Topology and Groups - MATH0074

Based on lectures by Dr Lars Louder
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Notes based on the Autumn 2021 Topology and Groups lectures by Dr Lars Louder.

Contents

1 Review of metric topology

A metric space is a pair (X, d) where X is a set and d is a metric on X. Recall the notions of an open and closed ball around a point $x \in X$

$$B_\epsilon(x) = \{x' \in X : d(x, x') < \epsilon\}$$

$$\overline{B_\epsilon(x)} = \{x' \in X : d(x, x') < \epsilon\}$$

Definition. A map $f : X \to Y$ (X, Y metric space) is continuous at $x \in X$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$f(B_\delta(x)) \subseteq B_\epsilon(f(x))$$

We say f is continuous if it is continuous at every point in X.

If X is a metric space then $U \subseteq X$ is said to be *open* if

$$\forall x \in U, \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U$$

2 Topological spaces

Definition (Topological space)**.** A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $T \subseteq 2^X (= \mathcal{P}(X))$ satisfying the following

1. ϕ, $X \in \mathcal{T}$
2. $\mathcal{T}$ is closed under finite intersections and closed under arbitrary unions

Example.

1. A metric space (X, d) can be considered as a topological space in its own right with the topology $\mathcal{T}$ (called the metric topology) on X being the collection of open sets determined by d.
2. $(X, 2^X)$, the topology on X being the collection of all subsets of X is called the discrete topology.

3. $(X, \{\emptyset, X\}$, with the topology on X consisting of only ϕ and X is called the indiscrete topology.

4. $(X, \{U \subseteq X : |X \setminus U| < \infty \textit{ or } U = \emptyset\})$, the cofinite topology on X.

Definition. If $(X, \mathcal{T})$ is a topological space, $x \in X$, $U \in \mathcal{T}$ then we say U is an *open neighbourhood* of x if $x \in U$.

Definition (Hausdorff space)**.** A topological space is *Hausdorff* if given $x, y \in X$, $x \neq y$, $\exists U, V \in \mathcal{T}$ such that

$$U \cap V = \emptyset$$

Definition (Continuous map)**.** A map $f : X \to Y$ is *continuous* if $f^{-1}(U)$ is open in X, for U open in Y, i.e., the preimage of every open set is open.

Definition (Homeomorphism)**.** A continuous map $f : X \to Y$ is a *homeomorphism* if

1. f is a bijection

2. $f^{-1} : Y \to X$ is continuous

We say that a property of a space is topological if it is preserved under homeomorphism.