

# Multivariable Analysis - MATH0019

**Based on lectures by Prof Yiannis Petridis**

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Notes based on the Autumn 2021 Multivariable Analysis lectures  
by Prof Yiannis Petridis.

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# 1 Review of Euclidean space and some linear algebra

## 1.1 Euclidean space

Recall the *Euclidean n-space*  $\mathbb{R}^n$

$$\mathbb{R}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R}\}$$

Note that use of superscripts instead of subscripts. We also have the *Euclidean norm* given by

$$|x| = \left( (x^1)^2 + \dots + (x^n)^2 \right)^{\frac{1}{2}}$$

and the *inner product*

$$x \cdot y = \sum_{i=1}^n x^i \cdot y^i$$

where

$$\begin{aligned} x &= (x^1, \dots, x^n) \\ y &= (y^1, \dots, y^n) \end{aligned}$$

Recall the *standard basis*

$$\{e_1, e_2, \dots, e_n\}$$

where  $e_i \in \mathbb{R}^n$  where the only non-zero component is the  $i^{\text{th}}$  component whose value is 1. Hence we can represent  $x \in \mathbb{R}^n$  as

$$x = \sum_{i=1}^n x^i e_i$$

**Proposition.**

1.  $|x| \geq 0, |x| = 0 \iff x = 0$
2.  $|x \cdot y| \leq |x||y|$  (Cauchy-Schwarz inequality)

$$3. \quad |x + y| \leq |x| + |y| \quad (\text{Triangle inequality})$$

$$4. \quad |a \cdot x| = |a||x| \quad (\text{for } a \in \mathbb{R})$$

$$5. \quad x \cdot y = y \cdot x$$

We may also write  $x \cdot y = \langle x, y \rangle$  as the inner product is a bilinear form. Recall the properties of a bilinear form which are

$$1. \quad \langle x + 1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

$$2. \quad \langle a \cdot x, y \rangle = a \langle x, y \rangle$$

Also note that

$$\langle x, x \rangle = |x|^2$$

## 1.2 Some linear algebra

Recall that a mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be *linear* if and only if, for  $x, y \in \mathbb{R}^n$ ,  $a \in \mathbb{R}$ ,

$$T(x + y) = T(x) + T(y)$$

$$T(a \cdot x) = a \cdot T(x)$$

Note that in both equations, the operations taking place on the left hand side is done in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  on the right hand side.

Recall that we can recover  $M$ , the matrix representation of a linear mapping  $T$  by applying  $T$  to each of the standard basis. So if

$$T(e_j) = a_{1j}e_1 + a_{2j}e_2 + \dots + a_{mj}e_m$$

then

$$\begin{aligned} M &= (a_{ij}) \\ &= [T] \end{aligned}$$

where  $M$  (or as we shall denote  $[T]$ ) is the  $m \times n$  matrix representing the linear transformation  $T$ .

$$\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} = M \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

If we have two mappings  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$  where  $[S]_{p \times m}$  represents the  $p \times m$  matrix representing the linear transformation  $S$ , then  $S \circ T$  is a linear mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^p$ .

$$S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^p$$

and

$$[S \circ T] = [S][T]$$

## 2 Functions and continuity

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called a *vector field* if  $m > 1$ , and a *scalar field* if  $m = 1$ .

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  then we have

$$f = (f^1, f^2, \dots, f^m)$$

where

$$f^i : \mathbb{R}^n \rightarrow \mathbb{R}$$

so we can write

$$\begin{aligned} f(x) &= f^1(x)e_1 + \dots + f^m(x)e_m \\ &= (f^1(x), \dots, f^m(x)) \end{aligned}$$

and we call each of these  $f^i$ 's the components of  $f$ .

Now we define a function  $\pi^i : \mathbb{R}^m \rightarrow \mathbb{R}$  given by  $\pi^i(y) = y^i$ , so that

$$\pi^i(y^1, y^2, \dots, y^m) = y^i$$

and we call this the *projection in the  $i^{\text{th}}$  direction* (or the *projection function*). This function is a linear transformation.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \\ & \searrow f^i = \pi^i \circ f & \downarrow \pi^i \\ & & \mathbb{R} \end{array}$$

Sometimes instead of  $\mathbb{R}^n$  we may define  $f$  on a subset of  $A \subset \mathbb{R}^n$ , usually where  $A$  is open, i.e., sometimes we have  $f : A \rightarrow \mathbb{R}^m$ .

## 2.1 Limits

**Definition.** Let  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ . We write  $\lim_{x \rightarrow a} f(x) = b$  to mean

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - b| < \epsilon$$

We say  $f$  is continuous on a set  $A$  if  $f$  is continuous at  $a$ , for all  $a \in A$ .

**Theorem** (Combination theorem). Suppose that

$$\lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c$$

then,

1.  $\lim_{x \rightarrow a} (f(x) + g(x)) = b + c$
2. If  $\lambda \in \mathbb{R}$ , then  $\lim_{x \rightarrow a} (\lambda \cdot f(x)) = \lambda \cdot b$
3.  $\lim_{x \rightarrow a} f(x) \cdot g(x) = b \cdot c$
4.  $\lim_{x \rightarrow a} |f(x)| = |b|$