Algebraic Topology - MATH0023

Based on lectures by Prof FEA Johnson
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Notes based on the Autumn 2021 Algebraic Topology lectures by Prof FEA Johnson.

Contents

1 Simplicial complexes

Definition (Simplicial complex)**.** A *simplicial complex* X is a pair $(V_X, \mathcal{S}_X)$ where V_X denotes the vertex set of X and $\mathcal{S}_X$ is the set of *finite, non-empty* subsets of V_X satisfying

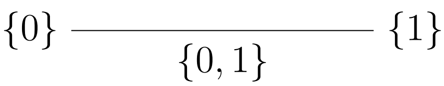
1. $\forall v \in V_X$, then $\{v\} \in \mathcal{S}_X$
2. If $\sigma \in \mathcal{S}_X$, $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{S}_X$.

$\mathcal{S}_X$ is called the set of *simplices* of X.

Example. A *standard 1-simplex*, denoted by Δ^1 is simply the line segment (or usually denoted by I).

$$V_{\Delta^1} = \{0, 1\}$$

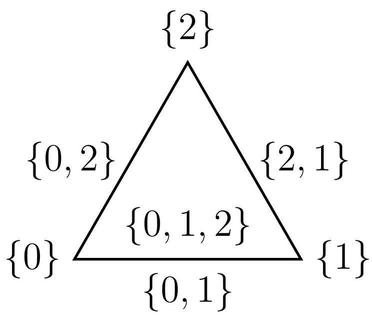
$$\mathcal{S}_{\Delta^1} = \{\{0\}, \{1\}, \{0, 1\}\}$$



A *standard 2-simplex*, denoted by Δ^2 is the equilateral triangle.

$$V_{\Delta^2} = \{0, 1, 2\}$$

$$\mathcal{S}_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$



In general, the *standard n-simplex* Δ^n, is $\Delta^n = (V_{\Delta^n}, \mathcal{S}_{\Delta^n})$ where

$$V_{\Delta^n} = \{0, 1, \ldots, n\}$$

$$\mathcal{S}_{\Delta^n} = \{\alpha : \alpha \subset \{0, \ldots, n\},\, \alpha \neq \emptyset\}$$

If $X = (V_x, \mathcal{S}_X)$ is a simplicial complex, we now want to pick a field $\mathbb{F}$, usually $\mathbb{Q}$ or $\mathbb{F}_2$ (in this course) and want to produce a sequence of vector spaces (over $\mathbb{F}$)

$$C_n(X)_{0 \leq n}$$

$C_0(X)$ is the vector space whose basis elements are simply the vertices of the simplicial complex, and this has dimension 0.

Definition (k-simplex of a simplicial complex)**.** If X is a simplicial complex then a *k-simplex of* X is a simplex $\sigma \in \mathcal{S}_X$ such that $|\sigma| = k + 1$.

$C_k(X)$ is the vector space whose basis elements are the *oriented* k-simplices of X which are the following symbols,

$$[v_0, v_1, \ldots, v_n]$$

(where $\{v_0, \ldots, v_n\}$ is an n-simplex of X) subject to the rules

$$[v_{\rho(0)}, v_{\rho(1)}, \ldots, v_{\rho(n)}] = \mathrm{sign}(\rho)[v_0, \ldots, v_n]$$

Definition.

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is a linear map defined on basis elements as follows;

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_o, \ldots, \hat{v_r}, \ldots, v_n]$$

where $\hat{v_r}$ indincates omission of v_r.

Example.

$$\partial_2[0,1,2] = [1,2] - [0,2] + [0,1]$$

$$\partial_1[v_0, v_2] = [v_1] - [v_0]$$

$$\begin{aligned}\partial_1\partial_2[0,1,2] &= \partial_1([1,2] - [0,2] + [0,1]) \\ &= ([2]-[1]) - ([2]-[0]) + ([1]-[0]) \\ &= 0\end{aligned}$$

Proposition (Poincaré lemma)**.** Let X be a simplicial complex. Consider

$$\partial_r : C_r(X) \to C_{r-1}(X)$$

for $r \geq 1$, then

$$\partial_{n-1}\partial_n \equiv 0$$

Proof.

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \ldots, \hat{v_r}, \ldots, v_n]$$

$$\begin{aligned}\partial_{n-1}[v_0, \ldots, \hat{v_r}, \ldots, v_n] &= \sum_{s<r} (-1)^s [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\ &\quad + \sum_{s>r} (-1)^{s-1} [v_0 \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n]\end{aligned}$$

$$\begin{aligned}\partial_{n-1}\partial_n[v_0, \ldots, v_n] &= \sum_{s<r} (-1)^{r+s} [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\ &\quad + \sum_{s>r} (-1)^{r+s-1} [v_0, \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n] \\ &= 0\end{aligned}$$

□

Proposition. If

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

then

$$\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$$

Proof. By previous lemma. $\square$

2 Homology

2.1 Quotient spaces

Let V be a vector space over a field $\mathbb{F}$, and $U \subset V$ a vector subspace.

Definition. The following set

$$x + U = \{x + u : u \in U\}$$

is called the (left) coset of U in V. Note that

$$x + U = x' + U \iff x - x' \in U$$

Definition (Quotient space)**.** The quotient space V/U is the set

$$V/U = \{x + U : x \in V\}$$

where addition and scalar multiplication is defined by

$$(x + U) + (y + U) = x + y + U$$

$$\lambda \cdot (x + U) = \lambda x + U$$

and 0 is represented by

$$0 + U$$

Note that V/U is a vector space.

Proposition.

$$\dim(V/U) = \dim(V) - \dim(U)$$

Proof. There exists a natural linear map

$$\eta : V \to V/U$$

given by

$$\eta(x) = x + U$$

Clearly this map is surjective so

$$\dim(V/U) = \dim(\operatorname{im}(\eta))$$

Now,

$$\begin{aligned} \ker(\eta) &= \{x \in V : \eta(x) = U\} \\ &= \{x \in V : x + U = U\} \end{aligned}$$

and

$$x + U = U \iff x - 0 \in U \iff x \in U$$

so $\ker(\eta) = U$. Then,

$$\dim(V) = \dim\ker(\eta) + \dim\operatorname{im}(\eta)$$

so

$$\dim(V/U) = \dim\operatorname{im}(\eta) = \dim(V) - \dim(U)$$

□

Definition.

$$H_n(X; \mathbb{F}) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

We call $H_n(X;\mathbb{F})$ the n^{th} *homology group* of X with coefficients in $\mathbb{F}$. If $\mathbb{F} = \mathbb{Q}$, then $\dim H_n(X;\mathbb{Q})$ is called the n^{th} *Betti* number of X.

Consider Δ^3. The set $\{0, 1, 2, 3\}$ represents the 'middle' of the tetrahedron (inside, interior). If we exclude the middle and simply take its boundary, we have

$$\partial\Delta^n = S^{n-1}$$

It happens that S^2 (middle excluded) is the simplest simplicial model of the 2-sphere.

Example. Consider

$$H_k(S^2; \mathbb{F})$$

Note that

$$C_n(S^2) = 0 \text{ for } n \geq 3$$

as there are no 3-simplices, so we only have to worry about

$$H_2(S^2; \mathbb{F}), H_1(S^2; \mathbb{F}), H_0(S^2; \mathbb{F})$$

We proceed to calculate these from first principles. First note that $C_3(S^2) = 0$. Now, (noting the order of these bases) $C_2(S^2)$ has basis

$$[0,1,2], [0,1,3], [0,2,3], [1,2,3]$$

$C_1(S^2)$ has basis

$$[0,1], [0,2], [0,3], [1,2], [1,3], [2,3]$$

and lastly $C_0(S^2)$ has basis

$$[0], [1], [2], [3]$$

The linear maps

$$\partial_2 : C_2(S^2) \to C_1(S^2)$$
$$\partial_1 : C_1(S^2) \to C_0(S^2)$$

can both be represented by a 6×4 matrix and a 4×6 matrix respectively.

We apply ∂_2 and ∂_1 to the bases to obtain the entries to the matrices, so for example

$$\partial_2([0,1,2]) = [1,2] - [0,2] + [0,1]$$

so the first column of the matrix representing ∂_2 is $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Proceeding, we will obtain that

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Notice that $\partial_1 \partial_2 = 0$, which further confirms the lemma from before. Now reducing both the matrices to row reduced echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim \ker \partial_2 = 1$, $\dim \operatorname{im} \partial_2 = 3$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim \ker \partial_1 = 3$, $\dim \operatorname{im} \partial_1 = 3$

$$0 \underset{\partial_3}{\longrightarrow} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

so now

$$H_2(S^3) = \ker(\partial_2)/\operatorname{im}(\partial_3) = \ker(\partial_2) \cong \mathbb{F}$$

as $\mathrm{im}(\partial_3) = 0$, so in total,

$$H_2(S^2; \mathbb{F}) \cong \mathbb{F}$$

Next,

$$H_1(S^2) = \ker(\partial_1)/\mathrm{im}(\partial_2)$$

Now note that

$$\dim H_1(S^2) = \dim \ker(\partial_1) - \dim \mathrm{im}(\partial_2) = 3 - 3 = 0$$

thus

$$H_1(S^2; \mathbb{F}) = 0$$

Next,

$$H_0(S^2) = \ker(\partial_0)/\mathrm{im}(\partial_1) = C_0/\mathrm{im}(\partial_1)$$

and

$$\dim H_0(S^2) = \dim C_0 - \dim \mathrm{im}(\partial_1) = 4 - 3 = 1$$

thus

$$H_0(S^2; \mathbb{F}) \cong \mathbb{F}$$

We've shown

$$H_k(S^2; \mathbb{F}) \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

We will soon see that this theorem generalises if

$$S^n = \Delta^{n+1}$$

then

$$H_k(S^n) = \begin{cases} \mathbb{F} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

2.2 Chain complex

Definition (Chain complex)**.** Let $\mathbb{F}$ be a field. A *chain complex* over $\mathbb{F}$ is

$$C_* = (C_r, \partial_r)_{r\in\mathbb{N}}$$

where

1. Each C_r is a vector space over $\mathbb{F}$
2. $\partial_r : C_r \to C_{r-1}$ is a linear map such that $\partial_r\partial_{r+1} = 0$ for all r.

If $X = (V_X, \mathcal{S}_X)$, we have defined a chain complex

$$C_*(X) = (C_r(X), \partial_r)$$

Given a chain complex

$$C_*(C_r, \partial_r)_{r\geq 0}$$

we define its *homology* $H_*(C_*)$ by

$$H_k(C_*) = \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$$

If $X = (V_X, \mathcal{S}_X)$ is a simplicial complex, we define

$$H_k(X, \mathbb{F}) = H_k(C_*(X; \mathbb{F}))$$

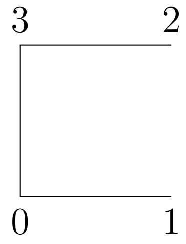
2.3 Simplicial mapping

Definition (Simplicial mapping)**.** Let X, Y be simplicial complexes, i.e., $X = (V_X, \mathcal{S}_X)$ and $Y = (V_Y, \mathcal{S}_Y)$. A *simplicial mapping* $f : X \to Y$ is a mapping of vertex sets $f : V_X \to V_Y$ such that

$$\sigma \in \mathcal{S}_X \implies f(\sigma) \in \mathcal{S}_Y$$

Example. Let $X = Y = \Delta^2$. Then defining f by $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, it is obvious that this mapping is simplicial.

Consider the following simplicial complex



and consider

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 0$$

This mapping is *not* simplicial as $f(\{0,1\})$ is *not* a simplex.

Given a simplicial mapping $f : X \to Y$, we are going to produce linear maps

$$H_k(f) : H_k(X) \to H_k(Y)$$

such that if

$$g : Y \to Z$$

then

$$g \circ f : X \to Z$$

and

1. $H_k(g \circ f) = H_k(g) \circ H_k(f)$
2. $H_k(\mathrm{id}_X) = \mathrm{id}_{H_k(X)}$

2.4 Chain mapping

Definition. Let

$$C_* = (C_r, \partial_r^C)$$
$$D_* = (D_r, \partial_r^D)$$

be chain complexes. A *chain mapping* $f_* : C_* \to D_*$ is a collection of linear maps

$$f* = (f_r)_{r \geq 0}$$

where $f_r : C_r \to D_r$ and the following commutes

$$\begin{tikzcd} C_r \arrow[r, "\partial_r^C"] \arrow[d, "f_r"'] & C_{r-1} \arrow[d, "f_{r-1}"] \\ D_r \arrow[r, "\partial_r^D"'] & D_{r-1} \end{tikzcd}$$

Notice from the diagram that

$$\partial_n^D \circ f_n = f_{n-1} \partial_n^C$$

If $g : D_* \to E_*$ is also a chain mapping, then

$$(g \circ f)_n = g_n \circ f_n : C_* \to E_*$$

is also a chain mapping.

$$\mathrm{id} : C_* \to C_*, \;\; \mathrm{id}_n = \mathrm{id}_{C_n}$$

is also a chain mapping.

Proposition. If $f : X \to Y$ is a simplicial mapping, define

$$C_n(f) : C_n(X) \to C_n(Y)$$

by action on a basis as follows

$$C_n(f)[v_0, \ldots, v_n] = [f(v_0), \ldots, f(v_n)]$$

then

$$C_*(f) : C_*(X) \to C_*(Y)$$

is also a chain mapping.

Proof.

$$\begin{aligned}
\partial d_n^D C_n(f)[v_0, \ldots, v_n] &= \partial_n^D([f(v_0), \ldots, f(v_n)]) \\
&= \sum_{r=0}^{n} (-1)^r [f(v_0), \ldots, f(\hat{v_0}), \ldots, f(v_n)] \\
&= C_{n-1}(f) \sum_{r=0}^{n} (-1)^r [v_0, \ldots, \hat{v_r}, \ldots, v_n] \\
&= C_{n-1}(f) \partial_n^C [v_0, \ldots, v_n]
\end{aligned}$$

□

We will often write $f_n[v_0, \ldots, v_n$ rather than $C_n(f)[v_0, \ldots, v_n]$.

Proposition. If $f : X \to Y$, $g : Y \to Z$ are simplicial maps, then

$$C_n(g \circ f) = C_n(g) \circ C_n(f)$$

which sometimes we will write as

$$(g \circ f)_n = g_n \circ f_n$$

instead.

Proof.

$$\begin{aligned}
(g \circ f)[v_0, \ldots, v_n] &= [(g \circ f)(v_0), \ldots, (g \circ f)(v_n)] \\
&= g_n[f(v_0), \ldots, f(v_n)] \\
&= g_n \circ f_n[v_0, \ldots, v_n]
\end{aligned}$$

□

Proposition. Let

$$\mathrm{id} : X \to X$$

then $C_*(\mathrm{id}) : C_*(X) \to C_*(X)$ is a chain mapping.

If $C_* = (C_n, \partial_n)$ is a chain complex, define

$$H_n(C_*) = \ker \partial_n / \mathrm{im}(\partial_{n+1})$$

It is usual to write

$$Z_n(C) = \ker(\partial_n) \ \text{(cycles)}$$

$$B_n(C) = \mathrm{im}(\partial_{n+1}) \ \text{(boundaries)}$$

thus by this notation,

$$H_n(C) = Z_n(C)/B_n(C)$$

If $f = (f_n)$, $C_* \to D_*$ is a chain mapping, we now want to show f induces a mapping

$$H_n(F) : H_n(C_*) \to H_n(D_*)$$

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(Z_n(C_*)) \subset Z_n(D_*)$$

Proof. Recall that

$$f_{n-1}\partial_n^C(z) = \partial_n^D f_n(z)$$

If

$$z \in Z_n(C_*),\ \partial_n^C(z) = 0$$

then we have

$$f_{n-1}\partial_n^C(z) = 0$$

and so

$$\partial_n^D f_n(z) = 0$$

and thus

$$f_n(z) \in Z_n(D_*)$$

□

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(B_n(C_*)) \subset B_n(D_*)$$

Proof. Note that

$$f_n \partial^C_{n+1}(x) = \partial^D_{n+1} f_{n+1}(x)$$

If $\beta \in B_n(C_*)$, we can write $\beta = \partial^C_{n+1}(x)$ for some x and then

$$f_n(\beta) = \partial^D_{n+1}(k)$$

where $k = f_{n+1}(x)$ so

$$f_n(\beta) \in B_n(D_*)$$

□

Corollary. If $f : C_* \to D_*$ is a chain mapping, then f induces a (linear) mapping

$$H_n(f) : H_n(C_*) \to H_n(D_*)$$

Proof. An element of $H_n(C_*)$ has form

$$[z] = z + B_n(C_*),\ z \in Z_n(C_*)$$

Now define

$$H_n(f)[z] = f_n(z) + B_n(D_*) \in H_n(D_*)$$

and now note that

$$f_n(z) \in Z_n(D_*)$$

□

By now it is clear if $g : D_* \to E_*$, $f : C_* \to D_*$ are chain mappings, then

$$H_n(g \circ f) = H_n(g) \circ H_n(f)$$

and also if $\mathrm{id} : C_* \to C_*$ we have

$$H_n(\mathrm{id} = \mathrm{id}_{H_n}$$

We now formally have

$$H_n(X) = H_n(C_*(X))$$

Corollary. If X is a *non-empty* simplicial complex, then $H_0(X; \mathbb{F} \neq 0$ (for any field $\mathbb{F}$).

Proof. As $X \neq \emptyset$, we have that $V_X \neq \emptyset$. Let $v \in V_X$ be a vertex and $*$ be the simplicial complex

$$* = (\{v\}, \{\{v\}\})$$

so $*$ consists of one vertex v, and one 0-simplex $\{v\}$. Now define a constant mapping

$$c : X \to *, \, c(x) = v, \, \forall x \in V_X$$

We also have a simplicial mapping

$$\iota : * \to X, \, \iota(v) = v$$

so now

$$c \circ \iota = id_*$$

and so

$$H_0(C) \circ H_n(\iota) = H \circ (\mathrm{id}_*)$$

but notice that

$$H_0(*) = \mathbb{F}$$

since we know

$$C_0(*) = \mathbb{F}, \, C_r(*) = 0, \, r \geq 1$$

and thus

$$H_0(C) \circ H_0(\iota) = \mathrm{id}_{\mathbb{F}}$$

$$r \circ \iota = \mathrm{id} \neq 0$$

and now note that r is surjective, and ι is injective. In particular

$$H_0(C) : H_0(X) \to \mathbb{F} = H_0(*)$$

is surjective, so

$$H_0(X) \neq 0$$

□