Solutions

(a) **Prove that optimal heuristics (i.e., $h^*(n)$) are consistent.**

Proof:

The optimal heuristic $h^*(n)$ represents the actual minimal cost from node n to the goal.

A heuristic $h(n)$ is consistent if, for every node n and each successor n', the following inequality holds:

$$h(n) \leq c(n, n') + h(n')$$

where $c(n, n')$ is the cost of reaching n' from n.

Since $h^*(n)$ is the true minimal cost from n to the goal, we consider the path from n to the goal that goes through n'. Then:

$$h^*(n) = c(n, n') + h^*(n')$$

if the optimal path passes through n', or

$$h^*(n) \leq c(n, n') + h^*(n')$$

if it does not (since $h^*(n)$ is the minimal cost).

In both cases, we have:

$$h^*(n) \leq c(n, n') + h^*(n')$$

Therefore, $h^*(n)$ satisfies the consistency condition and is a consistent heuristic.

(b) **Prove: If $h_1(n), \ldots, h_k(n)$ are admissible, so is $h(n) = \max(h_1(n), \ldots, h_k(n))$.**

Proof:

Since each $h_i(n)$ is admissible, we have:

$$h_i(n) \leq h^*(n) \quad \text{for all } i = 1, 2, \ldots, k.$$

Therefore,

$$h(n) = \max\{h_1(n), \ldots, h_k(n)\} \leq h^*(n).$$

Thus, $h(n)$ does not overestimate the true cost to reach the goal and is admissible.

(c) **Prove: If $h_1(n), \ldots, h_k(n)$ are consistent, so is $h(n) = \max(h_1(n), \ldots, h_k(n))$.**

Proof:

For each $h_i(n)$, the consistency condition holds:

$$h_i(n) \leq c(n, n') + h_i(n') \quad \text{for all } i.$$

Taking the maximum over all $h_i(n)$:

$$h(n) = \max\{h_1(n), \ldots, h_k(n)\} \leq c(n, n') + \max\{h_1(n'), \ldots, h_k(n')\} = c(n, n') + h(n').$$

Therefore, $h(n)$ satisfies the consistency condition and is consistent.

(d) **Disprove: If $h_1(n), \ldots, h_k(n)$ are admissible, so is $h(n) = h_1(n) + \ldots + h_k(n)$.**

Counterexample:

Let $k = 2$ and consider a node n where:

$$h_1(n) = 1, \quad h_2(n) = 1.$$

Both $h_1(n)$ and $h_2(n)$ are admissible since they do not overestimate the true cost $h^*(n) = 1$.

Compute the combined heuristic:

$$h(n) = h_1(n) + h_2(n) = 1 + 1 = 2.$$

Since $h(n) = 2 > h^*(n) = 1$, the heuristic $h(n)$ overestimates the true cost and is not admissible.

Therefore, $h(n) = h_1(n) + \ldots + h_k(n)$ is not necessarily admissible.

(e) **Prove: If $h_1(n), \ldots, h_k(n)$ are admissible, so is $h(n) = \dfrac{h_1(n) + \ldots + h_k(n)}{k}$.**

Proof:

Since each $h_i(n)$ is admissible:

$$h_i(n) \leq h^*(n) \quad \text{for all } i.$$

Summing over all heuristics:

$$h_1(n) + \ldots + h_k(n) \leq k \cdot h^*(n).$$

Dividing both sides by k:

$$h(n) = \frac{h_1(n) + \ldots + h_k(n)}{k} \leq h^*(n).$$

Therefore, $h(n)$ does not overestimate the true cost to reach the goal and is admissible.