector is the length of the shadow of the given vector over another vector.

The vector projection formula in vector algebra for the projection of vector $\vec{a}$ on vector $\vec{b}$ is equal to the dot product of vector $\vec{a}$ and vector $\vec{b}$, divided by the magnitude of vector $\vec{b}$ and multiplied by the unit vector of the vector $\vec{b}$, i.e. $\hat{b}$.

This implies that the new vector is going in the direction of $\vec{b}$.

Graphically, it can be represented as



Projection of vector $\vec{\boldsymbol{a}}$ onto vector $\vec{\boldsymbol{b}}$ is given by the formula

$$Proj_{\vec{\boldsymbol{b}}}\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\hat{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Projection of vector $\vec{\boldsymbol{b}}$ onto vector $\vec{\boldsymbol{a}}$ is given by the formula

$$Proj_{\vec{\boldsymbol{a}}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\hat{a} = \left(\frac{\vec{a}\cdot b}{|\vec{a}|}\right)\left(\frac{\vec{a}}{|\vec{a}|}\right) = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Example: Find the vector projection of $\vec{a}$= $6\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k}$ onto $\vec{b}$=$\hat{\imath} - 2\,\hat{\jmath} - 2\hat{k}$.

Solution:

$$Proj_{\vec{b}}\vec{a} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

$$\vec{a}.\vec{b} = \left(6\hat{\imath} + 3\,\hat{\jmath} + 2\hat{k}\right).\left(\hat{\imath} - 2\,\hat{\jmath} - 2\hat{k}\right) = 6 - 6 - 4 = -4$$

$\overrightarrow{|b|}$=$\sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9}$ =3

$\overrightarrow{|b|}^2 = 3^2 = 9$

$Proj_{\vec{b}}\vec{a} = \frac{-4}{9}(\hat{\imath} - 2\,\hat{\jmath} - 2\hat{k})$

<u>Practice Problems:</u>

Question 1: Find $\vec{a}.\vec{b}$, $|\vec{a}|$, $|\vec{b}|$ and $Proj_{\vec{b}}\vec{a}$, if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} + 10\,\hat{\jmath} - 11\hat{k}$

Question 2: Find the angle between the vectors $\vec{a} = 2\hat{\imath} + \hat{\jmath}$, and $\vec{b} = \hat{\imath} + 2\,\hat{\jmath} - \hat{k}$.

Question 3: Find the measures of the angles of the triangle having vertices

$A = (-1,0), B = (2,1)$ and $C = (1,-20)$.

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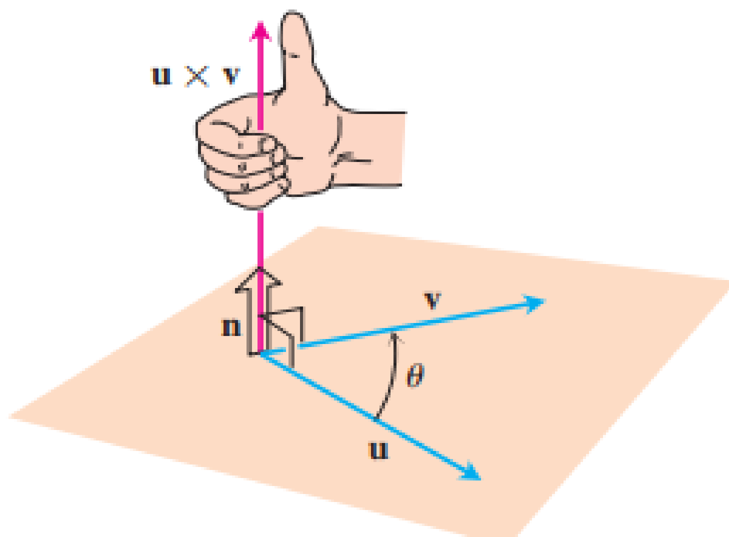
Ex. 12.3: 1-13.

Cross Product of Two vectors

Cross Product is also called a Vector Product. Cross product is a form of vector multiplication, performed between two vectors.

When two vectors are multiplied with each other and the product is also a vector quantity, then the resultant vector is called the cross product of two vectors or the vector product. The resultant vector is perpendicular to the plane containing the two given vectors.

Let $\vec{u}$ and $\vec{v}$ be two vectors, if $\vec{u}$ and $\vec{v}$ are not parallel, they determine a plane. We select a unit vector n̂ perpendicular to the plane by the right-hand rule.



This means that we choose $\hat{n}$ to be unit (normal) vector that points the way your right thumb points when your finger curl through the angle $\boldsymbol{\theta}$ from $\vec{u}$ to $\vec{v}$. The cross-product $\vec{u} \times \vec{v}$ is a vector defined as follows:

- **Geometric Definition**

$$\vec{u} \times \vec{v} = |\vec{u}|\,|\vec{v}|\,sin\,\theta\,\hat{n}$$

where $\mathbf{0} \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$ is the angle between $\vec{u}$ & $\vec{v}$ and $\hat{n}$ is the unit vector perpendicular to $\vec{u}$ & $\vec{v}$ pointing in the direction given by the right-hand rule.

- **Algebraic Definition**

$$\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}$$

$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\imath}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$\boxed{\begin{gathered}\vec{u} \times \vec{v} = \hat{\imath}(u_2v_3 - v_2u_3) - \hat{\jmath}(u_1v_3 - v_1u_3) \\ + \hat{k}(u_1v_2 - v_1u_2)\end{gathered}}$$

Example 1:

- Find the cross product of $\vec{\boldsymbol{u}} = \mathbf{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \mathbf{2}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{v}} = \mathbf{3}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$.
- Check that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{u}}$ or not**.**
- Check that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{v}}$ or not.

Solution:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \hat{\boldsymbol{\imath}}(1-0) - \hat{\boldsymbol{\jmath}}(2+6) + \hat{\boldsymbol{k}}(0-3)$$

$$\vec{u} \times \vec{v} = \hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$$

To check $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$, we will take dot product

$$\vec{u}\,.(\vec{u} \times \vec{v}\,) = \left(2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}\right).(\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}})$$

$$\vec{u}\,.(\vec{u} \times \vec{v}\,) = 2 - 8 + 6 = 0$$

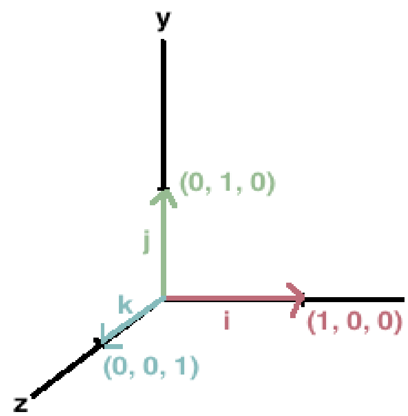
Since $\vec{u}\,.(\vec{u} \times \vec{v}\,) = 0$, represents that the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{u}}$**.**

Similarly, to check $\vec{u} \times \vec{v}$ is perpendicular to $\vec{v}$, we will take dot product

$$\vec{v}\,.\,(\vec{u}\times\vec{v}) = \left(3\hat{\boldsymbol{\imath}} + \widehat{\boldsymbol{k}}\right).\,(\hat{\boldsymbol{\imath}} - 8\hat{\boldsymbol{\jmath}} - 3\widehat{\boldsymbol{k}})$$

$$\vec{v}\,.\,(\vec{u}\times\vec{v}) = 3 - 3 = 0$$

Since $\vec{v}\,.\,(\vec{u}\times\vec{v}\,) = 0$, represents that the cross product $\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}}$ is perpendicular to $\vec{\boldsymbol{v}}$**.**



Definition: Parallel vector

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are **paralle**l if and only

$$\vec{\boldsymbol{u}}\times\vec{\boldsymbol{v}} = \mathbf{0}$$

For Example: $\hat{\imath}\times\hat{\imath} = 0,\ \hat{\jmath}\times\hat{\jmath} = 0,\ \hat{k}\times\hat{k} = 0$

Example 1:

Find $\hat{\imath}\times\hat{\jmath}$ and $\hat{\jmath}\times\hat{\imath}$.

Solution:

- For $\hat{\imath}\times\hat{\jmath}$

  As $\vec{u}\times\vec{v} = |\vec{u}|\,|\vec{v}|\,sin\,\theta\,\hat{n}$

$$|\hat{\imath}| = 1$$

$$|\hat{\jmath}| = 1$$

  Angle between $\hat{\imath}$ & $\hat{\jmath}$ is $\boldsymbol{\frac{\pi}{2}}$.

  By right hand rule, the vector $\hat{\imath}\times\hat{\jmath}$ is in the direction of vector $\hat{k}$ so

$$\hat{n} = \hat{k}$$

  So,

$$\hat{\imath}\times\hat{\jmath} = \left(|\hat{\imath}|\,|\hat{\jmath}|\,sin\,\left(\frac{\pi}{2}\right)\right)\hat{k}$$

$$= (1.1.1)\,\hat{k} = \hat{k}$$

$$\hat{\imath}\times\hat{\jmath} = \hat{k}$$

- For $\hat{\jmath}\times\hat{\imath}$

The right-hand rule says that the direction of $\hat{j} \times \hat{\imath}$ is $-\hat{k}$. So

$$\hat{j} \times \hat{\imath} = \left(|\hat{j}|\ |\hat{\imath}|\ sin\ \frac{\pi}{2}\right)(-\hat{k}) = (1.1.1)(-\hat{k})$$

$$\hat{j} \times \hat{\imath} = -\hat{k}$$

Hence proved.

Practice question: **Prove these yourself** $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}}$, $\quad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{j}}$

Example 2: For any vector $\vec{\boldsymbol{v}}$ find $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}}$.

Solution:

As $\vec{\boldsymbol{v}}$ is parallel to itself so $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}} = \mathbf{0}$.

Remarks:

- The dot product of unit vectors $\hat{\imath}$, $\hat{j}$ and $\hat{k}$ follows similar rules as the dot product of vectors. The angle between the same vectors is equal to 0º, and hence their dot product is equal to 1. And the angle between two perpendicular vectors is 90º, and their dot product is equal to $\mathbf{0}$.
  - $\hat{\boldsymbol{\imath}}.\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{j}}.\hat{\boldsymbol{j}} = \widehat{\boldsymbol{k}}.\widehat{\boldsymbol{k}} = \mathbf{1}$
  - $\hat{\boldsymbol{\imath}}.\hat{\boldsymbol{j}} = \hat{\boldsymbol{j}}.\widehat{\boldsymbol{k}} = \widehat{\boldsymbol{k}}.\hat{\boldsymbol{\imath}} = \mathbf{0}$
- The cross product of unit vectors $\hat{\imath}$, $\hat{j}$ and $\hat{k}$ follows similar rules as the cross product of vectors. The angle between the same vectors is equal to $\mathbf{0}^{\boldsymbol{o}}$, and hence their cross product is equal to $\mathbf{0}$. The angle between two perpendicular vectors is $\mathbf{90}^{\mathbf{0}}$, and their cross product gives a vector, which is perpendicular to the
  - $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{j}} \times \hat{\boldsymbol{j}} = \widehat{\boldsymbol{k}} \times \widehat{\boldsymbol{k}} = \mathbf{0}$
  - $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{j}} = \widehat{\boldsymbol{k}},\ \hat{\boldsymbol{j}} \times \widehat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}},\ \widehat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{j}}$
  - $\hat{\boldsymbol{j}} \times \hat{\boldsymbol{\imath}} = -\widehat{\boldsymbol{k}},\ \widehat{\boldsymbol{k}} \times \hat{\boldsymbol{j}} = -\hat{\boldsymbol{\imath}},\ \hat{\boldsymbol{\imath}} \times \widehat{\boldsymbol{k}} = -\hat{\boldsymbol{j}}$

For concept revision check the following links

- https://www.cuemath.com/algebra/dot-product/
- https://www.cuemath.com/algebra/product-of-vectors/

Practice Problems

Question: Find the cross product of the following vectors

1) $\vec{u} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
   $\vec{v} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
2) $\vec{u} = -3\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{\jmath}} + 4\widehat{\boldsymbol{k}}$
   $\vec{v} = \hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
3) $\vec{u} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}$
   $\vec{v} = -6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 3\widehat{\boldsymbol{k}}$

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Ex# 12.4: Q # 23, 24