Topology and Groups - MATH0074

Based on lectures by Dr. Lars Louder
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Contents

1 Point-set Topology

1.1 Preliminaries

Definition (Topological space)**.** A topological space is a pair $(X, \mathcal{T})$ such that

1. X is a set
2. $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X
3. $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$
4. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Definition (Open neighbourhood)**.** If $x \in X$, U open in X, and $x \in U$, then U is an *open neighbourhood* of x.

Definition (Hausdorff spaces)**.** A topological space $(X, \mathcal{T})$ is *Hausdorff* if $\forall\, x, y \in X$, there exists U, V open neighbourhoods of x, y respectively such that $U \cap V = \emptyset$.

Definition (Homeomorphisms)**.** A map $f : X \to Y$ is a *homeomorphism* if

1. f is bijective
2. f is continuous
3. f^{-1} is continuous

Definition (Continuous maps)**.** A map $f : X \to Y$ is continuous if $\forall\, U$ (open) $\subset Y$, $f^{-1}(U)$ is open in X.

Definition. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X such that $\mathcal{T} \subsetneq \mathcal{T}'$ then $\mathcal{T}'$ is *finer* than $\mathcal{T}$, and $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Proposition. $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$.

Definition (Subspace topology)**.** If X is a topological space, $Y \subset X$, the subspace topology on Y is defined by

$$U \text{ open in } Y \iff \exists V \text{ open in } X \text{ such that } U = Y \cap V$$

Definition. If a map $f : X \to Y$ is continuous, the *image* of f is the set

$$f(X) = \{f(x) \,|\, x \in X\} \subset Y$$

with the subspace topology.

Definition (Product topology)**.** Let X, Y be spaces. The *product topology* on $X \times Y$ is the smallest (coarsest) topology making the projections

$$p_X : X \times Y \to X, \;\; p_Y : X \times Y \to Y$$

continuous.

Proposition. Product of Hausdorff spaces is Hausdorff.

1.2 Connectedness

Definition (Connectedness)**.** A space X is *disconnected* if there exists a surjective continuous map $f : X \to \{p_1, p_2\}$. A space is *connected* if every continuous function $f : X \to \{p_1, p_2\}$ is constant.

Definition. A pair of sets $U, V \subset X$ is said to disconnect X if they are non-empty, disjoint, $U \cup V = X$ and both are open.

Definition. X is disconnected if there exists U, V which disconnect X.

Definition (Path)**.** A *path* in X is a continuous map $\gamma : [0, 1] \to X$. γ is a path from $\gamma(0)$ to $\gamma(1)$. $a, b \in X$ are said to be connected by a path if there is a path from a to b.

Definition (Path-connectedness)**.** A space X is *path-connected* if for all x, y, there exists

$$\gamma : [0, 1] \to X \text{ such that } \gamma(0) = x, \, \gamma(1) = y$$

or equivalently,

Definition. We say X is path-connected if there exists a unique equivalence class, where the equivalence relation $\sim$ is defined $a \sim b$ if and only if there exists a path from a to b.

Proposition. Suppose X is connected. Then, if $f : X \to Y$, then $f(X) \subset Y$ is connected.

Proposition. $[0, 1]$ is connected.

Corollary. If X is path-connected, then X is connected.

Definition. $X \subset \mathbb{R}$ is an *interval* if $a \leq b \leq c$, $a, c \in X \implies b \in X$.

Proposition. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Definition (Locally (path) connected)**.** A space X is locally (path) connected at a point p if for every open neighbourhood U of p, there exists a (path) connected open neighbourhood V of p such that $p \in V \subset U$.

Proposition. If X is locally path-connected then the path components of X are open.

Proposition. If X is connected and locally path-connected, then X is path connected.

1.3 Compactness

Definition (Open cover)**.** An *open cover* of a space X is a collection of open sets $\mathcal{U}$ such that

$$X = \bigcup_{U \in \mathcal{U}} U$$

Definition. A space X is *compact* if every open cover has a finite subcover.

Lemma. Closed subsets of compact spaces are compact.

Theorem. If X, Y are compact, then $X \times Y$ is compact.

Theorem (Heine-Borel theorem)**.** $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.

Theorem. $[0, 1]$ is compact.

Theorem. If $f : X \to Y$ is continuous, X compact, then $f(X) \subset Y$ is compact with respect to the subspace topology.

Proposition. If $C \subset Y$ is compact, Y Hausdorff, then C is closed.

Proposition. If $f : X \to Y$ is a continuous bijection, X compact, Y Hausdorff, then f is a homeomorphism

1.4 Quotient spaces

Definition (Quotient map)**.** Let $q : X \to Y$ be a continuous surjection. Then q is a *quotient map* if $q^{-1}(Y)$ is open if and only if U is open. (A bijective quotient map is a homeomorphism)

Definition (Quotient space)**.** Let X be a space, and $\sim$ an equivalence relation on X, and $q : X \to X/\sim = Y$ the quotient map. The quotient topology on Y is defined by U open in Y if and only if $q^{-1}(U)$ is open in X.

Lemma.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & Z \\ & Y \arrow[u, "h"'] \end{tikzcd}$$

Let f be continuous, and suppose f factors through $q : X \to Y$, a quotient map, i.e., $\exists h : Y \to Z$ such that $h \circ q = f$. Then h is continuous.

Proposition. Let $f : X \to Y$ be a continuous surjection with X compact, Y Hausdorff. Then f is a quotient map.

Definition (Disjoint union)**.** Let X_1, X_2 be topological spaces. The *disjoint union* of X_1 and X_2, $X_1 \sqcup X_2$, is the space with the underlying set $X_1 \sqcup X_2$, with U open in $X_1 \sqcup X_2$ if and only if $U \cap X_1$ is open in X_1, and $U \cap X_2$ is open in X_2.

Definition (Cell complex)**.** A *cell complex* is a space built up inductively, as follows

1. ($n = 0$) We start with a discrete set $X^{(0)}$ consisting of points, which we call 0-cells $\{e_i^0 \mid i \in I_0\}$, $e_i^0 \cong pt$. $X^{(0)} = \bigsqcup_i e_i^0$ is called the 0-skeleton.

2. ($n > 0$) We add a (possibly empty) subset of n-cells $\{e_i^n \mid i \in I_n\}$ $e_i^n \cong D^n$, the n-dimensional disk, and a continuous map

$$\phi_i^n : \partial e_i^n \cong S^{n-1} \to X^{(n-1)}$$

and here the n-skeleton is

$$X^{(n)} = X^{(n-1)} \sqcup \bigsqcup e_i^n / \sim$$

A space X is a cell complex if there exists $X^{(0)} \subset X^{(1)} \subset \dots$ as above, with the condition that U is open in X if and only if $X^{(n)} \cap U$ is open for all n.

$X^{(0)} \subseteq X^{(1)} \subseteq \dots$ is called the *cell decomposition* of X.

Definition. The suspension SX of a space X is the space

$$SX = X \times I / \sim$$

where $(x, t) \sim (x', t')$ if and only if $(x, t) = (x', t')$ or $t = t' = 1$ or $t = t' = 0$.

Proposition. SS^n is homeomorphic to S^{n+1}. SD^n is homeomorphic to D^{n+1}.

Definition (Presentation complex)**.** Given a group G and its presentation, the presentation complex of G with respect to the given presentation is the 2-dimension cell complex with 1 vertex, obtained by attaching a loop (1-cell) at the vertex for each generator of G, and attaching a 2-cell along every relation in the presentation, where the boundary of the 2-cell is attached according to the appropriate word.

Definition (Cayley graph)**.** Given a group G and its presentation, and S a (possibly generating) set of G, then the Cayley graph $C(G, S) = (G \sqcup G \times I \times S)/\sim$ where the equivalence relation $\sim$ is given by $g \sim (g, 0, s)$, $gs \sim (g, 1, s)$.

2 Homotopy

2.1 Homotopy

Definition. Let (X, A) be a pair of spaces, where $A \subseteq X$, $f_0, f_1 : X \to Y$. We say f_0 and f_1 are *homotopic relative* to A if there exists a continuous function F where $F : X \times I \to Y$ such that $F(-, 0) = f_0$, $F(-, 1) = f_1$ and $F(a, t) = f_0(a) = f_1(a)$ for all t. In this case we write $f_0 \simeq_A f_1$.

If $A = \emptyset$ then we say f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$.

Lemma (*)**.** A function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately.

Proposition. Any two continuous maps $f_0, f_1 : X \to \mathbb{R}^n$ are homotopic via the homotopy

$$F(x, t) = tf_1(x) + (1 - t)f_0(x)$$

Definition (Homotopy equivalence)**.** Two spaces X and Y are *homotopy equivalent* if there exists $f : X \to Y$, $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. In this case, we write $X \simeq Y$.

Proposition. Homotopy equivalence is an equivalence relation on (topological) spaces.

Proposition. $\mathbb{R}^n \simeq pt$

Definition. A space X is *contractible* if $X \simeq pt$, or in other words, $\mathrm{id} : X \to X$ is homotopic to a constant map. In this case the map id_X is said to be *null-homotopic*.

Proposition. $\mathbb{R}^n \setminus pt \simeq S^{n-1}$

Proposition. If $f : X \to S^2$ is a non-surjective map then f is homotopic to a constant map.

Definition. The cone CX on a space X is the space

$$CX = X \times I/\sim$$

where $(x,t) \sim (x',t')$ if and only if $(x,t) = (x',t')$ or $t = t' = 1$.

Proposition. CX is always contractible.

Proposition. If X is contractible then X is path-connected.

Definition (Retract)**.** Let $A \subseteq X$ be a subspace. A is a *retract* of X if there exists a continuous map $r : X \to A$ (retraction) such that $r|_A = \mathrm{id}_A$. A is a *deformation retract* of X if there exists such a function r such that r is homotopic to id_X relative to A.

Proposition. If A is a deformation retract of X then $X \simeq A$.

2.2 Paths and path-homotopy

Definition (Path-homotopy)**.** Two paths γ_0 and γ_1 are *path-homotopic* if they are homotopic relative to $\{0,1\} \subseteq I$. In particular $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$. If F is a homotopy from γ_0 to γ_1,

$$F(-,0) = \gamma_0(0),\ F(-,1) = \gamma_1(1)$$

F is a family of paths connecting $\gamma_0(0)$ and $\gamma_0(1)$

Proposition. Path-homotopy is an equivalence relation on the set of paths in (a topological space) X.

Definition (Based loop)**.** A *based loop* at $x_0 \in X$ is a path $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = x_0$.

Definition (Fundamental group of a space)**.** The *fundamental group* of X at x_0 is the set (group)

$$\{[\gamma] \mid \gamma \text{ is a loop based at } x_0\}$$

which is denoted by $\pi_1(X, x_0)$.

Definition (n^{th} homotopy group)**.** The n^{th} homotopy group of a space X at x_0 is the set (group)

$$\pi_n(X, x_0) = \{[f : I^n \to X \mid f(\partial I^n) \to x_0]\}$$

Definition. A loop based at x_0 is null-homotopic if it is path-homotopic to a constant path.

Definition (Free homotopy)**.** If γ_0 and γ_1 are based loops (not necessarily at the same point), then γ_0 and γ_1 are *freely homotopic* if they are homotopic through based loops, so if F is a free homotopy between γ_0 and γ_1, then,

$$F(x_0) = \gamma_0,\ F(x, 1) = \gamma_1$$
$$F(0, t) = F(1, t) \text{ for all } t$$

Proposition. Free homotopy is an equivalence relation on the set of based loops in (a topological space) X.

Definition. A based loop *bounds a disk* if the induced map

$$\bar{\gamma} : [0, 1]/_{0=1} \cong S^1 \subseteq D^2$$

extends to a continuous function $D^2 \to X$.

Lemma. The following are equivalent

1. γ bounds a disk.
2. γ is null-homotopic.
3. γ is freely homotopic to a constant path.

3 Covering spaces

Definition (Covering map)**.** A map $p : X' \to X$ is a *covering map* if $\forall x \in X$, there exists U, an open neighbourhood of x, and a discrete set Δ and a homeomorphism $h_U : U \times \Delta \to p^{-1}(U)$ such that

$$p \circ h_u = \pi_U : U \times \Delta \to U$$

and such a neighbourhood U is called a *covering neighbourhood.*

Definition (Lift)**.** Let $f : Y \to X$ and $g : Z \to X$ be two maps,

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

a lift of f is a map $\tilde{f} : Y \to Z$ such that

$$g \circ \tilde{f} = f$$

3.1 Path/Homotopy lifting lemma

Lemma (Path/Homotopy lifting lemma)**.** Let $p : X' \to X$ be a covering map and $f : I^n \to X$ a continuous map. Then for any $x' \in p^{-1}(f(U))$, there exists a unique lift $\tilde{f}$ of f to X', where $\tilde{f}(0) = x'$.

Definition. A covering space $p : X' \to X$ is *trivial* if X is a covering neighbourhood.

Lemma. Suppose $p : X' \to X$ is a trivial covering map, and $f : Y \to X$ is continuous, Y connected, the for any $y_0 \in Y$ and $x' \in p^{-1}(f(y_0))$, there exists a unique lift $\tilde{f} : Y \to X'$ such that $\tilde{f}(y_0) = x'$.

Lemma. Let X be a compact metric space. Then a continuous function $f : X \to \mathbb{R}$ attains a maximum and minimum value on X.

Lemma (Lebesgue's number lemma)**.** Let X be a compact metric space, $\mathcal{U}$ an open cover of X, then there exists $\epsilon > 0$ such that for all $x \in X$, there exists $U \in \mathcal{U}$ such that $B_\epsilon(x) \subseteq U$.
Such an ϵ is called the Lebesgue number for $\mathcal{U}$.

Lemma ((+))**.** Let $p : X' \to X$ be a covering space, and $f : Y \to X$ a continuous map, Y connected. Then two lifts $\tilde{f_1}, \tilde{f_2} : Y \to X'$ are equal for all $y \in Y$ if and only if they are equal for some $y \in Y$.

Corollary. If $[\gamma] \in \pi_1(X, x_0)$ and there exists a covering space X' of X so that γ lifts to a non-closed path then $[\gamma] \neq 1 \in \pi_1(X, x_0)$.

Corollary.

$$\pi_1(S^1) \neq 1$$

Corollary. $\mathrm{id}_{S^1} : S^1 \to S^1$ is *not* null-homotopic. In particular S^1 is not contractible.

3.2 Winding numbers

Definition. Let γ be a closed path in S^1. The *winding number* of γ, $\omega(\gamma)$ is the integer $\tilde{\gamma(1)} - \tilde{\gamma(0)}$ where $\tilde{\gamma}$ is any lift of γ to $\mathbb{R}$.

Proposition. $\omega(\gamma)$ is well-defined, and only depends on the free homotopy classes of γ.

Proposition. If $\gamma \simeq \gamma'$ (freely homotopic) then $\omega(\gamma) = \omega(\gamma')$.

3.3 Covering transformations

Definition (Covering transformation)**.** Let $p : X' \to X$ be a covering map. A *covering transformation* is a homeomorphism $h : X' \to X'$ such that $p \circ h = p$

Theorem. If X' is the universal cover of space X, then

$$\pi_1(X, x_0) = \{h : X' \to X' \,|\, p \circ h = p\}$$

4 More on fundamental groups and covering spaces

Definition. Let α, β be paths in a space X. We say α, β are *composable* if

$$\alpha(1) = \beta(0) \text{ (note that the order matters)}$$

If α, β are composable, their product $\alpha \cdot \beta$ is the path

$$\alpha \cdot \beta = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq 1 \end{cases}$$

i.e., we traverse through α, β with twice the speed. Note that in some sources they may write $\beta \cdot \alpha$ instead of $\alpha \cdot \beta$ to mean the same thing.

Definition. If α is a path (not necessarily closed), then $\bar{\alpha}$ is the pathdefined by $\bar{\alpha} = \alpha(1-t)$, i.e., traversing through α in the backwards direction.

Theorem. Let X be a *path-connected* space, x_0 a basepoint. The set

$$\pi(X, x_0) = \{\text{loops based at } x_0\}/\text{path homotopy}$$

with multiplication given by $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$, for any $[\alpha], [\beta] \in \pi(X, x_0)$, and inverses given by $[\alpha]^{-1} = [\bar{\alpha}]$, and the identity given by $1 = [x_0]$, defines a group, called the *fundamental group* (of X).

We note that it makes sense to talk about *the* fundamental group, if we restrict the spaces in question to path-connected spaces, so we shall restrict our attention to path-connected spaces from now on.

Theorem. Let α be a path from x_0 to x_1. The map

$$[\alpha]_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

defined by

$$[\gamma] \mapsto [\bar{\alpha} \cdot \gamma \cdot \alpha]$$

is an isomorphism.

Theorem. Let (X, x_0), (Y, y_0) be two (pointed) path-connected spaces, and $f : (X, x_0) \to (Y, y_0)$ be continuous and such that $f(x_0) = y_0$. Then this map induces the map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by $f_*([\gamma]) = [f \circ \gamma]$.

Theorem. With conditions as the previous theorem, along with another path-connected space (Z, z_0) and another continuous $g : (Y, y_0) \to (Z, z_0)$, we have

$$(g \circ f)_* = g_* \circ f_*$$

In other words, the previous two theorems show that π is a functor taking the category of topological spaces to the category of groups.

Theorem. Let $f_t : X \times I \to Y$ be a homotopy of f_0 and f_1, and let α be the path $f_t(x_0)$ from $f_0(x_0) = y_0$ to $f_1(x_0) = y_1$. Then,

$$[\alpha]_* \circ (f_0)_* = (f_1)_*$$

Corollary. Let X, Y be path-connected and homotopy equivalent. Then,

$$\pi_1(X) \cong \pi_1(Y)$$

Theorem (Brouwer's no retraction theorem)**.** There does not exist $r : D^2 \to S^1$ such that $r|_{S^1} = \mathrm{id}_{S^1}$.

Theorem (Brouwer's fixed point theorem)**.** Let $f : D^2 \to D^2$ be continuous. Then f has a fixed point, i.e.,

$$\exists x \in D^2 \text{ such that } f(x) = x$$

4.1 Classification of covering spaces

Definition. Let Y_0, Y_1 be two covers of a space X. We say that Y_0 and Y_1 are *equivalent* if there exists a homeomorphism $h : Y_0 \to Y_1$ such that

$$p_1 \circ h = p_0$$

(and $h(y_0) = y_1$ if Y_0, Y_1 have base points).

If $Y_0 = Y_1$, an equivalence $h : Y_0 \to Y_0$ is a *covering transformation.* It also follows that

$$Aut(Y) = \{h : Y \to Y \mid p \circ h = p\}$$

Lemma. Let $p : (Y, y_0) \to (X, x_0)$ be a cover. Then

$$p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$$

is injective.

This way we get a map from the pointed covering spaces of (X, x_0) (quotient according to equivalences) to the subgroups of $\pi_1(X, x_0)$, where X is path-connected and locally contractible.

Proposition. $p_*(\pi_1(Y, y_0)) < \pi_1(X, x_0)$

Theorem (Classification of covering spaces)**.** Let X be path-connected and locally contractible.

1. The map from the set of connected covering spaces under the quotient of equivalence,

   $$\{(Y, y_0)\}/\text{equiv} \to \{\text{subgroups of } \pi_1(X, x_0)\}\}$$

   defined by
   $$(Y, y_0) \mapsto p_*(\pi(Y, y_0))$$
   is *bijective.*

2. Given (1), $H < \pi_1(X, x_0)$ let (X_H, x_H) be the cover of $\pi_1(X, x_0)$ corresponding to H. $H < K \iff \exists (X_H, x_H) \xrightarrow{h} (X_K, x_K)$

   i. h is a cover

   ii. $[H : K] = \#h^{-1}(X_K)$

3.

$$Aut(X_H) = \{h : X_H \to X_H \,|h \text{ a covering transformation}\}$$

Then

$$X_H \cong N(H)/H$$

In particular if $H = 1 < \pi_1(X, x_0)$, then

$$Aut(X_H) \cong \pi_1(X, x_0)$$

in which case X_H is the *universal cover.*