Algebraic Topology - MATH0023

Based on lectures by Prof FEA Johnson
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Notes based on the Autumn 2021 Algebraic Topology lectures by Prof FEA Johnson.

Contents

1 Simplicial complexes

Definition (Simplicial complex)**.** A *simplicial complex* X is a pair $(V_X, \mathcal{S}_X)$ where V_X denotes the vertex set of X and $\mathcal{S}_X$ is the set of *finite, non-empty* subsets of V_X satisfying

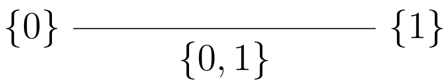
1. $\forall v \in V_X$, then $\{v\} \in \mathcal{S}_X$
2. If $\sigma \in \mathcal{S}_X$, $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{S}_X$.

$\mathcal{S}_X$ is called the set of *simplices* of X.

Example. A *standard 1-simplex*, denoted by Δ^1 is simply the line segment (or usually denoted by I).

$$V_{\Delta^1} = \{0, 1\}$$

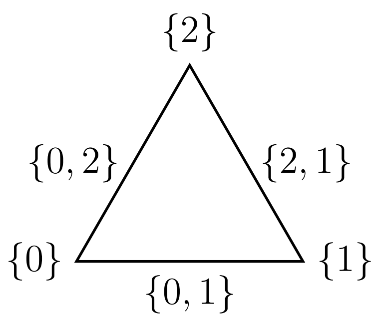
$$\mathcal{S}_{\Delta^1} = \{\{0\}, \{1\}, \{0, 1\}\}$$



A *standard 2-simplex*, denoted by Δ^2 is the equilateral triangle.

$$V_{\Delta^2} = \{0, 1, 2\}$$

$$\mathcal{S}_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$



In general, the *standard n-simplex* Δ^n, is $\Delta^n = (V_{\Delta^n}, \mathcal{S}_{\Delta^n})$ where

$$V_{\Delta^n} = \{0, 1, \dots, n\}$$

$$\mathcal{S}_{\Delta^n} = \{\alpha : \alpha \subset \{0, \dots, n\},\, \alpha \neq \emptyset\}$$

If $X = (V_x, \mathcal{S}_X)$ is a simplicial complex, we now want to pick a field $\mathbb{F}$, usually $\mathbb{Q}$ or $\mathbb{F}_2$ (in this course) and want to produce a sequence of vector spaces (over $\mathbb{F}$)

$$C_n(X)_{0 \leq n}$$

$C_0(X)$ is the vector space whose basis elements are simply the vertices of the simplicial complex, and this has dimension 0.

Definition (k-simplex of a simplicial complex)**.** If X is a simplicial complex then a *k-simplex of* X is a simplex $\sigma \in \mathcal{S}_X$ such that $|\sigma| = k + 1$.

$C_k(X)$ is the vector space whose basis elements are the *oriented* k-simplices of X. If $\{v_0, \dots, v_n\}$ is an n-simplex of X, consider the symbols

$$[v_0, v_1, \dots, v_n]$$

such that

$$[v_{\rho(0)}, v_{\rho(1)}, \dots, v_{\rho(n)}] = \operatorname{sign}(\rho)[v_0, \dots, v_n]$$

Definition.

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is a linear map defined on basis elements as follows;

$$\partial_n[v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^r [v_o, \dots, \hat{v_r}, \dots, v_n]$$

where $\hat{v_r}$ indincates omission of v_r.

Example.

$$\partial_2[0,1,2] = [1,2] - [0,2] + [0,1]$$

$$\partial_1[v_0, v_2] = [v_1] - [v_0]$$

$$\begin{aligned}
\partial_1\partial_2[0,1,2] &= \partial_1([1,2] - [0,2] + [0,1]) \\
&= ([2] - [1]) - ([2] - [0]) + ([1] - [0]) \\
&= 0
\end{aligned}$$

Proposition (Poincaré lemma)**.** Let X be a simplicial complex. Consider

$$\partial_r : C_r(X) \to C_{r-1}(X)$$

for $r \geq 1$, then

$$\partial_{n-1}\partial_n \equiv 0$$

Proof.

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \ldots, \hat{v_r}, \ldots, v_n]$$

$$\begin{aligned}
\partial_{n-1}[v_0, \ldots, \hat{v_r}, \ldots, v_n] &= \sum_{s<r} (-1)^s [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\
&\quad + \sum_{s>r} (-1)^{s-1} [v_0 \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n]
\end{aligned}$$

$$\begin{aligned}
\partial_{n-1}\partial_n[v_0, \ldots, v_n] &= \sum_{s<r} (-1)^{r+s} [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\
&\quad + \sum_{s>r} (-1)^{r+s-1} [v_0, \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n] \\
&= 0
\end{aligned}$$

□