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GRADUATION THESIS

Tensors

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1 Preliminaries

1.1 Vector space

Before we dive into the notion of tensors, first we need to remember fundamental concepts that are necessary into understanding the tensors.

Let us begin with the definition of an abstract vector space.

Definition 1. *A* ***vector space*** *V over $\mathbb{C}$ is a set of objects called* ***vectors****, with the following properties:*

1. *To every pair of vectors x and y in V there corresponds a vector $x + y$ also in V, called the sum of x and y such that:*

   a) $x + y = y + x$

   b) $x + (y + z) = (x + y) + z$

   c) There exists a unique vector $0 \in V$, called the ***zero vector****, such that $x + 0 = x$ for every vector x*

   d) To every vector $x \in V$ there corresponds a unique vector $-x \in V$ such that $x + (-x) = 0$

2. *To every complex number α(also called a* ***scalar****) and every vector x there corresponds a vector αx in V such that:*

   a) $\alpha(\beta x) = (\alpha\beta)x$

   b) $1x = x$

3. *Multiplication involving vectors and scalars is distributive:*

   a) $\alpha(x + y) = \alpha x + \alpha y$

   b) $(\alpha + \beta)x = \alpha x + \beta x$

Example 1.1. $\mathbb{C}^2$ *is a vector space with multiplication and addition having their usual meaning.*

Now we shall remember the inner product as it is defined, so we can revisit it later on when we talk about tensors.

Remark 2. *The inner product generalizes the dot product to abstract vector spaces over a field of scalars, being either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. It is usually denoted by $\langle x, y\rangle$.*

Definition 3. *The* ***inner product*** *of two vectors, x and y, in a vector space V is a complex number, $\langle x, y\rangle \in \mathbb{C}$ such that*

1. $\langle x, y\rangle = \langle y, x\rangle^*$
2. $\langle x, (\beta y + \gamma z)\rangle = \beta\langle x, y\rangle + \gamma\langle x, z\rangle$
3. $\langle x, x\rangle \geqslant 0$ *and* $\langle x, x\rangle = 0$ *if and only if $x = 0$.*

The last relation is called the ***positive definite*** *property of the inner product. A positive definite real inner product is also called a* ***Euclidian*** *inner product, otherwise it is called* ***pseudo-Euclidian****.*

Definition 4. *The vectors $x_1,x_2,\ldots,x_n$ are said to be* ***linearly independent*** *if for $\alpha_i \in \mathbb{C}$, the relation $\sum_{i=1}^{n}\alpha_i x_i = 0$ implies $\alpha_i = 0$ for all i. The vectors are called* ***linearly dependent*** *otherwise.*

Definition 5. *A* ***subspace*** *W of a vector space V is a nonempty subset of V with the property that if $x,y \in W$, then $\alpha x + \beta y$ also belongs to W, $\forall\ \alpha,\beta \in \mathbb{C}$*

Remark 6. *The subspace is a vector space in its own right and the intersection of two subspaces is also a subspace.*

Definition 7. *A* ***basis*** *of a vector space V is a set B of linearly independent vectors that spans all of V. A vector space that has a finite basis is called* ***finite-dimensional*** *and* ***infinite-dimensional*** *otherwise. We call the cardinality of the set B the* ***dimension*** *of V.*

Definition 8. *A* ***linear map***(*or* ***transformation***) *from the complex vector space V to the complex vector space W is a mapping $\boldsymbol{T}$:V→W such that:*
$\boldsymbol{T}(\alpha x + \beta y) = \alpha\boldsymbol{T}(x) + \beta\boldsymbol{T}(y)\ \forall\ x,y \in V$ and $\alpha,\beta \in \mathbb{C}$
A linear transformation $\boldsymbol{T}$:V→V is called an ***endomorphism*** *of V or a* ***linear operator*** *on V.*

An important example of linear transformations occurs when the second vector space W, happens to be the set of scalars, $\mathbb{C}$ or $\mathbb{R}$, in which case the linear transformation is called a **linear functional**. The set of linear functionals $\mathcal{L}$(V,$\mathbb{C}$) or $\mathcal{L}$(V,$\mathbb{R}$) if V is a real vector space is denoted by V* and is called the **dual space** of V.

Definition 9. *A vector space V is said to be isomorphic to another vector space W and written $V \cong W$, if there exists a bijective linear map $\boldsymbol{T}$: V → W. Then $\boldsymbol{T}$ is called an* ***isomorphism****. A bijective linear map of V onto itself is called an* ***automorphism*** *of V.An automorphism is also called an* ***invertible*** *linear map.*

Definition 10. *Let A be an $N \times N$ matrix. The mapping tr:$M^{(N\times N)} \to \mathbb{C}$ (or $\mathbb{R}$) given by tr A $= \sum_{i=1}^{n}\alpha_{ii}$ is called the* ***trace*** *of A.*

Theorem 11. *The trace is a linear mapping. Furthermore, tr(AB) = tr(BA) and trA^t = trA.*

Proof. To prove the first identity, we use the definitions of trace and matrix product:
tr(AB) $= \sum_{i=1}^{n}(\mathrm{AB})_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{A})_{ij}(\mathrm{B})_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathrm{B})_{ji}(\mathrm{A})_{ij} = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}(\mathrm{B})_{ji}(\mathrm{A})_{ij}\right) = \sum_{j=1}^{n}(\mathrm{BA})_{jj} = \mathrm{tr(BA)}$.
The linearity of the trace and the second identity follow directly from the definition. □

1.2 Multilinear maps

There is a very useful generalization of the linear functionals that becomes essential in the treatment of tensors. However, a limited version of its application is used in the discussion of determinants, which we shall start here.

Definition 12. *Let V and U be vector spaces. Let V^P denote the p-fold Cartesian product of V. A* ***p-linear map*** *from V to U is a map Θ:$V^P \rightarrow U$ which is linear with respect to each of its arguments:*
$\Theta(x_1,...,\alpha x_j + \beta y_j,...,x_p) = \alpha\Theta(x_1,....,x_j,....,x_p) + \beta\Theta(x_1,....,y_j,...,x_p)$.
A p-linear map from V to $\mathbb{C}$ or $\mathbb{R}$ is called a ***p-linear function*** *in V.*

Example 1.2. *Let $\{\Phi_i\}_i^p$ be a linear functionals on V. Define Θ by $\Theta(x_1,...,x_p) = \Phi_1(x_1),...,\Phi_P(x_P)$, $x_i \in V$.*
Clearly Θ is p-linear.

Let σ denote the permutation of 1, 2,..., p. Define the p-linear map $\sigma\omega$ by $\sigma\omega(x_1,...,x_p) = \omega(x_{\sigma(1)},....,x_{\sigma(p)})$

Definition 13. *A p-linear map ω from V to U is* ***skew-symmetric*** *if $\sigma\omega = \epsilon_\sigma$ if:*

$$\omega(x_{\sigma(1)},...,x_{\sigma(p)}) = \epsilon_\sigma\omega(x_1,...,x_p)$$

where ϵ_σ is the sign of σ, which is +1 if σ is even and -1 if it is odd. The set of p-linear skew-symmetric maps from V to U is denoted by $\Lambda^P(V,U)$. The set of p-linear skew-symmetric functions in V is denoted by $\Lambda^P(V)$.
The permutation sign ϵ_ω is sometimes written as

$$\epsilon_\sigma = \epsilon_{\sigma(1)\sigma(2)...\sigma(p)} \equiv \epsilon_{i_1 i_2 ... i_p}, \text{ where } i_k \equiv \sigma(k).$$

Definition 14. *A skew symmetric N-linear function in V,i.e., a member of $\Lambda^N(V)$ is called a* ***determinant function*** *in V.*
Let $B = \{|e_k\}_{k=1}^N$ be a basis of V and $B^* = \{\epsilon_k\}_{j=1}^N$ a basis of V^*, dual to B. For any set of N vector $\{x_k\}_{k=1}^N$ in V, define the N-linear function Θ by

$$\Theta(x_1,...,x_N) = \epsilon_1(x_1)...\epsilon_N(x_N),$$

Now let Δ be defined by $\Delta \equiv \sum_\pi \epsilon_\pi \cdot \theta$ Then, $\Delta \in \Lambda^N(V)$,i.e., Δ is a determinant function.

Let **A** be a linear operator on an N-dimensional vector space V. Choose a nonzero determinant function Δ. For a basis $\{\mathrm{v}_i\}_{i=1}^N$ define the function Δ_A by

$$\Delta_A(\mathrm{v}_1,...,\mathrm{v}_N) \equiv \Delta(\mathrm{Av}_1,...,\mathrm{Av}_N).$$

Definition 15. *Let $A \in End(V)$. Let Δ be a nonzero determinant function in V, and let Δ_A be as mentioned before. Then*

$$\Delta_A = det\ A \cdot \Delta$$

defines the ***determinant of A****.*

2 Tensors

From here on,we will consider only real vector spaces and the basis vectors of a vector space V will be distinguished by a subscript and those of its dual space by a superscript.For example, if $\{e_i\}_{i=1}^N$ is a basis in V, then the basis in V^* will be $\{\epsilon^j\}_{j=1}^N$ so we can avoid confusions. **Einstein's summation convention** will also be used:
Repeated indices, of which one is an upper and the other a lower index, are assumed to be summed over: $a_i^k b_j^i$ means $\sum_{i=1}^N a_i^k b_j^i$.
It is more natural to label the elements of a matrix representation of an operator **A** by α_j^i (rather than α_{ji}, because then $Ae_i = \alpha_i^j e_j$.

2.1 Tensors as Multilinear Maps

Since tensors are special kinds of linear operators on vector spaces, let us reconsider $\mathcal{L}$(V,W), the space of all linear mappings from the real vector space V to the real vector space W.

Definition 16. *A map* $\boldsymbol{T}$*:*$V_1 \times V_2 \times ... \times V_r \to W$ *is called r-**linear** if it is linear in all its variables:*

$$\boldsymbol{T}(v_1,....,\alpha v_i + \alpha' v_i',....,v_r) = \alpha \boldsymbol{T}(v_1,....,v_i,....,v_r) + \alpha' \boldsymbol{T}(v_1,....,v'_i,....,v_r) \text{ for all } i.$$

Definition 17. *Let* $\tau_1 \in V_1^*$ *and* $\tau_2 \in V_2^*$*. We construct the bilinear map* $\tau_1 \otimes \tau_2 : V_1 \times V_2 \to \mathbb{R}$ *by* $\tau_1 \otimes \tau_2(v_1, v_2) = \tau_1(v_1)\tau_2(v_2)$*. The expression* $\tau_1 \otimes \tau_2$ *is called the **tensor product** of* τ_1 *and* τ_2

An r-linear map can be multiplied by a scalar, and two r-linear maps can be added; in each case the result is an r-linear map. Thus, the set of r-linear maps from $V_1 \times \cdots \times V_r$ into W forms a vector space that is denoted by $\mathcal{L}(V_1,...,V_r;W)$.

We can also construct multilinear maps on the dual space. First, we note that we can define a natural linear functional on V^* as follows. We let $\tau \in V^*$ and $v \in V$; then $\tau(v) \in \mathbb{R}$. Now we twist this around and define a mapping $v : V_* \to \mathbb{R}$ given by $v(\tau) \equiv \tau(v)$. We have naturally constructed a linear functional on V^* by identifying $(V^*)^*$ with V.

Definition 18. *Let V be a vector space with dual space V*. Then a **tensor of type (r,s)** is a multilinear mapping*

$$\boldsymbol{T}^r_s : \underbrace{V^* \times V^* \times ... \times V^*}_{r \text{ times}} \times \underbrace{V \times V \times ... \times V}_{s \text{ times}}$$

The set of all such mappings for fixed r and s forms a vector space denoted by $T_s^r(V)$*. The number r is called the **contravariant degree** of the tensor, and s is called the **covariant degree** of the tensor.*

Example 2.1.

a) A tensor of type (0, 0) is defined to be a scalar, so $T_0^0(V) = (R)$*.*

b) A tensor of type (1, 0), an ordinary vector, is called a contravariant vector, and one of type (0, 1), a dual vector (or a linear functional), is called a covariant vector.

c) A tensor of type (r, 0) is called a contravariant tensor of rank r, and one of type (0,s) is called a covariant tensor of rank s.

The union of T^r_s (V) for all possible r and s can be made into an (infinite-dimensional) algebra called **algebra of tensors**.
First we define the following product on it:

Definition 19. *The **tensor product** of a tensor **T** of type (r,s) and a tensor **U** of type (k,l) is a tensor **T**⊗**U** of type (r+k,s+l),defined, as an operator on $(V^*)^{r+k} \times V^{s+l}$, by*

$$\boldsymbol{T}\otimes \boldsymbol{U}(\theta^1,\ldots,\theta^{r+k},u_1,\ldots,u_{s+l}) = \boldsymbol{T}(\theta^1,\ldots,\theta^r,u_1,\ldots,u_s)\boldsymbol{U}(\theta^{r+1},\ldots,\theta^{r+k},u_{s+l},\ldots,u_{s+l}).$$

*This product turns the (infinite-dimensional) vector space of all tensors into an associative algebra called a **tensor algebra**.*

Example 2.2. *What is the tensor product of* $\boldsymbol{A} = 2e_x - e_y + 3e_z$ *with itself?*
$\boldsymbol{A}\otimes\boldsymbol{A}$ = (2,-1,3)⊗(2-1,3)
Using the formula from above, we can compute the tensor product
(2,-1,3)⊗(2-1,3)=(2 · 2,2 · -1,2 · 3,-1 · 2,-1 · -1,-1 · 3,3 · 2,3 · -1,3 · 3)
(2,-1,3)⊗(2-1,3) = (4, -2, 6, -2, 1, -3, 6, -3, 9)

Difference between direct sum and tensor product
We'll assume V and W are finite dimensional vector spaces. That means we can think of V as $\mathbb{R}^n$ and W as $\mathbb{R}^m$ for some positive integers n and m.I will use the following simple example to show the difference between direct sum and tensor product:
Let x.y be two vectors, x ∈ $\mathbb{R}^3$ and y ∈ $\mathbb{R}^2$

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix}, y = \begin{pmatrix}4\\5\end{pmatrix}$$

We call x ⊕ y the **direct sum** .In this case, x⊕y is:

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$$

We shall see that the direct sum give a list of m + n numbers, this gives us a way to build a space where the dimensions **add**.
Now lets see the tensor product of x and y, denoted x⊗y.

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\cdot 4\\1\cdot 5\\2\cdot 4\\2\cdot 5\\3\cdot 4\\3\cdot 5\end{pmatrix}$$

We see that the tensor product gives a list of m * n numbers, this gives us a way to build a space where the dimensions **multiply**.

Definition 20. *A* ***contraction*** *of a tensor* $A \in T^r_s(V)$ *with respect to a contravariant index at position p and covariant index at position q is a linear mapping* $C^p_q(V) \rightarrow T^{r-1}_{s-1}(V)$ *given in component form by*

$$[C^p_q(A)]^{i_1 \ldots i_{r-1}}_{j_1 \ldots j_{s-1}} = A^{\ i_1 \ldots i_{p-1} k i_{p+1} \ldots i_r}_{\ j_1 \ldots j_{q-1} k j_{q+1} \ldots j_s} \equiv \sum_k A^{\ i_1 \ldots i_{p-1} k i_{p+1} \ldots i_r}_{\ j_1 \ldots j_{q-1} k j_{q+1} \ldots j_s}$$

2.2 Symmetries of Tensors

Many applications demand tensors that have some kind of symmetry property. One symmetric tensor is the metric 'tensor' of an inner product: If V is a vector space and $v_1, v_2 \in V$, then $g(v_1,v_2) = g(v_2,v_1)$. The following generalizes this property.

Definition 21. *A tensor A is* ***symmetric*** *in the ith and jth variables if its value as a multilinear function is unchanged when these variables are interchanged.Clearly, the two variables must be of the same kind.*

From this definition. it follows that in any basis, the components of a symmetric tensor do not change when the ith and jth indices are interchanged.

Definition 22. *A tensor is* ***contravariant-symmetric*** *if it is symmetric in every pair of its covariant indices. A tensor is* ***symmetric*** *if it is both contravariant-symmetric and covariant-symmetric.*

An immediate consequence of this definition is the following theorem:

Theorem 23. *A tensor S of type(r,0) is symmetric if and only if for any permutation* π *of 1,2,...r, and any* $\tau^1, \tau^2, ..., \tau^r$ *in* V^* *we have*

$$S(\tau^{\pi(1)}, \tau^{\pi(2)}, ..., \tau^{\pi(r)}) = S(\tau^1, \tau^2, ..., \tau^n)$$

Definition 24. *A* ***symmetrizer*** *is an operator* $S{:}T^r_0 \rightarrow S^r$ *given by:*

$[S(A)](\tau^1,...,\tau^r) = \frac{1}{r!}\sum_\pi A(\tau^{\pi(1)}, ...\tau^{\pi(r)})$, *where the sum is taken over the r! permutations of the integers 1, 2,...., r, and* $\tau^1,....\tau^r$ *are all in* V^*. *S(A) is often denoted by* A_s.

A_s is a symmetric tensor. In fact, $A_s\ (\tau^{\sigma(1)},...,\tau^{\sigma(r)}) = [S(A)](\tau^{\sigma(1)},...,\tau^{\sigma(r)})$

$$= \frac{1}{r!}\sum_\pi A(\tau^{\pi(\sigma(1))}, ..., \tau^{\pi(\sigma(r))})$$
$$= \frac{1}{r!}\sum_{\pi\sigma} A(\tau^{\pi\sigma(1)}, ..., \tau^{\pi\sigma(r)})$$
$$= A_s(\tau^1, \tau^2, ...\tau^r),$$

where we have used the fact that the sum over π is equal to the sum over the product (or composition) $\pi\sigma$, because they both include all permutations. Furthermore,if A is symmetric, then $S(A) = A$:

$$[S(A)](\tau^1, ..., \tau^r) = \frac{1}{r!}\sum_\pi A(\tau^{\pi(1)},...,\tau^{\pi(r)}) = \frac{1}{r!}\sum_\pi A(\tau^1, ..., \tau^r) = \frac{1}{r!}\underbrace{(\sum_\pi 1)}_{=r!}A(\tau^1, ..., \tau^r)$$

$= A(\tau^1, ..., \tau^r)$.

A similar definition gives the symmetrizer $S{:}T^0_s \rightarrow S_s$. Instead of $\tau^1, ...\tau^r$ in the definition, we would have $v_1,...,v_s$.

Example 2.3. *For $r = 2$, we only have two permutations, and*

$$A_s(\tau^1, \tau^2) = \frac{1}{2}\ [A(\tau^1, \tau^2) + A(\tau^2, \tau^1)].$$

For $r = 3$, we have six permutations 1, 2, 3, and the definition gives

$$A_s(\tau^1, \tau^2, \tau^3) = \frac{1}{6}[A(\tau^1, \tau^2, \tau^3) + A(\tau^2, \tau^1, \tau^3) + A(\tau^1, \tau^3, \tau^2) + A(\tau^3, \tau^1, \tau^2) + A(\tau^3, \tau^2, \tau^1) + A(\tau^2, \tau^3, \tau^1)].$$

It is clear that the interchanging any pair of τ's on the right hand side of the above two equations does not change the sum. Thus, A_s is indeed a symmetric tensor.

We are now ready to define a product on the collection of symmetric tensors and make it an algebra, called the **symmetric algebra**.

Definition 25. *The **symmetric product** of symmetric tensors $A \in S^r(V)$ and $B \in S^s(V)$ is denoted by AB and defined as*
$AB(\tau^1,\ldots,\tau^{r+s}) \equiv \frac{(r+s)!}{r!s!} S(A\otimes B)(\tau^1, \ldots, \tau^{r+s})$
$= \frac{1}{r!s!} \sum_\pi A(\tau^{\pi(1)}, \ldots, \tau^{\pi(r)}) B(\tau^{\pi(r+1)}, \ldots, \tau^{\pi(r+s)})$, *where the sum is over all permutations of 1 2,...,r+s. The symmetric product of $A \in S_r(V)$ and $B \in S_s(V)$ is defined similarly.*

Example 2.4. *Let us construct the symmetric tensor products of vectors. Fist we find the symmetric product of v_1 and v_2 both belonging to $V = T^1_0(V)$:*
$(v_1,v_2)(\tau^1,\tau^2) \equiv v_1(\tau^1)v_2(\tau^2) + v_1(\tau^2)v_2(\tau^1)$
$= v_1(\tau^1)v_2(\tau^2) + v_2(\tau^1)v_1(\tau^2)$
$= (v_1\otimes v_2 + v_2\otimes v_1)(\tau^1,\tau^2)$.
Since it is true for any pair τ^1 and τ^2, we have

$$v_1 v_2 = v_1\otimes v_2 + v_2\otimes v_1.$$

In general $v_1,v_2\ldots v_r = \sum_\pi v_{\pi(1)} \otimes v_{\pi(2)} \otimes \ldots \otimes v_{\pi(r)}$.
It is clear from the definition that the symmetric multiplication is commutative, associative, and distributive. If we choose a basis $\{e_i\}_{i=1}^N$ for V and express all symmetric tensors in terms of symmetric products of e_i using the above properties, then any symmetric tensor can be expressed as a linear combination of terms of the form $(e_1)^{n_1}\ldots(e_N)^{n_N}$.

Skew-symmetry or **antisymmetry** is the same as symmetry except that in the interchange of variables the tensor changes sign.

Definition 26. *A **covariant (contravariant)** skew-symmetric (or anti-symmetric) tensor is one that is skew-symmetric in all pairs of covariant (contravariant) variables. A tensor is skew-symmetric if it is both covariant and contravariant skew-symmetric.*

Theorem 27. *A tensor A of type(r, 0) is skew if and only if for any permutations π of 1, 2, ... r, and any$\tau^1, \tau^2, \ldots, \tau^r$ in V^*. we have*

$$A(\tau^{\pi(1)}, \tau^{\pi(2)}, \ldots., \tau^{\pi(r)} = \epsilon_\pi\ A(\tau^1, \tau^2, \ldots, \tau^r).$$

Definition 28. *An **antisymmetrizer** is a linear operator A on T^r_0, given by*

$$[A(T)](\tau^1, ..., \tau^r) = \frac{1}{r!} \sum_\pi T(\tau^{\pi(1)}, ..., \tau^{\pi(r)}.$$

A(T) is denoted by T_a.
Clearly, T_a is an antisymmetric tensor. In fact, using $(\epsilon_\sigma)^2 = 1$, which holds for any permutation, we have
$T_a \tau^{\sigma(1)}, ..., \tau^{\sigma(r)} = [A(T)](\tau^{\sigma(1)}, ..., \tau^{\sigma(r)})$
$=(\epsilon_\sigma)^2 \frac{1}{r!} \sum_\pi \epsilon_\pi A(\tau^{\pi\sigma(1)}, ..., \tau^{\pi\sigma(r)})$
$=\epsilon_\sigma \frac{1}{r!} \sum_{\pi\sigma} \epsilon_\pi \epsilon_\sigma T(\tau^{\pi\sigma(1)}, ..., \tau^{\pi\sigma(r)})$
$=\epsilon_\sigma T_a(\tau^1, \tau^2, ...\tau^r)$.
where we used the fact that $\epsilon_\pi \epsilon_\sigma = \epsilon_{\pi\sigma}$. If T is antisymmetric, then A(T) = T:
$[A(T)](\tau^1, ..., \tau^r) = \frac{1}{r!} \sum_\pi \epsilon_\pi T(\tau^{\pi(1)}, ..., \tau^{\pi(r)})$
$=\frac{1}{r!} \sum_\pi (\epsilon_\pi)^2 T(\tau^1, ...\tau^r)$
$=\frac{1}{r!} \sum_\pi 1 \; T(\tau^1, ..., \tau^r) = T(\tau^1, ...\tau^r)$.
A similar definition gives the antisymmetrizer A on T^0_s. Instead of $\tau^1, ...\tau^r$ we would have used $v_1, ...v_s$.

Example 2.5. *Let us write the equation of an antisymmetrizer for r = 3.*
$T_a(\tau^1, \tau^2, \tau^3) = \frac{1}{6}[\epsilon_{123} A(\tau^1, \tau^2, \tau^3) + \epsilon_{213} A(\tau^2, \tau^1, \tau^3) + \epsilon_{132} A(\tau^1, \tau^3, \tau^2) + \epsilon_{312} A(\tau^3, \tau^1, \tau^2)$
$+ \epsilon_{321} A(\tau^3, \tau^2, \tau^1) + \epsilon_{231} A(\tau^2, \tau^3, \tau^1)]$
$=\frac{1}{6}[A(\tau^1, \tau^2, \tau^3) - A(\tau^2, \tau^1, \tau^3) - A(\tau^1, \tau^3, \tau^2) + A(\tau^3, \tau^1, \tau^2) - A(\tau^3, \tau^2, \tau^1) + A(\tau^2, \tau^3, \tau^1)$
$]$.

2.3 Exterior algebra

The following discussion will concentrate on tensors of type (r, 0). However, interchanging the roles of V and V* makes all definitions, theorems, propositions, and conclusions valid for tensors of type (0,s) as well. The set of all skew-symmetric tensors of type (p, 0) forms a subspace of T^p_0. This subspace is denoted by $\Lambda^p(V^*)$ and its members are called **p-vectors** It is not, however, an algebra unless we define a skew-symmetric product analogous to that for the symmetric case. This is done in the following definition:

Definition 29. *The* ***exterior product*** *(also called the wedge, Grassmann, alternating, or veck product) of two skew-symmetric tensors* $\boldsymbol{A} \in \Lambda^p(V^*)$ *and* $\boldsymbol{B} \in \Lambda^q(V^*)$ *is a skew-symmetric tensor belonging to* $\Lambda^{p+q}(V^*)$ *and given by*

$$A \wedge B \equiv \frac{(r+s)!}{r!s!} \mathbb{A} \, (A \otimes B) = \frac{(r+s)!}{r!s!} (A \otimes B)_a.$$

More explicitly,
$A \wedge B = (\tau^1, ..., \tau^{r+s})$
$= \frac{1}{r!s!} \sum_\pi \epsilon_\pi A(\tau^{\pi(1)}, ..., \tau^{\pi(r)}) B(\tau^{\pi(r+1)}, ..., \tau^{\pi(r+s)})$

Theorem 30. *The exterior product is associative and distributive with respect to the addition of tensors. Furthermore, it satisfies the following anticommutativity property:*

$$A \wedge B = (-1)^{pq} B \wedge A$$

whenever $A \in \Lambda^p(V^*)$ *and* $B \in \Lambda^q(V^*)$ *. In particular,* $v_1 \wedge v_2 = - v_2 \wedge v_1$ *for* $v_1,\ v_2 \in V$

Definition 31. *The elements of* $\Lambda^p(V^*)$ *are called* ***p-forms***

A linear transformation T : V $\rightarrow$ W induces a transformation T* : $\Lambda^p(W) \rightarrow \Lambda^p(V)$ defined by

$$(T^*\rho(v_1, ..., v_p) \equiv \rho(Tv_1, ..., Tv_p)\ ,\ \rho \in \Lambda^p(W),\ v_i \in V$$

T$^*\rho$ is is called the **pullback** of ρ by T. The most important properties of pullback maps are given in the following:
Let T : V $\rightarrow$ W and S : W $\rightarrow$ U. Then

1. T* : $\Lambda^p(W) \rightarrow \Lambda^p(V)$
2. (S $\circ$ T)* = $T^* \circ S^*$
3. If T is the identity map, so is T^*
4. If T is an isomorphism, so is T^* and $(T^*)^{-1} = (T^{-1})^*$
5. If $\rho \in \Lambda^p(W)$ and $\sigma \in \Lambda^p(W)$, then T*($\rho \wedge \sigma) = T^*\rho \wedge T^*\sigma$.

If $\{e_i\}_{i=1}^N$ is a basis of V, we can form a basis for $\Lambda^p(V^*)$ by constructing all products of the form $e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{1_p}$. The number of linearly independent such vectors, which is the dimension of $\Lambda^p(V^*)$, , is equal to the number of ways p numbers can be chosen from among N distinct numbers in such a way that no two of them are equal. This is simply the combination of N objects taken p at a time. Thus, we have

$$\dim\ (\Lambda^p(V^*)) = \binom{N}{p} = \frac{N!}{p!(N-p)!}$$

In particular dim $(\Lambda^p(V^*)) = 1$.

Any A $\in \Lambda^p(V^*)$ can be written as

$$\begin{aligned} A &= \textstyle\sum_{i_1<i_2<...<i_p}^N\ A^{i_1...i_p}\ e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{1_p} \\ &= \frac{1}{p!} \textstyle\sum_{i_1,i_2...i_p}^N\ A^{i_1...i_p}\ e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{1_p} \end{aligned}$$

where $A^{i_1...i_p}$ are the components of A, which are assumed completely antisymmetric in all $i_1, i_2...i_p$. In the second sum, all i's run from 1 to N.

Theorem 32. *Set* $\Lambda^0(V^*) = \mathbb{R}$ *and let* $\Lambda(V^*)$ *denote the direct sum of all* $\Lambda^p(V^*)$*:*

$$\Lambda(V^*) = \oplus_{p=0}^N\ \Lambda(V^*) \equiv \mathbb{R} \oplus V \oplus \Lambda^2(V^*) \oplus \Lambda^N(V^*)$$

Then $\Lambda(V^*)$ *is a* 2^N*-dimensional algebra with exterior product defining its multiplication rule.*

Given two vector spaces V and U, one can construct a tensor product of $\Lambda(V^*)$ and $\Lambda(U^*)$ and define a product $\odot$ on it as follows. Let $A_i \in \Lambda^{p_i}(V^*)$, i = 1, 2 and $B_j \in \Lambda^{q_j}(U^*)$, j = 1, 2. Then

$$(A_1 \otimes B_1) \odot (A_2 \otimes B_2) \equiv (-1)^{p_2 q_1} (A_1 \wedge A_2) \otimes (B_1 \wedge B_2).$$

Definition 33. *The tensor product of the two vector spaces $\Lambda(V^*)$ and $\Lambda(U^*)$ together with the product given before is called **skew tensor product** of $\Lambda(V^*)$ and $\Lambda(U^*)$ and denoted by $\Lambda(V^*) \overset{\wedge}{\otimes} \Lambda(U^*)$*

An elegant way of determining the linear independence of vectors using the formalism developed so far is given in the following proposition.

Proposition 1. *A set of vectors $v_1, ..., v_p$ is linearly independent if and only if $v_1 \wedge ... \wedge v_p$ $\neq$ 0.*

Proof. If $\{v_i\}_{i=1}^p$ are independent, then they span a p-dimensional subspace M of V. Considering M as a vector space in its own right, we have dim $\Lambda^p(M^*) = 1$. A basis for $\Lambda^p(M^*)$ is simply $v_1 \wedge ... \wedge v_p$, which cannot be zero.

Conversely, suppose that $\alpha_1 v_1 + ...\alpha_p v_p = 0$. Then taking the exterior product of the left hand side with $v_2 \wedge v_3 \wedge ... \wedge v_p$ makes all terms vanish (because each will have two factors of a vector in the wedge product) except the first one. Thus, we have $\alpha_1 v_1 \wedge ... \wedge v_p$. The fact that the wedge product is not zero forces α_1 to be zero. Similarly, multiplying by $v_1 \wedge v_3 \wedge ... \wedge v_p$ shows that $\alpha_2 = 0$, and so on. □

Example 2.6. *Let $\{e_i\}_{i=1}^N$ be a basis for V. Let $v_1 = e_1 + 2e_2 - e_3$ $v_2 = 3e_1 + e_2 + 2e_3$, $v_3 = -e_1 - 3e_2 + 2e_3$.*

$$\begin{aligned} v_1 \wedge v_2 &= (e_1 + 2e_2 - e_3) \wedge (3e_1 + e_2 + 2e_3) \\ &= -5e_1 \wedge e_2 + 5e_1 \wedge e_3 + 5\, e_2 \wedge e_3. \end{aligned}$$

All the wedge products that have repeated factors vanish. Now we multiply by v_3:

$$\begin{aligned} v_1 \wedge v_2 \wedge v_3 &= -5e_1 \wedge e_2 \wedge (-e_1 - 3e_2 + 2e_3) \\ &\quad +5e_1 \wedge e_3 \wedge (-e_1 - 3e_2 + 2e_3) \\ &\quad +5e_2 \wedge e_3 \wedge (-e_1 - 3e_2 + 2e_3) \\ &= -10e_1 \wedge e_2 \wedge e_3 - 15e_1 \wedge e_3 \wedge e_2 - 5e_2 \wedge e_3 \wedge e_1 = 0. \end{aligned}$$

We conclude that the three vectors are linearly dependent.

As an aplication of Proposition 1 let us prove the following.

Theorem 34. *(Cartan's lemma) Suppose that $\{e_i\}_{i=1}^p$, $p \leq$ dim V form a linearly independent set of vectors in V and that $\{v_i\}_{i=1}^p$ are also vectors in V such that $\sum_{i=1}^p e_i \wedge v_i$ = 0. Then all v_i are linear combinations of only the set $\{e_i\}_{i=1}^p$. Furthermore, if $v_i = \sum_{j=1}^p A_{ij} e_j$, then $A_{ij} = A_{ji}$.*

Proof. Multiplying both sides of $\sum_{i=1}^p e_i \wedge v_i = 0$ by $e_2 \wedge ...e_p$ gives

$$-v_1 \wedge e_1 \wedge e_2 \wedge ... \wedge e_p = 0.$$

By Proposition 1, v_1 and the e_i are linearly dependent. Similarly, by multiplying $\sum_{i=1}^p e_i \wedge v_i = 0$ by the wedge product with e_k we show that v_k and the e_i are linearly dependent. Thus, $v_k = \sum_{i=1}^p A_{ki} e_i$, for all k. Furthermore, we have

$$0 = \sum_{k=1}^{p} e_k \wedge v_k = \sum_{k=1}^{p} \sum_{i=1}^{p} e_k \wedge (A_{ki} e_i) = \sum_{k<i} (A_{ki} - A_{ik}) e_k \wedge e_i$$

where the last sum is over both k and i with k<i. Clearly, $\{e_k \wedge e_i\}$ with k $<$ i are linearly independent . Therefore, their coefficients must vanish. □

Definition 35. *The symbol $\epsilon_{i_1,i_2,...,i_n}$ called the **Levi-Civita tensor**, can be defined by*

$$\epsilon^{i_1} \wedge ... \wedge \epsilon^{i_N} = \epsilon_{i_1 ... i_N} \epsilon^1 \wedge ... \epsilon^N.$$

Proposition 2. *The Levi-Civita tensor $\epsilon_{i_1,i_2,...,i_n}$ takes the same value in all coordinate systems.*

Definition 36. *A U-valued p-form, is a linear machine that takes p vectors from V and produces a vector in U. The space of U-valued p-forms is denoted by $\Lambda^p(V, U)$. In this new context $\Lambda^p(V) = \Lambda^p(V, \mathbb{R})$.*

2.4 Orientation

Definition 37. *An **oriented basis** of an N-dimensional vector space is an ordered collection of N linearly independent vectors.*

If $\{v_i\}_{i=1}^N$ is one oriented basis and $\{u_i\}_{i=1}^N$ is a second one, then
$$u_1 \wedge u_2 \wedge ... \wedge u_N = (detR) v_1 \wedge v_2 \wedge ... \wedge v_n,$$

where R is the transformation matrix and det R is a nonzero number (R is invertible), which can be positive or negative

Accordingly, we have the following definition

Definition 38. *An **orientation** is the collection of all oriented bases related by a transformation matrix having a positive determinant. A vector space for which an orientation is specified is called an **oriented** vector space.*

Clearly, there are only two orientations in any vector space. Each oriented basis is positively related to any oriented basis belonging to the same orientation and negatively related to any oriented basis belonging to the other orientation. For example, in $\mathbb{R}^3$, the bases $\{e_x, e_y, e_z\}$ and $\{e_y, e_x, e_z\}$ belong to different orientations because

$$e_x \wedge e_y \wedge e_z = \text{-}e_y \wedge e_x \wedge e_z.$$

The first basis is (by convention) called a right-handed coordinate system, and the second is called a left-handed coordinate system. Any other basis is either right-handed or left-handed. There is no third alternative!

Definition 39. *Let V be a vector space. Let V^* have the oriented basis $\{\epsilon_i\}_{i=1}^N$. The oriented **volume element** $\mu \in \Lambda^N(V)$ of V is defined as*

$$\mu \equiv \epsilon^1 \wedge \epsilon^2 \wedge ... \wedge \epsilon^N.$$

Note that if $\{e_i\}$ is as ordered as $\{\epsilon^j\}$ then $\mu(e_1, e_2, ..., e_N) = +1/N!$ and we say that $\{e_i\}$ is positively oriented with respect to μ. In general, $\{v_i\}$ is positively oriented with respect to μ if $\mu(v_1, v_2, ..., v_N) > 0$.
The volume element of V is defined in terms of a basis for V*

2.5 Symplectic Vector Spaces

Definition 40. *A 2-form $\omega \in \Lambda^2(V)$ is* ***nondegenerate*** *if $\omega(v_1, v_2) = 0$ for all $v_1 \in V$ implies $v_2 = 0$.*
A ***symplectic form*** *on V is a nondegenerate 2-form $\omega \in \Lambda^2(V)$.*
The pair(V,ω) is called a ***symplectic vector space****.*
If (V,ω) and (W,ρ) are symplectic vector spaces, a linear transformation $T : V \to W$ is called a ***symplectic transformation*** *or a* ***symplectic map*** *if $T^*\rho = \omega$.*

Any 2-form (degenerate or nondegenerate) leads to other quantities that are also of interest. For instance, given any basis $\{v_i\}$ in V, one defines the **matrix** of the 2-form $\omega \in \Lambda^2(V)$ by $\omega_{ij} \equiv \omega(\mathrm{v}_i, \mathrm{v}_j)$. Similarly, one can define the useful linear map $\omega^\flat$: V→ V* by

$$[\omega^\flat(v)]v' \equiv \omega(v, v').$$

where $[\omega^\flat(v)]v' \in \mathbb{R}$ and $[\omega^\flat(v)] \in V^*$
The rank of $\omega^\flat$ is called the **rank** of ω.

Remark 41. *A 2-form ω is nondegenerate if and only if the determinant of (ω_{ij}) is nonzero, if and only if $\omega^\flat$ is an isomorphism, in which case the inverse of $\omega^\flat$ is denoted by $\omega^\sharp$.*

Proposition 3. *Let (V,ω) be a symplectic vector space. Then the set of symplectic mappings T : (V,ω) → (V,ω) forms a group under composition, called the* ***symplectic group*** *and denoted by Sp(V,ω).*

2.6 Inner Product

Definition 42. *A* ***symmetric bilinear form*** *b on V is a symmetric tensor of type (0, 2).*

If $\{\mathrm{e}_j\}_{j=1}^N$ is a basis of V and $\{\epsilon^i\}_{i=1}^N$ is its dual basis, then b $= \frac{1}{2}b_{ij}\epsilon^i\epsilon^j$, because $\epsilon^i\epsilon^j = \epsilon^i \otimes \epsilon^j + \epsilon^j \otimes \epsilon^i$ form a basis of S_2(V). If v and u are any two vectors in V, then

$$\begin{aligned}\mathrm{b(v,u)} &= \frac{1}{2}b_{ij}(\epsilon^i \otimes \epsilon^j + \epsilon^j \otimes \epsilon^i)(v^k e_k, u^m e_m)\\ &= \frac{1}{2}b_{ij}v^k u^m[\epsilon^i(e_k)\epsilon^j(e_m) + \epsilon^j(e_k) + \epsilon^i(e_m)]\\ &= \frac{1}{2}b_{ij}v^k u^m[\zeta^i_k\zeta^j_m + \zeta^j_k\zeta^i_m]\\ &= \frac{1}{2}b_{ij}(v^i u^j + v^j u^i)\\ &= b_{ij}v^i u^j\end{aligned}$$

For any vector v ∈ V, we can write

$$\begin{aligned}\mathrm{b(v)} &= \frac{1}{2}b_{ij}\epsilon^i\epsilon^j(v)\\ &= \frac{1}{2}b_{ij}(\epsilon^i \otimes \epsilon^j + \epsilon^j \otimes \epsilon^i)(v^k e_k)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} b_{ij} v^k [\epsilon^i \epsilon^j (e_k) \epsilon^j \epsilon^i (e_k)] \\
&= \frac{1}{2} b_{ij} v^k [\epsilon^i \zeta^j_k (e_k) \epsilon^j \zeta^i_k (e_k)] \\
&= \frac{1}{2} b_{ij} [v^j \epsilon^i + v^i \epsilon^j] \\
&= b_{ij} v^j \epsilon^i \\
&= b_{ij} v^i \epsilon^j .
\end{aligned}
$$

Thus, b(v)$\in$ V*. This shows that b can be thought of as a mapping from V to V*, which we denote by b_* and write b_* : V $\rightarrow$ V*. For this mapping to make sense, it should not matter which factor in the symmetric product v contracts with. But this is a trivial consequence of the symmetries b$_{ij}$ = b$_{ji}$ and $\epsilon^i \epsilon^j = \epsilon^j \epsilon^i$.

Let v and u be any two vectors in V. Let $\{e_j\}_{j=1}^N$ be a basis of V and $\{\epsilon^i\}_{i=1}^N$ its dual basis. The natural pairing of v and b$_*$(u) is given by

$$
\begin{aligned}
&\langle b_*(u), v\rangle = \\
&= \langle b_{ij} u^j \epsilon^i, v^k e_k \rangle \\
&= b_{ij} u^j v^k \langle \epsilon^i, e_k \rangle \\
&= b_{ij} u^j v^k \zeta^i_k \\
&= b_{ij} u^j v^i \\
&= b(u, v) \\
&= b(v, u)
\end{aligned}
$$

where we used Proposition **??** in the last step.

The components $b_{ij} v^j$ of b_*(v) in the basis of $\{\epsilon^i\}_{i=1}^N$ of V^* are denoted by v$_i$, so

$$
b_*(v) = v_i \epsilon^i
$$

where $v_i = b_{ij} v^j$.

We have thus **lowered** the index of v^j by the use of the symmetric bilinear form b. In applications v$_i$ is uniquely defined; furthermore, there is a one-to-one correspondence between v$_i$ and v^i . This can happen if and only if the mapping b$_*$: V $\rightarrow$ V* is invertible, in which case b is usually denoted by g. If g$_*$ is invertible, there must exist a unique $(g_*)^-1 \equiv (g^-1)_* : V^* \rightarrow V$, or $g^-1 \in S_2(V^*) = S_2(V)$, such that

$$
\begin{aligned}
&v^j e_j = \mathrm{v} = \\
&= (g_*)^{-1} g_*(v) \\
&= (g_*)^{-1} (v_i \epsilon^i) \\
&= v_i (g_*)^{-1} (\epsilon^i) \\
&= v_i [(g^{-1})^{jk} e_j e_k] (\epsilon^i) \\
&= v_i (g^{-1})^{jk} e_j \underbrace{e_k(\epsilon^i)}_{=\zeta^i_k} \\
&= v_i (g^{-1})^{ji} e_j .
\end{aligned}
$$

Comparison of the LHS and the RHS yield $v^j = v_i (g^{-1})^{ji}$. It is customary to omit the -1 and simply write

$$v^j = g^{ji} v_i,$$

where it is understood that g with upper indices is the inverse of g (with lower indices).

Definition 43. *An invertible bilinear form is called* ***nondegenerate****.*

Definition 44. *A symmetric bilinear form g that is nondegenerate is called an* ***inner product****. When there is no danger of confusion, we write $\langle u, v \rangle$ instead of $g(u,\, v)$.*

Bibliografie

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