Topology and Groups - MATH0074

Based on lectures by Dr. Lars Louder

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(Revision) notes based on the Autumn 2021 Topology and Groups lectures by Dr. Lars Louder. Some parts marked with (*) (+) are taken from Hatcher's Algebraic Topology and the site on Topology and Groups by Prof. Jonny Evans.

Contents

1 Point-set Topology

1.1 Preliminaries

Definition (Topological space)**.** A topological space is a pair $(X, \mathcal{T})$ such that

1. X is a set
2. $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X
3. $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$
4. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Definition (Open neighbourhood)**.** If $x \in X$, U open in X, and $x \in U$, then U is an *open neighbourhood* of x.

Definition (Hausdorff spaces)**.** A topological space $(X, \mathcal{T})$ is *Hausdorff* if $\forall\, x, y \in X$, there exists U, V open neighbourhoods of x, y respectively such that $U \cap V = \emptyset$.

Definition (Homeomorphisms)**.** A map $f : X \to Y$ is a *homeomorphism* if

1. f is bijective
2. f is continuous
3. f^{-1} is continuous

Definition (Continuous maps)**.** A map $f : X \to Y$ is continuous if $\forall\, U$ (open) $\subset Y$, $f^{-1}(U)$ is open in X.

Definition. If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X such that $\mathcal{T} \subsetneq \mathcal{T}'$ then $\mathcal{T}'$ is *finer* than $\mathcal{T}$, and $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Proposition. $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous if and only if $\mathcal{T}$ is finer than $\mathcal{T}'$.

Definition (Subspace topology)**.** If X is a topological space, $Y \subset X$, the subspace topology on Y is defined by

$$U \text{ open in } Y \iff \exists V \text{ open in } X \text{ such that } U = Y \cap V$$

Definition. If a map $f : X \to Y$ is continuous, the *image* of f is the set

$$f(X) = \{f(x) \,|\, x \in X\} \subset Y$$

with the subspace topology.

Definition (Product topology)**.** Let X, Y be spaces. The *product topology* on $X \times Y$ is the smallest (coarsest) topology making the projections

$$p_X : X \times Y \to X, \;\; p_Y : X \times Y \to Y$$

continuous.

Proposition. Product of Hausdorff spaces is Hausdorff.

1.2 Connectedness

Definition (Connectedness)**.** A space X is *disconnected* if there exists a surjective continuous map $f : X \to \{p_1, p_2\}$. A space is *connected* if every continuous function $f : X \to \{p_1, p_2\}$ is constant.

Definition. A pair of sets $U, V \subset X$ is said to disconnect X if they are non-empty, disjoint, $U \cup V = X$ and both are open.

Definition. X is disconnected if there exists U, V which disconnect X.

Definition (Path)**.** A *path* in X is a continuous map $\gamma : [0, 1] \to X$. γ is a path from $\gamma(0)$ to $\gamma(1)$. $a, b \in X$ are said to be connected by a path if there is a path from a to b.

Definition (Path-connectedness)**.** A space X is *path-connected* if for all x, y, there exists

$$\gamma : [0, 1] \to X \text{ such that } \gamma(0) = x, \, \gamma(1) = y$$

or equivalently,

Definition. We say X is path-connected if there exists a unique equivalence class, where the equivalence relation $\sim$ is defined $a \sim b$ if and only if there exists a path from a to b.

Proposition. Suppose X is connected. Then, if $f : X \to Y$, then $f(X) \subset Y$ is connected.

Proposition. $[0, 1]$ is connected.

Corollary. If X is path-connected, then X is connected.

Definition. $X \subset \mathbb{R}$ is an *interval* if $a \leq b \leq c$, $a, c \in X \implies b \in X$.

Proposition. A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Definition (Locally (path) connected)**.** A space X is locally (path) connected at a point p if for every open neighbourhood U of p, there exists a (path) connected open neighbourhood V of p such that $p \in V \subset U$.

Proposition. If X is locally path-connected then the path components of X are open.

Proposition. If X is connected and locally path-connected, then X is path connected.

1.3 Compactness

Definition (Open cover)**.** An *open cover* of a space X is a collection of open sets $\mathcal{U}$ such that
$$X = \bigcup_{U \in \mathcal{U}} U$$

Definition. A space X is *compact* if every open cover has a finite subcover.

Lemma. Closed subsets of compact spaces are compact.

Theorem. If X, Y are compact, then $X \times Y$ is compact.

Theorem (Heine-Borel theorem)**.** $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.

Theorem. $[0, 1]$ is compact.

Theorem. If $f : X \to Y$ is continuous, X compact, then $f(X) \subset Y$ is compact with respect to the subspace topology.

Proposition. If $C \subset Y$ is compact, Y Hausdorff, then C is closed.

Proposition. If $f : X \to Y$ is a continuous bijection, X compact, Y Hausdorff, then f is a homeomorphism

1.4 Quotient spaces

Definition (Quotient map)**.** Let $q : X \to Y$ be a continuous surjection. Then q is a *quotient map* if $q^{-1}(Y)$ is open if and only if U is open. (A bijective quotient map is a homeomorphism)

Definition (Quotient space)**.** Let X be a space, and $\sim$ an equivalence relation on X, and $q : X \to X/\sim = Y$ the quotient map. The quotient topology on Y is defined by U open in Y if and only if $q^{-1}(U)$ is open in X.

Lemma.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "q"'] & Z \\ & Y \arrow[u, "h"'] \end{tikzcd}$$

Let f be continuous, and suppose f factors through $q : X \to Y$, a quotient map, i.e., $\exists h : Y \to Z$ such that $h \circ q = f$. Then h is continuous.

Proposition. Let $f : X \to Y$ be a continuous surjection with X compact, Y Hausdorff. Then f is a quotient map.

Definition (Disjoint union)**.** Let X_1, X_2 be topological spaces. The *disjoint union* of X_1 and X_2, $X_1 \sqcup X_2$, is the space with the underlying set $X_1 \sqcup X_2$, with U open in $X_1 \sqcup X_2$ if and only if $U \cap X_1$ is open in X_1, and $U \cap X_2$ is open in X_2.

Definition (Cell complex)**.** A *cell complex* is a space built up inductively, as follows

1. ($n = 0$) We start with a discrete set $X^{(0)}$ consisting of points, which we call 0-cells $\{e_i^0 \mid i \in I_0\}$, $e_i^0 \cong pt$. $X^{(0)} = \bigsqcup_i e_i^0$ is called the 0-skeleton.

2. ($n > 0$) We add a (possibly empty) subset of n-cells $\{e_i^n \mid i \in I_n\}$ $e_i^n \cong D^n$, the n-dimensional disk, and a continuous map

$$\phi_i^n : \partial e_i^n \cong S^{n-1} \to X^{(n-1)}$$

and here the n-skeleton is

$$X^{(n)} = X^{(n-1)} \sqcup \bigsqcup e_i^n / \sim$$

A space X is a cell complex if there exists $X^{(0)} \subset X^{(1)} \subset \ldots$ as above, with the condition that U is open in X if and only if $X^{(n)} \cap U$ is open for all n.

$X^{(0)} \subseteq X^{(1)} \subseteq \ldots$ is called the *cell decomposition* of X.

Definition. The suspension SX of a space X is the space

$$SX = X \times I / \sim$$

where $(x, t) \sim (x', t')$ if and only if $(x, t) = (x', t')$ or $t = t' = 1$ or $t = t' = 0$.

Proposition. SS^n is homeomorphic to S^{n+1}. SD^n is homeomorphic to D^{n+1}.

Definition (Presentation complex)**.** Given a group G and its presentation, the presentation complex of G with respect to the given presentation is the 2-dimension cell complex with 1 vertex, obtained by attaching a loop (1-cell) at the vertex for each generator of G, and attaching a 2-cell along every relation in the presentation, where the boundary of the 2-cell is attached according to the appropriate word.

Definition (Cayley graph)**.** Given a group G and its presentation, and S a (possibly generating) set of G, then the Cayley graph $C(G, S) = (G \sqcup G \times I \times S)/\sim$ where the equivalence relation $\sim$ is given by $g \sim (g, 0, s)$, $gs \sim (g, 1, s)$.

2 Homotopy

2.1 Homotopy

Definition. Let (X, A) be a pair of spaces, where $A \subseteq X$, $f_0, f_1 : X \to Y$. We say f_0 and f_1 are *homotopic relative* to A if there exists a continuous function F where $F : X \times I \to Y$ such that $F(-, 0) = f_0$, $F(-, 1) = f_1$ and $F(a, t) = f_0(a) = f_1(a)$ for all t. In this case we write $f_0 \simeq_A f_1$.

If $A = \emptyset$ then we say f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$.

Lemma (*)**.** A function defined on the union of two closed sets is continuous if it is continuous when restricted to each of the closed sets separately.

Proposition. Any two continuous maps $f_0, f_1 : X \to \mathbb{R}^n$ are homotopic via the homotopy

$$F(x, t) = tf_1(x) + (1 - t)f_0(x)$$

Definition (Homotopy equivalence)**.** Two spaces X and Y are *homotopy equivalent* if there exists $f : X \to Y$, $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. In this case, we write $X \simeq Y$.

Proposition. Homotopy equivalence is an equivalence relation on (topological) spaces.

Proposition. $\mathbb{R}^n \simeq pt$

Definition. A space X is *contractible* if $X \simeq pt$, or in other words, $\mathrm{id} : X \to X$ is homotopic to a constant map. In this case the map id_X is said to be *null-homotopic*.

Proposition. $\mathbb{R}^n \setminus pt \simeq S^{n-1}$

Proposition. If $f : X \to S^2$ is a non-surjective map then f is homotopic to a constant map.

Definition. The cone CX on a space X is the space

$$CX = X \times I / \sim$$

where $(x, t) \sim (x', t')$ if and only if $(x, t) = (x', t')$ or $t = t' = 1$.

Proposition. CX is always contractible.

Proposition. If X is contractible then X is path-connected.

Definition (Retract)**.** Let $A \subseteq X$ be a subspace. A is a *retract* of X if there exists a continuous map $r : X \to A$ (retraction) such that $r|_A = \mathrm{id}_A$. A is a *deformation retract* of X if there exists such a function r such that r is homotopic to id_X relative to A.

Proposition. If A is a deformation retract of X then $X \simeq A$.

Definition (Homotopy extension property)**.** A pair of spaces (X, A) has the *homotopy extension property* if for any $f : X \to Y$, and homotopy H of $h = f|_A$, this homotopy extends to a homotopy F of f.

Theorem. If $A \subseteq X$, X a cell-complex and A a subcomplex, then (X, A) has the homotopy extension property.

Corollary. If $A \subseteq X$, X a CW complex, A a subcomplex, A contractible, then the quotient map $q : X \to X/A$ (collapse A to a single point) is a homotopy equivalence.

2.2 Paths and path-homotopy

Definition (Path-homotopy)**.** Two paths γ_0 and γ_1 are *path-homotopic* if they are homotopic relative to $\{0,1\} \subseteq I$. In particular $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$. If F is a homotopy from γ_0 to γ_1,

$$F(-,0) = \gamma_0(0),\ F(-,1) = \gamma_1(1)$$

F is a family of paths connecting $\gamma_0(0)$ and $\gamma_0(1)$

Proposition. Path-homotopy is an equivalence relation on the set of paths in (a topological space) X.

Definition (Based loop)**.** A *based loop* at $x_0 \in X$ is a path $\gamma : I \to X$ such that $\gamma(0) = \gamma(1) = x_0$.

Definition (Fundamental group of a space)**.** The *fundamental group* of X at x_0 is the set (group)

$$\{[\gamma] \,|\, \gamma \text{ is a loop based at } x_0\}$$

which is denoted by $\pi_1(X, x_0)$.

Definition (n^{th} homotopy group)**.** The n^{th} homotopy group of a space X at x_0 is the set (group)

$$\pi_n(X, x_0) = \{[f : I^n \to X \,|\, f(\partial I^n) \to x_0]\}$$

Definition. A loop based at x_0 is null-homotopic if it is path-homotopic to a constant path.

Definition (Free homotopy)**.** If γ_0 and γ_1 are based loops (not necessarily at the same point), then γ_0 and γ_1 are *freely homotopic* if they are homotopic through based loops, so if F is a free homotopy between γ_0 and γ_1, then,

$$F(x_0) = \gamma_0,\ F(x,1) = \gamma_1$$
$$F(0,t) = F(1,t) \text{ for all } t$$

Proposition. Free homotopy is an equivalence relation on the set of based loops in (a topological space) X.

Definition. A based loop *bounds a disk* if the induced map

$$\bar{\gamma} : [0,1]/_{0=1} \cong S^1 \subseteq D^2$$

extends to a continuous function $D^2 \to X$.

Lemma. The following are equivalent

1. γ bounds a disk.
2. γ is null-homotopic.
3. γ is freely homotopic to a constant path.

3 Covering spaces

Definition (Covering map)**.** A map $p : X' \to X$ is a *covering map* if $\forall x \in X$, there exists U, an open neighbourhood of x, and a discrete set Δ and a homeomorphism $h_U : U \times \Delta \to p^{-1}(U)$ such that

$$p \circ h_u = \pi_U : U \times \Delta \to U$$

and such a neighbourhood U is called a *covering neighbourhood.*

Definition (Lift)**.** Let $f : Y \to X$ and $g : Z \to X$ be two maps,

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

a lift of f is a map $\tilde{f} : Y \to Z$ such that

$$g \circ \tilde{f} = f$$

3.1 Path/Homotopy lifting lemma

Lemma (Path/Homotopy lifting lemma)**.** Let $p : X' \to X$ be a covering map and $f : I^n \to X$ a continuous map. Then for any $x' \in p^{-1}(f(U))$, there exists a unique lift $\tilde{f}$ of f to X', where $\tilde{f}(0) = x'$.

Definition. A covering space $p : X' \to X$ is *trivial* if X is a covering neighbourhood.

Lemma. Suppose $p : X' \to X$ is a trivial covering map, and $f : Y \to X$ is continuous, Y connected, the for any $y_0 \in Y$ and $x' \in p^{-1}(f(y_0))$, there exists a unique lift $\tilde{f} : Y \to X'$ such that $\tilde{f}(y_0) = x'$.

Lemma. Let X be a compact metric space. Then a continuous function $f : X \to \mathbb{R}$ attains a maximum and minimum value on X.

Lemma (Lebesgue's number lemma)**.** Let X be a compact metric space, $\mathcal{U}$ an open cover of X, then there exists $\epsilon > 0$ such that for all $x \in X$, there exists $U \in \mathcal{U}$ such that $B_\epsilon(x) \subseteq U$.
Such an ϵ is called the Lebesgue number for $\mathcal{U}$.

Lemma ((+))**.** Let $p : X' \to X$ be a covering space, and $f : Y \to X$ a continuous map, Y connected. Then two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to X'$ are equal for all $y \in Y$ if and only if they are equal for some $y \in Y$.

Corollary. If $[\gamma] \in \pi_1(X, x_0)$ and there exists a covering space X' of X so that γ lifts to a non-closed path then $[\gamma] \neq 1 \in \pi_1(X, x_0)$.

Corollary.

$$\pi_1(S^1) \neq 1$$

Corollary. $\mathrm{id}_{S^1} : S^1 \to S^1$ is *not* null-homotopic. In particular S^1 is not contractible.

3.2 Winding numbers

Definition. Let γ be a closed path in S^1. The *winding number* of γ, $\omega(\gamma)$ is the integer $\tilde{\gamma(1)} - \tilde{\gamma(0)}$ where $\tilde{\gamma}$ is any lift of γ to $\mathbb{R}$.

Proposition. $\omega(\gamma)$ is well-defined, and only depends on the free homotopy classes of γ.

Proposition. If $\gamma \simeq \gamma'$ (freely homotopic) then $\omega(\gamma) = \omega(\gamma')$.

3.3 Covering transformations

Definition (Covering transformation)**.** Let $p : X' \to X$ be a covering map. A *covering transformation* is a homeomorphism $h : X' \to X'$ such that $p \circ h = p$

Theorem. If X' is the universal cover of space X, then

$$\pi_1(X, x_0) = \{h : X' \to X' \,|\, p \circ h = p\}$$

4 More on fundamental groups and covering spaces

Definition. Let α, β be paths in a space X. We say α, β are *composable* if

$$\alpha(1) = \beta(0) \text{ (note that the order matters)}$$

If α, β are composable, their product $\alpha \cdot \beta$ is the path

$$\alpha \cdot \beta = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \leq 1 \end{cases}$$

i.e., we traverse through α, β with twice the speed. Note that in some sources they may write $\beta \cdot \alpha$ instead of $\alpha \cdot \beta$ to mean the same thing.

Definition. If α is a path (not necessarily closed), then $\bar{\alpha}$ is the pathdefined by $\bar{\alpha} = \alpha(1-t)$, i.e., traversing through α in the backwards direction.

Theorem. Let X be a *path-connected* space, x_0 a basepoint. The set

$$\pi(X, x_0) = \{\text{loops based at } x_0\}/\text{path homotopy}$$

with multiplication given by $[\alpha]\cdot[\beta] = [\alpha\cdot\beta]$, for any $[\alpha], [\beta] \in \pi(X, x_0)$, and inverses given by $[\alpha]^{-1} = [\bar{\alpha}]$, and the identity given by $1 = [x_0]$, defines a group, called the *fundamental group* (of X).

We note that it makes sense to talk about *the* fundamental group, if we restrict the spaces in question to path-connected spaces, so we shall restrict our attention to path-connected spaces from now on.

Theorem. Let α be a path from x_0 to x_1. The map

$$[\alpha]_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

defined by

$$[\gamma] \mapsto [\bar{\alpha} \cdot \gamma \cdot \alpha]$$

is an isomorphism.

Theorem. Let (X, x_0), (Y, y_0) be two (pointed) path-connected spaces, and $f : (X, x_0) \to (Y, y_0)$ be continuous and such that $f(x_0) = y_0$. Then this map induces the map

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

defined by $f_*([\gamma]) = [f \circ \gamma]$.

Theorem. With conditions as the previous theorem, along with another path-connected space (Z, z_0) and another continuous $g : (Y, y_0) \to (Z, z_0)$, we have

$$(g \circ f)_* = g_* \circ f_*$$

In other words, the previous two theorems show that π is a functor taking the category of topological spaces to the category of groups.

Theorem. Let $f_t : X \times I \to Y$ be a homotopy of f_0 and f_1, and let α be the path $f_t(x_0)$ from $f_0(x_0) = y_0$ to $f_1(x_0) = y_1$. Then,

$$[\alpha]_* \circ (f_0)_* = (f_1)_*$$

Corollary. Let X, Y be path-connected and homotopy equivalent. Then,

$$\pi_1(X) \cong \pi_1(Y)$$

Theorem (Brouwer's no retraction theorem)**.** There does not exist $r : D^2 \to S^1$ such that $r|_{S^1} = \mathrm{id}_{S^1}$.

Theorem (Brouwer's fixed point theorem)**.** Let $f : D^2 \to D^2$ be continuous. Then f has a fixed point, i.e.,

$$\exists x \in D^2 \text{ such that } f(x) = x$$

4.1 Classification of covering spaces

Definition. Let Y_0, Y_1 be two covers of a space X. We say that Y_0 and Y_1 are *equivalent* if there exists a homeomorphism $h : Y_0 \to Y_1$ such that

$$p_1 \circ h = p_0$$

(and $h(y_0) = y_1$ if Y_0, Y_1 have base points).

If $Y_0 = Y_1$, an equivalence $h : Y_0 \to Y_0$ is a *covering transformation.* It also follows that

$$Aut(Y) = \{h : Y \to Y \,|\, p \circ h = p\}$$

Lemma. Let $p : (Y, y_0) \to (X, x_0)$ be a cover. Then

$$p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$$

is injective.

This way we get a map from the pointed covering spaces of (X, x_0) (quotient according to equivalences) to the subgroups of $\pi_1(X, x_0)$, where X is path-connected and locally contractible.

Proposition. $p_*(\pi_1(Y, y_0)) < \pi_1(X, x_0)$

Theorem (Classification of covering spaces)**.** Let X be path-connected and locally contractible.

1. The map from the set of connected covering spaces under the quotient of equivalence,

$$\{(Y, y_0)\}/\text{equiv} \to \{\text{subgroups of } \pi_1(X, x_0)\}\}$$

defined by

$$(Y, y_0) \mapsto p_*(\pi(Y, y_0))$$

is *bijective.*

2. Given (1), $H < \pi_1(X, x_0)$ let (X_H, x_H) be the cover of $\pi_1(X, x_0)$ corresponding to H. $H < K \iff \exists (X_H, x_H) \xrightarrow{h} (X_K, x_K)$

   i. h is a cover

   ii. $[H : K] = \#h^{-1}(X_K)$

3.

$$Aut(X_H) = \{h : X_H \to X_H \,| h \text{ a covering transformation}\}$$

Then

$$X_H \cong N(H)/H$$

In particular if $H = 1 < \pi_1(X, x_0)$, then

$$Aut(X_H) \cong \pi_1(X, x_0)$$

in which case X_H is the *universal cover.*

Proposition. Define $f : \pi_1(X, x_0) \to p^{-1}(x)$ by

$$[\gamma] \mapsto \tilde{\gamma}(1)$$

Then,

$$f([\gamma_1]) = f([\gamma_2]) \iff [\gamma_1][\gamma_2]^{-1} \in H,\ [\gamma_1] \in H[\gamma_2]$$

i.e., $[\gamma_1], [\gamma_2]$ determine the same right coset of H, and the map f, factors through $H \backslash G$ (through the map $\tilde{f}$)

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr] & p^{-1}(x_0) \\ & H\backslash G \arrow[u, "\tilde{f}"] \end{tikzcd}$$

and by a similar argument, we get a bijection

$$\text{f} \to p^{-1}(x_0), \;\; \tilde{f}(H) = y_0$$

Corollary.

$$[H : G] = |H\backslash G| = |p^{-1}(x_0)|$$

i.e., degree of the cover is equal to the index of the subgroup.

Theorem (General lifting lemma)**.**

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Suppose Z is path-connected and locally contractible. Then there exists a unique lift

$$\tilde{f} : (Z, z_0) \to (Y, y_0) \iff f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$$

Corollary.

$$\begin{tikzcd} (Y, y_0) \arrow[rr, shift left, "h_0"] \arrow[dr, "p_0"'] & & (Y_1, y_1) \arrow[ll, shift left, "h_1"] \arrow[dl, "p_1"] \\ & (X, x_0) & \end{tikzcd}$$

Suppose $p_{0_*}(\pi_1(Y_0, y_0)) = p_{1_*}(\pi_1(Y_1, y_1))$, then,

$$\exists! h : (Y_0, y_0) \xrightarrow{\cong} (Y_1, y_1)$$

so any two path-connected locally contractible pointed covers which correspond to the same subgroup are equivalent. In particular this proves the injectivity claim in statement (1) of the theorem for classification of covering spaces.

Theorem. Let $G = \pi_1(X, x_0)$, $g \in G$, and $H = p_*(\pi_1(X_H, x_H))$, so $g^{-1}Hg = p_*(\pi_1(X_H, x_H \cdot g))$.

$$\begin{tikzcd}
(X_H, x_H) \arrow[rr, "h"] \arrow[dr, "p"'] & & (X_H, x_H \cdot g) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

By the general lifting lemma, we know $\exists h \iff H < g^{-1}Hg$. If such an h exists, then it is also a covering transformation (isomorphism), so equivalently we have

$$\# h^{-1}(x_H \cdot g) = 1$$

i.e., $g^{-1}Hg = H$, which is equivalent to $g \in N(H)$. Hence we have, the following,

$$\exists h_g : (X_H, x_H) \xrightarrow{\cong} (X_H, x_H \cdot g) \iff g \in N(H)$$

which also defines a surjective map

$$N(H) \to \{h : X_H \to X_H \mid h \text{ a covering transformation}\}$$

$$g \mapsto h_g : X_H \to X_H$$

Theorem.

$$Aut(X_H) \cong N(H)/H$$

Definition (Normal cover, universal cover and the deck group)**.** If $H \triangleleft G$, then X_H is said to be a *normal cover*. If $H = 1$, then X_H is simply connected, and is denoted $\tilde{X}$, called the *universal cover*. We call $Aut(\tilde{X})$ the *deck group*.

If such an $\tilde{X}$ exists, then

$$Aut(\tilde{X}) \cong G \text{ and } X_H = \tilde{X}/H$$

In particular, $X = \tilde{X}/G$.

5 Free groups

Definition. A group F is free on $S \subseteq F$ if for every group G and map $\phi : S \to G$ there exists a unique homomorphism $f : F \to G$ such that $f|_S = \phi$. S is said to be a *basis* for F.

Given S, we construct F_S as follows,

1. S is a *set*, thought of as a collection of symbols.
2. We extend S to $S^{\pm} = S \cup S^{-}$, where $S^{-} = \{s^{-1} \mid s \in S\}$.
3. Define $(S^{\pm})^* = \bigcup_{n \in \mathbb{N}} \{w : n \to S^{\pm}\}$. An element of $(S^{\pm})^*$ is a *word*.
4. There is a special word $\epsilon : 0 \to S^{\pm}$, the unique string of length 0.

Definition. A word w is *reduced* if $w(i) = s \implies w(i-1) \neq s^{-1} =\neq w(i+1)$ for all $s \in S$, and also $w(i) = s^{-1} \implies w(i-1) \neq s \neq w(i+1)$ for all $s^{-1} \in S$.

We write $w \to w'$ is w' is obtained from w by deleting an adjacent pair of ss^{-1} or $s^{-1}s$ from w.

Definition. w reduces to w' if there exists a sequence

$$w = w_0 \to w_1 \to w_2 \to \ldots \to w_k = w'$$

where in this case, we write $w \implies w'$.

Lemma. If $w \implies w'$ and $w \implies w''$, then $w' = w''$.

Definition. $F_S = \{\text{reduced words in } (S^{\pm})^*\}$

Proposition. F_S is a group, where $e = \epsilon$ is the empty string, w^-1 is w in reverse, and swapping the signs of each letter. $w_1 \cdot w_2$ is the unique word that $w_1 w_2$ reduces to.

<u>Universal property</u>
Identify $s \in S$ with the string $s : 1 \to S^{\pm}$, $s(1) = s$. Given $\phi : S \to G$ (group), extend this to a homomorphism,

$$f(e) = e$$

$$f(s) = \phi(s), f(s^{-1}) = f(s)^{-1}$$

Now define $f(w) = f(w(0)) \cdot f(w(1)) \cdot f(w(2)) \cdot \ldots \cdot f(w(n-1))$ when $w : n \to S^{\pm}$. It can be checked that this indeed defines a homomorphism.

Definition. w and w' are homotopic if there exists $w = w_0, w_1, \ldots, w_n = w'$ such that either $w_i \to w_{i+1}$ or $w_i \leftarrow w_{i+1}$ for all i.

Theorem. If w and w' are homotopic and reduced then $w = w'$.

Alternatively we could have also defined F_S to be the set of homotopy classes of words. The real important thing is that reduced representative is *unique*.

Definition. A presentation is a pair $\langle S \mid R \rangle$, where S is a set and R is a collection of words in $S^{\pm}$. $\langle S \mid R \rangle$ is a presentation of G if $G \cong F_S / \langle\langle R' \rangle\rangle$ where $R' = \{\text{reduced words equivalent to elements in } R\}$. Recall that $\langle\langle R' \rangle\rangle$ is the smallest normal subgroup containing R'.

Theorem. Let X be a connected graph, then $\pi_1(X)$ is free.

Definition. A *graph* is a 1-dimensional CW-complex.

Corollary. Subgroups of free groups are free.

Corollary. Every group is a quotient of a free group by a free group.

Definition. A group G is finitely generated if $\exists f : F_S \to G, |S| < \infty$.

Definition. A *forest* is a graph with no embedded S^1's (1-complex homeomorphic to S^1). A *tree* is a connected forest. A forest $F \subseteq X$ is *maximal* if for any $F \subsetneq Y \subseteq X$, Y is not a forest.

Proposition. Any graph X contains a maximal forest, and if X is connected, F is free.