Measure Theory - MATH0017

Contents

1 Preliminaries

Definition 1.

1. $f : X \to Y$ is an injective map if and only if $f(x) = f(y) \implies x = y$.

2. $f : X \to Y$ is a surjective map if and only if $\{f(x) \mid x \in X\}$ is the whole of Y.

3. $f : X \to Y$ is a bijective map if and only if it is both injective and surjective.

Definition 2. Given two (finite or infinite) sets X and Y, if there exists an injective map $f : X \to Y$ then we say that the cardinality of X is less than or equal to the cardinality of Y (and in this case we write $\#X \leq \#Y$).

Definition 3. If $\#X \leq \#Y$ and $\#Y \leq \#X$ then we say that X and Y have the same cardinality (and in this case we write $\#X = \#Y$).

Definition 4. A set X is called countable if $\#X \leq \#\mathbb{N}$.

Definition 5. The cardinality of $(0, 1)$ is called continuum.

Definition 6. A set $D \subset \mathbb{R}$ has (Lebesgue) measure zero if for every $\epsilon > 0$ there is a countable collection of open intervals $I_j = (a_j, b_j)$ such that $D \subset \cup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \epsilon$, where $|I_j| = b_j - a_j$ denotes the length of the interval I_j.

2 σ-algebras and measures

Definition 7. A σ-algebra on $\mathcal{A}$ on a set X is a family of subsets of X with the following properties:

1. $X \in \mathcal{A}$
2. **(closedness under complementation)**
   $A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}$
3. **(closedness under countable unions)**
   $\{A_j\}_{j\in\mathbb{N}} \subset A \implies \cup_{j\in\mathbb{N}} \in \mathcal{A}$

Definition 8. A (positive) measure μ on a space X is an assignment

$$\mu : A \to [0, +\infty]$$

defined on a σ-algebra $\mathcal{A}$, such that

1. $\mu(\emptyset) = 0$
2. **(σ-additivity)** $\{A_j\}_{j=1}^{\infty} \subset \mathcal{A}$, A_j pairwise disjoint $\implies \mu(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$

The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. On the other hand $(X, \mathcal{A})$ is called a *measurable space*.

Other properties of σ-algebras and measures.

1. Let $A \subset X, B \subset X$. If $A, B \in \mathcal{A}$ then $B \setminus A \in \mathcal{A}$
2. Let $A \in \mathcal{A}$ with $\mu(A)$ finite. We then also have $\mu(X \setminus A) = \mu(X) - \mu(A)$.
3. Let $A, B \in \mathcal{A}, A \subset B$. Then $\mu(A) \leq \mu(B)$.

Proposition 1. Whenever $\{\mathcal{A}_j\}_{j\in J}$ is a family of σ-algebras in X, then the intersection $\cap_{j\in J}\mathcal{A}_j$ is again a σ-algebra in X.

Definition 9. Given any family $\mathcal{G} \subset \mathcal{P}(X)$ (i.e., any family of subsets of X) there exists a 'smallest' σ-algebra containing $\mathcal{G}$. This is called the σ-algebra generated by $\mathcal{G}$ and denoted by $\sigma(\mathcal{G})$.

3 Outer measures

Definition 10. An outer measure μ on a space X is an assignment (defined on all subsets of X) $\mu : \mathcal{P}(X) \to [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$
2. **(monotonicity)** $A \subset B \implies \mu(A) \le \mu(B)$
3. **(σ-subadditivity)** $A_j \subset X, \{A_j\}_{j=1}^\infty \implies \mu(\cup_{j=1}^\infty A_j) \le \sum_{j=1}^\infty \mu(A_j)$

2. and 3. can equivalently be phrased as

$$A \subset \cup_{j=1}^\infty A_j \implies \mu(A) \le \sum_{j=1}^\infty \mu(A_j)$$

Definition 11. Let μ be an outer measure on a space X. A set $A \subset X$ is said to be μ-measurable if for every $B \subset X$ we have

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

Theorem 3.1.1. Let μ be an outer measure on a space X. The family

$$\Sigma = \{A \subset X \mid A \text{ is } \mu\text{-measurable}\}$$

is a σ-algebra on X.

Proposition 2. Let μ be an outer measure on X and let Σ be the σ-algebra of μ-measurable sets. Then whenever $\{A_k\}_{k=1}^\infty \subset \Sigma$ and A_k are pairwise disjoint we have $\mu(\cup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k)$.

Proposition 3. Let μ be an outer measure on X and let $A \subset X$ be such that $\mu(A) = 0$. Then A is μ-measurable.

Definition 12. A measure space $(X, \mathcal{A}, \mu)$ is complete if, whenever $D \in \mathcal{A}$ is such that $\mu(D) = 0$, and $N \subset D$, then $N \in \mathcal{A}$ (it follows that $\mu(N) = 0$ as well).

Definition 13. We say that $\mathcal{S} \subset \mathcal{P}(X)$ is a covering class (for X) when the following two properties hold:

1. $\emptyset \in \mathcal{S}$
2. for any $A \subset X$ there exists a countable collection of sets in $\mathcal{S}$ that 'covers A', i.e., for any $A \subset X$, there exists $\{S_j\}_{j=1}^{\infty}$ such that $S_j \in \mathcal{S}$ for every j and $A \subset \cup_{j=1}^{\infty} S_j$. (Note that for a given A there might be plenty of choices of $\{S_j\}_{j=1}^{\infty}$ that cover A. Each such choice $\{S_j\}_{j=1}^{\infty}$ is a possible 'cover', or 'covering', of A.)

Theorem 3.2.1. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a covering class of X and $\lambda : \mathcal{S} \to [0, \infty]$ be given, with $\lambda(\emptyset) = 0$. Then, the assignment $\mu : \mathcal{P}(X) \to [0, +\infty]$ defined by

$$\mu(A) := \inf\Big\{\sum_{j=1}^{\infty} \lambda(S_j) \mid S_j \in \mathcal{S} \text{ and } A \subset \cup_{j=1}^{\infty} S_j\Big\}$$

is an outer measure on X.

4 From pre-measures to outer measures - Lebesgue measure in $\mathbb{R}^n$

Definition 14. A collection $\mathcal{S} \subset \mathcal{P}(X)$ is a semi-ring (or semi-algebra) on X if it satisfies the three properties below

1. $\mathcal{S}$ contains $\emptyset$
2. if $R_1, R_2 \in \mathcal{S}$, then $R_1 \cap R_2 \in \mathcal{S}$
3. if $R_1, R_2 \in \mathcal{S}$, then there exists finitely many disjoint $T_1, \ldots, T_S \subset \mathcal{S}$ such that $R_1 \setminus R_2 = \cup_{s=1}^{S} T_s$

We denote the set of half-open rectangles in $\mathbb{R}^n$ by $\mathcal{I}^n$.

Definition 15. Let $\mathcal{S} \subset \mathcal{P}(X)$ be a semi-ring. An assignment $\lambda : \mathcal{S} \to [0, +\infty]$ is called a pre-measure if

1. $\lambda(\emptyset) = 0$
2. if $\{R_j\}_{j=1}^{\infty} \subset \mathcal{S}, R_j \cap R_k = \emptyset$ when $j \neq k$, $\cup_{j=1}^{\infty} R_j \in \mathcal{S}$ then $\lambda(\cup_{j=1}^{\infty} R_j) = \sum_{j=1}^{\infty} \lambda(R_j)$

Theorem 4.2.1. Let $\mathcal{S} = \mathcal{I}^n$, λ be defined as

$$\lambda([a, b)) = \Pi_{j=1}^{n}(b_j - a_j) \qquad (\star)$$

and let μ be the outer measure constructed from $\mathcal{S}$ and λ in Theorem 3.2.1. Then (i) λ and μ agree on $\mathcal{I}^n$ and moreover (ii) every $\mathcal{R} \in \mathcal{I}^n$ is μ-measurable.

Theorem 4.2.2 Let $\mathcal{S} \subset \mathcal{P}(X)$ be a covering class and a semi-ring and let $\lambda : \mathcal{S} \to [0, +\infty]$ be a pre-measure. Denote by μ the outer measure constructed from $\mathcal{S}$ and λ in Theorem 3.2.1. Then λ and μ agree on $\mathcal{S}$ and every $S \in \mathcal{S}$ is μ-measurable.

Lebesgue measure: We have obtained, in Theorem 4.2.1 an outer measure μ that extends the assignment λ defined in $(\star)$ and such that half-open rectangles are μ-measurable. Restricting the action of μ to the σ-algebra of the μ-measurable sets gives rise to the *n-dimensional Lebesgue measure* on $\mathbb{R}^n$, that we denote by λ^n.

Proposition 4. The σ-algebra generated by $\mathcal{I}^n$ is the same as the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ (i.e., the σ-algebra generated by open sets of $\mathbb{R}^n$). This implies in particular that $\mathcal{B}(\mathbb{R}^n) \subset \Sigma_n$, which is usually phrased by saying that every Borel set (i.e. every element of $\mathcal{B}(\mathbb{R}^n)$) is Lebesgue-measurable.

Lemma 4.3.1 Every open set in $\mathbb{R}^n$ is the countable union of half-open rectangles.

For $y \in \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$ we write $y + Z := \{x \in \mathbb{R}^n \mid x = y + z \text{ for some } z \in Z\}$ (the translation of Z by the vector y).

Proposition 5. The Lebesgue measure is translation invariant, namely, whenever $Z \in \Sigma_n$ then $y + Z \in \Sigma_n$ for every $y \in \mathbb{R}^n$ and moreover $\lambda^n(y + Z) = \lambda^n(Z)$

Theorem 4.4.1 Let $\mathcal{S}$ be a semi-algebra on X and λ a pre-measure defined on $\mathcal{S}$. Assume that there exists a countable collection of R_j in $\mathcal{S}$ such that $R_j \subset R_{j+1}$ for all j and $\cup_{j=1}^{\infty} R_j = X$ (this is called an exhausting sequence) with $\lambda(R_j) < \infty$ for all $j \in \mathbb{N}$. Let μ be the outer measure defined through Carathéodory's Theorem and Σ the σ-algebra of the μ-measurable sets.
Given any outer measure $\tilde{\mu} : \mathcal{P}(X) \to [0, \infty]$ with the property that $\tilde{\mu} = \lambda$ on $\mathcal{S}$, we have necessarily that $\tilde{\mu} = \mu$ on Σ.

Definition 16. A measure space $(X, \mathcal{A}, \mu)$ is said to be σ-finite if there exists $A_j \in \mathcal{A}$ such that $A_j \subset A_{j+1}$ for all j, $\cup_{j=1}^{\infty} A_j = X$ and $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$. (By abuse of language, the measure μ in this case is also called σ-finite.) If $\mu(X) < \infty$ then we say that μ is a finite measure on X and $(X, \mathcal{A}, \mu)$ is a finite measure space.

5 Hausdorff measures

Let $A \subset \mathbb{R}^n$ be arbitrary. For fixed $\alpha \geq 0$ and $\delta > 0$ we consider all possible coverings of A using a collection of balls with radii $\leq \delta$. Set

$$\mathcal{H}^\alpha_\delta(A) = \inf\Big\{\sum_{k=1}^{\infty}(r_k)^\alpha \mid A \subset \cup_{k=1}^{\infty} B_{r_k}(x_k), r_k \leq \delta \text{ for all } k\Big\}$$

We set

$$\mathcal{H}^\alpha(A) = \lim_{\delta\to 0}\mathcal{H}^\alpha_\delta(A) = \sup_{\delta>0}\mathcal{H}^\alpha_\delta(A)$$

Proposition 6. For any $\alpha \geq 0$, $\mathcal{H}^\alpha$ is an outer measure on $\mathbb{R}^n$.

Proposition 7. For $0 \leq \beta < \alpha < \infty$ and $A \subset \mathbb{R}^n$ it holds

$$\mathcal{H}^\beta(A) < \infty \implies \mathcal{H}^\alpha(A) = 0$$

$$\mathcal{H}^\alpha(A) > 0 \implies \mathcal{H}^\beta(A) = +\infty$$

Definition 17. Given $A \subset \mathbb{R}^n$ we define

$$\dim_H(A) = \inf\{\alpha > 0 \mid \mathcal{H}^\alpha(A) = 0\}$$

to be the Hausdorff dimension of A. Note that $\dim_H(A) \in [0, n]$.

Proposition 8. For any $\alpha \geq 0$ Borel sets are $\mathcal{H}^\alpha$-measurable.

Definition 18. An outer measure μ on $\mathbb{R}^n$ is called 'metric' if whenever $A, B \subset \mathbb{R}^n$ are such that $dist(A, B) = \inf\{|x-y| \mid x \in A, y \in B\} > 0$ (i.e. A and B are a positive distance apart) then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proposition 9. For $\alpha \geq 0$, $\mathcal{H}^\alpha$ is a metric outer measure (on $\mathbb{R}^n$).

Proposition 10 (Carathéodory criterium for Borel measures)**.** Given a metric outer measure μ, every Borel set is μ-measurable.

6 Measurable functions

Definition 19. Let $(X, \mathcal{A})$ be a measurable space and consider $f : X \to \mathbb{R}$, where the target space $\mathbb{R}$ is implicitly endowed with the Borel σ-algebra. We say that f is $\mathbb{A}$-measurable when $f^{-1}(A) \in \mathcal{A}$ whenever A is open in $\mathbb{R}$.

Given an outer measure μ on X it is common to speak of μ-measurable function $f : X \to \mathbb{R}$ with the implicit meaning that X is equipped with the σ-algebra Σ of μ-measurable sets and f is Σ-measurable.

Proposition 11. The collection $\{B \subset \mathbb{R} \mid f^{-1}(B) \in \mathcal{A}\}$ is a σ-algebra.

Definition 20. Let $(X, \mathcal{A})$ be a measurable space and consider $f : X \to \overline{\mathbb{R}}$. We say that f is $\mathcal{A}$-measurable when

$f^{-1}(\{+\infty\}) \in \mathcal{A}, f^{-1}(\{-\infty\}) \in \mathcal{A}, f^{-1}(A) \in \mathcal{A}$ whenever A is open in $\mathbb{R}$.

The condition

$$f^{-1}(A) \in \mathcal{A} \text{ whenever } A \text{ is open in } \mathbb{R}$$

in Definition 20 can be replaced by either of the following:

1. $f^{-1}(B) \in \mathcal{A}$ whenever $B \in \mathcal{B}(\mathbb{R})$
2. $f^{-1}((a, +\infty)) \in \mathcal{A}$ whenever $a \in \mathbb{R}$
3. $f^{-1}((-\infty, a)) \in \mathcal{A}$ whenever $a \in \mathbb{R}$.

Definition 21. A simple function $g : X \to \mathbb{R}$ on a measurable space $(X, \mathcal{A})$ is a function of the form

$$g(x) = \sum_{j=1}^{M} y_j \mathbb{1}_{A_j}(x), y_j \in \mathbb{R}, A_j \text{ pairwise disjoint}, A_j \in \mathcal{A} \text{ for } j \in \{1, \ldots, M\}$$

Theorem 6.2.1. Any $\mathcal{A}$-measurable function $f : X \to \overline{\mathbb{R}}$ on $(X, \mathcal{A})$ is the pointwise limit of simple functions f_j. Moreover if $f \geq 0$ then we can choose $f_j \geq 0$ and $f_j \leq f_{j+1}$ for all $j \in \mathbb{N}$ (so that $f = \sup_{j \in \mathbb{N}} f_j$).

When $f_j \to f$ pointwise and $f_j \leq f_{j+1}$ (increasing sequence) we write $f_j \uparrow f$

Theorem 6.2.2. Let $f, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable functions on $(X, \mathcal{A})$. Then the functions $f + g, f - g, fg$ and (when $g \neq 0$) f/g are $\mathcal{A}$-measurable. The same holds for the function $|f|$.

Theorem 6.2.3. Let $f_j : X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable functions on $(X, \mathcal{A})$ for $j \in \mathbb{N}$. Then the functions $\inf_j f_j, \sup_j f_j, \limsup_j f_j$ and $\liminf_j f_j$ are $\mathcal{A}$-measurable.

Corollary 6.2.1. Let $f, g : X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable functions on $(X, \mathcal{A})$. Then the functions $\min\{f, g\}$ and $\max\{f, g\}$ are $\mathcal{A}$-measurable.

7 Lebesgue integration

Definition 22. Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $g : X \to \mathbb{R}$ is a simple function when g admits a representation of the following form for some $M \in \mathbb{N}, y_j \in \mathbb{R}$ and $A_j \in \mathcal{A}$ (for $j = 1, 2, \ldots, M$) pairwise disjoint:

$$g = \sum_{j=1}^{M} y_j \mathbb{1}_{A_j}$$

We write, in the case that $g \geq 0, g \in \mathcal{E}^+$.

Lemma 7.1.1. Let $g \geq 0$ admit two distinct representation as a simple function, say $g = \sum_{j=1}^{M} y_j \mathbb{1}_{A_j} = \sum_{k=1}^{N} x_k \mathbb{1}_{B_k}$. Then

$$\sum_{j=1}^{M} y_j \mu(A_j) = \sum_{k=1}^{N} x_k \mu(B_k)$$

Definition 23. Given a measure space $(X, \mathcal{A}, \mu)$ and $f : X \to \overline{\mathbb{R}}$ $\mathcal{A}$-measurable and non-negative, we define the integral of f on X with respect to μ as follows:

$$\int_X f \, d\mu = \sup\{I_\mu(g) \mid g \in \mathcal{E}^+, g \leq f\} \in [0, +\infty]$$

Lemma 7.1.2. Let $g \geq 0$ be a simple function, say $g = \sum_{j=1}^{M} y_j \mathbb{1}_{A_j}$. Then

$$\int_X g \, d\mu = \sum_{j=1}^{M} y_j \mu(A_j)$$

Remark 7.1.1.

1. If $g, h \in \mathcal{E}^+$ then $g + h \in \mathcal{E}^+$ and $I_\mu(g + h) = I_\mu(g) + I_\mu(h)$
2. If $g \in \mathcal{E}^+$ and $\lambda \in [0, \infty)$ then $\lambda g \in \mathcal{E}^+$ and $I_\mu(\lambda g) = \lambda I_\mu(g)$

Proposition 12. Let $f, g : X \to \overline{\mathbb{R}}$ be $\mathcal{A}$-measurable and non-negative functions on $(X, \mathcal{A}, \mu)$. Assume that $f \geq g$. Then $\in_X f \, d\mu \geq \int_X g \, d\mu$.

Theorem 7.2.1. (Fatou's lemma). Let $(X, \mathcal{A}, \mu)$ be a measure space and consider a sequence $\{f_k\}_{k=1}^{\infty}$ of measurable functions $f_k : X \to \overline{\mathbb{R}}$ with $f_k \geq 0$ for all k. Then

$$\int_X (\lim_{k\to\infty} f_k)\, d\mu \leq \lim_{k\to\infty} \Big(\int_X f_k\, d\mu \Big)$$

Theorem 7.2.2. (Beppo Levi's theorem, or 'monotone convergence' theorem). Let $(X, \mathcal{A}, \mu)$ be a measure space and consider a sequence $\{f_j\}_{j=1}^{\infty}$ of measurable functions $f_j : X \to \overline{\mathbb{R}}$ with $0 \leq f_1 \leq f_2 \leq \ldots \leq f_m \leq f_{m+1} \leq \ldots$. Then

$$\int_X (\sup_j f_j)\, d\mu = \sup_j (\int_X f_j\, d\mu)$$

or equivalently, by Proposition 12,

$$\int_X (\lim_j f_j)\, d\mu = \lim_j (\int_X f_j\, d\mu)$$

Corollary 7.2.1. Let $f_j : X \to \overline{\mathbb{R}}$ be an increasing sequence of non-negative $\mathcal{A}$-measurable functions, and let $f_j \uparrow f$ pointwise. Then $\int_X f_j\, d\mu \to \int_X f\, d\mu$ ($f : X \to \overline{\mathbb{R}}$ is $\mathcal{A}$-measurable by Theorem 6.2.3 and clearly $f \geq 0$).

Proposition 13. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable and non-negative on $(X, \mathcal{A}, \mu)$, let $\alpha \in (0, +\infty)$. Then

1. $\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$
2. $\int_X (\alpha f)\, d\mu = \alpha \int_X f\, d\mu$

Corollary 7.2.2. Let $u_j \geq 0$ be measurable, $u_j : X \to \overline{\mathbb{R}}$. Then $\int_X (\sum_{j\in\mathbb{N}} u_j)\, d\mu = \sum_{j\in\mathbb{N}} \int_X u_j\, d\mu$.

Theorem 7.2.3. ('reverse Fatou's lemma' and 'dominated convergence' for non-negative functions). Let $u : X \to \overline{\mathbb{R}}$ be a non-negative $\mathcal{A}$-measurable function on the measure space $(X, \mathcal{A}, \mu)$ with $\int_X u\, d\mu < \infty$.

Let $f_j : X \to \mathbb{R}$ be a sequence of non-negative $\mathcal{A}$-measurable functions such that $f_j \leq u$ for all $j \in \mathbb{N}$. Assume that $f_j \to f$ pointwise, where $f : X \to \overline{\mathbb{R}}$ (this function is $\mathcal{A}$-measurable by Theorem 6.2.3.). Then

$$\int_X f \, d\mu = \lim_{j \to \infty} f_j \, d\mu$$

Definition 24. Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$ be a $\mathcal{A}$-measurable function. Whenever $\int_X f^+ \, d\mu$ and $\int_X f^{-1} \, d\mu$ are not both $+\infty$ we set

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^{-1} \, d\mu$$

This value is in $[-\infty, +\infty]$.

Definition 25 (summable functions)**.** With the same notations and conditions as in Definition 24, assume in addition that $\int_X f + \, d\mu < \infty$ and $\int_X f^{-1} \, d\mu < \infty$. Then we say that f is μ-summable (note that in this case $\int_X f \, d\mu$ is finite).

Remark 7.3.1. Whenever $f : X \to \overline{\mathbb{R}}$ is μ-measurable, the μ-summability of f is equivalent to the finiteness of $\int_X |f| \, d\mu$ (recall from Theorem 6.2.2 that the function $|f|$ is measurable if f is measurable). This follows upon noticing that $|f| = f^+ + f^{-1}$ and so by Proposition 13(i) we have $\int_X |f| \, d\mu = \int_X f^+ \, d\mu + \int_X f^{-1} \, d\mu$.

Theorem 7.3.1. ((Lebesgue's) dominated convergence theorem (for signed functions) - weak statement). Let $u : X \to \overline{\mathbb{R}}$ be a non-negative $\mathcal{A}$-measurable function on the measure space $(X, \mathcal{A}, \mu)$ with $\int_X u \, d\mu < \infty$ (i.e. u is μ-summable). Let $f_j : X \to \mathbb{R}$ be a sequence of $\mathcal{A}$-measurable functions usch that $|f_j| \leq u$ for all $j \in \mathbb{N}$. Assume that $f_j \to f$ pointwise (where $f : X \to \overline{\mathbb{R}}$). Then

$$\int_X f \, d\mu = \lim_{j \to \infty} \int_X f_j \, d\mu$$

Proposition 14. Let $f, g : X \to \overline{\mathbb{R}}$ be μ-summable, let $\alpha \in \mathbb{R}$. Then

1. If $f+g$ is well-defined then it is μ-summable and $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$
2. αf is μ-summable and $\int_X (\alpha f)\,d\mu = \alpha \int_X f\,d\mu$

Proposition 15. Let $f, g : X \to \overline{\mathbb{R}}$ be μ-summable. Then

1. $\max\{f, g\}$ and $\min\{f, g\}$ are μ-summable
2. $f \leq g \implies \int_X f\,d\mu \leq \int_X g\,d\mu$
3. $|\int_X f\,d\mu| \leq \int_X |f|\,d\mu$

Proposition 16. Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let $f, g : X \to \overline{\mathbb{R}}$ be μ-summable. Then $f+g$ is μ-summable and we have $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$.

Theorem 7.3.2 ((Lebesgue's) dominated convergence theorem - strong statement). Let $u : X \to \overline{\mathbb{R}}$ be a non-negative $\mathcal{A}$-measurable function on the complete measure space $(X, \mathcal{A}, \mu)$ with $\int_X u\,d\mu < \infty$ (i.e. u is μ-summable). Let $f_j : X \to \overline{\mathbb{R}}$ be a sequence of $\mathcal{A}$-measurable functions such that $|f_j| \leq u$ for all $j \in \mathbb{N}$. Assume that $f_j \to f$ pointwise almost everywhere (that is, there exists $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f_j(X) \to f(x)$ for all $x \in X \setminus N$), where $f : X \to \overline{\mathbb{R}}$. Then

$$\lim_{j\to\infty} \int_X |f - f_j|\,d\mu = 0$$

Definition 26. Given a measure space $(X, \mathcal{A}, \mu)$, let $A \in \mathcal{A}$ and let $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Assume that $\mathbb{1}_A f$ is μ-integrable (note that $f\mathbb{1}_A$ is automatically $\mathcal{A}$-measurable). Then the μ-integral of f on A is defined as

$$\int_A f\,d\mu = \int_X f\mathbb{1}_A\,d\mu$$

(if f is assumed to be μ-summable, then $\mathbb{1}_A f$ is also μ-summable, in particular it is μ-integrable.)

Proposition 17 (Markov's inequality)**.** Let $f : X \to \overline{\mathbb{R}}$ be μ-summable on the measure space $(X, \mathcal{A}, \mu)$. For every $c > 0$ and $A \in \mathcal{A}$

$$\mu(\{x \in X \mid |f(x) \geq c\} \cap A) \leq \frac{1}{c} \int_A |f| \, d\mu$$

In particular choosing $A = X$ we have

$$\mu(\{x \in X \mid |f(x)| \geq c\}) \leq \frac{1}{c} \int_X |f| \, d\mu$$

Proposition 18. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a non-negative Riemann-integrable function with $\int_a^b f \, dx = R$. Denote with λ the Lebesgue 1-dimensionaly measure on $\mathbb{R}$. Then f is Lebesgue integrable and the two integrals yield the same value, i.e. $\int_I f \, d\lambda = R$.

Proposition 19. Let $f : R \times [0, 1] \to \mathbb{R}$ be such that

1. for every $x \in \mathbb{R}$ the function $y \to f(x, y)$ is summable on $[0, 1]$ (with respect to the 1-dimensional Lebesgue measure)
2. the partial derivative $\frac{\partial f}{\partial x}$ exists everywhere and is bounded, i.e. there is $G > 0$ such that $\left|\frac{\partial f}{\partial x}\right| \leq G$.

Then for every $x \in \mathbb{R}$ the function $y \to \frac{\partial f}{\partial x}(x, y)$ is summable on $[0, 1]$ and (note that in the following both sides are functions of the variable x)

$$\frac{d}{dx}\left(\int_0^1 f(x, y) \, dy\right) = \int_0^1 \frac{\partial f}{\partial x}(x, y) \, dy$$

8 Product measures

We define a measure $(\mu \times \nu)$ on $X \times Y$ by setting, for $A \in \mathcal{A}$ and $S \in \mathcal{S}$,

$$(\mu \times \nu)(A \times S) := \mu(A)\nu(S)$$

Proposition 20. Let $\mathcal{A}$ be a σ-algebra on X and let $\mathcal{S}$ be a σ-algebra on Y. Then the collection $\mathcal{A} \times \mathcal{S} := \{A \times S \subset X \times Y \mid A \in \mathcal{A}, S \in \mathcal{S}\}$ is a semi-ring (or semi-algebra, see Chapter 4) on $X \times Y$.

Proposition 21. The assignment $(\mu \times nu)$ is a pre-measure on the semi-ring $\mathcal{A} \times \mathcal{B}$.

Definition 27. The σ-algebra generated by the semi-ring $\mathcal{A} \times \mathcal{S}$ is denoted by $\mathcal{A} \otimes \mathcal{S}$ and it is called the product σ-algebra of $\mathcal{A}$ and $\mathcal{S}$.

Definition 28. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be σ-finite measure spaces. The unique triple $(X \times Y, \mathcal{A} \otimes \mathcal{S}, \mu \times \nu)$ constructed above is called the product measure space of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{S}, \nu)$.

Corollary 8.1.1. The Lebesgue measure on $\mathbb{R}^n$ is the product measure of the Lebesgue measures on $\mathbb{R}^k$ and $\mathbb{R}^d$ whenever $k + d = n$. Moreover $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$ is the product measure space of the following two factors:

$$(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n) = (\mathbb{R}^k \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d), \lambda^k \times \lambda^d)$$

Lemma 8.2.1. Let $\mathcal{A} \otimes \mathcal{S}$ on $X \times Y$ be the product σ-algebra defined in the previous section. For $D \subset X \times Y$ we look at 'slices':

1. for every $y \in Y$ let D_y denote the set $\{x \in X \mid (x, y) \in D$
2. for every $x \in X$ let D_x denote the set $\{y \in Y \mid (x, y) \in D\}$

Let $D \in \mathcal{A} \otimes \mathcal{S}$; then for every $y \in Y$ the set D_y is in $\mathcal{A}$ and for every $x \in X$ the set D_x is in $\mathcal{S}$.

Theorem 8.2.1 (Tonelli's theorem). let $(X \times Y, \mathcal{A} \otimes \mathcal{S}, \mu \times \nu)$ be the product measure space of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{S}, \nu)$, where all of these measure spaces are σ-finite. If $f : X \times Y \to \overline{\mathbb{R}}$ is $(\mathcal{A} \otimes \mathcal{S})$-measurable and non-negative, then

1. for every $y \in Y$ the function from X to $\overline{\mathbb{R}}$ defined by the assignment $x \to f(x, y)$ is $\mathcal{A}$-measurable;
   for every $x \in X$ the function from Y to $\overline{\mathbb{R}}$ defined by the assignment $y \to f(x, y)$ is $\mathcal{S}$-measurable

2. the function (from X to $\overline{\mathbb{R}}$) defined by the assignment $x \to \int_Y f(x, y)\, d\nu(y)$ is $\mathcal{A}$-measurable;
   the function (from Y to $\overline{\mathbb{R}}$) defined by the assignnment $y \to \int_X f(x, y)\, d\mu(x)$ is $\mathcal{S}$-measurable

3. $\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \left(\int_Y f(x, y)\, d\nu(y)\right)\, d\mu(x) = \int_Y \left(\int_X f(x, y)\, d\mu(x)\right)\, d\nu(y)$ where the values are in $[0, +\infty]$

Theorem 8.2.2 (Fubini's theorem). Let $(X \times Y, \mathcal{A} \otimes \mathcal{S}, \mu \times \nu)$ be the product measure space of $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{S}, \nu)$, where all of these measure spaces are σ-finite and let $f : X \times Y \to \overline{\mathbb{R}}$ be $(\mathcal{A} \otimes \mathcal{S})$-measurable. Assume that atleast one of the following integrals is finite:

$$\int_{X\times Y} |f|\, d(\mu\times\nu), \int_X \left(\int_Y |f|(x, y)\, d\nu(y)\right)\, d\mu(x), \int_Y \left(\int_X |f|(x, y)\, d\mu(x)\right)\, d\nu(y)$$

, then f is $(\mu \times \nu)$-summable on $X \times Y$ and all three aboe integrals are finite and coincide. Moreover:

1. for ν-almost everywhere $y \in Y$ the function (from X to $\mathbb{R}$) defined by the assignment $x \to f(x, y)$ is $\mathcal{A}$-measurable and μ-summable;
   for μ-almost everywhere $x \in X$ the function (from Y to $\mathbb{R}$) defined by the assignment $y \to f(x, y)$ is $\mathcal{S}$-measurable and ν-summable

2. the function defined by the assignment $x \to \int_Y f(x, y)\, d\nu(y)$ is $\mathcal{A}$-measurable and μ-summable;
   the function defined by the assignment $y \to \int_X f(x, y)\, d\nu(y)$ is $\mathcal{S}$-measurable and ν-summable

3. $\int_{X\times Y} f\, d(\mu\times\nu) = \int_X \left(\int_Y f(x, y)\, d\nu(y)\right)\, d\mu(x) = \int_Y \left(\int_X f(x, y)\, d\mu(x)\right)\, d\nu(y)$, where the values are in $(-\infty, +\infty)$

Theorem 8.2.3 (Fubini's theorem for Lebesgue-measurable functions). Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be Lebesgue-measurable. Assume that $k + d = n$ $(d, k \in \mathbb{N}*)$. Then f is λ^n-summable if and only if either one of the following integrals is finite:

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^k} |f|(x,y)\, d\lambda^k(y) \right) \, d\lambda^d(x), \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^d} |f|(x,y)\, d\lambda^d(x) \right) \, d\lambda^k(y)$$

In that case

$$\int_{\mathbb{R}^n} f(x,y)\, d\lambda^n = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^k} |f|(x,y)\, d\lambda^k(y) \right) \, d\lambda^d(x), \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^d} |f|(x,y)\, d\lambda^d(x) \right) \, d\lambda^k(y)$$

where all the inner integrals are well-defined and finite almost-everywhere.