Algebraic Topology - MATH0023

Based on lectures by Prof FEA Johnson
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Notes based on the Autumn 2021 Algebraic Topology lectures by Prof FEA Johnson.

Contents

1 Simplicial complexes

Definition (Simplicial complex)**.** A *simplicial complex* X is a pair $(V_X, \mathcal{S}_X)$ where V_X denotes the vertex set of X and $\mathcal{S}_X$ is the set of *finite, non-empty* subsets of V_X satisfying

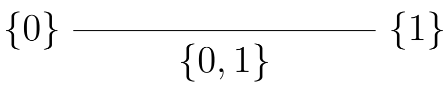
1. $\forall v \in V_X$, then $\{v\} \in \mathcal{S}_X$
2. If $\sigma \in \mathcal{S}_X$, $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{S}_X$.

$\mathcal{S}_X$ is called the set of *simplices* of X.

Example. A *standard 1-simplex*, denoted by Δ^1 is simply the line segment (or usually denoted by I).

$$V_{\Delta^1} = \{0, 1\}$$

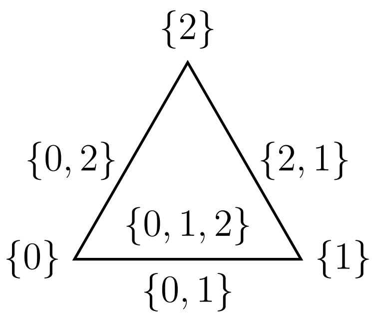
$$\mathcal{S}_{\Delta^1} = \{\{0\}, \{1\}, \{0, 1\}\}$$



A *standard 2-simplex*, denoted by Δ^2 is the equilateral triangle.

$$V_{\Delta^2} = \{0, 1, 2\}$$

$$\mathcal{S}_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$



In general, the *standard n-simplex* Δ^n, is $\Delta^n = (V_{\Delta^n}, \mathcal{S}_{\Delta^n})$ where

$$V_{\Delta^n} = \{0, 1, \ldots, n\}$$

$$\mathcal{S}_{\Delta^n} = \{\alpha : \alpha \subset \{0, \ldots, n\},\ \alpha \neq \emptyset\}$$

If $X = (V_x, \mathcal{S}_X)$ is a simplicial complex, we now want to pick a field $\mathbb{F}$, usually $\mathbb{Q}$ or $\mathbb{F}_2$ (in this course) and want to produce a sequence of vector spaces (over $\mathbb{F}$)

$$C_n(X)_{0 \leq n}$$

$C_0(X)$ is the vector space whose basis elements are simply the vertices of the simplicial complex, and this has dimension 0.

Definition (k-simplex of a simplicial complex)**.** If X is a simplicial complex then a *k-simplex of* X is a simplex $\sigma \in \mathcal{S}_X$ such that $|\sigma| = k + 1$.

$C_k(X)$ is the vector space whose basis elements are the *oriented* k-simplices of X which are the following symbols,

$$[v_0, v_1, \ldots, v_n]$$

(where $\{v_0, \ldots, v_n\}$ is an n-simplex of X) subject to the rules

$$[v_{\rho(0)}, v_{\rho(1)}, \ldots, v_{\rho(n)}] = \mathrm{sign}(\rho)[v_0, \ldots, v_n]$$

Definition.

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is a linear map defined on basis elements as follows;

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_o, \ldots, \hat{v_r}, \ldots, v_n]$$

where $\hat{v_r}$ indincates omission of v_r.

Example.

$$\partial_2[0,1,2] = [1,2] - [0,2] + [0,1]$$

$$\partial_1[v_0, v_2] = [v_1] - [v_0]$$

$$\begin{aligned}
\partial_1\partial_2[0,1,2] &= \partial_1([1,2] - [0,2] + [0,1]) \\
&= ([2] - [1]) - ([2] - [0]) + ([1] - [0]) \\
&= 0
\end{aligned}$$

Proposition (Poincaré lemma)**.** Let X be a simplicial complex. Consider

$$\partial_r : C_r(X) \to C_{r-1}(X)$$

for $r \geq 1$, then

$$\partial_{n-1}\partial_n \equiv 0$$

Proof.

$$\partial_n[v_0, \dots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \dots, \hat{v_r}, \dots, v_n]$$

$$\begin{aligned}
\partial_{n-1}[v_0, \dots, \hat{v_r}, \dots, v_n] &= \sum_{s<r} (-1)^s [v_0, \dots, \hat{v_s}, \dots, \hat{v_r}, \dots, v_n] \\
&\quad + \sum_{s>r} (-1)^{s-1} [v_0 \dots, \hat{v_r}, \dots, \hat{v_s}, \dots, v_n]
\end{aligned}$$

$$\begin{aligned}
\partial_{n-1}\partial_n[v_0, \dots, v_n] &= \sum_{s<r} (-1)^{r+s} [v_0, \dots, \hat{v_s}, \dots, \hat{v_r}, \dots, v_n] \\
&\quad + \sum_{s>r} (-1)^{r+s-1} [v_0, \dots, \hat{v_r}, \dots, \hat{v_s}, \dots, v_n] \\
&= 0
\end{aligned}$$

□

Proposition. If

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

then

$$\operatorname{im}(\partial_{n+1}) \subset \ker(\partial_n)$$

Proof. By previous lemma. □

2 Homology

2.1 Quotient spaces

Let V be a vector space over a field $\mathbb{F}$, and $U \subset V$ a vector subspace.

Definition. The following set

$$x + U = \{x + u : u \in U\}$$

is called the (left) coset of U in V. Note that

$$x + U = x' + U \iff x - x' \in U$$

Definition (Quotient space)**.** The quotient space V/U is the set

$$V/U = \{x + U : x \in V\}$$

where addition and scalar multiplication is defined by

$$(x + U) + (y + U) = x + y + U$$

$$\lambda \cdot (x + U) = \lambda x + U$$

and 0 is represented by

$$0 + U$$

Note that V/U is a vector space.

Proposition.

$$\dim(V/U) = \dim(V) - \dim(U)$$

Proof. There exists a natural linear map

$$\eta : V \to V/U$$

given by

$$\eta(x) = x + U$$

Clearly this map is surjective so

$$\dim(V/U) = \dim(\mathrm{im}(\eta))$$

Now,

$$\begin{aligned} \ker(\eta) &= \{x \in V : \eta(x) = U\} \\ &= \{x \in V : x + U = U\} \end{aligned}$$

and

$$x + U = U \iff x - 0 \in U \iff x \in U$$

so $\ker(\eta) = U$. Then,

$$\dim(V) = \dim \ker(\eta) + \dim \mathrm{im}(\eta)$$

so

$$\dim(V/U) = \dim \mathrm{im}(\eta) = \dim(V) - \dim(U)$$

□

Definition.

$$H_n(X; \mathbb{F}) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

We call $H_n(X; \mathbb{F})$ the n^{th} *homology group* of X with coefficients in $\mathbb{F}$. If $\mathbb{F} = \mathbb{Q}$, then $\dim H_n(X; \mathbb{Q})$ is called the n^{th} *Betti* number of X.

Consider Δ^3. The set $\{0, 1, 2, 3\}$ represents the 'middle' of the tetrahedron (inside, interior). If we exclude the middle and simply take its boundary, we have

$$\partial \Delta^n = S^{n-1}$$

It happens that S^2 (middle excluded) is the simplest simplicial model of the 2-sphere.

Example. Consider
$$H_k(S^2;\mathbb{F})$$
Note that
$$C_n(S^2) = 0 \text{ for } n \geq 3$$
as there are no 3-simplices, so we only have to worry about
$$H_2(S^2;\mathbb{F}), H_1(S^2;\mathbb{F}), H_0(S^2;\mathbb{F})$$
We proceed to calculate these from first principles. First note that $C_3(S^2) = 0$. Now, (noting the order of these bases) $C_2(S^2)$ has basis
$$[0,1,2],[0,1,3],[0,2,3],[1,2,3]$$
$C_1(S^2)$ has basis
$$[0,1],[0,2],[0,3],[1,2],[1,3],[2,3]$$
and lastly $C_0(S^2)$ has basis
$$[0],[1],[2],[3]$$
The linear maps
$$\partial_2 : C_2(S^2) \to C_1(S^2)$$
$$\partial_1 : C_1(S^2) \to C_0(S^2)$$
can both be represented by a 6×4 matrix and a 4×6 matrix respectively.

We apply ∂_2 and ∂_1 to the bases to obtain the entries to the matrices, so for example
$$\partial_2([0,1,2]) = [1,2] - [0,2] + [0,1]$$
so the first column of the matrix representing ∂_2 is $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Proceeding,

we will obtain that

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Notice that $\partial_1 \partial_2 = 0$, which further confirms the lemma from before. Now reducing both the matrices to row reduced echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim \ker \partial_2 = 1$, $\dim \operatorname{im} \partial_2 = 3$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim \ker \partial_1 = 3$, $\dim \operatorname{im} \partial_1 = 3$

$$0 \underset{\partial_3}{\longrightarrow} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

so now

$$H_2(S^3) = \ker(\partial_2)/\operatorname{im}(\partial_3) = \ker(\partial_2) \cong \mathbb{F}$$

as $\mathrm{im}(\partial_3) = 0$, so in total,

$$H_2(S^2; \mathbb{F}) \cong \mathbb{F}$$

Next,

$$H_1(S^2) = \ker(\partial_1)/\mathrm{im}(\partial_2)$$

Now note that

$$\dim H_1(S^2) = \dim \ker(\partial_1) - \dim \mathrm{im}(\partial_2) = 3 - 3 = 0$$

thus

$$H_1(S^2; \mathbb{F}) = 0$$

Next,

$$H_0(S^2) = \ker(\partial_0)/\mathrm{im}(\partial_1) = C_0/\mathrm{im}(\partial_1)$$

and

$$\dim H_0(S^2) = \dim C_0 - \dim \mathrm{im}(\partial_1) = 4 - 3 = 1$$

thus

$$H_0(S^2; \mathbb{F}) \cong \mathbb{F}$$

We've shown

$$H_k(S^2; \mathbb{F}) \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

We will soon see that this theorem generalises if

$$S^n = \Delta^{n+1}$$

then

$$H_k(S^n) = \begin{cases} \mathbb{F} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

2.2 Chain complex

Definition (Chain complex)**.** Let $\mathbb{F}$ be a field. A *chain complex* over $\mathbb{F}$ is

$$C_* = (C_r, \partial_r)_{r\in\mathbb{N}}$$

where

1. Each C_r is a vector space over $\mathbb{F}$
2. $\partial_r : C_r \to C_{r-1}$ is a linear map such that $\partial_r\partial_{r+1} = 0$ for all r.

If $X = (V_X, \mathcal{S}_X)$, we have defined a chain complex

$$C_*(X) = (C_r(X), \partial_r)$$

Given a chain complex

$$C_*(C_r, \partial_r)_{r\geq 0}$$

we define its *homology* $H_*(C_*)$ by

$$H_k(C_*) = \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$$

If $X = (V_X, \mathcal{S}_X)$ is a simplicial complex, we define

$$H_k(X;\mathbb{F}) = H_k(C_*(X;\mathbb{F}))$$

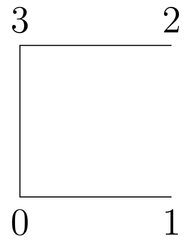
2.3 Simplicial mapping

Definition (Simplicial mapping)**.** Let X, Y be simplicial complexes, i.e., $X = (V_X, \mathcal{S}_X)$ and $Y = (V_Y, \mathcal{S}_Y)$. A *simplicial mapping* $f : X \to Y$ is a mapping of vertex sets $f : V_X \to V_Y$ such that

$$\sigma \in \mathcal{S}_X \implies f(\sigma) \in \mathcal{S}_Y$$

Example. Let $X = Y = \Delta^2$. Then defining f by $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, it is obvious that this mapping is simplicial.

Consider the following simplicial complex



and consider

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 0$$

This mapping is *not* simplicial as $f(\{0, 1\})$ is *not* a simplex.

Given a simplicial mapping $f : X \to Y$, we are going to produce linear maps

$$H_k(f) : H_k(X) \to H_k(Y)$$

such that if

$$g : Y \to Z$$

then

$$g \circ f : X \to Z$$

and

1. $H_k(g \circ f) = H_k(g) \circ H_k(f)$
2. $H_k(\mathrm{id}_X) = \mathrm{id}_{H_k(X)}$

Remark. (Look up on functors for a more general treatment of the above concept.)

2.4 Chain mapping

Definition. Let

$$C_* = (C_r, \partial_r^C)$$

$$D_* = (D_r, \partial_r^D)$$

be chain complexes. A *chain mapping* $f_* : C_* \to D_*$ is a collection of linear maps

$$f* = (f_r)_{r\geq 0}$$

where $f_r : C_r \to D_r$ and the following commutes

$$\begin{tikzcd}
C_r \arrow[r, "\partial_r^C"] \arrow[d, "f_r"'] & C_{r-1} \arrow[d, "f_{r-1}"] \\
D_r \arrow[r, "\partial_r^D"'] & D_{r-1}
\end{tikzcd}$$

Notice from the diagram that

$$\partial_n^D \circ f_n = f_{n-1}\partial_n^C$$

If $g : D_* \to E_*$ is also a chain mapping, then

$$(g \circ f)_n = g_n \circ f_n : C_* \to E_*$$

is also a chain mapping.

$$\mathrm{id} : C_* \to C_*,\ \ \mathrm{id}_n = \mathrm{id}_{C_n}$$

is also a chain mapping.

Proposition. If $f : X \to Y$ is a simplicial mapping, define

$$C_n(f) : C_n(X) \to C_n(Y)$$

by action on a basis as follows

$$C_n(f)[v_0, \ldots, v_n] = [f(v_0), \ldots, f(v_n)]$$

then

$$C_*(f) : C_*(X) \to C_*(Y)$$

is also a chain mapping.

Proof.

$$\begin{aligned}
\partial_n^D C_n(f)[v_0, \ldots, v_n] &= \partial_n^D([f(v_0), \ldots, f(v_n)]) \\
&= \sum_{r=0}^{n} (-1)^r [f(v_0), \ldots, f(\hat{v_0}), \ldots, f(v_n)] \\
&= C_{n-1}(f) \sum_{r=0}^{n} (-1)^r [v_0, \ldots, \hat{v_r}, \ldots, v_n] \\
&= C_{n-1}(f)\partial_n^C [v_0, \ldots, v_n]
\end{aligned}$$

□

We will often write $f_n[v_0, \ldots, v_n$ rather than $C_n(f)[v_0, \ldots, v_n]$.

Proposition. If $f : X \to Y$, $g : Y \to Z$ are simplicial maps, then

$$C_n(g \circ f) = C_n(g) \circ C_n(f)$$

which sometimes we will write as

$$(g \circ f)_n = g_n \circ f_n$$

instead.

Proof.

$$\begin{aligned}
(g \circ f)[v_0, \ldots, v_n] &= [(g \circ f)(v_0), \ldots, (g \circ f)(v_n)] \\
&= g_n[f(v_0), \ldots, f(v_n)] \\
&= g_n \circ f_n[v_0, \ldots, v_n]
\end{aligned}$$

□

Proposition. Let

$$\mathrm{id} : X \to X$$

then $C_*(\mathrm{id}) : C_*(X) \to C_*(X)$ is a chain mapping.

If $C_* = (C_n, \partial_n)$ is a chain complex, define

$$H_n(C_*) = \ker \partial_n / \mathrm{im}(\partial_{n+1})$$

It is usual to write

$$Z_n(C) = \ker(\partial_n) \ \ \text{(cycles)}$$

$$B_n(C) = \mathrm{im}(\partial_{n+1}) \ \ \text{(boundaries)}$$

thus by this notation,

$$H_n(C) = Z_n(C)/B_n(C)$$

If $f = (f_n)$, $C_* \to D_*$ is a chain mapping, we now want to show f induces a mapping

$$H_n(F) : H_n(C_*) \to H_n(D_*)$$

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(Z_n(C_*)) \subset Z_n(D_*)$$

Proof. Recall that

$$f_{n-1}\partial_n^C(z) = \partial_n^D f_n(z)$$

If

$$z \in Z_n(C_*),\ \partial_n^C(z) = 0$$

then we have

$$f_{n-1}\partial_n^C(z) = 0$$

and so

$$\partial_n^D f_n(z) = 0$$

and thus

$$f_n(z) \in Z_n(D_*)$$

□

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(B_n(C_*)) \subset B_n(D_*)$$

Proof. Note that

$$f_n \partial^C_{n+1}(x) = \partial^D_{n+1} f_{n+1}(x)$$

If $\beta \in B_n(C_*)$, we can write $\beta = \partial^C_{n+1}(x)$ for some x and then

$$f_n(\beta) = \partial^D_{n+1}(k)$$

where $k = f_{n+1}(x)$ so

$$f_n(\beta) \in B_n(D_*)$$

□

Corollary. If $f : C_* \to D_*$ is a chain mapping, then f induces a (linear) mapping

$$H_n(f) : H_n(C_*) \to H_n(D_*)$$

Proof. An element of $H_n(C_*)$ has form

$$[z] = z + B_n(C_*),\ z \in Z_n(C_*)$$

Now define

$$H_n(f)[z] = f_n(z) + B_n(D_*) \in H_n(D_*)$$

and now note that

$$f_n(z) \in Z_n(D_*)$$

□

By now it is clear if $g : D_* \to E_*$, $f : C_* \to D_*$ are chain mappings, then

$$H_n(g \circ f) = H_n(g) \circ H_n(f)$$

and also if $\mathrm{id} : C_* \to C_*$ we have

$$H_n(\mathrm{id}) = \mathrm{id}_{H_n}$$

We now formally have

$$H_n(X) = H_n(C_*(X))$$

Corollary. If X is a *non-empty* simplicial complex, then $H_0(X;\mathbb{F}) \neq 0$ (for any field $\mathbb{F}$).

Proof. As $X \neq \emptyset$, we have that $V_X \neq \emptyset$. Let $v \in V_X$ be a vertex and $*$ be the simplicial complex

$$* = (\{v\}, \{\{v\}\})$$

so $*$ consists of one vertex v, and one 0-simplex $\{v\}$. Now define a constant mapping

$$c : X \to *,\ c(x) = v,\ \forall x \in V_X$$

We also have a simplicial mapping

$$\iota : * \to X,\ \iota(v) = v$$

so now

$$c \circ \iota = \mathrm{id}_*$$

and so

$$H_0(c) \circ H_0(\iota) = H_0(\mathrm{id}_*)$$

but notice that

$$H_0(*) = \mathbb{F}$$

since we know

$$C_0(*) = \mathbb{F},\ C_r(*) = 0,\ r \geq 1$$

and thus

$$H_0(c) \circ H_0(\iota) = \mathrm{id}_{\mathbb{F}}$$

$$c \circ \iota = \mathrm{id} \neq 0$$

and now note that c is surjective, and ι is injective. In particular

$$H_0(c) : H_0(X) \to \mathbb{F} = H_0(*)$$

is surjective, so

$$H_0(X) \neq 0$$

□

So we now know if $H_0(X) \neq 0$ if $X \neq \emptyset$.

Now let X be a simplicial complex. If $v, w \in V_X$, then by a path from v to w, we mean a sequence of 1-simplices

$$[v_0, v_1], [v_1, v_2], \ldots, [v_{n-1}, v_{n-1}], [v_{n-1}, v_n]$$

such that $v_0 = v$ and $v_n = w$.

Proposition. If X is non-empty and connected, then

$$H_0(X; \mathbb{F}) \cong \mathbb{F}$$

Proof.

$$C_1(X) \xrightarrow{\partial_1} C_0(X)$$

If $v, w \in V_X$, then $[w] - [v] \in \operatorname{im}(\partial_1)$. To see this, choose a path

$$v = v_0 < v_1 < \ldots < v_{n-1} < v_n = w$$

i.e., $[v_{i+1}, v_i]$ is a 1-simplex for $0 \leq i \leq n-1$.

$$\partial_1[v_i, v_{i+1}] = [v_{i+1}] - [v_i] \in \operatorname{im}(\partial_1)$$

so then,

$$[w] - [v] = \sum_{i=0}^{n-1} [v_{i+1}, v_i] \in \operatorname{im}(\partial_1)$$

Now $\{[v] : v \in V_X\}$ is a basis for C_0. Choose a specific $v \in V_X$. By elementary basis change,

$$\{[v]\} \cup \{[w] - [v] : w \in V_x, w \neq v\}$$

is a basis for C_0. However $[w]-[v] \in \operatorname{im}(\partial_1)$ $(w \neq v)$. So $C_0(X)/\operatorname{im}(\partial_1)$ has dimension ≤ 1, and then $\dim H_0(X) \leq 1$ if X is connected. But $X \neq 0$, hence $\dim H_0(X) = 1$, hence

$$H_0(X) \cong \mathbb{F}$$

when X is connected. $\square$

Proposition. In general, $\dim H_0(X)$ is equal to the number of connected components in X

If X is a simplicial complex, then define a relation $\sim$ on V_X by $v \sim w$ if and only if there exists a path from v to w.

$\sim$ defines an equivalence relation, where the number of connected components is equal to the number of equivalence classes.

If X consists of a single point,

$$H_k(\text{pt.}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

2.5 Cone

Definition. Let X be a simplicial complex. A *cone* on X, $C(X)$, is defined as follows, choose $*$ (cone point) such that $* \notin V_X$

$$V_{C(X)} = \{*\} \cup V_X$$

$$\mathcal{S}_{C(X)} = \mathcal{S}_X \cup \{\{*\} \cup \{\sigma \cup \{*\} : \sigma \in S_X\}\}$$

i.e., join everything in X to the cone point.

Theorem. If X is a simplicial complex, then,

$$H_k(C(X); \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

i.e., $C(X)$ behaves just like a point (homologically).

Proof. First note that $C(X)$ is connected. Take $v, w \in V_{C(X)}$, $v \neq w$. Either one of them is the cone point, or none of them are the cone point.

(1) Without loss of generality, suppose w is the cone point. ($w = *$). By definition, $[v, w] = [v, *]$ is a 1-simplex of $C(X)$. So we've joined v to w.

(2) If neither are the cone point, then, $[v, *]$ and $[w, *]$ are both 1-simplices, so again, we've joined v to w. So

$$H_0(C(X); \mathbb{F}) \cong \mathbb{F}$$

Now we must show

$$H_k(C(X)) = 0,\ k \geq 0$$

We define, for each $k > 0$, a linear map

$$\mathcal{H}_k : C_k(C(X)) \to C_{k+1}(C(X))$$

(called a contracting homotopy) $\mathcal{H}_k$ is defined on a basis by

$$\mathcal{H}_k[v_o, \dots, v_k] = [*, v_0, \dots, v_k]$$

Then,

$$\begin{aligned} \partial_{k+1}\mathcal{H}_k[v_0, \dots, v_k] &= \partial_{k+1}[*, v_0, \dots, v_k] \\ &= [v_0, \dots, v_k] + \sum_{r=0}^{k} (-1)^{r+1}[*, v_0, \dots, \hat{v_r}, \dots, v_k] \end{aligned}$$

$$\partial_{k+1}\mathcal{H}_k([v_0, \dots, v_k] + \sum_{r=0}^{k} (-1)^{r}[*, v_0, \dots, \hat{v_r}, \dots, v_k]) = [v_0, \dots, v_k]$$

However,

$$\mathcal{H}_{k-1}[v_0, \dots, \hat{v_r}, \dots, v_k] = [*, v_0, \dots, \hat{v_r}, \dots, v_k]$$

and

$$(\partial_{k+1}\mathcal{H}_k + \mathcal{H}_{k-1}\partial_k)[v_0, \dots, v_k] = [v_0, \dots, v_k]$$

i.e.,

$$\partial_{k+1}\mathcal{H}_k + \mathcal{H}_{k-1}\partial_k = \mathrm{id}$$

(we call the above a homotopy relation)

$$H_k(C(X)) = Z_k(C(X))/B_k(C(X))$$

and if $z \in Z_k(C(X))$, $\partial_k(z) = 0$, so if $z \in Z_k(C(X))$, $z = \partial_{k+1}\mathcal{H}_k(z)$ so $z \in \mathrm{im}(\partial_{k+1})$, i.e., $Z_k(C(X)) \subset B_k(C(X)) (\subset Z_k(X))$ so if $C(X)$ is a cone and $k > 0$,

$$Z_k(C(X)) = B_k(C(X))$$

and $H_k(C(X); \mathbb{F}) = 0$ $\square$

Corollary.

$$H_k(\Delta^n; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

where $\Delta^n = n$-simplex

Proof. Δ^n is a cone. $\Delta^n = (C(\Delta^{n-1}))$ $\square$

Let X be a simplicial complex, $n \geq 0$. Then the n-skeleton $X^{(n)}$ of X is defined by

$$V_{X^{(n)}} = V_X$$

$$\mathcal{S}_{X^{(n)}} = \{\sigma \in \mathcal{S}_X : |\sigma| \leq n+1\}$$

i.e., $\dim(\sigma) \leq n$.

The standard model S^n of the n-sphere

$$V_{S^n} = \{0, \ldots, n+1\}$$

$$\mathcal{S}_{S^n} = \{\sigma \subset \{0, \ldots, n+1\} | \sigma \neq 0, |\sigma| \leq n+1\}$$

i.e., $S^n = n$-skeleton of Δ^{n+1}

Theorem.

$$H_k(X^{(n)}) \equiv H_k(X), \text{ for } 0 \leq k \leq n-1$$

(and there exists a natural surjection $H_n(X^{(n)}) \to H_n(X)$) (note this is not an isomorphism)

Proof. From definition, $C_k(X^{(n)}) \equiv C_k(X)$, $0 \le k \le n$

$$C_*(X^{(n)})\ 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$C_*(X)\ C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$H_k(X^{(n)}) \equiv H_k(X) \text{ for } k \le n-1$$

$$\begin{aligned} H_n(X^{(n)}) &\equiv \ker(\partial_n : C_n(X) \to C_{n-1}(X)) \\ &= Z_n(X) \end{aligned}$$

but $B_n(X^{(n)}) = 0$. In general $B_n(X) \neq 0$. □

As $S^n = (\Delta^{n+1})^{(n)}$, $(n \neq 0, n \ge 1)$ we see that

$$H_k(S^{(n)}; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & 1 \le k \le n-1 \end{cases}$$

We now still need to compute $H_n(S^n)$.

2.6 Exact sequences

Definition. Let $U \xrightarrow{f} V \xrightarrow{g} W$ be linear maps. We say sequence is *exact* at V when

$$\ker(g) = \mathrm{im}(f)$$

In general if

$$V_{n+1} \xrightarrow{f_{n+1}} V_n \to \dots \to V_{r+1} \xrightarrow{f_{r+1}} V_r \xrightarrow{f_r} V_{r-1} \to \dots \to V_1 \xrightarrow{f_1} V_0$$

is a sequence of linear maps, we say a sequence is *exact* at V_r when

$$\ker f_r = \mathrm{im} f_{r+1}$$

We say the sequence is *exact* when it is *exact* at each possible V_r.

<u>4 term exact sequence</u>

$$0 \to U \xrightarrow{f} V \to 0$$

is exact if and only if f is an isomorphism.

Proof. The sequence is exact at V, so

$$\mathrm{im}(f) = \ker(V \to 0) = V$$

so f is surjective. The sequence is exact at U, so

$$\ker(f) = \mathrm{im}(0 \to U) = 0$$

so f is injective. □

<u>Short exact sequence</u>

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

Exactness here means

1. g is surjective, $\mathrm{im}(g) = \ker(W \to 0)$
2. f is injective, $\ker(f) = \mathrm{im}(0 \to V) = 0$
3. $\ker(g) = \mathrm{im}(f)$

Example. <u>Kernel-rank theorem</u>
Suppose we have the exact sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

if U, V, W are finite dimensional, then

$$\dim(V) = \dim(U) + \dim(W)$$

by the kernel-rank theorem. To see this, note that

$$\mathrm{im}(g) = W$$

by exactness.

$\dim\ker(g)+\dim\operatorname{im}(g) = \dim(V) \implies \dim\ker(g)+\dim(W) = \dim(V)$

$$\ker(g) = \operatorname{im}(f) \cong U$$

(since f is injective) and so

$$\dim\ker(g) = \dim(U)$$

so

$$\dim(U) + \dim(W) = \dim(V)$$

$$H_k(X) = Z_k(X)/B_k(X)$$

$$0 \to B_k(X) \hookrightarrow Z_k(X) \to H_k(X) \to 0$$

is a short exact sequence, $z \mapsto [z]$, $z + B_k(X)$, so

$$\dim H_k(X) = \dim Z_k(X) - \dim B_k(X)$$

<u>Exact sequences of chain complexes</u> Let A_*, B_*, C_* be chain complexes and '

$$f : A_* \to B_*,\ g : B_* \to C_*$$

Consider the following sequence of chain maps

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

so for each n we have a sequence of linear maps

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

We say that this is exact when for each n, this sequence is exact.

3 Mayer-Vietoris Theorem

3.1 Algebraic Mayer-Vietoris Theorem

Theorem (Algebraic Mayer-Vietoris Theorem)**.** Suppose

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

is an exact sequence of chain complexes, then there exists a long exact sequence of the following type

$$\to H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B) \dots$$

$$\to H_1(A) \xrightarrow{i_*} H_1(B) \xrightarrow{p_*} H_1(C) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(B) \xrightarrow{p_*} H_0(C) \to 0$$

where in our case, $A_n = B_n = C_n = 0$ for $n < 0$.

This requires

$$A_* = (A_n, \partial_n),\ A_n = 0, n < 0$$

$$B_* = (B_n, \partial_n),\ B_n = 0, n < 0$$
$$C_* = (C_n, \partial_n),\ C_n = 0, n < 0$$

The connecting homomorphisms have the following *naturality property*: Suppose we have the following exact sequences of chain complexes,

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

$$0 \to A'_* \xrightarrow{i} B'_* \xrightarrow{p} C'_* \to 0$$

and suppose the following commutes,

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "\alpha"'] & B_* \arrow[r, "p"] \arrow[d, "\beta"'] & C_* \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A'_* \arrow[r] & B'_* \arrow[r] & C'_* \arrow[r] & 0
\end{tikzcd}$$

Compare the two long exact sequences,

$$\begin{array}{ccccccccc}
H_{n+1}(B) & \xrightarrow{p_*} & H_{n+1}(C) & \xrightarrow{\delta} & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{p_*} & H_n(0) \\
\downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\
H_{n+1}(B') & \xrightarrow{q_*} & H_{n+1}(C') & \xrightarrow{\delta'} & H_n(A') & \xrightarrow{j_*} & H_n(B') & \xrightarrow{q_*} & H_n(0)
\end{array}$$

this diagram commutes.

The Algebraic Mayer-Vietoris Theorem implies the *Geometric* Mayer-Vietoris Theorem.

3.2 Subcomplexes

Let $X = (V_X, \mathcal{S}_X)$, $Y = (V_Y, \mathcal{S}_Y)$ be simplicial complexes. Then we say that Y is a *subcomplex* of X if,

1. $V_Y \subset V_X$

2. $\mathcal{S}_Y \subset \mathcal{S}_X$

Proposition.

1. Let X_1, X_2 be subcomplexes of Z. Then $(V_{X_1} \cup V_{X_2}, \mathcal{S}_{X_1} \cup \mathcal{S}_{X_2})$ is also a subcomplex of Z. This is called the union $X_1 \cup X_2$.

2. $(V_{X_1} \cap V_{X_2}, \mathcal{S}_{X_1} \cap \mathcal{S}_{X_2})$ is also a subcomplex of Z. This is called the intersection $X_1 \cap X_2$.

We are interested in the case $Z = X_1 \cup X_2$.

Definition. Let Δ, Δ' be chain complexes. $\Delta = (\Delta_n, \partial_n)$, $\Delta' = (\Delta'_n, \partial'_n)$. Then the *direct sum*:

$$\Delta \oplus \Delta' = \left(\Delta \oplus \Delta', \begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix}\right)$$

$$\begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix} \begin{pmatrix} \partial_{n+1} & 0 \\ 0 & \partial_{n'+1} \end{pmatrix} = \begin{pmatrix} \partial_n \partial_{n+1} & 0 \\ 0 & \partial'_n \partial'_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3.3 The Geometric Mayer-Vietoris Theorem: Chain Version

Suppose X is a simplicial complex decomposed as a union $X = X_+ \cup X_-$, where X_+, X_- are subcomplexes. Then there exists an exact sequence of chain complexes like this,

$$0 \to C_*(X_+ \cap X_-) \xrightarrow{i} C_*(X_+ \oplus X_-) \xrightarrow{p} C_*(X) \to 0$$

If we apply the algebraic Mayer-Vietoris Theorem, we get the homological version, namely the long exact sequence,

$$H_{n+1}(X_+) \oplus H_{n+1}(X_-) \to H_{n+1}(X) \xrightarrow{\delta} H_n(X + \cap X_-)$$

$$\to H_n(X_+) \oplus H_n(X_-) \to H_n(X) \xrightarrow{\delta} H_{n-1}(X_+ \cap X_-)$$

and finishes

$$\xrightarrow{\delta} H_1(X_+ \cap X_-) \to H_1(X_+) \oplus H_1(X_-) \to H_1(X) \xrightarrow{\delta} H_0(X_+ \cap X_-)$$

$$\to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0$$

Let S^n = standard model of n-sphere,

$$S^n = (\Delta^{n+1})^{(n)}$$

We've shown for $n \geq 1$,

$$H_r(S^n; \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & 0 < r < n \\ ? & r = n \\ 0 & n < r \end{cases}$$

We've shown that $H_2(S^2; \mathbb{F}) = \mathbb{F}$.

Proposition. For $n \geq 2$, S^n can be written as $S^n = X_+ \cup X_-$ where $X_+ \cap X_- = S^{n-1}$ and X_+, X_- are *cones.*

$\Delta^{n+1} = (\{0, 1, \ldots, n{+}1\}, \{\text{all non-empty subsets of } \{0, 1, \ldots, n+1\}\})$

$S^n = (\{0, 1, \ldots, n{+}1\}, \{\text{all } \textit{proper} \text{ non-empty subsets of } \{0, 1, \ldots, n+1\}\})$
In particular every non-empty subset of $\{0, 1, \ldots, n\}$ is a simplex of S^n so,

1. $\Delta^n \subset S^n$. But as $S^{n-1} \subset \Delta^n$, then,
2. $S^{n-1} \subset S^n$ (note that $n + 1 \notin V_{S^{n-1}}$) and,
3. Taking $n + 1$ to be the cone point $C(S^{n-1}) \subset S^n$. ($C(S^{n-1})$ is sometimes called the *Witches hat*)
4. $$S^n = \Delta^n \cup C(S^{n-1})$$ $$S^{n-1} = \Delta^n \cap C(S^{n-1})$$

So we can write,

$$S^n = X_+ \cup X_-, \text{ where}$$

$$X_+ = C(S^{n-1})$$
$$X_- = \Delta^n$$
$$X_+ \cap X_- = S^{n-1}$$

Corollary. $H_n(S^n; \mathbb{F}) \cong \mathbb{F}$ for all $n \geq 2$.

Proof. By induction on n. We know this is true for $n = 2$. Suppose we've proven the hypothesis for $n - 1$ and consider the exact sequence,

$$H_n(X_+) \oplus H_n(X_-) \longrightarrow H_n(S^n) \xrightarrow{\delta} H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(X_+) \oplus H_{n-1}(X_-)$$

$$0 \oplus 0 \longrightarrow H_n(S^n) \xrightarrow{\cong} H_{n-1}(S^{n-1}) \longrightarrow 0 \oplus 0$$

which is isomorphic by the very short exact sequence. □

Let W be a vector space over $\mathbb{F}$ and suppose we have two vector subspaces of W, say U and V.

3.4 External and internal sum

Definition. External sum (coproduct)

$$U \oplus V = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in U, v \in V \right\}$$

$U \oplus V$ is a vector space. We define sums, scalar multiplication and zero as follows,

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$$

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

If U and V have finite dimensions, then

$$\dim(U \oplus V) = \dim(U) + \dim(V)$$

where U, V are subspaces of W.

Definition. Internal sum

$$U + V = \{u + v : u \in U, v \in V\}$$

Note that $U + V$ is a vector subspace of W.

What is the relationship between $U + V$ and $U \oplus V$? There is an exact sequence

$$\to U \oplus V \xrightarrow{\mu} U + V$$

$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = u + v$$

μ is linear and surjective by the definition of $U + V$.

Proposition.

$$\mu\begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff u + v = 0 \iff v = -u,\ u \in U,\ v \in V \text{ so } v \in U \cap V$$

We get an exact sequence,

$$0 \to U \cap V \xrightarrow{i} U \oplus V \xrightarrow{\mu} U + V \to 0$$

$$i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

As a consequence,

$$\dim(U \cap V) + \dim(U + V) = \dim(U) + \dim(V)$$

Theorem. (Chain version of the Geometric Mayer-Vietoris Theorem) Let $X = X_+ \cup X_-$ be the union of subcomplexes. For each n, there exists an exact sequence,

$$0 \to C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \to 0$$

$$\mu\begin{pmatrix} x \\ y \end{pmatrix} = x + y,\ i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

Proof. $C_n(X)$ has basis $\{[v_0, v_1, \ldots, v_n] : [v_0, \ldots, v_n] \in \mathcal{S}_X\}$

$$\mathcal{S}_X = \mathcal{S}_{X_+} \cup \mathcal{S}_{X_-}$$

$$C_n(X_+) \oplus C_n(X_-) \to C_n(X) \to 0$$

$$\begin{pmatrix} e \\ f \end{pmatrix} \mapsto e + f$$

The map is surjective because a basis element of $C_n(X)$ is either in $C_n(X_+)$ or $C_n(X_-)$. As a basis for the kernel, we have

$$\begin{pmatrix} [v_0, \ldots, v_n] \\ -[v_0, \ldots, v_n] \end{pmatrix}$$

where $\{v_0, \ldots, v_n\} \subset \mathcal{S}_{X_+} \cap \mathcal{S}_{X_-} = \mathcal{S}_{X+\cap X_-}$ so we have an exact sequence,

$$0 \to C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \to 0$$

This is an exact sequence of chain complexes because boundary formula is the same in every case. $\square$

Corollary. (of the geometric Mayer-Vietoris Theorem) Let X be a finite simplicial complex. Then,

$$\dim H_0(X; \mathbb{F}) = \{\text{number of connected components of } X\}$$

Proof. Let n be the number of connected components. This is true for $n = 1$. Suppose this is true for $n - 1$, and X has n connected components $X_1, X_2, \ldots, X_n$. Put

$$X_- = X_1 \cup X_2 \cup \ldots \cup X_{n-1}$$

$$X_+ = X_n$$

$$X_+ \cup X_- = X,\ X_+ \cap X_- = \emptyset(\text{ by definition})$$

Look at the following

$$H_0(X_+ \cap X_-) \to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0$$

(note that $H_0(X_+ \cap X_-) = 0)$). So

$$\dim H_0(X) = \dim H_0(X_+) + \dim H_0(X_-) = 1 + n - 1 = n$$

$\square$

Example.

$$S^0 = 0\text{-sphere} = 2 \text{ distinct points } \{-1, +1\}$$

So $H_0(S^0; \mathbb{F}) \cong \mathbb{F} \oplus \mathbb{F}$

$$H_n(S^0; \mathbb{F}) = 0,\ n \neq 0 \text{ (no higher simplices)}$$

On the other hand, the standard model of S^1 is,

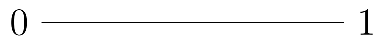
$$V_{S^1} = \{0, 1, 2\}$$

$$\mathcal{S}_{S^1} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$$

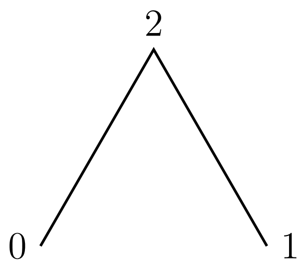
Proposition.

$$H_n(S^1; \mathbb{F}) = \begin{cases} \mathbb{F} & n = 0 \\ \mathbb{F} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Proof. Decompose $S^1 = X_1 \cup X_+$, where X_- is equal to



and X_+ is equal to



i.e.,

$$X_- = C(0),\ \ X_+ = \text{cone on } S^0 = \{\{0\}, \{1\}\}$$

$X_+ \cap X_- = S^0$. Use the Mayer-Vietoris Theorem, so,

$$H_1(X_+) \oplus H_1(X_-) \to H_1(S^1) \to H_0(S^0) \to H_0(X_+) \oplus H_0(X_-) \to H_0(S^1)$$

$$0 \to H_1(S) \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$$

$\dim(H_1(S^1)) = 1$ follows from Whitehead's lemma. $\square$

Lemma. Let

$$0 \to V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \ldots \to V_1 \xrightarrow{f_1} V_0 \to 0$$

be an exact sequence of finite dimensional vector spaces. Then,

$$\sum_{n \geq 0} \dim(V_{2n}) = \sum_{n \geq 0} \dim(V_{2n+1})$$

Proof. Let $P(n)$ denote the induction hypothesis on n.

$$0 \to V_1 \to V_0 \to 0$$

then $P(1)$ holds. The sequence is exact which implies $V_1 \cong V_0$. Now suppose we have an exact sequence,

$$0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$$

then by the kernel-rank theorem, this implies that

$$\dim(V_0) + \dim(V_2) = \dim(V_1)$$

and so $P(2)$ is true. Now we prove that $P(2n) \implies P(2n+1)$. Suppose that $P(2n)$ is true, and take the following exact sequence,

$$0 \to V_{2n+1} \xrightarrow{f_{2n+1}} V_{2n} \xrightarrow{f_{2n}} V_{2n-1} \to \ldots \to V_0 \to 0$$

Split the sequence and define $f = \mathrm{im}(f_{2n}) = \ker(f_{2n-1})$. Now we have two exact sequences,

$$0 \to V_{2n+1} \to V_{2n} \to f \to 0$$

and

$$0 \to f \to V_{2n-1} \to \ldots \to V_0 \to 0$$

By $P(2n)$,

$$\dim(f) + \sum_{r=0}^{n-1} \dim(V_{2r}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

and $\dim(f) = \dim(V_{2n}) - \dim(V_{2n+1})$. Substitute this into the previous expression and we get,

$$\sum_{r=0}^{n} \dim(V_{2r}) - \dim(V_{2n+1}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

This proves that $P(2n) \implies P(2n+1)$. To prove that $P(2n+1) \implies P(2n+2)$, take

$$0 \to V_{2n+2} \to V_{2n+1} \to V_{2n} \to \ldots$$

Split the exact sequence as before and proceed as before. (Set $f = \operatorname{im}(f_{2n+1}) = \ker(f_{2n})$) $\square$

Lemma. (Five lemma) Suppose we have a commutative diagram of abelian groups and homomorphisms,

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_0"] \arrow[d, "f_0"] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[d, "f_4"] \\
B_0 \arrow[r, "\beta_0"] & B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4
\end{tikzcd}$$

in which both rows are exact, and f_0, f_1, f_3, f_4 are isomorphisms. Then f_2 is also an isomorphism.

Proof. We first show that f_2 is injective. Suppose $x \in A_2$ such that $f_2(x) = 0$. We want to show that $x = 0$.

$$\beta_2 f_2(x) = 0 \implies f_3 \alpha_2(x) = 0$$

but f_3 is an isomorphism, which implies that $\alpha_2(x) = 0$. But then $x \in \ker(\alpha_2) = \operatorname{im}(\alpha_2)$, so $x = \alpha_1(y)$ for some $y \in A_1$.

$$f_2 \alpha_1(y) = 0 \implies \beta_1 f_1(y) = 0$$

so $f_1(y) \in \ker(\beta_1) = \operatorname{im}(\beta_0)$. Thus there exists $w \in \beta_0$ such that $\alpha_0(w) = f_1(y)$. But f_0 is surjective so write

$$w = f_0(z),\ \alpha_0 f_0(z) = f(y) \implies f_1 \alpha_0(z) = f_1(y)$$

but now f_1 is an isomorphism so $y = \alpha_0(z)$, $x = \alpha_1(y) = \alpha_1\alpha_0(z)$. By exactness, $\alpha_1\alpha_0 = 0$, so $\alpha = 0$

Now we show that f_2 is surjective. Take $b \in \beta_2$. We want to find $a \in A_2$ such that $f_2(a) = b$. Now, $\beta_2(b) \in B_3$. f_2 is an isomorphism so choose $x \in A_3$ so that

$$f_3(x) = \beta_2(b) \implies \beta_3 f_3(x) = \beta_3 \beta_2(b)$$

However by exactness, $\beta_3\beta_2 = 0$, so $\beta_3 f_3(x) = 0 \implies f_4\alpha_3(x) = 0$. Now f_4 is an isomorphism thus $\alpha_3(x) = 0$, $x \in \ker(\alpha_3) = \ker(\alpha_2)$. Now there exists $y \in A_2$ such that $\alpha_2(y) = x$. Consider $b - f_2(y)$. Then

$$\beta_2(b - f_2(y)) = \beta_2(b) - \beta_2 f_2(y) = \beta_2(b) - f_3\alpha_2(y) = \beta_2(b) - f_3(x) = 0$$

Thus $b - f_2(y) \in \ker(\beta_2) = \ker(\beta_1)$ so there exists $w \in \beta_1$ such that $\beta_1(w) = b - f_2(y)$. f_1 is an isomorphism implies that there exists $z \in A_1$ such that $f_1(z) = w$. So

$$\beta_1 f_1(z) = b - f_2(y)$$

$$f_2\alpha_1(z) = b - f_2(y) \implies b = f_2(y + \alpha_1(z))$$

Let $a = y + \alpha_1(z)$ which implies $b = f_2(a)$. Thus f_2 is surjective. $\square$

4 Subdivision

We will now show that homology is invariant under 'subdivision'. We first have to illustrate what 'subdivision' means.

Take for example Δ^2 (the triangle), and add a point at its barycenter, adding edges from the barycenter to each three of the vertices of Δ^2. We end up with an additional point (vertex), two additional regions and three additional edges. This is an example of an easy subdivision.

Definition. Let $X = (V_X, \mathcal{S}_X)$ be a finite simplicial complex, and let $\tau \in \mathcal{S}_X$. $\hat{\tau}$ will denote the subcomplex of X determined by τ.

$$V_{\hat{\tau}} = \tau, \;\; \mathcal{S}_{\hat{\tau}} = \{p \in \mathcal{S}_X, \, p \subset \tau\}$$

We say that $\sigma \in \mathcal{S}_X$ is *principal* (or maximal) when σ is not contained properly in any other simplex.

Proposition. If $\sigma_1, \ldots, \sigma_N$ are the principal simplices of X then

$$X = \hat{\sigma_1} \cup \hat{\sigma_2} \cup \ldots \cup \hat{\sigma_N}$$

4.1 Subdivision at a principal simplex

Let σ be a principal simplex of X and let $\sigma_1, \ldots \sigma_N$ be the remaining principal simplices such that

$$X = \hat{\sigma} \cup \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

Put $X_+ = \hat{\sigma}$, $X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$. Then $X = X_+ \cup X_-$ and $X_+ \cap X_- \subset \partial\hat{\sigma}$ (boundary of $\hat{\sigma}$)

Definition.

$$Sd(X, \sigma) = C(\partial\sigma) \cup \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

i.e.,

$$Sd(X\sigma) = X'_+ \cup X'_-$$

where X'_+ is the cone on the boundary of σ and

$$X'_- = X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

and

$$X'_+ \cap X'_- = X_+ \cap X_-$$

Taking our Δ^2 example earlier, letting $\sigma = \Delta^2$, $Sd(\Delta^2, \sigma)$ is exactly the resulting simplex we get by performing our subdivision earlier.

4.2 Squash mapping

Let σ be an n-simplex and consider $C(\partial\sigma)$. We construct simplicial mappings $C(\partial\sigma) \to \sigma$ as follows,

$$Sq|_{\partial\sigma} = \mathrm{id}_{\partial\sigma}$$

$$Sq(*) = \text{some (arbitrarily chosen) vertex in } \partial\sigma$$

where $*$ is our cone point.

Proposition. $Sq : H_k(C(\partial\sigma)) \to H_k(\sigma)$ is an isomorphism for all k.

Proof. $C(\partial\sigma)$ and σ are both cones, so $H_k(C(\partial\sigma)) = H_k(\sigma) = 0$ if $k > 0$. For $k = 0$, any vertex V in $C(\partial\sigma)$ gives a basis $[v]$ for $H_0(C(\partial\sigma))$ (any two vertices differ by a boundary). Likewise, any vertex w in σ gives basis element $[w]$ in $H_0(\sigma)$ and $Sq([v]) = [w]$, so now

$$Sq : H_0(C(\partial\sigma)) \xrightarrow{\cong} H_0(\sigma)$$

$\square$

Theorem. Let K be a finite complex. Let σ be a principal complex, and let $\sigma_1, \ldots, \sigma_N$ be the remaining principal simplices and define an extended squash map $Sq : Sd(X, \sigma) \to X$ by

$$Sq : C(\delta\sigma) \to \sigma \text{ is a squash mapping}$$

$$Sq : \sigma_i \to \sigma_i \text{ identity } i = 1, \ldots, N$$

Then $Sq : H_k(Sd(X, \sigma)) \to H_k(X)$ is an isomorphism for all k.

Proof. Put

$$X_+ = \hat{\sigma},\ X'_+ = C(\partial\sigma)$$

$$X'_- = X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

so $X'_+ \cap X'_- = X_+ \cap X_-$ and $Sq : X'_- \to X_-$ is the identity. Consider the Mayer-Vietoris sequences

$$\begin{tikzcd}
H_n(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n(X'_+) \oplus H_n(X'_-) \arrow[r] \arrow[d, "M"] & H_n(Sd(X,\sigma)) \arrow[r] \arrow[d, "Sq"] & H_{n-1}(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \arrow[d, "M"] \\
H_n(X_+ \cap X_-) \arrow[r] & H_n(X_+) \oplus H_n(X_-) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(X_+ \cap X_-) \arrow[r] & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{tikzcd}$$

where $M = \begin{pmatrix} Sq & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. id is clearly an isomorphism, as well as M, since $Sq : H_n(X'_+) \to H_n(X_+)$ is an isomorphism. By the five lemma, Sq is an isomorphism. $\square$

We have now shown that if $Sd(X, \sigma)$ is the subdivision of X at a principal simplex, then $H_*(Sd(X, \sigma)) \cong H_*(X)$. Now we have to show that this also holds for non-principal simplices.

4.3 Subdivision at a non-principal simplex

We first describe an example of a non-principal simplex. Take Δ^2. Then take $\{0,1\}$. This is contained within $\{0,1,2\}$, hence this is a non-principal simplex. We wish to perform subdivisions at simplices such as these.

Definition (Join)**.** Let $K = (V_K, \mathcal{S}_K)$ and $L = (V_L, \mathcal{S}_L)$ be simplicial complexes such that $V_K \cap V_L = \emptyset$. Define

$$K * L = (V_K \cup V_L,\ \mathcal{S}_K \cup \mathcal{S}_L \cup \{p \cup \tau, p \in \mathcal{S}_K, \tau \in \mathcal{S}_L\}$$

A special case is where $K =$ point, so then $K * L = C(L)$.

Proposition.

$$\Delta^{m+n+1} \cong \Delta^m * \Delta^n$$

Proof. Vertex set of Δ^{m+n+1} is

$$\{0, \ldots, m+n+1\} = \{0, \ldots, m\} \cup \{m+1, \ldots, m+n+1\}$$

There is a 1-1 correspondence between the last set and

$$0, \ldots, n$$

so if we take as our model of Δ^n the vertex set $\{m+1, \ldots, m+n+1\}$ and simplices to be all the non-empty subsets, we get $\Delta^{m+n+1} = \Delta^m * \Delta^n$ (the dimension goes up by 1). □

Note also that $S^m * S^n \cong S^{m+n+1}$. If k is a single point pt, then $C(L) = \{pt\} * L$.

Join is associative. If K, L, M are simplicial complexes with no vertices in common, then

$$(K * L) * M \equiv K * (L * M)$$

Corollary. If K, L are disjoint complexes, then $C(K) * L \cong C(K * L)$

So the join of a cone to anything is a cone.

4.4 Star and Link

Definition (Star neighbourhood)**.** Let τ be a simplex of X, and let $\sigma_1, \ldots, \sigma_N$ be the principal simplices which contain τ. Then

$$St(\tau, X) = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N} = \textit{star neighbourhood} \text{ of } \tau \text{ in } X$$

Definition (Link)**.** Let X be a simplicial complex and ρ, τ be simplices of X such that $rho \cap \tau = \emptyset$. We say that ρ is joinable to τ in X (i.e., $\rho \cup \tau = p * \tau$). The *link* of τ in X, $Lk(\tau, X)$ consists of all these simplices of ρ of X such that $\rho \cap \tau = \emptyset$ and $\rho \cap \tau$ is a simplex, i.e., ρ is joinable to τ.

Example. star and link example

Proposition. If τ is a simplex of X, then $St(\tau, X) = \hat{\tau} * Lk(\tau, X)$

Proof. The case where τ is principal is empty here. So suppose τ is not principal. Let σ be a principal simplex with $\tau \subset \sigma$. Write

$$\tau = \{v_0, \ldots, v_m\} \;\; m < n$$

$$\sigma = \{v_0, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$

Put $\rho = \{v_{m+1}, \ldots, v_n\}$ so then

$$\sigma = \tau * \rho$$

Do this for every principal simplex which contains τ. Each $\sigma_1 = \tau * \rho_i$ for some ρ_i, so

$$\bigcup \sigma_i = \tau * (\cup \rho_i) = \tau * Lk(\tau, X)$$

□

Definition (Subdivision at a non-principal simplex)**.** Let X be a finite simplicial complex, and τ a non-principal simplex. Let $\sigma_1, \ldots, \sigma_m$

be the principal simplices which contain τ. Let $\sigma_{m+1}, \ldots, \sigma_N$ be the remaining principal simplices. Put

$$X_+ = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_m} = St(\tau, X)$$

$$X_- = \sigma_{\hat{m+1}} \cup \ldots \cup \hat{\sigma_N}$$

$$X = X_+ + X_- \ \ (X_+ \cap X_- \cap \tau \subset \partial\sigma)$$

and put

$$X'_+ = C(\partial\tau) * Lk(\tau, X)$$

$$X'_- = X_-$$

Define

$$Sd(X,\tau) = X'_+ \cup X'_-$$

$$Sd = (C(\partial\tau) * Lk) \cup X'_-$$

We have $Sq : C(\partial\tau) \to \tau$. Extend by identity to $Sq : C(\partial\tau) * Lk \to \tau * Lk$ by identity on Lk. Extend again by identity on $X'_- = X_-$, $Sq : Sd(X,\tau) \to X$

Proposition. $Sq : Sd(X,\tau) \to X$ induces an isomorphism on homology.

Proof.

$$\begin{tikzcd}
H_n(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n(X'_+) \oplus H_n(X'_-) \arrow[r] \arrow[d, "M"] & H_n(Sd(X,\tau)) \arrow[r] \arrow[d, "Sq"] & H_{n-1}(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \arrow[d, "M"] \\
H_n(X_+ \cap X_-) \arrow[r] & H_n(X_+) \oplus H_n(X_-) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(X_+ \cap X_-) \arrow[r] & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{tikzcd}$$

where $M = \begin{pmatrix} Sq & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. Apply the five lemma to show Sq induces an isomorphism. □

So now we've proved the following,

Theorem. Homology is invariant under subdivision.

Example. subdivision isomorphism example

We now have a functor H_n which takes simplicial complexes to vector spaces, and simplicial maps to linear maps, e.g., if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

then

$$\begin{tikzcd} H_n(X) \arrow[r, "H_n(f)"] \arrow[rr, bend right, "H_n(g \circ f)"'] & H_n(Y) \arrow[r, "H_n(g)"] & H_n(Z) \end{tikzcd}$$

Properties of functors:

1. $H_n(g \circ f) = H_n(g) \circ H_n(f)$
2. $H_n(\mathrm{id}) = \mathrm{id}_{H_n}$ i.e.,

$$\mathrm{id} : X \to X,\ H_n(\mathrm{id}) : H_n(X) \to H_n(X)$$

As a consequence, if $f : X \to Y$ is an isomorphism, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.

Proof. If $g = f^{-1} : Y \to X$, $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$ then

$$H_n(g) \circ H_n(f) = \mathrm{id},\ H_n(f) \circ H_n(g) = \mathrm{id}$$

so

$$H_n(g) = H_n(f)^{-1}$$

□

But we have established a stronger property, that is, H_n is invariant under subdivision, i.e., if Y subdivides X, then $H_n(Y) \cong H_n(X)$.

Definition. Let X, Y be simplicial complexes. We say that X, Y are *combinatorially equivalent* (written $X \sim Y$) if and only if there exists a finite sequence $(X_r)_{0 \leq r \leq N}$ of complexes X_r such that $X_0 = X$, $X_N = Y$ and for each r, $0 \leq r \leq N - 1$, either X_{r+1} is a subdivision of X_r or X_r is a subdivision of X_{r+1}.

Corollary. If $X \sim Y$ then $H_n(X) \cong H_n(Y)$.

Thus we won't worry too much about how we triangulate things.