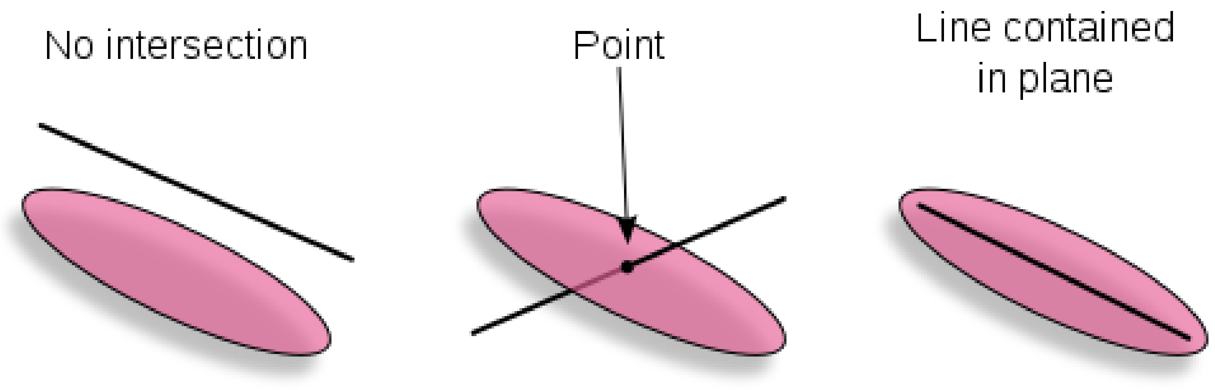
Topics:

- **Interaction of a Line and a Plane**
- **Interaction of Two Planes**

Interaction of a Line and a Plane:

Graphically we have the observe that for a line and plane to interact, the following case



We observe that *Interaction* of line & plane is either a line or a point.

Methodology:

- Substitute the line's **parametric equations** into the **standard form** of the plane. Solve for the parameter we will get one of the following cases:

1. ***Case 1*: No intersection** $\Longrightarrow$ line is on a different parallel plane, therefore it is a skew line to the plane

Method I:

Collecting like terms on the left side causes the variable $\boldsymbol{t}$ to cancel out and leaves us with a contradiction in which $\boldsymbol{t}$ **gets eliminated and resultantly** $\boldsymbol{L.H.S \neq R.H.S}$, then there is no value of $\boldsymbol{t}$ that makes this equation true, and thus there is no value of $\boldsymbol{t}$ that will give us a point on the line that is also on the plane. This means that **this line does not intersect with this plane** and there will be **no point of intersection**.

Method II:

Find the vector $\vec{n}$, which is the normal to the plane and the vector $\vec{v}$ the parallel vector for the line. If $\vec{n} \cdot \vec{v} = 0$ $i.e.$, both are orthogonal. So, if the dot product is zero, this implies that the line is either on the plane or parallel to the plane.

We can prove that the line does not lie on the plane by putting that the point (x_0, y_0, z_0) into the equation of plane, for which we will get $L.H.S \neq R.H.S$, the point does not satisfy the equation of the plane, the point is not on the plane. The line and the plane are parallel and do not intersect.

2. ***Case 2:*** **One (Unique) point of intersection**

If there is a valid solution i.e., we find a value of the parameter say t. Furthermore, substitute the value of the parameter back into the equation of the line to find the point of intersection.

3. ***Case 3:*** **Infinitely many points of intersection** $\Longrightarrow$ line lies on the plane

Method I:

Collecting like terms on the left side causes the variable t to cancel out and leaves us with a contradiction in which t **gets eliminated but resultantly** $L.H.S = R.H.S$, then this means that **every** value of t **will produce a point on the line that is also on the plane, telling us that the line is contained in the plane whose equation is** given.

Method II:

Find the vector $\vec{n}$, which is the normal to the plane and the vector $\vec{v}$ the parallel vector for the line. If $\vec{n} \cdot \vec{v} = 0$ $i.e.$, if the dot product is **zero**, this implies that that the **line and plane are parallel**.

Now by putting that the point of the line (x_0, y_0, z_0) into the equation of plane, for which we will get $L.H.S = R.H.S$, the point does satisfy the equation of the plane, So, this point lies on the plane as well. Since the line and plane are parallel and (x_0, y_0, z_0) lies on the plane, the entire line lies on the plane.

Case 1 Example: No Intersection

a) Determine whether the following line having parametric equations

$$L: \quad x = 1 + 2t, \quad y = -2 + 3t, \quad z = 1 + 4t$$

intersects with the given plane having equation

$$x + 2y - 2z = 5.$$

b) If they do intersect, determine whether the line is contained in the plane or intersects it in a single point.

c) If the line intersects the plane in a single point, then determine this point of intersection.

Solution:

Step 1: Substitute the values of parametric equation into the equation of plane

The given equation of the plane is

$$x + 2y - 2z = 5$$

Substitute the values of x, y, and z given in the parametric equations of the line into the given equation of plane, we have

$$x + 2y - 2z = 5$$

$$(1 + 2t) + 2(-2 + 3t) - 2(1 + 4t) = 5$$

Step 2: Solving this equation for t

$$(1 + 2t) + 2(-2 + 3t) - 2(1 + 4t) = 5$$

$$1 + 2t - 4 + 6t - 2 - 8t = 5$$

$$2t + 6t - 8t + 1 - 4 - 2 = 5$$

$$8t - 8t + 1 - 6 = 5$$

$$0\,t-5= \ 5$$

$$-5\neq \ 5$$

Since this is not true, we know that there is no value of $\boldsymbol{t}$ that makes this equation true, and thus there is no value of $\boldsymbol{t}$ that will give us a point on the line that is also on the plane. This means that **this line does not intersect with this plane** and there will be **no point of intersection**.

Case 2 Example: One (Unique) point of intersection

a) Determine whether the following line having parametric equations

$$\boldsymbol{L}: \qquad \boldsymbol{x}=\frac{\mathbf{8}}{\mathbf{3}}+\mathbf{2}\boldsymbol{t}, \qquad \boldsymbol{y}=-\mathbf{2}\boldsymbol{t}, \qquad \boldsymbol{z}=\mathbf{1}+\boldsymbol{t}$$

intersects with the given plane having equation

$$\mathbf{3}\boldsymbol{x}+\mathbf{2}\boldsymbol{y}+\mathbf{6}\boldsymbol{z}=\mathbf{6}.$$

b) If they do intersect, determine whether the line is contained in the plane or intersects it in a single point.

c) If the line intersects the plane in a single point, then determine this point of intersection.

Solution:

Step 1: Substitute the values of parametric equation into the equation of plane

The given equation of the plane is

$$\mathbf{3}\boldsymbol{x}+\mathbf{2}\boldsymbol{y}+\mathbf{6}\boldsymbol{z}=\mathbf{6}$$

Substitute the values of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ given in the parametric equations of the line into the given equation of plane, we have

$$3\left(\frac{8}{3}+2\,t\right)+\ 2\,(-2\,t)+\ 6\,(1\ +\ t)\ =\ 6$$

Step 2: Solving this equation for $\boldsymbol{t}$

$$3\left(\frac{8}{3}\right)\ +3\,(2\,t)-\ 4\,t\ +\ 6\ +\ 6\,t\ =\ 6$$

$$8\ +6\,t-\ 4\,t\ +\ 6\ +\ 6\,t\ =\ 6$$

$$6t + 6t - 4t = 6 - 6 - 8$$

$$12t - 4t = -8$$

$$8t = -8$$

$$t = -1$$

Since we found a single value of $\boldsymbol{t}$ from this process, we know that the line should intersect the plane in a single point, here where $\boldsymbol{t = -1}$.

Step 3: Find the point of Intersection

The point of intersection can be determined by plugging this value of $\boldsymbol{t}$ in the parametric equations of the line.

$$x = \frac{8}{3} + 2t = \frac{8}{3} + 2(-1) = \frac{8}{3} - 2 = \frac{8-6}{3} = \boldsymbol{\frac{2}{3}}$$

$$y = -2t = -2(-1) = \boldsymbol{2}$$

$$z = 1 + t = 1 + (-1) = 1 - 1 = 0$$

Hence, the point of intersection of line with the plane is

$$\boldsymbol{(x, y, z) = \left(\frac{2}{3}, 2, 0\right)}$$

Verification (Optional):

We can verify this by putting the coordinates of this point into the plane equation and checking to see that it is satisfied.

$$\boldsymbol{3x + 2y + 6z = 6}$$

$$\boldsymbol{3\left(\frac{2}{3}\right) + 2(2) + 6(0) = 6}$$

$$\boldsymbol{2 + 4 + 0 = 6}$$

$$\boldsymbol{6 = 6}$$

$$\boldsymbol{L.H.S = R.H.S}$$

Hence verified.

Case 3 Example: Infinitely many points of intersection

a) Determine whether the following line having parametric equations

$$L: \quad x = 1 + 2t, \qquad y = -2 + 3t, \qquad z = 1 + 4t$$

intersects with the given plane having equation

$$x + 2y - 2z = -5.$$

b) If they do intersect, determine whether the line is contained in the plane or intersects it in a single point.

c) If the line intersects the plane in a single point, then determine this point of intersection.

Solution:

Step 1: Substitute the values of parametric equation into the equation of plane

The given equation of the plane is

$$x + 2y - 2z = -5$$

Substitute the values of x, y, and z given in the parametric equations of the line into the given equation of plane, we have

$$x + 2y - 2z = -5$$

$$(1 + 2t) + 2(-2 + 3t) - 2(1 + 4t) = -5$$

Step 2: Solving this equation for t

$$(1 + 2t) + 2(-2 + 3t) - 2(1 + 4t) = -5$$

$$1 + 2t - 4 + 6t - 2 - 8t = -5$$

$$2t + 6t - 8t + 1 - 4 - 2 = -5$$

$$8t - 8t + 1 - 6 = -5$$

$$0t - 5 = -5$$

$$-5 = -5$$

This means that **every** value of t will produce a point on the line that is also on the plane, telling us that the line is contained in the plane whose equation is

$$x + 2y - 2z = -5$$

Practice Questions for students:

1. Find the point of intersection (if any), where the line $\boldsymbol{L}$ having parametric equations
$$\boldsymbol{L}: \quad \boldsymbol{x = 3 + 14\, s}, \quad \boldsymbol{y = -2 - 5\, s}, \quad \boldsymbol{z = 1 - 3\, s}$$
intersects the plane having equation
$$\boldsymbol{x + y - 3\, z - 4 = 0}.$$
2. Find the point of intersection (if any), where the line $\boldsymbol{L}$ having parametric equations
$$\boldsymbol{L}: \quad \boldsymbol{x = 2 + q}, \quad \boldsymbol{y = 2 + 2\, q}, \quad \boldsymbol{z = 9 + 8\, q}$$
intersects the plane having equation
$$\boldsymbol{2\, x - 5\, y + z - 6 = 0}.$$

Practice Questions for students:

(Book: Thomas Calculus Ex # 12.5: Q # 53-56)

Find the point (if any) in which the line meets the plane

1) $\boldsymbol{Line: \; x = 1 - t, \;\; y = 3t, \;\; z = 1 + t, \qquad Plane: \; 2x - y + 3z = 6}$

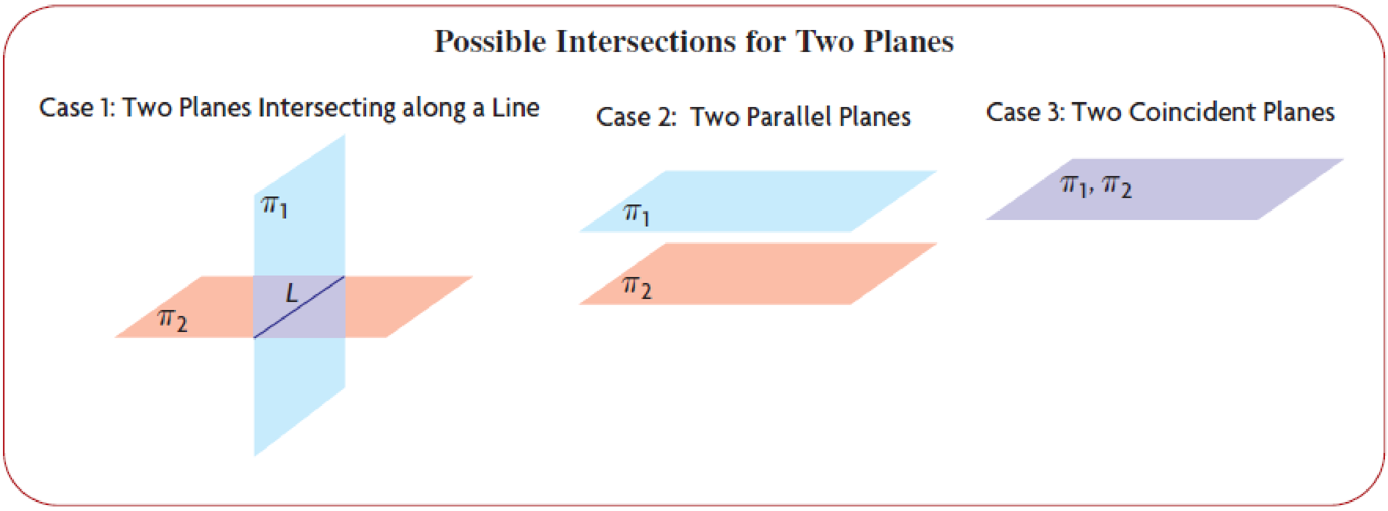
2) $\boldsymbol{Line: \; x = 2, \;\; y = 3 + 2t, \;\; z = -2 - 2t, \qquad Plane: \; 6x + 3y - 4z = -12}$

3) $\boldsymbol{Line: \; x = 1 + 2t, \;\; y = 1 + 5t, \;\; z = 3t, \qquad Plane: \; x + y + z = 2}$

4) $\boldsymbol{Line: \; x = -1 + 3t, \;\; y = -2, \;\; z = 5t, \qquad Plane: \; 2x - 3z = 7}$

Interaction between Two Planes

Observation: Interaction of two planes is either a a line or a plane.



We were looking at two planes $P_1$and P_2 ,with normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$. We know that two planes were parallel if and only if their normal vectors were scalar multiples of each other but what if two planes are not parallel? Then they intersect, but instead of intersecting at a single point, the set of points where they intersect form a line.

Methodology

Step 1: If the normal vectors $\overrightarrow{n_1}$ **and** $\overrightarrow{n_2}$ of plane 1 and plane 2 respectively, are scalar multiple of each other, then $\overrightarrow{n_1} \,||\, \overrightarrow{n_2}$. Thus plane 1 is parallel to plane 2. If $\overrightarrow{n_1}$ **and** $\overrightarrow{n_2}$ are not parallel, then find $\vec{v}$.

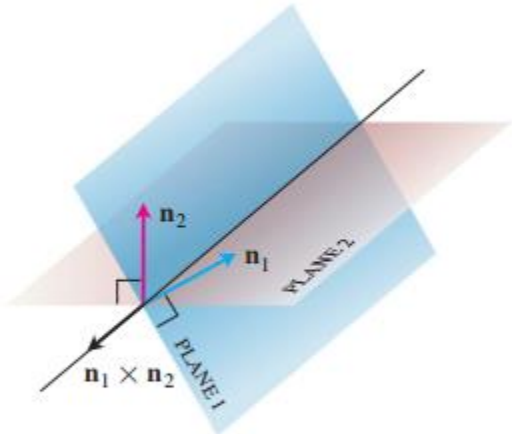
Step 2: Find $\vec{v} = \overrightarrow{n_1} \times \overrightarrow{n_2}$. Here $\vec{v}$ is parallel to the line L, formed by the intersection of the plane.

Step 3: Put any variable x or y or z equals to 0, i.e., put either $x = 0$ **or** $y = 0$ **or** $z = 0$, in both equations of the planes. We will get two equations having two variables, then find the values of the remaining variables. Thus, finding the point on the line L.

Step 4: Using the point and the vector $\vec{v}$, form the parametric equation of line L.

EXAMPLE:

Find the parametric equations for the line in which the given planes intersect.



$$\textit{Plane } P_1: \; 3x - 6y - 2z = 15$$

$$\textit{Plane } P_2: \; 2x + y - 2z = 5$$

Solution:

Step 1:

The normal vector $\vec{n}_1$ to the equation of plane 1 is

$$\overrightarrow{n_1} = 3\,\hat{i} - 6\hat{j} - 2\,\hat{k}$$

The normal vector $\vec{n}_2$ to the equation of plane 2 is

$$\overrightarrow{n_2} = 2\,\hat{i} + \hat{j} - 2\,\hat{k}$$

Since the vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are **not scalar multiple** of each other. That is $\overrightarrow{n_1} \neq k\,\overrightarrow{n_2}$. **Therefore,** $\overrightarrow{n_1}$ & $\overrightarrow{n_2}$ are **not parallel**, implies that the **planes P_1 and P_2 are not parallel.** As two planes are not parallel, so they will intersect with each other.

Step 2:

To find the line, we need a **vector** and the **point** of intersection of these two planes. The line of intersection of two planes is **perpendicular** to both the plane normal vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ and therefore, parallel to $\vec{n}_1 \times \vec{n}_2$.

So, $\overrightarrow{n_1} = 3\,\hat{i} - 6\hat{j} - 2\,\hat{k}$

$\overrightarrow{n_2} = 2\,\hat{i} + \hat{j} - 2\,\hat{k}$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \hat{i}\begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix}$$

$\vec{v} = \hat{\imath}(12+2) - \hat{\jmath}(-6+4) + \hat{k}(3+12)$

$$\vec{\boldsymbol{v}} = \mathbf{14}\hat{\boldsymbol{\imath}} + \mathbf{2}\hat{\boldsymbol{\jmath}} + \mathbf{15}\hat{\boldsymbol{k}} = <\mathbf{14}, \mathbf{2}, \mathbf{15}>$$

<u>**Step 3:**</u> To find the point of intersection we take the equations of these two planes.

$\boldsymbol{Plane\ P_1}$: $3x - 6y - 2z = 15$

$\boldsymbol{Plane\ P_2}$: $2x + y - 2z = 5$

As the number of equations is less than the number of variables, therefore the given system is **under-determined system**, and hence the system of equations has **infinite many solutions**. From these infinitely many solutions, we need only one solution.

Let us substitute $z = \mathbf{0}$, then the equations of planes become

$$3x - 6y = 15 - - - - (\mathbf{1})$$

$$2x + y = 5 \quad - - - - (\mathbf{2})$$

Multiply equation (2) by "$\mathbf{6}$" and **adding** it in equation (1), we get

$$\mathbf{3x - 6y = 15} \ - - - - (\mathbf{1})$$

$$\mathbf{12x + 6y = 30} \ - - - - (\mathbf{2})$$

$$15x \quad = \ 45$$

$$\Longrightarrow x = \frac{45}{15} = 3$$

$$\Longrightarrow \boldsymbol{x = 3}$$

Put $\boldsymbol{x} = \mathbf{3}$ in equation **(1)** to find the value of variable $\boldsymbol{y}$, we have

As equation (1) is $\quad \mathbf{3x - 6y = 15} \ - - - - (\mathbf{1})$

$$\mathbf{3(3) - 6y = 15} \Longrightarrow \mathbf{9 - 6y = 15} \Longrightarrow \mathbf{9 - 15 = 6y} \Longrightarrow \mathbf{-6 = 6y} \Longrightarrow \frac{\mathbf{-6}}{\mathbf{6}} = \boldsymbol{y}$$

This implies that

$$\boldsymbol{y = -1}$$

Hence, we get the values of the variables given as

$$\boldsymbol{x = 3} \quad \& \quad \boldsymbol{y = -1} \quad \& \quad \boldsymbol{z = 0}$$

So, the point of intersection is

$$\boldsymbol{P(x_0, y_0, z_0) = (x, y, z) = (3, -1, 0)}$$

And we have already calculated the vector given as

$$\boldsymbol{\vec{v} = 14\hat{\imath} + 2\hat{\jmath} + 15\hat{k} = < 14, 2, 15 > = < a, b, c >}$$

Step 4:

The parametric equations of the line are

$$\begin{cases} \boldsymbol{x = x_0 + a\,t = 3 + 14\,t} \\ \boldsymbol{y = y_0 + b\,t = -1 + 2\,t} \\ \boldsymbol{z = z_0 + c\,t = 0 + 15\,t} \end{cases}, \qquad -\infty < \boldsymbol{t} < \infty$$

Note:

Two planes are parallel, if their normal vectors are parallel. For example,

$\boldsymbol{Plane\ 1: x + 2y - 3z = 4}$ and $\boldsymbol{Plane\ 2: 2x + 4y - 6z = 3}$ are parallel because their normal vectors $\boldsymbol{\vec{n_1} = < 1, 2, -3 >}$ corresponding to **plane 1** and $\boldsymbol{\vec{n_2} = < 2, 4, -6 >}$ are parallel.

Practice question for students

(Book: Thomas calculus, 12^{th} Edition, Ex # 12.5, Q # 57 - 60)

Find the parametrizations for the lines in which the planes intersect.

1) $\boldsymbol{Plane\ 1}$: $\boldsymbol{x + y + z = 1}$, $\boldsymbol{Plane\ 2}$: $\boldsymbol{x + y = 2}$
2) $\boldsymbol{Plane\ 1}$: $\boldsymbol{3x - 6y - 2z = 3}$, $\boldsymbol{Plane\ 2}$: $\boldsymbol{2x + y - 2z = 2}$
3) $\boldsymbol{Plane\ 1}$: $\boldsymbol{x - 2y + 4z = 2}$, $\boldsymbol{Plane\ 2}$: $\boldsymbol{x + y - 2z = 5}$
4) $\boldsymbol{Plane\ 1}$: $\boldsymbol{5x - 2y = 11}$, $\boldsymbol{Plane\ 2}$: $\boldsymbol{4y - 5z = -17}$