

# Algebraic Topology - MATH0023

Based on lectures by Prof FEA Johnson

Notes taken by Imran Radzi

Notes based on the Autumn 2021 Algebraic Topology lectures by Prof FEA Johnson.

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# 1 Simplicial complexes

**Definition** (Simplicial complex). A *simplicial complex*  $X$  is a pair  $(V_X, \mathcal{S}_X)$  where  $V_X$  denotes the vertex set of  $X$  and  $\mathcal{S}_X$  is the set of *finite, non-empty* subsets of  $V_X$  satisfying

1.  $\forall v \in V_X$ , then  $\{v\} \in \mathcal{S}_X$
2. If  $\sigma \in \mathcal{S}_X$ ,  $\tau \subset \sigma$ ,  $\tau \neq \emptyset$ , then  $\tau \in \mathcal{S}_X$ .

$\mathcal{S}_X$  is called the set of *simplices* of  $X$ .

**Example.** A *standard 1-simplex*, denoted by  $\Delta^1$  is simply the line segment (or usually denoted by  $I$ ).

$$V_{\Delta^1} = \{0, 1\}$$

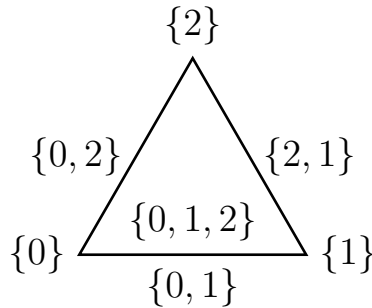
$$\mathcal{S}_{\Delta^1} = \{\{0\}, \{1\}, \{0, 1\}\}$$

$$\{0\} \xrightarrow{\{0, 1\}} \{1\}$$

A *standard 2-simplex*, denoted by  $\Delta^2$  is the equilateral triangle.

$$V_{\Delta^2} = \{0, 1, 2\}$$

$$\mathcal{S}_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$



In general, the *standard  $n$ -simplex*  $\Delta^n$ , is  $\Delta^n = (V_{\Delta^n}, \mathcal{S}_{\Delta^n})$  where

$$V_{\Delta^n} = \{0, 1, \dots, n\}$$

$$\mathcal{S}_{\Delta^n} = \{\alpha : \alpha \subset \{0, \dots, n\}, \alpha \neq \emptyset\}$$

If  $X = (V_x, \mathcal{S}_X)$  is a simplicial complex, we now want to pick a field  $\mathbb{F}$ , usually  $\mathbb{Q}$  or  $\mathbb{F}_2$  (in this course) and want to produce a sequence of vector spaces (over  $\mathbb{F}$ )

$$C_n(X)_{0 \leq n}$$

$C_0(X)$  is the vector space whose basis elements are simply the vertices of the simplicial complex, the 0-dimensional bits.

**Definition** ( $k$ -simplex of a simplicial complex). If  $X$  is a simplicial complex then a  $k$ -simplex of  $X$  is a simplex  $\sigma \in \mathcal{S}_X$  such that  $|\sigma| = k + 1$ .

$C_k(X)$  is the vector space whose basis elements are the *oriented  $k$ -simplices* of  $X$  which are the following symbols,

$$[v_0, v_1, \dots, v_n]$$

(where  $\{v_0, \dots, v_n\}$  is an  $n$ -simplex of  $X$ ) subject to the rules

$$[v_{\rho(0)}, v_{\rho(1)}, \dots, v_{\rho(n)}] = \text{sign}(\rho)[v_0, \dots, v_n]$$

**Definition.**

$$\partial_n : C_n(X) \rightarrow C_{n-1}(X)$$

is a linear map defined on basis elements as follows;

$$\partial_n[v_0, \dots, v_n] = \sum_{r=0}^n (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n]$$

where  $\hat{v}_r$  indicates omission of  $v_r$ .

**Example.**

$$\begin{aligned}\partial_2[0, 1, 2] &= [1, 2] - [0, 2] + [0, 1] \\ \partial_1[v_0, v_2] &= [v_1] - [v_0]\end{aligned}$$

$$\begin{aligned}\partial_1\partial_2[0, 1, 2] &= \partial_1([1, 2] - [0, 2] + [0, 1]) \\ &= ([2] - [1]) - ([2] - [0]) + ([1] - [0]) \\ &= 0\end{aligned}$$

**Proposition** (Poincaré lemma). Let  $X$  be a simplicial complex. Consider

$$\partial_r : C_r(X) \rightarrow C_{r-1}(X)$$

for  $r \geq 1$ , then

$$\partial_{n-1}\partial_n \equiv 0$$

*Proof.*

$$\partial_n[v_0, \dots, v_n] = \sum_{r=0}^n (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n]$$

$$\begin{aligned}\partial_{n-1}[v_0, \dots, \hat{v}_r, \dots, v_n] &= \sum_{s < r} (-1)^s [v_0, \dots, \hat{v}_s, \dots, \hat{v}_r, \dots, v_n] \\ &\quad + \sum_{s > r} (-1)^{s-1} [v_0, \dots, \hat{v}_r, \dots, \hat{v}_s, \dots, v_n]\end{aligned}$$

$$\begin{aligned}\partial_{n-1}\partial_n[v_0, \dots, v_n] &= \sum_{s < r} (-1)^{r+s} [v_0, \dots, \hat{v}_s, \dots, \hat{v}_r, \dots, v_n] \\ &\quad + \sum_{s > r} (-1)^{r+s-1} [v_0, \dots, \hat{v}_r, \dots, \hat{v}_s, \dots, v_n] \\ &= 0\end{aligned}$$

□

**Proposition.** If

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

then

$$\text{im}(\partial_{n+1}) \subset \ker(\partial_n)$$

*Proof.* By previous lemma. □

## 2 Homology

### 2.1 Quotient spaces

Let  $V$  be a vector space over a field  $\mathbb{F}$ , and  $U \subset V$  a vector subspace.

**Definition.** The following set

$$x + U = \{x + u : u \in U\}$$

is called the (left) coset of  $U$  in  $V$ . Note that

$$x + U = x' + U \iff x - x' \in U$$

**Definition** (Quotient space). The quotient space  $V/U$  is the set

$$V/U = \{x + U : x \in V\}$$

where addition and scalar multiplication is defined by

$$(x + U) + (y + U) = x + y + U$$

$$\lambda \cdot (x + U) = \lambda x + U$$

and 0 is represented by

$$0 + U$$

Note that  $V/U$  is a vector space.

**Proposition.**

$$\dim(V/U) = \dim(V) - \dim(U)$$

*Proof.* There exists a natural linear map

$$\eta : V \rightarrow V/U$$

given by

$$\eta(x) = x + U$$

Clearly this map is surjective so

$$\dim(V/U) = \dim(\text{im}(\eta))$$

Now,

$$\begin{aligned} \ker(\eta) &= \{x \in V : \eta(x) = U\} \\ &= \{x \in V : x + U = U\} \end{aligned}$$

and

$$x + U = U \iff x - 0 \in U \iff x \in U$$

so  $\ker(\eta) = U$ . Then,

$$\dim(V) = \dim \ker(\eta) + \dim \text{im}(\eta)$$

so

$$\dim(V/U) = \dim \text{im}(\eta) = \dim(V) - \dim(U)$$

□

**Definition.**

$$H_n(X; \mathbb{F}) = \ker(\partial_n) / \text{im}(\partial_{n+1})$$

We call  $H_n(X; \mathbb{F})$  the  $n^{\text{th}}$  *homology group* of  $X$  with coefficients in  $\mathbb{F}$ . If  $\mathbb{F} = \mathbb{Q}$ , then  $\dim H_n(X; \mathbb{Q})$  is called the  $n^{\text{th}}$  *Betti* number of  $X$ .

Consider  $\Delta^3$ . The set  $\{0, 1, 2, 3\}$  represents the 'middle' of the tetrahedron (inside, interior). If we exclude the middle and simply take its boundary, we have

$$\partial \Delta^3 = S^2$$

It happens that  $S^2$  (middle excluded) is the simplest simplicial model of the 2-sphere.

**Example.** Consider

$$H_k(S^2; \mathbb{F})$$

Note that

$$C_n(S^2) = 0 \text{ for } n \geq 3$$

as there are no 3-simplices, so we only have to worry about

$$H_2(S^2; \mathbb{F}), H_1(S^2; \mathbb{F}), H_0(S^2; \mathbb{F})$$

We proceed to calculate these from first principles. First note that  $C_3(S^2) = 0$ . Now, (noting the order of these bases)  $C_2(S^2)$  has basis

$$[0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]$$

$C_1(S^2)$  has basis

$$[0, 1], [0, 2], [0, 3], [1, 2], [1, 3], [2, 3]$$

and lastly  $C_0(S^2)$  has basis

$$[0], [1], [2], [3]$$

The linear maps

$$\partial_2 : C_2(S^2) \rightarrow C_1(S^2)$$

$$\partial_1 : C_1(S^2) \rightarrow C_0(S^2)$$

can both be represented by a  $6 \times 4$  matrix and a  $4 \times 6$  matrix respectively.

We apply  $\partial_2$  and  $\partial_1$  to the bases to obtain the entries to the matrices, so for example

$$\partial_2([0, 1, 2]) = [1, 2] - [0, 2] + [0, 1]$$

so the first column of the matrix representing  $\partial_2$  is  $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$  Proceeding,



we will obtain that

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Notice that  $\partial_1 \partial_2 = 0$ , which further confirms the lemma from before. Now reducing both the matrices to row reduced echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

thus  $\dim \ker \partial_2 = 1$ ,  $\dim \operatorname{im} \partial_2 = 3$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus  $\dim \ker \partial_1 = 3$ ,  $\dim \operatorname{im} \partial_1 = 3$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0$$

so now

$$H_2(S^2) = \ker(\partial_2) / \operatorname{im}(\partial_3) = \ker(\partial_2) \cong \mathbb{F}$$

as  $\text{im}(\partial_3) = 0$ , so in total,

$$H_2(S^2; \mathbb{F}) \cong \mathbb{F}$$

Next,

$$H_1(S^2) = \ker(\partial_1)/\text{im}(\partial_2)$$

Now note that

$$\dim H_1(S^2) = \dim \ker(\partial_1) - \dim \text{im}(\partial_2) = 3 - 3 = 0$$

thus

$$H_1(S^2; \mathbb{F}) = 0$$

Next,

$$H_0(S^2) = \ker(\partial_0)/\text{im}(\partial_1) = C_0/\text{im}(\partial_1)$$

and

$$\dim H_0(S^2) = \dim C_0 - \dim \text{im}(\partial_1) = 4 - 3 = 1$$

thus

$$H_0(S^2; \mathbb{F}) \cong \mathbb{F}$$

We've shown

$$H_k(S^2; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

We will soon see that this theorem generalises if

$$S^n = \partial(\Delta^{n+1})$$

then

$$H_k(S^n) = \begin{cases} \mathbb{F} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

## 2.2 Chain complex

**Definition** (Chain complex). Let  $\mathbb{F}$  be a field. A *chain complex* over  $\mathbb{F}$  is

$$C_* = (C_r, \partial_r)_{r \in \mathbb{N}}$$

where

1. Each  $C_r$  is a vector space over  $\mathbb{F}$
2.  $\partial_r : C_r \rightarrow C_{r-1}$  is a linear map such that  $\partial_r \partial_{r+1} = 0$  for all  $r$ .

If  $X = (V_X, \mathcal{S}_X)$ , we have defined a chain complex

$$C_*(X) = (C_r(X), \partial_r)$$

Given a chain complex

$$C_* = (C_r, \partial_r)_{r \geq 0}$$

we define its *homology*  $H_*(C_*)$  by

$$H_k(C_*) = \ker(\partial_k) / \text{im}(\partial_{k+1})$$

If  $X = (V_X, \mathcal{S}_X)$  is a simplicial complex, we define

$$H_k(X; \mathbb{F}) = H_k(C_*(X; \mathbb{F}))$$

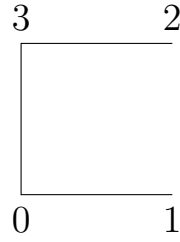
## 2.3 Simplicial mapping

**Definition** (Simplicial mapping). Let  $X, Y$  be simplicial complexes, i.e.,  $X = (V_X, \mathcal{S}_X)$  and  $Y = (V_Y, \mathcal{S}_Y)$ . A *simplicial mapping*  $f : X \rightarrow Y$  is a mapping of vertex sets  $f : V_X \rightarrow V_Y$  such that

$$\sigma \in \mathcal{S}_X \implies f(\sigma) \in \mathcal{S}_Y$$

**Example.** Let  $X = Y = \Delta^2$ . Then defining  $f$  by  $f(0) = 1, f(1) = 2, f(2) = 0$ , it is obvious that this mapping is simplicial.

Consider the following simplicial complex



and consider

$$f(0) = 1, f(1) = 2, f(2) = 3, f(3) = 0$$

This mapping is *not* simplicial as  $f(\{0, 1\})$  is *not* a simplex.

Given a simplicial mapping  $f : X \rightarrow Y$ , we are going to produce linear maps

$$H_k(f) : H_k(X) \rightarrow H_k(Y)$$

such that if

$$g : Y \rightarrow Z$$

then

$$g \circ f : X \rightarrow Z$$

and

1.  $H_k(g \circ f) = H_k(g) \circ H_k(f)$
2.  $H_k(\text{id}_X) = \text{id}_{H_k(X)}$

**Remark.** (Look up on functors for a more general treatment of the above concept.)

## 2.4 Chain mapping

**Definition.** Let

$$C_* = (C_r, \partial_r^C)$$

$$D_* = (D_r, \partial_r^D)$$

be chain complexes. A *chain mapping*  $f_* : C_* \rightarrow D_*$  is a collection of linear maps

$$f_* = (f_r)_{r \geq 0}$$

where  $f_r : C_r \rightarrow D_r$  and the following commutes

$$\begin{array}{ccc} C_r & \xrightarrow{\partial_r^C} & C_{r-1} \\ f_r \downarrow & & \downarrow f_{r-1} \\ D_r & \xrightarrow{\partial_r^D} & D_{r-1} \end{array}$$

i.e.,

$$\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C$$

If  $g : D_* \rightarrow E_*$  is also a chain mapping, then

$$(g \circ f)_n = g_n \circ f_n : C_* \rightarrow E_*$$

is also a chain mapping.

$$\text{id} : C_* \rightarrow C_*, \quad \text{id}_n = \text{id}_{C_n}$$

is also a chain mapping.

**Proposition.** If  $f : X \rightarrow Y$  is a simplicial mapping, define

$$C_n(f) : C_n(X) \rightarrow C_n(Y)$$

by action on a basis as follows

$$C_n(f)[v_0, \dots, v_n] = [f(v_0), \dots, f(v_n)]$$

then

$$C_*(f) : C_*(X) \rightarrow C_*(Y)$$

is also a chain mapping.

*Proof.*

$$\begin{aligned}
\partial_n^D C_n(f)[v_0, \dots, v_n] &= \partial_n^D([f(v_0), \dots, f(v_n)]) \\
&= \sum_{r=0}^n (-1)^r [f(v_0), \dots, f(\hat{v}_r), \dots, f(v_n)] \\
&= C_{n-1}(f) \sum_{r=0}^n (-1)^r [v_0, \dots, \hat{v}_r, \dots, v_n] \\
&= C_{n-1}(f) \partial_n^C[v_0, \dots, v_n]
\end{aligned}$$

□

We will often write  $f_n[v_0, \dots, v_n]$  rather than  $C_n(f)[v_0, \dots, v_n]$ .

**Proposition.** If  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  are simplicial maps, then

$$C_n(g \circ f) = C_n(g) \circ C_n(f)$$

which sometimes we will write as

$$(g \circ f)_n = g_n \circ f_n$$

instead.

*Proof.*

$$\begin{aligned}
(g \circ f)[v_0, \dots, v_n] &= [(g \circ f)(v_0), \dots, (g \circ f)(v_n)] \\
&= g_n[f(v_0), \dots, f(v_n)] \\
&= g_n \circ f_n[v_0, \dots, v_n]
\end{aligned}$$

□

**Proposition.** Let

$$\text{id} : X \rightarrow X$$

then  $C_*(\text{id}) : C_*(X) \rightarrow C_*(X)$  is a chain mapping.

If  $C_* = (C_n, \partial_n)$  is a chain complex, define

$$H_n(C_*) = \ker \partial_n / \text{im}(\partial_{n+1})$$

It is usual to write

$$Z_n(C) = \ker(\partial_n) \quad (\text{cycles})$$

$$B_n(C) = \text{im}(\partial_{n+1}) \quad (\text{boundaries})$$

thus by this notation,

$$H_n(C) = Z_n(C) / B_n(C)$$

If  $f = (f_n)$ ,  $C_* \rightarrow D_*$  is a chain mapping, we now want to show  $f$  induces a mapping

$$H_n(f) : H_n(C_*) \rightarrow H_n(D_*)$$

**Proposition.** If  $f : C_* \rightarrow D_*$  is a chain mapping, then

$$f_n(Z_n(C_*)) \subset Z_n(D_*)$$

*Proof.* Recall that

$$f_{n-1}\partial_n^C(z) = \partial_n^D f_n(z)$$

If

$$z \in Z_n(C_*), \partial_n^C(z) = 0$$

then we have

$$f_{n-1}\partial_n^C(z) = 0$$

and so

$$\partial_n^D f_n(z) = 0$$

and thus

$$f_n(z) \in Z_n(D_*)$$

□

**Proposition.** If  $f : C_* \rightarrow D_*$  is a chain mapping, then

$$f_n(B_n(C_*)) \subset B_n(D_*)$$

*Proof.* Note that

$$f_n \partial_{n+1}^C(x) = \partial_{n+1}^D f_{n+1}(x)$$

If  $\beta \in B_n(C_*)$ , we can write  $\beta = \partial_{n+1}^C(x)$  for some  $x$  and then

$$f_n(\beta) = \partial_{n+1}^D(k)$$

where  $k = f_{n+1}(x)$  so

$$f_n(\beta) \in B_n(D_*)$$

□

**Corollary.** If  $f : C_* \rightarrow D_*$  is a chain mapping, then  $f$  induces a (linear) mapping

$$H_n(f) : H_n(C_*) \rightarrow H_n(D_*)$$

*Proof.* An element of  $H_n(C_*)$  has form

$$[z] = z + B_n(C_*), \quad z \in Z_n(C_*)$$

Now define

$$H_n(f)[z] = f_n(z) + B_n(D_*) \in H_n(D_*)$$

and now note that

$$f_n(z) \in Z_n(D_*)$$

□

By now it is clear if  $g : D_* \rightarrow E_*$ ,  $f : C_* \rightarrow D_*$  are chain mappings, then

$$H_n(g \circ f) = H_n(g) \circ H_n(f)$$

and also if  $\text{id} : C_* \rightarrow C_*$  we have

$$H_n(\text{id}) = \text{id}_{H_n}$$

We now formally have

$$H_n(X) = H_n(C_*(X))$$



**Corollary.** If  $X$  is a *non-empty* simplicial complex, then  $H_0(X; \mathbb{F}) \neq 0$  (for any field  $\mathbb{F}$ ).

*Proof.* As  $X \neq \emptyset$ , we have that  $V_X \neq \emptyset$ . Let  $v \in V_X$  be a vertex and  $*$  be the simplicial complex

$$* = (\{v\}, \{\{v\}\})$$

so  $*$  consists of one vertex  $v$ , and one 0-simplex  $\{v\}$ . Now define a constant simplicial mapping

$$c : X \rightarrow *, c(x) = v, \forall x \in V_X$$

We also have a simplicial mapping

$$\iota : * \rightarrow X, \iota(v) = v$$

so now

$$c \circ \iota = \text{id}_*$$

and so (since both maps are simplicial, hence chain mappings)

$$H_0(c) \circ H_0(\iota) = H_0(\text{id}_*)$$

but notice that

$$H_0(*) = \mathbb{F}$$

since we know

$$C_0(*) = \mathbb{F}, C_r(*) = 0, r \geq 1$$

and thus

$$H_0(c) \circ H_0(\iota) = \text{id}_{\mathbb{F}}$$

$$c \circ \iota = \text{id} \neq 0$$

and now note that  $c$  is surjective, and  $\iota$  is injective. In particular

$$H_0(c) : H_0(X) \rightarrow \mathbb{F} = H_0(*)$$

is surjective, so

$$H_0(X) \neq 0$$

□

So we now know that  $H_0(X) \neq 0$  if  $X \neq \emptyset$ .

**Definition.** Let  $X$  be a simplicial complex. If  $v, w \in V_X$ , then by a path from  $v$  to  $w$ , we mean a sequence of 1-simplices

$$[v_0, v_1], [v_1, v_2], \dots, [v_{n-2}, v_{n-1}], [v_{n-1}, v_n]$$

such that  $v_0 = v$  and  $v_n = w$ .

**Proposition.** If  $X$  is non-empty and connected, then

$$H_0(X; \mathbb{F}) \cong \mathbb{F}$$

*Proof.*

$$C_1(X) \xrightarrow{\partial_1} C_0(X)$$

If  $v, w \in V_X$ , then  $[w] - [v] \in \text{im}(\partial_1)$ . To see this, choose a path

$$v = v_0 < v_1 < \dots < v_{n-1} < v_n = w$$

i.e.,  $[v_{i+1}, v_i]$  is a 1-simplex for  $0 \leq i \leq n-1$ .

$$\partial_1[v_{i+1}, v_i] = [v_{i+1}] - [v_i] \in \text{im}(\partial_1)$$

so then,

$$[w] - [v] = \sum_{i=0}^{n-1} [v_{i+1}, v_i] \in \text{im}(\partial_1)$$

Now  $\{[v] : v \in V_X\}$  is a basis for  $C_0$ . Choose a specific  $v \in V_X$ . By elementary basis change,

$$\{[v]\} \cup \{[w] - [v] : w \in V_X, w \neq v\}$$

is a basis for  $C_0$ . However  $[w] - [v] \in \text{im}(\partial_1)$  ( $w \neq v$ ). So  $C_0(X)/\text{im}(\partial_1)$  has dimension  $\leq 1$ , and then  $\dim H_0(X) \leq 1$  if  $X$  is connected. But  $X \neq \emptyset$ , so  $H_0(X) \neq 0$ , hence  $\dim H_0(X) = 1$ , hence

$$H_0(X) \cong \mathbb{F}$$

when  $X$  is connected. □

**Proposition.** In general,  $\dim H_0(X)$  is equal to the number of connected components in  $X$

If  $X$  is a simplicial complex, then define a relation  $\sim$  on  $V_X$  by  $v \sim w$  if and only if there exists a path from  $v$  to  $w$ .

$\sim$  defines an equivalence relation, where the number of connected components is equal to the number of equivalence classes.

If  $X$  consists of a single point,

$$H_k(\text{pt.}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

## 2.5 Cone

**Definition.** Let  $X$  be a simplicial complex. A *cone* on  $X$ ,  $C(X)$ , is defined as follows, choose  $*$  (cone point) such that  $*$   $\notin V_X$

$$V_{C(X)} = \{*\} \cup V_X$$

$$\mathcal{S}_{C(X)} = \mathcal{S}_X \cup \{\{*\} \cup \{\sigma \cup \{*\} : \sigma \in S_X\}\}$$

i.e., join everything in  $X$  to the cone point.

**Theorem.** If  $X$  is a simplicial complex, then,

$$H_k(C(X); \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

i.e.,  $C(X)$  behaves just like a point (homologically).

*Proof.* First note that  $C(X)$  is connected. Take  $v, w \in V_{C(X)}$ ,  $v \neq w$ . Either one of them is the cone point, or none of them are the cone point.

(1) Without loss of generality, suppose  $w$  is the cone point. ( $w = *$ ). By definition,  $[v, w] = [v, *]$  is a 1-simplex of  $C(X)$ . So we've joined  $v$  to  $w$ .

(2) If neither are the cone point, then,  $[v, *]$  and  $[*, w]$  are both 1-simplices, so again, we've joined  $v$  to  $w$ . So

$$H_0(C(X); \mathbb{F}) \cong \mathbb{F}$$

Now we must show

$$H_k(C(X)) = 0, k > 0$$

We define, for each  $k > 0$ , a linear map

$$\mathcal{H}_k : C_k(C(X)) \rightarrow C_{k+1}(C(X))$$

(called a contracting homotopy)  $\mathcal{H}_k$  is defined on a basis by

$$\mathcal{H}_k[v_0, \dots, v_k] = [* , v_0, \dots, v_k]$$

Then,

$$\begin{aligned} \partial_{k+1} \mathcal{H}_k[v_0, \dots, v_k] &= \partial_{k+1}[* , v_0, \dots, v_k] \\ &= [v_0, \dots, v_k] + \sum_{r=0}^k (-1)^{r+1} [* , v_0, \dots, \hat{v}_r, \dots, v_k] \end{aligned}$$

$$\partial_{k+1} \mathcal{H}_k[v_0, \dots, v_k] + \sum_{r=0}^k (-1)^r [* , v_0, \dots, \hat{v}_r, \dots, v_k] = [v_0, \dots, v_k]$$

However,

$$\mathcal{H}_{k-1}[v_0, \dots, \hat{v}_r, \dots, v_k] = [* , v_0, \dots, \hat{v}_r, \dots, v_k]$$

and

$$(\partial_{k+1} \mathcal{H}_k + \mathcal{H}_{k-1} \partial_k)[v_0, \dots, v_k] = [v_0, \dots, v_k]$$

i.e.,

$$\partial_{k+1} \mathcal{H}_k + \mathcal{H}_{k-1} \partial_k = \text{id}$$

(we call the above a homotopy relation)

$$H_k(C(X)) = Z_k(C(X))/B_k(C(X))$$

and if  $z \in Z_k(C(X))$ ,  $\partial_k(z) = 0$ , so if  $z \in Z_k(C(X))$ ,  $z = \partial_{k+1}\mathcal{H}_k(z)$  so  $z \in \text{im}(\partial_{k+1})$ , i.e.,  $Z_k(C(X)) \subset B_k(C(X)) (\subset Z_k(X))$  so if  $C(X)$  is a cone and  $k > 0$ ,

$$Z_k(C(X)) = B_k(C(X))$$

and  $H_k(C(X); \mathbb{F}) = 0$  □

**Corollary.**

$$H_k(\Delta^n; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

where  $\Delta^n = n$ -simplex

*Proof.*  $\Delta^n$  is a cone.  $\Delta^n = (C(\Delta^{n-1}))$  □

Let  $X$  be a simplicial complex,  $n \geq 0$ . Then the  $n$ -skeleton  $X^{(n)}$  of  $X$  is defined by

$$V_{X^{(n)}} = V_X$$

$$\mathcal{S}_{X^{(n)}} = \{\sigma \in \mathcal{S}_X : |\sigma| \leq n + 1\}$$

i.e.,  $\dim(\sigma) \leq n$ .

The standard model  $S^n$  of the  $n$ -sphere is

$$V_{S^n} = \{0, \dots, n + 1\}$$

$$\mathcal{S}_{S^n} = \{\sigma \subset \{0, \dots, n + 1\} | \sigma \neq \emptyset, |\sigma| \leq n + 1\}$$

i.e.,  $S^n = n$ -skeleton of  $\Delta^{n+1}$

**Theorem.**

$$H_k(X^{(n)}) \equiv H_k(X), \text{ for } 0 \leq k \leq n - 1$$

(and there exists a natural surjection  $H_n(X^{(n)}) \rightarrow H_n(X)$ ) (note this is not an isomorphism)

*Proof.* From definition,  $C_k(X^{(n)}) \equiv C_k(X)$ ,  $0 \leq k \leq n$

$$C_*(X^{(n)}) \quad 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$C_*(X) \quad C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$H_k(X^{(n)}) \equiv H_k(X) \text{ for } k \leq n-1$$

$$\begin{aligned} H_n(X^{(n)}) &\equiv \ker(\partial_n : C_n(X) \rightarrow C_{n-1}(X)) \\ &= Z_n(X) \end{aligned}$$

but  $B_n(X^{(n)}) = 0$ . In general  $B_n(X) \neq 0$ . □

As  $S^n = (\Delta^{n+1})^{(n)}$ , ( $n \neq 0, n \geq 1$ ) we see that

$$H_k(S^n; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & 1 \leq k \leq n-1 \end{cases}$$

We now still need to compute  $H_n(S^n)$ .

## 2.6 Exact sequences

**Definition.** Let  $U \xrightarrow{f} V \xrightarrow{g} W$  be linear maps. We say sequence is *exact* at  $V$  when

$$\ker(g) = \text{im}(f)$$

In general if

$$V_{n+1} \xrightarrow{f_{n+1}} V_n \rightarrow \dots \rightarrow V_{r+1} \xrightarrow{f_{r+1}} V_r \xrightarrow{f_r} V_{r-1} \rightarrow \dots \rightarrow V_1 \xrightarrow{f_1} V_0$$

is a sequence of linear maps, we say a sequence is *exact* at  $V_r$  when

$$\ker f_r = \text{im} f_{r+1}$$

We say the sequence is *exact* when it is *exact* at each possible  $V_r$ .

#### 4 term exact sequence

$$0 \rightarrow U \xrightarrow{f} V \rightarrow 0$$

is exact if and only if  $f$  is an isomorphism.

*Proof.* The sequence is exact at  $V$ , so

$$\text{im}(f) = \ker(V \rightarrow 0) = V$$

so  $f$  is surjective. The sequence is exact at  $U$ , so

$$\ker(f) = \text{im}(0 \rightarrow U) = 0$$

so  $f$  is injective. Thus  $f$  is bijective and so an isomorphism.  $\square$

#### Short exact sequence

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

Exactness here means

1.  $g$  is surjective,  $\text{im}(g) = \ker(W \rightarrow 0)$
2.  $f$  is injective,  $\ker(f) = \text{im}(0 \rightarrow V) = 0$
3.  $\ker(g) = \text{im}(f)$

#### **Example.** Kernel-rank theorem

Suppose we have the exact sequence

$$0 \rightarrow U \xrightarrow{f} V \xrightarrow{g} W \rightarrow 0$$

if  $U, V, W$  are finite dimensional, then

$$\dim(V) = \dim(U) + \dim(W)$$

by the kernel-rank theorem. To see this, note that

$$\text{im}(g) = W$$

by exactness.

$$\dim \ker(g) + \dim \operatorname{im}(g) = \dim(V) \implies \dim \ker(g) + \dim(W) = \dim(V)$$

$$\ker(g) = \operatorname{im}(f) \cong U$$

(since  $f$  is injective) and so

$$\dim \ker(g) = \dim(U)$$

so

$$\dim(U) + \dim(W) = \dim(V)$$

**Example.**

$$\begin{aligned} H_k(X) &= Z_k(X)/B_k(X) \\ 0 \rightarrow B_k(X) &\hookrightarrow Z_k(X) \rightarrow H_k(X) \rightarrow 0 \end{aligned}$$

is a short exact sequence,  $z \mapsto [z]$ ,  $z + B_k(X)$ , so

$$\dim H_k(X) = \dim Z_k(X) - \dim B_k(X)$$

Exact sequences of chain complexes Let  $A_*, B_*, C_*$  be chain complexes and

$$f : A_* \rightarrow B_*, g : B_* \rightarrow C_*$$

Consider the following sequence of chain maps

$$0 \rightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \rightarrow 0$$

so for each  $n$  we have a sequence of linear maps

$$0 \rightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \rightarrow 0$$

We say that this is exact when for each  $n$ , this sequence is exact.



### 3 Mayer-Vietoris Theorem

#### 3.1 Algebraic Mayer-Vietoris Theorem

**Theorem** (Algebraic Mayer-Vietoris Theorem). Suppose

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

is an exact sequence of chain complexes, then there exists a long exact sequence of the following type

$$\rightarrow H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B) \dots$$

$$\rightarrow H_1(A) \xrightarrow{i_*} H_1(B) \xrightarrow{p_*} H_1(C) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(B) \xrightarrow{p_*} H_0(C) \rightarrow 0$$

with  $\delta$  called the connecting homomorphism, where in our case,  $A_n = B_n = C_n = 0$  for  $n < 0$ , i.e.,

$$A_* = (A_n, \partial_n), A_n = 0, n < 0$$

$$B_* = (B_n, \partial_n), B_n = 0, n < 0$$

$$C_* = (C_n, \partial_n), C_n = 0, n < 0$$

The connecting homomorphisms have the following *naturality property*:

Suppose we have the following exact sequences of chain complexes,

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

$$0 \rightarrow A'_* \xrightarrow{i} B'_* \xrightarrow{p} C'_* \rightarrow 0$$

and suppose the following commutes,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* \longrightarrow 0 \end{array}$$

(where  $\alpha, \beta, \gamma$ ) are chain maps). Compare the two long exact sequences,

$$\begin{array}{ccccccccc}
H_{n+1}(B) & \xrightarrow{p_*} & H_{n+1}(C) & \xrightarrow{\delta} & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{p_*} & H_n(0) \\
\downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\
H_{n+1}(B') & \xrightarrow{q_*} & H_{n+1}(C') & \xrightarrow{\delta'} & H_n(A') & \xrightarrow{j_*} & H_n(B') & \xrightarrow{q_*} & H_n(0)
\end{array}$$

this diagram commutes.

The Algebraic Mayer-Vietoris Theorem implies the *Geometric* Mayer-Vietoris Theorem.

### 3.2 Subcomplexes

Let  $X = (V_X, \mathcal{S}_X)$ ,  $Y = (V_Y, \mathcal{S}_Y)$  be simplicial complexes. Then we say that  $Y$  is a *subcomplex* of  $X$  if,

1.  $V_Y \subset V_X$
2.  $\mathcal{S}_Y \subset \mathcal{S}_X$

**Proposition.**

1. Let  $X_1, X_2$  be subcomplexes of  $Z$ . Then  $(V_{X_1} \cup V_{X_2}, \mathcal{S}_{X_1} \cup \mathcal{S}_{X_2})$  is also a subcomplex of  $Z$ . This is called the union  $X_1 \cup X_2$ .
2.  $(V_{X_1} \cap V_{X_2}, \mathcal{S}_{X_1} \cap \mathcal{S}_{X_2})$  is also a subcomplex of  $Z$ . This is called the intersection  $X_1 \cap X_2$ .

We are interested in the case  $Z = X_1 \cup X_2$ .

**Definition.** Let  $\Delta, \Delta'$  be chain complexes.  $\Delta = (\Delta_n, \partial_n)$ ,  $\Delta' = (\Delta'_n, \partial'_n)$ . Then the *direct sum*:

$$\begin{aligned}
\Delta \oplus \Delta' &= \left( \Delta \oplus \Delta', \begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix} \right) \\
\begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix} \begin{pmatrix} \partial_{n+1} & 0 \\ 0 & \partial'_{n+1} \end{pmatrix} &= \begin{pmatrix} \partial_n \partial_{n+1} & 0 \\ 0 & \partial'_n \partial'_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

### 3.3 The Geometric Mayer-Vietoris Theorem: Chain Version

**Theorem.** Suppose  $X$  is a simplicial complex decomposed as a union  $X = X_+ \cup X_-$ , where  $X_+, X_-$  are subcomplexes. Then there exists an exact sequence of chain complexes as follows,

$$0 \rightarrow C_*(X_+ \cap X_-) \xrightarrow{i} C_*(X_+ \oplus X_-) \xrightarrow{p} C_*(X) \rightarrow 0$$

If we apply the algebraic Mayer-Vietoris Theorem, we get the homological version, namely the long exact sequence,

$$\begin{aligned} H_{n+1}(X_+) \oplus H_{n+1}(X_-) &\rightarrow H_{n+1}(X) \xrightarrow{\delta} H_n(X_+ \cap X_-) \\ &\rightarrow H_n(X_+) \oplus H_n(X_-) \rightarrow H_n(X) \xrightarrow{\delta} H_{n-1}(X_+ \cap X_-) \end{aligned}$$

and finishes

$$\begin{aligned} \xrightarrow{\delta} H_1(X_+ \cap X_-) &\rightarrow H_1(X_+) \oplus H_1(X_-) \rightarrow H_1(X) \xrightarrow{\delta} H_0(X_+ \cap X_-) \\ &\rightarrow H_0(X_+) \oplus H_0(X_-) \rightarrow H_0(X) \rightarrow 0 \end{aligned}$$

**Example.** Let  $S^n$  = standard model of  $n$ -sphere,

$$S^n = (\Delta^{n+1})^{(n)}$$

We've shown for  $n \geq 1$ ,

$$H_r(S^n; \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & 0 < r < n \\ ? & r = n \\ 0 & n < r \end{cases}$$

We've shown that  $H_2(S^2; \mathbb{F}) = \mathbb{F}$ .

**Proposition.** For  $n \geq 2$ ,  $S^n$  can be written as  $S^n = X_+ \cup X_-$  where  $X_+ \cap X_- = S^{n-1}$  and  $X_+$ ,  $X_-$  are *cones*.

$$\Delta^{n+1} = (\{0, 1, \dots, n+1\}, \{\text{all non-empty subsets of } \{0, 1, \dots, n+1\}\})$$

$$S^n = (\{0, 1, \dots, n+1\}, \{\text{all proper non-empty subsets of } \{0, 1, \dots, n+1\}\})$$

In particular every non-empty subset of  $\{0, 1, \dots, n\}$  is a simplex of  $S^n$  so,

1.  $\Delta^n \subset S^n$ . But as  $S^{n-1} \subset \Delta^n$ , then,
2.  $S^{n-1} \subset S^n$  (note that  $n+1 \notin V_{S^{n-1}}$ ) and,
3. Taking  $n+1$  to be the cone point  $C(S^{n-1}) \subset S^n$ . ( $C(S^{n-1})$  is sometimes called the *Witches hat*)
- 4.

$$\begin{aligned} S^n &= \Delta^n \cup C(S^{n-1}) \\ S^{n-1} &= \Delta^n \cap C(S^{n-1}) \end{aligned}$$

So we can write,

$$S^n = X_+ \cup X_-, \text{ where}$$

$$X_+ = C(S^{n-1})$$

$$X_- = \Delta^n$$

$$X_+ \cap X_- = S^{n-1}$$

**Corollary.**  $H_n(S^n; \mathbb{F}) \cong \mathbb{F}$  for all  $n \geq 2$ .

*Proof.* By induction on  $n$ . We know this is true for  $n = 2$ . Suppose we've proven the hypothesis for  $n-1$  and consider the exact sequence,

$$H_n(X_+) \oplus H_n(X_-) \longrightarrow H_n(S^n) \xrightarrow{\delta} H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(X_+) \oplus H_{n-1}(X_-)$$

$$0 \oplus 0 \longrightarrow H_n(S^n) \xrightarrow{\cong} H_{n-1}(S^{n-1}) \longrightarrow 0 \oplus 0$$

which is isomorphic by the very short exact sequence.  $\square$

### 3.4 External and internal sum

Let  $W$  be a vector space over  $\mathbb{F}$  and suppose we have two vector subspaces of  $W$ , say  $U$  and  $V$ .

**Definition.** External sum (coproduct)

$$U \oplus V = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in U, v \in V \right\}$$

$U \oplus V$  is a vector space. We define sums, scalar multiplication and zero as follows,

$$\begin{aligned} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} &= \begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix} \\ \lambda \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= 0 \end{aligned}$$

If  $U$  and  $V$  have finite dimensions, then

$$\dim(U \oplus V) = \dim(U) + \dim(V)$$

where  $U, V$  are subspaces of  $W$ .

**Definition.** Internal sum

$$U + V = \{u + v : u \in U, v \in V\}$$

Note that  $U + V$  is a vector subspace of  $W$ .

What is the relationship between  $U + V$  and  $U \oplus V$ ? There is an exact sequence

$$\begin{aligned} \rightarrow U \oplus V &\xrightarrow{\mu} U + V \\ \mu \begin{pmatrix} u \\ v \end{pmatrix} &= u + v \end{aligned}$$

$\mu$  is linear and surjective by the definition of  $U + V$ .

**Proposition.**

$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff u + v = 0 \iff v = -u, u \in U, v \in V \text{ so } v \in U \cap V$$

We get an exact sequence,

$$0 \rightarrow U \cap V \xrightarrow{i} U \oplus V \xrightarrow{\mu} U + V \rightarrow 0$$

$$i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

As a consequence,

$$\dim(U \cap V) + \dim(U + V) = \dim(U) + \dim(V)$$

**Theorem.** (Chain version of the Geometric Mayer-Vietoris Theorem)  
Let  $X = X_+ \cup X_-$  be the union of subcomplexes. For each  $n$ , there exists an exact sequence,

$$0 \rightarrow C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \rightarrow 0$$

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} = x + y, i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

*Proof.*  $C_n(X)$  has basis  $\{[v_0, v_1, \dots, v_n] : [v_0, \dots, v_n] \in \mathcal{S}_X\}$

$$\mathcal{S}_X = \mathcal{S}_{X_+} \cup \mathcal{S}_{X_-}$$

$$C_n(X_+) \oplus C_n(X_-) \rightarrow C_n(X) \rightarrow 0$$

$$\begin{pmatrix} e \\ f \end{pmatrix} \mapsto e + f$$

The map is surjective because a basis element of  $C_n(X)$  is either in  $C_n(X_+)$  or  $C_n(X_-)$ . As a basis for  $\ker(\mu)$ , we have

$$\begin{pmatrix} [v_0, \dots, v_n] \\ -[v_0, \dots, v_n] \end{pmatrix}$$

where  $\{v_0, \dots, v_n\} \subset \mathcal{S}_{X_+} \cap \mathcal{S}_{X_-} = \mathcal{S}_{X_+ \cap X_-}$  so we have an exact sequence,

$$0 \rightarrow C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \rightarrow 0$$

This is an exact sequence of chain complexes because boundary formula is the same in every case.  $\square$

**Corollary.** (of the Geometric Mayer-Vietoris Theorem) Let  $X$  be a finite simplicial complex. Then,

$$\dim H_0(X; \mathbb{F}) = \{\text{number of connected components of } X\}$$

*Proof.* Let  $n$  be the number of connected components. This is true for  $n = 1$ . Suppose this is true for  $n - 1$ , and  $X$  has  $n$  connected components  $X_1, X_2, \dots, X_n$ . Put

$$X_- = X_1 \cup X_2 \cup \dots \cup X_{n-1}$$

$$X_+ = X_n$$

$$X_+ \cup X_- = X, X_+ \cap X_- = \emptyset \text{ (by definition)}$$

Look at the following

$$H_0(X_+ \cap X_-) \rightarrow H_0(X_+) \oplus H_0(X_-) \rightarrow H_0(X) \rightarrow 0$$

(note that  $H_0(X_+ \cap X_-) = 0$ ). So

$$\dim H_0(X) = \dim H_0(X_+) + \dim H_0(X_-) = 1 + n - 1 = n$$

$\square$

**Example.**

$$S^0 = 0\text{-sphere} = 2 \text{ distinct points } \{-1, +1\}$$

So  $H_0(S^0; \mathbb{F}) \cong \mathbb{F} \oplus \mathbb{F}$

$$H_n(S^0; \mathbb{F}) = 0, n \neq 0 \text{ (no higher simplices)}$$

On the other hand, the standard model of  $S^1$  is,

$$V_{S^1} = \{0, 1, 2\}$$

$$\mathcal{S}_{S^1} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$$

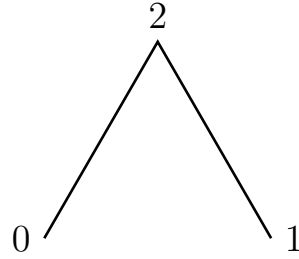
**Proposition.**

$$H_n(S^1; \mathbb{F}) = \begin{cases} \mathbb{F} & n = 0 \\ \mathbb{F} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

*Proof.* Decompose  $S^1 = X_1 \cup X_+$ , where  $X_-$  is equal to

$$0 \text{ ————— } 1$$

and  $X_+$  is equal to



i.e.,

$$X_- = C(0), \quad X_+ = \text{cone on } S^0 = \{\{0\}, \{1\}\}$$

$X_+ \cap X_- = S^0$ . Use the Mayer-Vietoris Theorem, so,

$$H_1(X_+) \oplus H_1(X_-) \rightarrow H_1(S^1) \rightarrow H_0(S^0) \rightarrow H_0(X_+) \oplus H_0(X_-) \rightarrow H_0(S^1)$$

$$0 \rightarrow H_1(S) \rightarrow \mathbb{F} \oplus \mathbb{F} \rightarrow \mathbb{F} \oplus \mathbb{F} \rightarrow \mathbb{F}$$

$\dim(H_1(S^1)) = 1$  follows from Whitehead's lemma. □

**Lemma.** Let

$$0 \rightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \rightarrow V_1 \xrightarrow{f_1} V_0 \rightarrow 0$$

be an exact sequence of finite dimensional vector spaces. Then,

$$\sum_{n \geq 0} \dim(V_{2n}) = \sum_{n \geq 0} \dim(V_{2n+1})$$



*Proof.* Let  $P(n)$  denote the induction hypothesis on  $n$ .

$$0 \rightarrow V_1 \rightarrow V_0 \rightarrow 0$$

then  $P(1)$  holds. The sequence is exact which implies  $V_1 \cong V_0$ . Now suppose we have an exact sequence,

$$0 \rightarrow V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \rightarrow 0$$

then by the kernel-rank theorem, this implies that

$$\dim(V_0) + \dim(V_2) = \dim(V_1)$$

and so  $P(2)$  is true. For  $n = 3$ ,

$$0 \rightarrow V_3 \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \rightarrow 0$$

is an exact sequence. Put  $K = \ker(f_1) = \text{im}(f_2)$  so we have two exact sequences

$$0 \rightarrow K \rightarrow V_1 \rightarrow V_0 \rightarrow 0$$

$$0 \rightarrow V_3 \rightarrow V_2 \rightarrow K \rightarrow 0$$

so by the kernel-rank theorem,

$$\dim V_0 + \dim V_2 = \dim V_1 + \dim V_3$$

Now we prove that  $P(2n) \implies P(2n + 1)$ . Suppose that  $P(2n)$  is true, and take the following exact sequence,

$$0 \rightarrow V_{2n+1} \xrightarrow{f_{2n+1}} V_{2n} \xrightarrow{f_{2n}} V_{2n-1} \rightarrow \dots \rightarrow V_0 \rightarrow 0$$

Split the sequence and define  $f = \text{im}(f_{2n}) = \ker(f_{2n-1})$ . Now we have two exact sequences,

$$0 \rightarrow V_{2n+1} \rightarrow V_{2n} \rightarrow f \rightarrow 0$$

and

$$0 \rightarrow f \rightarrow V_{2n-1} \rightarrow \dots \rightarrow V_0 \rightarrow 0$$

By  $P(2n)$ ,

$$\dim(f) + \sum_{r=0}^{n-1} \dim(V_{2r}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

and  $\dim(f) = \dim(V_{2n}) - \dim(V_{2n+1})$ . Substitute this into the previous expression and we get,

$$\sum_{r=0}^n \dim(V_{2r}) - \dim(V_{2n+1}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

This proves that  $P(2n) \implies P(2n+1)$ . To prove that  $P(2n+1) \implies P(2n+2)$ , take

$$0 \rightarrow V_{2n+2} \rightarrow V_{2n+1} \rightarrow V_{2n} \rightarrow \dots$$

Split the exact sequence as before and proceed as before. (Set  $f = \text{im}(f_{2n+1}) = \ker(f_{2n})$ )  $\square$

**Lemma.** (Five lemma) Suppose we have a commutative diagram of abelian groups and homomorphisms,

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 \\ \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\ B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 \end{array}$$

in which both rows are exact, and  $f_0, f_1, f_3, f_4$  are isomorphisms. Then  $f_2$  is also an isomorphism.

*Proof.* We first show that  $f_2$  is injective. Suppose  $x \in A_2$  such that  $f_2(x) = 0$ . We want to show that  $x = 0$ .

$$\beta_2 f_2(x) = 0 \implies f_3 \alpha_2(x) = 0$$

but  $f_3$  is an isomorphism, which implies that  $\alpha_2(x) = 0$ . But then  $x \in \ker(\alpha_2) = \text{im}(\alpha_1)$ , so  $x = \alpha_1(y)$  for some  $y \in A_1$ .

$$f_2 \alpha_1(y) = 0 \implies \beta_1 f_1(y) = 0$$

so  $f_1(y) \in \ker(\beta_1) = \text{im}(\beta_0)$ . Thus there exists  $w \in \beta_0$  such that  $\alpha_0(w) = f_1(y)$ . But  $f_0$  is surjective so write

$$w = f_0(z), \beta_0 f_0(z) = f_1(y) \implies f_1 \alpha_0(z) = f_1(y)$$

but now  $f_1$  is an isomorphism so  $y = \alpha_0(z)$ ,  $x = \alpha_1(y) = \alpha_1 \alpha_0(z)$ . By exactness,  $\alpha_1 \alpha_0 = 0$ , so  $x = 0$

Now we show that  $f_2$  is surjective. Take  $b \in \beta_2$ . We want to find  $a \in A_2$  such that  $f_2(a) = b$ . Now,  $\beta_2(b) \in B_3$ .  $f_2$  is an isomorphism so choose  $x \in A_3$  so that

$$f_3(x) = \beta_2(b) \implies \beta_3 f_3(x) = \beta_3 \beta_2(b)$$

However by exactness,  $\beta_3 \beta_2 = 0$ , so  $\beta_3 f_3(x) = 0 \implies f_4 \alpha_3(x) = 0$ . Now  $f_4$  is an isomorphism thus  $\alpha_3(x) = 0$ ,  $x \in \ker(\alpha_3) = \ker(\alpha_2)$ . Now there exists  $y \in A_2$  such that  $\alpha_2(y) = x$ . Consider  $b - f_2(y)$ . Then

$$\beta_2(b - f_2(y)) = \beta_2(b) - \beta_2 f_2(y) = \beta_2(b) - f_3 \alpha_2(y) = \beta_2(b) - f_3(x) = 0$$

Thus  $b - f_2(y) \in \ker(\beta_2) = \ker(\beta_1)$  so there exists  $w \in \beta_1$  such that  $\beta_1(w) = b - f_2(y)$ .  $f_1$  is an isomorphism implies that there exists  $z \in A_1$  such that  $f_1(z) = w$ . So

$$\beta_1 f_1(z) = \beta_1(w) = b - f_2(y)$$

$$f_2 \alpha_1(z) = b - f_2(y) \implies b = f_2(y + \alpha_1(z))$$

Let  $a = y + \alpha_1(z)$  which implies  $b = f_2(a)$ . Thus  $f_2$  is surjective.  $\square$

## 4 Subdivision

We will now show that homology is invariant under 'subdivision'. We first have to illustrate what 'subdivision' means.

Take for example  $\Delta^2$  (the triangle), and add a point at its barycenter, adding edges from the barycenter to each three of the vertices

of  $\Delta^2$ . We end up with an additional point (vertex), two additional regions and three additional edges. This is an example of an easy subdivision.

**Definition.** Let  $X = (V_X, \mathcal{S}_X)$  be a finite simplicial complex, and let  $\tau \in \mathcal{S}_X$ .  $\hat{\tau}$  will denote the subcomplex of  $X$  determined by  $\tau$ .

$$V_{\hat{\tau}} = \tau, \quad \mathcal{S}_{\hat{\tau}} = \{p \in \mathcal{S}_X, p \subset \tau\}$$

We say that  $\sigma \in \mathcal{S}_X$  is *principal* (or maximal) when  $\sigma$  is not contained properly in any other simplex.

**Proposition.** If  $\sigma_1, \dots, \sigma_N$  are the principal simplices of  $X$  then

$$X = \hat{\sigma}_1 \cup \hat{\sigma}_2 \cup \dots \cup \hat{\sigma}_N$$

## 4.1 Subdivision at a principal simplex

Let  $\sigma$  be a principal simplex of  $X$  and let  $\sigma_1, \dots, \sigma_N$  be the remaining principal simplices such that

$$X = \hat{\sigma} \cup \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N$$

Put  $X_+ = \hat{\sigma}$ ,  $X_- = \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N$ . Then  $X = X_+ \cup X_-$  and  $X_+ \cap X_- \subset \partial \hat{\sigma}$  (boundary of  $\hat{\sigma}$ )

**Definition.**

$$Sd(X, \sigma) = C(\partial \sigma) \cup \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N$$

i.e.,

$$Sd(X, \sigma) = X'_+ \cup X'_-$$

where  $X'_+$  is the cone on the boundary of  $\sigma$  and

$$X'_- = X_- = \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N$$

and

$$X'_+ \cap X'_- = X_+ \cap X_-$$

Taking our  $\Delta^2$  example earlier, letting  $\sigma = \Delta^2$ ,  $Sd(\Delta^2, \sigma)$  is exactly the resulting simplex we get by performing our subdivision earlier.

## 4.2 Squash mapping

Let  $\sigma$  be an  $n$ -simplex and consider  $C(\partial\sigma)$ . We construct simplicial mappings  $C(\partial\sigma) \rightarrow \sigma$  as follows,

$$Sq|_{\partial\sigma} = \text{id}_{\partial\sigma}$$

$$Sq(*) = \text{some (arbitrarily chosen) vertex in } \partial\sigma$$

where  $*$  is our cone point.

**Proposition.**  $Sq : H_k(C(\partial\sigma)) \rightarrow H_k(\sigma)$  is an isomorphism for all  $k$ .

*Proof.*  $C(\partial\sigma)$  and  $\sigma$  are both cones, so  $H_k(C(\partial\sigma)) = H_k(\sigma) = 0$  if  $k > 0$ . For  $k = 0$ , any vertex  $V$  in  $C(\partial\sigma)$  gives a basis  $[v]$  for  $H_0(C(\partial\sigma))$  (any two vertices differ by a boundary). Likewise, any vertex  $w$  in  $\sigma$  gives basis element  $[w]$  in  $H_0(\sigma)$  and  $Sq([v]) = [w]$ , so now

$$Sq : H_0(C(\partial\sigma)) \xrightarrow{\cong} H_0(\sigma)$$

□

**Theorem.** Let  $K$  be a finite complex. Let  $\sigma$  be a principal complex, and let  $\sigma_1, \dots, \sigma_N$  be the remaining principal simplices and define an extended squash map  $Sq : Sd(X, \sigma) \rightarrow X$  by

$$Sq : C(\partial\sigma) \rightarrow \sigma \text{ is a squash mapping}$$

$$Sq : \sigma_i \rightarrow \sigma_i \text{ identity } i = 1, \dots, N$$

Then  $Sq : H_k(Sd(X, \sigma)) \rightarrow H_k(X)$  is an isomorphism for all  $k$ .

*Proof.* Put

$$X_+ = \hat{\sigma}, X'_+ = C(\partial\sigma)$$

$$X'_- = X_- = \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N$$

so  $X'_+ \cap X'_- = X_+ \cap X_-$  and  $Sq : X'_- \rightarrow X_-$  is the identity. Consider the Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
H_n(X'_+ \cap X'_-) & \longrightarrow & H_n(X'_+) \oplus H_n(X'_-) & \longrightarrow & H_n(Sd(X, \sigma)) & \longrightarrow & H_{n-1}(X'_+ \cap X'_-) & \longrightarrow & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \\
\downarrow \text{id} & & \downarrow M & & \downarrow Sq & & \downarrow \text{id} & & \downarrow M \\
H_n(X_+ \cap X_-) & \longrightarrow & H_n(X_+) \oplus H_n(X_-) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(X_+ \cap X_-) & \longrightarrow & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{array}$$

where  $M = \begin{pmatrix} Sq & 0 \\ 0 & \text{id} \end{pmatrix}$ .  $\text{id}$  is clearly an isomorphism, as well as  $M$ ,

since  $Sq : H_n(X'_+) \rightarrow H_n(X_+)$  is an isomorphism. By the five lemma,  $Sq$  is an isomorphism.  $\square$

We have now shown that if  $Sd(X, \sigma)$  is the subdivision of  $X$  at a principal simplex, then  $H_*(Sd(X, \sigma)) \cong H_*(X)$ . Now we have to show that this also holds for non-principal simplices.

### 4.3 Subdivision at a non-principal simplex

We first describe an example of a non-principal simplex. Take  $\Delta^2$ . Then take  $\{0, 1\}$ . This is contained within  $\{0, 1, 2\}$ , hence this is a non-principal simplex. We wish to perform subdivisions at simplices such as these.

**Definition** (Join). Let  $K = (V_K, \mathcal{S}_K)$  and  $L = (V_L, \mathcal{S}_L)$  be simplicial complexes such that  $V_K \cap V_L = \emptyset$ . Define

$$K * L = (V_K \cup V_L, \mathcal{S}_K \cup \mathcal{S}_L \cup \{p \cup \tau, p \in \mathcal{S}_K, \tau \in \mathcal{S}_L\})$$

A special case is where  $K = \text{point}$ , so then  $K * L = C(L)$ .

**Proposition.**

$$\Delta^{m+n+1} \cong \Delta^m * \Delta^n$$

*Proof.* Vertex set of  $\Delta^{m+n+1}$  is

$$\{0, \dots, m+n+1\} = \{0, \dots, m\} \cup \{m+1, \dots, m+n+1\}$$

There is a 1-1 correspondence between the last set and

$$\{0, \dots, n\}$$

so if we take as our model of  $\Delta^n$  the vertex set  $\{m+1, \dots, m+n+1\}$  and simplices to be all the non-empty subsets, we get  $\Delta^{m+n+1} = \Delta^m * \Delta^n$  (the dimension goes up by 1).  $\square$

Note also that  $S^m * S^n \cong S^{m+n+1}$ . If  $k$  is a single point  $pt$ , then  $C(L) = \{pt\} * L$ .

Join is associative. If  $K, L, M$  are simplicial complexes with no vertices in common, then

$$(K * L) * M \equiv K * (L * M)$$

**Corollary.** If  $K, L$  are disjoint complexes, then  $C(K) * L \cong C(K * L)$

So the join of a cone to anything is a cone.

## 4.4 Star and Link

**Definition** (Star neighbourhood). Let  $\tau$  be a simplex of  $X$ , and let  $\sigma_1, \dots, \sigma_N$  be the principal simplices which contain  $\tau$ . Then

$$St(\tau, X) = \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_N = \text{star neighbourhood of } \tau \text{ in } X$$

**Definition** (Link). Let  $X$  be a simplicial complex and  $\rho, \tau$  be simplices of  $X$  such that  $\rho \cap \tau = \emptyset$ . We say that  $\rho$  is joinable to  $\tau$  in  $X$  when  $\rho \cup \tau = p * \tau$ . The *link* of  $\tau$  in  $X$ ,  $Lk(\tau, X)$  consists of all these simplices of  $\rho$  of  $X$  such that  $\rho \cap \tau = \emptyset$  and  $\rho \cup \tau$  is a simplex, i.e.,  $\rho$  is joinable to  $\tau$ .

$$Lk(\tau, X) = \{\rho \in X \mid \rho \cap \tau = \emptyset, \rho \cup \tau \in \mathcal{S}_X\}$$

**Proposition.** If  $\tau$  is a simplex of  $X$ , then  $St(\tau, X) = \hat{\tau} * Lk(\tau, X)$

*Proof.* The case where  $\tau$  is principal is empty here. So suppose  $\tau$  is not principal. Let  $\sigma$  be a principal simplex with  $\tau \subset \sigma$ . Write

$$\tau = \{v_0, \dots, v_m\} \quad m < n$$

$$\sigma = \{v_0, \dots, v_m, v_{m+1}, \dots, v_n\}$$

Put  $\rho = \{v_{m+1}, \dots, v_n\}$  so then

$$\sigma = \tau * \rho$$

Do this for every principal simplex which contains  $\tau$ . Each  $\sigma_i = \tau * \rho_i$  for some  $\rho_i$ , so

$$\bigcup \sigma_i = \tau * (\bigcup \rho_i) = \tau * Lk(\tau, X)$$

□

**Definition** (Subdivision at a non-principal simplex). Let  $X$  be a finite simplicial complex, and  $\tau$  a non-principal simplex. Let  $\sigma_1, \dots, \sigma_m$  be the principal simplices which contain  $\tau$ . Let  $\sigma_{m+1}, \dots, \sigma_N$  be the remaining principal simplices. Put

$$X_+ = \hat{\sigma}_1 \cup \dots \cup \hat{\sigma}_m = St(\tau, X)$$

$$X_- = \hat{\sigma}_{m+1} \cup \dots \cup \hat{\sigma}_N$$

$$X = X_+ + X_- \quad (X_+ \cap X_- \cap \tau \subset \partial\sigma)$$

and put

$$X'_+ = C(\partial\tau) * Lk(\tau, X)$$

$$X'_- = X_-$$

Define

$$Sd(X, \tau) = X'_+ \cup X'_-$$

$$Sd = (C(\partial\tau) * Lk) \cup X'_-$$

We have  $Sq : C(\partial\tau) \rightarrow \tau$ . Extend by identity to  $Sq : C(\partial\tau) * Lk \rightarrow \tau * Lk$  by identity on  $Lk$ . Extend again by identity on  $X'_- = X_-$ ,  $Sq : Sd(X, \tau) \rightarrow X$

**Proposition.**  $Sq : Sd(X, \tau) \rightarrow X$  induces an isomorphism on homology.



*Proof.*

$$\begin{array}{ccccccccc}
H_n(X'_+ \cap X'_-) & \longrightarrow & H_n(X'_+) \oplus H_n(X'_-) & \longrightarrow & H_n(Sd(X, \tau)) & \longrightarrow & H_{n-1}(X'_+ \cap X'_-) & \longrightarrow & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \\
\downarrow \text{id} & & \downarrow M & & \downarrow Sq & & \downarrow \text{id} & & \downarrow M \\
H_n(X_+ \cap X_-) & \longrightarrow & H_n(X_+) \oplus H_n(X_-) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(X_+ \cap X_-) & \longrightarrow & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{array}$$

where  $M = \begin{pmatrix} Sq & 0 \\ 0 & \text{id} \end{pmatrix}$ . By the five lemma,  $Sq$  induces an isomorphism.

□

So now we've proved the following,

**Theorem.** Homology is invariant under subdivision.

We now have a functor  $H_n$  which takes simplicial complexes to vector spaces, and simplicial maps to linear maps, e.g., if

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
& \searrow & & \nearrow & \\
& & g \circ f & & 
\end{array}$$

then

$$\begin{array}{ccccc}
H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) & \xrightarrow{H_n(g)} & H_n(Z) \\
& \searrow & & \nearrow & \\
& & H_n(g \circ f) & & 
\end{array}$$

Properties of functors:

1.  $H_n(g \circ f) = H_n(g) \circ H_n(f)$
2.  $H_n(\text{id}) = \text{id}_{H_n}$  i.e.,

$$\text{id} : X \rightarrow X, H_n(\text{id}) : H_n(X) \rightarrow H_n(X)$$

As a consequence, if  $f : X \rightarrow Y$  is an isomorphism, then  $H_n(f) : H_n(X) \rightarrow H_n(Y)$  is an isomorphism.

*Proof.* If  $g = f^{-1} : Y \rightarrow X$ ,  $g \circ f = \text{id}_X$ ,  $f \circ g = \text{id}_Y$  then

$$H_n(g) \circ H_n(f) = \text{id}, H_n(f) \circ H_n(g) = \text{id}$$

so

$$H_n(g) = H_n(f)^{-1}$$

□

But we have established a stronger property, that is,  $H_n$  is invariant under subdivision, i.e., if  $Y$  subdivides  $X$ , then  $H_n(Y) \cong H_n(X)$ .

**Definition.** Let  $X, Y$  be simplicial complexes. We say that  $X, Y$  are *combinatorially equivalent* (written  $X \sim Y$ ) if and only if there exists a finite sequence  $(X_r)_{0 \leq r \leq N}$  of complexes  $X_r$  such that  $X_0 = X$ ,  $X_N = Y$  and for each  $r$ ,  $0 \leq r \leq N - 1$ , either  $X_{r+1}$  is a subdivision of  $X_r$  or  $X_r$  is a subdivision of  $X_{r+1}$ .

**Corollary.** If  $X \sim Y$  then  $H_n(X) \cong H_n(Y)$ .

So we won't worry too much about how we triangulate things.

Consider  $S^2 = \partial\Delta^3$ . This is the minimal model of  $S^2$ . The dodecahedron is also a model of  $S^2$  obtained from the minimal model by a sequence of subdivisions, hence for any model of  $S^2$ ,

$$H_k(S^2; \mathbb{F}) \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k > 2 \end{cases}$$

We note that the usual definition of  $S^2$  is given by

$$S^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

which is harder to compute homology with.

Now we define  $S^1(n)$  to be the model of the circle  $S^1$  with  $n$ -subdivision points ( $n \geq 3$ ), so

$$S^1(n) \sim S^1(m) \quad \forall m, n \geq 3$$

so for example,  $S^1(3)$  is the triangle,  $S^1(4)$  is the square,  $S^1(5)$  the pentagon, and so on.

## 5 Orientation Theorem

**Definition** (Orientability). We say that a surface  $\Sigma$  is orientable if and only if it is possible to orient each 2-simplex in such a way that every 1-simplex receives the opposite orientations from its containing 2-simplices.

### 5.1 Euler characteristic

**Definition.** Let  $X = (V_X, \mathcal{S}_X)$  be a finite simplicial complex. Let  $c_n$  be the number of  $n$ -simplices of  $X$ ,

$$c_n(X) = c_n = \text{no. of } n\text{-simplices of } X$$

we define

$$\chi_{\text{geom}}(X) = \sum_n (-1)^n c_n(X)$$

This is known as the *geometric* Euler characteristic.

Put  $h_n^{\mathbb{F}}(X) = \dim H_n(X; \mathbb{F}) (= h_n)$ , and define

$$\chi_{\text{hom}}^{\mathbb{F}}(X) = \sum_n (-1)^n h_n^{\mathbb{F}}(X)$$

This is known as the *homological* Euler characteristic.

We will show that

**Theorem.**

$$\chi_{\text{hom}}^{\mathbb{F}}(X) = \chi_{\text{geom}}(X)$$

In particular  $\chi_{\text{hom}}^{\mathbb{F}}$  is independent of  $\mathbb{F}$ , so we'll ignore  $\mathbb{F}$ .

*Proof.* Fix a field  $\mathbb{F}$ .  $c_n = c_n(X) = \text{no. of } n\text{-simplices of } X$

$$c_n = \dim C_n(X; \mathbb{F})$$

so put  $h_n = \dim H_n(X; \mathbb{F})$  and look at the sequence

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

$$H_n(X) = \ker \partial_n / \text{im } \partial_{n+1} = Z_n(X) / B_n(X)$$

Put  $z_n = \dim \ker \partial_n$ ,  $b_n = \dim \text{im}(\partial_{n+1})$  so

$$z_n = h_n + b_n$$

However by the kernel-rank theorem,

$$c_n = z_n + b_{n-1}$$

hence

$$c_n = h_n + b_n + b_{n-1}$$

Now take the alternating sum

$$\sum_n (-1)^n c_n = \sum_n (-1)^n h_n + \sum_n (-1)^n (b_n + b_{n-1})$$

The last term on the RHS evaluates to 0, and recognising what the other two sums are, we have

$$\chi_{\text{hom}}^{\mathbb{F}}(X) = \chi_{\text{geom}}(X)$$

□

As  $H_*(X; \mathbb{F})$  is invariant under subdivision, it follows that  $\chi_{\text{geom}}(X)$  is the same as well, so from now on, we will usually just write  $\chi(X)$ .

**Definition** (Connected sum of surfaces). Let  $\Sigma, \Sigma'$  be surfaces. Let  $\sigma$  be a 2-simplex in  $\Sigma$ ,  $\sigma'$  be a 2-simplex in  $\Sigma'$ . Let  $\Sigma_0$  be the complex obtained from  $\Sigma$  by removing  $\sigma$ . Likewise for  $\Sigma'_0$ . Formally,

$$\Sigma \# \Sigma' = \Sigma_0 \bigcup_{\partial=\partial'} \Sigma'_0$$

(i.e., we glue the boundaries of  $\Sigma_0$  and  $\Sigma'_0$  together.)

**Proposition.**

$$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$$

**Example.** Some examples of orientable surfaces, are the following,

$$\Sigma_+^0 = S^2$$

$$\Sigma_+^1 = T^2$$

$$\Sigma_+^2 = T^2 \# T^2$$

$$\Sigma_+^g = T^2 \# \dots \# T^2 \text{ (g times)}$$

These are orientable surfaces of genus  $g$ .

**Proposition.**  $\chi(\Sigma_+^g) = 2 - 2g$

*Proof.* We proceed by induction on  $g$ . This is clearly true for  $g = 0$  ( $\chi(S^2) = 2$ ) and  $g = 1$  ( $\chi(T^2) = 0$ ). Suppose this is true for some  $g \geq 1$ , then,

$$\Sigma_+^{g+1} = \Sigma_+^g \# \Sigma_+^1$$

$$\begin{aligned} \chi(\Sigma_+^{g+1}) &= \chi(\Sigma_+^g) + \chi(\Sigma_+^1) - 2 \\ &= 2 - 2g + 0 - 2 \\ &= 2 - 2(g + 1) \end{aligned}$$

□

**Corollary.** Over any field  $\mathbb{F}$ ,

$$H_k(\Sigma_+^g; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^{2g} & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k > 2 \end{cases}$$

**Example.** The following are examples of non-orientable surfaces,

$$\Sigma_-^0 = \mathbb{R}P(2)$$

$$\Sigma_-^1 = \mathbb{R}P(2) \# \mathbb{R}P(2) \ (\cong) \text{ Klein bottle}$$

$$\Sigma_-^g = \mathbb{R}P(2) \# \dots \# \mathbb{R}P(2) \text{ (g+1 times)}$$

**Proposition.**  $\chi(\Sigma_-^g) = 1 - g$

*Proof.* We proceed by induction.

$$\Sigma_-^0 = \mathbb{R}P(2), \chi(\Sigma_-^0) = 1$$

hence is true for  $g = 0$ . Suppose this is true for  $g \geq 0$ , then,

$$\Sigma_-^{g+1} = \Sigma_-^g \# \mathbb{R}P(2)$$

and so on. Then,

$$\begin{aligned} \chi(\Sigma_-^{g+1}) &= \chi(\Sigma_-^g) + \chi(\mathbb{R}P(2)) - 2 \\ &= (1 - g) + 1 - 2 \\ &= 1 - (g + 1) \end{aligned}$$

□

**Proposition.** If  $1 + 1 \neq 0$  in  $\mathbb{F}$ , then,

$$H_k(\Sigma_-^g; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^g & k = 1 \\ 0 & k \geq 2 \end{cases}$$

*Proof.*  $H_2(\Sigma_-^g; \mathbb{F}) = 0$  by the Orientation theorem. We know

$$H_0(\Sigma_-^g; \mathbb{F}) = \mathbb{F} \text{ (connected)}$$

Then,

$$\chi_{\text{hom}}(\Sigma_-^g) = h_0^{\mathbb{F}} - h_1^{\mathbb{F}} + h_2^{\mathbb{F}}$$

$$1 - g = 1 - h_1^{\mathbb{F}} + 0$$

$$h_1^{\mathbb{F}} = g$$

hence

$$H_1(\Sigma_-^g; \mathbb{F}) = \mathbb{F}^g$$

□

**Proposition.** If  $1 + 1 = 0$  in  $\mathbb{F}$ , then,

$$H_k(\Sigma_-^g; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^{g+1} & k = 1 \\ \mathbb{F} & k = 2 \end{cases}$$

*Proof.*  $\chi = 1 - g$  but now

$$h_0^{\mathbb{F}} = 1, h_2^{\mathbb{F}} = 1 \text{ (} 1 + 1 = 0 \text{)}$$

and thus

$$h_1^{\mathbb{F}} = g + 1$$

□

**Theorem** (Classification Theorem for Surfaces). Let  $\Sigma$  be a finite connected surface.

1. If  $\Sigma$  is orientable, then,

$$\Sigma \sim \Sigma_+^g \text{ for some } g \geq 0$$

2. If  $\Sigma$  is non-orientable, then,

$$\Sigma \sim \Sigma_-^g \text{ for some } g \geq 0$$

The homology groups of the surfaces distinguishes them, i.e.,

$$\Sigma_+^g \sim \Sigma_+^h \iff g = h$$

$$\Sigma_-^g \sim \Sigma_-^h \iff g = h$$

$$\Sigma_+^g \not\sim \Sigma_-^g$$

One non-trivial relation is

$$\mathbb{R}P(2) \# T^2 \sim \mathbb{R}P(2) \# \mathbb{R}P(2) \# \mathbb{R}P(2)$$

Recall that if  $\sigma$  is a simplex of  $X$ , then  $Lk(\sigma, X)$  is equal to the complex where simplices  $\tau$  satisfy  $\sigma \cap \tau = \emptyset$ , where  $\sigma \cup \tau$  is a simplex of  $X$ .

**Definition** (Simplicial surface). A *simplicial surface*  $\Sigma$  is a complex in which

$$Lk(v, \Sigma) \cong S^1(N)$$

where  $v$  is a vertex of  $\Sigma$  ( $N \geq 3$ ). Recall that  $S^1(N)$  is the circle with  $N$  subdivision points, for example,  $S^1(5)$  is 'the' pentagon.

Observe that in  $S^1(N)$ , every vertex belongs to exactly *two* 1-simplices.

**Proposition.** If  $\Sigma$  is a simplicial surface then every 1-simplex lies in exactly two 2-simplices.

*Proof.* Let  $\rho = [v_0, v_1]$  be a 1-simplex.  $Lk(v_0, \Sigma) \cong S^1(N)$ ,  $v_1 \in Lk(v_0, \Sigma)$  and  $v_1$  belongs to exactly two 1-simplices, say  $\tau_0, \tau_1$ , so then

$$\tau_0 * \{v_0\}, \tau_1 * \{v_0\}$$

are the two 2-simplices which contain  $\rho$ . □



## 5.2 Copath

**Definition.** Let  $X$  be a simplicial complex of dimension 2. Let  $\sigma, \sigma'$  be 2-simplices in  $X$ . A *copath* from  $\sigma$  to  $\sigma'$  is a collection of 2-simplices

$$\{\sigma_0, \sigma_1, \dots, \sigma_N\}$$

such that  $\sigma_0 = \sigma$ ,  $\sigma_N = \sigma'$  and  $\sigma_r \cap \sigma_{r+1}$  is a 1-simplex for  $0 \leq r \leq N-1$ .

**Theorem.** If  $\Sigma$  is a connected simplicial surface and  $\sigma, \sigma'$ ,  $\sigma \neq \sigma'$  are 2-simplices in  $\Sigma$  then there exists a copath  $(\sigma_0, \dots, \sigma_N)$  from  $\sigma$  to  $\sigma'$ .

*Proof.* Consider  $\sigma \cap \sigma'$ . A priori we have 4 cases

1.  $|\sigma \cap \sigma'| = 3$ . This is impossible, as this implies  $\sigma = \sigma'$
2.  $|\sigma \cap \sigma'| = 2$ . Put  $\rho = \sigma \cap \sigma'$  which is a 1-simplex, where  $\rho$  lies in exactly two 2-simplices which are  $\sigma, \sigma'$  and now  $(\sigma, \sigma')$  is a copath from  $\sigma$  to  $\sigma'$
3.  $|\sigma \cap \sigma'| = 1$ .  $\sigma \cap \sigma' = \{v\}$  for some vertex  $v$ . Look at  $Lk(v, \Sigma)$ . Write

$$\sigma = \{v, u, u_0\}$$

$$\sigma' = \{v, w, w_0\}$$

We know  $v, w \in Lk(v, \Sigma) \cong S^1(N)$  which is connected. So choose a path in  $Lk(v, \Sigma)$  from  $u$  to  $w$

$$\xi = (\xi_0, \dots, \xi_N)$$

$$\xi_0 = u, \dots, \xi_N = w$$

Then  $[\xi_i, \xi_{i+1}]$  is a 1-simplex in  $Lk(v, \Sigma)$ . Define  $\sigma_i = \{v, \xi_i, \xi_{i+1}\}$  which is a 2-simplex, and then we have that  $\sigma_0, \dots, \sigma_N$  is a copath from  $\sigma$  to  $\sigma'$ .

4.  $\sigma \cap \sigma' = \emptyset$ . Let  $N$  be a shortest path from a vertex  $v$  of  $\sigma$  to a vertex  $v'$  of  $\sigma'$ . We proceed by induction on  $N$ . The induction base case here is when  $N = 1$ .  $(v, v')$  sits inside two 2-simplices.

Choose one of them and call it  $\tau$ .  $\sigma \cap \tau = \{v\}$  so there exists a copath from  $\sigma$  to  $\tau$ . Similarly,  $\sigma' \cap \tau = \{v'\}$  so there exists a copath from  $\tau$  to  $\sigma'$ . Compare the two copaths to get a copath from  $\sigma$  to  $\sigma'$ .

Now for our induction step (assume hypothesis proved for  $N-1$ ), let  $v \in \sigma$ ,  $v' \in \sigma'$ . Let  $(w_0, \dots, w_M)$  be a shortest path from  $v$  to  $v'$ . Let  $\tau$  be any 2-simplex such that  $w_{m-1} \in \tau$ . By our induction hypothesis, there exists a copath from  $\sigma$  to  $\tau$ . By the induction base, there exists a copath from  $\tau$  to  $\sigma'$ . Compose the two copaths to get a copath from  $\sigma$  to  $\sigma'$ .

□

### 5.3 Orientation Theorem

**Theorem** (Orientation Theorem). Let  $\Sigma$  be a connected simplicial surface, and let  $\mathbb{F}$  be a field.

1. If  $1 + 1 \neq 0$  in  $\mathbb{F}$  and  $\Sigma$  is orientable then

$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$

2. If  $1 + 1 \neq 0$  in  $\mathbb{F}$  and  $\Sigma$  is non-orientable then

$$H_2(\Sigma; \mathbb{F}) = 0$$

3. If  $1 + 1 = 0$  then

$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$

regardless if  $\Sigma$  is orientable or not.

**Definition** (Intersection number).

$$\langle [v_0, v_1, v_2], [v_0, v_1] \rangle = +1$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\langle [v_0, v_1, v_2], [v_0, v_2] \rangle = -1$$

$$\langle [v_0, v_1, v_2], [v_1, v_2] \rangle = +1$$

More generally,

$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_0, v_1] \rangle = \text{sgn}(\sigma)$$

$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_0, v_2] \rangle = -\text{sgn}(\sigma)$$

$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_1, v_2] \rangle = \text{sgn}(\sigma)$$

*Proof.* (of Orientation Theorem) For each 1-simplex  $\rho$  of  $\Sigma$ , fix once and for all a specific orientation  $\hat{\rho}$  of  $\rho$ .

Let  $\sigma_1, \dots, \sigma_N$  be a list of the 2-simplices of  $\Sigma$  and  $\hat{\sigma}_i$  fixed orientation of  $\sigma_i$ . To change the orientations on 2-simplices, we need a function

$$\eta : \{1, \dots, N\} \rightarrow \{\pm 1\}$$

$\eta(i)\hat{\sigma}_i$  is the oriented 2-simplex which is

$$\begin{cases} \hat{\sigma}_i & \eta(i) = 1 \\ \text{opposite orientation of } \hat{\sigma}_i & \eta(i) = -1 \end{cases}$$

We shall consider elements of  $C_2(\Sigma; \mathbb{F})$  of the form

$$[\eta] = \sum_{i=1}^N \eta(i)\hat{\sigma}_i \in C_2(\Sigma; \mathbb{F})$$

We want to calculate  $\partial[\eta]$ .

Fix a 1-simplex  $\rho$  and let  $\sigma_s, \sigma_t$  be the adjacent 2-simplices which contain  $\rho$ . The coefficient of  $\hat{\rho}$  in  $\partial[\eta]$  is simply

$$[\eta(s)\hat{\sigma}_s, \hat{\rho}] + [\eta(t)\hat{\sigma}_t, \hat{\rho}] = \begin{cases} 2 \\ 0 \\ -2 \end{cases}$$

To ensure that  $\partial[\eta] = 0$  we require  $\eta$  to satisfy

$$[\eta(s)\hat{\sigma}_s, \hat{\rho}] + [\eta(t)\hat{\sigma}_t, \hat{\rho}] = 0$$

i.e.,

$$\langle \eta(s)\hat{\sigma}_s, \hat{\rho} \rangle + \langle \eta(t)\hat{\sigma}_t, \hat{\rho} \rangle = 0 \quad (*)$$

whenever  $\sigma_s, \sigma_t$  are adjacent, and now we know  $\Sigma$  is orientable if and only if there exists a function

$$\eta : \{1, \dots, N\} \rightarrow \{\pm 1\}$$

such that we have  $(*)$  whenever  $\sigma_s, \sigma_t$  are adjacent.

So when  $\Sigma$  is orientable and  $\eta : \{1, \dots, N\} \rightarrow \{\pm 1\}$  is an orientation, then  $[\eta] \in Z_2(\Sigma)$  and so defines a non-zero element of  $H_2(\Sigma; \mathbb{F})$ .

Now we show that when  $\Sigma$  is orientable,

$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$

and  $[\eta]$  is a *generator*.

Let us consider the elements

$$\sum_{s=1}^N a_s \hat{\sigma}_s \in C_2(\Sigma; \mathbb{F})$$

where  $a_s \in \mathbb{F}$ . Suppose that  $\rho$  is a 1-simplex and  $\sigma_s, \sigma_t$  are adjacent 2-simplices which contain  $\rho$ .

We calculate  $\partial(\sum a_s \sigma_s)$ . The coefficients of  $\hat{\rho}$  is simply  $a_s \langle \hat{\sigma}_t, \hat{\rho} \rangle + a_t \langle \hat{\sigma}_s, \hat{\rho} \rangle$ . If we want  $\partial(\sum a_s \sigma_s) = 0$  then

$$a_s \langle \hat{\sigma}_t, \hat{\rho} \rangle + a_t \langle \hat{\sigma}_s, \hat{\rho} \rangle = 0$$

for adjacent  $s, t$ .

$$\pm a_s \pm a_t = 0$$

so  $a_t = \pm a_s$  if  $\sigma_s, \sigma_t$  are adjacent. So going along a copath, coefficients  $a_s$  are constant up to sign.

Fix a "base 2-simplex"  $\sigma_0$  and suppose

$$\partial(\sum a_s \hat{\sigma}_s) = 0$$

Then going along a copath from  $\sigma_0$  to  $\sigma_s$ , we find that  $a_s = \pm a_0$ . Define  $\eta : \{1, \dots, N\} \rightarrow \{\pm 1\}$  by

$$\eta(s) = \begin{cases} +1 & a_s = a_0 \\ -1 & a_s = -a_0 \end{cases}$$

then  $\alpha = a_0[\eta]$  if  $\partial\alpha = 0$  which shows that  $\dim H_2(\Sigma; \mathbb{F}) \leq 1$ .

If  $\Sigma$  is orientable, there exists global orientation  $\eta : \{1, \dots, N\} \rightarrow \{\pm 1\}$  and  $[\eta]$  generates  $H_2(\Sigma; \mathbb{F})$

If  $\Sigma$  is non-orientable,  $\partial[\eta] \neq 0$  for any such  $\eta$  and if  $\alpha \in Z_2(\Sigma; \mathbb{F})$ ,  $\alpha = \sum a_s \hat{\sigma}_s$ ,

$$\alpha = a_0[\eta] \text{ , } \partial(\alpha) = a_0\partial[\eta]$$

$$\partial(\alpha) = 0 \implies a_0\partial[\eta] = 0$$

but  $\partial[\eta] \neq 0$  and  $\alpha = 0$ , so  $a_0 = 0$ , so then

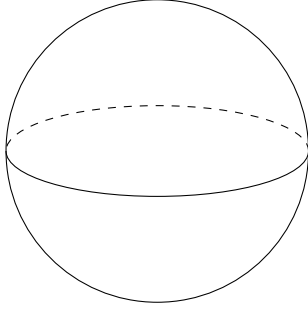
$$H_2(\Sigma; \mathbb{F}) = 0$$

However if  $1 + 1 = 0$ , then

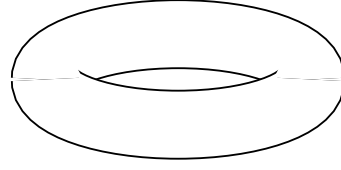
$$\partial(\sum_{s=1}^N \hat{\sigma}_s) = 0$$

as  $\pm 2 = 0$  and  $H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$ . □

For surfaces,  $H_0$  tells us whether the surface is connected or not.  $H_2$  tells us whether the surface is orientable or not.  $H_1$  in a sense tells us how 'big' the surface is.



$$\Sigma_0^+, S^2, \text{ genus} = 0$$



$$\Sigma_1^+, T^2, \text{ genus} = 1$$

$$H_k = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \end{cases}$$

$$H_k = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F} \oplus \mathbb{F} & k = 1 \\ \mathbb{F} & k = 2 \end{cases}$$

In general for  $\Sigma_g^+ = n$ -fold torus, we have

$$H_k = \begin{cases} \mathbb{F} & k = 0 \\ H_1 = \mathbb{F} \oplus \dots \oplus \mathbb{F} \text{ (} 2g \text{ times)} & k = 1 \\ \mathbb{F} & k = 2 \end{cases}$$

We also have

$$\Sigma_0^- = \mathbb{R}P(2), \Sigma_g^- = \mathbb{R}P(2) \# \dots \# \mathbb{R}P(2) \text{ (} g + 1 \text{ times)}$$

If  $1 + 1 \neq 0$ , then

$$H_k = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^g & k = 1 \\ 0 & k = 2 \end{cases}$$

but if  $1 + 1 = 0$ ,

$$H_k = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^{g+1} & k = 1 \\ \mathbb{F} & k = 2 \end{cases}$$

As  $H_*(-; \mathbb{F})$  is invariant under combinatorial equivalence ( $\sim$ ), we have

$$\begin{aligned}\Sigma_g^+ &\sim \Sigma_h^+ \implies g = h \\ \Sigma_g^- &\sim \Sigma_h^- \implies g = h \\ \Sigma_g^+ &\not\sim \Sigma_h^- \text{ for any } g, h\end{aligned}$$

## 6 Some linear algebra

**Proposition.** If  $A, B$  are  $n \times n$  matrices over  $\mathbb{F}$ , then

$$\text{Tr}(AB) = \text{Tr}(BA)$$

*Proof.* First write  $A = (a_{kj})$ ,  $B = (b_{ji})$ . Then,

$$\begin{aligned}(AB)_{ki} &= \sum_{j=1}^n a_{kj} b_{ji} \\ (AB)_{kk} &= \sum_j a_{kj} b_{jk}\end{aligned}$$

$$\begin{aligned}\text{Tr}(AB) &= \sum_{k=1}^n \sum_{j=1}^n a_{kj} b_{jk} \\ &= \sum_{j=1}^n \sum_{k=1}^n a_{kj} b_{jk}\end{aligned}$$

and we know

$$a_{kj} b_{jk} = b_{jk} a_{kj}$$

as  $\mathbb{F}$  is a field, so now we have

$$= \sum_{j=1}^n \sum_{k=1}^n b_{jk} a_{kj} = \text{Tr}(BA)$$

□

**Remark.** Note that in general  $\text{Tr}(AB) \neq \text{Tr}(A)\text{Tr}(B)$ . For example, take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$$

so that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\text{Tr}(A) = \text{Tr}(B) = 0, \quad \text{Tr}(AB) = 2$$

**Corollary.** If  $A, P$  are  $n \times n$  matrices over  $\mathbb{F}$  and  $P$  is invertible then

$$\text{Tr}(PAP^{-1}) = \text{Tr}(A)$$

*Proof.*

$$\text{Tr}((PA)P^{-1}) = \text{Tr}(P^{-1}PA) = \text{Tr}(A)$$

□

## 6.1 Trace of a linear map

Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$ . Let  $X : V \rightarrow V$  be a linear map. Take a basis  $\{e_1, \dots, e_n\}$  for  $V$ , and write

$$X(e_i) = \sum_{j=1}^n e_j \xi_{ji}$$

$$X \sim (\xi_{ji}) = \xi$$

We would like to define  $\text{Tr}(X) = \text{Tr}(\xi) = \sum_{j=1}^n \xi_{ji}$

However,  $\xi$  depends on the choice of basis  $\{e_1, \dots, e_n\}$ . Suppose we take another basis  $\{f_1, \dots, f_n\}$  so that  $X(f_i) = \sum_{j=1}^n f_j \eta_{ji}$ ,

$$X \sim (\eta)$$



$\eta, \xi$  are related by  $\eta = P\xi P^{-1}$  where  $P$  is the change of basis matrix,

$$P = M(\text{id})_{\xi}^{\eta}, \quad P^{-1} = M(\text{id})_{\eta}^{\xi}$$

Consequently

$$\text{Tr}(\eta) = \text{Tr}(P\xi P^{-1}) = \text{Tr}(\xi)$$

so the trace is independent of the particular basis so we can legitimately define

$$\text{Tr}(X) = \text{Tr}(\xi)$$

when  $X(e_i) = \sum_{j=1}^n e_j \xi_{ji}$ .

**Proposition** (Additivity of  $\text{Tr}$ ). Let

$$0 \rightarrow U \rightarrow V \xrightarrow{p} W \rightarrow 0$$

be an exact sequence of finite dimensional vector spaces over  $\mathbb{F}$  and suppose there exists linear maps  $T_U, T_V, T_W$  such that the following commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \xrightarrow{p} & W \longrightarrow 0 \\ & & \downarrow T_U & & \downarrow T_V & & \downarrow T_W \\ 0 & \longrightarrow & U & \longrightarrow & V & \xrightarrow{p} & W \longrightarrow 0 \end{array}$$

then

$$\text{Tr}(T_V) = \text{Tr}(T_U) + \text{Tr}(T_W)$$

**Lemma.** Let  $T : V \rightarrow V$  be a linear map over  $\mathbb{F}$ . Suppose  $\dim(V) = n$ , and let  $U \subset V$  be a subspace;  $T(U) \subset U$ , and  $\dim(U) = k$ . Then  $T$  can be represented by a matrix

$$T \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where  $A$  is a  $k \times k$  matrix,  $B$  is a  $k \times (n - k)$  matrix and  $D$  is a  $(n - k) \times (n - k)$  matrix.

*Proof.* (of lemma) Let  $\{e_1, \dots, e_k\}$  be a basis for  $U$ . Extend to a basis  $\{e_1, \dots, e_k, f_1, \dots, f_q\}$  for  $V$  ( $q = n - k$ ). With respect to this basis,

$$T(e_i) = \sum_{j=1}^k e_j a_{ji} \quad (T(U) \subset U)$$

$T(f_r)$  is a linear combination in  $\{e_1, \dots, e_k, f_1, \dots, f_q\}$ . Write

$$T(f_r) = \sum_{s=1}^k e_s b_{sr} + \sum_{t=1}^q f_t d_{tr} \quad (1 \leq r \leq q)$$

so the matrix of  $T$  has block form,

$$T \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where  $A = (a_{ji})$ ,  $B = (b_{sr})$ ,  $D = (d_{tr})$ . □

Note that  $Tr(T) = Tr(A) + Tr(D)$ .

*Proof.* (of proposition) Let  $\{e_1, \dots, e_k\}$  be a basis for  $U$ , and  $\{\phi_1, \dots, \phi_q\}$  be basis for  $W$ . For all  $r$ , choose  $f_r \in V$ ,  $p(f_r) = \phi_r$ . Then,  $\{e_1, \dots, e_k\} \cup \{f_1, \dots, f_q\}$  is a basis for  $V$ . By the previous lemma,  $T_V$  is represented by a block matrix

$$T_V \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where  $A$  = matrix of  $T_U$ , and factoring out  $U$ ,

$$D = \text{matrix } T_W \text{ with respect to } \{f_1, \dots, f_q\}$$

hence

$$\begin{aligned} Tr(T_V) &= Tr(A) + Tr(D) \\ &= Tr(T_U) + Tr(T_W) \end{aligned}$$

□

## 7 Lefschetz Fixed Simplex Theorem

**Theorem** (Lefschetz Fixed Simplex Theorem). Let  $f : K \rightarrow K$  be a simplicial map where  $K$  is a finite simplicial complex. Define

$$\lambda(f) = \sum_k (-1)^k \text{Tr}(H_k(f))$$

If  $\lambda(f) \neq 0$  then there exists a simplex  $\sigma$  of  $K$  such that  $f(\sigma) = \sigma$ .  $\lambda(f)$  is called the *Lefschetz number* of  $f$  (pick a field  $\mathbb{F}$ )

**Definition** (Geometrical Lefschetz index).

$$\lambda_{\text{geom}}(f) = \sum_k (-1)^k \text{Tr}(C_k(f))$$

where  $C_k(f) : C_k(K; \mathbb{F}) \rightarrow C_k(K; \mathbb{F})$  is the mapping induced by  $f$ .

**Definition** (Homological Lefschetz index).

$$\lambda_{\text{hom}}(f) = \sum_k (-1)^k \text{Tr}(H_k(f))$$

**Proposition.**

$$\lambda_{\text{geom}}(f) = \lambda_{\text{hom}}(f)$$

*Proof.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_k(K) & \longrightarrow & C_k(K) & \longrightarrow & B_{k-1}(K) \longrightarrow 0 \\ & & \downarrow Z_k(f) & & \downarrow C_k(f) & & \downarrow B_{k-1}(f) \\ 0 & \longrightarrow & Z_k(K) & \longrightarrow & C_k(K) & \longrightarrow & B_{k-1}(K) \longrightarrow 0 \end{array}$$

$$Z_k(K) = \ker \partial_k, \quad B_{k-1}(K) = \text{im}(\partial_k)$$

so

$$\text{Tr}(C_k(f)) = \text{Tr}(Z_k(K)) + \text{Tr}(B_{k-1}(K)) \quad (1)$$

and we also have

$$\begin{array}{ccccccc}
0 & \longrightarrow & B_k(K) & \longrightarrow & Z_k(K) & \longrightarrow & H_k(K) \longrightarrow 0 \\
& & \downarrow B_k(f) & & \downarrow Z_k(F) & & \downarrow H_k(f) \\
0 & \longrightarrow & B_k(K) & \longrightarrow & Z_k(K) & \longrightarrow & H_k(K) \longrightarrow 0
\end{array}$$

where both the rows are exact, so

$$Tr(Z_k(f)) = Tr(H_k(f)) + Tr(B_k(f)) \quad (2)$$

Substituting (2) into (1), we have

$$Tr C_k(f) = Tr H_k(f) + Tr B_k(f) + Tr(B_{k-1}(f))$$

but

$$\sum_k (-1)^k Tr B_k(f) + Tr B_{k-1}(f) = 0$$

so

$$\sum_k (-1)^k Tr C_k(f) = \sum_k (-1)^k Tr H_k(f)$$

thus

$$\lambda_{\text{geom}}(f) = \lambda_{\text{hom}}(f)$$

□

**Remark.** Note that  $\lambda_{\text{hom}}(f)$  is easier to compute, but  $\lambda_{\text{geom}}(f)$  carries geometric information.

Now consider  $C_k(f) : C_k(K) \rightarrow C_k(K)$ . If  $\sigma$  is a  $k$ -simplex of  $K$ , either  $f(\sigma)$  is a  $k$ -simplex or  $f(\sigma)$  is an  $l$ -simplex, where  $l < k$ .

In the first case,  $C_k(f)(\sigma)$  is a basis element of  $C_k$ . In the second case,  $C_k(f)(\sigma) = 0$ , so representing  $C_k(f)$  as a matrix, in a column there is at most one non-zero entry.

List the  $k$ -simplices of  $K$ ,  $\sigma_1, \dots, \sigma_N$ .  $C_k(f)$  is an  $N \times N$  matrix. The  $(i, i)$  entry of  $C_k(f)$  is non-zero if and only if  $f(\sigma) = \pm\sigma$ , so if no  $k$ -simplex is fixed by  $f$  then the diagonal of  $C_k(f)$  is 0 and  $Tr(C_k(f)) = 0$ .

*Proof.* (of Lefschetz fixed simplex theorem) Formally put,

$$f \text{ fixes no } k\text{-simplex} \implies \text{Tr } C_k(f) = 0$$

$$\text{so } f \text{ fixes no simplex} \implies \text{Tr } C_k(f) = 0 \text{ for all } k$$

$$\text{so } f \text{ fixes no simplex} \implies \sum_k (-1)^k \text{Tr } C_k(f) = (\lambda_{\text{geom}}(f) =) 0$$

In the contrapositive,

$$\lambda_{\text{geom}}(f) \neq 0 \implies f \text{ fixes some simplex (up to sign, it may change local orientation)}$$

hence

$$\lambda_{\text{hom}}(f) \neq 0 \implies f \text{ fixes some simplex}$$

□

Recall that

**Proposition.** If  $f : K \rightarrow K$  is a simplicial map and  $K$  is connected then

$$H_0(f) = \text{id} : H_0(K) \rightarrow H_0(K)$$

*Proof.* If  $v, w$  are vertices of  $K$ , then  $|v| - |w| \in \text{im } \partial_1$ , so  $[v] = [w]$  in  $H_0(K)$ . Hence  $[f(v)] = [v]$  in  $H_0(K)$  for any vertex  $v$ . But any vertex  $v$  generates  $H_0(K)$  ( $K$  connected), so

$$H_0(f) = \text{id} : \text{generator} \rightarrow \text{itself}$$

□

**Corollary.** If  $K$  is a connected simplicial complex and  $f : K \rightarrow K$  is simplicial, then

$$\text{Tr } H_0(K) = 1$$

**Corollary.** Let  $f : K \rightarrow K$  be a simplicial map, where  $K$  is a finite connected simplicial complex and

$$H_k(K; \mathbb{F}) = 0 \text{ for } k > 0$$

then  $\lambda(f) = 1$ .

**Corollary.** If  $f : K \rightarrow K$  is a simplicial map,  $K$  a finite connected complex such that  $H_k(K; \mathbb{F}) = 0$  for  $k > 0$ , then there exists a simplex  $\sigma$  of  $K$  such that  $f(\sigma) = \sigma$  (up to orientation.)

*Proof.*

$$\lambda(f) = 1 \neq 0$$

□

**Corollary.** If  $K$  is a finite simplicial complex and  $K \sim CX$  (combinatorially equivalent) where  $CX$  is the cone on  $X$ , for some  $X$ , then any simplicial map  $f : K \rightarrow K$  fixes a simplex.

**Example.**  $K \sim \Delta^n$ , any simplicial  $f : K \rightarrow K$  fixes a simplex. (Brouwer Fixed Simplex)

To transform a finite simplicial complex into a metric space, replace the formal  $n$ -simplex by standard geometric  $n$ -simplex,

$$|\Delta^n| = \{t_0 e_0 + t_1 e_1 + \dots + t_n e_n \mid t_i \geq 0, \sum_{i=0}^n t_i = 1\}$$

where  $e_0, \dots, e_n$  are the standard basis for  $\mathbb{R}^{n+1}$ .

$f : \Delta^n \rightarrow \Delta^n$  (formal simplicial mapping) gives a continuous mapping

$$\begin{aligned} |f| : |\Delta^n| &\rightarrow |\Delta^n| \\ |f|(\sum t_i e_i) &\rightarrow \sum t_i f(e_i) \end{aligned}$$

$f$  permutes  $e_0, \dots, e_n$ ,

$$|f|(\frac{1}{n} \sum e_i) = \frac{1}{n} \sum e_i \text{ (fixed point)}$$

so if  $g : K \rightarrow K$  is a simplicial map,  $K$  a finite complex, we get a continuous mapping (!)  $|g| : |K| \rightarrow |K|$ . If  $g$  fixes a simplex, then  $|g|$  fixes a point.

**Theorem** (Brouwer Fixed Point Theorem). Let  $X = |K|$  where  $K$  is some finite simplicial complex. Suppose any simplicial map  $g(m) : K(m) \rightarrow K(m)$  has a fixed simplex  $K(m)$  (subdivision of  $K$ ), then any continuous  $f : X \rightarrow X$  has a fixed point.

*Proof.* Suppose  $f : X \rightarrow X$  does not have a fixed point.  $X$  compact, so there exists  $\epsilon > 0$  such that  $\epsilon \leq \|f(x) - x\|$  for all  $x$ . Suppose  $g(m) : K(m) \rightarrow K(m)$ , then

$$\|f(x) - x\| \leq \|f(x) - g_m(x)\| + \|g_m(x) - x\|$$

so

$$\forall \eta \exists m \|f(x) - g_m(x)\| < \eta \quad \forall x$$

Choose  $m$  so that  $\|f(x) - g_m(x)\| < \frac{\epsilon}{2}$  so then

$$\frac{\epsilon}{2} \leq \|g_m(x) - x\| \quad \forall x$$

which is a contradiction, thus  $f$  has a fixed point. □

## 8 Posets and products

**Theorem** (Künneth theorem (will not be proven here)).

$$H_n(X \times Y; \mathbb{F}) = \bigoplus_{r=0}^n H_r(X; \mathbb{F}) \otimes H_{n-r}(Y; \mathbb{F})$$

however we will prove the following,

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

where  $X, Y$  are finite simplicial complexes. To see applications of this, consider  $S^2 \times S^2$  and  $S^4$ , both of which are simply connected compact 4-manifolds. However,  $S^2 \times S^2 \not\cong S^4$  since

$$\chi(S^2 \times S^2) = \chi(S^2)\chi(S^2) = 4$$

and

$$\chi(S^4) = 2$$

**Definition** (Posets (Partially ordered sets)). A *poset*  $(X, \leq)$  consists of a set  $X$  and a relation  $\leq$  in  $X \times X$

1.  $x \leq x \quad \forall x$
2.  $x \leq y \wedge y \leq z \implies x \leq z$

In a total ordering we also have  $\forall x, y \in X$  either  $x \leq y$  or  $y \leq x$ .

Now let  $(X, \leq)$  be a finite poset. Construct a simplicial complex  $N(X, \leq)$  the *nerve* of  $(X, \leq)$ . The vertex set of  $N(X, \leq)$  is  $X$  and simplex set of  $N(X, \leq)$  is the set of totally ordered non-empty subsets.

If  $(X, \leq)$ ,  $(Y, \leq')$  are finite posets then the product poset is  $(X \times Y, \preceq)$  where

$$(x, y) \preceq (x', y') \iff (x \leq x') \wedge (y \leq' y')$$

Notice that if  $(X, \leq)$ ,  $(Y, \leq')$  are totally ordered then  $(X \times Y, \preceq)$  isnt, except trivially.

From now on we will always use the symbol ' $\leq$ '.

**Proposition.** If  $X$  is a finite simplicial complex, we can write

$$X = N(\mathcal{X}, \leq)$$

for some  $\mathcal{X}$ .

*Proof.* Take an arbitrary ordering on the vertices of  $X$ ,  $\{v_0, \dots, v_n\}$ .  $X$  embeds in  $\Delta^N$ ,

$$v_i \mapsto i, \quad \mathcal{X} = \text{im}(v_i \mapsto i)$$

$$\Delta^N = \text{nerve on totally ordered set } 0 \leq 1 \leq \dots \leq N$$

so each simplex  $\sigma$  of  $X$  is totally ordered. □



So now to define  $X \times Y$ , we write

$$X = N(\mathcal{X}), \quad Y = N(\mathcal{Y})$$

where  $\mathcal{X}, \mathcal{Y}$  are posets. Then, we define

$$X \times Y = N(\mathcal{X} \times \mathcal{Y})$$

By an *ordered* simplex complex, we mean a simplicial complex

$$X = (V_X, \mathcal{S}_X)$$

together with a partial ordering on  $V_X$  such that for all  $\sigma \in \mathcal{S}_X$ ,  $\sigma$  is totally ordered, so any simplicial complex can be regarded as an ordered simplicial complex,

$$X = N(\mathcal{X}) \text{ for some } \mathcal{X}$$

If  $X, Y$  are ordered simplicial complexes, then so is  $X \times Y$ .

**Example.**

$$\Delta^n = N(\{0, \dots, n\})$$

**Definition.**

$$\Delta^m \times \Delta^n = N(\{0, \dots, m\} \times \{0, \dots, n\})$$

$\Delta^m \times \Delta^n$  has dimension  $m + n$  and has  $\frac{(m+n)!}{m!n!}$  principal simplices.

**Example.**  $\Delta^1 \times \Delta^1$  as defined before is equivalent to  $I \times I$  but triangulated.

Every finite simplicial complex  $K$  can be represented as an *ordered simplicial complex*, as shown before.

## 8.1 Product structure on $X \times Y$

Let  $X, Y$  be finite simplicial complexes. Represent them as ordered simplicial complexes. If  $\sigma \in \mathcal{S}_X$ ,  $\tau \in \mathcal{S}_Y$ , we make the isomorphisms

$$\sigma \cong \Delta^m, \quad \tau \cong \Delta^n$$

and triangulate  $\sigma \times \tau$  by corresponding triangulation on  $\Delta^m \times \Delta^n$ . Write

$$X = \bigcup_{\sigma \in \mathcal{S}_X} \sigma, \quad Y = \bigcup_{\tau \in \mathcal{S}_Y} \tau$$

and

$$X \times Y = \bigcup_{\sigma \in \mathcal{S}_X, \tau \in \mathcal{S}_Y} \sigma \times \tau$$

and each  $\sigma \times \tau$  is triangulated as explained before.

It can be shown up to combinatorial equivalence that this is independent of orderings chosen on  $X, Y$ .

Now we wish to prove

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

Recall the Künneth theorem,

$$H_n(X \times Y; \mathbb{F}) = \bigoplus_{r=0}^n H_r(X; \mathbb{F}) \otimes H_{n-r}(Y; \mathbb{F})$$

In the special case where  $Y = \Delta^m$ ,

$$H_{n-k}(Y) = \begin{cases} \mathbb{F} & n - k = 0 \\ 0 & \text{otherwise} \end{cases}$$

so as a special case, we do have

$$H_*(X \times \Delta^m) \cong H_*(X)$$

which we will prove.

Fix  $\Delta^m$ . For any simplicial complex  $X$ , we have simplicial maps

$$i : X \rightarrow X \times \Delta^m, \quad i(v) = (v, 0)$$

$$\pi : X \times \Delta^m \rightarrow X, \quad \pi(x, y) = x$$

Both maps induce isomorphisms on  $H_*$ , so we will just prove this for  $i$ . Let  $P(n, k)$  denote the statement,

If  $X$  is finite complex of  $\dim(X) \leq n$  having at most  $k$  simplices of dimension  $n$ ,

then  $i : H_*(X; \mathbb{F}) \rightarrow H_*(X \times \Delta^m; \mathbb{F})$  is an isomorphism

We first show that  $P(n, 0)$  is true for all  $n$ , i.e.,

$i : \Delta^n \rightarrow \Delta^n \times \Delta^m$  induces isomorphisms

$$i_* : H_*(\Delta^n) \rightarrow H_*(\Delta^n \times \Delta^m)$$

Let  $(A, \leq)$  be a poset. We say that  $A$  has a *maximum* when there exists  $a \in A$  such that for all  $b \in A$ ,  $b \leq a$ .

**Proposition.** If  $(A, \leq)$  has a maximum then  $N(A, \leq)$  is a cone.

*Proof.* Let  $a$  be a maximum element. Define  $A' = A - \{a\}$ .  $(A', \leq)$  is still a poset, and  $N(A, \leq)$  is a cone on  $N(A', \leq)$  with cone point  $a$ .  $\square$

Now note that  $\Delta^m \times \Delta^n$  has maximum  $(m, n)$ , so  $\Delta^m \times \Delta^n$  is a cone.

**Corollary.**

$$i_* : H_*(\Delta^n) \xrightarrow{\cong} H_*(\Delta^n \times \Delta^m)$$

and  $P(n, 0)$  is true for all  $n$ .

**Proposition.**  $P(0, k)$  is true for all  $k$ .

*Proof.* Here  $X$  is a 0-dimensional complex with  $k$ -points. Note that for  $k = 0$ , this is true, so  $P(0, 0)$  is true. We proceed by induction.

Suppose that the statement is proven for  $k - 1$ . Then

$$X = X_0 \sqcup \{x_k\} \quad X_0 = \{x_1, \dots, x_{k-1}\}$$

By hypothesis

$$H_*(X_0 \times \Delta^m) \cong \bigoplus_{r=1}^{k-1} H_*(\{x_r\} \times \Delta^m) = \begin{cases} \mathbb{F}^{k-1} & * = 0 \\ 0 & * \neq 0 \end{cases}$$

$$X \cap X_0 = \emptyset, \quad H_*(X \cap X_0) = 0 \text{ for all } *$$

Using the Mayer-Vietories sequence,

$$H_*((X_0 \sqcup \{x_k\}) \times \Delta^m) \cong H_*(X_0 \times \Delta^m) \oplus H_*(\{x_k\} \times \Delta^m)$$

We know

$$H_*((X_0 \sqcup \{x_k\}) \times \Delta^m) \cong H_*(X)$$

and

$$H_*(X_0 \times \Delta^m) \oplus H_*(\{x_k\} \times \Delta^m) \cong H_*(X_0) \oplus H_*(\{x_k\})$$

so  $P(0, k)$  for all  $k$ . □

Now let  $P(n) = \bigwedge_{k \geq 0} P(n, k)$ . So now we know  $P(0)$  is true. Also,  $P(n, 0) \equiv P(n - 1)$  ( $P(n, 0)$ ; no  $n$ -simplices).

$P(n, 1)$  is true for all  $n$ . It is enough to prove

$$P(n, k - 1) \wedge P(n - 1) \implies P(n, k)$$

*Proof.* Take  $X$  to be a finite complex of dimension less than or equal to  $n$  with exactly  $k$  simplices of dimension  $n$ .

Let  $X_0$  be the complex obtained by removing an  $n$ -simplex  $\sigma \cong \Delta^n$ ,

$$X = X_0 \cup \sigma$$

$X_0 \cap \sigma$  has dimension  $\leq n - 1$ .

$$\begin{array}{ccccccc}
H_p(X_0 \cap \sigma) & \longrightarrow & H_p(X_0) \oplus H_p(\sigma) & \longrightarrow & H_p(X) & \longrightarrow & \\
\downarrow i_* & & \downarrow \mathcal{I} & & \downarrow i_* & & \\
H_p(X_0 \cap \sigma \times \Delta^n) & \longrightarrow & H_p(X \times \Delta^m) \oplus H_p(\sigma \times \Delta^m) & \longrightarrow & H_p(X \times \Delta^m) & \longrightarrow & \\
& & & & & & \\
& \longrightarrow & H_{p-1}(X_0 \cap \sigma) & \longrightarrow & H_{p-1}(X_0) \oplus H_{p-1}(\sigma) & & \\
& & \downarrow i_* & & \downarrow \mathcal{I} & & \\
& \longrightarrow & H_{p-1}(X_0 \cap \sigma \times \Delta^m) & \longrightarrow & H_{p-1}(X_0 \times \Delta^m) \oplus H_{p-1}(\sigma \times \Delta^m) & & 
\end{array}$$

where

$$\mathcal{I} = \begin{pmatrix} i_* & 0 \\ 0 & i_* \end{pmatrix}$$

Both rows are exact. The outer arrows are isomorphisms. Hence  $i_* : H_p(X) \xrightarrow{\cong} H_p(X \times \Delta^m)$  is isomorphic for all  $p$ , so

$$P(n, k - 1) \wedge P(n - 1) \implies P(n, k)$$

□