Algebraic Topology - MATH0023

Based on lectures by Prof FEA Johnson

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Notes based on the Autumn 2021 Algebraic Topology lectures by Prof FEA Johnson.

Contents

1 Simplicial complexes

Definition (Simplicial complex)**.** A *simplicial complex* X is a pair $(V_X, \mathcal{S}_X)$ where V_X denotes the vertex set of X and $\mathcal{S}_X$ is the set of *finite, non-empty* subsets of V_X satisfying

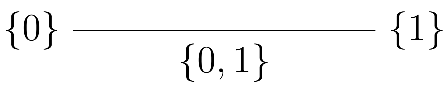
1. $\forall v \in V_X$, then $\{v\} \in \mathcal{S}_X$
2. If $\sigma \in \mathcal{S}_X$, $\tau \subset \sigma$, $\tau \neq \emptyset$, then $\tau \in \mathcal{S}_X$.

$\mathcal{S}_X$ is called the set of *simplices* of X.

Example. A *standard 1-simplex*, denoted by Δ^1 is simply the line segment (or usually denoted by I).

$$V_{\Delta^1} = \{0, 1\}$$

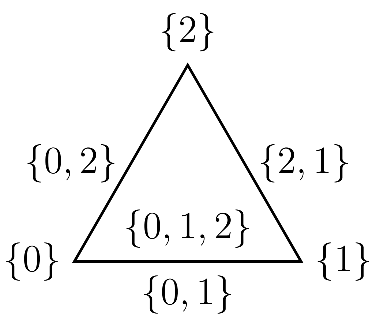
$$\mathcal{S}_{\Delta^1} = \{\{0\}, \{1\}, \{0, 1\}\}$$



A *standard 2-simplex*, denoted by Δ^2 is the equilateral triangle.

$$V_{\Delta^2} = \{0, 1, 2\}$$

$$\mathcal{S}_{\Delta^2} = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$



In general, the *standard n-simplex* Δ^n, is $\Delta^n = (V_{\Delta^n}, \mathcal{S}_{\Delta^n})$ where

$$V_{\Delta^n} = \{0, 1, \ldots, n\}$$

$$\mathcal{S}_{\Delta^n} = \{\alpha : \alpha \subset \{0, \ldots, n\},\ \alpha \neq \emptyset\}$$

If $X = (V_x, \mathcal{S}_X)$ is a simplicial complex, we now want to pick a field $\mathbb{F}$, usually $\mathbb{Q}$ or $\mathbb{F}_2$ (in this course) and want to produce a sequence of vector spaces (over $\mathbb{F}$)

$$C_n(X)_{0 \leq n}$$

$C_0(X)$ is the vector space whose basis elements are simply the vertices of the simplicial complex, the 0-dimensional bits.

Definition (k-simplex of a simplicial complex)**.** If X is a simplicial complex then a *k-simplex of* X is a simplex $\sigma \in \mathcal{S}_X$ such that $|\sigma| = k + 1$.

$C_k(X)$ is the vector space whose basis elements are the *oriented* k-simplices of X which are the following symbols,

$$[v_0, v_1, \ldots, v_n]$$

(where $\{v_0, \ldots, v_n\}$ is an n-simplex of X) subject to the rules

$$[v_{\rho(0)}, v_{\rho(1)}, \ldots, v_{\rho(n)}] = \mathrm{sign}(\rho)[v_0, \ldots, v_n]$$

Definition.

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is a linear map defined on basis elements as follows;

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_o, \ldots, \hat{v_r}, \ldots, v_n]$$

where $\hat{v_r}$ indincates omission of v_r.

Example.

$$\partial_2[0,1,2] = [1,2] - [0,2] + [0,1]$$

$$\partial_1[v_0, v_2] = [v_1] - [v_0]$$

$$\begin{aligned} \partial_1\partial_2[0,1,2] &= \partial_1([1,2] - [0,2] + [0,1]) \\ &= ([2]-[1]) - ([2]-[0]) + ([1]-[0]) \\ &= 0 \end{aligned}$$

Proposition (Poincaré lemma)**.** Let X be a simplicial complex. Consider

$$\partial_r : C_r(X) \to C_{r-1}(X)$$

for $r \geq 1$, then

$$\partial_{n-1}\partial_n \equiv 0$$

Proof.

$$\partial_n[v_0, \ldots, v_n] = \sum_{r=0}^{n} (-1)^r [v_0, \ldots, \hat{v_r}, \ldots, v_n]$$

$$\begin{aligned} \partial_{n-1}[v_0, \ldots, \hat{v_r}, \ldots, v_n] &= \sum_{s<r} (-1)^s [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\ &\quad + \sum_{s>r} (-1)^{s-1} [v_0 \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n] \end{aligned}$$

$$\begin{aligned} \partial_{n-1}\partial_n[v_0, \ldots, v_n] &= \sum_{s<r} (-1)^{r+s} [v_0, \ldots, \hat{v_s}, \ldots, \hat{v_r}, \ldots, v_n] \\ &\quad + \sum_{s>r} (-1)^{r+s-1} [v_0, \ldots, \hat{v_r}, \ldots, \hat{v_s}, \ldots, v_n] \\ &= 0 \end{aligned}$$

$\square$

Proposition. If

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

then

$$\operatorname{im}(\partial_{n+1}) \subset \ker(\partial_n)$$

Proof. By previous lemma. □

2 Homology

2.1 Quotient spaces

Let V be a vector space over a field $\mathbb{F}$, and $U \subset V$ a vector subspace.

Definition. The following set

$$x + U = \{x + u : u \in U\}$$

is called the (left) coset of U in V. Note that

$$x + U = x' + U \iff x - x' \in U$$

Definition (Quotient space)**.** The quotient space V/U is the set

$$V/U = \{x + U : x \in V\}$$

where addition and scalar multiplication is defined by

$$(x + U) + (y + U) = x + y + U$$

$$\lambda \cdot (x + U) = \lambda x + U$$

and 0 is represented by

$$0 + U$$

Note that V/U is a vector space.

Proposition.

$$\dim(V/U) = \dim(V) - \dim(U)$$

Proof. There exists a natural linear map

$$\eta : V \to V/U$$

given by

$$\eta(x) = x + U$$

Clearly this map is surjective so

$$\dim(V/U) = \dim(\mathrm{im}(\eta))$$

Now,

$$\begin{aligned} \ker(\eta) &= \{x \in V : \eta(x) = U\} \\ &= \{x \in V : x + U = U\} \end{aligned}$$

and

$$x + U = U \iff x - 0 \in U \iff x \in U$$

so $\ker(\eta) = U$. Then,

$$\dim(V) = \dim\ker(\eta) + \dim\mathrm{im}(\eta)$$

so

$$\dim(V/U) = \dim\mathrm{im}(\eta) = \dim(V) - \dim(U)$$

□

Definition.

$$H_n(X; \mathbb{F}) = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

We call $H_n(X; \mathbb{F})$ the n^{th} *homology group* of X with coefficients in $\mathbb{F}$. If $\mathbb{F} = \mathbb{Q}$, then $\dim H_n(X; \mathbb{Q})$ is called the n^{th} *Betti* number of X.

Consider Δ^3. The set $\{0, 1, 2, 3\}$ represents the 'middle' of the tetrahedron (inside, interior). If we exclude the middle and simply take its boundary, we have

$$\partial\Delta^n = S^{n-1}$$

It happens that S^2 (middle excluded) is the simplest simplicial model of the 2-sphere.

Example. Consider

$$H_k(S^2; \mathbb{F})$$

Note that

$$C_n(S^2) = 0 \text{ for } n \geq 3$$

as there are no 3-simplices, so we only have to worry about

$$H_2(S^2; \mathbb{F}), H_1(S^2; \mathbb{F}), H_0(S^2; \mathbb{F})$$

We proceed to calculate these from first principles. First note that $C_3(S^2) = 0$. Now, (noting the order of these bases) $C_2(S^2)$ has basis

$$[0,1,2], [0,1,3], [0,2,3], [1,2,3]$$

$C_1(S^2)$ has basis

$$[0,1], [0,2], [0,3], [1,2], [1,3], [2,3]$$

and lastly $C_0(S^2)$ has basis

$$[0], [1], [2], [3]$$

The linear maps

$$\partial_2 : C_2(S^2) \to C_1(S^2)$$
$$\partial_1 : C_1(S^2) \to C_0(S^2)$$

can both be represented by a 6×4 matrix and a 4×6 matrix respectively.

We apply ∂_2 and ∂_1 to the bases to obtain the entries to the matrices, so for example

$$\partial_2([0,1,2]) = [1,2] - [0,2] + [0,1]$$

so the first column of the matrix representing ∂_2 is $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ Proceeding, we will obtain that

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Notice that $\partial_1\partial_2 = 0$, which further confirms the lemma from before. Now reducing both the matrices to row reduced echelon form, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim\ker\partial_2 = 1$, $\dim\operatorname{im}\,\partial_2 = 3$

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

thus $\dim\ker\partial_1 = 3$, $\dim\operatorname{im}\,\partial_1 = 3$

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

so now

$$H_2(S^2) = \ker(\partial_2)/\operatorname{im}(\partial_3) = \ker(\partial_2) \cong \mathbb{F}$$

as $\operatorname{im}(\partial_3) = 0$, so in total,

$$H_2(S^2; \mathbb{F}) \cong \mathbb{F}$$

Next,

$$H_1(S^2) = \ker(\partial_1)/\operatorname{im}(\partial_2)$$

Now note that

$$\dim H_1(S^2) = \dim \ker(\partial_1) - \dim \operatorname{im}(\partial_2) = 3 - 3 = 0$$

thus

$$H_1(S^2; \mathbb{F}) = 0$$

Next,

$$H_0(S^2) = \ker(\partial_0)/\operatorname{im}(\partial_1) = C_0/\operatorname{im}(\partial_1)$$

and

$$\dim H_0(S^2) = \dim C_0 - \dim \operatorname{im}(\partial_1) = 4 - 3 = 1$$

thus

$$H_0(S^2; \mathbb{F}) \cong \mathbb{F}$$

We've shown

$$H_k(S^2; \mathbb{F}) \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

We will soon see that this theorem generalises if

$$S^n = \partial(\Delta^{n+1})$$

then

$$H_k(S^n) = \begin{cases} \mathbb{F} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

2.2 Chain complex

Definition (Chain complex)**.** Let $\mathbb{F}$ be a field. A *chain complex* over $\mathbb{F}$ is

$$C_* = (C_r, \partial_r)_{r \in \mathbb{N}}$$

where

1. Each C_r is a vector space over $\mathbb{F}$
2. $\partial_r : C_r \to C_{r-1}$ is a linear map such that $\partial_r \partial_{r+1} = 0$ for all r.

If $X = (V_X, \mathcal{S}_X)$, we have defined a chain complex

$$C_*(X) = (C_r(X), \partial_r)$$

Given a chain complex

$$C_*(C_r, \partial_r)_{r \geq 0}$$

we define its *homology* $H_*(C_*)$ by

$$H_k(C_*) = \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$$

If $X = (V_X, \mathcal{S}_X)$ is a simplicial complex, we define

$$H_k(X; \mathbb{F}) = H_k(C_*(X; \mathbb{F}))$$

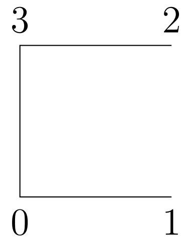
2.3 Simplicial mapping

Definition (Simplicial mapping)**.** Let X, Y be simplicial complexes, i.e., $X = (V_X, \mathcal{S}_X)$ and $Y = (V_Y, \mathcal{S}_Y)$. A *simplicial mapping* $f : X \to Y$ is a mapping of vertex sets $f : V_X \to V_Y$ such that

$$\sigma \in \mathcal{S}_X \implies f(\sigma) \in \mathcal{S}_Y$$

Example. Let $X = Y = \Delta^2$. Then defining f by $f(0) = 1$, $f(1) = 2$, $f(2) = 0$, it is obvious that this mapping is simplicial.

Consider the following simplicial complex



and consider

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 0$$

This mapping is *not* simplicial as $f(\{0, 1\})$ is *not* a simplex.

Given a simplicial mapping $f : X \to Y$, we are going to produce linear maps

$$H_k(f) : H_k(X) \to H_k(Y)$$

such that if

$$g : Y \to Z$$

then

$$g \circ f : X \to Z$$

and

1. $H_k(g \circ f) = H_k(g) \circ H_k(f)$
2. $H_k(\mathrm{id}_X) = \mathrm{id}_{H_k(X)}$

Remark. (Look up on functors for a more general treatment of the above concept.)

2.4 Chain mapping

Definition. Let

$$C_* = (C_r, \partial_r^C)$$
$$D_* = (D_r, \partial_r^D)$$

be chain complexes. A *chain mapping* $f_* : C_* \to D_*$ is a collection of linear maps

$$f* = (f_r)_{r \geq 0}$$

where $f_r : C_r \to D_r$ and the following commutes

$$\begin{tikzcd} C_r \arrow[r, "\partial_r^C"] \arrow[d, "f_r"'] & C_{r-1} \arrow[d, "f_{r-1}"] \\ D_r \arrow[r, "\partial_r^D"'] & D_{r-1} \end{tikzcd}$$

i.e.,

$$\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C$$

If $g : D_* \to E_*$ is also a chain mapping, then

$$(g \circ f)_n = g_n \circ f_n : C_* \to E_*$$

is also a chain mapping.

$$\mathrm{id} : C_* \to C_*, \;\; \mathrm{id}_n = \mathrm{id}_{C_n}$$

is also a chain mapping.

Proposition. If $f : X \to Y$ is a simplicial mapping, define

$$C_n(f) : C_n(X) \to C_n(Y)$$

by action on a basis as follows

$$C_n(f)[v_0, \dots, v_n] = [f(v_0), \dots, f(v_n)]$$

then

$$C_*(f) : C_*(X) \to C_*(Y)$$

is also a chain mapping.

Proof.

$$\begin{aligned}
\partial_n^D C_n(f)[v_0,\ldots,v_n] &= \partial_n^D([f(v_0),\ldots,f(v_n)]) \\
&= \sum_{r=0}^{n}(-1)^r[f(v_0),\ldots,f(\hat{v_0}),\ldots,f(v_n)] \\
&= C_{n-1}(f)\sum_{r=0}^{n}(-1)^r[v_0,\ldots,\hat{v_r},\ldots,v_n] \\
&= C_{n-1}(f)\partial_n^C[v_0,\ldots,v_n]
\end{aligned}$$

□

We will often write $f_n[v_0,\ldots,v_n]$ rather than $C_n(f)[v_0,\ldots,v_n]$.

Proposition. If $f : X \to Y$, $g : Y \to Z$ are simplicial maps, then

$$C_n(g \circ f) = C_n(g) \circ C_n(f)$$

which sometimes we will write as

$$(g \circ f)_n = g_n \circ f_n$$

instead.

Proof.

$$\begin{aligned}
(g \circ f)[v_0,\ldots,v_n] &= [(g \circ f)(v_0),\ldots,(g \circ f)(v_n)] \\
&= g_n[f(v_0),\ldots,f(v_n)] \\
&= g_n \circ f_n[v_0,\ldots,v_n]
\end{aligned}$$

□

Proposition. Let

$$\mathrm{id} : X \to X$$

then $C_*(\mathrm{id}) : C_*(X) \to C_*(X)$ is a chain mapping.

If $C_* = (C_n, \partial_n)$ is a chain complex, define

$$H_n(C_*) = \ker \partial_n / \mathrm{im}(\partial_{n+1})$$

It is usual to write

$$Z_n(C) = \ker(\partial_n) \ \text{(cycles)}$$

$$B_n(C) = \mathrm{im}(\partial_{n+1}) \ \text{(boundaries)}$$

thus by this notation,

$$H_n(C) = Z_n(C)/B_n(C)$$

If $f = (f_n)$, $C_* \to D_*$ is a chain mapping, we now want to show f induces a mapping

$$H_n(f) : H_n(C_*) \to H_n(D_*)$$

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(Z_n(C_*)) \subset Z_n(D_*)$$

Proof. Recall that

$$f_{n-1}\partial_n^C(z) = \partial_n^D f_n(z)$$

If

$$z \in Z_n(C_*),\ \partial_n^C(z) = 0$$

then we have

$$f_{n-1}\partial_n^C(z) = 0$$

and so

$$\partial_n^D f_n(z) = 0$$

and thus

$$f_n(z) \in Z_n(D_*)$$

□

Proposition. If $f : C_* \to D_*$ is a chain mapping, then

$$f_n(B_n(C_*)) \subset B_n(D_*)$$

Proof. Note that

$$f_n \partial^C_{n+1}(x) = \partial^D_{n+1} f_{n+1}(x)$$

If $\beta \in B_n(C_*)$, we can write $\beta = \partial^C_{n+1}(x)$ for some x and then

$$f_n(\beta) = \partial^D_{n+1}(k)$$

where $k = f_{n+1}(x)$ so

$$f_n(\beta) \in B_n(D_*)$$

□

Corollary. If $f : C_* \to D_*$ is a chain mapping, then f induces a (linear) mapping

$$H_n(f) : H_n(C_*) \to H_n(D_*)$$

Proof. An element of $H_n(C_*)$ has form

$$[z] = z + B_n(C_*),\ z \in Z_n(C_*)$$

Now define

$$H_n(f)[z] = f_n(z) + B_n(D_*) \in H_n(D_*)$$

and now note that

$$f_n(z) \in Z_n(D_*)$$

□

By now it is clear if $g : D_* \to E_*$, $f : C_* \to D_*$ are chain mappings, then

$$H_n(g \circ f) = H_n(g) \circ H_n(f)$$

and also if $\mathrm{id} : C_* \to C_*$ we have

$$H_n(\mathrm{id}) = \mathrm{id}_{H_n}$$

We now formally have

$$H_n(X) = H_n(C_*(X))$$

Corollary. If X is a *non-empty* simplicial complex, then $H_0(X;\mathbb{F}) \neq 0$ (for any field $\mathbb{F}$).

Proof. As $X \neq \emptyset$, we have that $V_X \neq \emptyset$. Let $v \in V_X$ be a vertex and $*$ be the simplicial complex

$$* = (\{v\}, \{\{v\}\})$$

so $*$ consists of one vertex v, and one 0-simplex $\{v\}$. Now define a constant simplicial mapping

$$c : X \to *,\ c(x) = v,\ \forall x \in V_X$$

We also have a simplicial mapping

$$\iota : * \to X,\ \iota(v) = v$$

so now

$$c \circ \iota = \mathrm{id}_*$$

and so (since both maps are simplicial, hence chain mappings)

$$H_0(c) \circ H_0(\iota) = H_0(\mathrm{id}_*)$$

but notice that

$$H_0(*) = \mathbb{F}$$

since we know

$$C_0(*) = \mathbb{F},\ C_r(*) = 0,\ r \geq 1$$

and thus

$$H_0(c) \circ H_0(\iota) = \mathrm{id}_{\mathbb{F}}$$

$$c \circ \iota = \mathrm{id} \neq 0$$

and now note that c is surjective, and ι is injective. In particular

$$H_0(c) : H_0(X) \to \mathbb{F} = H_0(*)$$

is surjective, so

$$H_0(X) \neq 0$$

□

So we now know that $H_0(X) \neq 0$ if $X \neq \emptyset$.

Definition. Let X be a simplicial complex. If $v, w \in V_X$, then by a path from v to w, we mean a sequence of 1-simplices

$$[v_0, v_1], [v_1, v_2], \ldots, [v_{n-2}, v_{n-1}], [v_{n-1}, v_n]$$

such that $v_0 = v$ and $v_n = w$.

Proposition. If X is non-empty and connected, then

$$H_0(X; \mathbb{F}) \cong \mathbb{F}$$

Proof.

$$C_1(X) \xrightarrow{\partial_1} C_0(X)$$

If $v, w \in V_X$, then $[w] - [v] \in \mathrm{im}(\partial_1)$. To see this, choose a path

$$v = v_0 < v_1 < \ldots < v_{n-1} < v_n = w$$

i.e., $[v_{i+1}, v_i]$ is a 1-simplex for $0 \leq i \leq n-1$.

$$\partial_1[v_i, v_{i+1}] = [v_{i+1}] - [v_i] \in \mathrm{im}(\partial_1)$$

so then,

$$[w] - [v] = \sum_{i=0}^{n-1} [v_{i+1}, v_i] \in \mathrm{im}(\partial_1)$$

Now $\{[v] : v \in V_X\}$ is a basis for C_0. Choose a specific $v \in V_X$. By elementary basis change,

$$\{[v]\} \cup \{[w] - [v] : w \in V_x, w \neq v\}$$

is a basis for C_0. However $[w] - [v] \in \mathrm{im}(\partial_1)$ $(w \neq v)$. So $C_0(X)/\mathrm{im}(\partial_1)$ has dimension ≤ 1, and then $\dim H_0(X) \leq 1$ if X is connected. But $X \neq \emptyset$, so $H_0(X) \neq 0$, hence $\dim H_0(X) = 1$, hence

$$H_0(X) \cong \mathbb{F}$$

when X is connected. $\square$

Proposition. In general, $\dim H_0(X)$ is equal to the number of connected components in X

If X is a simplicial complex, then define a relation $\sim$ on V_X by $v \sim w$ if and only if there exists a path from v to w.

$\sim$ defines an equivalence relation, where the number of connected components is equal to the number of equivalence classes.

If X consists of a single point,

$$H_k(\text{pt.}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

2.5 Cone

Definition. Let X be a simplicial complex. A *cone* on X, $C(X)$, is defined as follows, choose $*$ (cone point) such that $* \notin V_X$

$$V_{C(X)} = \{*\} \cup V_X$$

$$\mathcal{S}_{C(X)} = \mathcal{S}_X \cup \{\{*\} \cup \{\sigma \cup \{*\} : \sigma \in S_X\}\}$$

i.e., join everything in X to the cone point.

Theorem. If X is a simplicial complex, then,

$$H_k(C(X); \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

i.e., $C(X)$ behaves just like a point (homologically).

Proof. First note that $C(X)$ is connected. Take $v, w \in V_{C(X)}$, $v \neq w$. Either one of them is the cone point, or none of them are the cone point.

(1) Without loss of generality, suppose w is the cone point. $(w = *)$. By definition, $[v, w] = [v, *]$ is a 1-simplex of $C(X)$. So we've joined v to w.

(2) If neither are the cone point, then, $[v, *]$ and $[*, w]$ are both 1-simplices, so again, we've joined v to w. So

$$H_0(C(X); \mathbb{F}) \cong \mathbb{F}$$

Now we must show

$$H_k(C(X)) = 0,\ k > 0$$

We define, for each $k > 0$, a linear map

$$\mathcal{H}_k : C_k(C(X)) \to C_{k+1}(C(X))$$

(called a contracting homotopy) $\mathcal{H}_k$ is defined on a basis by

$$\mathcal{H}_k[v_0, \ldots, v_k] = [*, v_0, \ldots, v_k]$$

Then,

$$\begin{aligned}
\partial_{k+1}\mathcal{H}_k[v_0, \ldots, v_k] &= \partial_{k+1}[*, v_0, \ldots, v_k] \\
&= [v_0, \ldots, v_k] + \sum_{r=0}^{k} (-1)^{r+1}[*, v_0, \ldots, \hat{v}_r, \ldots, v_k]
\end{aligned}$$

$$\partial_{k+1}\mathcal{H}_k[v_0, \ldots, v_k] + \sum_{r=0}^{k} (-1)^{r}[*, v_0, \ldots, \hat{v}_r, \ldots, v_k] = [v_0, \ldots, v_k]$$

However,

$$\mathcal{H}_{k-1}[v_0, \ldots, \hat{v}_r, \ldots, v_k] = [*, v_0, \ldots, \hat{v}_r, \ldots, v_k]$$

and

$$(\partial_{k+1}\mathcal{H}_k + \mathcal{H}_{k-1}\partial_k)[v_0, \ldots, v_k] = [v_0, \ldots, v_k]$$

i.e.,

$$\partial_{k+1}\mathcal{H}_k + \mathcal{H}_{k-1}\partial_k = \mathrm{id}$$

(we call the above a homotopy relation)

$$H_k(C(X)) = Z_k(C(X))/B_k(C(X))$$

and if $z \in Z_k(C(X))$, $\partial_k(z) = 0$, so if $z \in Z_k(C(X))$, $z = \partial_{k+1}\mathcal{H}_k(z)$ so $z \in \mathrm{im}(\partial_{k+1})$, i.e., $Z_k(C(X)) \subset B_k(C(X))(\subset Z_k(X))$ so if $C(X)$ is a cone and $k > 0$,

$$Z_k(C(X)) = B_k(C(X))$$

and $H_k(C(X); \mathbb{F}) = 0$ $\square$

Corollary.

$$H_k(\Delta^n; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

where $\Delta^n = n$-simplex

Proof. Δ^n is a cone. $\Delta^n = (C(\Delta^{n-1}))$ $\square$

Let X be a simplicial complex, $n \geq 0$. Then the n-skeleton $X^{(n)}$ of X is defined by

$$V_{X^{(n)}} = V_X$$

$$\mathcal{S}_{X^{(n)}} = \{\sigma \in \mathcal{S}_X : |\sigma| \leq n + 1\}$$

i.e., $\dim(\sigma) \leq n$.

The standard model S^n of the n-sphere is

$$V_{S^n} = \{0, \ldots, n + 1\}$$

$$\mathcal{S}_{S^n} = \{\sigma \subset \{0, \ldots, n + 1\} | \sigma \neq 0, |\sigma| \leq n + 1\}$$

i.e., $S^n = n$-skeleton of Δ^{n+1}

Theorem.

$$H_k(X^{(n)}) \equiv H_k(X), \text{ for } 0 \leq k \leq n - 1$$

(and there exists a natural surjection $H_n(X^{(n)}) \to H_n(X)$) (note this is not an isomorphism)

Proof. From definition, $C_k(X^{(n)}) \equiv C_k(X)$, $0 \leq k \leq n$

$$C_*(X^{(n)})\ 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$C_*(X)\ C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$H_k(X^{(n)}) \equiv H_k(X) \text{ for } k \leq n-1$$

$$\begin{aligned} H_n(X^{(n)}) &\equiv \ker(\partial_n : C_n(X) \to C_{n-1}(X)) \\ &= Z_n(X) \end{aligned}$$

but $B_n(X^{(n)}) = 0$. In general $B_n(X) \neq 0$. $\square$

As $S^n = (\Delta^{n+1})^{(n)}$, $(n \neq 0, n \geq 1)$ we see that

$$H_k(S^n; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ 0 & 1 \leq k \leq n-1 \end{cases}$$

We now still need to compute $H_n(S^n)$.

2.6 Exact sequences

Definition. Let $U \xrightarrow{f} V \xrightarrow{g} W$ be linear maps. We say sequence is *exact* at V when

$$\ker(g) = \mathrm{im}(f)$$

In general if

$$V_{n+1} \xrightarrow{f_{n+1}} V_n \to \dots \to V_{r+1} \xrightarrow{f_{r+1}} V_r \xrightarrow{f_r} V_{r-1} \to \dots \to V_1 \xrightarrow{f_1} V_0$$

is a sequence of linear maps, we say a sequence is *exact* at V_r when

$$\ker f_r = \mathrm{im} f_{r+1}$$

We say the sequence is *exact* when it is *exact* at each possible V_r.

<u>4 term exact sequence</u>

$$0 \to U \xrightarrow{f} V \to 0$$

is exact if and only if f is an isomorphism.

Proof. The sequence is exact at V, so

$$\mathrm{im}(f) = \ker(V \to 0) = V$$

so f is surjective. The sequence is exact at U, so

$$\ker(f) = \mathrm{im}(0 \to U) = 0$$

so f is injective. Thus f is bijective and so an isomorphism. □

<u>Short exact sequence</u>

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

Exactness here means

1. g is surjective, $\mathrm{im}(g) = \ker(W \to 0)$
2. f is injective, $\ker(f) = \mathrm{im}(0 \to V) = 0$
3. $\ker(g) = \mathrm{im}(f)$

Example. <u>Kernel-rank theorem</u>
Suppose we have the exact sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

if U, V, W are finite dimensional, then

$$\dim(V) = \dim(U) + \dim(W)$$

by the kernel-rank theorem. To see this, note that

$$\mathrm{im}(g) = W$$

by exactness.

$$\dim\ker(g)+\dim\operatorname{im}(g)=\dim(V) \implies \dim\ker(g)+\dim(W)=\dim(V)$$

$$\ker(g)=\operatorname{im}(f)\cong U$$

(since f is injective) and so

$$\dim\ker(g)=\dim(U)$$

so

$$\dim(U)+\dim(W)=\dim(V)$$

Example.

$$H_k(X)=Z_k(X)/B_k(X)$$

$$0\to B_k(X)\hookrightarrow Z_k(X)\to H_k(X)\to 0$$

is a short exact sequence, $z\mapsto[z]$, $z+B_k(X)$, so

$$\dim H_k(X)=\dim Z_k(X)-\dim B_k(X)$$

Exact sequences of chain complexes Let A_*, B_*, C_* be chain complexes and

$$f: A_*\to B_*,\ g: B_*\to C_*$$

Consider the following sequence of chain maps

$$0\to A_*\xrightarrow{f} B_*\xrightarrow{g} C_*\to 0$$

so for each n we have a sequence of linear maps

$$0\to A_n\xrightarrow{f_n} B_n\xrightarrow{g_n} C_n\to 0$$

We say that this is exact when for each n, this sequence is exact.

3 Mayer-Vietoris Theorem

3.1 Algebraic Mayer-Vietoris Theorem

Theorem (Algebraic Mayer-Vietoris Theorem)**.** Suppose

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

is an exact sequence of chain complexes, then there exists a long exact sequence of the following type

$$\to H_{n+1}(A) \xrightarrow{i_*} H_{n+1}(B) \xrightarrow{p_*} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{i_*} H_n(B) \dots$$

$$\to H_1(A) \xrightarrow{i_*} H_1(B) \xrightarrow{p_*} H_1(C) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(B) \xrightarrow{p_*} H_0(C) \to 0$$

with δ called the connecting homomorphism, where in our case, $A_n = B_n = C_n = 0$ for $n < 0$, i.e.,

$$\begin{aligned} A_* &= (A_n, \partial_n),\ A_n = 0, n < 0 \\ B_* &= (B_n, \partial_n),\ B_n = 0, n < 0 \\ C_* &= (C_n, \partial_n),\ C_n = 0, n < 0 \end{aligned}$$

The connecting homomorphisms have the following *naturality property*:

Suppose we have the following exact sequences of chain complexes,

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

$$0 \to A'_* \xrightarrow{i} B'_* \xrightarrow{p} C'_* \to 0$$

and suppose the following commutes,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

(where α, β, γ) are chain maps). Compare the two long exact sequences,

$$\begin{array}{ccccccccc}
H_{n+1}(B) & \xrightarrow{p_*} & H_{n+1}(C) & \xrightarrow{\delta} & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{p_*} & H_n(0) \\
\downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} \\
H_{n+1}(B') & \xrightarrow{q_*} & H_{n+1}(C') & \xrightarrow{\delta'} & H_n(A') & \xrightarrow{j_*} & H_n(B') & \xrightarrow{q_*} & H_n(0)
\end{array}$$

this diagram commutes.

The Algebraic Mayer-Vietoris Theorem implies the *Geometric* Mayer-Vietoris Theorem.

3.2 Subcomplexes

Let $X = (V_X, \mathcal{S}_X)$, $Y = (V_Y, \mathcal{S}_Y)$ be simplicial complexes. Then we say that Y is a *subcomplex* of X if,

1. $V_Y \subset V_X$
2. $\mathcal{S}_Y \subset \mathcal{S}_X$

Proposition.

1. Let X_1, X_2 be subcomplexes of Z. Then $(V_{X_1} \cup V_{X_2}, \mathcal{S}_{X_1} \cup \mathcal{S}_{X_2})$ is also a subcomplex of Z. This is called the union $X_1 \cup X_2$.
2. $(V_{X_1} \cap V_{X_2}, \mathcal{S}_{X_1} \cap \mathcal{S}_{X_2})$ is also a subcomplex of Z. This is called the intersection $X_1 \cap X_2$.

We are interested in the case $Z = X_1 \cup X_2$.

Definition. Let Δ, Δ' be chain complexes. $\Delta = (\Delta_n, \partial_n)$, $\Delta' = (\Delta'_n, \partial'_n)$. Then the *direct sum*:

$$\Delta \oplus \Delta' = \left(\Delta \oplus \Delta', \begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix} \right)$$

$$\begin{pmatrix} \partial_n & 0 \\ 0 & \partial'_n \end{pmatrix} \begin{pmatrix} \partial_{n+1} & 0 \\ 0 & \partial_{n'+1} \end{pmatrix} = \begin{pmatrix} \partial_n \partial_{n+1} & 0 \\ 0 & \partial'_n \partial'_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3.3 The Geometric Mayer-Vietoris Theorem: Chain Version

Theorem. Suppose X is a simplicial complex decomposed as a union $X = X_+ \cup X_-$, where X_+, X_- are subcomplexes. Then there exists an exact sequence of chain complexes as follows,

$$0 \to C_*(X_+ \cap X_-) \xrightarrow{i} C_*(X_+ \oplus X_-) \xrightarrow{p} C_*(X) \to 0$$

If we apply the algebraic Mayer-Vietoris Theorem, we get the homological version, namely the long exact sequence,

$$H_{n+1}(X_+) \oplus H_{n+1}(X_-) \to H_{n+1}(X) \xrightarrow{\delta} H_n(X + \cap X_-)$$

$$\to H_n(X_+) \oplus H_n(X_-) \to H_n(X) \xrightarrow{\delta} H_{n-1}(X_+ \cap X_-)$$

and finishes

$$\xrightarrow{\delta} H_1(X_+ \cap X_-) \to H_1(X_+) \oplus H_1(X_-) \to H_1(X) \xrightarrow{\delta} H_0(X_+ \cap X_-)$$

$$\to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0$$

Example. Let $S^n =$ standard model of n-sphere,

$$S^n = (\Delta^{n+1})^{(n)}$$

We've shown for $n \geq 1$,

$$H_r(S^n; \mathbb{F}) = \begin{cases} \mathbb{F} & r = 0 \\ 0 & 0 < r < n \\ ? & r = n \\ 0 & n < r \end{cases}$$

We've shown that $H_2(S^2; \mathbb{F}) = \mathbb{F}$.

Proposition. For $n \geq 2$, S^n can be written as $S^n = X_+ \cup X_-$ where $X_+ \cap X_- = S^{n-1}$ and X_+, X_- are *cones*.

$\Delta^{n+1} = (\{0, 1, \ldots, n{+}1\}, \{\text{all non-empty subsets of } \{0, 1, \ldots, n+1\}\})$

$S^n = (\{0, 1, \ldots, n{+}1\}, \{\text{all } \textit{proper} \text{ non-empty subsets of } \{0, 1, \ldots, n+1\}\})$

In particular every non-empty subset of $\{0, 1, \ldots, n\}$ is a simplex of S^n so,

1. $\Delta^n \subset S^n$. But as $S^{n-1} \subset \Delta^n$, then,
2. $S^{n-1} \subset S^n$ (note that $n + 1 \notin V_{S^{n-1}}$) and,
3. Taking $n + 1$ to be the cone point $C(S^{n-1}) \subset S^n$. ($C(S^{n-1})$ is sometimes called the *Witches hat*)
4. $$S^n = \Delta^n \cup C(S^{n-1})$$ $$S^{n-1} = \Delta^n \cap C(S^{n-1})$$

So we can write,

$$S^n = X_+ \cup X_-, \text{ where}$$

$$X_+ = C(S^{n-1})$$

$$X_- = \Delta^n$$

$$X_+ \cap X_- = S^{n-1}$$

Corollary. $H_n(S^n; \mathbb{F}) \cong \mathbb{F}$ for all $n \geq 2$.

Proof. By induction on n. We know this is true for $n = 2$. Suppose we've proven the hypothesis for $n - 1$ and consider the exact sequence,

$$H_n(X_+) \oplus H_n(X_-) \longrightarrow H_n(S^n) \xrightarrow{\delta} H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(X_+) \oplus H_{n-1}(X_-)$$

$$0 \oplus 0 \longrightarrow H_n(S^n) \xrightarrow{\cong} H_{n-1}(S^{n-1}) \longrightarrow 0 \oplus 0$$

which is isomorphic by the very short exact sequence. $\square$

3.4 External and internal sum

Let W be a vector space over $\mathbb{F}$ and suppose we have two vector subspaces of W, say U and V.

Definition. External sum (coproduct)

$$U \oplus V = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in U, v \in V \right\}$$

$U \oplus V$ is a vector space. We define sums, scalar multiplication and zero as follows,

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ v_1 + v_2 \end{pmatrix}$$

$$\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

If U and V have finite dimensions, then

$$\dim(U \oplus V) = \dim(U) + \dim(V)$$

where U, V are subspaces of W.

Definition. Internal sum

$$U + V = \{u + v : u \in U, v \in V\}$$

Note that $U + V$ is a vector subspace of W.

What is the relationship between $U + V$ and $U \oplus V$? There is an exact sequence

$$\to U \oplus V \xrightarrow{\mu} U + V$$

$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = u + v$$

μ is linear and surjective by the definition of $U + V$.

Proposition.

$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = 0 \iff u + v = 0 \iff v = -u,\ u \in U,\ v \in V \text{ so } v \in U \cap V$$

We get an exact sequence,

$$0 \to U \cap V \xrightarrow{i} U \oplus V \xrightarrow{\mu} U + V \to 0$$

$$i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

As a consequence,

$$\dim(U \cap V) + \dim(U + V) = \dim(U) + \dim(V)$$

Theorem. (Chain version of the Geometric Mayer-Vietoris Theorem) Let $X = X_+ \cup X_-$ be the union of subcomplexes. For each n, there exists an exact sequence,

$$0 \to C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \to 0$$

$$\mu \begin{pmatrix} x \\ y \end{pmatrix} = x + y,\ i(u) = \begin{pmatrix} u \\ -u \end{pmatrix}$$

Proof. $C_n(X)$ has basis $\{[v_0, v_1, \ldots, v_n] : [v_0, \ldots, v_n] \in \mathcal{S}_X\}$

$$\mathcal{S}_X = \mathcal{S}_{X_+} \cup \mathcal{S}_{X_-}$$

$$C_n(X_+) \oplus C_n(X_-) \to C_n(X) \to 0$$

$$\begin{pmatrix} e \\ f \end{pmatrix} \mapsto e + f$$

The map is surjective because a basis element of $C_n(X)$ is either in $C_n(X_+)$ or $C_n(X_-)$. As a basis for $\ker(\mu)$, we have

$$\begin{pmatrix} [v_0, \ldots, v_n] \\ -[v_0, \ldots, v_n] \end{pmatrix}$$

where $\{v_0, \dots, v_n\} \subset \mathcal{S}_{X_+} \cap \mathcal{S}_{X_-} = \mathcal{S}_{X+\cap X_-}$ so we have an exact sequence,

$$0 \to C_n(X_+ \cap X_-) \xrightarrow{i} C_n(X_+) \oplus C_n(X_-) \xrightarrow{\mu} C_n(X) \to 0$$

This is an exact sequence of chain complexes because boundary formula is the same in every case. □

Corollary. (of the Geometric Mayer-Vietoris Theorem) Let X be a finite simplicial complex. Then,

$$\dim H_0(X; \mathbb{F}) = \{\text{number of connected components of } X\}$$

Proof. Let n be the number of connected components. This is true for $n = 1$. Suppose this is true for $n - 1$, and X has n connected components $X_1, X_2, \dots, X_n$. Put

$$X_- = X_1 \cup X_2 \cup \ldots \cup X_{n-1}$$

$$X_+ = X_n$$

$$X_+ \cup X_- = X,\ X_+ \cap X_- = \emptyset (\text{ by definition})$$

Look at the following

$$H_0(X_+ \cap X_-) \to H_0(X_+) \oplus H_0(X_-) \to H_0(X) \to 0$$

(note that $H_0(X_+ \cap X_-) = 0)$). So

$$\dim H_0(X) = \dim H_0(X_+) + \dim H_0(X_-) = 1 + n - 1 = n$$

□

Example.

$$S^0 = 0\text{-sphere} = 2 \text{ distinct points } \{-1, +1\}$$

So $H_0(S^0; \mathbb{F}) \cong \mathbb{F} \oplus \mathbb{F}$

$$H_n(S^0; \mathbb{F}) = 0,\ n \neq 0 \text{ (no higher simplices)}$$

On the other hand, the standard model of S^1 is,

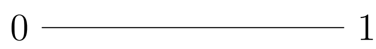
$$V_{S^1} = \{0, 1, 2\}$$

$$\mathcal{S}_{S^1} = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$$

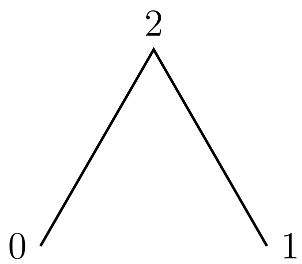
Proposition.

$$H_n(S^1; \mathbb{F}) = \begin{cases} \mathbb{F} & n = 0 \\ \mathbb{F} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Proof. Decompose $S^1 = X_1 \cup X_+$, where X_- is equal to



and X_+ is equal to



i.e.,

$$X_- = C(0),\ X_+ = \text{cone on } S^0 = \{\{0\}, \{1\}\}$$

$X_+ \cap X_- = S^0$. Use the Mayer-Vietoris Theorem, so,

$$H_1(X_+) \oplus H_1(X_-) \to H_1(S^1) \to H_0(S^0) \to H_0(X_+) \oplus H_0(X_-) \to H_0(S^1)$$

$$0 \to H_1(S) \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F} \oplus \mathbb{F} \to \mathbb{F}$$

$\dim(H_1(S^1)) = 1$ follows from Whitehead's lemma. $\square$

Lemma. Let

$$0 \to V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \ldots \to V_1 \xrightarrow{f_1} V_0 \to 0$$

be an exact sequence of finite dimensional vector spaces. Then,

$$\sum_{n \geq 0} \dim(V_{2n}) = \sum_{n \geq 0} \dim(V_{2n+1})$$

Proof. Let $P(n)$ denote the induction hypothesis on n.

$$0 \to V_1 \to V_0 \to 0$$

then $P(1)$ holds. The sequence is exact which implies $V_1 \cong V_0$. Now suppose we have an exact sequence,

$$0 \to V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$$

then by the kernel-rank theorem, this implies that

$$\dim(V_0) + \dim(V_2) = \dim(V_1)$$

and so $P(2)$ is true. For $n = 3$,

$$0 \to V_3 \xrightarrow{f_3} V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \to 0$$

is an exact sequence. Put $K = \ker(f_1) = \operatorname{im}(f_2)$ so we have two exact sequences

$$0 \to K \to V_1 \to V_0 \to 0$$
$$0 \to V_3 \to V_2 \to K \to 0$$

so by the kernel-rank theorem,

$$\dim V_0 + \dim V_2 = \dim V_1 + \dim V_3$$

Now we prove that $P(2n) \implies P(2n+1)$. Suppose that $P(2n)$ is true, and take the following exact sequence,

$$0 \to V_{2n+1} \xrightarrow{f_{2n+1}} V_{2n} \xrightarrow{f_{2n}} V_{2n-1} \to \ldots \to V_0 \to 0$$

Split the sequence and define $f = \operatorname{im}(f_{2n}) = \ker(f_{2n-1})$. Now we have two exact sequences,

$$0 \to V_{2n+1} \to V_{2n} \to f \to 0$$

and

$$0 \to f \to V_{2n-1} \to \ldots \to V_0 \to 0$$

By $P(2n)$,

$$\dim(f) + \sum_{r=0}^{n-1} \dim(V_{2r}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

and $\dim(f) = \dim(V_{2n}) - \dim(V_{2n+1})$. Substitute this into the previous expression and we get,

$$\sum_{r=0}^{n} \dim(V_{2r}) - \dim(V_{2n+1}) = \sum_{r=0}^{n-1} \dim(V_{2r+1})$$

This proves that $P(2n) \implies P(2n+1)$. To prove that $P(2n+1) \implies P(2n+2)$, take

$$0 \to V_{2n+2} \to V_{2n+1} \to V_{2n} \to \dots$$

Split the exact sequence as before and proceed as before. (Set $f = \mathrm{im}(f_{2n+1}) = \ker(f_{2n})$) $\square$

Lemma. (Five lemma) Suppose we have a commutative diagram of abelian groups and homomorphisms,

$$\begin{array}{ccccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4
\end{array}$$

in which both rows are exact, and $f_0,\ f_1,\ f_3, f_4$ are isomorphisms. Then f_2 is also an isomorphism.

Proof. We first show that f_2 is injective. Suppose $x \in A_2$ such that $f_2(x) = 0$. We want to show that $x = 0$.

$$\beta_2 f_2(x) = 0 \implies f_3\alpha_2(x) = 0$$

but f_3 is an isomorphism, which implies that $\alpha_2(x) = 0$. But then $x \in \ker(\alpha_2) = \mathrm{im}(\alpha_2)$, so $x = \alpha_1(y)$ for some $y \in A_1$.

$$f_2\alpha_1(y) = 0 \implies \beta_1 f_1(y) = 0$$

so $f_1(y) \in \ker(\beta_1) = \operatorname{im}(\beta_0)$. Thus there exists $w \in \beta_0$ such that $\alpha_0(w) = f_1(y)$. But f_0 is surjective so write

$$w = f_0(z),\, \beta_0 f_0(z) = f_1(y) \implies f_1\alpha_0(z) = f_1(y)$$

but now f_1 is an isomorphism so $y = \alpha_0(z)$, $x = \alpha_1(y) = \alpha_1\alpha_0(z)$. By exactness, $\alpha_1\alpha_0 = 0$, so $x = 0$

Now we show that f_2 is surjective. Take $b \in \beta_2$. We want to find $a \in A_2$ such that $f_2(a) = b$. Now, $\beta_2(b) \in B_3$. f_2 is an isomorphism so choose $x \in A_3$ so that

$$f_3(x) = \beta_2(b) \implies \beta_3 f_3(x) = \beta_3\beta_2(b)$$

However by exactness, $\beta_3\beta_2 = 0$, so $\beta_3 f_3(x) = 0 \implies f_4\alpha_3(x) = 0$. Now f_4 is an isomorphism thus $\alpha_3(x) = 0$, $x \in \ker(\alpha_3) = \ker(\alpha_2)$. Now there exists $y \in A_2$ such that $\alpha_2(y) = x$. Consider $b - f_2(y)$. Then

$$\beta_2(b - f_2(y)) = \beta_2(b) - \beta_2 f_2(y) = \beta_2(b) - f_3\alpha_2(y) = \beta_2(b) - f_3(x) = 0$$

Thus $b - f_2(y) \in \ker(\beta_2) = \ker(\beta_1)$ so there exists $w \in \beta_1$ such that $\beta_1(w) = b - f_2(y)$. f_1 is an isomorphism implies that there exists $z \in A_1$ such that $f_1(z) = w$. So

$$\beta_1 f_1(z) = \beta_1(w) = b - f_2(y)$$

$$f_2\alpha_1(z) = b - f_2(y) \implies b = f_2(y + \alpha_1(z))$$

Let $a = y + \alpha_1(z)$ which implies $b = f_2(a)$. Thus f_2 is surjective. $\square$

4 Subdivision

We will now show that homology is invariant under 'subdivision'. We first have to illustrate what 'subdivision' means.

Take for example Δ^2 (the triangle), and add a point at its barycenter, adding edges from the barycenter to each three of the vertices of Δ^2. We end up with an additional point (vertex), two additional regions and three additional edges. This is an example of an easy subdivision.

Definition. Let $X = (V_X, \mathcal{S}_X)$ be a finite simplicial complex, and let $\tau \in \mathcal{S}_X$. $\hat{\tau}$ will denote the subcomplex of X determined by τ.

$$V_{\hat{\tau}} = \tau, \;\; \mathcal{S}_{\hat{\tau}} = \{p \in \mathcal{S}_X, \, p \subset \tau\}$$

We say that $\sigma \in \mathcal{S}_X$ is *principal* (or maximal) when σ is not contained properly in any other simplex.

Proposition. If $\sigma_1, \ldots, \sigma_N$ are the principal simplices of X then

$$X = \hat{\sigma_1} \cup \hat{\sigma_2} \cup \ldots \cup \hat{\sigma_N}$$

4.1 Subdivision at a principal simplex

Let σ be a principal simplex of X and let $\sigma_1, \ldots \sigma_N$ be the remaining principal simplices such that

$$X = \hat{\sigma} \cup \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

Put $X_+ = \hat{\sigma}$, $X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$. Then $X = X_+ \cup X_-$ and $X_+ \cap X_- \subset \partial \hat{\sigma}$ (boundary of $\hat{\sigma}$)

Definition.

$$Sd(X, \sigma) = C(\partial \sigma) \cup \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

i.e.,

$$Sd(X, \sigma) = X'_+ \cup X'_-$$

where X'_+ is the cone on the boundary of σ and

$$X'_- = X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

and

$$X'_+ \cap X'_- = X_+ \cap X_-$$

Taking our Δ^2 example earlier, letting $\sigma = \Delta^2$, $Sd(\Delta^2, \sigma)$ is exactly the resulting simplex we get by performing our subdivision earlier.

4.2 Squash mapping

Let σ be an n-simplex and consider $C(\partial\sigma)$. We construct simplicial mappings $C(\partial\sigma) \to \sigma$ as follows,

$$Sq|_{\partial\sigma} = \mathrm{id}_{\partial\sigma}$$

$$Sq(*) = \text{some (arbitrarily chosen) vertex in } \partial\sigma$$

where $*$ is our cone point.

Proposition. $Sq : H_k(C(\partial\sigma)) \to H_k(\sigma)$ is an isomorphism for all k.

Proof. $C(\partial\sigma)$ and σ are both cones, so $H_k(C(\partial\sigma)) = H_k(\sigma) = 0$ if $k > 0$. For $k = 0$, any vertex V in $C(\partial\sigma)$ gives a basis $[v]$ for $H_0(C(\partial\sigma))$ (any two vertices differ by a boundary). Likewise, any vertex w in σ gives basis element $[w]$ in $H_0(\sigma)$ and $Sq([v]) = [w]$, so now

$$Sq : H_0(C(\partial\sigma)) \xrightarrow{\cong} H_0(\sigma)$$

$\square$

Theorem. Let K be a finite complex. Let σ be a principal complex, and let $\sigma_1, \ldots, \sigma_N$ be the remaining principal simplices and define an extended squash map $Sq : Sd(X, \sigma) \to X$ by

$$Sq : C(\partial\sigma) \to \sigma \text{ is a squash mapping}$$

$$Sq : \sigma_i \to \sigma_i \text{ identity } i = 1, \ldots, N$$

Then $Sq : H_k(Sd(X, \sigma)) \to H_k(X)$ is an isomorphism for all k.

Proof. Put

$$X_+ = \hat{\sigma},\ X'_+ = C(\partial\sigma)$$

$$X'_- = X_- = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N}$$

so $X'_+ \cap X'_- = X_+ \cap X_-$ and $Sq : X'_- \to X_-$ is the identity. Consider the Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
H_n(X'_+ \cap X'_-) & \longrightarrow & H_n(X'_+) \oplus H_n(X'_-) & \longrightarrow & H_n(Sd(X,\sigma)) & \longrightarrow & H_{n-1}(X'_+ \cap X'_-) & \longrightarrow & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle M} & & \Big\downarrow{\scriptstyle Sq} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle M} \\
H_n(X_+ \cap X_-) & \longrightarrow & H_n(X_+) \oplus H_n(X_-) & \longrightarrow & H_n(X) & \longrightarrow & H_{n-1}(X_+ \cap X_-) & \longrightarrow & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{array}$$

where $M = \begin{pmatrix} Sq & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. id is clearly an isomorphism, as well as M, since $Sq : H_n(X'_+) \to H_n(X_+)$ is an isomorphism. By the five lemma, Sq is an isomorphism. $\square$

We have now shown that if $Sd(X,\sigma)$ is the subdivision of X at a principal simplex, then $H_*(Sd(X,\sigma)) \cong H_*(X)$. Now we have to show that this also holds for non-principal simplices.

4.3 Subdivision at a non-principal simplex

We first describe an example of a non-principal simplex. Take Δ^2. Then take $\{0,1\}$. This is contained within $\{0,1,2\}$, hence this is a non-principal simplex. We wish to perform subdivisions at simplices such as these.

Definition (Join)**.** Let $K = (V_K, \mathcal{S}_K)$ and $L = (V_L, \mathcal{S}_L)$ be simplicial complexes such that $V_K \cap V_L = \emptyset$. Define

$$K * L = (V_K \cup V_L,\, \mathcal{S}_K \cup \mathcal{S}_L \cup \{p \cup \tau, p \in \mathcal{S}_K, \tau \in \mathcal{S}_L\}$$

A special case is where $K =$ point, so then $K * L = C(L)$.

Proposition.

$$\Delta^{m+n+1} \cong \Delta^m * \Delta^n$$

Proof. Vertex set of Δ^{m+n+1} is

$$\{0,\dots,m+n+1\} = \{0,\dots,m\} \cup \{m+1,\dots,m+n+1\}$$

There is a 1-1 correspondence between the last set and

$$\{0,\dots,n\}$$

so if we take as our model of Δ^n the vertex set $\{m+1, \dots, m+n+1\}$ and simplices to be all the non-empty subsets, we get $\Delta^{m+n+1} = \Delta^m * \Delta^n$ (the dimension goes up by 1). $\square$

Note also that $S^m * S^n \cong S^{m+n+1}$. If k is a single point pt, then $C(L) = \{pt\} * L$.

Join is associative. If K, L, M are simplicial complexes with no vertices in common, then

$$(K * L) * M \equiv K * (L * M)$$

Corollary. If K, L are disjoint complexes, then $C(K) * L \cong C(K * L)$

So the join of a cone to anything is a cone.

4.4 Star and Link

Definition (Star neighbourhood)**.** Let τ be a simplex of X, and let $\sigma_1, \dots, \sigma_N$ be the principal simplices which contain τ. Then

$$St(\tau, X) = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_N} = \textit{star neighbourhood} \text{ of } \tau \text{ in } X$$

Definition (Link)**.** Let X be a simplicial complex and ρ, τ be simplices of X such that $\rho \cap \tau = \emptyset$. We say that ρ is joinable to τ in X when $\rho \cup \tau = p * \tau$. The *link* of τ in X, $Lk(\tau, X)$ consists of all these simplices of ρ of X such that $\rho \cap \tau = \emptyset$ and $\rho \cup \tau$ is a simplex, i.e., ρ is joinable to τ.

$$Lk(\tau, X) = \{\rho \in X \mid \rho \cap \tau = \emptyset, \rho \cup \tau \in \mathcal{S}_X\}$$

Proposition. If τ is a simplex of X, then $St(\tau, X) = \hat{\tau} * Lk(\tau, X)$

Proof. The case where τ is principal is empty here. So suppose τ is not principal. Let σ be a principal simplex with $\tau \subset \sigma$. Write

$$\tau = \{v_0, \dots, v_m\} \;\; m < n$$

$$\sigma = \{v_0, \dots, v_m, v_{m+1}, \dots, v_n\}$$

Put $\rho = \{v_{m+1}, \ldots, v_n\}$ so then

$$\sigma = \tau * \rho$$

Do this for every principal simplex which contains τ. Each $\sigma_i = \tau * \rho_i$ for some ρ_i, so

$$\bigcup \sigma_i = \tau * (\cup \rho_i) = \tau * Lk(\tau, X)$$

$\square$

Definition (Subdivision at a non-principal simplex)**.** Let X be a finite simplicial complex, and τ a non-principal simplex. Let $\sigma_1, \ldots, \sigma_m$ be the principal simplices which contain τ. Let $\sigma_{m+1}, \ldots, \sigma_N$ be the remaining principal simplices. Put

$$X_+ = \hat{\sigma_1} \cup \ldots \cup \hat{\sigma_m} = St(\tau, X)$$

$$X_- = \hat{\sigma_{m+1}} \cup \ldots \cup \hat{\sigma_N}$$

$$X = X_+ + X_- \ \ (X_+ \cap X_- \cap \tau \subset \partial\sigma)$$

and put

$$X'_+ = C(\partial\tau) * Lk(\tau, X)$$

$$X'_- = X_-$$

Define

$$Sd(X, \tau) = X'_+ \cup X'_-$$

$$Sd = (C(\partial\tau) * Lk) \cup X'_-$$

We have $Sq : C(\partial\tau) \to \tau$. Extend by identity to $Sq : C(\partial\tau) * Lk \to \tau * Lk$ by identity on Lk. Extend again by identity on $X'_- = X_-$, $Sq : Sd(X, \tau) \to X$

Proposition. $Sq : Sd(X, \tau) \to X$ induces an isomorphism on homology.

Proof.

$$\begin{tikzcd}
H_n(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n(X'_+) \oplus H_n(X'_-) \arrow[r] \arrow[d, "M"] & H_n(Sd(X,\tau)) \arrow[r] \arrow[d, "Sq"] & H_{n-1}(X'_+ \cap X'_-) \arrow[r] \arrow[d, "\mathrm{id}"] & H_{n-1}(X'_+) \oplus H_{n-1}(X'_-) \arrow[d, "M"] \\
H_n(X_+ \cap X_-) \arrow[r] & H_n(X_+) \oplus H_n(X_-) \arrow[r] & H_n(X) \arrow[r] & H_{n-1}(X_+ \cap X_-) \arrow[r] & H_{n-1}(X_+) \oplus H_{n-1}(X_-)
\end{tikzcd}$$

where $M = \begin{pmatrix} Sq & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. By the five lemma, Sq induces an isomorphism.

$\square$

So now we've proved the following,

Theorem. Homology is invariant under subdivision.

We now have a functor H_n which takes simplicial complexes to vector spaces, and simplicial maps to linear maps, e.g., if

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

then

$$\begin{tikzcd}
H_n(X) \arrow[r, "H_n(f)"] \arrow[rr, bend right, "H_n(g\circ f)"'] & H_n(Y) \arrow[r, "H_n(g)"] & H_n(Z)
\end{tikzcd}$$

<ins>Properties of functors:</ins>

1. $H_n(g \circ f) = H_n(g) \circ H_n(f)$
2. $H_n(\mathrm{id}) = \mathrm{id}_{H_n}$ i.e.,

$$\mathrm{id} : X \to X,\ H_n(\mathrm{id}) : H_n(X) \to H_n(X)$$

As a consequence, if $f : X \to Y$ is an isomorphism, then $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism.

Proof. If $g = f^{-1} : Y \to X$, $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$ then

$$H_n(g) \circ H_n(f) = \mathrm{id},\ H_n(f) \circ H_n(g) = \mathrm{id}$$

so

$$H_n(g) = H_n(f)^{-1}$$

$\square$

But we have established a stronger property, that is, H_n is invariant under subdivision, i.e., if Y subdivides X, then $H_n(Y) \cong H_n(X)$.

Definition. Let X, Y be simplicial complexes. We say that X, Y are *combinatorially equivalent* (written $X \sim Y$) if and only if there exists a finite sequence $(X_r)_{0 \le r \le N}$ of complexes X_r such that $X_0 = X$, $X_N = Y$ and for each r, $0 \le r \le N-1$, either X_{r+1} is a subdivision of X_r or X_r is a subdivision of X_{r+1}.

Corollary. If $X \sim Y$ then $H_n(X) \cong H_n(Y)$.

So we won't worry too much about how we triangulate things.

Consider $S^2 = \partial\Delta^3$. This is the minimal model of S^2. The dodecahedron is also a model of S^2 obtained from the minimal model by a sequece of subdivisions, hence for any model of S^2,

$$H_k(S^2; \mathbb{F}) \begin{cases} \mathbb{F} & k = 0 \\ 0 & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k > 2 \end{cases}$$

We note that the usual definition of S^2 is given by

$$S^2 = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 + x_3^2 = 1 \}$$

which is harder to compute homology with.

Now we define $S^1(n)$ to be the model of the circle S^1 with n-subdivision points ($n \geq 3$), so

$$S^1(n) \sim S^1(m) \ \forall m, n \geq 3$$

so for example, $S^1(3)$ is the triangle, $S^1(4)$ is the square, $S^1(5)$ the pentagon, and so on.

5 Orientation Theorem

Definition (Orientability)**.** We say that a surface Σ is orientable if and only if it is possible to orient each 2-simplex in such a way that every 1-simplex receives the opposite orientations from its containing 2-simplices.

5.1 Euler characteristic

Definition. Let $X = (V_X, \mathcal{S}_X)$ be a finite simplicial complex. Let c_n be the number of n-simplices of X,

$$c_n(X) = c_n = \text{no. of } n\text{-simplices of } X$$

we define

$$\chi_{\text{geom}}(X) = \sum_n (-1)^n c_n(X)$$

This is known as the *geometric* Euler characteristic.

Put $h_n^{\mathbb{F}}(X) = \dim H_n(X; \mathbb{F}) (= h_n)$, and define

$$\chi_{\text{hom}}^{\mathbb{F}}(X) = \sum_n (-1)^n h_n^{\mathbb{F}}(X)$$

This is known as the *homological* Euler characteristic.

We will show that

Theorem.

$$\chi^{\mathbb{F}}_{\text{hom}}(X) = \chi_{\text{geom}}(X)$$

In particular $\chi^{\mathbb{F}}_{\text{hom}}$ is independent of $\mathbb{F}$, so we'll ignore $\mathbb{F}$.

Proof. Fix a field $\mathbb{F}$. $c_n = c_n(X) =$ no. of n-simplices of X

$$c_n = \dim C_n(X; \mathbb{F})$$

so put $h_n = \dim H_n(X; \mathbb{F})$ and look at the sequence

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

$$H_n(X) = \ker \partial_n / \text{im } \partial_{n+1} = Z_n(X)/B_n(X)$$

Put $z_n = \dim \ker \partial_n$, $b_n = \dim \text{im}(\partial_{n+1})$ so

$$z_n = h_n + b_n$$

However by the kernel-rank theorem,

$$c_n = z_n + b_{n-1}$$

hence

$$c_n = h_n + b_n + b_{n-1}$$

Now take the alternating sum

$$\sum_n (-1)^n c_n = \sum_n (-1)^n h_n + \sum_n (-1)^n (b_n + b_{n-1})$$

The last term on the RHS evaluates to 0, and recognising what the other two sums are, we have

$$\chi^{\mathbb{F}}_{\text{hom}}(X) = \chi_{\text{geom}}(X)$$

□

As $H_*(X; \mathbb{F})$ is invariant under subdivision, it follows that $\chi_{\text{geom}}(X)$ is the same as well, so from now on, we will usually just write $\chi(X)$.

Definition (Connected sum of surfaces)**.** Let Σ, Σ' be surfaces. Let σ be a 2-simplex in Σ, σ' be a 2-simplex in Σ'. Let Σ_0 be the complex obtained from Σ by removing σ. Likewise for Σ_0'. Formally,

$$\Sigma \# \Sigma' = \Sigma_0 \bigcup_{\partial = \partial'} \Sigma_0'$$

(i.e., we glue the boundaries of Σ_0 and Σ_0' together.)

Proposition.

$$\chi(\Sigma \# \Sigma') = \chi(\Sigma) + \chi(\Sigma') - 2$$

Example. Some examples of orientable surfaces, are the following,

$$\Sigma_+^0 = S^2$$

$$\Sigma_+^1 = T^2$$

$$\Sigma_+^2 = T^2 \# T^2$$

$$\Sigma_+^g = T^2 \# \ldots \# T^2 \text{ (g times)}$$

These are orientable surfaces of genus g.

Proposition. $\chi(\Sigma_+^g) = 2 - 2g$

Proof. We proceed by induction on g. This is clearly true for $g = 0$ ($\chi(S^2) = 2$) and $g = 1$ ($\chi(T^2) = 0$). Suppose this is true for some $g \geq 1$, then,

$$\Sigma_+^{g+1} = \Sigma_+^g \# \Sigma_+^1$$

$$\begin{aligned} \chi(\Sigma_+^{g+1}) &= \chi(\Sigma_+^g) + \chi(\Sigma_+^1) - 2 \\ &= 2 - 2g + 0 - 2 \\ &= 2 - 2(g+1) \end{aligned}$$

□

Corollary. Over any field $\mathbb{F}$,

$$H_k(\Sigma_+^g; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^{2g} & k = 1 \\ \mathbb{F} & k = 2 \\ 0 & k > 2 \end{cases}$$

Example. The following are examples of non-orientable surfaces,

$$\Sigma_-^0 = \mathbb{R}P(2)$$

$$\Sigma_-^1 = \mathbb{R}P(2)\#\mathbb{R}P(2) \;\; (\cong) \text{ Klein bottle)}$$
$$\Sigma_-^g = \mathbb{R}P(2)\#\ldots\#\mathbb{R}P(2) \text{ (g+1 times)}$$

Proposition. $\chi(\Sigma_-^g) = 1 - g$

Proof. We proceed by induction.

$$\Sigma_-^0 = \mathbb{R}P(2),\ \chi(\Sigma_-^0) = 1$$

hence is true for $g = 0$. Suppose this is true for $g \geq 0$, then,

$$\Sigma_-^{g+1} = \Sigma_-^g\#\mathbb{R}P(2)$$

and so on. Then,

$$\begin{aligned} \chi(\Sigma_-^{g+1}) &= \chi(\Sigma_-^g) + \chi(\mathbb{R}P(2)) - 2 \\ &= (1-g) + 1 - 2 \\ &= 1 - (g+1) \end{aligned}$$

□

Proposition. If $1 + 1 \neq 0$ in $\mathbb{F}$, then,

$$H_k(\Sigma_-^g; \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \\ \mathbb{F}^g & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Proof. $H_2(\Sigma^g_-;\mathbb{F}) = 0$ by the Orientation theorem. We know

$$H_0(\Sigma^g_-;\mathbb{F}) = \mathbb{F} \text{ (connected)}$$

Then,

$$\begin{aligned} \chi_{\text{hom}}(\Sigma^g_-) &= h_0^{\mathbb{F}} - h_1^{\mathbb{F}} + h_2^{\mathbb{F}} \\ 1 - g &= 1 - h_1^{\mathbb{F}} + 0 \\ h_1^{\mathbb{F}} &= g \end{aligned}$$

hence

$$H_1(\Sigma^g_-;\mathbb{F}) = \mathbb{F}^g$$

□

Proposition. If $1+1=0$ in $\mathbb{F}$, then,

$$H_k(\Sigma^g_-;\mathbb{F}) = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F}^{g+1} & k=1 \\ \mathbb{F} & k=2 \end{cases}$$

Proof. $\chi = 1-g$ but now

$$h_0^{\mathbb{F}} = 1,\ h_2^{\mathbb{F}} = 1\ \ (1+1=0)$$

and thus

$$h_1^{\mathbb{F}} = g+1$$

□

Theorem (Classification Theorem for Surfaces)**.** Let Σ be a finite connected surface.

1. If Σ is orientable, then,

$$\Sigma \sim \Sigma^g_+ \text{ for some } g \geq 0$$

2. If Σ is non-orientable, then,

$$\Sigma \sim \Sigma^g_- \text{ for some } g \geq 0$$

The homology groups of the surfaces distinguishes them, i.e.,

$$\Sigma_+^g \sim \Sigma_+^h \iff g = h$$

$$\Sigma_-^g \sim \Sigma_-^h \iff g = h$$

$$\Sigma_+^g \not\sim \Sigma_-^g$$

One non-trivial relation is

$$\mathbb{R}P(2)\#T^2 \sim \mathbb{R}P(2)\#\mathbb{R}P(2)\#\mathbb{R}P(2)$$

Recall that if σ is a simplex of X, then $Lk(\sigma, X)$ is equal to the complex where simplices τ satisfy $\sigma \cap \tau = \emptyset$, where $\sigma \cup \tau$ is a simplex of X.

Definition (Simplicial surface)**.** A *simplicial surface* Σ is a complex in which

$$Lk(v, \Sigma) \cong S^1(N)$$

where v is a vertex of Σ ($N \geq 3$). Recall that $S^1(N)$ is the circle with N subdivision points, for example, $S^1(5)$ is 'the' pentagon.

Observe that in $S^1(N)$, every vertex belongs to exactly *two* 1-simplices.

Proposition. If Σ is a simplicial surface then every 1-simplex lies in exactly two 2-simplices.

Proof. Let $\rho = [v_0, v_1]$ be a 1-simplex. $Lk(v_0, \Sigma) \cong S^1(N)$, $v_1 \in Lk(v_0, \Sigma)$ and v_1 belongs to exactly two 1-simplices, say τ_0, τ_1, so then

$$\tau_0 * \{v_0\},\ \tau_1 * \{v_0\}$$

are the two 2-simplices which contain ρ. $\square$

5.2 Copath

Definition. Let X be a simplicial complex of dimension 2. Let σ, σ' be 2-simplices in X. A *copath* from σ to σ' is a collection of 2-simplices

$$\{\sigma_0, \sigma_1, \dots, \sigma_N\}$$

such that $\sigma_0 = \sigma$, $\sigma_N = \sigma'$ and $\sigma_r \cap \sigma_{r+1}$ is a 1-simplex for $0 \leq r \leq N-1$.

Theorem. If Σ is a connected simplicial surface and σ, σ', $\sigma \neq \sigma'$ are 2-simplices in Σ then there exists a copath $(\sigma_0, \dots, \sigma_N)$ from σ to σ'.

Proof. Consider $\sigma \cap \sigma'$. A priori we have 4 cases

1. $|\sigma \cap \sigma'| = 3$. This is impossible, as this implies $\sigma = \sigma'$
2. $|\sigma \cap \sigma'| = 2$. Put $\rho = \sigma \cap \sigma'$ which is a 1-simplex, where ρ lies in exactly two 2-simplices which are σ, σ' and now (σ, σ') is a copath from σ to σ'
3. $|\sigma \cap \sigma'| = 1$. $\sigma \cap \sigma' = \{v\}$ for some vertex v. Look at $Lk(v, \Sigma)$. Write
$$\sigma = \{v, u, u_0\}$$
$$\sigma' = \{v, w, w_0\}$$
We know $v, w \in Lk(v, \Sigma) \cong S^1(N)$ which is connected. So choose a path in $Lk(v, \Sigma)$ from u to w
$$\xi = (\xi_0, \dots, \xi_N)$$
$$\xi_0 = u, \dots, \xi_N = w$$
Then $[\xi_i, \xi_{i+1}]$ is a 1-simplex in $Lk(v, \Sigma)$. Define $\sigma_i = \{v, \xi_i, \xi_{i+1}\}$ which is a 2-simplex, and then we have that $\sigma_0, \dots, \sigma_N$ is a copath from σ to σ'.
4. $\sigma \cap \sigma' = \emptyset$. Let N be a shortest path from a vertex v of σ to a vertex v' of σ'. We proceed by induction on N. The induction base case here is when $N = 1$. (v, v') sits inside two 2-simplices.

Choose one of them and call it τ. $\sigma \cap \tau = \{v\}$ so there exists a copath from σ to τ. Similarly, $\sigma' \cap \tau = \{v'\}$ so there exists a copath from τ to σ'. Compare the two copaths to get a copath from σ to σ'.

Now for our induction step (assume hypothesis proved for $N-1$), let $v \in \sigma$, $v' \in \sigma'$. Let $(w_0, \ldots, w_M)$ be a shortest path from v to v'. Let τ be any 2-simplex such that $w_{m-1} \in \tau$. By our induction hypothesis, there exists a copath from σ to τ. By the induction base, there exists a copath from τ to σ'. Compose the two copaths to get a copath from σ to σ'.

□

5.3 Orientation Theorem

Theorem (Orientation Theorem)**.** Let Σ be a connected simplicial surface, and let $\mathbb{F}$ be a field.

1. If $1+1 \neq 0$ in $\mathbb{F}$ and Σ is orientable then
$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$
2. If $1+1 \neq 0$ in $\mathbb{F}$ and Σ is non-orientable then
$$H_2(\Sigma; \mathbb{F}) = 0$$
3. If $1+1 = 0$ then
$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$
regardless if Σ is orientable or not.

Definition (Intersection number)**.**

$$\langle [v_0, v_1, v_2], [v_0, v_1] \rangle = +1$$

$$\partial [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\langle [v_0, v_1, v_2], [v_0, v_2] \rangle = -1$$
$$\langle [v_0, v_1, v_2], [v_1, v_2] \rangle = +1$$

More generally,

$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_0, v_1] \rangle = sgn(\sigma)$$
$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_0, v_2] \rangle = -sgn(\sigma)$$
$$\langle [v_{\sigma(0)}, v_{\sigma(1)}, v_{\sigma(2)}], [v_1, v_2] \rangle = sgn(\sigma)$$

Proof. (of Orientation Theorem) For each 1-simplex ρ of Σ, fix once and for all a specific orientation $\hat{\rho}$ of ρ.

Let $\sigma_1, \ldots, \sigma_N$ be a list of the 2-simplices of Σ and $\hat{\sigma_i}$ fixed orientation of σ_i. To change the orientations on 2-simplices, we need a function

$$\eta : \{1, \ldots, N\} \to \{\pm 1\}$$

$\eta(i)\hat{\sigma_i}$ is the oriented 2-simplex which is

$$\begin{cases} \hat{\sigma_i} & \eta(i) = 1 \\ \text{opposite orientation of } \hat{\sigma_i} & \eta(i) = -1 \end{cases}$$

We shall consider elements of $C_2(\Sigma; \mathbb{F})$ of the form

$$[\eta] = \sum_{i=1}^{N} \eta(i)\hat{\sigma_i} \in C_2(\Sigma; \mathbb{F})$$

We want to calculate $\partial[\eta]$.

Fix a 1-simplex ρ and let σ_s, σ_t be the adjacent 2-simplices which contain ρ. The coefficient of $\hat{\rho}$ in $\partial[\eta]$ is simply

$$[\eta(s)\hat{\sigma_s}, \hat{\rho}] + [\eta(t)\hat{\sigma_t}, \hat{\rho}] = \begin{cases} 2 \\ 0 \\ -2 \end{cases}$$

To ensure that $\partial[\eta] = 0$ we require η to satisfy

$$[\eta(s)\hat{\sigma}_s, \hat{\rho}] + [\eta(t)\hat{\sigma}_t, \hat{\rho}] = 0$$

i.e.,

$$\langle \eta(s)\hat{\sigma_s}, \hat{\rho}\rangle + \langle \eta(t)\hat{\sigma}_t, \hat{\rho}\rangle = 0 \ \ (*)$$

whenever σ_s, σ_t are adjacent, and now we know Σ is orientable if and only if there exists a function

$$\eta : \{1, \ldots, N\} \to \{\pm 1\}$$

such that we have $(*)$ whenever σ_s, σ_t are adjacent.

So when Σ is orientable and $\eta : \{1, \ldots, N\} \to \{\pm 1\}$ is an orientation, then $[\eta] \in Z_2(\Sigma)$ and so defines a non-zero element of $H_2(\Sigma; \mathbb{F})$.

Now we show that when Σ is orientable,

$$H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$$

and $[\eta]$ is a *generator*.

Let us consider the elements

$$\sum_{s=1}^{N} a_s\hat{\sigma}_s \in C_2(\Sigma; \mathbb{F})$$

where $a_s \in \mathbb{F}$. Suppose that ρ is a 1-simplex and σ_s, σ_t are adjacent 2-simplices which contain ρ.

We calculate $\partial(\sum a_s\sigma_s)$. The coefficients of $\hat{\rho}$ is simply $a_s\langle\hat{\sigma}_t, \hat{\rho}\rangle + a_t\langle\hat{\sigma}_t, \hat{\rho}\rangle$. If we want $\partial(\Sigma a_s\sigma_s) = 0$ then

$$a_s\langle\hat{\sigma}_t, \hat{\rho}\rangle + a_t\langle\hat{\sigma}_t, \hat{\rho}\rangle = 0$$

for adjacent s, t.

$$\pm a_s \pm a_t = 0$$

so $a_t = \pm a_s$ if σ_s, σ_t are adjacent. So going along a copath, coefficients a_s are constant up to sign.

Fix a "base 2-simplex" σ_0 and suppose

$$\partial(\sum a_s \hat{\sigma_s}) = 0$$

Then going along a copath from σ_0 to σ_s, we find that $a_s = \pm a_0$. Define $\eta : \{1, \ldots, N\} \to \{\pm 1\}$ by

$$\eta(s) = \begin{cases} +1 & a_s = a_0 \\ -1 & a_s = -a_0 \end{cases}$$

then $\alpha = a_0[\eta]$ if $\partial\alpha = 0$ which shows that $\dim H_2(\Sigma; \mathbb{F}) \leq 1$.

If Σ is orientable, there exists global orientation $\eta : \{1, \ldots, N\} \to \{\pm 1\}$ and $[\eta]$ generates $H_2(\Sigma; \mathbb{F})$

If Σ is non-orientable, $\partial[\eta] \neq 0$ for any such η and if $\alpha \in Z_2(\Sigma; \mathbb{F})$, $\alpha = \sum a_s \hat{\sigma_s}$,

$$\alpha = a_0[\eta] \ , \partial(\alpha) = a_0 \partial[\eta]$$

$$\partial(\alpha) = 0 \implies a_0 \partial[\eta] = 0$$

but $\partial[\eta] \neq 0$ and $\alpha = 0$, so $a_0 = 0$, so then

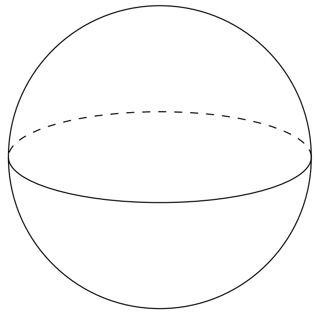
$$H_2(\Sigma; \mathbb{F}) = 0$$

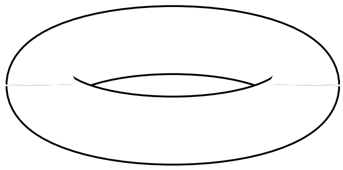
However if $1 + 1 = 0$, then

$$\partial(\sum_{s=1}^{N} \hat{\sigma_s}) = 0$$

as $\pm 2 = 0$ and $H_2(\Sigma; \mathbb{F}) \cong \mathbb{F}$. $\square$

For surfaces, H_0 tells us whether the surface is connected or not. H_2 tells us whether the surface is orientable or not. H_1 in a sense tells us how 'big' the surface is.





$\Sigma_0^+, S^2,$ genus $= 0$ $\qquad\qquad$ $\Sigma_1^+, T^2,$ genus $= 1$

$$H_k = \begin{cases} \mathbb{F} & k=0 \\ 0 & k=1 \\ \mathbb{F} & k=2 \end{cases} \qquad\qquad H_k = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F}\oplus\mathbb{F} & k=1 \\ \mathbb{F} & k=2 \end{cases}$$

In general for $\Sigma_g^+ = n$-fold torus, we have

$$H_k = \begin{cases} \mathbb{F} & k=0 \\ H_1 = \mathbb{F}\oplus\ldots\oplus\mathbb{F}\ (2g \text{ times}) & k=1 \\ \mathbb{F} & k=2 \end{cases}$$

We also have

$$\Sigma_0^- = \mathbb{R}P(2),\ \Sigma_g^- = \mathbb{R}P(2)\#\ldots\#\mathbb{R}P(2)\ (g+1 \text{ times})$$

If $1+1\neq 0$, then

$$H_k = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F}^g & k=1 \\ 0 & k=2 \end{cases}$$

but if $1+1=0$,

$$H_k = \begin{cases} \mathbb{F} & k=0 \\ \mathbb{F}^{g+1} & k=1 \\ \mathbb{F} & k=2 \end{cases}$$

As $H_*(-;\mathbb{F})$ is invariant under combinatorial equivalence ($\sim$), we have

$$\Sigma_g^+ \sim \Sigma_h^+ \implies g = h$$
$$\Sigma_g^- \sim \Sigma_h^- \implies g = h$$
$$\Sigma_g^+ \not\sim \Sigma_h^- \text{ for any } g, h$$

6 Some linear algebra

Proposition. If A, B are $n \times n$ matrices over $\mathbb{F}$, then

$$Tr(AB) = Tr(BA)$$

Proof. First write $A = (a_{kj})$, $B = (b_{ji})$. Then,

$$(AB)_{ki} = \sum_{j=1}^{n} a_{kj} b_{ji}$$

$$(AB)_{kk} = \sum_{j} a_{kj} b_{jk}$$

$$\begin{aligned} Tr(AB) &= \sum_{k=1}^{n} \sum_{j=1}^{n} a_{kj} b_{jk} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} a_{kj} b_{jk} \end{aligned}$$

and we know

$$a_{kj} b_{jk} = b_{jk} a_{kj}$$

as $\mathbb{F}$ is a field, so now we have

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} a_{jk} = Tr(BA)$$

$\square$

Remark. Note that in general $Tr(AB) \neq Tr(A)Tr(B)$. For example, take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$$

so that

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$Tr(A) = Tr(B) = 0, \;\; Tr(AB) = 2$$

Corollary. If A, P are $n \times n$ matrices over $\mathbb{F}$ and P is invertible then

$$Tr(PAP^{-1}) = Tr(A)$$

Proof.

$$Tr((PA)P^{-1}) = Tr(P^{-1}PA) = Tr(A)$$

□

6.1 Trace of a linear map

Let V be a finite dimensional vector space over $\mathbb{F}$. Let $X : V \to V$ be a linear map. Take a basis $\{e_1, \ldots, e_n\}$ for V, and write

$$X(e_i) = \sum_{j=1}^{n} e_j \xi_{ji}$$

$$X \sim (\xi_{ji}) = \xi$$

We would like to define $Tr(X) = Tr(\xi) = \sum_{j=1}^{n} \xi_{ji}$

However, ξ depends on the choice of basis $\{e_1, \ldots, e_n\}$. Suppose we take another basis $\{f_1, \ldots, f_n\}$ so that $X(f_i) = \sum_{j=1}^{n} f_i \eta_{ji}$,

$$X \sim (\eta)$$

η, ξ are related by $\eta = P\xi P^{-1}$ where P is the change of basis matrix,

$$P = M(\mathrm{id})_{\xi}^{\eta}, \ \ P^{-1} = M(\mathrm{id})_{\eta}^{\xi}$$

Consequently

$$Tr(\eta) = Tr(P\xi P^{-1}) = Tr(\xi)$$

so the trace is independent of the particular basis so we can legitimately define

$$Tr(X) = Tr(\xi)$$

when $X(e_i) = \sum_{j=1}^{n} e_j \xi_{ji}$.

Proposition (Additivity of Tr)**.** Let

$$0 \to U \to V \xrightarrow{p} W \to 0$$

be an exact sequence of finite dimensional vector spaces over $\mathbb{F}$ and suppose there exists linear maps T_U, T_V, T_W such that the following commutes

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d, "T_U"] & V \arrow[r, "p"] \arrow[d, "T_V"] & W \arrow[r] \arrow[d, "T_W"] & 0 \\
0 \arrow[r] & U \arrow[r] & V \arrow[r, "p"] & W \arrow[r] & 0
\end{tikzcd}$$

then

$$Tr(T_V) = Tr(T_U) + Tr(T_W)$$

Lemma. Let $T : V \to V$ be a linear map over $\mathbb{F}$. Suppose $\dim(V) = n$, and let $U \subset V$ be a subspace; $T(U) \subset U$, and $\dim(U) = k$. Then T can be represented by a matrix

$$T \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix and D is a $(n-k) \times (n-k)$ matrix.

Proof. (of lemma) Let $\{e_1, \ldots, e_k\}$ be a basis for U. Extend to a basis $\{e_1, \ldots, e_k, f_1, \ldots, f_q\}$ for V $(q = n - k)$. With respect to this basis,

$$T(e_i) = \sum_{j=1}^{k} e_j a_{ji} \quad (T(U) \subset U)$$

$T(f_r)$ is a linear combination in $\{e_1, \ldots, e_k, f_1, \ldots, f_q\}$. Write

$$T(f_r) = \sum_{s=1}^{k} e_s b_{sr} + \sum_{t=1}^{q} f_t d_{tr} \ \ (1 \leq r \leq q)$$

so the matrix of T has block form,

$$T \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A = (a_{ji})$, $B = (b_{sr})$, $D = (d_{tr})$. □

Note that $Tr(T) = Tr(A) + Tr(D)$.

Proof. (of proposition) Let $\{e_1, \ldots, e_k\}$ be a basis for U, and $\{\phi_1, \ldots, \phi_q\}$ be basis for W. For all r, choose $f_r \in V$, $p(f_r) = \phi_r$. Then, $\{e_1, \ldots, e_k\} \cup \{f_1, \ldots, f_q\}$ is a basis for V. By the previous lemma, T_V is represented by a block matrix

$$T_V \sim \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A =$ matrix of T_U, and factoring out U,

$$D = \text{matrix } T_W \text{ with respect to } \{f_1, \ldots, f_q\}$$

hence

$$\begin{aligned} Tr(T_V) &= Tr(A) + Tr(D) \\ &= Tr(T_U) + Tr(T_W) \end{aligned}$$

□

7 Lefschetz Fixed Simplex Theorem

Theorem (Lefschetz Fixed Simplex Theorem)**.** Let $f : K \to K$ be a simplicial map where K is a finite simplicial complex. Define

$$\lambda(f) = \sum_k (-1)^k Tr(H_k(f))$$

If $\lambda(f) \neq 0$ then there exists a simplex σ of K such that $f(\sigma) = \sigma$. $\lambda(f)$ is called the *Lefschetz number* of f (pick a field $\mathbb{F}$)

Definition (Geometrical Lefschetz index)**.**

$$\lambda_{\text{geom}}(f) = \sum_k (-1)^k Tr(C_k(f))$$

where $C_k(f) : C_k(K;\mathbb{F}) \to C_{k-1}(K;\mathbb{F})$ is the mapping induced by f.

Definition (Homological Lefschetz index)**.**

$$\lambda_{\text{hom}}(f) = \sum_k (-1)^k Tr(H_k(f))$$

Proposition.

$$\lambda_{\text{geom}}(f) = \lambda_{\text{hom}}(f)$$

Proof.

$$\begin{tikzcd}
0 \arrow[r] & Z_k(K) \arrow[r] \arrow[d, "Z_k(f)"] & C_k(K) \arrow[r] \arrow[d, "C_k(f)"] & B_{k-1}(K) \arrow[r] \arrow[d, "B_{k-1}(f)"] & 0 \\
0 \arrow[r] & Z_k(K) \arrow[r] & C_k(K) \arrow[r] & B_{k-1}(K) \arrow[r] & 0
\end{tikzcd}$$

$$Z_k(K) = \ker \partial_k,\ B_{k-1}(K) = \operatorname{im}(\partial_k)$$

so

$$Tr(C_k(f)) = Tr(Z_k(K)) + Tr(B_{k-1}(K)) \;\; (1)$$

and we also have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_k(K) & \longrightarrow & Z_k(K) & \longrightarrow & H_k(K) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle B_k(f)} & & \big\downarrow{\scriptstyle Z_k(F)} & & \big\downarrow{\scriptstyle H_k(f)} & & \\ 0 & \longrightarrow & B_k(K) & \longrightarrow & Z_k(K) & \longrightarrow & H_k(K) & \longrightarrow & 0 \end{array}$$

where both the rows are exact, so

$$Tr(Z_k(f)) = Tr(H_k(f)) + Tr(B_k(f)) \quad (2)$$

Substituting (2) into (1), we have

$$Tr\,C_k(f) = Tr\,H_k(f) + Tr\,B_k(f) + Tr(B_{k-1}(f))$$

but

$$\sum_k (-1)^k Tr\,B_k(f) + Tr\,B_{k-1}(f) = 0$$

so

$$\sum_k (-1)^k Tr\,C_k(f) = \sum (-1)^k Tr\,H_k(f)$$

thus

$$\lambda_{\text{geom}}(f) = \lambda_{\text{hom}}(f)$$

□

Remark. Note that $\lambda_{\text{hom}}(f)$ is easier to compute, but $\lambda_{\text{geom}}(f)$ carries geometric information.

Now consider $C_k(f) : C_k(K) \to C_k(K)$. If σ is a k-simplex of K, either $f(\sigma)$ is a k-simplex or $f(\sigma)$ is an l-simplex, where $l < k$.

In the first case, $C_k(f)(\sigma)$ is a basis element of C_k. In the second case, $C_k(f)(\sigma) = 0$, so representing $C_k(f)$ as a matrix, in a column there is at most one non-zero entry.

List the k-simplices of K, $\sigma_1, \ldots, \sigma_N$. $C_k(f)$ is an $N \times N$ matrix. The (i,i) entry of $C_k(f)$ is non-zero if and only if $f(\sigma) = \pm\sigma$, so if no k-simplex if fixed by f then the diagonal of $C_k(f)$ is 0 and $Tr(C_k(f)) = 0$.

Proof. (of Lefschetz fixed simplex theorem) Formally put,

$$f \text{ fixes no } k\text{-simplex} \implies Tr\, C_k(f) = 0$$

$$\text{so } f \text{ fixes no simplex} \implies Tr\, C_k(f) = 0 \text{ for all } k$$

$$\text{so } f \text{ fixes no simplex} \implies \sum_k (-1)^k Tr\, C_k(f) = (\lambda_{\text{geom}}(f) =)0$$

In the contrapositive,

$\lambda_{\text{geom}}(f) \neq 0 \implies f$ fixes some simplex (up to sign, it may change local orientation

hence

$$\lambda_{\text{hom}}(f) \neq 0 \implies f \text{ fixes some simplex}$$

□

Recall that

Proposition. If $f : K \to K$ is a simplicial map and K is connected then

$$H_0(f) = \text{id} : H_0(K) \to H_0(K)$$

Proof. If v, w are vertices of K, then $|v| - |w| \in \text{im } \partial_1$, so $[v] = [w]$ in $H_0(K)$. Hence $[f(v)] = [v]$ in $H_0(K)$ for any vertex v. But any vertex v generates H_0 (K connected), so

$$H_0(f) = \text{id} : \text{generator} \to \text{itself}$$

□

Corollary. If K is a connected simplicial complex and $f : K \to K$ is simplicial, then

$$Tr\, H_0(K) = 1$$

Corollary. Let $f : K \to K$ be a simplicial map, where K is a finite connected simplicial complex and

$$H_k(K; \mathbb{F}) = 0 \text{ for } k > 0$$

then $\lambda(f) = 1$.

Corollary. If $f : K \to K$ is a simplicial map, K a finite connected complex such that $H_k(K; \mathbb{F}) = 0$ for $k > 0$, then there exists a simplex σ of K such that $f(\sigma) = \sigma$ (up to orientation.)

Proof.

$$\lambda(f) = 1 \neq 0$$

$\square$

Corollary. If K is a finite simplicial complex and $K \sim CX$ (combinatorially equivalent) where CX is the cone on X, for some X, then any simplicial map $f : K \to K$ fixes a simplex.

Example. $K \sim \Delta^n$, any simplicial $f : K \to K$ fixes a simplex. (Brouwer Fixed Simplex)

To transform a finite simplicial complex into a metric space, replace the formal n-simplex by standard geometric n-simplex,

$$|\Delta^n| = \{t_0 e_0 + t_1 e_1 + \ldots + t_n e_n \,|\, t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

where $e_0, \ldots, e_n$ are the standard basis for $\mathbb{R}^{n+1}$.

$f : \Delta^n \to \Delta^n$ (formal simplicial mapping) gives a continuous mapping

$$|f| : |\Delta^n| \to |\Delta^n|$$

$$|f|(\sum t_i e_i) \to \sum t_i f(e_i)$$

f permutes $e_0, \ldots, e_n$,

$$|f|(\frac{1}{n} \sum e_i) = \frac{1}{n} \sum e_i \text{ (fixed point)}$$

so if $g : K \to K$ is a simplicial map, K a finite complex, we get a continuous mapping (!) $|g| : |K| \to |K|$. If g fixes a simplex, then $|g|$ fixes a point.

Theorem (Brouwer Fixed Point Theorem)**.** Let $X = |K|$ where K is some finite simplicial complex. Suppose any simplicial map $g(m) : K(m) \to K(m)$ has a fixed simplex $K(m)$ (subdivision of K), then any continuous $f : X \to X$ has a fixed point.

Proof. Suppose $f : X \to X$ does not have a fixed point. X compact, so there exists $\epsilon > 0$ such that $\epsilon \leq ||f(x) - x||$ for all x. Suppose $g(m) : K(m) \to K(m)$, then

$$||f(x) - x|| \leq ||f(x) - g_m(x)|| + ||g_m(x) - x||$$

so

$$\forall \eta \ \exists m \, ||f(x) - g_m(x)|| < \eta \ \forall x$$

Choose m so that $||f(x) - g_m(x)|| < \frac{\epsilon}{2}$ so then

$$\frac{\epsilon}{2} \leq ||g_m(x) - x|| \ \forall x$$

which is a contradiction, thus f has a fixed point. □

8 Posets and products

Theorem (Künneth theorem (will not be proven here))**.**

$$H_n(X \times Y; \mathbb{F}) = \bigoplus_{r=0}^{n} H_r(X; \mathbb{F}) \otimes H_{n-r}(Y; \mathbb{F})$$

however we will prove the following,

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

where X, Y are finite simplicial complexes. To see applications of this, consider $S^2 \times S^2$ and S^4, both of which are simply connected compact 4-manifolds. However, $S^2 \times S^2 \ncong S^4$ since

$$\chi(S^2 \times S^2) = \chi(S^2)\chi(S^2) = 4$$

and

$$\chi(S^4) = 2$$

Definition (Posets (Partially ordered sets))**.** A *poset* $(X, \leq)$ consists of a set X and a relation $\leq$ in $X \times X$

1. $x \leq x \ \forall x$
2. $x \leq y \wedge y \leq 2 \implies x \leq z$

In a total ordering we also have $\forall x, y \in X$ either $x \leq y$ or $y \leq x$.

Now let $(X, \leq)$ be a finite poset. Construct a simplicial complex $N(X, \leq)$ the *nerve* of $(X, \leq)$. The vertex set of $N(X, \leq)$ is X and simplex set of $N(X, \leq)$ is the set of totally ordered non-empty subsets.

If $(X, \leq)$, $(Y, \leq')$ are finite posets then the product poset is $(X \times Y, \preccurlyeq)$ where

$$(x, y) \preccurlyeq (x', y') \iff (x \leq x') \wedge (y \leq' y')$$

Notice that if $(X, \leq)$, $(Y, \leq')$ are totally ordered then $(X \times Y, \preccurlyeq)$ isnt, except trivially.

From now on we will always use the symbol '$\leq$'.

Proposition. If X is a finite simplicial complex, we can write

$$X = N(\mathcal{X}, \leq)$$

for some $\mathcal{X}$.

Proof. Take an arbitrary ordering on the vertices of X, $\{v_0, \ldots, v_n\}$. X embeds in Δ^N,

$$v_i \mapsto i, \ \ \mathcal{X} = \mathrm{im}(v_i \mapsto i)$$

$$\Delta^N = \text{nerve on totally ordered set } 0 \leq 1 \leq \ldots N$$

so each simplex σ of X is totally ordered. $\square$

So now to define $X \times Y$, we write

$$X = N(\mathcal{X}), \;\; Y = N(\mathcal{Y})$$

where $\mathcal{X}$, $\mathcal{Y}$ are posets. Then, we define

$$X \times Y = N(\mathcal{X} \times \mathcal{Y})$$

By an *ordered* simplex complex, we mean a simplicial complex

$$X = (V_X, \mathcal{S}_X)$$

together with a partial ordering on V_X such that for all $\sigma \in \mathcal{S}_X$, σ is totally ordered, so any simplicial complex can be regarded as an ordered simplicial complex,

$$X = N(\mathcal{X}) \text{ for some } \mathcal{X}$$

If X, Y are ordered simplicial complexes, then so is $X \times Y$.

Example.

$$\Delta^n = N(\{0, \ldots, n\})$$

Definition.

$$\Delta^m \times \Delta^n = N(\{0, \ldots, m\} \times \{0, \ldots, n\})$$

$\Delta^m \times \Delta^n$ has dimension $m + n$ and has $\frac{(m+n)!}{m!n!}$ principal simplices.

Example. $\Delta^1 \times \Delta^1$ as defined before is equivalent to $I \times I$ but triangulated.

Every finite simplicial complex K can be represented as an *ordered simplicial complex*, as shown before.

8.1 Product structure on $X \times Y$

Let X, Y be finite simplicial complexes. Represent them as ordered simplicial complexes. If $\sigma \in \mathcal{S}_X$, $\tau \in \mathcal{S}_Y$, we make the isomorphisms

$$\sigma \cong \Delta^m, \ \ \tau \cong \Delta^n$$

and triangulate $\sigma \times \tau$ by corresponding triangulation on $\Delta^m \times \Delta^n$. Write

$$X = \bigcup_{\sigma \in \mathcal{S}_X} \sigma, \ \ Y = \bigcup_{\tau \in \mathcal{S}_Y} \tau$$

and

$$X \times Y = \bigcup_{\sigma \in \mathcal{S}_X, \tau \in \mathcal{S}_Y} \sigma \times \tau$$

and each $\sigma \times \tau$ is triangulated as explained before.

It can be shown up to combinatorial equivalence that this is independent of orderings chosen on X, Y.

Now we wish to prove

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

Recall the Künneth theorem,

$$H_n(X \times Y; \mathbb{F}) = \bigoplus_{r=0}^{n} H_r(X; \mathbb{F}) \otimes H_{n-r}(Y; \mathbb{F})$$

In the special case where $Y = \Delta^m$,

$$H_{n-k}(Y) = \begin{cases} \mathbb{F} & n-k=0 \\ 0 \text{ otherwise} & \end{cases}$$

so as a special case, we do have

$$H_*(X \times \Delta^m) \cong H_*(X)$$

which we will prove.

Fix Δ^m. For any simplicial complex X, we have simplicial maps

$$i : X \to X \times \Delta^m, \;\; i(v) = (v, 0)$$

$$\pi : X \times \Delta^m \to X, \;\; \pi(x, y) = x$$

Both maps induce isomorphisms on H_*, so we will just prove this for i. Let $P(n, k)$ denote the statement,

If X is finite complex of $\dim(X) \leq n$ having at most k simplices of dimension n,

then $i : H_*(X; \mathbb{F}) \to H_*(X \times \Delta^m; \mathbb{F})$ is an isomorphism

We first show that $P(n, 0)$ is true for all n, i.e.,

$$i : \Delta^n \to \Delta^n \times \Delta^m \text{ induces isomorphisms}$$

$$i_* : H_*(\Delta^n) \to H_*(\Delta^n \times \Delta^m)$$

Let $(A, \leq)$ be a poset. We say that A has a *maximum* when there exists $a \in A$ such that for all $b \in A$, $b \leq a$.

Proposition. If $(A, \leq)$ has a maximum then $N(A, \leq)$ is a cone.

Proof. Let a be a maximum element. Define $A' = A - \{a\}$. $(A', \leq)$ is still a poset, and $N(A, \leq)$ is a cone on $N(A', \leq)$ with cone point a. □

Now note that $\Delta^m \times \Delta^n$ has maximum (m, n), so $\Delta^m \times \Delta^n$ is a cone.

Corollary.

$$i_* : H_*(\Delta^n) \xrightarrow{\cong} H_*(\Delta^n \times \Delta^m)$$

and $P(n, 0)$ is true for all n.

Proposition. $P(0, k)$ is true for all k.

Proof. Here X is a 0-dimensional complex with k-points. Note that for $k = 0$, this is true, so $P(0,0)$ is true. We proceed by induction.

Suppose that the statement is proven for $k-1$. Then

$$X = X_0 \sqcup \{x_k\} \;\; X_0 = \{x_1, \dots, x_{k-1}\}$$

By hypothesis

$$H_*(X_0 \times \Delta^m) \cong \bigoplus_{r=1}^{k-1} H_*(\{x_r\} \times \Delta^m) = \begin{cases} \mathbb{F}^{k-1} & * = 0 \\ 0 & * \neq 0 \end{cases}$$

$$X \cap X_0 = \emptyset, \;\; H_*(X \cap X_0) = 0 \text{ for all } *$$

Using the Mayer-Vietories sequence,

$$H_*((X_0 \sqcup \{x_k\}) \times \Delta^m) \cong H_*(X_0 \times \Delta^m) \oplus H_*(\{x_k\} \times \Delta^m)$$

We know

$$H_*((X_0 \sqcup \{x_k\}) \times \Delta^m) \cong H_*(X)$$

and

$$H_*(X_0 \times \Delta^m) \oplus H_*(\{x_k\} \times \Delta^m) \cong H_*(X_0) \oplus H_*(\{x_k\})$$

so $P(0,k)$ for all k. □

Now let $P(n) = \wedge_{k \geq 0} P(n,k)$. So now we know $P(0)$ is true. Also, $P(n,0) \equiv P(n-1)$ ($P(n,0)$; no n-simplices).

$P(n,1)$ is true for all n. It is enough to prove

$$P(n,k-1) \wedge P(n-1) \implies P(n,k)$$

Proof. Take X to be a finite complex of dimension less than or equal to n with exactly k simplices of dimension n.

Let X_0 be the complex obtained by removing an n-simplex $\sigma \cong \Delta^n$,

$$X = X_0 \cup \sigma$$

$X_0 \cap \sigma$ has dimension $\leq n - 1$.

$$\begin{tikzcd}
H_p(X_0 \cap \sigma) \arrow[r] \arrow[d, "i_*"] & H_p(X_0) \oplus H_p(\sigma) \arrow[r] \arrow[d, "\mathcal{I}"] & H_p(X) \arrow[r] \arrow[d, "i_*"] & {} \\
H_p(X_0 \cap \sigma \times \Delta^n) \arrow[r] & H_p(X \times \Delta^m) \oplus H_p(\sigma \times \Delta^m) \arrow[r] & H_p(X \times \Delta^m) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & H_{p-1}(X_0 \cap \sigma) \arrow[r] \arrow[d, "i_*"] & H_{p-1}(X_0) \oplus H_{p-1}(\sigma) \arrow[d, "\mathcal{I}"] \\
{} \arrow[r] & H_{p-1}(X_0 \cap \sigma \times \Delta^m) \arrow[r] & H_{p-1}(X_0 \times \Delta^m) \oplus H_{p-1}(\sigma \times \Delta^m)
\end{tikzcd}$$

where

$$\mathcal{I} = \begin{pmatrix} i_* & 0 \\ 0 & i_* \end{pmatrix}$$

Both rows are exact. The outer arrows are isomorphisms. Hence $i_* : H_p(X) \xrightarrow{\cong} H_p(X \times \Delta^m)$ is isomorphic for all p, so

$$P(n, k-1) \wedge P(n-1) \implies P(n, k)$$

□

So now we know that

$$H_*(X \times \Delta^m) \cong H_*(X)$$

for any finite complex X and any m.

As χ is expressable in terms of H_*, we know,

Theorem.

$$\chi(X \times \Delta^m) = \chi(X)$$

for any finite complex X.

Next we will prove that

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

for any finite complexes X, Y.

Lemma. Suppose $X = X_+ \cup X_-$, where X is a finite complex, X_+, X_- subcomplexes. Then,

$$\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$$

Proof. We know that there exists an exact sequence of chain complexes,

$$0 \to C_*(X_+ \cap X_-) \to C_*(X_+) \oplus C_*(X_-) \to C_*(X) \to 0$$

In particular, for each n, we have an exact sequence

$$0 \to C_n(X_+ \cap X_-) \to C_n(X_+) \oplus C_n(X_-) \to C_n(X) \to 0$$

so

$$\dim C_n(X) + \dim C_n(X_+ \cap X_-) = \dim C_n(X_+) + \dim C_n(X_-)$$

and so if we take alternating sums,

$$\chi(X) + \chi(X_+ \cap X_-) = \chi(X_+) + \chi(X_-)$$

□

Theorem. Let X, Y be finite simplicial complexes, then

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

Proof. Fix X. Define statements $Q(n, k)$ as follows,

$$Q(n, k); \chi(X \times Y) = \chi(X)\chi(Y)$$

when Y is a finite complex of dimension $\leq n$ having precisely k n-simplices. We know that $\chi(X \times \Delta^m) = \chi(X) = \chi(X)\chi(\Delta^n)$ because

$\chi(\Delta^n) = 1$. So $Q(n,1)$ is true for all n. Next we prove that $Q(0,k)$ is true for all k. This says that if $X = \{y_1\} \cup \ldots \cup \{y_k\}$, where $y_1, \ldots, y_k$ are points, then

$$\chi(X \times Y) = \chi(X)k$$

because $\chi(Y) = k$.

We prove this by induction on k. $Q(0,0) = \emptyset$, $Q(0,1)$ is true since

$$X \times \{y_1\} \cong X$$

so

$$\chi(X \times \{y_1\}) = \chi(X) = \chi(X)\chi(\{y_1\})$$

Now suppose this is true for some k, and

$$Y = \{y_1\} \cup \ldots \cup \{y_{k+1}\}$$

where $y_1, \ldots, y_{k+1}$ are points,

$$Y' = \{y_1\} \cup \ldots \cup \{y_k\}$$

$$Y = Y' \cup \{y_{k+1}\}, \;\; Y' \cup \{y_{k+1}\} = \emptyset$$

so then

$$X \times Y = (X \times Y') \cup (X \times \{y_{k+1}\})$$

and applying χ, we have

$$\chi(X \times Y) + \chi(X \times (Y' \cap \{y_{k+1}\})) = \chi(X \times Y') + \chi(X)\chi(\{y_{k+1}\})$$

Given that $Y' \cap \{y_{k+1}\} = \emptyset$, we have

$$X \times (Y' \times \{y_{k+1}\}) = \emptyset$$

$$\chi(X \times (Y' \times \{y_{k+1}\})) = 0$$

$$\chi(X \times Y) = \chi(X \times Y') + \chi(X)$$

because $\chi(\{y_{k+1}\}) = 1$. By hypothesis $P(0,k)$,

$$\chi(X \times Y') = \chi(X)\chi(Y')$$

so

$$\begin{aligned}\chi(X \times Y) &= \chi(X)\chi(Y') + \chi(X) \\ &= \chi(X)[\chi(Y') + 1] \\ &= \chi(X)\chi(Y)\end{aligned}$$

so $Q(0,k) \implies Q(0,k+1)$.

Now define $Q(n-1) = \wedge_k Q(n-1,k)$. We now know that $Q(0)$ is true. Also,

$$Q(1,0) \equiv Q(0)$$

We also know that $Q(n-1)$ is true.

For our final induction step, we prove that

$$Q(n,k) \wedge Q(n-1) \implies Q(n,k+1)(*)$$

This then shows each $Q(n,k)$ is true so $Q(n)$ is true, so then we proceed with

$$Q(n+1,k) \wedge Q(n) \implies Q(n+1,k+1)$$

and so on. So to prove $(*)$, let Y be a finite complex of dimension $\leq n$ having precisely $k+1$ n-simplices. In particular, the n-simplices of Y are principal simplices. Let $\sigma_1, \ldots, \sigma_k, \sigma_{k+1}$ be the n-simplices of Y.

Let Y' be the subcomplex of Y where the principal n-simplices are $\sigma_1, \ldots, \sigma_k$. Write

$$Y = Y' \cup \sigma_{\hat{k+1}}$$

where $\sigma_{\hat{k+1}}$ is the subcomplex of Y consisting of σ_{k+1} and all of its faces. Observe $Y' \cap \sigma_{\hat{k+1}}$ has dimension $\leq n-1$.

$$X \times Y = (X \times Y') \cup X \times \sigma_{\hat{k+1}}$$

$$\chi(X \times Y) = \chi(X \times Y') + \chi(X \times \hat{\sigma_{k+1}}) - \chi(X \times (Y' \cap \hat{\sigma_{k+1}}))$$

By hypothesis $Q(n, k)$

$$\chi(X \times Y') = \chi(X)\chi(Y')$$

By $Q(n, 1)$

$$\chi(X \times \hat{\sigma_{k+1}}) = \chi(X)$$

$$\chi(X \times \hat{\sigma_{k+1}}) = \chi(X)\chi(\hat{\sigma_{k+1}})$$

By $Q(n - 1)$

$$\chi(X \times (Y' \cap \hat{\sigma_{k+1}})) = \chi(X)\chi(Y' \cap \hat{\sigma_{k+1}})$$

Thus by $Q(n, k)$, $Q(n, 1)$, $Q(n - 1)$, together we get

$$\begin{aligned}
\chi(X \times Y) &= \chi(X)\chi(Y') + \chi(X)\chi(\hat{\sigma_{k+1}}) - \chi(X)\chi(Y' \cap \hat{\sigma_{k+1}}) \\
&= \chi(X)[\chi(Y') + \chi(\hat{\sigma_{k+1}} - \chi(Y' \cap \hat{\sigma_{k+1}}))] \\
&= \chi(X)\chi(Y)
\end{aligned}$$

hence

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

for any finite complexes X, Y. □

Definition (n-manifold)**.** An n-manifold X is a complex in which

$$Lk(v, X) \sim S^{n-1}$$

for any vertex.

If X is an m-manifold, Y an n-manifold, then $X \times Y$ is an $m + n$-manifold.

Example. S^n is an n-manifold. S^4 is a 4-manifold, and so is $S^2 \times S^2$. We have that

$$\chi(S^4) = 2$$

since

$$H_k(S^4) = \begin{cases} \mathbb{F} & k = 0, 4 \\ 0 & \text{otherwise} \end{cases}$$

We also know $\chi(S^2) = 2$, and so $\chi(S^2 \times S^2) = 4$. $S^1 \times S^3$ is also a 4-manifold, however

$$\chi(S^1 \times S^3) = \chi(S^1) \times \chi(S^3) = 0$$

since χ of any odd dimenisonal compact manifold is equal to 0.

Consider $S^1 \times S^7$, $S^3 \times S^5$. Both are 8-manifolds, and both have $\chi = 0$, however both are distinguished by the Künneth theorem.