

- 4 The points A and B have position vectors, relative to the origin O , given by

$$\vec{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \vec{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line l has vector equation

$$\mathbf{r} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}.$$

- (i) Show that l does not intersect the line passing through A and B .

- (ii) The point P lies on l and is such that angle PAB is equal to 60° . Given that the position vector of P is $(1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of P .

① Eq. of Line AB $\mathbf{r} = \vec{p} + s\vec{d}$

$$\mathbf{r}_2 = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$$

$$\begin{aligned} \vec{r}_1 &= \vec{OA} + s\vec{AB} \\ &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+s \\ 2-s \\ 3+0s \end{pmatrix} \end{aligned}$$

$$\vec{r}_1 = \vec{r}_2$$

$$x: \quad 1-2t = 1+s \quad \text{--- (1)}$$

$$y: \quad 5+t = 2-s \quad \text{--- (2)}$$

$$z: \quad 2-t = 3 \quad \text{--- (3)}$$

$$\textcircled{1} + \textcircled{2}$$

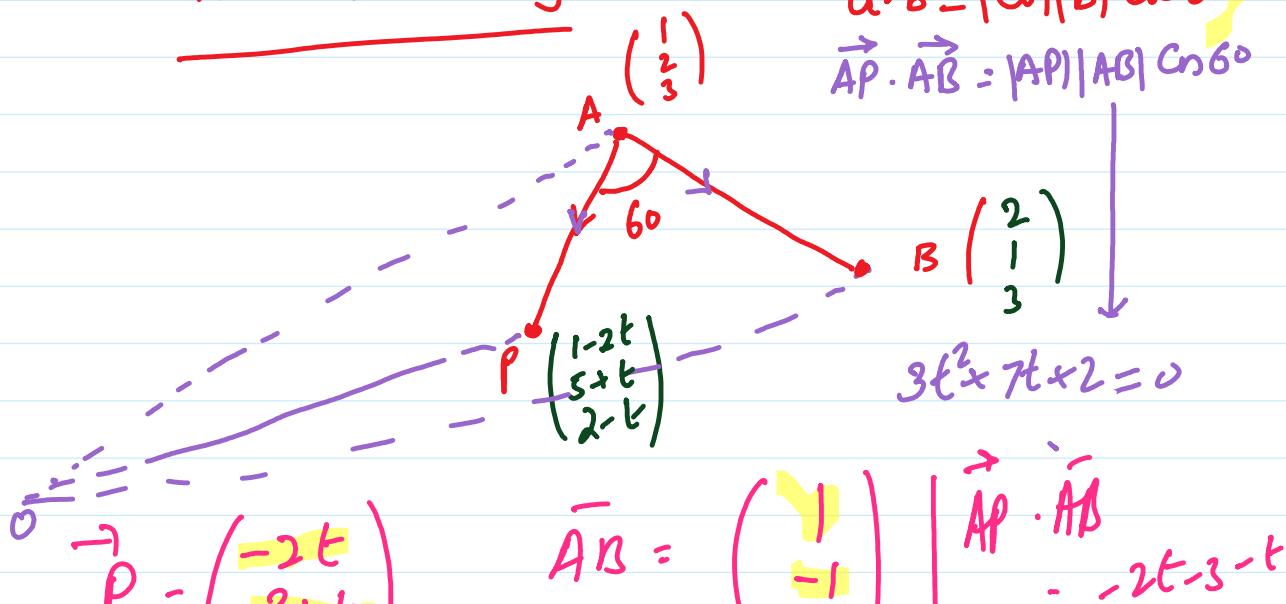
$$6-t = 3$$

$$t = 3$$

Not intersecting

$$\begin{aligned} \vec{a} \cdot \vec{b} &= |\vec{a}| |\vec{b}| \cos \theta \\ \vec{AP} \cdot \vec{AB} &= |AP| |AB| \cos 60^\circ \end{aligned}$$

b)



$$\vec{AP} = \begin{pmatrix} -2t \\ 3+t \\ -1-t \end{pmatrix} \quad \vec{AB} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \vec{AP} \cdot \vec{AB} = -2t - 3 - t = -3t - 3$$

$$|\vec{AP}| = \sqrt{4t^2 + (3+t)^2 + (1+t)^2} \quad ; \quad |\vec{AB}| = \sqrt{1+1}$$

$$\left[\frac{-3t-3}{\sqrt{4t^2 + (3+t)^2 + (1+t)^2}} \right]^2 = \left[\frac{-3t-3}{\sqrt{6t^2 + 8t + 10}} \right]^2 \quad \sqrt{2} \cdot \frac{1}{2}$$

$$18(1+t)^2 = (6t^2 + 8t + 10) \cdot 2$$

$$18(1+t^2+2t) = 6t^2 + 8t + 10$$

$$18 + 18t^2 + 36t - 6t^2 - 8t - 10 = 0$$

$$12t^2 + 28t + 8 = 0 \quad \div 4$$

$$3t^2 + 7t + 2 = 0$$

$$3t^2 + 6t + t + 2 = 0$$

$$3t(t+2) + 1(t+2) = 0$$

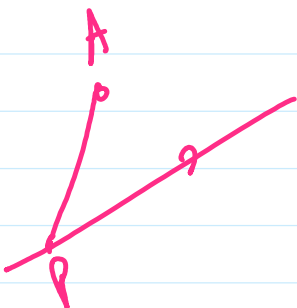
$$t = -2, \quad t = -\frac{1}{3}$$

$$t = -2, \quad \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 + \frac{2}{3} \\ 5 - \frac{1}{3} \\ 2 + \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$$

$$\begin{pmatrix} 5/3 \\ 14/3 \end{pmatrix}$$

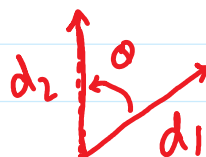


✓

$$\begin{pmatrix} 14/3 \\ 7/3 \end{pmatrix}$$

Angle between Lines

$$d_1 \cdot d_2 = |d_1| |d_2| \cos \alpha$$



6 With respect to the origin O , the points A and B have position vectors given by $\vec{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\vec{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line AB and OP is perpendicular to AB .

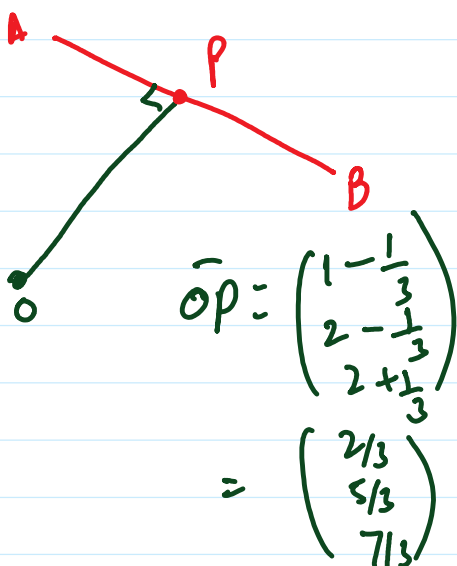
(i) Find a vector equation for the line AB . $\rightarrow r = \vec{OA} + t \vec{AB}$

[1]

(ii) Find the position vector of P .

[4]

Known



Wanted

$$r = \begin{pmatrix} 1+2t \\ 2+2t \\ 2-2t \end{pmatrix}$$

$$\vec{OP} \cdot d_{AB} = 0$$

$$\begin{pmatrix} 1+2t \\ 2+2t \\ 2-2t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 0$$

$$2 + 4t + 4 + 4t - 4 + 4t = 0$$

$$12t + 2 = 0 \quad \boxed{t = -\frac{1}{6}}$$

8 With respect to the origin O , the position vectors of two points A and B are given by $\vec{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\vec{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point P lies on the line through A and B , and $\vec{AP} = \lambda \vec{AB}$.

(i) Show that $\vec{OP} = (1 + 2\lambda)\mathbf{i} + (2 + 2\lambda)\mathbf{j} + (2 - 2\lambda)\mathbf{k}$.

[2]

(ii) By equating expressions for $\cos AOP$ and $\cos BOP$ in terms of λ , find the value of λ for which OP bisects the angle AOB .

[5]

(iii) When λ has this value, verify that $AP : PB = OA : OB$.

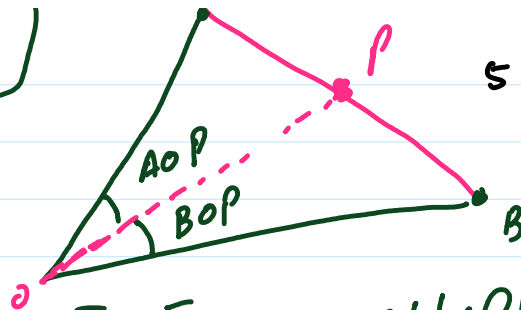
[1]

$$\cos AOP = \cos BOP$$



$$5 + 45 = 50 = 42\lambda + 33$$

$$\boxed{\cos AOP = \cos BOP}$$



$$5 + 45 = 42\lambda + 33$$

$$9 =$$

$$\vec{OA} \cdot \vec{OP} = |\vec{OA}| |\vec{OP}| \cos AOP$$

$$\frac{\vec{OA} \cdot \vec{OP}}{|\vec{OA}| |\vec{OP}|} = \cos AOP$$

$$\vec{OB} \cdot \vec{OP} = |\vec{OB}| |\vec{OP}| \cos BOP$$

$$\frac{\vec{OB} \cdot \vec{OP}}{|\vec{OB}| |\vec{OP}|} = \cos BOP$$

$$\frac{\vec{OA} \cdot \vec{OP}}{|\vec{OA}| |\vec{OP}|} = \frac{\vec{OB} \cdot \vec{OP}}{|\vec{OB}| |\vec{OP}|}$$

$$\vec{OA} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{OP} = \begin{pmatrix} 1+2\lambda \\ 2+2\lambda \\ 2-2\lambda \end{pmatrix}$$

$$\vec{OB} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

$$\vec{OP} = \begin{pmatrix} 7/4 \\ 11/4 \\ 5/4 \end{pmatrix}$$

$$\vec{OA} \cdot \vec{OP} = 1+2\lambda+4+4\lambda+4-4\lambda = 9+2\lambda$$

$$|\vec{OA}| = 3$$

$$\vec{OB} \cdot \vec{OP} = 3+6\lambda+8+8\lambda = 11+14\lambda$$

$$|\vec{OB}| = 5$$

$$\vec{AP} = \begin{pmatrix} 3/4 \\ 3/4 \\ -3/4 \end{pmatrix}$$

$$|\vec{AP}| = \sqrt{\frac{27}{16}} = \frac{\sqrt{27}}{4}$$

$$\frac{9+2\lambda}{3} = \frac{11+14\lambda}{5}$$

$$\Rightarrow 5(9+2\lambda) = 3(11+14\lambda)$$

$$45+10\lambda = 33+42\lambda$$

$$12 = 32\lambda$$

$$\lambda = \frac{3}{8}$$

$$\lambda = \frac{12}{32} = \frac{3}{8}$$

$$\vec{BP} = \begin{pmatrix} 7/4-3 \\ 11/4-4 \\ 5/4 \end{pmatrix} = \begin{pmatrix} -5/4 \\ -5/4 \\ 5/4 \end{pmatrix}$$

$$\frac{\sqrt{27}}{4} : \frac{\sqrt{75}}{4}$$

$$\frac{\sqrt{27}}{\sqrt{9 \times 3}} : \frac{\sqrt{75}}{\sqrt{25 \times 3}}$$

$$|\vec{BP}| = \frac{\sqrt{75}}{4}$$

$$3:5$$

$$3:5$$

$$\begin{aligned} \sqrt{27} &= \sqrt{9 \times 3} \\ &= 3\sqrt{3} \end{aligned}$$

5:5

- 10 The lines l and m have equations $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} - \mathbf{k})$ respectively, where a and b are constants.

(i) Given that l and m intersect, show that

$$2a - b = 4. \quad [4]$$

(ii) Given also that l and m are perpendicular, find the values of a and b . [4]

(iii) When a and b have these values, find the position vector of the point of intersection of l and m . [2]