

# Multivariable Analysis - MATH0019

**Based on lectures by Prof Yiannis Petridis**

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Notes based on the Autumn 2021 Multivariable Analysis lectures  
by Prof Yiannis Petridis.

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# 1 Review of Euclidean space and some linear algebra

## 1.1 Euclidean space

Recall the *Euclidean n-space*  $\mathbb{R}^n$

$$\mathbb{R}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R}\}$$

Note that use of superscripts instead of subscripts. We also have the *Euclidean norm* given by

$$|x| = \left( (x^1)^2 + \dots + (x^n)^2 \right)^{\frac{1}{2}}$$

and the *inner product*

$$x \cdot y = \sum_{i=1}^n x^i \cdot y^i$$

where

$$\begin{aligned} x &= (x^1, \dots, x^n) \\ y &= (y^1, \dots, y^n) \end{aligned}$$

Recall the *standard basis*

$$\{e_1, e_2, \dots, e_n\}$$

where  $e_i \in \mathbb{R}^n$  where the only non-zero component is the  $i^{\text{th}}$  component whose value is 1. Hence we can represent  $x \in \mathbb{R}^n$  as

$$x = \sum_{i=1}^n x^i e_i$$

**Proposition.**

1.  $|x| \geq 0, |x| = 0 \iff x = 0$
2.  $|x \cdot y| \leq |x||y|$  (Cauchy-Schwarz inequality)

$$3. \quad |x + y| \leq |x| + |y| \quad (\text{Triangle inequality})$$

$$4. \quad |a \cdot x| = |a| \cdot |x| \quad (\text{for } a \in \mathbb{R})$$

$$5. \quad x \cdot y = y \cdot x$$

We may also write  $x \cdot y = \langle x, y \rangle$  as the inner product is a bilinear form. Recall the properties of a bilinear form which are

$$1. \quad \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$$

$$2. \quad \langle ax, y \rangle = a \langle x, y \rangle$$

Also note that

$$\langle x, x \rangle = |x|^2$$

## 1.2 Some linear algebra

Recall that a mapping  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is said to be *linear* if and only if, for  $x, y \in \mathbb{R}^n$ ,  $a \in \mathbb{R}$ ,

$$T(x + y) = T(x) + T(y)$$

$$T(a \cdot x) = a \cdot T(x)$$

Note that in both equations, the operations taking place on the left hand side is done in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  on the right hand side.

Recall that we can recover  $M$ , the matrix representation of a linear mapping  $T$  by applying  $T$  to each of the standard basis. So if

$$T(e_j) = a_{1j}e_1 + a_{2j}e_2 + \dots + a_{mj}e_m$$

then

$$\begin{aligned} M &= (a_{ij}) \\ &= [T] \end{aligned}$$

where  $M$  (or as we shall denote  $[T]$ ) is the  $m \times n$  matrix representing the linear transformation  $T$ .

$$\begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^m \end{pmatrix} = M \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}$$

If we have two mappings  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$  where  $[S]_{p \times m}$  represents the  $p \times m$  matrix representing the linear transformation  $S$ , then  $S \circ T$  is a linear mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^p$ .

$$S \circ T : \mathbb{R}^n \rightarrow \mathbb{R}^p$$

and

$$[S \circ T] = [S][T]$$

## 2 Functions and continuity

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is called a *vector field* if  $m > 1$ , and a *scalar field* if  $m = 1$ .

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  then we have

$$f = (f^1, f^2, \dots, f^m)$$

where

$$f^i : \mathbb{R}^n \rightarrow \mathbb{R}$$

so we can write

$$\begin{aligned} f(x) &= f^1(x)e_1 + \dots + f^m(x)e_m \\ &= (f^1(x), \dots, f^m(x)) \end{aligned}$$

and we call each of these  $f^i$ 's the components of  $f$ .

Now we define a function  $\pi^i : \mathbb{R}^m \rightarrow \mathbb{R}$  given by  $\pi^i(y) = y^i$ , so that

$$\pi^i(y^1, y^2, \dots, y^m) = y^i$$

and we call this the *projection in the  $i^{\text{th}}$  direction* (or the *projection function*). This function is a linear transformation.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^m \\ & \searrow f^i = \pi^i \circ f & \downarrow \pi^i \\ & & \mathbb{R} \end{array}$$

Sometimes instead of  $\mathbb{R}^n$  we may define  $f$  on a subset of  $A \subset \mathbb{R}^n$ , usually where  $A$  is open, i.e., sometimes we have  $f : A \rightarrow \mathbb{R}^m$ .

## 2.1 Limits and continuity

**Definition.** Let  $a \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^m$ . We write  $\lim_{x \rightarrow a} f(x) = b$  to mean

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - b| < \epsilon$$

We say  $f$  is continuous at  $a$  if and only if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

$f$  is said to be continuous on a set  $A$  if  $f$  is continuous at  $a$ , for all  $a \in A$ .

**Theorem** (Combination theorem). Suppose that

$$\lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c$$

then,

$$1. \lim_{x \rightarrow a} (f(x) + g(x)) = b + c$$

$$2. \text{ If } \lambda \in \mathbb{R}, \text{ then } \lim_{x \rightarrow a} (\lambda \cdot f(x)) = \lambda \cdot b$$

$$3. \lim_{x \rightarrow a} f(x) \cdot g(x) = b \cdot c$$

$$4. \lim_{x \rightarrow a} |f(x)| = |b|$$

*Proof.*

3.

$$\begin{aligned} f(x) \cdot g(x) - b \cdot c &= f(x) \cdot g(x) - b \cdot g(x) + b \cdot g(x) - b \cdot c \\ &= (f(x) - b) \cdot g(x) + b \cdot (g(x) - c) \end{aligned}$$

$$\begin{aligned} |f(x) \cdot g(x) - b \cdot c| &\leq |(f(x) - b) \cdot g(x)| + |b \cdot (g(x) - c)| \\ &\leq |f(x) - b| \cdot |g(x)| + |b| \cdot |g(x) - c| \end{aligned}$$

and  $|f(x) - b|, |g(x) - c|$  both tend to 0 as  $x \rightarrow a$ , while  $|g(x)|$  is bounded close to  $a$ , so the right hand side tends to 0.

4. By the reverse triangle inequality,

$$||f(x)| - |b|| \leq |f(x) - b|$$

□

Now note that a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is continuous everywhere.

**Lemma.** Given  $T$  linear  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , there exists  $M > 0$  such that

$$\forall x \in \mathbb{R}^n, |T(x)| \leq M|x|$$

*Proof.* Given that  $x = (x^1, x^2, \dots, x^n)$ , we can rewrite

$$x = \sum_{i=1}^n x^i e_i$$

and so as  $T$  is linear,

$$T(x) = \sum_{i=1}^n x^i T(e_i)$$

and then,

$$\begin{aligned} |T(x)| &= \left| \sum_{i=1}^n x^i T(e_i) \right| \\ &\leq \sum_{i=1}^n |x^i| |T(e_i)| \leq \left( \sum_{i=1}^n |x^i|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^n |T(e_i)|^2 \right)^{\frac{1}{2}} \\ &= |x| \left( \sum_{i=1}^n |T(e_i)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

and so set

$$M = |x| \left( \sum_{i=1}^n |T(e_i)|^2 \right)^{\frac{1}{2}}$$

□

Using this lemma we can prove  $T$  to be continuous at  $y \in \mathbb{R}^n$  as follows,

$$|T(x) - T(y)| = |T(x - y)| \leq M|x - y|$$

Given  $\epsilon > 0$ , we can simply take  $\delta = \frac{\epsilon}{M} > 0$ .

**Remark.**

1. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Then,

$$f \text{ is continuous} \iff f^i \text{ is continuous for } i = 1, \dots, m$$

Sufficiency follows by noting that  $f^i = \pi^i \circ f$  and the fact that  $\pi^i$  is continuous. Necessity is a simple exercise to prove.

2. Polynomials in  $n$ -variables are continuous, for example, the function

$$1024(x^4)^5(x^2)^3(x^5)^{29}$$

is continuous. The same holds for rational functions

$$R(x^1, \dots, x^n) = \frac{P(x^1, \dots, x^n)}{Q(x^1, \dots, x^n)}$$

where  $P, Q$  are polynomials, continuous, where denominator is not equal to 0.

### 3 Derivatives

**Definition.** We define

$$D_i f(a) = \lim_{h \rightarrow 0} \frac{f(a^1, a^2, \dots, a^{i-1}, a^i + h, a^{i+1}, \dots, a^n) - f(a)}{h}$$

**Example.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , so  $f(x, y)$  is a function of two variables  $x, y$ . Then,

$$D_1 f(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h} = \frac{\partial f}{\partial x}(a, b)$$

$$D_2 f(a, b) = \frac{\partial f}{\partial y}(a, b)$$

and similarly for  $\mathbb{R}^3 \rightarrow \mathbb{R}$ .

**Definition.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  (or replace  $\mathbb{R}^n$  with  $A$  open in  $\mathbb{R}^n$ ), and let  $a \in \mathbb{R}^n$ . We say  $f$  is *differentiable* at  $a \in \mathbb{R}^n$  if we can find a linear transformation  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{h \rightarrow 0} \frac{|f(a + h) - f(a) - \lambda(h)|}{|h|} = 0 \tag{1}$$

We call  $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m$  the *derivative* of  $f$  at  $a$ , and we denote it by  $Df(a)$ .



**Theorem.**  $\lambda$  as defined above is unique.

*Proof.* If  $f$  is differentiable at  $a$  and  $\lambda, \mu : \mathbb{R}^n \rightarrow \mathbb{R}^m$  both linear transformations that satisfy (1), then

$$\mu = \lambda$$

□

*Proof.* We need to prove that  $\forall x \in \mathbb{R}^n$ , that  $\mu(x) = \lambda(x)$ . We can assume  $x \neq 0$  since as  $\mu, \lambda$  are linear (so they map 0 to 0).

$$\begin{aligned} \frac{|\lambda(h) - \mu(h)|}{|h|} &= \frac{|\lambda(h) + f(a) - f(a+h) + f(a+h) - f(a) - \mu(h)|}{|h|} \\ &= \frac{|\lambda(h) + f(a) - f(a+h)|}{|h|} + \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} \end{aligned}$$

Now if we take limits as  $h \rightarrow 0$ , then

$$\lim_{h \rightarrow 0} \frac{|\lambda(h) - \mu(h)|}{|h|} = 0$$

If we take  $t$  small, and let  $tx = h$ , then since  $h \rightarrow 0 \iff t \rightarrow 0$  we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} &= \lim_{t \rightarrow 0} \frac{|t\lambda(x) - t\mu(x)|}{|t||x|} \\ &= \lim_{t \rightarrow 0} \frac{|t||\lambda(x) - \mu(x)|}{|t||x|} \\ &= \lim_{t \rightarrow 0} \frac{|\lambda(x) - \mu(x)|}{|x|} \\ &= \frac{|\lambda(x) - \mu(x)|}{|x|} \end{aligned}$$

which is equal to 0 as

$$\lim_{t \rightarrow 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} = 0$$

hence

$$|\lambda(x) - \mu(x)| = 0$$

i.e.,

$$\lambda(x) = \mu(x)$$

□

Note that for  $h \in \mathbb{R}^n$ , that

$$\lambda(h) = Df(a)(h)$$

so  $Df(a)$  is a linear transformation,  $Df(a) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

We call the matrix representation of  $Df(a) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  the *Jacobian* of  $f$  at  $a$  (here we use the standard basis) and we denote it by  $f'(a)$ . So,

$$Df(a)(h) = f'(a) \begin{pmatrix} h^1 \\ h^2 \\ \vdots \\ h^n \end{pmatrix}$$

Note that the product of this operation is column  $m$ -vector ( $m \times 1$  matrix), an element of  $\mathbb{R}^m$ .

**Definition.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  (or replace  $\mathbb{R}^n$  with any  $A$  open in  $\mathbb{R}^n$ ), and let  $u \in \mathbb{R}^n$ , where  $u \neq 0$ . We define the *directional derivative* of  $f$  at  $a \in \mathbb{R}^n$  in the  $u$ -direction by

$$D_u f(a) = \lim_{h \rightarrow 0} \frac{f(a + hu) - f(a)}{h}$$

(if it exists), where  $h \in \mathbb{R}$ . If  $u = e_i$ , then  $D_u f(a) = D_i f(a)$ .

**Theorem.** If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is differentiable at  $a \in \mathbb{R}^n$  then it is continuous at  $a$ .

*Proof.* We want to show that

$$\lim_{h \rightarrow 0} f(a + h) = f(a)$$

Hence we proceed

$$\begin{aligned} \lim_{h \rightarrow 0} |f(a + h) - f(a)| &= \lim_{h \rightarrow 0} |f(a + h) - f(a) - \lambda(h) + \lambda(h)| \\ &\leq \lim_{h \rightarrow 0} |f(a + h) - f(a) - \lambda(h)| + \lim_{h \rightarrow 0} |\lambda(h)| \\ &\leq \lim_{h \rightarrow 0} \frac{|f(a + h) - f(a) - \lambda(h)|}{|h|} |h| + \lim_{h \rightarrow 0} |\lambda(h)| \\ &= 0 \end{aligned}$$

so

$$\lim_{h \rightarrow 0} |f(a + h) - f(a)| = 0$$

□

**Remark.** If

$$\lim_{h \rightarrow 0} \frac{|f(a + h) - f(a) - \lambda(h)|}{|h|}$$

then letting  $h = x - a$ , we can write

$$\lim_{x \rightarrow a} \frac{|f(x) - f(a) - \lambda(x - a)|}{|x - a|} = 0$$

If we let  $\phi(x) = f(x) - f(a) - \lambda(x - a)$ , then we have that  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and we can rewrite

$$f(x) = f(a) + \lambda(x - a) + \phi(x)$$

or, recalling that  $\lambda = Df(a)$ ,

$$f(x) = f(a) + Df(a)(x - a) + \phi(x)$$

By definition, we have

$$\lim_{x \rightarrow a} \frac{|\phi(x)|}{|x - a|} = 0$$