

12.5 | Lines and Planes in Space

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space. We will use these representations throughout the rest of the book.

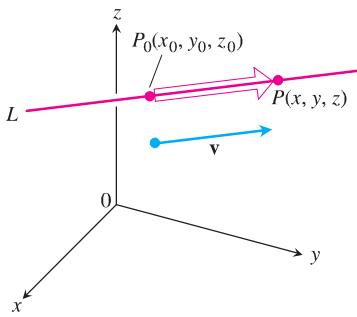


FIGURE 12.35 A point P lies on L through P_0 parallel to \mathbf{v} if and only if $\overrightarrow{P_0P}$ is a scalar multiple of \mathbf{v} .

Lines and Line Segments in Space

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a *vector* giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to \mathbf{v} (Figure 12.35). Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter t . The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \quad (1)$$

If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and \mathbf{r}_0 is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty, \quad (2)$$

where \mathbf{r} is the position vector of a point $P(x, y, z)$ on L and \mathbf{r}_0 is the position vector of $P_0(x_0, y_0, z_0)$.

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter t :

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \quad (3)$$

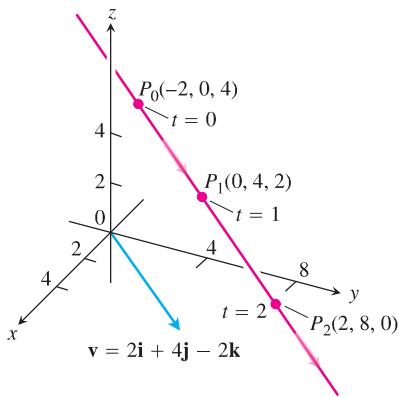


FIGURE 12.36 Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing \$t\$.

EXAMPLE 1 Find parametric equations for the line through \$(-2, 0, 4)\$ parallel to \$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}\$ (Figure 12.36).

Solution With \$P_0(x_0, y_0, z_0)\$ equal to \$(-2, 0, 4)\$ and \$v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\$ equal to \$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}\$, Equations (3) become

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t. \quad \blacksquare$$

EXAMPLE 2 Find parametric equations for the line through \$P(-3, 2, -3)\$ and \$Q(1, -1, 4)\$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with \$(x_0, y_0, z_0) = (-3, 2, -3)\$ give

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We could have chosen \$Q(1, -1, 4)\$ as the “base point” and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of \$t\$. \blacksquare

Notice that parametrizations are not unique. Not only can the “base point” change, but so can the parameter. The equations \$x = -3 + 4t^3\$, \$y = 2 - 3t^3\$, and \$z = -3 + 7t^3\$ also parametrize the line in Example 2.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the \$t\$-values for the endpoints and restrict \$t\$ to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.

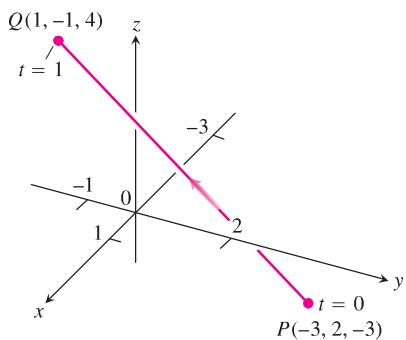


FIGURE 12.37 Example 3 derives a parametrization of line segment \$PQ\$. The arrow shows the direction of increasing \$t\$.

EXAMPLE 3 Parametrize the line segment joining the points \$P(-3, 2, -3)\$ and \$Q(1, -1, 4)\$ (Figure 12.37).

Solution We begin with equations for the line through \$P\$ and \$Q\$, taking them, in this case, from Example 2:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through \$P(-3, 2, -3)\$ at \$t = 0\$ and \$Q(1, -1, 4)\$ at \$t = 1\$. We add the restriction \$0 \leq t \leq 1\$ to parametrize the segment:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1. \quad \blacksquare$$

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position \$P_0(x_0, y_0, z_0)\$ and moving in the direction of vector \$\mathbf{v}\$. Rewriting Equation (2), we have

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \mathbf{r}_0 + t|\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (4)\end{aligned}$$

Initial position Time Speed Direction

In other words, the position of the particle at time t is its initial position plus its distance moved (speed \times time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle.\end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60 \text{ ft/sec})(10 \text{ sec}) = 600 \text{ ft}$, which is the length of the vector $\mathbf{r}(10)$. ■

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector \mathbf{v} , we find the absolute value of the scalar component of \overrightarrow{PS} in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}| \sin \theta$, which is $\frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

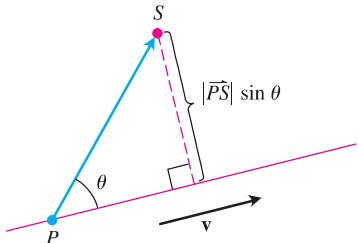


FIGURE 12.38 The distance from S to the line through P parallel to \mathbf{v} is $|\overrightarrow{PS}| \sin \theta$, where θ is the angle between \overrightarrow{PS} and \mathbf{v} .

Distance from a Point S to a Line Through P Parallel to \mathbf{v}

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}. \blacksquare$$

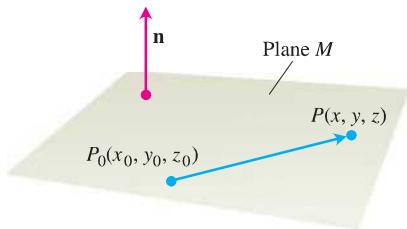


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to \mathbf{n} if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to \mathbf{n} (Figure 12.39). Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where

$$D = Ax_0 + By_0 + Cz_0$$

EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22. \blacksquare$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x , y , and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6. \quad \blacksquare$$

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k . Two planes that are not parallel intersect in a line.

EXAMPLE 8 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to both planes' normal vectors \mathbf{n}_1 and \mathbf{n}_2 (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■

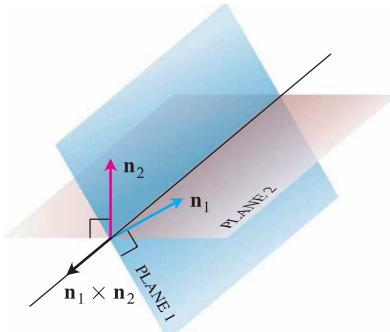


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

EXAMPLE 9 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z . The different choices would simply give different parametrizations of the same line. ■

Sometimes we want to know where a line and a plane intersect. For example, if we are looking at a flat plate and a line segment passes through it, we may be interested in knowing what portion of the line segment is hidden from our view by the plate. This application is used in computer graphics (Exercise 74).

EXAMPLE 10 Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t \right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1 \right) = \left(\frac{2}{3}, 2, 0 \right). \quad \blacksquare$$

The Distance from a Point to a Plane

If P is a point on a plane with normal \mathbf{n} , then the distance from any point S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} . That is, the distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of \overrightarrow{PS} onto a vector \mathbf{n} normal to the plane (Figure 12.41). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

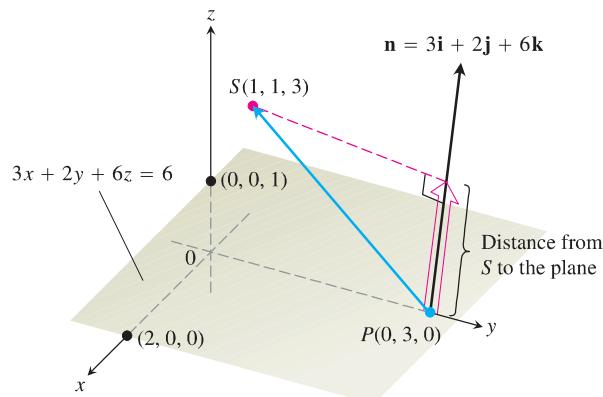


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} (Example 11).

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y -intercept $(0, 3, 0)$, then

$$\begin{aligned}\overrightarrow{PS} &= (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} \\ &= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \\ |\mathbf{n}| &= \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.\end{aligned}$$

The distance from S to the plane is

$$\begin{aligned}d &= \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| && \text{length of proj}_{\mathbf{n}} \overrightarrow{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}.\end{aligned}$$
■

Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure 12.42).

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned}\theta &= \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{4}{21} \right) \\ &\approx 1.38 \text{ radians.} && \text{About 79 deg}\end{aligned}$$
■

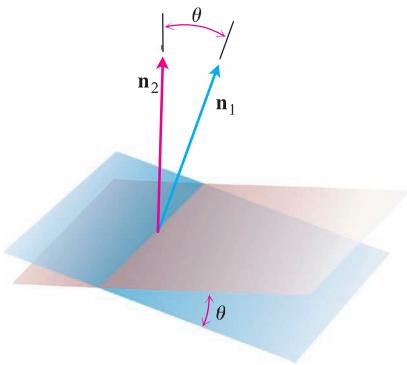


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

Exercises 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.

1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z -axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$

9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$

10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

11. The x -axis
12. The z -axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.

13. $(0, 0, 0), (1, 1, 3/2)$
14. $(0, 0, 0), (1, 0, 0)$
15. $(1, 0, 0), (1, 1, 0)$
16. $(1, 1, 0), (1, 1, 1)$
17. $(0, 1, 1), (0, -1, 1)$
18. $(0, 2, 0), (3, 0, 0)$
19. $(2, 0, 2), (0, 2, 0)$
20. $(1, 0, -1), (0, 3, 0)$

Planes

Find equations for the planes in Exercises 21–26.

21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
 22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
 24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
 25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
 27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
 28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane determined by the intersecting lines.

29. L1: $x = -1 + t$, $y = 2 + t$, $z = 1 - t$; $-\infty < t < \infty$
 L2: $x = 1 - 4s$, $y = 1 + 2s$, $z = 2 - 2s$; $-\infty < s < \infty$
 30. L1: $x = t$, $y = 3 - 3t$, $z = -2 - t$; $-\infty < t < \infty$
 L2: $x = 1 + s$, $y = 4 + s$, $z = -1 + s$; $-\infty < s < \infty$
 31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.
 32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.

33. $(0, 0, 12)$; $x = 4t$, $y = -2t$, $z = 2t$
 34. $(0, 0, 0)$; $x = 5 + 3t$, $y = 5 + 4t$, $z = -3 - 5t$
 35. $(2, 1, 3)$; $x = 2 + 2t$, $y = 1 + 6t$, $z = 3$
 36. $(2, 1, -1)$; $x = 2t$, $y = 1 + 2t$, $z = 2t$
 37. $(3, -1, 4)$; $x = 4 - t$, $y = 3 + 2t$, $z = -5 + 3t$
 38. $(-1, 4, 3)$; $x = 10 + 4t$, $y = -3$, $z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4)$, $x + 2y + 2z = 13$
 40. $(0, 0, 0)$, $3x + 2y + 6z = 6$
 41. $(0, 1, 1)$, $4y + 3z = -12$
 42. $(2, 2, 3)$, $2x + y + 2z = 4$
 43. $(0, -1, 0)$, $2x + y + 2z = 4$
 44. $(1, 0, -1)$, $-4x + y + z = 4$
 45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.
 46. Find the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1$, $2x + y - 2z = 2$
 48. $5x + y - z = 10$, $x - 2y + 3z = -1$

- T Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3$, $2x - 2y - z = 5$
 50. $x + y + z = 1$, $z = 0$ (the xy -plane)
 51. $2x + 2y - z = 3$, $x + 2y + z = 2$
 52. $4y + 3z = -12$, $3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t$, $y = 3t$, $z = 1 + t$; $2x - y + 3z = 6$
 54. $x = 2$, $y = 3 + 2t$, $z = -2 - 2t$; $6x + 3y - 4z = -12$
 55. $x = 1 + 2t$, $y = 1 + 5t$, $z = 3t$; $x + y + z = 2$
 56. $x = -1 + 3t$, $y = -2$, $z = 5t$; $2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1$, $x + y = 2$
 58. $3x - 6y - 2z = 3$, $2x + y - 2z = 2$
 59. $x - 2y + 4z = 2$, $x + y - 2z = 5$
 60. $5x - 2y = 11$, $4y - 5z = -17$

Given two lines in space, either they are parallel, or they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky). Exercises 61 and 62 each give three lines. In each exercise, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection.

61. L1: $x = 3 + 2t$, $y = -1 + 4t$, $z = 2 - t$; $-\infty < t < \infty$
 L2: $x = 1 + 4s$, $y = 1 + 2s$, $z = -3 + 4s$; $-\infty < s < \infty$
 L3: $x = 3 + 2r$, $y = 2 + r$, $z = -2 + 2r$; $-\infty < r < \infty$
 62. L1: $x = 1 + 2t$, $y = -1 - t$, $z = 3t$; $-\infty < t < \infty$
 L2: $x = 2 - s$, $y = 3s$, $z = 1 + s$; $-\infty < s < \infty$
 L3: $x = 5 + 2r$, $y = 1 - r$, $z = 8 + 3r$; $-\infty < r < \infty$

7. (a) $10 + \sqrt{17}, \sqrt{26}, \sqrt{21}$ (b) $\frac{10 + \sqrt{17}}{\sqrt{546}}$

(c) $\frac{10 + \sqrt{17}}{\sqrt{26}}$ (d) $\frac{10 + \sqrt{17}}{26}(5\mathbf{i} + \mathbf{j})$

9. 0.75 rad 11. 1.77 rad

13. Angle at $A = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435$ degrees, angle at $B = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130$ degrees, angle at $C = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435$ degrees.

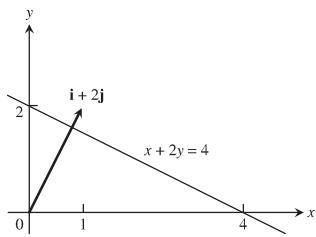
23. Horizontal component: ≈ 1188 ft/sec, vertical component: ≈ 167 ft/sec

25. (a) Since $|\cos \theta| \leq 1$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\cos \theta| \leq |\mathbf{u}||\mathbf{v}|(1) = |\mathbf{u}||\mathbf{v}|$.

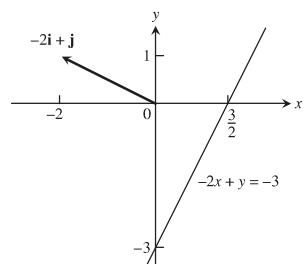
(b) We have equality precisely when $|\cos \theta| = 1$ or when one or both of \mathbf{u} and \mathbf{v} are $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π , that is, when the vectors are parallel.

27. a

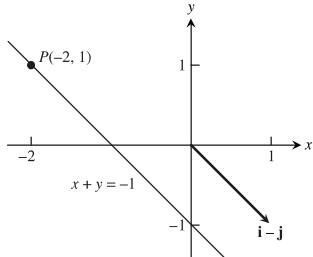
33. $x + 2y = 4$



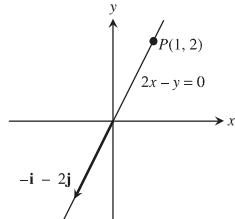
35. $-2x + y = -3$



37. $x + y = -1$



39. $2x - y = 0$



41. 5 J 43. 3464 J 45. $\frac{\pi}{4}$ 47. $\frac{\pi}{6}$ 49. 0.14

Section 12.4, pp. 686–688

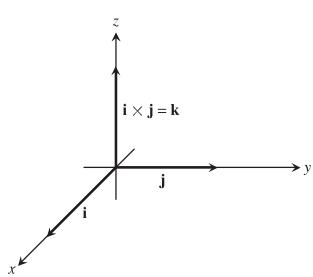
1. $|\mathbf{u} \times \mathbf{v}| = 3$, direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 3$, direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

3. $|\mathbf{u} \times \mathbf{v}| = 0$, no direction; $|\mathbf{v} \times \mathbf{u}| = 0$, no direction

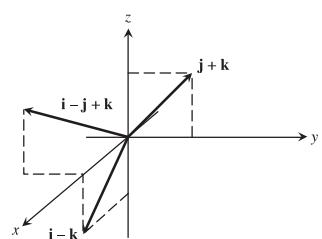
5. $|\mathbf{u} \times \mathbf{v}| = 6$, direction is $-\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 6$, direction is \mathbf{k}

7. $|\mathbf{u} \times \mathbf{v}| = 6\sqrt{5}$, direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$; $|\mathbf{v} \times \mathbf{u}| = 6\sqrt{5}$, direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

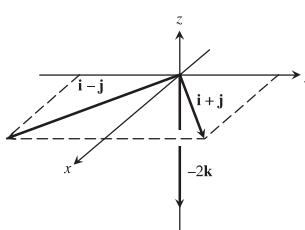
9.



11.



13.



15. (a) $2\sqrt{6}$ (b) $\pm \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

17. (a) $\frac{\sqrt{2}}{2}$ (b) $\pm \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

19. 8 21. 7 23. (a) None (b) \mathbf{u} and \mathbf{w} 25. $10\sqrt{3}$ ft-lb

27. (a) True (b) Not always true (c) True (d) True (e) Not always true (f) True (g) True (h) True

29. (a) $\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}$ (b) $\pm \mathbf{u} \times \mathbf{v}$ (c) $\pm (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ (d) $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ (e) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w})$ (f) $|\mathbf{u}| \frac{\mathbf{v}}{|\mathbf{v}|}$

31. (a) Yes (b) No (c) Yes (d) No

33. No, \mathbf{v} need not equal \mathbf{w} . For example, $\mathbf{i} + \mathbf{j} \neq -\mathbf{i} + \mathbf{j}$, but $\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = -\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.

35. 2 37. 13 39. $\sqrt{129}$ 41. $\frac{11}{2}$ 43. $\frac{25}{2}$

45. $\frac{3}{2}$ 47. $\frac{\sqrt{21}}{2}$

49. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

and the triangle's area is

$$\frac{1}{2} \left| \mathbf{A} \times \mathbf{B} \right| = \pm \frac{1}{2} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

The applicable sign is (+) if the acute angle from \mathbf{A} to \mathbf{B} runs counterclockwise in the xy -plane, and (-) if it runs clockwise.

Section 12.5, pp. 694–696

1. $x = 3 + t, y = -4 + t, z = -1 + t$

3. $x = -2 + 5t, y = 5t, z = 3 - 5t$

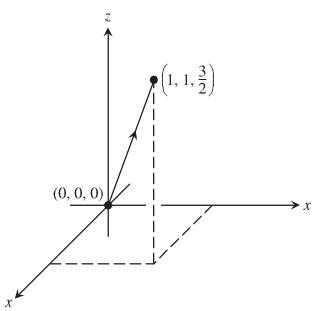
5. $x = 0, y = 2t, z = t$

7. $x = 1, y = 1, z = 1 + t$

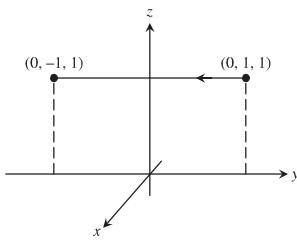
9. $x = t, y = -7 + 2t, z = 2t$

11. $x = t, y = 0, z = 0$

13. $x = t, y = t, z = \frac{3}{2}t, 0 \leq t \leq 1$

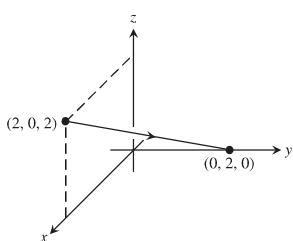


17. $x = 0, y = 1 - 2t, z = 1, 0 \leq t \leq 1$



21. $3x - 2y - z = -3$

19. $x = 2 - 2t, y = 2t, z = 2 - 2t, 0 \leq t \leq 1$



23. $7x - 5y - 4z = 6$

25. $x + 3y + 4z = 34$

27. $(1, 2, 3), -20x + 12y + z = 7$

29. $y + z = 3$

31. $x - y + z = 0$

33. $2\sqrt{30}$

35. $\frac{9\sqrt{42}}{7}$

37. $39. 3$

41. $19/5$

43. $5/3$

45. $9/\sqrt{41}$

47. $\pi/4$

49. 1.38 rad

51. 0.82 rad

53. $\left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$

55. $(1, 1, 0)$

57. $x = 1 - t, y = 1 + t, z = -1$

59. $x = 4, y = 3 + 6t, z = 1 + 3t$

61. L_1 intersects L_2 ; L_2 is parallel to L_3 ; L_1 and L_3 are skew.

63. $x = 2 + 2t, y = -4 - t, z = 7 + 3t; x = -2 - t, y = -2 + (1/2)t, z = 1 - (3/2)t$

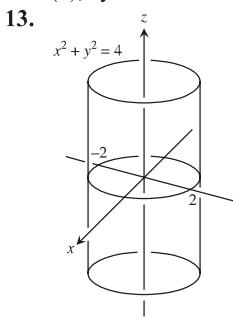
65. $\left(0, -\frac{1}{2}, -\frac{3}{2}\right), (-1, 0, -3), (1, -1, 0)$

69. Many possible answers. One possibility: $x + y = 3$ and $2y + z = 7$.

71. $(x/a) + (y/b) + (z/c) = 1$ describes all planes except those through the origin or parallel to a coordinate axis.

Section 12.6, pp. 700–701

1. (d), ellipsoid 3. (a), cylinder 5. (l), hyperbolic paraboloid
 7. (b), cylinder 9. (k), hyperbolic paraboloid 11. (h), cone



15.

