Multivariable Analysis - MATH0019

Based on lectures by Prof Yiannis Petridis

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Notes based on the Autumn 2021 Multivariable Analysis lectures by Prof Yiannis Petridis.

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1 Review of Euclidean space and some linear algebra

1.1 Euclidean space

Recall the Euclidean n-space \mathbb{R}^n

$$\mathbb{R}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R}\}$$

Note that use of superscripts instead of subscripts. We also have the *Euclidean norm* given by

$$|x| = ((x^1)^2 + \dots + (x^n)^2)^{\frac{1}{2}}$$

and the inner product

$$x \cdot y = \sum_{i=1}^{n} x^{i} \cdot y^{i}$$

where

$$x = (x^1, \dots, x^n)$$

$$y = (y^1, \dots, y^n)$$

Recall the standard basis

$$\{e_1, e_2, \ldots, e_n\}$$

where $e_i \in \mathbb{R}^n$ where the only non-zero component is the i^{th} component whose value is 1. Hence we can represent $x \in \mathbb{R}^n$ as

$$x = \sum_{i=1}^{n} x^{i} e_{i}$$

Proposition.

1.
$$|x| \ge 0, |x| = 0 \iff x = 0$$

2.
$$|x \cdot y| \le |x||y|$$
 (Cauchy-Schwarz inequality)

3.
$$|x+y| \le |x| + |y|$$
 (Triangle inequality)

4.
$$|a \cdot x| = |a||x|$$
 (for $a \in \mathbb{R}$)

$$5. \ x \cdot y = y \cdot x$$

We may also write $x \cdot y = \langle x, y \rangle$ as the inner product is a bilinear form. Recall the properties of a bilinear form which are

1.
$$\langle x+1+x_2,y\rangle = \langle x_1,y\rangle + \langle x_2,y\rangle$$

2.
$$\langle a \cdot x, y \rangle = a \langle x, y \rangle$$

Also note that

$$\langle x, x \rangle = |x|^2$$

1.2 Some linear algebra

Recall that a mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be *linear* if and only if, for $x, y \in \mathbb{R}^n$, $a \in \mathbb{R}$,

$$T(x+y) = T(x) + T(y)$$

$$T(a \cdot x) = a \cdot T(x)$$

Note that in both equations, the operations taking place on the left hand side is done in \mathbb{R}^n and \mathbb{R}^m on the right hand side.

Recall that we can recover M, the matrix representation of a linear mapping T by applying T to each of the standard basis. So if

$$T(e_i) = a_{ij}e_1 + a_{2j}e_2 + \ldots + a_{mj}e_m$$

then

$$M = (a_{ij})$$
$$= [T]$$

where M (or as we shall denote [T]) is the $m \times n$ matrix representing the linear transformation T.

$$\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} = M \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

If we have two mappings $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^p$ where $[S]_{p \times m}$ represents the $p \times m$ matrix representing the linear transformation S, then $S \circ T$ is a linear mapping from \mathbb{R}^n to \mathbb{R}^p .

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^p$$

and

$$[S \circ T] = [S][T]$$

2 Functions and continuity

A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is called a vector field if m > 1, and a scalar field if m = 1.

If $f: \mathbb{R}^n \to \mathbb{R}^m$ then we have

$$f = (f^1, f^2, \dots, f^m)$$

where

$$f^i: \mathbb{R}^n \to \mathbb{R}$$

so we can write

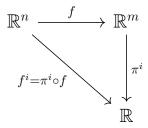
$$f(x) = f^{1}(x)e_{1} + \ldots + f^{m}(x)e_{m}$$
$$= \left(f^{1}(x), \ldots, f^{m}(x)\right)$$

and we call each of these f^{i} 's the components of f.

Now we define a function $\pi^i: \mathbb{R}^m \to \mathbb{R}$ given by $\pi^i(y) = y^i$, so that

$$\pi^i(y^1, y^2, \dots, y^m) = y^i$$

and we call this the projection in the i^{th} direction (or the projection function). This function is a linear transformation.



Sometimes instead of \mathbb{R}^n we may define f on a subset of $A \subset \mathbb{R}^n$, usually where A is open, i.e., sometimes we have $f: A \to \mathbb{R}^m$.

2.1 Limits

Definition. Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. We write $\lim_{x \to a} f(x) = b$ to mean

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - b| < \epsilon$$

We say f is continuous on a set A us f is continuous at a, for all $a \in A$.

Theorem (Combination theorem). Suppose that

$$\lim_{x \to a} f(x) = b \text{ and } \lim_{x \to a} g(x) = c$$

then,

1.
$$\lim_{x \to a} \left(f(x) + g(x) \right) = b + c$$

2. If
$$\lambda \in \mathbb{R}$$
, then $\lim_{x \to a} (\lambda \cdot f(x)) = \lambda \cdot b$

3.
$$\lim_{x \to a} f(x) \cdot g(x) = b \cdot c$$

4.
$$\lim_{x \to a} |f(x)| = |b|$$