Feuerbach's Theorem and Points Revisited

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Abstract

In this paper we give a proof of the famous Theorem of Feuerbach based on a direct computation which uses some standard well known metric relations in a triangle. Along the proof, we also establish some properties and metric relations that the Points of Feuerbach satisfy.

1 Introduction and some History

In the geometry of the triangle one among the fascinating theorems is the Theorem of Feuerbach[1]. This is the following.

Theorem (Feuerbach's Beautiful Theorem) *In every triangle ABC, the inscribed circle is internally tangent to the nine-point circle and the nine-point circle is externally tangent to the three exscribed circles.*

The points of tangency are called the **Points of Feuerbach**. The first enunciation of Feuerbach's Theorem, including the first published proof, appears in Karl Wilhelm Feuerbach's classic memoir of 1822 "*Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks*", along with many other interesting proofs relating to the nine point circle and other metric relations in a triangle. Feuerbach proved this theorem by using metric relations of the triangle itself and metric relations of its *orthic triangle*, i.e., *the triangle with vertices the feet of the heights of the given triangle*. When he proved it, he did not pay the due attention to this beautiful result that gave him eternal fame. He did not also live long to see the recognition of its beauty, some extensions of it and at least three different proofs that follow after him. {Here, we cite the references [2], [3], [4], [5], [6], [7]. There are many more references that can be found in the internet and the libraries. The theorem was rediscovered by Steiner[2] a few years after Feuerbach. Our proof in this paper may be the fourth proof of this theorem.}

[1]Karl Wilhelm Feuerbach, German mathematician (high school teacher), 1800-1834.
[2]Jakob Steiner, Swiss mathematician, great geometer, 1796-1863.

The theorem is immediately observed to be valid for an isosceles but not equilateral triangle, in which case the midpoint of the base is the interior Feuerbach point and one of the three exterior Feuerbach points, since median, height and angle bisector to the base coincide. (If the triangle is equilateral, the nine point circle and the inscribed circle coincide.)

The proofs of this theorem that many books in circulation present use the transformation of inversion and or some ideas difficult to think about {for instance see [1], pp. 117-119, [3], pp. 117-119, pp. 200-203}. In this paper we prove this theorem by using some standard known metric relations and some metric relations established by Leibniz[3], Euler[4] and Stewart[5]. In this way, we avoid inversion (used in most proofs), extraneous ideas, and the properties of the orthic triangle which are convenient when the triangle is scalene (all angles are acute). (If the triangle is not scalene the properties of the orthic triangle need adjustments depending if the triangle has a right or an obtuse angle.) Along these lines and using the aforementioned relations, we also establish some properties and metric relations of the Points of Feuerbach.

2 The Primary Known Relations

First, we list the metric relations in a triangle that we will use in the proof below. They are standard and can be found in many books. So, we take them for granted and give some references. These relations can be used by any reader in proving many different results of the geometry of the triangle. The terms and symbols that we use must be familiar to the readers while studying this paper.

(a) Let us symbolize a circle with center O and radius $c > 0$ by (O, c).
Now, two circles (O_1, c_1) and (O_2, c_2) are:
(1) **internally tangent** iff $O_1O_2 = |c_1 - c_2|$.
(2) **externally tangent** iff $O_1O_2 = c_1 + c_2$.
(These relations are easy to observe.)

For any triangle ABC, we use the following notation:
(1) $a = BC$, $b = CA$, and $c = AB$ the **sides** of the triangle and O_a, O_b, and O_c the corresponding mid-points.
(2) $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ the **angles** of the triangle.
(3) G **center of gravity** (**centroid**), i.e., the common point of its medians.
(4) H its **orthocenter**, i.e., the common points of its heights. H_a, H_b and H_c are the feet of the heights the on the sides $a = BC$, $b = CA$, and $c = AB$ correspondingly.
(5) O the **circumcenter**, i.e., the center of the circumscribed circle which is the common point of the perpendicular bisectors of the sides.

[3]Gottfried Wilhelm (von) Leibniz, German polymath, philosopher and mathematician, 1646-1716.
[4]Leonhard Euler, Swiss mathematician, 1707-1783.
[5]Matthew Stewart, Scottish mathematician, 1717-1785.

(6) R the **circumradius**, i.e., the radius of the circumcircle.
(7) I the **incenter**, i.e., the center of the inscribed circle, which is the common point of the angle bisectors.
(8) r the **inradius**, i.e., the radius of the inscribed circle.
(9) I_a, I_b, and I_c the **centers of the exscribed circles** apposite to the vertices A, B, and C, called **excenters**.
(10) r_a, r_b, and r_c the **exradii**, i.e., the radii of the exscribed circles.
(11) E the **center of the nine point circle**.
(12) ρ the **radius of the nine point circle**.
(13) (ABC) the **area** of the triangle.
(14) $s = \dfrac{a+b+c}{2}$ the **semiperimeter** of the triangle.
(15) T_a, T_a, T_a the **points of tangency** of the incircle (I, r) with the sides $a = BC$, $b = CA$, and $c = AB$ of the triangle, respectively. These are interior points of the segments BC, CA, and AB.
(16) T_{aa}, T_{ab}, T_{ac} the **points of tangency** of the excircle (I_a, r_a) with the sides $a = BC$, $b = CA$, and $c = AB$ of the triangle, respectively.
The point T_{aa} is interior point of the segments BC, and the points T_{ab}, T_{ac} are exterior points of CA, and AB.
Analogously, we have the points T_{ba}, T_{bb}, T_{bc} and T_{ca}, T_{cb}, T_{cc}.

(b) **Leibniz Relation**. Let P be any point of the plane of the triangle ABC. Then

$$PA^2 + PB^2 + PC^2 = 3PG^2 + \frac{1}{3}\left(a^2 + b^2 + c^2\right), \qquad (LR).$$

[So the minimum of $PA^2 + PB^2 + PC^2$ is $\frac{1}{3}\left(a^2+b^2+c^2\right)$, obtained when $P = G$.]

Applying this to $P = O$, we have $OA = OB = OC = R$ and then we obtain

$$GO^2 = R^2 - \frac{a^2+b^2+c^2}{9}, \qquad (LG).$$

{See [3], pp. 174-175.}

(c) **Euler Relations**. In any triangle we have:

1. $$OI^2 = R^2 - 2Rr, \qquad \text{(ER1)}.$$
   Hence, $\dfrac{R}{2} \geq r$. (Equality holds iff the triangle is equilateral iff $O = I$.)
2. $OI_a^2 = R^2 + 2Rr_a, \quad OI_b^2 = R^2 + 2Rr_b, \quad OI_c^2 = R^2 + 2Rr_c, \quad$ (ER2).
3. O, G, E, H are on the Euler Line and appear in this order as we move from O to H and the following relations hold:

   $$HG = 2GO, \qquad 2EG = OG, \qquad \text{and} \qquad HO = 2HE = 2EO, \qquad \text{(ER3)}.$$

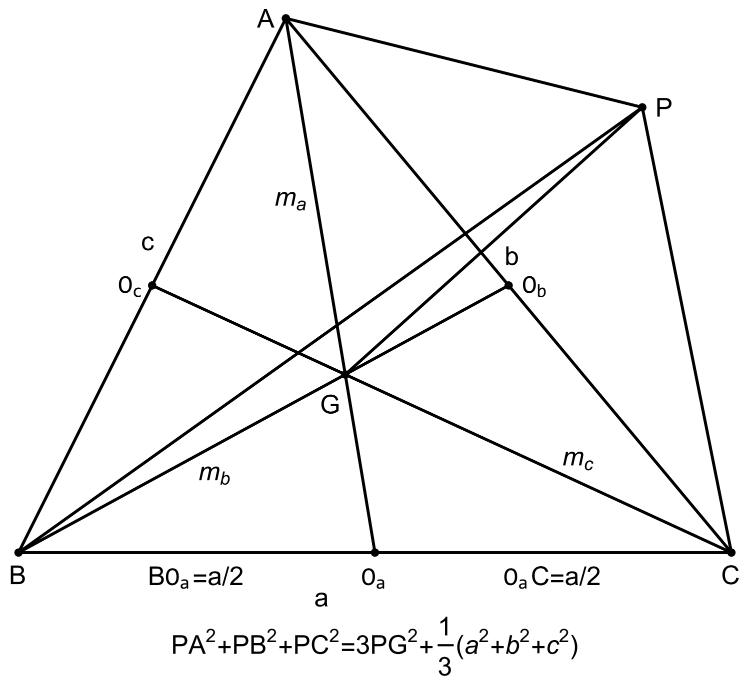

Figure 1: Leibniz Relation

Therefore, by (LG), we have $\qquad GH^2 = 4GO^2 = 4R^2 - \dfrac{4}{9}(a^2+b^2+c^2),$

and then, by (LR), we find $\qquad HA^2+HB^2+CH^2 = 12R^2-(a^2+b^2+c^2).$

Also, $\quad HO^2 = 4EO^2 = 4(EG+GO)^2 = 4\left(\dfrac{1}{2}+1\right)^2 GO^2 =$
$9GO^2 = 9R^2 - \left(a^2+b^2+c^2\right), \qquad$ (ER4).

4. **Radius of the nine point circle** $\qquad \rho = \dfrac{1}{2}R, \qquad$ (ER5).

{See [7], pp. 153-154, 162, [3], pp. 165, 187, 205.}

(d) **Stewart Relation**. In a triangle ABC we pick any interior or boundary point D of the side BC and draw the segment AD. Then

$$AB^2 \cdot DC + AC^2 \cdot BD = AD^2 \cdot BC + BC \cdot BD \cdot DC, \qquad (SR).$$

(This can be generalized with axis and signed segments.) {See [7], pp. 96-98.}

(e) **Heron's Formula**.

$$(ABC) = \sqrt{s(s-a)(s-b)(s-c)}, \qquad (HF).$$

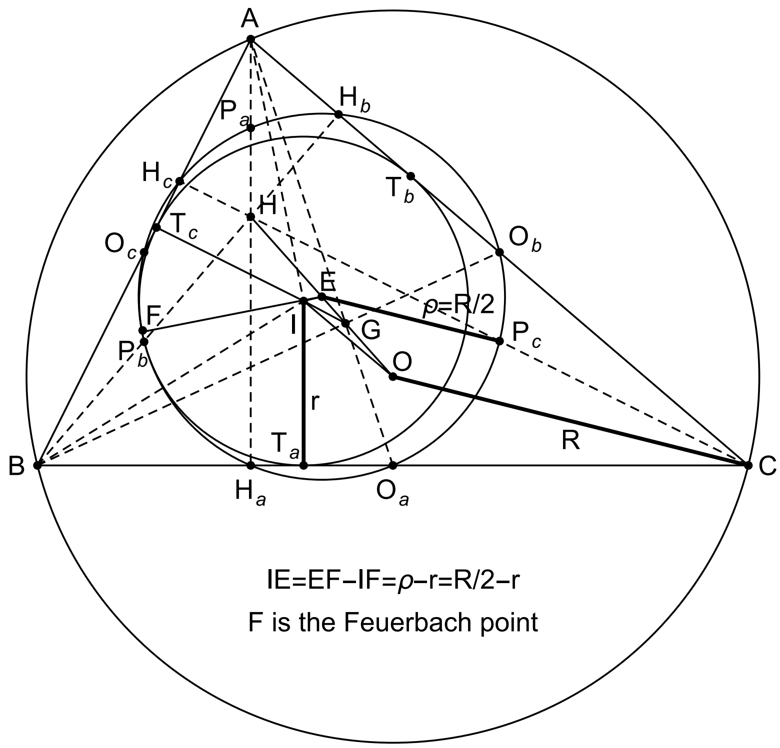

Figure 2: For Euler's Relations

{See [1], p.58, [7], p. 123, [3], p. 11.}

(f) **Other Standard Metric Relations**.

1. $R = \dfrac{abc}{4(ABC)} = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$, (OR1).

2. $r = \dfrac{(ABC)}{s} = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$, (OR2).

3. $Rr = \dfrac{abc}{4s}$, (follows from the previous two relations), (OR3).

4. $r_a = \dfrac{(ABC)}{s-a}$, $r_b = \dfrac{(ABC)}{s-b}$, $r_c = \dfrac{(ABC)}{s-c}$, (OR4)

5. $AT_b = AT_c = BT_{cc} = BT_{ca} = CT_{bb} = CT_{ba} = s - a$,
$BT_c = BT_a = CT_{aa} = CT_{ab} = AT_{cc} = AT_{cb} = s - b$,
$CT_a = CT_b = AT_{bb} = AT_{bc} = BT_{aa} = BT_{ac} = s - c$,
$AT_{ab} = AT_{ac} = BT_{ba} = BT_{bc} = CT_{cb} = CT_{ca} = s$, (OR5)

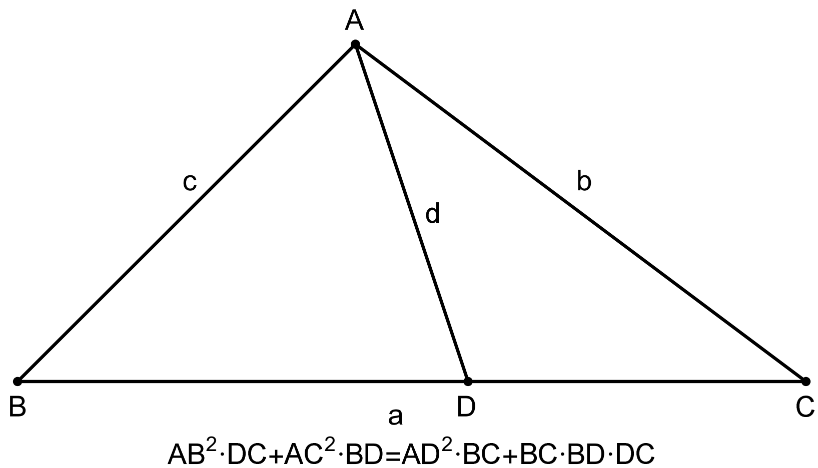

Figure 3: Stewart's Relation

{There are many more relations that the points of tangency satisfy. See [1], pp.10-13, [3], p. 184, [7], pp. 143-147, 155-156.}

3 Proof of Feuerbach's Theorem

We are going to prove that the inscribed circle is internally tangent to the nine-point circle in a triangle ABC, by proving relation **(a), (1)**, for these two circles. That is, we are going to prove that $EI = \rho - r = \frac{R}{2} - r$ (≥ 0).

[In similar way by proving relation **(a), (2)**, i.e., $EI_a = \rho + r_a = \frac{R}{2} + r_a$, etc., one proves that the nine-point circle is externally tangent to the three exscribed circles, as any reader can check through the analogous computation.]

The triangles T_bIA and T_cIA are equal right triangles. Then, using (OR5), we have $IA^2 = AT_b^2 + r^2 = (s-a)^2 + r^2$ and similarly $IB^2 = (s-b)^2 + r^2$ and $IC^2 = (s-c)^2 + r^2$. Therefore, by (LR) with $P = I$, we obtain

$$(s-a)^2 + (s-b)^2 + (s-c)^2 + 3r^2 = 3\,GI^2 + \frac{1}{3}\left(a^2 + b^2 + c^2\right).$$

Solving for GI^2, using $s = \frac{1}{2}(a+b+c)$ and simplifying, we find

$$GI^2 = -\frac{1}{3}s^2 + r^2 + \frac{2}{9}\left(a^2 + b^2 + c^2\right), \qquad (LI)$$

Now we apply (SR) in the triangle IEO with IG the intermediate segment (G is always between E and O) and we have

$$IE^2 \cdot GO + IO^2 \cdot GE = IG^2 \cdot OE + OE \cdot EG \cdot GO.$$

We use (ER3) and simplify to obtain

$$IE^2 + \frac{1}{2}IO^2 = \frac{3}{2}IG^2 + \frac{3}{4}OG^2 \quad \text{and so} \quad IE^2 = -\frac{1}{2}IO^2 + \frac{3}{2}IG^2 + \frac{3}{4}OG^2.$$

We substitute IO^2, IG^2 and OG^2, [as found in (ER1), (LI), (LG)] and we find

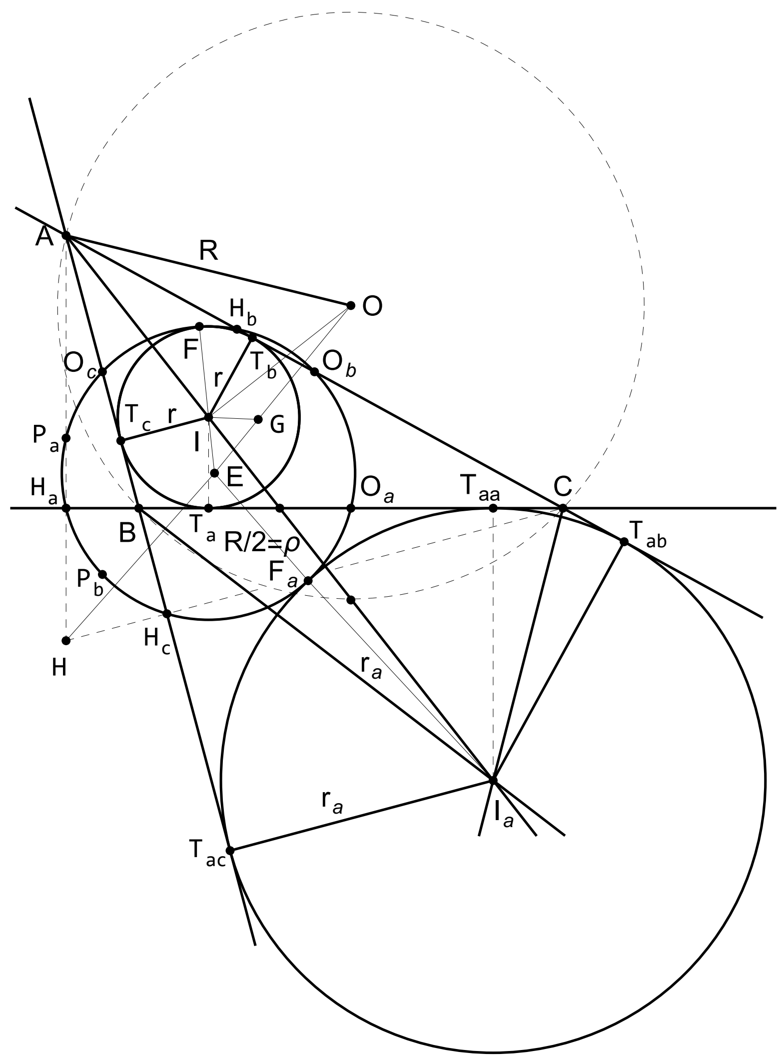

Figure 4: For Feuerbach's Theorem

$$IE^2 = \frac{1}{4}R^2 + Rr + \frac{3}{2}r^2 - \frac{1}{2}s^2 + \frac{1}{4}\left(a^2+b^2+c^2\right) =$$
$$\left(\frac{R}{2}\right)^2 + r^2 + Rr + \left[\frac{1}{2}r^2 - \frac{1}{2}s^2 + \frac{1}{4}\left(a^2+b^2+c^2\right)\right]$$

We substitute R, r and s in terms of a, b and c (from our list of the triangle relations), use (OR3) and we find that the bracket is equal to

$$\frac{1}{2}r^2 - \frac{1}{2}s^2 + \frac{1}{4}\left(a^2+b^2+c^2\right) = \frac{-abc}{2s} = -2Rr.$$

Therefore,

$$IE^2 = \left(\frac{R}{2}\right)^2 + r^2 + Rr - 2Rr = \left(\frac{R}{2} - r\right)^2 \quad \text{and so} \quad IE = \frac{R}{2} - r.$$

This finishes the proof.

Historical Note: In proving this theorem, Feuerbach used the circumcircle (O, R) and the inscribed circle (I, r) of the given triangle ABC. Then, he considered the orthic triangle of ABC, whose circumcircle is the nine point circle of ABC, $\left(E, \frac{R}{2}\right)$, and so its circumcenter is E and its circumradius is $\frac{R}{2}$. When ABC is scalene, its orthocenter H is the incenter of its orthic triangle. Here we denote the inradius of the orthic triangle by ρ and so the incircle of the orthic triangle is (H, ρ). (See **Figure 5**.) Feuerbach also used Euler's relation $OI^2 = R^2 - 2Rr$ and proved the following two interesting relations

$$IH^2 = 2r^2 - 2R\rho \qquad \text{and} \qquad OH^2 = R^2 - 4R\rho.$$

Then, as E is the midpoint of OH, by the theorem of the medians in a triangle, he obtained the result that the median IE of the triangle IOH satisfies

$$EI^2 = \frac{1}{2}[OI^2 + HI^2] - EH^2 = \frac{1}{4}R^2 - Rr + r^2 = \left(\frac{1}{2}R - r\right)^2$$

and so

$$EI = \frac{1}{2}R - r,$$

which proves the theorem.

The relation $OH^2 = R^2 - 4R\rho$ follows immediately by writing Euler's relation for the orthic triangle whose circumcenter is E, its incenter is H, its circumradius is $\frac{R}{2}$, and its inradius ρ. So,

$$HE^2 = \left(\frac{R}{2}\right)^2 - 2\left(\frac{R}{2}\right)\rho = \frac{R^2}{4} - R\rho.$$

Then, we use the fact that E is the midpoint of $OH = 2 \cdot HE$ and we obtain the relation $OH^2 = R^2 - 4R\rho$.

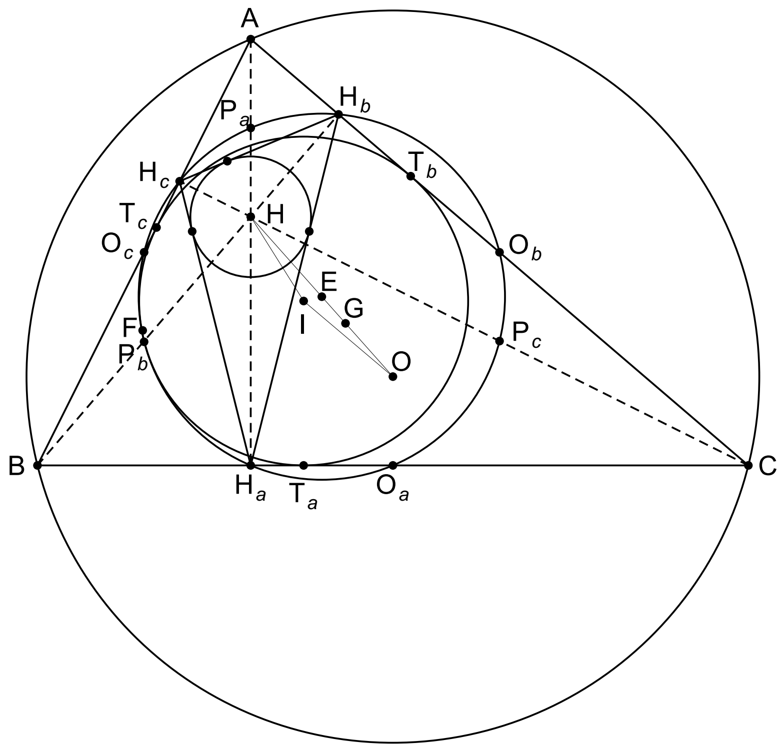

Figure 5: $H_aH_bH_c$ the Orthic Triangle of ABC

If the triangle ABC is scalene, then $a^2+b^2+c^2 > 8R^2$ and by (ER4) and $OH^2 = R^2 - 4R\rho$, we find that the inradius of the orthic triangle is

$$\rho = \frac{a^2+b^2+c^2-8R^2}{4R}$$

If ABC is right, then $a^2+b^2+c^2-8R^2 = 0$, the orthic triangle is degenerate and $\rho = 0$.

If ABC is obtuse, then $a^2+b^2+c^2-8R^2 < 0$ and

$$\rho = \frac{\left|a^2+b^2+c^2-8R^2\right|}{4R},$$

a formula valid in all three cases.

The equation $IH^2 = 2r^2 - 2R\rho \quad (\Longrightarrow \quad r^2 \geq R\rho)$ can be proved by using the relation

$$IH^2 = 2r^2 + 4R^2 - \frac{1}{2}\left(a^2+b^2+c^2\right),$$

which is the first equation that we prove in part **(e)** of the next section, and this is also valid for any triangle. Then we use the expression of ρ obtained above to set this equation in the form $IH^2 = 2r^2 - 2R\rho$.

{**Note**: Using trigonometric formulae we find that $\rho = 2R \cdot |\cos(\widehat{A}) \cdot \cos(\widehat{B}) \cdot \cos(\widehat{C})|$. Trigonometry combined with geometry can prove the two Feuerbach relations. Note that $a = 2R \cdot \sin(\widehat{A})$, $b = 2R \cdot \sin(\widehat{B})$, $c = 2R \cdot \sin(\widehat{C})$, and $\sin^2(\widehat{A}) + \sin^2(\widehat{B}) + \sin^2(\widehat{C}) = 2 + 2\cos(\widehat{A}) \cdot \cos(\widehat{B}) \cdot \cos(\widehat{C})$.}

4 Properties and Metric Relations of Feuerbach's Points

The points F, F_a, F_b, and F_c, are called the **Points of Feuerbach** and the nine point circle is also at times call **Circle of Feuerbach**. F is an interior or boundary point of the triangle and F_a, F_b, and F_c are exterior or boundary points. Their many properties have been studied and published in the pertinent bibliography.

(a) Here we give an easy direct proof of the **following property**, called the **distance property of the Feuerbach point**. Let us suppose that $a \geq b \geq c$ but **the triangle is not equilateral**. Then $FO_b = FO_a + FO_c$. We also compute these distances in terms of the sides. (Analogous equations, if the sides satisfy other inequalities obtained by a cyclic rotations of the letter.)

We apply Stewart relation to the triangle EFO_c with intermediate segment the IT_c. We have

$$FO_c^2 \cdot EI + EO_c^2 \cdot IF = IO_c^2 \cdot EF + EF \cdot EI \cdot IF$$

and we have seen

$$EO_c = \rho = \frac{R}{2} = EF, \qquad EI = \rho - r = \frac{R}{2} - r, \qquad IF = r$$

and

$$IO_c^2 = r^2 + O_cT_c^2 = r^2 + \left(s - a - \frac{c}{2}\right)^2 = r^2 + \left(\frac{b-a}{2}\right)^2.$$

We substitute and simplify to obtain

$$FO_c^2 = \frac{R}{4(R-2r)} \cdot (b-a)^2, \qquad \text{and so} \qquad FO_c = \sqrt{\frac{R}{R-2r}} \cdot \frac{|b-a|}{2}.$$

Similarly,

$$FO_b = \sqrt{\frac{R}{R-2r}} \cdot \frac{|a-c|}{2}, \qquad \text{and} \qquad FO_a = \sqrt{\frac{R}{R-2r}} \cdot \frac{|c-b|}{2}.$$

If now $a \geq b \geq c$, we properly eliminate the absolute values from the last three equations and we find:

$$a \geq b \geq c \quad \Longrightarrow \quad FO_b = FO_a + FO_c.$$

Conclusion: If $a > b > c$, then FO_b is the greatest among the three segments and $FO_b = FO_a + FO_c$.

If the triangle is isosceles with $b = c \neq a$, then $F = O_a$ the midpoint of the base, and so, $FO_a = 0$ and $FO_b = FO_c = \dfrac{b}{2} = \dfrac{c}{2}$, fact that agrees with computing and simplifying the

$$FO_c^2 = \frac{R}{4(R-2r)} \cdot (b-a)^2 = FO_b^2 = \frac{R}{4(R-2r)} \cdot (a-c)^2 = \left(\frac{b}{2}\right)^2 = \left(\frac{c}{2}\right)^2.$$

Remark: When the triangle is equilateral, $a = b = c$, the nine-point circle and the inscribed circle coincide ($O = E = I$, $R = 2r = 2\rho$), and F is indeterminate. Also, the fractions in the computation of the above conclusion become indeterminate $\dfrac{0}{0}$. We then can say that, as $a \neq b = c \longrightarrow a$, we have $FO_a = 0$ and

$$\lim_{a \neq b = c \to a} \frac{R}{4(R-2r)} \cdot (b-a)^2 = \lim_{a \neq b = c \to a} \frac{R}{4(R-2r)} \cdot (a-c)^2 = \left(\frac{b}{2}\right)^2 = \left(\frac{c}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

(b) Next, we will establish the analogous formulae for the points F_a, F_b, and F_c. We see that

$$I_aO_a^2 = r_a^2 + \left[\frac{a}{2} - (s-b)\right]^2 = r_a^2 + \left(\frac{b-c}{2}\right)^2.$$

We then apply Stewart's relation to the triangle I_aEO_a and intermediate segment F_aO_a and we find that

$$F_aO_a = \sqrt{\frac{R}{R+2r_a}} \cdot \frac{|b-c|}{2}$$

and similarly,

$$F_bO_b = \sqrt{\frac{R}{R+2r_b}} \cdot \frac{|c-a|}{2}, \qquad \text{and} \qquad F_cO_c = \sqrt{\frac{R}{R+2r_c}} \cdot \frac{|a-b|}{2}.$$

To compute F_aO_c, we observe that

$$I_aO_c^2 = \left(s - \frac{c}{2}\right)^2 + r_a^2 = \left(\frac{a+b}{2}\right)^2 + r_a^2,$$

and we apply Stewart's relation to the triangle I_aEO_c and intermediate segment F_aO_c. Finally we find

$$F_aO_c = \sqrt{\frac{R}{R+2r_c}} \cdot \frac{a+b}{2}, \qquad \text{and similarly,} \qquad F_aO_b = \sqrt{\frac{R}{R+2r_c}} \cdot \frac{c+a}{2}.$$

Now

$$a \geq b \geq c \quad \Longrightarrow \quad F_aO_c = F_aO_b + F_aO_a,$$

and we have the analogous formulae for the points F_b and F_c with respect to the other possible inequalities that the sides of the triangle may satisfy.

(c) Now, we are going to compute FG^2. Knowing that the centroid G of ABC is also the centroid of $O_aO_bO_c$ and $O_bO_c = \dfrac{a}{2}$, $O_cO_a = \dfrac{b}{2}$ and $O_aO_b = \dfrac{c}{2}$, we apply the Leibniz relation to $O_aO_bO_c$ and F and we find

$$FO_a^2 + FO_b^2 + FO_c^2 = 3FG^2 + \frac{1}{12}(a^2 + b^2 + c^2).$$

Using the expressions for FO_a^2, FO_b^2, FO_c^2, found in **(a)** above, and solving for FG^2 we find

$$FG^2 = \frac{R}{12(R-2r)}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] - \frac{1}{36}\left(a^2 + b^2 + c^2\right).$$

Similar work shows

$$F_aG^2 = \frac{R}{12(R+2r_a)}\left[(a+b)^2 + (b-c)^2 + (c+a)^2\right] - \frac{1}{36}\left(a^2 + b^2 + c^2\right).$$

We leave the corresponding relations for F_b, and F_c to the readers.

(d) We are going to compute AF^2, BF^2, and CF^2. Since FO_c is the median of the triangle ABF, by the know relation for the medians of a triangle, we have that

$$AF^2 + BF^2 = 2FO_c^2 + \frac{1}{2}AB^2 = 2FO_c^2 + \frac{1}{2}c^2.$$

Similarly,

$$BF^2 + CF^2 = 2FO_a^2 + \frac{1}{2}a^2,$$

and

$$CF^2 + AF^2 = 2FO_b^2 + \frac{1}{2}b^2.$$

Solving these three equations we find

$$CF^2 = FO_a^2 + FO_b^2 - FO_c^2 + \frac{1}{4}\left(a^2 + b^2 + c^2\right) - \frac{1}{2}c^2 =$$
$$\frac{R}{4(R-2r)}\left[-(a-b)^2 + (b-c)^2 + (c-a)^2\right] + \frac{1}{4}\left(a^2 + b^2 - c^2\right).$$

Similarly,

$$BF^2 = \frac{R}{4(R-2r)}\left[-(c-a)^2 + (a-b)^2 + (b-c)^2\right] + \frac{1}{4}\left(a^2 - b^2 + c^2\right),$$

and

$$AF^2 = \frac{R}{4(R-2r)}\left[-(b-c)^2+(c-a)^2+(a-b)^2\right]+\frac{1}{4}\left(-a^2+b^2+c^2\right).$$

Remark: These results agree with the FG^2 computed in **(c)**. Also, when $b = c \neq a$, then $F = O_c$ and the brackets in the equations for CF^2 and BF^2 are zero and and $CF = BF = \frac{a}{2}$. This result agree with the **previous Remark**. The third equation gives $AF^2 = \frac{R}{2(R-2r)} \cdot (a-b)^2 + \frac{1}{4}\left(-a^2+2b^2\right)$, which is the square of the height AO_c, i.e., $AO_a^2 = b^2 - \frac{a^2}{4}$ of the isosceles triangle ABC.

Next, similar work shows

$$CF_a^2 = \frac{R}{4(R+2r_c)}\left[(c+a)^2-(a+b)^2+(b-c)^2\right]+\frac{1}{4}\left(a^2+b^2-c^2\right),$$
$$BF_a^2 = \frac{R}{4(R+2r_b)}\left[(b-c)^2-(c+a)^2+(a+b)^2\right]+\frac{1}{4}\left(a^2-b^2+c^2\right),$$
$$AF_a^2 = \frac{R}{4(R+2r_a)}\left[(a+b)^2-(b-c)^2+(c+a)^2\right]+\frac{1}{4}\left(-a^2+b^2+c^2\right).$$

We leave the corresponding relations for F_b, and F_c to the readers.

(e) **Other metric relations.**

With the help of the Leibniz, Euler, Stewart and the other relations listed before, we can compute the segments, or their squares, defined by any two points among H, E, G, O, I, I_a, I_b, I_c, F, F_a, F_b, F_c, (and possibly other points that we have indicated before) in terms of the sides a, b and c of the given triangle ABC. We have already computed several of them in the previous paragraphs. Next, we are going to compute some more segments out of the 66 possible segments and leave the remaining metric relations for practice and exercise to the interested readers.

Applying Stewart's relation to triangle IHG and intermediate segment IE or the theorem of medians to triangle HIO we find

$$HI^2 = 4R^2 - 4rR + r^2 + s^2 - \left(a^2+b^2+c^2\right) =$$
$$(2R-r)^2 + s^2 - \left(a^2+b^2+c^2\right) = 4R^2 + 2r^2 - \frac{1}{2}\left(a^2+b^2+c^2\right).$$

From (OR5) we find that $T_cT_{ac} = a = T_bT_{ab}$, etc. Also $AI^2 = r^2 + (s-a)^2$, etc. Then, from the similar triangles AT_cI and $AT_{ac}I_a$

$$II_a^2 = \frac{a^2[r^2+(s-a)^2]}{(s-a)^2} = \frac{a^2bc}{s(s-a)} = \frac{4arR}{s-a}$$

and analogously,

$$II_b^2 = \frac{ab^2c}{s(s-b)} = \frac{4brR}{s-b} \qquad \text{and} \qquad II_c^2 = \frac{abc^2}{s(s-c)} = \frac{4crR}{s-c}.$$

Applying Stewart's relation to triangle EII_a and intermediate segment IF_a, we find

$$IF_a^2 = \frac{R \cdot II_a^2 - r_a(r_aR + 2rR - 2r^2)}{R + 2r_a}, \qquad \text{etc.}$$

Applying Stewart's relation to the triangle IEI_a and intermediate segment II_a, we find

$$FI_a^2 = \frac{R \cdot II_a^2 - r^2R - 2rr_aR - 2rr_a^2}{R - 2r}, \qquad \text{etc.}$$

Similarly, applying Stewart's relation to the triangle IFF_a and intermediate segment IF_a, found above, we find

$$FF_a^2 = \frac{R\left(IF_a^2 - r^2\right)}{R-2r} = \frac{R^2\left[II_a^2 - (r+r_a)^2\right]}{(R+2r_a)(R-2r)} = \frac{R^2(II_a - r - r_a)(II_a + r + r_a)}{(R+2r_a)(R-2r)},$$

and analogously,

$$FF_b^2 = \frac{R^2(II_b - r - r_b)(II_b + r + r_b)}{(R+2r_b)(R-2r)}, \quad FF_c^2 = \frac{R^2(II_c - r - r_c)(II_c + r + r_c)}{(R+2r_c)(R-2r)}.$$

Next, we would like to compute F_aF_b, F_bF_c, and F_cF_a. For this we notice that the points E, F_a and I_a are on the sam straight line and similarly E, F_b and I_b and the E, F_c and I_c.

We will work out the F_aF_b and analogous work for the other two segment. We notice the identity of the angles $\widehat{F_aEF_b} = \widehat{I_aEI_b} := \phi$ and the triangle F_aEF_b is isosceles with base F_aF_b and

$$EF_a = EF_b = \frac{R}{2}.$$

Then by the law of cosines we find that

$$F_aF_b^2 = \frac{R^2}{2}[1 - \cos(\phi)].$$

To compute the $\cos(\phi)$, we must use the triangle I_aEI_b for which we have

$$EI_a = \frac{R}{2} + r_a, \qquad EI_b = \frac{R}{2} + r_a$$

and the known relation

$$I_aI_b = 4R\cos\left(\frac{\widehat{C}}{2}\right)$$

{e.g., see Theorem in [3] p. 186}, and so

$$I_aI_b^2 = 16R^2\cos^2\left(\frac{\widehat{C}}{2}\right) = 16R^2 \cdot \frac{1+\cos(\widehat{C})}{2} = 16R^2 \cdot \frac{s(s-c)}{ab}.$$

Then, from this and

$$I_aI_b^2 = \left(\frac{R}{2}+r_a\right)^2 + \left(\frac{R}{2}+r_b\right)^2 - 2\left(\frac{R}{2}+r_a\right)\left(\frac{R}{2}+r_b\right)\cos(\phi).$$

we find that

$$\cos(\widehat{F_aEF_b}) = \cos(\phi) = \frac{ab\left[R^2+2R(r_a+r_b)+2\left(r_a^2+r_b^2\right)^2\right]-32R^2s(s-c)}{ab(R+2r_a)(R+2r_b)}.$$

Substituting this into $F_aF_b^2$ and simpifying, we find

$$F_aF_b^2 = R^2 \cdot \frac{16R^2s(s-c)-ab(r_a-r_b)^2}{ab(R+2r_a)(R+2r_b)},$$

which can be written in terms of a, b and c, by the formulae listed in **Section 2** of this work.

(Now, write the formulae for $F_bF_c^2$ and $F_cF_a^2$ and the cosines of the angles $\widehat{F_bEF_c}$ and $\widehat{F_cEF_a}$.)

We also compute the squares of OF and OF_a. For OF we apply Stewart's relation to the triangle OFE with intermediate segment OI. So we have

$$OF^2 \cdot EI + OE^2 \cdot IF = OI^2 \cdot EF + EF \cdot EI \cdot IF.$$

Using

$$\begin{aligned} EO^2 &= \frac{9}{4} - \frac{1}{4}\left(a^2+b^2+c^2\right), \\ OI^2 &= R^2 - 2Rr, \\ EI &= \frac{R}{2} - r, \\ EF &= \frac{R}{2}, \\ IF &= r, \end{aligned}$$

solving and simplifying, we find

$$OF^2 = \frac{2R^3-12R^2r+r\left(a^2+b^2+c^2\right)-2Rr^2}{2(R-2r)}.$$

For OF_a we apply Stewart's relation to triangle OEI_a with intermediate segment OF_a. Then we have

$$OE^2 \cdot F_aI_a + OI_a^2 \cdot EF_a = OF_a \cdot EF + EF \cdot EI \cdot IF.$$

Then making the appropriate substitutions and solving, we find

$$OF_a^2 = \frac{12R^2r_a - r_a\left(a^2+b^2+c^2\right)+2R^3-2Rr_a^2}{2(R+2r_a)}.$$

5 An Exercise

In **Figure 6** the sides of the triangle ABC satisfy $b > a > c$ and $AX \parallel FO_a$ with X on the circumcircle. Find the hints that the figure suggests and prove that:

(a) F, G, X are collinear.
(b) $GX = 2\,GF$.
(c) $AX \parallel= 2FO_a = FO_a + FO_b + FO_c$.
(d) $BX \parallel= 2FO_b$ and $CX \parallel= 2FO_c$.

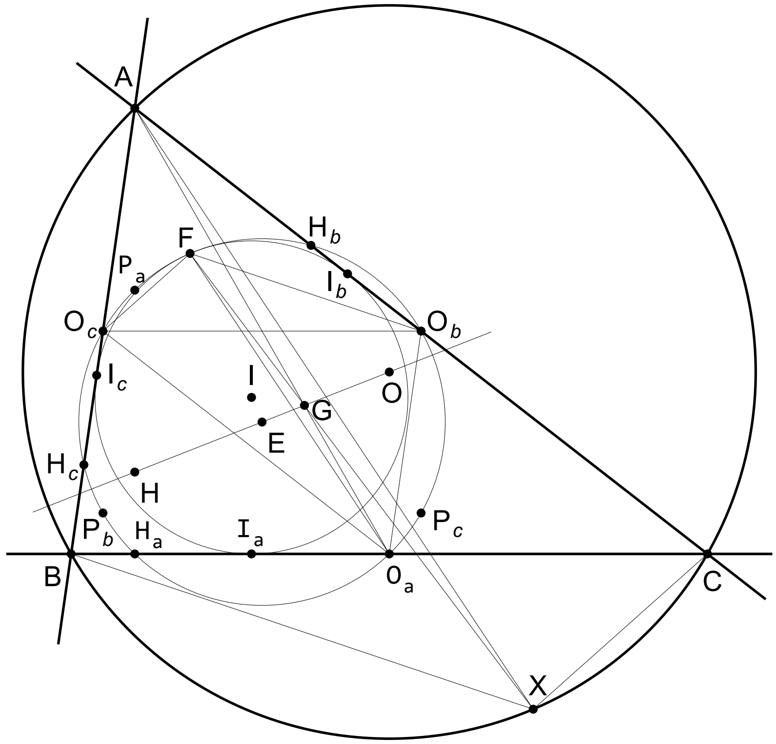

Figure 6: $AX = FO_a + FO_b + FO_c$

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