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GRADUATION THESIS

Tensors

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1 Preliminaries

1.1 Vector space

Before we dive into the notion of tensors, first we need to remember fundamental concepts that are necessary into understanding the tensors.

Let us begin with the definition of an abstract vector space.

Definition 1. *A* ***vector space*** *V over $\mathbb{C}$ is a set of objects called* ***vectors****, with the following properties:*

1. To every pair of vectors x and y in V there corresponds a vector $x + y$ also in V, called the sum of x and y such that:

a) $x + y = y + x$

b) $x + (y + z) = (x + y) + z$

c) There exists a unique vector $0 \in V$, called the ***zero vector****, such that $x + 0 = x$ for every vector x*

d) To every vector $x \in V$ there corresponds a unique vector $-x \in V$ such that $x + (-x) = 0$

2. To every complex number α(also called a ***scalar****) and every vector x there corresponds a vector αx in V such that:*

a) $\alpha(\beta x) = (\alpha\beta)x$

b) $1x = x$

3. Multiplication involving vectors and scalars is distributive:

a) $\alpha(x + y) = \alpha x + \alpha y$

b) $(\alpha + \beta)x = \alpha x + \beta x$

Example 1.1. $\mathbb{C}^2$ *is a vector space with multiplication and addition having their usual meaning.*

Now we shall remember the inner product as it is defined, so we can revisit it later on when we talk about tensors.

Remark 2. *The inner product generalizes the dot product to abstract vector spaces over a field of scalars, being either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. It is usually denoted by $\langle x, y\rangle$.*

Definition 3. *The* ***inner product*** *of two vectors, x and y, in a vector space V is a complex number, $\langle x, y\rangle \in \mathbb{C}$ such that*

1. $\langle x, y\rangle = \langle y, x\rangle^*$

2. $\langle x, (\beta y + \gamma z)\rangle = \beta\langle x, y\rangle + \gamma\langle x, z\rangle$

3. $\langle x, x\rangle \geqslant 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

The last relation is called the ***positive definite*** *property of the inner product. A positive definite real inner product is also called a* ***Euclidian*** *inner product, otherwise it is called* ***pseudo-Euclidian****.*

Definition 4. *The vectors x_1,x_2,....,x_n are said to be **linearly independent** if for $\alpha_i \in \mathbb{C}$, the relation $\sum_{i=1}^{n} \alpha_i x_i = 0$ implies $\alpha_i = 0$ for all i. The vectors are called **linearly dependent** otherwise.*

Definition 5. *A **subspace** W of a vector space V is a nonempty subset of V with the property that if x,y $\in$ W, then $\alpha x + \beta y$ also belongs to W, $\forall\ \alpha,\beta \in \mathbb{C}$*

Remark 6. *The subspace is a vector space in its own right and the intersection of two subspaces is also a subspace.*

Definition 7. *A **basis** of a vector space V is a set B of linearly independent vectors that spans all of V. A vector space that has a finite basis is called **finite-dimensional** and **infinite-dimensional** otherwise. We call the cardinality of the set B the **dimension** of V.*

Definition 8. *A **linear map**(or **transformation**) from the complex vector space V to the complex vector space W is a mapping $\boldsymbol{T}$:V$\rightarrow$W such that:*
$\boldsymbol{T}(\alpha x + \beta y) = \alpha \boldsymbol{T}(x) + \beta \boldsymbol{T}(y)\ \forall\ x,y \in V$ and $\alpha,\beta \in \mathbb{C}$
*A linear transformation $\boldsymbol{T}$:V$\rightarrow$V is called an **endomorphism** of V or a **linear operator** on V.*

An important example of linear transformations occurs when the second vector space W, happens to be the set of scalars, $\mathbb{C}$ or $\mathbb{R}$, in which case the linear transformation is called a **linear functional**. The set of linear functionals $\mathcal{L}$(V,$\mathbb{C}$) or $\mathcal{L}$(V,$\mathbb{R}$) if V is a real vector space is denoted by V* and is called the **dual space** of V.

Definition 9. *A vector space V is said to be isomorphic to another vector space W and written $V \cong W$, if there exists a bijective linear map $\boldsymbol{T}$: V $\rightarrow$ W. Then $\boldsymbol{T}$ is called an **isomorphism**. A bijective linear map of V onto itself is called an **automorphism** of V.An automorphism is also called an **invertible** linear map.*

Definition 10. *Let A be an $N \times N$ matrix. The mapping tr:$M^{(N \times N)} \rightarrow \mathbb{C}$ (or $\mathbb{R}$) given by tr A $= \sum_{i=1}^{n} \alpha_{ii}$ is called the **trace** of A.*

Theorem 11. *The trace is a linear mapping. Furthermore, tr(AB) = tr(BA) and trA^t = trA.*

Proof. To prove the first identity, we use the definitions of trace and matrix product:
$\mathrm{tr(AB)} = \sum_{i=1}^{n} (\mathrm{AB})_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathrm{A})_{ij} (\mathrm{B})_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} (\mathrm{B})_{ji} (\mathrm{A})_{ij} = \sum_{j=1}^{n} (\sum_{i=1}^{n} (\mathrm{B})_{ji} (\mathrm{A})_{ij}) = \sum_{j=1}^{n} (\mathrm{BA})_{jj} = \mathrm{tr(BA)}$.
The linearity of the trace and the second identity follow directly from the definition. $\square$

1.2 Multilinear maps

There is a very useful generalization of the linear functionals that becomes essential in the treatment of tensors. However, a limited version of its application is used in the discussion of determinants, which we shall start here.

Definition 12. *Let V and U be vector spaces. Let V^P denote the p-fold Cartesian product of V. A **p-linear map** from V to U is a map Θ:$V^P \rightarrow U$ which is linear with respect to each of its arguments:*
$\Theta(x_1,...,\alpha x_j + \beta y_j,...,x_p) = \alpha\Theta(x_1,....,x_j,....,x_p) + \beta\Theta(x_1,....,y_j,...,x_p)$.
*A p-linear map from V to $\mathbb{C}$ or $\mathbb{R}$ is called a **p-linear function** in V.*

Example 1.2. *Let $\{\Phi_i\}_i^p$ be a linear functionals on V. Define Θ by $\Theta(x_1,...,x_p) = \Phi_1(x_1),...,\Phi_P(x_P)$, $x_i \in V$.*
Clearly Θ is p-linear.

Let σ denote the permutation of 1, 2,..., p. Define the p-linear map $\sigma\omega$ by $\sigma\omega(x_1,...,x_p) = \omega(x_{\sigma(1)},....,x_{\sigma(p)})$

Definition 13. *A p-linear map ω from V to U is **skew-symmetric** if $\sigma\omega = \epsilon_\sigma$ if:*

$$\omega(x_{\sigma(1)},...,x_{\sigma(p)}) = \epsilon_\sigma \omega(x_1,...,x_p)$$

where ϵ_σ is the sign of σ, which is +1 if σ is even and -1 if it is odd. The set of p-linear skew-symmetric maps from V to U is denoted by $\Lambda^P(V,U)$. The set of p-linear skew-symmetric functions in V is denoted by $\Lambda^P(V)$.
The permutation sign ϵ_ω is sometimes written as

$$\epsilon_\sigma = \epsilon_{\sigma(1)\sigma(2)...\sigma(p)} \equiv \epsilon_{i_1 i_2 ... i_p}, \text{ where } i_k \equiv \sigma(k).$$

Definition 14. *A skew symmetric N-linear function in V,i.e., a member of $\Lambda^N(V)$ is called a **determinant function** in V.*
Let $B = \{|e_k\}_{k=1}^N$ be a basis of V and $B^* = \{\epsilon_k\}_{j=1}^N$ a basis of V^*, dual to B. For any set of N vector $\{x_k\}_{k=1}^N$ in V, define the N-linear function Θ by

$$\Theta(x_1,...,x_N) = \epsilon_1(x_1)...\epsilon_N(x_N),$$

Now let Δ be defined by $\Delta \equiv \sum_\pi \epsilon_\pi \cdot \theta$ Then, $\Delta \in \Lambda^N(V)$,i.e., Δ is a determinant function.

Let **A** be a linear operator on an N-dimensional vector space V. Choose a nonzero determinant function Δ. For a basis $\{\mathrm{v}_i\}_{i=1}^N$ define the function Δ_A by

$$\Delta_A(\mathrm{v}_1,...,\mathrm{v}_N) \equiv \Delta(\mathrm{Av}_1,...,\mathrm{Av}_N).$$

Definition 15. *Let $A \in End(V)$. Let Δ be a nonzero determinant function in V, and let Δ_A be as mentioned before. Then*

$$\Delta_A = det\ A \cdot \Delta$$

*defines the **determinant of A**.*

2 Tensors

From here on,we will consider only real vector spaces and the basis vectors of a vector space V will be distinguished by a subscript and those of its dual space by a superscript.For example, if $\{e_i\}_{i=1}^N$ is a basis in V, then the basis in V^* will be $\{\epsilon^j\}_{j=1}^N$ so we can avoid confusions. **Einstein's summation convention** will also be used:
Repeated indices, of which one is an upper and the other a lower index, are assumed to be summed over: $a_i^k b_j^i$ means $\sum_{i=1}^N a_i^k b_j^i$.
It is more natural to label the elements of a matrix representation of an operator **A** by α_j^i (rather than α_{ji}, because then $Ae_i = \alpha_i^j e_j$.

2.1 Tensors as Multilinear Maps

Since tensors are special kinds of linear operators on vector spaces, let us reconsider $\mathcal{L}$(V,W), the space of all linear mappings from the real vector space V to the real vector space W.

Definition 16. *A map $\boldsymbol{T}$:$V_1\times\ V_2\times ...\times\ V_r \to W$ is called r-**linear** if it is linear in all its variables:*

$$\boldsymbol{T}(v_1,....,\alpha v_i + \alpha' v_i',....,v_r) = \alpha\boldsymbol{T}(v_1,....,v_i,....,v_r) + \alpha'\boldsymbol{T}(v_1,....,v'_i,....,v_r) \text{ for all } i.$$

Definition 17. *Let $\tau_1 \in V_1^*$ and $\tau_2 \in V_2^*$. We construct the bilinear map $\tau_1 \otimes \tau_2 : V_1 \times V_2 \to \mathbb{R}$ by $\tau_1 \otimes \tau_2(v_1,v_2) = \tau_1(v_1)\tau_2(v_2)$. The expression $\tau_1 \otimes \tau_2$ is called the **tensor product** of τ_1 and τ_2*

An r-linear map can be multiplied by a scalar, and two r-linear maps can be added; in each case the result is an r-linear map. Thus, the set of r-linear maps from $V_1 \times \cdots \times V_r$ into W forms a vector space that is denoted by $\mathcal{L}(V_1,...,V_r;W)$.

We can also construct multilinear maps on the dual space. First, we note that we can define a natural linear functional on V^* as follows. We let $\tau \in V^*$ and v $\in$ V; then τ(v) $\in \mathbb{R}$. Now we twist this around and define a mapping v : $V_* \to \mathbb{R}$ given by v(τ) $\equiv \tau$(v). We have naturally constructed a linear functional on V^* by identifying $(V^*)^*$ with V.

Definition 18. *Let V be a vector space with dual space V^*. Then a **tensor of type (r,s)** is a multilinear mapping*

$$\boldsymbol{T}^{\,r}_{\,s}:\underbrace{V^* \times V^* \times ... \times V^*}_{r\ times} \times \underbrace{V \times V \times ... \times V}_{s\ times}$$

*The set of all such mappings for fixed r and s forms a vector space denoted by $T_s^r(V)$. The number r is called the **contravariant degree** of the tensor, and s is called the **covariant degree** of the tensor.*

Example 2.1.

a) A tensor of type (0, 0) is defined to be a scalar, so $T_0^0(V) = (R)$.

b) A tensor of type (1, 0), an ordinary vector, is called a contravariant vector, and one of type (0, 1), a dual vector (or a linear functional), is called a covariant vector.

c) A tensor of type (r, 0) is called a contravariant tensor of rank r, and one of type (0,s) is called a covariant tensor of rank s.

The union of T_s^r (V) for all possible r and s can be made into an (infinite-dimensional) algebra called **algebra of tensors**.
First we define the following product on it:

Definition 19. *The* ***tensor product*** *of a tensor* $\boldsymbol{T}$ *of type (r,s) and a tensor* $\boldsymbol{U}$ *of type (k,l) is a tensor* $\boldsymbol{T}\otimes\boldsymbol{U}$ *of type (r+k,s+l),defined, as an operator on* $(V^*)^{r+k} \times V^{s+l}$*, by*

$$\boldsymbol{T}\otimes\boldsymbol{U}(\theta^1,\ldots,\theta^{r+k},u_1,\ldots,u_{s+l}) = \boldsymbol{T}(\theta^1,\ldots,\theta^r,u_1,\ldots,u_s)\boldsymbol{U}(\theta^{r+1},\ldots,\theta^{r+k},u_{s+l},\ldots,u_{s+l}).$$

This product turns the (infinite-dimensional) vector space of all tensors into an associative algebra called a ***tensor algebra****.*

Example 2.2. *What is the tensor product of* $\boldsymbol{A} = 2e_x - e_y + 3e_z$ *with itself?*
$\boldsymbol{A}\otimes\boldsymbol{A} = (2,-1,3)\otimes(2-1,3)$
Using the formula from above, we can compute the tensor product
$(2,-1,3)\otimes(2-1,3)=(2 \cdot 2,2 \cdot -1,2 \cdot 3,-1 \cdot 2,-1 \cdot -1,-1 \cdot 3,3 \cdot 2,3 \cdot -1,3 \cdot 3)$
$(2,-1,3)\otimes(2-1,3) = (4, -2, 6, -2, 1, -3, 6, -3, 9)$

Difference between direct sum and tensor product
We'll assume V and W are finite dimensional vector spaces. That means we can think of V as $\mathbb{R}^n$ and W as $\mathbb{R}^m$ for some positive integers n and m.I will use the following simple example to show the difference between direct sum and tensor product:
Let x.y be two vectors, x $\in \mathbb{R}^3$ and y $\in \mathbb{R}^2$

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix}, y = \begin{pmatrix}4\\5\end{pmatrix}$$

We call x $\oplus$ y the **direct sum** .In this case, x$\oplus$y is:

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$$

We shall see that the direct sum give a list of m + n numbers, this gives us a way to build a space where the dimensions **add**.
Now lets see the tensor product of x and y, denoted x$\otimes$y.

$$x = \begin{pmatrix}1\\2\\3\end{pmatrix} \oplus \begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}1\cdot 4\\1\cdot 5\\2\cdot 4\\2\cdot 5\\3\cdot 4\\3\cdot 5\end{pmatrix}$$

We see that the tensor product gives a list of m * n numbers, this gives us a way to build a space where the dimensions **multiply**.

Definition 20. *A* ***contraction*** *of a tensor* $A \in T^r_s(V)$ *with respect to a contravariant index at position p and covariant index at position q is a linear mapping* $C^p_q(V) \rightarrow T^{r-1}_{s-1}(V)$ *given in component form by*

$$[C^p_q(A)]^{i_1 \dots i_{r-1}}_{j_1 \dots j_{s-1}} = A^{\ i_1 \dots i_{p-1} k i_{p+1} \dots i_r}_{\ j_1 \dots j_{q-1} k j_{q+1} \dots j_s} \equiv \sum_k A^{\ i_1 \dots i_{p-1} k i_{p+1} \dots i_r}_{\ j_1 \dots j_{q-1} k j_{q+1} \dots j_s}$$

2.2 Symmetries of Tensors

Many applications demand tensors that have some kind of symmetry property. One symmetric tensor is the metric 'tensor' of an inner product: If V is a vector space and $v_1, v_2 \in V$, then $g(v_1,v_2) = g(v_2,v_1)$. The following generalizes this property.

Definition 21. *A tensor A is* ***symmetric*** *in the ith and jth variables if its value as a multilinear function is unchanged when these variables are interchanged.Clearly, the two variables must be of the same kind.*

From this definition. it follows that in any basis, the components of a symmetric tensor do not change when the ith and jth indices are interchanged.

Definition 22. *A tensor is* ***contravariant-symmetric*** *if it is symmetric in every pair of its covariant indices. A tensor is* ***symmetric*** *if it is both contravariant-symmetric and covariant-symmetric.*

An immediate consequence of this definition is the following theorem:

Theorem 23. *A tensor S of type(r,0) is symmetric if and only if for any permutation* π *of 1,2,...r, and any* $\tau^1, \tau^2, \dots, \tau^r$ *in* V^* *we have*

$$S(\tau^{\pi(1)}, \tau^{\pi(2)}, \dots, \tau^{\pi(r)}) = S(\tau^1, \tau^2, \dots, \tau^n)$$

Definition 24. *A* ***symmetrizer*** *is an operator* $S: T^r_0 \rightarrow S^r$ *given by:*

$[S(A)](\tau^1,\dots,\tau^r) = \frac{1}{r!}\sum_\pi A(\tau^{\pi(1)}, \dots \tau^{\pi(r)})$, *where the sum is taken over the r! permutations of the integers 1, 2,...., r, and* $\tau^1, \dots. \tau^r$ *are all in* V^**. S(A) is often denoted by* A_s*.*

A_s is a symmetric tensor. In fact, $A_s\ (\tau^{\sigma(1)},\dots,\tau^{\sigma(r)}) = [S(A)](\tau^{\sigma(1)},\dots,\tau^{\sigma(r)})$

$$= \frac{1}{r!}\sum_\pi A(\tau^{\pi(\sigma(1))}, \dots, \tau^{\pi(\sigma(r))})$$
$$= \frac{1}{r!}\sum_{\pi\sigma} A(\tau^{\pi\sigma(1)}, \dots, \tau^{\pi\sigma(r)})$$
$$= A_s(\tau^1, \tau^2, \dots \tau^r),$$

where we have used the fact that the sum over π is equal to the sum over the product (or composition) $\pi\sigma$, because they both include all permutations. Furthermore,if A is symmetric, then S(A) = A:

$$[S(A)](\tau^1, \dots, \tau^r) = \frac{1}{r!}\sum_\pi A(\tau^{\pi(1)},\dots,\tau^{\pi(r)}) = \frac{1}{r!}\sum_\pi A(\tau^1, \dots, \tau^r) = \frac{1}{r!}\underbrace{(\sum_\pi 1)}_{=r!}A(\tau^1, \dots, \tau^r)$$

$= A(\tau^1, \dots, \tau^r)$.

A similar definition gives the symmetrizer $S: T^0_s \rightarrow S_s$. Instead of $\tau^1, \dots \tau^r$ in the definition, we would have $v_1, \dots, v_s$.

Example 2.3. *For $r = 2$, we only have two permutations, and*

$$A_s(\tau^1, \tau^2) = \frac{1}{2} [A(\tau^1, \tau^2) + A(\tau^2, \tau^1)].$$

For $r = 3$, we have six permutations 1, 2, 3, and the definition gives

$$A_s(\tau^1, \tau^2, \tau^3) = \frac{1}{6}[A(\tau^1, \tau^2, \tau^3) + A(\tau^2, \tau^1, \tau^3) + A(\tau^1, \tau^3, \tau^2) + A(\tau^3, \tau^1, \tau^2) + A(\tau^3, \tau^2, \tau^1) + A(\tau^2, \tau^3, \tau^1)].$$

It is clear that the interchanging any pair of τ's on the right hand side of the above two equations does not change the sum. Thus, A_s is indeed a symmetric tensor.

We are now ready to define a product on the collection of symmetric tensors and make it an algebra, called the **symmetric algebra**.

Definition 25. *The* ***symmetric product*** *of symmetric tensors $A \in S^r(V)$ and $B \in S^s(V)$ is denoted by AB and defined as*
$AB(\tau^1,...,\tau^{r+s}) \equiv \frac{(r+s)!}{r!s!} S(A \otimes B)(\tau^1, ..., \tau^{r+s})$
$= \frac{1}{r!s!} \sum_\pi A(\tau^{\pi(1)}, ..., \tau^{\pi(r)}) B(\tau^{\pi(r+1)}, ..., \tau^{\pi(r+s)})$, *where the sum is over all permutations of 1 2,...,r+s. The symmetric product of $A \in S_r(V)$ and $B \in S_s(V)$ is defined similarly.*

Example 2.4. *Let us construct the symmetric tensor products of vectors. Fist we find the symmetric product of v_1 and v_2 both belonging to $V = T^1_0(V)$:*
$(v_1,v_2)(\tau^1,\tau^2) \equiv v_1(\tau^1)v_2(\tau^2) + v_1(\tau^2)v_2(\tau^1)$
$= v_1(\tau^1)v_2(\tau^2) + v_2(\tau^1)v_1(\tau^2)$
$= (v_1 \otimes v_2 + v_2 \otimes v_1)(\tau^1,\tau^2)$.
Since it is true for any pair τ^1 and τ^2, we have

$$v_1 v_2 = v_1 \otimes v_2 + v_2 \otimes v_1.$$

In general $v_1, v_2 ... v_r = \sum_\pi v_{\pi(1)} \otimes v_{\pi(2)} \otimes ... \otimes v_{\pi(r)}$.
It is clear from the definition that the symmetric multiplication is commutative, associative, and distributive. If we choose a basis $\{e_i\}_{i=1}^N$ for V and express all symmetric tensors in terms of symmetric products of e_i using the above properties, then any symmetric tensor can be expressed as a linear combination of terms of the form $(e_1)^{n_1} ... (e_N)^{n_N}$.

Skew-symmetry or **antisymmetry** is the same as symmetry except that in the interchange of variables the tensor changes sign.

Definition 26. *A* ***covariant (contravariant)*** *skew-symmetric (or anti-symmetric) tensor is one that is skew-symmetric in all pairs of covariant (contravariant) variables. A tensor is skew-symmetric if it is both covariant and contravariant skew-symmetric.*

Theorem 27. *A tensor A of type$(r, 0)$ is skew if and only if for any permutations π of 1, 2, ... r, and any$\tau^1, \tau^2, ..., \tau^r$ in V^*. we have*

$$A(\tau^{\pi(1)}, \tau^{\pi(2)}, ..., \tau^{\pi(r)} = \epsilon_\pi A(\tau^1, \tau^2, ..., \tau^r).$$

Definition 28. *An* ***antisymmetrizer*** *is a linear operator A on T^r_0, given by*

$$[A(T)](\tau^1, ..., \tau^r) = \frac{1}{r!} \sum_\pi T(\tau^{\pi(1)}, ..., \tau^{\pi(r)}.$$

A(T) is denoted by T_a.
Clearly, T_a is an antisymmetric tensor. In fact, using $(\epsilon_\sigma)^2 = 1$, which holds for any permutation, we have
$T_a\tau^{\sigma(1)}, ..., \tau^{\sigma(r)} = [A(T)](\tau^{\sigma(1)}, ..., \tau^{\sigma(r)})$
$=(\epsilon_\sigma)^2 \frac{1}{r!} \sum_\pi \epsilon_\pi A(\tau^{\pi\sigma(1)}, ..., \tau^{\pi\sigma(r)})$
$=\epsilon_\sigma \frac{1}{r!} \sum_{\pi\sigma} \epsilon_\pi \epsilon_\sigma T(\tau^{\pi\sigma(1)}, ..., \tau^{\pi\sigma(r)})$
$=\epsilon_\sigma T_a(\tau^1, \tau^2, ...\tau^r)$.
where we used the fact that $\epsilon_\pi\epsilon_\sigma = \epsilon_{\pi\sigma}$. If T is antisymmetric, then A(T) = T:
$[A(T)](\tau^1, ..., \tau^r) = \frac{1}{r!} \sum_\pi \epsilon_\pi T(\tau^{\pi(1)}, ..., \tau^{\pi(r)})$
$=\frac{1}{r!} \sum_\pi (\epsilon_\pi)^2 T(\tau^1, ...\tau^r)$
$=\frac{1}{r!} \sum_\pi 1\ T(\tau^1, ..., \tau^r) = T(\tau^1, ...\tau^r)$.
A similar definition gives the antisymmetrizer A on T^0_s. Instead of $\tau^1, ...\tau^r$ we would have used $v_1, ...v_s$.

Example 2.5. *Let us write the equation of an antisymmetrizer for $r = 3$.*
$T_a(\tau^1, \tau^2, \tau^3) = \frac{1}{6}[\epsilon_{123}A(\tau^1, \tau^2, \tau^3) + \epsilon_{213}A(\tau^2, \tau^1, \tau^3) + \epsilon_{132}A(\tau^1, \tau^3, \tau^2) + \epsilon_{312}A(\tau^3, \tau^1, \tau^2)$
$+ \epsilon_{321}A(\tau^3, \tau^2, \tau^1) + \epsilon_{231}A(\tau^2, \tau^3, \tau^1)\ ]$
$=\frac{1}{6}[A(\tau^1, \tau^2, \tau^3) - A(\tau^2, \tau^1, \tau^3) - A(\tau^1, \tau^3, \tau^2) + A(\tau^3, \tau^1, \tau^2) - A(\tau^3, \tau^2, \tau^1) + A(\tau^2, \tau^3, \tau^1)$
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