Ioannis M. Roussos

Improper Riemann Integrals

Part II

Dedication

To the memory of:
My parents Markos Ioannou Roussos and Margaro Nikita Grispou,
and my first motivator and teacher in mathematics, my uncle,
Michael Ioannou Roussos, and his wife Evaggelia Louka Gavala.

Biography

Professor Ioannis Markos Roussos was born on November 5, 1954, at the village Katapola of the island of Amorgos, Greece. After the primary and secondary education he studied mathematics at the National and Kapodistrian University of Athens and received his BSc Degree (1972-1977). Then, he studied graduate mathematics and computer sciences at the University of Minnesota and received his Masters and PhD degrees (1977-1986). His specialization in mathematics was in Differential Geometry and Analysis. He has taught mathematics at the University of Minnesota (1977-1987), University of South Alabama (1987-1990) and Hamline University (1990-2022). Besides this book, he has published 17 research papers, 10 expository papers and the book *Basic Lessons on Isometries, Similarities and Inversions in the Euclidean Plane*. He has participated in meetings and has refereed papers and promotions of other professors. Other interests are classical music, history, international relations and travelling.

Contents

Acknowledgements

I would like to express my deep thanks to my colleague Professor of Mathematics, Dr. Arthur Guetter. Without him this work would not have been motivated, started and finished. He has also been my LaTeX teacher for the technical typing of this material. Secondly, I would like to thank my student Nathan Davis whose fluent expertise in LaTeX and figure making have been indispensable and very valuable in this work. He has spent a lot of time for this cause and I feel indebted to him for all his work and help. I also wish to thank some of my friends for their moral support and encouragement.

Prologue
of the First Edition of Improper Riemann Integrals (2013)

This book is written at the masters level to help students of mathematics, statistics, applied sciences and engineering. Its scope is the improper or generalized Riemann integral, its convergence, principal value, evaluation and its application to science and engineering. Questions, problems and applications involving various improper integrals and series of numbers often emerge in these subjects. At the undergraduate level, results concerning useful improper integrals are mostly taken for granted, provided by an authority or obtained through tables and computer programs or packages. Here we try to give students sufficient knowledge and tools to enable them to answer these questions by themselves and acquire a deeper understanding of this matter and/or prepare them to do so with some further study of the matter.

We try to achieve these goals by explaining the concepts involved, presenting sufficient theory and using a number of theorems, some with their proofs and others without. A complete, general and advanced exposition of this vast area of mathematics would contain a much greater number of theorems and proofs and involve advanced mathematical theories of real and complex analysis, integral transforms, special functions, etc., that lie far beyond the undergraduate and/or masters curriculum. We must add that the student, with this book at hand, is assumed to be fluent with the rules of antidifferentiation (indefinite integration, computing antiderivatives), the u-substitution, integration by parts and deriving recursive formulae, the change of variables with multiple integrals and the theorems of basic calculus, advanced calculus and mathematical analysis.

Whenever possible, we present the material in a self-contained manner. We have proved many results but not all. Sometimes our proofs are not established under the most general conditions that the more advanced theories can provide, but under conditions accessible to the undergraduate and sufficient for application. We also state and use a few advanced general theorems, results and tools from real and complex analysis without proofs. Their complete presentations and rigorous proofs would require taking the graduate level courses on these subjects. Here their statements are adjusted to a level that students can understand and are interpreted in a way so that the students can handle, manipulate and use them efficiently as powerful tools in our list of problems. In this way we avoid stating and proving a great number of criteria and partial results and thus avoid forcing the students into too much searching (a lot of times done by trial and error) for finding out the case they deal with each time and what criterion to apply. Thus, we try to render these advanced mathematical results and tools accessible and useful even to the undergraduate students with sufficient background so that they can use them in fairly straightforward manners in many pertinent problems they may come across in the subjects aforementioned. Moreover, our presentation and use of these advanced and general theorems and results give the undergraduate student a taste of the power of the graduate level mathematics and motivate the interested one to take these courses at the graduate level in due time. We also expose a great number of detailed examples in order to illustrate the concepts and practice a lot with the tools that check convergence of improper integrals and evaluate their exact value when this is possible.

We include many exercises and problems in every section. These are carefully chosen to serve both as practice and for further application. They are representative enough so that the student, on the basis of these, can solve many other exercises and problems not included in this book and also use them in many situations of application. We try to keep the number of exercises at a level so that on the one hand the student does not get lost in a vast sea of exercises and on the other hand the opportunity to practice and learn the material well and apply it is not compromised. A few problems that are lengthy, have several questions and may be hard could be assigned as projects to an individual student or a group of students. Also, the input and help from the teacher or pertinent bibliography may be significant.

Many examples are presented several times in different ways in order to see them from various points of view and see how different methods can give correct answers to the same questions. That is, their solutions are achieved in various ways depending on the context. We also repeat a few problems from section to section and we seek their solutions within the new context. In this way we try to show the students the interconnection of the whole matter, how a given question may be viewed in many ways and within various contexts, and that there are many ways to achieve a correct answer. This is something generally lacking in the undergraduate mathematics education.

This book includes many theorems and methods for checking the convergence and the computation of most improper integrals encountered in applications. The content is sufficient to provide answers to the majority of them. We briefly examine the Laplace transform, Mellin transform and Fourier transform. Except for a few results, we do not develop the theory behind these integral transforms, but we mostly concentrate on their evaluation and some applications. We have omitted other integral transforms, such as the Hankel transform, Hilbert transform, etc. At this level, we did not include many special and hard integrals such as improper integrals in several variables, elliptic and hyper-elliptic integrals,

integrals involving special functions such as Bessel functions, hypergeometric functions, asymptotic expansions, methods of steepest descents, etc., and some very special cases of contour integration (Cauchy, Legendre, Mellin, etc.). These are topics of the area of Special Functions at a more advanced level. However, a lot of concrete cases out of these special integrals can still be resolved by making appropriate use of the tools provided here. Also, in advanced mathematics we encounter the singular integrals (especially in higher dimensions), which is a whole subject in itself, very important in mathematics and application.

We must say that one will encounter several not fully explained points, indicated in the text by expressions like "justify this," etc. All of them, however, can be justified by the versatile, studious and knowledgeable master-level student. An undergraduate could also clarify all of them with the help of the teacher. The proofs of theorems and results omitted in the text can be found in real analysis, complex analysis and applied mathematics literature.

In the chapter of complex analysis methods, we have avoided the theory and the formulae that involve the index or winding number of a curve with respect to a point (or, of a point with respect to a curve). In this way, we do not get too far into the theory of complex variables and at the same time we do not lose anything much with respect to the computations of improper integrals. In an advanced complex analysis course, we see the local and global Cauchy theorems, the residue theorem and other theorems, and all the pertinent formulae stated and proved in the general context that involves some multiplications with the index (or rotation) number. This number assumes only integer values and in the development of our formulae we arrange the hypothesis so that its value is 1. The interested reader can consult a good book in complex analysis and study these topics in this generality. (A good number of such books has been listed in our bibliography.)

In conclusion, the useful and practical material of this book is accessible to and can be mastered by any student who has finished a calculus sequence and has taken some multi-variable calculus, basic ordinary differential equations, basic mathematical analysis, complex numbers and the basics of complex variables. Knowing this material, a student may not rely on authority, tables or computer packages to give and understand answers to questions related to this important material in theory and application. On account of all these and its whole content, this book can be used as the text for an undergraduate course or a supplementary text to other courses of mathematics, statistics, applied sciences and engineering. It can also become a very helpful manual and reference to students at the master level and even beyond.

At the senior undergraduate level, this material can be used for a capstone course of a program and also serve for a good review of calculus and basic mathematical and complex analysis. At the starting graduate level, we find many illustrations of several strong tools of real and complex analysis with numerous examples and problems, a good many of which are quite involved. We use these tools, results and theorems, not just in computing examples and solving problems, but also in justifying that our methods of various computations are legitimate.

A student who knows advanced calculus and has learnt the material and problems of this book must be able to verify at least all the integrals numbered $582 - 709$ that appear in the pages $448 - 455$ of the *CRC Standard Mathematical Tables and Formulae*, by Daniel Zwillinger, 31^{st} Edition, Chapman & Hall / CRC, 2003. Have a look at and practice with them after you have finished studying this book.

At the end of this book, we have collected in a list all the major integrals evaluated one way or another in the text and the major finite and infinite sums in a different list. We did not go into computing and collecting infinite products. That would have been another chapter in the book. As expected we have included a sufficient bibliography, but far from all the bibliography that circulates in the world on these subjects. For the convenience of students and readers, an index of terms and names is also included.

Finally we thank all the people who study and use this book and we kindly ask them, if they encounter a typo or error that has escaped out attention, to communicate it to the author for correcting it in a prospective next edition. Also, suggesting new interesting and pertinent problems is highly appreciated.

Dr. Ioannis M. Roussos,
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Additional Prologue on the Second Edition: Improper Riemann Integrals

The present book can be considered as an *improved second edition* of the book with title "*Improper Riemann Integrals*". Even though this new book is conceptually similar to the previous book, it constitutes an extended, more complete and detailed exposition of the subject matter. More experience, new bibliography, additional practice and knowledge with the subject matter along with readers' suggestions, reports and critiques have motivated me to put forward this newly supplemented version. Apart from having corrected typos and errors and having taken care of some cosmetics, the old material has been extended by more than 230 pages of new material. This new material is dispersed within the old almost uniformly. Every part of the old book has been touched and extended by something interesting or important. Since improper Riemann integrals and infinite sums are interconnected the extension "& Infinite Sums" in the title was also necessary in order to give a more precise description of the content. Apart from the fact of having computed and listed a large number of infinite sums, we have also developed various ways of evaluating infinite sums in detail. A chapter of mathematics that deals with the infinite sums a lot is the Fourier series. We do not explore this chapter since the volume of this book would increase by much and because the Fourier series is a subject well exposed in many books of the international literature.

The new material includes ADDITIONAL:

Theory
Theorems
Results and Formulae on improper integrals and infinite sums
Examples
Applications, some of which are in partial differential equations
Results on Laplace and Mellin Transforms
Problems
Hints
Inter-text references
Footnotes
Index references
Bibliography

Also, **we have corrected Problems 2.5.15, 3.7.55 and 3.7.81** from the old book.

The new material is interesting, challenging, important, useful and applicable in mathematics, engineering and science. It supplements the material of the old book in a substantial and useful way. Some of these items were added at the requests and suggestions of some readers of the first book, whom we thank greatly. This new edition will be much more helpful for the interested student, professor, scientist, engineer and reader.

Given that the material collected was large, this work was split into two parts as evenly as possible. **Part I** contains the first two chapters: (1) A preliminary chapter on Improper Riemann Integrals. (2) Real analysis techniques. It also contains the whole bibliography. **Part II** contains the complex analysis techniques with the necessary theory.

Professor Emeritus Ioannis Markos Roussos
Hamline University
Twin Cities, Minnesota, 2023.

Note for Readers

For Part II of this work, which pertains to complex analysis and improper integrals, you can download it from the link

imroussos/Improper_Integrals_Part2: Improper Riemann Integrals through Complex Analysis. Part II (github.com)

Or, you can also write to the author to send you a free electronic copy at the emails:

iroussos@hamline.edu infroussos@yahoo.com

Given that the whole material was extended to 1000 pages, we have published just the real analysis and improper integrals material in this regular print. For the complex analysis material send a message to the author. Complex analysis is a very powerful tool for computing improper integrals and infinite sums. We have also included the lists of the results obtained in the whole material.

The inter-text references inside this book that begin with "**II 1.**" refer to the material of the chapter in complex analysis. All the references are written in bold numbers and or letters. Similarly, in the complex analysis material the references that begin with "**I 1.**" refer to chapters 1 and with "**I 2.**" refer to chapters 2, 3, and 4 of this book, in which expose the calculus and real analysis techniques in improper integrals.

Chapter 1

Complex Analysis Techniques

Complex Analysis provides several powerful methods for proving existence (convergence) and evaluating proper and improper integrals. In this chapter, we are going to investigate the most important ones. But, in order to state and use these powerful tools, we need to know some necessary complex analysis preliminaries.

1.1 Basics of Complex Variables

1.1.1 Basic Definitions and Operations

A **complex number** z is an expression of the form

$$z = a + bi,$$

where i is a new symbol with the property $i^2 = -1$, and both a and b are real numbers.

Since $i^2 < 0$, the new symbol i cannot represent a real number. In fact, any expression of the form bi with b real number represents what we call an **imaginary number**. So, $i = 1 \cdot i$ is an imaginary number. Also, for any real numbers a and b, we require the legitimacy of the following computation: $(ai) \cdot (bi) = (ab) \cdot (ii) = ab \cdot i^2 = ab \cdot (-1) = -ab$, for the multiplication of imaginary numbers.

Then we have $(\pm i)^2 = (\pm i) \cdot (\pm i) = (\pm 1)^2 \cdot i^2 = 1(-1) = -1$. Hence, we say that the real number -1 has two imaginary square roots, the $\pm i$. In the context of the complex numbers, we write

$$\sqrt{-1} = \pm i, \quad \text{which means } +\sqrt{-1} = +i \text{ and } -\sqrt{-1} = -i.$$

The real number a is called the **real part** of the complex number $z = a + bi$, and the real number b is called the **imaginary part**. We write

$$a = \mathrm{Re}(z), \qquad b = \mathrm{Im}(z), \qquad \text{and} \qquad z = \mathrm{Re}(z) + i\mathrm{Im}(z).$$

Equality of two complex numbers z and w is defined by

$$z = w \quad \overset{def}{\Longleftrightarrow} \quad \mathrm{Re}(z) = \mathrm{Re}(w) \quad \text{and} \quad \mathrm{Im}(z) = \mathrm{Im}(w).$$

Notice:

$$\begin{aligned} 0 &= 0 + 0i = 0i, \\ 1 &= 1 + 0i, \\ z = a \in \mathbb{R} \quad \Longleftrightarrow \quad \mathrm{Im}(z) &= 0 \quad \Longleftrightarrow \quad z = a + 0i, \\ z = bi \quad \text{pure imaginary} \quad \Longleftrightarrow \quad \mathrm{Re}(z) &= 0 \quad \Longleftrightarrow \quad z = 0 + bi. \end{aligned}$$

The operations of **addition**, **subtraction** and **multiplication** of complex numbers are defined in the obvious way, as shown in the following examples.

Example 1.1.1 If $z = 2 + 3i$ and $w = -1 + 7i$, we have:

$$\begin{aligned} z + w &= (2 + 3i) + (-1 + 7i) = (2 - 1) + (3 + 7)i = 1 + 10i \\ z - w &= (2 + 3i) - (-1 + 7i) = (2 + 1) + (3 - 7)i = 3 - 4i \\ z \cdot w &= (2 + 3i) \cdot (-1 + 7i) = (-2 + 21i^2) + (-3 + 14)i = -23 + 11i. \end{aligned}$$

▲

That is, if in general $z = a + bi$ and $w = c + di$, then

$$\begin{aligned} z + w &= (a + bi) + (c + di) = (a + c) + (b + d)i \\ z - w &= (a + bi) - (c + di) = (a - c) + (b - d)i \\ z \cdot w &= (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i. \end{aligned}$$

In the notation of the multiplication, we may drop the dot "$\cdot$".

We observe:

$$\begin{aligned} \mathrm{Re}(z + w) &= \mathrm{Re}(z) + \mathrm{Re}(w), \\ \mathrm{Im}(z + w) &= \mathrm{Im}(z) + \mathrm{Im}(w), \\ \mathrm{Re}(z - w) &= \mathrm{Re}(z) - \mathrm{Re}(w), \\ \mathrm{Im}(z - w) &= \mathrm{Im}(z) - \mathrm{Im}(w), \\ z = \mathrm{Re}(z) - i\mathrm{Re}(iz) &= \mathrm{Im}(iz) + i\mathrm{Im}(z). \end{aligned}$$

For the **division**, we need first to define the conjugate of a complex number. The **conjugate** of a complex number $z = a + bi$ is given by the complex number

$$\bar{z} = a - bi.$$

Note that

$$\mathrm{Re}(\bar{z}) = \mathrm{Re}(z), \qquad \mathrm{Im}(\bar{z}) = -\mathrm{Im}(z),$$
$$z\bar{z} = (a+bi)(a-bi) = a^2 + b^2 \geq 0.$$

The function $z \in \mathbb{C} \longrightarrow \bar{z} \in \mathbb{C}$ is one-to-one and onto. We called it **conjugation of complex numbers**. Notice that if we compose it with itself, we get the identity function, i.e., $\bar{\bar{z}} = z$. Review the **basic properties of conjugation** listed in **Problem 1.1.11**. They are easy to verify. (See also **Problem 1.2.23**.)

We also define the **absolute value** of a complex number $z = a + bi$. This is defined by:

$$|z| = |a + ib| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

It is also called **modulus** or **magnitude** or **length** of the complex number z. For **example**,

$$|i| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1 \text{ and } |2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$

It follows immediately that

$$(1) \quad |\pm \bar{z}| = |\pm z| = |\pm (a \pm ib)| = \sqrt{a^2 + b^2},$$
$$(2) \quad \forall \;\; z \in \mathbb{C}, \;\; |z| \geq 0,$$
$$(3) \quad |z| = 0 \iff z = 0 \iff \bar{z} = 0,$$
$$(4) \quad |z|^2 = |\bar{z}|^2 = z\bar{z} = a^2 + b^2.$$

We now define the **division** of two complex numbers. We begin with an example:

Example 1.1.2 If $z = 2 + 3i$ and $w = -1 + 7i$, we have division in the following way:

$$\frac{z}{w} = \frac{2+3i}{-1+7i} = \frac{2+3i}{-1+7i} \cdot \frac{-1-7i}{-1-7i} = \frac{19-17i}{(-1)^2+7^2} = \frac{19-17i}{50} = \frac{19}{50} - \frac{17}{50}i.$$

▲

In general, when $z = a + bi$ and $w = c + di \neq 0 + 0i = 0$, the **division** $\frac{z}{w}$ $(w \neq 0)$ is defined by

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}} = \frac{z \cdot \bar{w}}{|w|^2} = \frac{ac + bd + (-ad + bc)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{(-ad + bc)}{c^2 + d^2}i.$$

So, the **reciprocal** of any non-zero complex number $z = a + ib \neq 0$ is

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{1}{|z|^2} \cdot \bar{z} = \frac{a - bi}{a^2 + b^2}.$$

All the operations thus defined in the complex numbers satisfy all the properties of the corresponding operations in the real numbers, as we can directly check. So, associativity, commutativity, distribution, opposite, reciprocal and neutral additive and multiplicative elements are legitimate to use when applicable. So, the set of complex numbers equipped with operations of addition and multiplication and their properties is, as we say in abstract algebra, an **algebraic field**

The following most important **properties of the absolute value** in the set of complex numbers are easily verified. (See **Problems 1.1.6, 1.1.7** and **1.1.8**.) We assume that z and w are any complex numbers and any denominators involved are not zero. Then we have:

$$(1) \quad |z \cdot w| = |z| \cdot |w|,$$

$$(2) \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|},$$

$$(3) \quad |z^n| = |z|^n, \ \forall \ n \in \mathbb{Z},$$

$$(4) \quad |z + w| \leq |z| + |w|,$$

$$(5) \quad \pm\left(|z| - |w|\right) \leq \big||z| - |w|\big| \leq |z \pm w| \leq |z| + |w|,$$

$$(6) \quad \max\{|\mathrm{Re}(z)|,\ |\mathrm{Im}(z)|\} \leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|,$$

$$(7) \quad |z \pm w|^2 = (z \pm w)(\overline{z \pm w}) = (z \pm w)(\bar{z} \pm \bar{w}) = \\ |z|^2 + |w|^2 \pm (z\bar{w} + \bar{z}w) = |z|^2 + |w|^2 \pm 2\,\mathrm{Re}(z\bar{w}) = \\ |z|^2 + |w|^2 \pm 2\,\mathrm{Re}(\bar{z}w).$$

The inequality $|z+w| \leq |z|+|w|$ is called **triangle inequality**. This holds as an equality if and only if $z = 0$, or $w = 0$, or z is a positive real multiple of w (or w is a positive real multiple of z). (See **Problem 1.1.7**.)

Remark 1: From these properties, we may produce others as corollaries. (See **Problems 1.1.14, 1.1.15** and **1.1.26**.)

Remark 2: For $r > 0$, the equation $|z| = |a + bi| = \sqrt{a^2 + b^2} = r$ is equivalent to $a^2 + b^2 = r^2$, which in the (a, b)-plane represents a circle with center the origin and radius r.

1.1.2 Representations and Roots of Complex Numbers

The usual form of a complex number $z = a + bi$, where $a,\ b \in \mathbb{R}$ is also called the **Cartesian**[1] **form** of the complex number. Below, we explore other forms of the complex numbers.

(1) Geometric and Vector Representations of Complex Numbers

We can give a **geometric interpretation of complex numbers** by associating the complex number $z = a + bi$ with the point (a, b) in the 2-dimensional Euclidean plane $\mathbb{R}^2$. We then write that $\mathbb{C} \approx \mathbb{R}^2$. The horizontal axis is called the **real axis**, and the vertical axis is called the **imaginary axis**. The **origin** is $O = (0, 0) = 0 + 0i$. See **Figure 1.1**.

Given a number $z = a + bi$, corresponding to the point (a, b), note that the **conjugate** $\bar{z} = a - bi$ corresponds to the point $(a, -b)$. Thus, geometrically, we get $\bar{z}$ from z by a reflection in the real axis.

The $|z| = |a + bi| = \sqrt{a^2 + b^2}$ is the distance of the point (a, b) from the origin.

Also, notice that the **distance of two points** $P = (a, b) = a+ib = z_1$ and $Q = (c, d) = c + id = z_2$ in the complex plane is given by

$$d(P, Q) = \sqrt{(c - a)^2 + (d - b)^2} = |z_2 - z_1| = |z_1 - z_2|.$$

In this way, using the complex numbers and their properties, we can describe and solve a lot of geometrical problems in the plane. There are books in the literature that have undertaken such a task. E.g., see: Dodge 1972, Pedoe 1988, Schwerdtfeger 1979 and Zwikker 2005.

For instance, a circle with center the point $C = (c, d)$ or the complex number $w = c + id$ and radius $r \geq 0$ is the set of complex numbers $z = x + iy$, such that

$$\{z \in \mathbb{C} \mid |z - w| = r\} = \left\{z = x + iy \in \mathbb{C} \mid \sqrt{(x - c)^2 + (y - d)^2} = r\right\} = \\ \{z \in \mathbb{C} \ \mid \ (z - w) \cdot (\bar{z} - \bar{w}) = r^2\} = \text{etc.}$$

A complex number $z = a + ib$ may also be viewed as the vector $\overrightarrow{OP}$, where $P = (a, b)$. If $w = c+id$ is another complex number viewed as the vector $\overrightarrow{OQ}$, where $Q = (c, d)$, then we have:

1. $z + w = (a + c) + i(b + d) = [(a + c), (b + d)] = \overrightarrow{OP} + \overrightarrow{OQ}$.
2. $z - w = (a - c) + i(b - d) = [(a - c), (b - d)] = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{QP}$.

[1] After René Descartes, French philosopher and mathematician, 1596-1650.

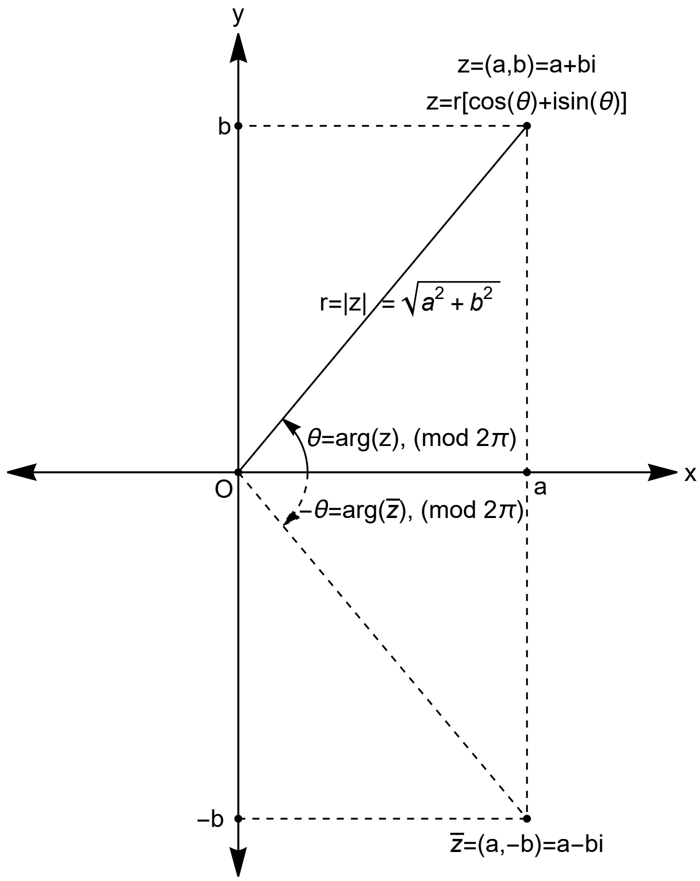

FIGURE 1.1: Complex Numbers and Trigonometric Form

3. The **length of the vector** is $\|\overrightarrow{OP}\| = \sqrt{a^2+b^2} = \sqrt{z\bar{z}} = |z|$.
4. The **inner product** is $\overrightarrow{OP} \cdot \overrightarrow{OQ} = ac + bd = \mathrm{Re}(z\bar{w}) = \mathrm{Re}(\bar{z}w)$. So, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are perpendicular iff $\mathrm{Re}(z\bar{w}) = \mathrm{Re}(\bar{z}w) = 0$.
5. If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are non-zero vectors (i.e., $z \neq 0$ and $w \neq 0$) and $0 \leq \alpha \leq \pi$ (mod 2π) is the angle between them, then

$$\cos(\alpha) =$$

$$\frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{\|\overrightarrow{OP}\| \cdot \|\overrightarrow{OQ}\|} = \frac{ac+bd}{\sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}} = \frac{\mathrm{Re}(z\bar{w})}{|z||w|} = \frac{\mathrm{Re}(\bar{z}w)}{|z||w|}.$$

6.

$$|z-w|^2 = (a-c)^2 + (b-d)^2 =$$
$$a^2 + c^2 + b^2 + d^2 - 2(ac+bd) =$$
$$|z|^2 + |w|^2 - 2\mathrm{Re}(z\overline{w}) = |z|^2 + |w|^2 - 2|z||w|\cos(\alpha),$$

$$\text{or} \quad \|\overrightarrow{PQ}\|^2 = \|\overrightarrow{QP}\|^2 = \|\overrightarrow{OP} - \overrightarrow{OQ}\|^2 =$$
$$\|\overrightarrow{OP}\|^2 + \|\overrightarrow{OQ}\|^2 - 2\|\overrightarrow{OP}\| \cdot \|\overrightarrow{OQ}\| \cos(\alpha),$$

which is the trigonometric **law of cosines** in the triangle OPQ.

7. If a, b, and c are real numbers such that $|a| + |b| \neq 0$, then the general Cartesian form of a **straight line** in the real plane $\mathbb{R}^2$,

$$ax + by + c = 0 \quad \text{is written as}$$

$$\mathrm{Re}(\bar{p}z) + c = \mathrm{Re}(\overline{\bar{p}z}) + c = \mathrm{Re}(p\bar{z}) + c = 0,$$

where $z = x + iy$ and $p = a + ib$, in the complex plane $\mathbb{C}$.

In this way, we can derive the various rules that pertain to vector and geometrical calculations by means of complex numbers.

(2) Trigonometric Form of Complex Numbers

Since each point (a, b) in the plane $\mathbb{R}^2$ is identified by the complex number $z = a+ib \in \mathbb{C}$, we can rewrite z, as we say, in the **trigonometric form of the complex number**. This is done as follows:

The point (a, b) has polar coordinate representation (r, θ), where $r = \sqrt{a^2 + b^2}$, and if $(a, b) \neq (0, 0)$, then θ is determined (mod 2π) by the following two relations:

$$\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{r} \quad \text{and} \quad \sin(\theta) = \frac{b}{\sqrt{a^2+b^2}} = \frac{b}{r}.$$

If $z = (a, b) = (0, 0)$, then $a = b = r = 0$ and θ is indeterminate.

From these relations we have

$$a = r\cos(\theta) \quad \text{and} \quad b = r\sin(\theta),$$

so that we can write the complex number z as

$$z = (a, b) = a + bi = r[\cos(\theta) + i\sin(\theta)].$$

This representation of z is called the **trigonometric form of** z.

Since θ is determined (mod 2π), in principle, we can use any of the infinitely values of θ in representing the polar coordinates of any point $(a, b) \in \mathbb{R}^2$. Similarly, in the representation of a complex number z in trigonometric form, we can use any of the value $\theta + 2k\pi$, with $k \in \mathbb{Z}$, without altering the number z.

In solving some problems and in other situations the use of the tangent of half of θ is more convenient. That is,

$$z = (a, b) = a + bi = r\left[\frac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} + i\frac{2\tan\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right].$$

We observe that the **radius** $r = |z| = \sqrt{a^2 + b^2}$ is the **absolute value** or **modulus** or **magnitude** or **length** of z.

We call the angle θ **argument** of z, and we write $\arg(z) = \theta$. See **Figure 1.1**.

Notice: $|\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = +\sqrt{1} = 1$, $\forall\ \theta \in \mathbb{R}$.

(3) Some Important Remarks on the Argument

As we have seen, the $\arg(z) = \theta$ for any $z \neq 0$ is determined (mod 2π). Then it is convenient to consider a fixed initial interval of values of the argument of complex numbers, such that all values in it are sufficient to represent any complex number $z \in \mathbb{C} - \{0\}$ in trigonometric form **uniquely**.

If $z = 0$, then $r = |z| = |0| = 0$, and any angle can serve as argument of $z = 0$. Thus, $z = 0$ cannot have unique trigonometric representation no matter how we restrict θ (unless we restrict it at just one value).

Many authors prefer this initial interval of values of the argument to be $0 \leq \theta < 2\pi$. Others prefer the $-\pi \leq \theta < \pi$ or $-\pi < \theta \leq \pi$. These assignments are convenient but arbitrary.

In general, we could use any half-open and half-closed interval of length 2π to begin with. Such an interval provides sufficient values for the angle θ to represent any complex number $z \neq 0$ in trigonometric form **uniquely**.

The argument that varies in such a fixed preassigned interval is called **principal argument**. Here, to write the principal argument, we use the notation $\mathbf{Arg}(z)$ or $\mathbf{Arg}_0(z)$.

Then, to find all possible values of $\arg(z)$ for any $0 \neq z \in \mathbb{C}$, we add $2k\pi$ to $\mathrm{Arg}(z)$, where $k \in \mathbb{Z}$. These values are precisely what we can call all possible arguments of $0 \neq z \in \mathbb{C}$. The notation $\mathbf{Arg}_0(z)$ for the principal argument is justified because if we use $k = 0$ we find $\mathrm{Arg}(z)$. Similarly, any other argument is obtained by using any integer $k \in \mathbb{Z}$, which we denote as $\mathbf{Arg}_k(z)$.

Hence, we define the **argument** (or we can also call it **total argument**) **of a complex number** as the set of values:

$$\arg(0) = \{\theta \mid \theta \in \mathbb{R}\}$$

and if $z \neq 0$, then

$$\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}(z) + 2k\pi \mid k \in \mathbb{Z}\}.$$

Any value singled out of these possible values is simply called **an argument of** z. (We will see more important remarks on the continuity and discontinuity of the argument in **Subsection 1.5.3**.)

Note: Pay attention to the fact that it is not always true that $\arg(z) = \theta = \arctan\left(\frac{b}{a}\right)$ even though $\tan(\theta) = \frac{b}{a}$. Also, notice if $a = 0$ and $b > 0$, then $\theta = \frac{\pi}{2} \pmod{2\pi}$. If $a = 0$ and $b < 0$, then $\theta = -\frac{\pi}{2}$ $\pmod{2\pi}$.

We illustrate the definition of the $\mathrm{Arg}(z)$ and the remarks on it by the following examples:

Example 1.1.3 Let $(a, b) = (-1, \sqrt{3}) = -1 + i\sqrt{3}$. Then $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2\,,\ \sin(\theta) = \frac{\sqrt{3}}{2} > 0$ and $\cos(\theta) = \frac{-1}{2} < 0$. Therefore, θ is a second quadrant angle. In fact, $\theta = \frac{2\pi}{3} \pmod{2\pi}$.

We observe that if the initial interval for the principal argument is $[0, 2\pi)$, or $(0, 2\pi]$, or $[-\pi, \pi)$, or $(-\pi, \pi]$ and $z = -1 + i\sqrt{3}$, then $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \frac{2\pi}{3}$, in all these cases.

We also have: $\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$ but $\arctan(-\sqrt{3}) = -\frac{\pi}{3} \neq \frac{2\pi}{3}$ $\pmod{2\pi}$.

Therefore, if we use arc-tangent or another inverse trigonometric function to determine the principal argument, **we may need to add a corrective constant**. In this example, we find that an argument of the complex number $z = -1 + i\sqrt{3}$ (or its principal argument, depending on what we declare as initial interval for the principal argument) is

$$\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \theta = \frac{2\pi}{3} = -\frac{\pi}{3} + \pi = \arctan(-\sqrt{3}) + \pi.$$

That is, the corrective constant is π.

▲

Example 1.1.4 If an argument is found to be $\theta = \dfrac{4\pi}{3}$, then in $[0, 2\pi)$ we have $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \dfrac{4\pi}{3}$. But in $[-\pi, \pi)$ or in $(-\pi, \pi]$, we have $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = -\dfrac{2\pi}{3}$.

Similarly, if $\theta = \pi$, then: In $[0, 2\pi)$ we have $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \pi$. In $[-\pi, \pi)$ we have $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \pi - 2\pi = -\pi$. In $(-\pi, \pi]$ we have $\mathrm{Arg}(z) = \mathrm{Arg}_0(z) = \pi$.

▲

(4) Some Important Usages of the Trigonometric Form

The trigonometric form of z gives useful interpretations of equality, multiplication, division, powers and roots of complex numbers and renders the pertinent computations easy. So, suppose

$$z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)] \quad \text{and} \quad z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)],$$

where $r_1 \geq 0$ and $r_2 \geq 0$ are two given complex numbers in trigonometric form.

In this form, we obtain the following basic results:

(1) For **equality**, we have:

$$\{\ z_1 = z_2,\ \text{ or }\ r_1[\cos(\theta_1) + i\sin(\theta_1)] = r_2[\cos(\theta_2) + i\sin(\theta_2)]\ \} \ \overset{\text{iff}}{\Longleftrightarrow}$$
$$\text{either} \quad \{r_1 = r_2 = 0 \quad \text{and} \quad \theta_1,\ \theta_2 \quad \text{are any angles}\},$$
$$\text{or} \quad \{r_1 = r_2 > 0 \quad \text{and} \quad \cos(\theta_1) + i\sin(\theta_1) = \cos(\theta_2) + i\sin(\theta_2)\ \}\,.$$

From this, we get that the following rule of **equality of complex numbers in their trigonometric forms**:

$$z_1 = z_2 \quad \overset{\text{iff}}{\Longleftrightarrow} \quad \text{either} \quad r_1 = r_2 = 0 \quad \text{and} \quad \theta_1,\ \theta_2 \quad \text{are any angles},$$
$$\text{or } r_1 = r_2 > 0 \text{ and } \theta_1 = \theta_2 \pmod{2\pi},\ \text{i.e., } \theta_1 = \theta_2 + 2k\pi \text{ with } k \in \mathbb{Z}.$$

(Prove these rules as an exercise!)

(2) For **conjugation**, we have:

$$\bar{z} = \overline{r[\cos(\theta) + i\sin(\theta)]} = \bar{r}\ \overline{[\cos(\theta) + i\sin(\theta)]} =$$
$$r[\cos(\theta) - i\sin(\theta)] = r[\cos(-\theta) + i\sin(-\theta)].$$

That is, to find the conjugate of $z = r[\cos(\theta) + i\sin(\theta)]$, we change the θ to $-\theta$ and simplify.

(3) For the **multiplication**, we have:

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1) + i\sin(\theta_1)\right] \cdot \left[\cos(\theta_2) + i\sin(\theta_2)\right] = r_1 r_2 \cdot$$
$$\{\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left[\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right]\} =$$
$$r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right].$$

Hence, the product $z_1 z_2$ in trigonometric form has absolute value the product of the absolute values of z_1 and z_2 and argument the sum of their arguments (angles) $\pmod{2\pi}$.

(4) For the **division**, when $z_2 \neq 0$, we have:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \frac{\cos(\theta_1) + i\sin(\theta_1)}{\cos(\theta_2) + i\sin(\theta_2)} = \frac{r_1}{r_2} \cdot$$
$$\{\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i[\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)]\} =$$
$$\frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

Hence, the division $\dfrac{z_1}{z_2}$ in trigonometric form has absolute value the quotient of the absolute values of z_1 and $z_2 \neq 0$ and argument the difference of the argument of the numerator minus the argument of the denominator $\pmod{2\pi}$.

(5) Powers with Integer Exponents and De Moivre Formula

To find the **powers** of a complex number $z = a + bi$ with integer exponents, we may use the **Binomial Theorem**: For all exponents $n = 0,\ 1,\ 2,\ 3,\ \ldots,$ we have

$$(a + bi)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k i^k, \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Remember that $0! = 1$ and so $\binom{n}{0} = 1 = \binom{n}{n}$. For $k = 1,\ 2,\ 3,\ \ldots,$ we simplify $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)(k-2)\ldots 3 \cdot 2 \cdot 1}$.

The powers of i with integer exponents k are simplified by the rule

$$i^k = \begin{cases} +1, & \text{if } k = 4m, \\ +i, & \text{if } k = 4m+1, \\ -1, & \text{if } k = 4m+2, \\ -i, & \text{if } k = 4m+3, \end{cases}$$

where m is the integer quotient of k divided by 4. (See **Problem 1.1.10**.)

Example 1.1.5 We have

$$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi = (a-b)(a+b) + 2abi,$$

$$z^3 = (a+bi)^3 =$$
$$(a^3 - 3ab^2) + (3a^2b - b^3)i = a(a^2 - 3b^2) + b(3a^2 - b^2)i =$$
$$a(a - \sqrt{3}\ b)(a + \sqrt{3}\ b) + b(\sqrt{3}\ a - b)(\sqrt{3}\ a + b)i,$$

$$z^4 = (a+bi)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k i^k = \ \dots,$$

and so on.

▲

The binomial method for computing powers of complex numbers is not efficient for "large" exponents. An efficient method uses the trigonometric form of the complex number z as follows:

A simple inductive argument using n multiplications of

$$z = r[\cos(\theta) + i\sin(\theta)]$$

by itself, for any positive integer n, results in the so-called **De Moivre**[2] **formula**

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)].$$

If $z \neq 0$, this formula is easily extended to $n = 0$, in which case both sides are equal to 1, and to n a negative integer, since $z^{-k} = \dfrac{1}{z^k}$ for k positive integer.

So, the **De Moivre formula** is valid $\forall\ n \in \mathbb{N} = \{1,\ 2,\ 3,\ 4, \dots\}$ regardless of z and if $z \neq 0$, then it is valid for all integers $n \in \mathbb{Z} = \{0,\ \pm1, \pm2,\ \pm3,\ \pm4, \dots\}$.

(6) Roots of Complex Numbers

Now we study the **roots** of complex numbers. First $\forall\, n \in \mathbb{N} = \{1,\ 2,\ 3,\ 4, \dots\}$ we consider the equation

$$\zeta^n = 1 = 1[\cos(0) + i\sin(0)].$$

Letting $\zeta = \rho[\cos(\phi) + i\sin(\phi)]$, in trigonometric form and so $\rho \geq 0$, by the De Moivre formula we find

$$\zeta^n = \rho^n[\cos(n\phi) + i\sin(n\phi)] = 1 = 1[\cos(0) + i\sin(0)].$$

[2] Abraham De Moivre, French mathematician, 1667-1754.

So, $\rho^n = 1$, $\cos(n\phi) = \cos(0)$ and $\sin(n\phi) = \sin(0)$. Therefore, $\rho = \sqrt[n]{1} = 1 > 0$ and $\phi = \dfrac{0 + 2k\pi}{n} = \dfrac{2k\pi}{n}$, with $k \in \mathbb{Z}$. So,

$$\zeta_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k \in \mathbb{Z}.$$

For any $k \in \mathbb{Z}$, by the division identity, we have $k = nq + l$ with $q \in \mathbb{Z}$ and $l = 0,\ 1,\ 2,\ \ldots,\ n-1$. So, for this $k \in \mathbb{Z}$ and $l = 0,\ 1,\ 2,\ \ldots,\ n-1$

$$\zeta_k = \zeta_{nq+l} = \cos\left(\frac{2l\pi}{n} + 2q\pi\right) + i\sin\left(\frac{2l\pi}{n} + 2q\pi\right).$$

Since the functions cosine and sine are 2π-periodic, we find that for any $k \in \mathbb{Z}$, we have that

$$\zeta_k = \zeta_l = \cos\left(\frac{2l\pi}{n}\right) + i\sin\left(\frac{2l\pi}{n}\right), \quad \text{for an} \quad l = 0,\ 1,\ 2,\ \ldots,\ n-1.$$

(Check that for $l = 0,\ 1,\ 2,\ \ldots,\ n-1$, these numbers are pairwise different, as an exercise!)

Finally, **the equation**

$$\zeta^n = 1$$

has n distinct solutions given by the formulae

$$\zeta_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0,\ 1,\ 2,\ \ldots,\ n-1.$$

These n different distinct solutions are called the n $\mathbf{n^{th}}$ **roots of unity**, that is, of the number 1.

Notice that

$$\zeta_k = \zeta_1^k, \text{ for } k = 0,\ 1,\ 2,\ \ldots,\ n-1 \quad (\zeta_0 = 1, \quad \forall \quad n \in \mathbb{N}).$$

Example 1.1.6 The **first root** of unity is trivially itself, a fact trivially true for any complex number.

The **second or square roots of unity** are 1 and -1 (easy).

The **third or cubic roots of unity** are easily computed to be 1, $-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ and $-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$.

The **fourth or quadric roots of unity** are easily computed to be 1, i, -1 and $-i$.

(Check these results and compute the five **fifth roots of unity**.)

▲

Next, we consider any complex number (set in trigonometric form)

$$w = a + ib = r[\cos(\theta) + i\sin(\theta)] \neq 0 + i0 = 0,$$

where $r = |w| > 0$ and θ are the magnitude and the argument of w, respectively, as defined earlier. θ could be any argument, but if it is convenient, we may use the unique one in the interval $[0, 2\pi)$, in radians.

Working as we did with the roots of unity and using the **De Moivre formula**, we find that the equation

$$z^n = \rho^n[\cos(n\phi) + i\sin(n\phi)] = w = a + ib = r[\cos(\theta) + i\sin(\theta)]$$

has again n different distinct solutions which are given by the formulae

$$z_k = \sqrt[n]{w} = \sqrt[n]{r}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right],$$
$$k = 0, 1, 2, \ldots, n-1,$$

where $\sqrt[n]{r} > 0$ is the positive n^{th} root of the positive number $r > 0$. (Work this out one more time as we did for $w = 1$ before, for practice!)

Using the rule of multiplication of complex numbers in trigonometric form and the De Moivre formula, we find

$$z_k = \sqrt[n]{w} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right] \cdot \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^k,$$
$$\text{or} \quad z_k = z_0 \cdot \zeta_1^k, \quad \text{for} \quad k = 0, 1, 2, \ldots, n-1,$$

that is, *the root z_k of w is obtained by multiplying the z_0 root of w with the k^{th} power of the root of unity ζ_1, for any $k = 0, 1, 2, \ldots, n-1$.* (Since $\zeta_l = \zeta_1^l$, we can also use any ζ_l raised to some power to obtain the roots of w.) These n different distinct solutions are called **the n $\mathbf{n^{th}}$ roots of $w \in \mathbb{C}$.**

Note: If $n \in \mathbb{N}$, then all the n^{th} roots of $w = 0 + i\,0 = 0$ are equal to $0 + i\,0 = 0$, since $|0 + i\,0| = 0$.

Also, notice that in calculus, $\forall\ n \in \mathbb{N}$ and $\forall\ a \geq 0$ the symbol $\sqrt[n]{a} = a^{\frac{1}{n}} = p$ represents the unique real value $p \geq 0$ such that $p^n = p \cdot p \cdot \ldots \cdot p = a$ (n times). But in the general complex analysis, the symbol $w^{\frac{1}{n}} = \sqrt[n]{w}$, for any complex number $w \neq 0$, represents n different distinct complex values, namely the n n^{th} toots of w. (Study and solve completely **Problem 1.1.27**!)

Example 1.1.7 Find the three cubic roots of $w = 2 - 3i$.

We have that $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. We may take θ to be the unique angle of the fourth quadrant in $[0, 2\pi)$ with

$$\cos(\theta) = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13} \qquad \text{and} \qquad \sin(\theta) = \frac{-3}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}.$$

Then we find that

$$\theta = 2\pi - \arccos\left(\frac{2}{\sqrt{13}}\right) = 2\pi + \arcsin\left(\frac{-3}{\sqrt{13}}\right) \simeq 5.3003916 \text{ radians}$$

(approximately).

So, we have that the three cubic roots of $w = 2 - 3i$ are:

$$z_0 = \sqrt[3]{\sqrt{13}}\left[\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right] = \sqrt[6]{13}\left[\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right],$$

and similarly

$$z_1 = \sqrt[6]{13}\left[\cos\left(\frac{2\pi+\theta}{3}\right) + i\sin\left(\frac{2\pi+\theta}{3}\right)\right],$$
$$z_2 = \sqrt[6]{13}\left[\cos\left(\frac{4\pi+\theta}{3}\right) + i\sin\left(\frac{4\pi+\theta}{3}\right)\right].$$

From these, we can evaluate the approximate values of z_0, z_1 and z_2, in seven decimal digits, to be:

$$z_0 \simeq -0.2986283 + i\ 1.5040465,$$
$$z_1 \simeq -1.1532283 - i\ 1.0106429,$$
$$z_2 \simeq +1.4518566 - i\ 0.4934035.$$

Notice that we can avoid the approximate answers by doing more work in the following way: We need to express the $\cos\left(\frac{\theta}{3}\right)$ and $\sin\left(\frac{\theta}{3}\right)$ in terms of the known numbers $\cos(\theta) = \frac{2\sqrt{13}}{13}$ and $\sin(\theta) = \frac{-3\sqrt{13}}{13}$. To do this, we use the trigonometric formulae:

$$\cos(\theta) = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right), \qquad \sin(\theta) = 3\sin\left(\frac{\theta}{3}\right) - 4\sin^3\left(\frac{\theta}{3}\right).$$

Then, we use the formulae for solving cubic polynomial equations by radicals (e.g., see bibliography: Dickson 1949, 46-51). After this, we use the expansion formulae of $\cos(\alpha \pm \beta)$ and $\sin(\alpha \pm \beta)$ in the three third roots of $w = 2 - 3i$ obtained and calculate them accurately in terms of radicals. (See **Problem 1.1.24**!)

This work is quite lengthy and cumbersome even when $n = 3$ or $n = 4$ and thus impractical. For $n \geq 5$, there are not even formulae for solving any general polynomial of degree $n \geq 5$.

Only for $n = 2$ is such a work quick and practical, as we shall see in the **next Subsection, 1.1.3**.

▲

(7) Argument Revisited

To accurately understand the properties of the argument, we must first define the following: Suppose $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$ non-empty sets and $z \in \mathbb{C}$. Then we define the following subsets of $\mathbb{C}$:

1. The **sum subset**
$$S + T = \{w = s + t \mid s \in S \text{ and } t \in T\}.$$
2. The **difference subset**
$$S - T = \{w = s - t \mid s \in S \text{ and } t \in T\}.$$
3. The **product subset**
$$S \cdot T = \{w = s \cdot t \mid s \in S \text{ and } t \in T\}.$$
4. The **multiple subset**
$$z \cdot S = \{z\} \cdot S = \{w = z \cdot s \mid s \in S\}.$$

(**Note**: If one of the sets S and T is empty, then these new sets are also empty.)

If $S = \{w\}$ (one element set), then we may simply write $w+T$, $w-T$ for $\{w\}+T$ and $\{w\}-T$, respectively (as we see in the multiple subset).

We notice that for the set of integers $\mathbb{Z}$, any $k \in \mathbb{Z}$ and any $\emptyset \neq M \subseteq \mathbb{Z}$, we have the easy relations:

$$\begin{gathered}
1 \cdot \mathbb{Z} = \mathbb{Z}, \\
(-1) \cdot \mathbb{Z} = -\mathbb{Z} = \mathbb{Z}, \\
\mathbb{Z} \cdot \mathbb{Z} = \mathbb{Z}, \\
k \cdot \mathbb{Z} \subseteq \mathbb{Z} \quad \text{and equality holds iff} \quad k = \pm 1, \\
M \cdot \mathbb{Z} \subseteq \mathbb{Z}, \\
k + \mathbb{Z} = \mathbb{Z}, \\
k - \mathbb{Z} = \mathbb{Z}, \\
\pm M \pm \mathbb{Z} = \mathbb{Z}, \\
\mathbb{Z} + \mathbb{Z} = \mathbb{Z}, \\
\mathbb{Z} - \mathbb{Z} = \mathbb{Z}, \\
\pm \mathbb{Z} \pm \mathbb{Z} \pm \ldots \pm \mathbb{Z} = \mathbb{Z}.
\end{gathered}$$

Using these definitions and notations, we can write the sets of $\arg(z)$ for any $z \in \mathbb{C}$ as

$$\arg(0) = \{\theta \mid \theta \in \mathbb{R}\} = \mathbb{R} \subset \mathbb{C},$$

and if $z \neq 0$ [using the notation $\mathrm{Arg}_0(z) = \mathrm{Arg}(z)$ for the designated principal argument], then we have

$$\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}(z) + 2k\pi \mid k \in \mathbb{Z}\} = \\ \mathrm{Arg}_0(z) + 2\pi\mathbb{Z} = \mathrm{Arg}(z) + 2\pi\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}.$$

We use the notation $\mathrm{Arg}_0(z)$ or $\mathrm{Arg}(z)$ for the principal argument of $z \in \mathbb{C}$. In view of what we have seen so far, for any $z,\ z_1,\ z_2 \in \mathbb{C}$, any $m \in \mathbb{Z}$ and any $n \in \mathbb{N}$, we can easily prove the following **set relations**:

1. $\arg(\bar{z}) = -\arg(z)$.
2. $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$.
3. If $z_2 \neq 0$, then $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.
4. $\arg\left(z^m\right) = m \cdot \mathrm{Arg}_0(z) + 2\pi\mathbb{Z} \supseteq m \cdot \arg(z)$ and equality holds if and only if $m = \pm 1$. (If $m = 0$ and $z = 0$, we consider $0^0 = 1$.)
5. If the n n^{th} roots of $z \neq 0$ are given by

   $$w_k = \sqrt[n]{|z|}\left\{\cos\left[\frac{\mathrm{Arg}(z) + 2k\pi}{n}\right] + i\,\sin\left[\frac{\mathrm{Arg}(z) + 2k\pi}{n}\right]\right\},$$

   with $k = 0,\ 1,\ 2,\ \ldots\ n-1$, then

   $$\arg(w_k) = \frac{\mathrm{Arg}(z) + 2k\pi}{n} + 2\pi\mathbb{Z}, \quad \text{for} \quad k = 0,\ 1,\ 2,\ \ldots\ n-1.$$

   The union of these sets is the set

   $$\arg\left(z^{\frac{1}{n}}\right) = \left\{\frac{\mathrm{Arg}(z) + 2k\pi}{n} \ \mid\ k = 0,\ 1,\ 2,\ \ldots\ n-1\right\} + 2\pi\mathbb{Z}.$$

   Notice that if $\mathrm{Arg}(z) \in [0, 2\pi)$, then $\dfrac{\mathrm{Arg}(z) + 2k\pi}{n} \in [0, 2\pi)$ for all $k = 0,\ 1,\ 2,\ 3\ \ldots\ n-1$.

Note: In the next **Section, 1.2**, we will the **exponential form** of any complex number $z \neq 0$.

Problems

1.1.1 For $z = 3 - 2i$ and $w = 3 + i$, compute $z + w,\ z - w,\ z \cdot w$ and $\dfrac{z}{w}$. Also compute $|z|,\ |w|,\ |z + w|,\ |z - w|,\ |z \cdot w|$ and $\left|\dfrac{z}{w}\right|$.

1.1.2 Find the real numbers x and y, if

$$\frac{2-i}{i} - \frac{3y(1-2i)}{3+i} + \frac{2xi - y}{1-i} = \frac{3x(2+i)^2}{-1+3i}.$$

1.1.3 Let z be complex number. Prove

$$z^2 = \overline{z}^2 \quad \Longleftrightarrow \quad \text{either } z \in \mathbb{R}, \quad \text{or } z \text{ is pure imaginary.}$$

1.1.4 Let $z \neq 0$ be complex number. Prove

$$\mathrm{Re}\ z^2 = -\frac{1}{2} \quad \Longleftrightarrow \quad |z| = \left|z + \frac{1}{z}\right| = \frac{|z^2+1|}{|z|} \quad \Longleftrightarrow \quad |z|^2 = |z^2 + 1|.$$

1.1.5 Let z be complex number. Prove

$$z^2 + z + 1 = 0 \quad \Longleftrightarrow \quad |z| = |z+1| = 1.$$

1.1.6 For any $z \in \mathbb{C}$, prove:

$$\text{(a)} \qquad \max\{|\mathrm{Re}\ z|,\ |\mathrm{Im}\ z|\} \leq |z| \leq |\mathrm{Re}\ z| + |\mathrm{Im}\ z|.$$

Under what conditions these inequalities hold as equalities.

In equivalent form with real numbers, this inequality says that for any two real numbers a and b, we have

$$\max\{|a|,\ |b|\} \leq \sqrt{a^2 + b^2} \leq |a| + |b|.$$

Give a geometric interpretation of this inequality.

$$\text{(b)} \qquad \frac{|\mathrm{Re}\ z| + |\mathrm{Im}\ z|}{\sqrt{2}} \leq |z| \leq |\mathrm{Re}\ z| + |\mathrm{Im}\ z|.$$

Under what conditions these inequalities hold as equalities? Write the equivalent form with real numbers.

1.1.7 Prove the **triangle inequality** $|z + w| \leq |z| + |w|$. Then prove that it holds as an equality if and only if $z = 0$, or $w = 0$, or z is a positive real multiple of w, or w is a positive real multiple of z.

1.1.8 For all complex numbers z and w, prove

$$\pm(|z| - |w|) \leq ||z| - |w|| \leq |z \pm w| \leq |z| + |w|.$$

1.1.9 For a complex number z prove:

$$\text{(a)} \quad \operatorname{Im} z = 0 \ \ (z \in \mathbb{R}) \quad \Longleftrightarrow \quad \left|\frac{z+|z|}{2}\right| + \left|\frac{z-|z|}{2}\right| = |z|.$$

$$\text{(b)} \quad \operatorname{Im} z \neq 0 \quad \Longleftrightarrow \quad \left|\frac{z+|z|}{2}\right| + \left|\frac{z-|z|}{2}\right| > |z|.$$

1.1.10 Show that for $n \in \mathbb{Z}$

$$(-1)^n = \begin{cases} +1, & \text{if } n = 2k, \text{ that is, } n \text{ is even,} \\ -1, & \text{if } n = 2k+1, \text{ that is, } n \text{ is odd,} \end{cases}$$

$$i^n = \begin{cases} +1, & \text{if } n = 4k, \\ +i, & \text{if } n = 4k+1, \\ -1, & \text{if } n = 4k+2, \\ -i, & \text{if } n = 4k+3, \end{cases}$$

$$\cos(n\pi) = \left.\begin{cases} +1, & \text{if } n = 2k, \\ -1, & \text{if } n = 2k+1, \end{cases}\right\} = (-1)^n,$$

$$\sin\left(n\pi + \frac{\pi}{2}\right) = \left.\begin{cases} +1, & \text{if } n = 2k, \\ -1, & \text{if } n = 2k+1, \end{cases}\right\} = (-1)^n.$$

1.1.11 For any complex numbers $z, w \in \mathbb{C}$, prove the following **basic properties of conjugation**:

1. $\bar{\bar{z}} = z$.
2. $\overline{z \pm w} = \bar{z} \pm \bar{w}$.
3. $\overline{zw} = \bar{z}\bar{w}$.
4. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.
5. $\forall \ n \in \mathbb{Z}, \quad \overline{z^n} = (\bar{z})^n$.
6. $|\bar{z}| = |z|$.
7. $\arg(\bar{z}) = -\arg(z), \quad (\text{mod } 2\pi)$.
8. $\operatorname{Re}(\bar{z}) = \frac{1}{2}(z + \bar{z}) = \operatorname{Re}(z)$.
9. $\operatorname{Im}(\bar{z}) = \frac{-1}{2i}(z - \bar{z}) = \frac{i}{2}(z - \bar{z}) = -\operatorname{Im}(z)$.

10. $z \in \mathbb{R} \Longleftrightarrow z = \bar{z}$.

11. z is pure imaginary $\Longleftrightarrow z = -\bar{z}$.

12. $\bar{z} = 0 \Longleftrightarrow z = 0 \Longleftrightarrow |z| = 0 \Longleftrightarrow |\bar{z}| = 0$.

13. $\bar{z} = z^{-1} \Longleftrightarrow |z| = 1$.

14. If $z \neq 0$, then $z \cdot w \in \mathbb{R} \iff w = \lambda \cdot \bar{z}$, with $\lambda \in \mathbb{R}$.

15. If $z \neq 0$ and $w \neq 0$, then $\dfrac{z}{w} \in \mathbb{R} \iff w = \lambda \cdot \dfrac{1}{\bar{z}}$, with $\lambda \in \mathbb{R} - \{0\}$.

16. Find the conditions such that $z \cdot w$ and/or $\dfrac{z}{w}$ are pure imaginary.

1.1.12 Give the geometric interpretation of the conjugate $\bar{z}$ of a complex number z.

1.1.13 Prove that: $[zw = a \in \mathbb{R} - \{0\} \iff w = t\bar{z}$ with $t \in \mathbb{R} - \{0\}.]$ Then $a = tz\bar{z}$. (What happens if $a = 0$?)

1.1.14 (a) Given any $z,\ w \in \mathbb{C}$, prove **the identity of the parallelogram** and give its geometric interpretation:

$$|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2.$$

(b) Now prove the identity
$|u - \sqrt{u^2 - v^2}| + |u + \sqrt{u^2 - v^2}| = |u + v| + |u - v|, \quad \forall\ u,\ v \in \mathbb{C}$.

1.1.15 Given any $z,\ w \in \mathbb{C}$, prove:

$$\text{(a)} \qquad |1 - z\bar{w}|^2 - |z - w|^2 = (1 - |z|^2)(1 - |w|^2).$$

$$\text{(b)} \qquad |w| = 1 \quad \Longrightarrow \quad \left|\frac{z - w}{1 - \overline{z}w}\right| = 1, \quad \text{etc.}$$

$$\text{(c)} \qquad |z| < 1 \text{ and } |w| < 1 \quad \Longrightarrow \quad |z - w| < |1 - \overline{z}w|, \quad \text{etc.}$$

$$\text{(d)} \qquad |z| \leq 1 \text{ and } |w| \leq 1 \quad \Longrightarrow \quad |z - w| \leq |1 - \overline{z}w|, \quad \text{etc.}$$

$$\text{(e)} \qquad |z| < 1 \text{ and } |w| \leq 1 \quad \Longrightarrow$$

$$\frac{||z| - |w||}{1 - |z||w|} \leq \frac{|z - w|}{|1 - \overline{z}w|} \leq \frac{|z| + |w|}{1 + |z||w|} \leq 1, \quad \text{etc.}$$

1.1.16 Let $z_1 = r_1[\cos(\theta_1) + i\sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i\sin(\theta_2)]$:
(a) Prove that $|z_1 \pm z_2|^2 = r_1^2 + r_2^2 \pm 2r_1r_2\cos(\theta_1 - \theta_2)$.
(b) Find the length and the argument of $z_1 - z_2$.

1.1.17 (a) Give the geometric interpretation of multiplying the complex number z by -1.

(b) Give the geometric interpretation of multiplying the complex number z by i.
[Hint: Note that for i, $r = 1$ and $\theta = \dfrac{\pi}{2}$.]

(c) Give the geometric interpretation of multiplying the complex number z by $-i$.

1.1.18 Given any $z,\ w \in \mathbb{C}$ and any $\lambda > 0$, prove

$$|z+w|^2 \leq (1+\lambda)|z|^2 + \left(1+\frac{1}{\lambda}\right)|w|^2.$$

1.1.19 Let three different points $A,\ B$ and C in the plane be represented by the complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$ respectively. Let $a = z_2 - z_1$ and $b = z_3 - z_1$.

(a) Show that the angle ϕ (mod 2π) from AB to AC satisfies

$$\left|\frac{a}{b}\right| \frac{b}{a} = \cos(\phi) + i\sin(\phi) \iff \cos(\phi) = \frac{\bar{a}b + a\bar{b}}{2|a||b|} \text{ and } \sin(\phi) = \frac{\bar{a}b - a\bar{b}}{2i|a||b|}.$$

(b) Prove that the points $A,\ B$ and C are on the same straight line $px+qy+r = 0$, where p, q, and r are real numbers such that $|p|+|q| \neq 0$,

$$\iff \quad \exists\ \lambda \in \mathbb{R}\ :\ a = z_2 - z_1 = \lambda b = \lambda(z_3 - z_1) \quad \iff$$

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \det\begin{pmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{pmatrix} = 0.$$

1.1.20 Consider the complex function $f(z) = z^2 + c$, where c is a complex constant. Prove that $|z| > |c| + 1 (\geq 1) \quad \Longrightarrow \quad |f(z)| = \left|z^2 + c\right| > |z|$.

1.1.21 Give the geometric interpretation of forming the reciprocals $\dfrac{1}{z}$ and $\dfrac{1}{\bar{z}}$ for the complex numbers z and $\bar{z}$. Distinguish between the cases $|z| < 1, \quad |z| = 1 \quad$ and $|z| > 1$. How about $\quad z = 0$?

1.1.22 Let $z = \sqrt{3} - i$ and $w = 2 + 2i$. Find the absolute values, the principal arguments in $[0, 2\pi)$ and the total arguments of:
$z, \quad w, \quad z+w, \quad z-w, \quad zw, \quad \dfrac{z}{w}, \quad z^{10}, \quad (zw)^{10}, \quad (zw)^{-10}, \quad$ and $z^{\frac{1}{10}}$.

1.1.23 Find all solutions to the equations $z^3 - 8 = 0$ and $z^4 = 1 - i$.

1.1.24 Find the exact answers in **Example 1.1.7** by following the steps we have described at the end of it.

1.1.25 Find all solutions to the equations:

(a) $z^4 - 30z^2 + 289 = 0$, (b) $(z+1)^n + (z-2)^n = 0$,
(c) $z^8 + z^4 + 1 = 0$, (d) $z^9 + z^6 + z^3 + 1 = 0$.

1.1.26 Prove that for all $z \in \mathbb{C}$ and any $n \in \mathbb{N}$, we have $\left|z^{\frac{1}{n}}\right| = |z|^{\frac{1}{n}}$.

1.1.27 (a) If $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$, then prove that

$$z^{\frac{p}{q}} = \sqrt[|q|]{z^{\text{sign}(q)\cdot p}}$$

and this has at most $|q|$ values.

(b) Show that this has exactly $|q|$ values iff p and q are relatively prime.

(c) Give an example in which the number of values is less than $|q|$.

1.1.28 Let $z \in \mathbb{C}$, $k \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots\ \omega_{k-1}$ be the k $\mathrm{k^{th}}$ roots of 1. Prove that the k $\mathrm{k^{th}}$ roots of z^k are given by $z\omega_0,\ z\omega_1,\ \ldots\ z\omega_{k-1}$.

1.1.29 Let $k \in \mathbb{N}$, $\omega_0,\ \omega_1,\ \ldots\ \omega_{k-1}$ be the k $\mathrm{k^{th}}$ roots of 1 and let ψ be a $\mathrm{k^{th}}$ root of -1. Prove that the k $\mathrm{k^{th}}$ roots of -1 are given by $\psi\omega_0,\ \psi\omega_1,\ \ldots\ \psi\omega_{k-1}$.

Generalize this problem to the roots of any $z \in \mathbb{C}$.

1.1.30 For any $z \in \mathbb{C} - \{0\}$, any $n \in \mathbb{Z} - \{0\}$ and $k \in \mathbb{N} - \{1\}$, we let A be the set of the n^{th} roots of z and B the set of the $(kn)^{\text{th}}$ roots of z^k. Prove that $A \subsetneq B$.

Use this to explain why the reasoning

$$-1 = \sqrt[3]{-1} = \sqrt[6]{(-1)^2} = \sqrt[6]{1} = 1$$

is false. Where is the mistake?

1.1.31 Find all solutions of: (a) $z^n = -1$, (b) $z^n = i$, and (c) $z^n = -i$.

1.1.32 (a) Show that all solutions to the equation $z^n = 1$ are vertices of a canonical (regular) n-gon inscribed in the unit circle, with one vertex the point $(1, 0)$.

(b) For any solution $z \neq 1$ of the equation $z^n = 1$, show that

$$1 + z + z^2 + \ldots + z^{n-1} = 0.$$

1.1.33 For an $\alpha \in \mathbb{R}$ fixed, find all solutions of the equation

$$(z^2 - 1)^4 = 16z^4[\cos(\alpha) + i\sin(\alpha)].$$

1.1.34 For a given point z_1 and an $r > 0$, what is the set of points $z \in \mathbb{C}$ satisfying one of the following relations: (a) $|z - z_1| = r$,
(b) $|z - z_1| < r$, (c) $|z - z_1| \leq r$, (d) $|z - z_1| > r$, (e) $|z - z_1| \geq r$?

1.1.35 (a) Prove that the straight line equation in the x-y plane, $ax + by = c$ with a, b and c real numbers, can be written as

$$\frac{1}{2}[wz + \bar{w}\bar{z}] = \mathrm{Re}(wz) = \mathrm{Re}(\bar{w}\bar{z}) = c, \text{ where } w = a - bi.$$

(b) Prove that the equation $|z - z_0| = r$, of a circle with center $z_0 \in \mathbb{C}$ and $r > 0$, can be written as

$$z\bar{z} - z\bar{z_0} - \bar{z}z_0 + z_0\bar{z_0} = r^2, \text{ or}$$
$$|z|^2 - 2\,\mathrm{Re}(z\bar{z_0}) + |z_0|^2 = |z|^2 - 2\,\mathrm{Re}(\bar{z}z_0) + |z_0|^2 = r^2.$$

(c) Consider two complex numbers $p \neq q$ and $k \geq 0$. Prove that the equation $|z - p| = k|z - q|$, in $z \in \mathbb{C}$, represents:

(1) A point, if $k = 0$. (Which one?)

(2) A straight line through $\dfrac{p+q}{2}$ and perpendicular to the straight segment $[p, q]$, if $k = 1$.

(3) A circle, if $0 \leq k \neq 1$. (This is an **Apollonius**[3] **circle** with respect to the points p and q and ratio k.) Find its center and radius.

1.1.36 For two different fixed points z_1 and z_2, what is the set of points z satisfying the equation $|z - z_1| + |z - z_2| = r$, where $r > 0$?

1.1.37 For two different fixed points z_1 and z_2, what is the set of points z satisfying the equation $|z - z_1| - |z - z_2| = r$, where $r > 0$?

1.1.38 (a) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Prove that the identity

$$|z_1|^2 \cdot |z_2|^2 = |z_1 \cdot z_2|^2 = |\overline{z_1} \cdot z_2|^2 = |z_1 \cdot \overline{z_2}|^2 = |\overline{z_1} \cdot \overline{z_2}|^2$$

can be written as

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 =$$
$$(a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2 = \dots, \text{ etc.}$$

I.e., the product of two sums of two squares of real numbers can always be written as the sum of two squares, in which only the given numbers are involved.

[3] Apollonius of Perga (Greek: Ἀπολλώνιος; Latin: Apollonius Pergaeus;) Greek geometer and astronomer, c. 262 - c. 190 BCE.

(b) Prove that for any real numbers a_i and b_i, $i = 1,\ 2,\ \ldots,\ n$, with $n \geq 1$ integer, the product

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots(a_{n-1}^2 + b_{n-1}^2)(a_n^2 + b_n^2)$$

can be written as the sum of two squares, in which only the given numbers are involved, in 2^n ways, up to the positions of $+$ and $-$ in general.

1.1.39 Prove that for any real numbers α and β

$$\left|e^{i\alpha} \pm e^{i\beta}\right| = \sqrt{2} \cdot \sqrt{1 \pm \cos(\alpha - \beta)} = \begin{cases} 2\left|\cos\left(\frac{\alpha-\beta}{2}\right)\right|, & \text{with } +, \\ \\ 2\left|\sin\left(\frac{\alpha-\beta}{2}\right)\right|, & \text{with } -. \end{cases}$$

1.1.40 (a) Prove that any four complex numbers z_1, z_2, z_3 and z_4 satisfy the identity

$$(z_1 - z_2)(z_3 - z_4) - (z_1 - z_3)(z_2 - z_4) + (z_1 - z_4)(z_2 - z_3) = 0.$$

(b) Use **(a)** to prove

$$|z_1 - z_3| \cdot |z_2 - z_4| \leq |z_1 - z_2| \cdot |z_3 - z_4| + |z_1 - z_4| \cdot |z_2 - z_3|$$

and write two more similar inequalities.

(c) Prove that if z_1, z_2, z_3 and z_4 lie on a straight line or on a circle in this order and in the same direction along the line or the circle, then the inequality in **(b)** holds as equality.

(d) Write and prove the converse of **(c)**.

[Hint: If you cannot do **(c)** and **(d)** directly, consult some bibliography about inversion and Ptolemy's[4] theorem.]

1.1.41 Let ω be a third root of 1 and $\omega \neq 1$. Prove that any three complex numbers v, w and z satisfy the following two identities:

$$(1) \qquad \begin{aligned} v^3 + w^3 + z^3 - 3vwz &= \\ (v + w + z)\left(v^2 + w^2 + z^2 - vw - wz - zv\right) &= \\ \frac{1}{2}(v + w + z)\left[(v - w)^2 + (w - z)^2 + (z - v)^2\right] &= \\ (v + w + z)\left(v + \omega\, w + \omega^2 z\right)\left(v + \omega^2 w + \omega\, z\right). & \end{aligned}$$

[4]Claudius Ptolemy or Clavdios Ptolemaios, 90-168, Alexandrian Greek mathematician.

$$(2) \qquad \left(v + \omega\, w + \omega^2 z\right)^3 + \left(v + \omega^2 w + \omega\, z\right)^3 = (2v - w - z)(-v + 2w - z)(-v - w + 2z).$$

1.1.42 Let $\omega \neq 1$ such that $\omega^3 = 1$ and a, b, c complex numbers.

(I) Prove

$$\text{(a)} \qquad \text{If} \quad n \in \mathbb{N}, \quad \text{then} \quad \omega^{2n} + \omega^n + 1 = \begin{cases} 3, & \text{if} \quad 3 \mid n, \\ 0, & \text{if} \quad 3 \nmid n. \end{cases}$$

$$\text{(b)} \qquad (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) = 9.$$

$$\text{(c)} \qquad (2 + 5\omega + 2\omega^2)^6 = (2 + 2\omega + 5\omega^2)^6 = 729.$$

(II) If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, and $X = ax + by + cz$, $Y = cx + by + az$, $Z = bx + ay + cz$, then show the four relations:

(1) $xyz = a^3 + b^3$, (2) $x^2 + y^2 + z^2 = 6ab$, (3) $x^3 + y^3 + z^3 = 3\left(a^3 + b^3\right)$,

$$(4) \ (a^2 + b^2 + c^2 - ab - bc - ca)(x^2 + y^2 + z^2 - xy - yz - zx) = X^2 + Y^2 + Z^2 - XY - YZ - ZX.$$

1.1.43 If the complex numbers v, w and z satisfy the two conditions $v + w + z = 0$ and $|v| = |w| = |z|$ then prove the following:

(a) $vw = z^2$, $wz = v^2$, and $zv = w^2$. (b) $v^2 + w^2 + z^2 = 0$.
(c) $vw + wz + zv = 0$. (d) $v^3 = w^3 = z^3 = vwz$.

1.1.44 Consider any $z_1, z_2, \ldots, z_n$ complex numbers such that for all $k = 1, 2, \ldots, n$, it holds: $\left|\dfrac{z_k - i}{z_k + i}\right| \leq 1$ or < 1. Prove

$$\left|\frac{z_1 + z_2 + \ldots + z_n - i}{z_1 + z_2 + \ldots + z_n + i}\right| \leq 1 \text{ or } < 1, \text{ respectively.}$$

1.1.45 Consider any $z_1, z_2, \ldots, z_n$ complex numbers in the complex plane. Prove that there is a subset of indices $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\left|\sum_{j \in S} z_j\right| \geq \frac{1}{4\sqrt{2}} \sum_{n=1}^{n} |z_j|.$$

[Hint: Let $w = |z_1| + \ldots + |z_n|$ and consider the four closed quadrants defined by the straight lines $y = \pm x$ in the complex plane. In one of these quadrants, if without loss of generality this quadrant is the

$$Q = \{z = x + iy = (x, y) \mid |y| \le x\}, \quad \text{we have the two conditions}$$

$$|z| \ge \text{Re }(z) = x \ge \frac{|z|}{\sqrt{2}} = \frac{\sqrt{x^2+y^2}}{\sqrt{2}} \ge 0 \quad \text{and} \quad \sum_{z_j \in Q} |z_j| \ge \frac{w}{4}, \quad \text{etc.]}$$

1.1.46 (I) Use the **De Moivre formula** along with the **Binomial Theorem**

$$[\cos(x) + i\sin(x)]^n = \cos(nx) + i\sin(nx) = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}(x) \cdot i^k \cdot \sin^k(x)$$

to find the $\cos(nx)$ and $\sin(nx)$ for $n = 2$, $n = 3$ and $n = 4$ in terms of $\cos(x)$ and $\sin(x)$.

(II) Now for any $n \ge 0$ integer, let $\left[\left[\frac{n}{2}\right]\right]$ be the integer part of $\frac{n}{2}$ and prove the following important general trigonometric formulae:

(a)

$$\cos(nx) = \sum_{k=0}^{[[\frac{n}{2}]]} (-1)^k \binom{n}{2k} \cos^{n-2k}(x) \sin^{2k}(x) =$$

$$\sin^n(x) \cdot \sum_{k=0}^{[[\frac{n}{2}]]} (-1)^k \binom{n}{2k} \cot^{n-2k}(x).$$

Use the first equality to prove that for any $n \ge 0$ integer, $\cos(nx)$ can be expressed as a polynomial of $\cos(x)$. Find this polynomial.[5]

(b) For any $m = 1,\ 2,\ 3,\ \ldots$ consider $n = 2m$ and $0 \le x < \frac{\pi}{2}$ in **(a)** and prove that the polynomial of degree m given by

$$P_m(y) = \sum_{k=0}^{m} (-1)^k \binom{2m}{2k} y^{m-k}$$

---

[5]These polynomials are the **Chebyshev** (or **Tschebyscheff**) **polynomials of the first kind**. They have many properties and are obtained by developing the formula

$$T_n(t) = \cos[n \cdot \arccos(t)] = \cos(nx), \text{ where } t = \cos(x) \text{ and } n \in \mathbb{N}_0.$$

They can also be derived by the recursive formula

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t), \text{ with } T_1(t) = t,\ T_0(t) = 1 \text{ and } n = 2,\ 3,\ 4,\ \ldots\ .$$

Notice that $T_n(t)$ is an even function when n is even and is odd when n is odd.

has m positive distinct roots which are:

$$y_k = \cot^2\left[\frac{(1+2k)\pi}{4m}\right], \quad k = 0,\ 2,\ 3,\ \ldots,\ m-1.$$

Then, use Vieta's[6] formulae for polynomials to prove the formulae:

$$\sum_{k=0}^{m-1} \cot^2\left[\frac{(1+2k)\pi}{4m}\right] = m(2m-1),$$

$$\sum_{k=0}^{m-1} \cot^4\left[\frac{(1+2k)\pi}{4m}\right] = \frac{m(2m-1)(4m^2+2m-3)}{3},$$

$$\text{and} \quad \prod_{k=0}^{m-1} \cot^2\left[\frac{(1+2k)\pi}{4m}\right] = 1.$$

From the last equality derive the results of **Problem I 2.1.22.**.

(c)

$$\sin(nx) = \sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(x)\sin^{2k+1}(x) =$$

$$\sin^n(x) \cdot \sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k+1} \cot^{n-2k-1}(x).$$

[For reverse formulae of see **Problem I 2.2.28, (a)** and **(e)**.]

Use the first equality to prove that for any $n \geq 1$ **odd** integer, $\sin(nx)$ can be expressed as a polynomial of $\sin(x)$. Find this polynomial. Then prove that for any $n \geq 2$ **even** integer, $\sin(nx)$ is a polynomial of both $\sin(x)$ and $\cos(x)$ [i.e., it cannot be a polynomial of just $\sin(x)$]. Find this polynomial.Use the first equality to prove that for any $n \geq 0$ integer, $\cos(nx)$ can be expressed as a polynomial of $\cos(x)$. Find this polynomial.[7]

[6] François Viéte (Latin: Franciscus Vieta), French mathematician, 1540-1603.

[7] These polynomials are immediately connected to the **Chebyshev** (or **Tschebyscheff**) **polynomials of the second kind**. They have many properties and are obtained by developing the formula

$$U_n(t) = \frac{\cos[(n+1)\arccos(t)]}{\sin[\arccos(t)]} \text{ or } U_n[\cos(x)] = \frac{\sin[(n+1)x]}{\sin(x)}, \ \ t = \cos(x), \ \ n \in \mathbb{N}_0.$$

(d) For any $m = 1,\ 2,\ 3,\ \dots$ consider $n = 2m+1$ and $0 < x \le \dfrac{\pi}{2}$ in **(c)** and prove that the polynomial of degree m given by

$$P_m(y) = \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k+1} y^{m-k}$$

has m positive distinct roots which are:

$$y_k = \cot^2\left(\frac{k\pi}{2m+1}\right), \quad k = 1,\ 2,\ 3,\ \dots,\ m.$$

Then, use Vieta's formulae for polynomials to prove the formulae:

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3},$$

$$\sum_{k=1}^{m} \cot^4\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45},$$

$$\text{and} \quad \prod_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{1}{2m+1}.$$

From the last equality derive the results of **Problem I 2.1.22**.

(e)

$$\tan(nx) = \frac{\sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k+1} \tan^{2k+1}(x)}{\sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k} \tan^{2k}(x)}.$$

(f) If $\arctan(x) = \sqrt{2}$, find $\tan(4x)$ and $\tan(5x)$.

(III) In the last formula, replace the "tan" with $\dfrac{1}{\cot}$ to find the corresponding formula for the $\cot(nx)$.

They can also be derived by the recursive formula

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \ \text{ with } U_1(t) = 2t,\ U_0(t) = 1 \text{ and } n = 2,\ 3,\ 4,\ \dots .$$

Notice that $U_n(t)$ is an even function when n is even and is odd when n is odd. They are related to the Chebyshev polynomials of the first kind in several ways. For instance, we easily prove that

$$\frac{d}{dt}\left[T_n(t)\right] = nU_{n-1}(t), \quad n \in \mathbb{N}.$$

1.1.3 Square Roots Without De Moivre

Here we will study the function of square root in $\mathbb{R}$ and $\mathbb{C}$ without using the De Moivre formula and the roots of complex numbers as they were expressed in trigonometric forms before. It is important to study this function in Cartesian coordinates and see its connection with some aspects discussed in **Subsections 1.5.3** and **1.5.5** later. So, we begin first with:

1: Square Roots in $\mathbb{R}$

Definition 1.1.1 *A square root of a non-negative real number $x \geq 0$ is defined to be any real number r such that $r^2 = x$.*

Since $\forall\ r \in \mathbb{R}$, we have that $r^2 \geq 0$ the condition $x \geq 0$, in this definition, is necessary.

If $x = 0$, then $r^2 = r \cdot r = 0$, and so $r = 0$ necessarily.

If $x > 0$, then by looking at the continuous polynomial function $f(r) = r^2 - x$, whose domain is all $\mathbb{R}$, we see that

$$f(0) = -x < 0 \quad \text{and} \quad f\left(x + \frac{1}{2}\right) = \left(x + \frac{1}{2}\right)^2 - x = x^2 + \frac{1}{4} > 0.$$

Therefore, when $x > 0$, by the **Intermediate Value Theorem** (study this theorem one more time), there is a real solution r of the equation $r^2 - x = 0$, which is equivalent to $r^2 = x(> 0)$ such that $0 < r < x + \frac{1}{2}$.

Since $(-1)^2 = +1$, if r is a square root of x, then $-r = -1 \cdot r$ is also a square root of x. These two square roots are equal if and only if

$$r = -r \Longleftrightarrow r = 0 \Longleftrightarrow x = 0.$$

So, if $r > 0$, then $-r < 0$, and so any real number $x > 0$ has at least two different square roots, one positive and the other its opposite negative.

By looking at $r^2 = x$ as $r^2 - x = 0$, i.e., as a polynomial equation of degree two in the variable (indeterminate) r over the real numbers, we know from algebra that this equation cannot have more than two solutions. In conclusion:

Any positive real number has exactly two different square roots, one positive and the other its opposite negative. As we have already said, **the square roots of 0 are both equal to 0**, which is also seen by the fact that the solutions of $r^2 = 0$, or $(r - 0)(r - 0) = 0$, are $r_1 = r_2 = 0$.

Since $r^2 \geq 0,\ \ \forall\, r \in \mathbb{R}$, a negative real number cannot have a square root which is a real number. The problem of the square root(s) of a negative real number is taken care of by introducing the symbol i such that

$$i^2 = -1 \ (< 0).$$

Therefore, $i \neq 0$ and i cannot be a real number, since for any real number $a \neq 0$ we always have: $a^2 > 0$. We say that i is an **imaginary number**.

Then for $x < 0$, we have that $-x = |x| > 0$. Hence by the discussion above, we have two opposite real numbers $\pm\, r \neq 0$ which are the two square roots of $-x = |x| > 0$, i.e., $(\pm\, r)^2 = |x| = -x > 0$.

Definition 1.1.2 *We define the square roots of a negative number $x < 0$ to be the imaginary numbers $i \cdot (\pm\, r) = \pm\, i \cdot r$, where r is the positive square root of $|x| = -x > 0$ (defined above).*

Now, when we use the **square root symbol** $\sqrt{}$ with any real number $x \in \mathbb{R}$ and we write

$$\sqrt{x},$$

then we mean the number $+r \geq 0$ **if** $x \geq 0$ and the imaginary number $+i \cdot r$, with $r > 0$, **if** $x < 0$, where r was the number found (or defined) above. So, this notation means

$$\sqrt{x} = +\sqrt{x}.$$

That is why when x is a real variable or an unknown real number, we have

$$\sqrt{x^2} = |x| \geq 0,$$

and it is not equal to simply x or $\pm x$.

To indicate the negative square root $-r$ when $x > 0$ or the imaginary root $-i \cdot r$ when $x < 0$, we explicitly write:

$$-\sqrt{x}.$$

To indicate both square roots together we write:

$$\pm\sqrt{x}.$$

In conclusion, we have obtained the **result**: For any real number $x \in \mathbb{R}$, we have:

$$\text{Square roots of } x = \begin{cases} \pm\sqrt{0} = 0, & \text{if} \quad x = 0, \\[2ex] \pm\sqrt{x} = \pm r, & \text{if} \quad x > 0, \\[2ex] \pm i\sqrt{-x} = \pm ir, & \text{if} \quad x < 0, \end{cases}$$

where $r \in \mathbb{R}$ was defined above. (For example: $\sqrt{-1} = i$, $-\sqrt{-1} = -i$ and $\pm\sqrt{-1} = \pm\, i$, respectively.)

(Remember, finally, that this is the way we must understand and use the **square root symbol** $\sqrt{}$ with real numbers.) Now we continue with:

2: Square Roots in $\mathbb{C}$

We consider that $\mathbb{R} \subset \mathbb{C}$ under the identification: If $x \in \mathbb{R}$, then $x = x + i0 \in \mathbb{C}$. Then we have

Definition 1.1.3 *For any $z = x + iy \in \mathbb{C}$, where $x,\ y \in \mathbb{R}$, we define a square root of z to be any complex number $w = u + iv$ such that*

$$w^2 = z \quad \text{or} \quad (u + iv)^2 = x + iy.$$

By developing the last equation, we find

$$u^2 - v^2 + 2iuv = x + iy,$$

and so we obtain the following quadratic system of two equations in two unknowns, u and v,

$$\left\{ \begin{array}{c} u^2 - v^2 = x \\ \\ 2uv = y \end{array} \right\}. \tag{1.1}$$

If $z = 0 = 0 + i0$, i.e., $x = y = 0$, then the system

$$\left\{ \begin{array}{c} u^2 - v^2 = 0 \\ \\ 2uv = 0 \end{array} \right\}$$

is easily solved, and we find the unique solution $u = v = 0$. That is, $w = 0 = 0 + i0$ is the only square root of $z = 0 = 0 + i0$.

Otherwise, when $z = x + iy \neq 0 + i0$, i.e., $x \neq 0$ or $y \neq 0$, we work as follows: We observe that

$$(u^2 + v^2)^2 = (u^2 - v^2)^2 + 4u^2v^2 = x^2 + y^2$$

and since $u^2 + v^2 \geq 0$, we get that

$$u^2 + v^2 = \sqrt{x^2 + y^2},$$

that is, we have only one equation with the $+$ sign in front of the square root. (The $-$ sign is excluded from this situation.)

Now, we solve the system

$$\left\{\begin{array}{c} u^2 - v^2 = x \\ \\ u^2 + v^2 = \sqrt{x^2 + y^2} \end{array}\right\},$$

and we find

$$\left\{\begin{array}{c} u^2 = \dfrac{1}{2}\left(x + \sqrt{x^2 + y^2}\right) \geq 0 \\ \\ v^2 = \dfrac{1}{2}\left(-x + \sqrt{x^2 + y^2}\right) \geq 0 \end{array}\right\}.$$

Therefore, we have

$$\left\{\begin{array}{c} u = \pm\sqrt{\dfrac{1}{2}\left(x + \sqrt{x^2 + y^2}\right)} \\ \\ v = \pm\sqrt{\dfrac{1}{2}\left(-x + \sqrt{x^2 + y^2}\right)} \end{array}\right\}.$$

Since $2uv = y$, the following conditions must be satisfied:

1. If $y > 0$, then $uv > 0$, i.e., ($u > 0$ and $v > 0$) or ($u < 0$ and $v < 0$).

2. If $y < 0$, then $uv < 0$, i.e., ($u > 0$ and $v < 0$) or ($u < 0$ and $v > 0$).

3. If $y = 0$, then $uv = 0$, i.e., $u = 0$ or $v = 0$ (inclusive or).

To incorporate the first two cases in one formula, we introduce the function

$$y \in \mathbb{R} - \{0\} \longrightarrow \frac{y}{|y|} = \begin{cases} +1, & \text{if } \; y > 0, \\ \\ -1, & \text{if } \; y < 0. \end{cases}$$

We observe that this function is not defined at $y = 0$. At $y = 0$, it has a jump equal to $|+1 - (-1)| = 2$, and therefore it cannot be extended (or defined) continuously at $y = 0$.

Now, taking into account what we have found for possible values of u and v together with the above three conditions, we find that the initial system **(1.1)** above has only **two solutions** (and not four, as someone prematurely may have thought) **which are**:

(a) **If** $y \neq 0$, then:

$$\left\{ \begin{array}{c} u = \pm\sqrt{\frac{1}{2}\left(x + \sqrt{x^2+y^2}\right)} \\ \\ v = \pm\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x + \sqrt{x^2+y^2}\right)} \end{array} \right\}$$

where the $+$ is coupled with the $+$ and the $-$ with the $-$.

(b) **If** $y = 0$, then:

1. $u = v = 0$, when $x = y = 0$,

2. $u = 0$, when $x < 0$ and $v = \pm\sqrt{-x}$,

3. $v = 0$, when $x > 0$ and $u = \pm\sqrt{x}$.

In conclusion, we have found the following result: The two square roots of a complex number: $z = x + iy$ are

$$\sqrt{z} = \sqrt{x+iy} = u + iv = u(x,y) + iv(x,y) =$$

$$\begin{cases} \pm\left[\sqrt{\frac{1}{2}\left(x+\sqrt{x^2+y^2}\right)} + i\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)}\right], & \text{if } y \neq 0, \\ \pm\sqrt{x}, & \text{if } \ y = 0,\ x \geq 0, \\ \pm i \cdot \sqrt{-x}, & \text{if } \ y = 0,\ x \leq 0. \end{cases}$$

So, a complex number $z = x + iy$ has two square roots. For $z = 0$, the two square roots coincide and are equal to 0.

Note: In complex analysis, we write the function

$$f(z) = \sqrt{z}, \quad \text{with} \quad z \in \mathbb{C}.$$

Unless otherwise declared, this usually means the two-valued-complex function with values the two values found above. So, we must keep in mind the context (real or complex) in which we use the **square root symbol** $\sqrt{}$.

Important remark: Since the function

$$H : \mathbb{R} \longrightarrow \mathbb{R} \quad : \quad y \longrightarrow H(y) = \frac{y}{|y|} = \begin{cases} +1, & \text{if } \; y > 0, \\ \text{undefined}, & \text{if } \; y = 0, \\ -1, & \text{if } \; y < 0, \end{cases}$$

cannot be defined continuously at $y = 0$ (i.e., no matter how we define it at $y = 0$, it will always be discontinuous at $y = 0$), we observe that the two square roots of $z = x + iy$ found above are discontinuous at

$$y = 0 \quad \text{when} \quad v(x, y) = \sqrt{\frac{1}{2}\left(-x + \sqrt{x^2 + y^2}\right)} \neq 0.$$

This happens when

$$v(x, 0) = \sqrt{\frac{1}{2}\left(-x + \sqrt{x^2}\right)} = \sqrt{\frac{1}{2}\left(-x + |x|\right)} \neq 0 \quad \Longleftrightarrow \quad x = -|x| < 0,$$

that is, when and only when x is negative.

In conclusion, the two square roots of $z = x + iy$ as functions of x and y are:

$$\textbf{(I)} \qquad f_+(z) = f_+(x + iy) = +\sqrt{z} = +\sqrt{x + iy} =$$

$$u_+ + iv_+ = u_+(x, y) + iv_+(x, y) =$$

$$= \begin{cases} + \left[\sqrt{\frac{1}{2}\left(x + \sqrt{x^2 + y^2}\right)} + i \frac{y}{|y|} \sqrt{\frac{1}{2}\left(-x + \sqrt{x^2 + y^2}\right)} \right], & \text{if } y \neq 0, \\ +\sqrt{x}, & \text{if } \; y = 0, \; x \geq 0, \\ +i \cdot \sqrt{-x}, & \text{if } \; y = 0, \; x \leq 0, \end{cases}$$

and

$$\textbf{(II)} \qquad f_-(z) = f_-(x+iy) = -\sqrt{z} = -\sqrt{x+iy} =$$

$$u_- + iv_- = u_-(x,y) + iv_-(x,y) =$$

$$= \begin{cases} -\left[\sqrt{\frac{1}{2}\left(x+\sqrt{x^2+y^2}\right)} + i\frac{y}{|y|}\sqrt{\frac{1}{2}\left(-x+\sqrt{x^2+y^2}\right)}\right], & \text{if } y \neq 0, \\ -\sqrt{x}, & \text{if } \ y=0,\ x \geq 0, \\ -i \cdot \sqrt{-x}, & \text{if } \ y=0,\ x \leq 0. \end{cases}$$

Both square root functions **(I)** and **(II)** are discontinuous along the negative x-axis (semi-axis). For example, with the square root **(I)** when $x < 0$ and $0 < y \longrightarrow 0^+$, we get

$$f_+(z) \longrightarrow i\sqrt{-x} = \sqrt{-x}\, e^{i\frac{\pi}{2}},$$

whereas when $x < 0$ and $0 > y \longrightarrow 0^-$, we get

$$f_+(z) \longrightarrow -i\sqrt{-x} = \sqrt{-x}\, e^{i\frac{3\pi}{2}}.$$

So, with $x < 0$ $(x \neq 0)$ these two limits are not equal, since $i\sqrt{-x} \neq -i\sqrt{-x}$. We also observe that these limits have the same magnitudes, equal to $\sqrt{-x}$, but different arguments $\dfrac{\pi}{2} \neq \dfrac{3\pi}{2}$. I.e., if we rewrite this square root function in trigonometric form, we observe a jump discontinuity in the argument along the negative x-axis. We observe that the absolute value of this jump is $\left|\dfrac{3\pi}{2} - \dfrac{\pi}{2}\right| = \pi$, which is half of 2π.

We have analogous results for the other square root $f_-(z)$. (Check them out in this case!)

Example 1.1.8: Find the square roots of the following complex numbers z : $10,\ -10,\ 100,\ -100,\ i,\ -i,\ -3+4i,\ -3-4i,\ 2-5i$ and $2+3i$.

Answers:

1. $z = 10 \Longrightarrow \sqrt{z} = \pm\sqrt{10}$.
2. $z = -10 \Longrightarrow \sqrt{z} = \pm i\sqrt{10}$.
3. $z = 100 \Longrightarrow \sqrt{z} = \pm\sqrt{100} = \pm 10$.

4. $z = -100 \Longrightarrow \sqrt{z} = \pm i\sqrt{100} = \pm i \cdot 10.$

5. Let $z = i$. Then $x = 0, y = 1, \sqrt{x^2+y^2} = \sqrt{0^2+1^2} = \sqrt{1} = 1$ and $\dfrac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{i} = \pm\left(\sqrt{\frac{1}{2}(0+1)} + i\sqrt{\frac{1}{2}(-0+1)}\right) =$$
$$\pm\left(\sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

6. Let $z = -i$. Then $x = 0,\ y = -1$, $\sqrt{x^2+y^2} = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$ and $\dfrac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{-i} = \pm\left(\sqrt{\frac{1}{2}(0+1)} - i\sqrt{\frac{1}{2}(-0+1)}\right) =$$
$$\pm\left(\sqrt{\frac{1}{2}} - i\sqrt{\frac{1}{2}}\right) = \pm\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right).$$

7. Let $z = -3+4i$. Then $x = -3,\ y = 4$, $\sqrt{x^2+y^2} = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$ and $\dfrac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{-3+4i} = \pm\left(\sqrt{\frac{1}{2}(-3+5)} + i\sqrt{\frac{1}{2}[-(-3)+5]}\right) =$$
$$\pm\left(\sqrt{1} + i\sqrt{4}\right) = \pm(1+2i).$$

8. Let $z = -3-4i$. Then $x = -3,\ y = -4$, $\sqrt{x^2+y^2} = \sqrt{(-3)^2+(-4)^2} = \sqrt{25} = 5$ and $\dfrac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{-3-4i} = \pm\left(\sqrt{\frac{1}{2}(-3+5)} - i\sqrt{\frac{1}{2}[-(-3)+5]}\right) =$$
$$\pm\left(\sqrt{1} - i\sqrt{4}\right) = \pm(1-2i).$$

9. Let $z = 2-5i$. Then $x = 2,\ y = -5$, $\sqrt{x^2+y^2} = \sqrt{2^2+(-5)^2} = \sqrt{29}$ and $\dfrac{y}{|y|} = -1$. So,

$$\sqrt{z} = \sqrt{2-5i} = \pm\left(\sqrt{\frac{2+\sqrt{29}}{2}} - i\sqrt{\frac{-2+\sqrt{29}}{2}}\right).$$

10. Let $z = -2 + 3i$. Then $x = -2,\ y = 3$,
$\sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ and $\dfrac{y}{|y|} = +1$. So,

$$\sqrt{z} = \sqrt{-2 + 3i} = \pm \left(\sqrt{\frac{-2 + \sqrt{13}}{2}} + i\sqrt{\frac{2 + \sqrt{13}}{2}} \right).$$

▲

Problems

1.1.47 Apply the square root formula twice to compute the four fourth roots of the complex numbers:
(a) $z = 3 - 2i$, (b) $z = -3 - 2i$, (c) $z = -3 + 2i$ and (d) $z = 3 + 2i$.

1.1.48 In a precalculus course, we learn that $\sqrt{x^2} = |x|,\ \forall\ x \in \mathbb{R}$.

What is valid with $\sqrt{z^2},\ \forall\ z \in \mathbb{C}$, according to the exposition of this **subsection**? Explain.

1.1.49 Solve and factor completely the polynomials:

(a) $3z^4 - 8z^2 - 5$, (b) $3z^4 - 9z^2 - 5$,
(c) $3z^4 - 2z^2 - 5$, (d) $3z^4 - 2z^2 + 5$.

1.1.50 Find the formulae for the four fourth roots of a complex number $z = x + iy$ in terms of the Cartesian coordinates x and y. (Thus, do not use the trigonometric form of z.)

1.1.51 Consider the **De Moivre formula** for the two square roots

$$\sqrt{z} = \sqrt{r}\left[\cos\left(\frac{\theta + k2\pi}{2}\right) + i\sin\left(\frac{\theta + k2\pi}{2}\right)\right]$$

of a complex number $z = x + iy \neq 0$, where $r = \sqrt{x^2 + y^2}$,
$\cos(\theta) = \dfrac{x}{r},\ \sin(\theta) = \dfrac{y}{r}$ and $k = 0,\ 1$.

Use appropriate half angle trigonometric formulae to show that these formulae can be transformed to the formulae **(I)** and **(II)** for the $\sqrt{z} = \sqrt{x + iy}$, found above in this **subsection**, in terms of x and y.

1.1.52 (a) Locate and explain the error made in

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \cdot \sqrt{-1} = i \cdot i = -1.$$

(b) Conclude when the rule $\sqrt{ab} = \sqrt{a}\ \sqrt{b}$ (the high school algebra rule about square roots) is valid with complex numbers a and b.

(See also **Problem 1.5.12**.)

1.2 Power Series, a Quick Review

A **real power series** in the real variable x and **based at** a, or **with center** a, is an infinite sum or a limit of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \lim_{K\to\infty} \sum_{n=0}^{K} c_n (x-a)^n.$$

The c_n's are real numbers called the **coefficients of the power series**.

Notice that any power series at $x = a$ attains the value c_0. In the following Theorem, we summarize the results on power series that we need in this text. These results can be found in calculus or real analysis.

Theorem 1.2.1 *Given a power series with real coefficients and of a real variable* x

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

there exists an $R \in [0, \infty]$, *called the* **radius of convergence of the power series**, *such that the power series converges absolutely (and therefore it converges) for all* x *such that* $|x - a| < R$, *if* $R > 0$, *or only at* $x = a$, *if* $R = 0$, *and diverges for all* x *such that* $|x-a| > R$. *We call the open interval* $(a-R, a+R)$ *the* ***open interval of convergence*** *of the power series.*

If the power series converges only for $x = a$, *attaining obviously the value* c_0, *then* $R = 0$.

If the power series converges for all real numbers $x \in \mathbb{R}$, *then* $R = \infty$.

If the power series converges for an $x_0 \neq a$, *then it converges absolutely for all* x *such that* $0 \le |x - a| < |x_0 - a|$ *and, therefore, it holds* $0 < |x_0 - a| \le R \le \infty$.

If $0 < R < \infty$, *the power series may or may not converge at either or both endpoints* $x = a \pm R$ *of its open interval of convergence. (We must check each endpoint individually for convergence.)*

If $R > 0$, *then on any closed subinterval* $[c, d] \subset (a - R, a + R)$ *the power series converges uniformly.*

If $R > 0$, *we define the function*

$$f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \textit{for} \quad a - R < x < a + R.$$

This function is continuous in the open interval $(a-R, a+R)$. *Moreover,* *if the power series converges at either endpoint $x = a \pm R$, then the function is also (right or left) continuous at this point. (**Abel's Lemma**[8].)*

The function $f(x)$ is also differentiable and integrable in the open interval $(a - R, a + R)$ and it holds

$$\text{(1)} \qquad f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1},$$

$$\textit{for every } x \textit{ such that} \quad a - R < x < a + R,$$

$$\textit{and (2)} \qquad \int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1},$$

$$\textit{for every } x \textit{ such that} \quad a - R < x < a + R.$$

*Hence a power series can be differentiated and integrated term by term in its **open** interval of convergence. (That is, we can switch the order of summation with differentiation or integration.) Both of the new series obtained by differentiation and/or integration have the same radius and center of convergence as the initially given power series.*

Remark: The term by term differentiation of a power series, as claimed in this Theorem, follows from **Theorem I 2.3.15**. The term by term integration of a power series, as claimed in this Theorem, follows from **Weierstraß M-Test, I 2.3.3**, and **Theorem I 2.3.9** or **Corollary I 2.3.2,**, etc. We apply these theorems to $S_k(x) := \sum_{n=0}^{k} c_n (x-a)^n$, $k \in \mathbb{N}_0$, which, as we can check, satisfies all the required hypotheses, etc. (Review again **Section I 2.3** and **Problem I 2.3.25**.)

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[8]**Abel's Lemma** (a simplified form): Suppose the power series $f(x) := \sum_{n=0}^{\infty} c_n x^n$, converges for $-1 < x < 1$ (without loss of generality the radius of convergence R is 1 and the center a is 0) and the number series $\sum_{n=0}^{\infty} c_n := s \in \mathbb{R}$ (converges to s). Then

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\sum_{n=0}^{\infty} c_n x^n \right) = s.$$

So, if we extend $f(x)$ at $x = 1$ as $f(1) := s$, then $f(x)$ is left continuous at $x = 1$. (E.g., see Apostol 1974, Theorem 9.31, 245, or Rudin 1976, Theorem 8.2, 174.)

The existence of $\sum_{n=0}^{\infty} c_n$ is necessary, even if $\lim_{x \to 1^-} \left(\sum_{n=0}^{\infty} c_n x^n \right)$ exists. E.g., consider the power series $\sum_{n=0}^{\infty} (-1)^n x^n = \dfrac{1}{1+x}$, which converges $\forall \ x : \ |x| < 1$.

Definition 1.2.1 *If a function f can be written as a real power series based at a and with radius of convergence $R > 0$, we say that f is* **real analytic at** a.

After some thinking, we can prove that if a function f is analytic at a, with radius of convergence $R > 0$, then it is analytic at every point of the open interval of convergence $(a - R, a + R)$ or, as we say, is analytic in $(a - R, a + R)$. I.e., we can write f as a power series with center any point $c \in (a - R, a + R)$ and radius of convergence $R_c > 0$.

The **Absolute Ratio Test, I 1.3.5**, can be used to find the radius of convergence of a power series as follows: Given the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, we find the limit

$$0 \leq \rho = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} \leq \infty,$$

provided this limit exists. Then the **radius of convergence** is equal to

$$R = \frac{1}{\rho} = \lim_{n \to \infty} \frac{|c_n|}{|c_{n+1}|}. \tag{1.2}$$

Also, by the **Absolute Root Test, I 1.3.4,**, we can prove that

$$R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{|c_n|}}, \tag{1.3}$$

provided again that the limit exists. This root expression of the radius of convergence is sometimes more convenient than the ratio one. (See **Problem 1.2.4**.)

Example 1.2.1 The **geometric series**

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

has radius of convergence 1 and so converges on the interval $(-1, 1)$. (It diverges at both 1 and -1.) We can manipulate this power series to get power series of other functions. For instance, for $-1 < x < 1$, we derive the following five results:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x},$$

$$f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots = \frac{1}{(1 - x)^2},$$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = -\ln(1-x),$$

$$f(-x^2) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1+x^2},$$

$$\int_0^x f(-t^2)\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \tan^{-1}(x).$$

In each of these examples, the convergence or divergence at each of the endpoints $x = -1$ and $x = 1$ is checked separately. (Check it!)

▲

Power series can be computed by the Taylor Power Series Theorem (as studied in a calculus or real analysis course) for many standard elementary functions. To do this for a real function of a real variable $y = f(x)$ at some point a of its domain as center, we compute the coefficients by

$$c_n = \frac{f^{(n)}(a)}{n!}, \quad \forall \;\; n = 0,\ 1,\ 2,\ 3, \ldots$$

and then we calculate the radius of convergence by **(1.2)** or **(1.3)**. If in particular the center $a = 0$, then the power series we obtain is called the **Maclaurin**[9] **series** of $f(x)$.

For instance, for $a = 0$ and $-\infty < x < \infty$, we have seen the following Maclaurin power series (compute them one more time!):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Product of Power Series: It can be shown that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R_1 and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ with radius of convergence R_2, then the product of the two functions also has a power series, given by the power series

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n, \quad \text{with } R = \min\{R_1, R_2\}.$$

[9] Colin Maclaurin, Scottish mathematician, 1698-1746.

(This power series has radius of convergence $R = \min\{R_1, R_2\}$.)

Everything that we have stated so far about real power series holds with no change if the real number a and the real variable x are replaced with a complex number c and the complex variable z and the coefficients c_n's are allowed to be complex numbers, in general, and thus producing complex power series.

The radius of convergence R of a complex power series, obtained in this way, is again computed by **(1.2)** or **(1.3)**. The open interval of convergence is now replaced with $D(c, R) := \{z \mid |z - c| < R\}$ the open disc of center c and radius R in the complex plane. The complex series converges absolutely (and therefore it converges) for any z in this open disc. Also, the convergence is uniform on any closed sub-disc $\overline{D(c, r)} := \{z \mid |z - c| \leq r\}$, where $0 < r < R$.

When $0 < R < \infty$, the boundary points of the disc, which make up $C(c, R) := \{z \mid |z - c| = R\}$ the circle of center c and radius R in the complex plane, must be checked separately for convergence, just like when we check separately the endpoints of the interval of converge in a calculus course.

[The theory of convergence of complex sequences and series of numbers or functions in the complex plane $\mathbb{C}$ or the extended complex plane $\overset{\smile}{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a whole fundamental chapter in a complete course of complex analysis. It involves several definitions, topological considerations and many criteria. (Here ∞ is the complex infinity. We talk about it a little in the next section.) In this book, we are going to use just what we have stated in order to practice with problem solving, instead of focusing on the underlying theory.]

In this way, the **previous Examples** of the real power series of the real functions $e^x, \sin(x)$ and $\cos(x)$ give the definitions of their extensions into the complex plane. I.e., for any complex number z, we define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

We observe that these three power series **converge absolutely** for any complex number z. That is, if in the power series we replace z with $|z|$ the new series still converge, for any $z \in \mathbb{C}$, as this follows from the absolute convergence of the obtained real series. (In the **problems that**

follow, we will see similar definitions of other functions and some results that we can prove easily.)

The **Series Rearrangement Theorem of B. G. F. Riemann** says that: *"The convergence of a series is absolute if and only if any rearrangement of the terms of the series, or any way of summing it up, does not change the final limit sum."* (For the proof of this, we need some advanced concepts and results on convergence from mathematical analysis. See, e.g., Rudin 1976, 75-78.)

Definition 1.2.2 *The complex function*

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

defined for all $z \in \mathbb{C}$ is called ***complex exponential function*** *or simply* ***exponential function****.*

We can easily prove:

$$\begin{aligned} &\text{(a)} \qquad e^{z+w} = e^z e^w, \ \forall \ z, \ w \in \mathbb{C} \\ &\text{(b)} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \ \forall \ z \in \mathbb{C} \\ &\text{(c)} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \ \forall \ z \in \mathbb{C}. \end{aligned}$$

The functions $\sin(z)$ and $\cos(z)$ are called the **complex sine** and **complex cosine**.

In the problems (**Problem 1.2.9**, **Problem 1.2.10**, **Problem 1.2.12**, **Problem 1.2.15**, etc.) we state and suggest to prove some important and rather immediate properties of the exponential function. Here we state the following with some auxiliary hints:

Using the rule for the powers of i (see **Problem 1.1.10**) and the power series of cosine and sine, we derive **Euler's formula**:

$$\forall \ \ y \in \mathbb{R} \qquad \text{we have:} \qquad e^{iy} = \sum_{m=0}^{\infty} \frac{i^m y^m}{m!} =$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!} = \cos(y) + i \sin(y).$$

Then, the **trigonometric form of** e^z is

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i \sin(y)], \quad \forall \ \ z = x + iy \in \mathbb{C}.$$

So,

$$
\begin{array}{ll}
\text{(a)} & \forall\ y \in \mathbb{R}, \quad \left|e^{iy}\right| = \sqrt{\cos^2(y) + \sin^2(y)} = 1,\\
\text{(b)} & \forall\ z = x + iy \in \mathbb{C}, \quad |e^z| = \left|e^{x+iy}\right| = e^x > 0,\\
\text{(c)} & \forall\ z = x + iy \in \mathbb{C}, \quad e^z \neq 0,\\
\text{(d)} & \forall\ z = x + iy \in \mathbb{C}, \quad \arg(e^z) = y + 2\pi\mathbb{Z}.
\end{array}
$$

Now, we can easily prove that

$$e^z = 1 \quad \Longleftrightarrow \quad z = 2k\pi\, i, \ \text{ with } \ k \in \mathbb{Z}.$$

Next, we observe that if $0 \neq w \in \mathbb{C}$ written in trigonometric form

$$w = r[\cos(\theta) + i\sin(\theta)],$$

for some argument θ, since $0 < r = e^{\ln(r)}$, using Euler's formula we get

$$0 \neq w = e^{\ln(r)}e^{i\theta} = e^{\ln(r)+i\theta} = e^{\ln(r)+i\theta+i2k\pi}, \quad \forall\ k \in \mathbb{Z}.$$

This expression of $w \in C - \{0\}$ is called the **exponential form of a complex number** $w \neq 0$. We see that $0 = 0 + i0$ does not have exponential form. (See also **Problem 1.2.32**.)

We have already seen that the **domain** of the exponential function is $\mathbb{C}$, and we now observe that its **range** is $\mathbb{C} - \{0\}$. In fact, the equation $e^z = w$, for any given $w \neq 0$, has infinitely many solutions in z, whereas $e^z = 0$ has no solution.

The **trigonometric functions** in the complex plane are defined by:

$$
\begin{aligned}
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = -i\,\frac{e^{iz} - e^{-iz}}{2}, \quad \forall\ z \in \mathbb{C},\\
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2}, \quad \forall\ z \in \mathbb{C},\\
\tan(z) &= \frac{\sin(z)}{\cos(z)} = -i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \quad \forall\ z \in \mathbb{C},\\
\cot(z) &= \frac{\cos(z)}{\sin(z)} = i\,\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}, \quad \forall\ z \in \mathbb{C},\\
\sec(z) &= \frac{1}{\cos(z)} = \frac{2}{e^{iz} + e^{-iz}}, \quad \forall\ z \in \mathbb{C},\\
\csc(z) &= \frac{1}{\sin(z)} = \frac{2\,i}{e^{iz} - e^{-iz}}, \quad \forall\ z \in \mathbb{C}.
\end{aligned}
$$

(When a denominator is zero, the answer is the complex infinity.)

The **trigonometric functions** as defined here for all complex numbers are the extensions of the real trigonometric functions to the complex plane. Therefore, they satisfy all the **trigonometric identities** that we learn in a trigonometry or a precalculus class. We can check the validity of the extension of each of these identities as an exercise. With more complex analysis, this becomes automatic.

Also, we can easily prove that the **range of the complex** $\sin(z)$ **and** $\cos(z)$, (etc.), is not just the interval $[-1, 1]$, which is the range of the real $\sin(x)$ and $\cos(x)$, (etc.), but the whole complex plane $\mathbb{C}$, (etc.). [See **Example 1.5.14** and **Problem 1.5.15, (b)**.] For instance:

If $z = iy$ with $y \in \mathbb{R}$, then $\quad \sin(z) = \sin(iy) = \dfrac{e^{-y} - e^{y}}{2i}$

and $\quad \cos(z) = \cos(iy) = \dfrac{e^{-y} + e^{y}}{2}$. But,

$$|\sin(iy)| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \frac{|e^{-y} - e^{y}|}{2} \longrightarrow \infty, \quad \text{as} \quad y \longrightarrow \pm\infty,$$

$$\text{and} \quad |\cos(iy)| = \left|\frac{e^{-y} + e^{y}}{2}\right| = \frac{e^{-y} + e^{y}}{2} \longrightarrow \infty, \quad \text{as} \quad y \longrightarrow \pm\infty.$$

The **hyperbolic functions** in the complex plane are defined by:

$$\begin{aligned}
\sinh(z) &= \frac{e^{z} - e^{-z}}{2}, \quad \forall\, z \in \mathbb{C}, \quad \text{the odd part of } e^{z},\\
\cosh(z) &= \frac{e^{z} + e^{-z}}{2}, \quad \forall\, z \in \mathbb{C}, \quad \text{the even part of } e^{z},\\
\tanh(z) &= \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}, \quad \forall\, z \in \mathbb{C},\\
\coth(z) &= \frac{\cosh(z)}{\sinh(z)} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}, \quad \forall\, z \in \mathbb{C},\\
\operatorname{sech}(z) &= \frac{1}{\cosh(z)} = \frac{2}{e^{z} + e^{-z}}, \quad \forall\, z \in \mathbb{C},\\
\operatorname{csch}(z) &= \frac{1}{\sinh(z)} = \frac{2}{e^{z} - e^{-z}}, \quad \forall\, z \in \mathbb{C}.
\end{aligned}$$

(When a denominator is zero, the answer is the complex infinity.)

All the identities that we learn in trigonometry, precalculus and calculus carry through to complex variables. Examine the problems that follow to find some additional identities and some **identities and relations** satisfied by the **trigonometric and hyperbolic functions**.

Definition 1.2.3 *A function that can be written as convergent complex power series at all points of its domain is called a **complex analytic function**.*

We observe that the exponential, trigonometric and hyperbolic functions are complex analytic.

1.2.1 Series Deduced from the Exponential Series

We have that for any $z = x + iy \in \mathbb{C}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Using the trigonometric from $z = r[\cos(\theta) + i\sin(\theta)]$ and the **De Moivre formula**, we have

$$e^{r[\cos(\theta)+i\sin(\theta)]} = \sum_{n=0}^{\infty} \frac{r^n[\cos(n\theta) + i\sin(n\theta)]}{n!},$$

and by the absolute convergence, we obtain

$$e^{r\cos(\theta)}\{\cos[r\sin(\theta)] + i\sin[r\sin(\theta)]\} = \sum_{n=0}^{\infty} \frac{r^n \cos(n\theta)}{n!} + i\sum_{n=0}^{\infty} \frac{r^n \sin(n\theta)}{n!}.$$

So, for all $r \in \mathbb{R}$ and all $\theta \in \mathbb{R}$, we have

$$e^{r\cos(\theta)}\cos[r\sin(\theta)] = \sum_{n=0}^{\infty} \frac{r^n \cos(n\theta)}{n!},$$

and

$$e^{r\cos(\theta)}\sin[r\sin(\theta)] = \sum_{n=0}^{\infty} \frac{r^n \sin(n\theta)}{n!}.$$

Example 1.2.2 We know that

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^4 + 2n^3 + n^2}{4}.$$

If we set this equal to

$$\frac{1}{4}[A + Bn + Cn(n-1) + Dn(n-1)(n-2) + En(n-1)(n-2)(n-3)]$$

and equate coefficients, we find $A = 0$, $B = 4$, $C = 14$, $D = 8$, and $E = 1$.

Therefore, for any $x \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!}x^n = x\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} + \frac{7}{2}x^2 \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} +$$
$$2x^3 \sum_{n=3}^{\infty} \frac{x^{n-3}}{(n-3)!} + \frac{1}{4}x^4 \sum_{n=4}^{\infty} \frac{x^{n-4}}{(n-4)!} = \left(x + \frac{7}{2}x^2 + 2x^3 + \frac{1}{4}x^4\right)e^x.$$

For example if $x = \sqrt{2}$, we get

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot 2^{\frac{n}{2}} = (8 + 5\sqrt{2})e^{\sqrt{2}}, \quad \text{etc.}$$

If instead of x, we use $z = r[\cos(\theta) + i\sin(\theta)]$, we get

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot r^n \cos(n\theta) =$$
$$e^{r\cos(\theta)} \left\{ r\cos[\theta + r\sin(\theta)] + \frac{7}{2}r^2 \cos[2\theta + r\sin(\theta)] + \right.$$
$$\left. 2r^3 \cos[3\theta + r\sin(\theta)] + \frac{1}{4}r^4 \cos[4\theta + r\sin(\theta)] \right\}$$

and

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot r^n \sin(n\theta) =$$
$$e^{r\cos(\theta)} \left\{ r\sin[\theta + r\sin(\theta)] + \frac{7}{2}r^2 \sin[2\theta + r\sin(\theta)] + \right.$$
$$\left. 2r^3 \sin[3\theta + r\sin(\theta)] + \frac{1}{4}r^4 \sin[4\theta + r\sin(\theta)] \right\}$$

▲

Problems

1.2.1 Use the **Ratio Test** to show that the interval of convergence for the geometric series is $(-1, 1)$. Show that the series diverges at either $x = 1$ or $x = -1$.

1.2.2 Use the **Ratio Test** to show that the open interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

is $(-1, 1)$. Does this series converge at either $x = 1$ or $x = -1$?

1.2.3 Use the **Ratio Test** to show that the interval of convergence for the series used to define e^x, $\sin(x)$, and $\cos(x)$ is $(-\infty, \infty)$.

1.2.4 Let (c_n) with $n \in \mathbb{N}_0$ be a sequence of complex numbers. Prove that if the limit $\lim_{n\to\infty} \dfrac{|c_{n+1}|}{|c_n|}$ exists, then the limit $\lim_{n\to\infty} \sqrt[n]{|c_n|}$ also exists, and both limits are equal.

[See also **Problem I 1.3.23, (a)**.]

1.2.5 (a) For $|z| \neq 1$ complex number, show that

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \dfrac{1}{(1-z)^2}\,, & \text{if } |z|<1, \\[2ex] \dfrac{1}{z(1-z)^2}\,, & \text{if } |z|>1. \end{cases}$$

[Hint: Note that

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{1}{z(1-z)}\left(\frac{1}{1-z^n} - \frac{1}{1-z^{n+1}}\right)$$

and use the limits of partial sums.]

(b) If $z = \pm 1$, the summation is obviously not defined. Study the same series for various $z \in \mathbb{C}$ such that $|z| = 1$ and $z \neq \pm 1$.

1.2.6 Study **Example 1.2.2** and prove $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{n!} = 5\,e.$

(For a generalization of this series see part (b) of problem 4.66 in *Furdui and Sîntămărian*, 2021, pp.138-139.)

1.2.7 Given that for $-1 < x < 1$,

$$\arctan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\,, \quad \text{prove:}$$

(a) $$\arctan^2(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)\frac{x^{2n}}{n} =$$

$$x^2 - \left(1+\frac{1}{3}\right)\frac{x^4}{2} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{3} - \ldots\,, \quad \forall\ -1<x<1.$$

(b) $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\,.$$

(c) $$\frac{\pi^2}{16} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(1+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)\frac{1}{n}.$$

[I.e., the power series of **(b)** and **(c)** are continuously valid at $x = 1$ and ± 1.]

1.2.8 (a) Prove that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty}(x_n + iy_n) = a + ib \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} x_n = a \text{ and } \sum_{n=1}^{\infty} y_n = b.$$

(b) If $z_n = x_n + iy_n$ with $x_n \geq 0$ (or $y_n \geq 0$), $\forall\ n \in \mathbb{N}$ (except possibly finitely many n's), and both $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_n^2$ converge, then prove:

(1) $\sum_{n=1}^{\infty} |z_n|^2 = \sum_{n=1}^{\infty}(x_n^2 + y_n^2)$ converges and

(2) $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely.

(c) Prove that the condition $x_n \geq 0$, $\forall\ n \in \mathbb{N}$ (except possibly finitely many n's), is necessary. Otherwise the two conclusions in **(b)** may fail.

[Hint on (c): For example use $z_n = \dfrac{e^{in}}{\sqrt{n}} = \dfrac{\cos(n)}{\sqrt{n}} + i\dfrac{\sin(n)}{\sqrt{n}}$, $\forall\ n \in \mathbb{N}$, as counterexample. (You will need: **Theorem 3.44** in Rudin 1976, p. 71, –or in any other book–, and/or the **Note** and **footnote** on pp. 89-90 of this book.)]

1.2.9 (a) Use the power series expansion of e^z, the power series product and the **Binomial Theorem** to show that for all $z \in \mathbb{C}$ and $w \in \mathbb{C}$, it holds $\quad e^{z+w} = e^z e^w \quad$ and so

$$e^z e^w = e^{z+w} = e^{w+z} = e^w e^z.$$

(b) Suppose that a function $f\ :\ \mathbb{C} \longrightarrow \mathbb{C}$ (or $\mathbb{R} \longrightarrow \mathbb{R}$) satisfies the following two properties:

(1) $f(z+w) = f(z)f(w)$ for all $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

(2) There is $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.

Prove that $f \equiv 0$ (f is identically zero).

Conclude from this that the equation $e^z = 0$ has no solution in $\mathbb{C}$.

1.2.10 (a) Use the appropriate power series and the **Riemann Absolute Convergence Rearrangement Theorem** to show that for y real

$$e^{iy} = \cos(y) + i\sin(y) \quad (\textbf{Euler's formula}).$$

(b) Now use the **previous Problem, (a),** to show that if $z = x + iy$, then

$$e^z = e^{x+iy} = e^x\left[\cos(y) + i\sin(y)\right].$$

(This relation can alternatively be used to extend the exponential function to the whole complex plane.)

(c) Prove $\overline{e^z} = e^{\bar{z}}$.

(d) Prove that the set

$$\{e^z \mid -\infty < x < \infty \text{ and } y \in \mathbb{R} \text{ fixed}\}$$

is the infinite open half line starting at the origin and forming angle y, (mod 2π), with the positive x-semi-axis. Each point of this half line is attained once.

(e) Prove that for any $a < b$ real such that $b - a = 2\pi$, either of the two following sets

$$\{e^z \mid x \in \mathbb{R} \text{ fixed and } a < y \leq b\} \text{ or } \{e^z \mid x \in \mathbb{R} \text{ fixed and } a \leq y < b\}$$

is a complete circle, each point of which is attained once. What is the center and the radius of this circle?

(f) For any $a < b$ real such that $b - a = 2\pi$, the horizontal strips

$$\{z \mid z = x + iy \text{ and } -\infty < x < \infty\text{and } a < y \leq b\},$$

or

$$\{z \mid z = x + iy \text{ and } -\infty < x < \infty\text{and } a \leq y < b\}$$

are mapped by the exponential function $f(z) = e^z$ in one-to-one manner onto the set $\mathbb{C} - \{0\}$.

So, prove that $f(\mathbb{C})$ is $\mathbb{C} - \{0\}$ obtained infinitely many times, once for every $k \in \mathbb{Z}$.

1.2.11 Using the power series definitions given above for $e^x, \sin(x)$ and $\cos(x)$, show that

$$\frac{d}{dx}\, e^x = e^x, \qquad \frac{d}{dx}\, \sin(x) = \cos(x), \qquad \frac{d}{dx}\, \cos(x) = -\sin(x).$$

(We can make a list of the derivatives of all the trigonometric and hyperbolic functions after we study the next section.)

1.2.12 (a) Prove that for z, complex number,

$$e^z = 1 \quad \Longleftrightarrow \quad z = 2k\pi i,\ \ k \in \mathbb{Z},$$

where $\mathbb{Z} = \{\, 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \dots \}$ is the set of integer numbers.

(b) Now set each of the functions $\cos(z), \sin(z), \cosh(z), \sinh(z)$ equal to 0 and find all the infinitely many solutions of the four equations.

(c) Prove that the equation $e^z = 0$ has no solution in the complex numbers.

1.2.13 Use **Euler's formula** to find the n n^{th}-roots of e^z for any complex number $z = x + iy$ and any $n \in \mathbb{N}$.

1.2.14 Write the n^{th} power and the n n^{th}-roots of any complex number $z = x + iy \neq 0$ in exponential forms for all $n \in \mathbb{N}$.

1.2.15 (a) Use **Euler's formula** to show that

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \text{and} \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

(b) These equations are used to define the trigonometric functions $\sin(z)$ and $\cos(z)$ in the whole complex plane, given by the same formulae by means of exponentials.

Write these two formulae for any complex number $z = a + ib$ by replacing x with $z = a + ib$ to obtain the **definitions of complex** $\sin(z)$ **and** $\cos(z)$ and show that they coincide with their power series representations stated earlier.

(c) Write the **definitions** of the remaining four **complex trigonometric functions** for any complex number $z = a+ib$ in terms of $\sin(z)$ and $\cos(z)$ and in terms of exponentials.

(d) Find: $|e^{iy}|, \ \forall \ y \in \mathbb{R}$ and $|e^z| = |e^{x+iy}|, \ \forall \ z = x+iy \in \mathbb{C}$.

(e) Prove: $|e^{iu}| = 1 \iff u \in \mathbb{R}$.

(f) Prove that for any integer $n \geq 0$ and any x real

$$\sum_{k=0}^{n} \sin(kx) = \sum_{k=1}^{n} \sin(kx) =$$
$$\frac{\sin\left[\frac{(n+1)x}{2}\right] \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

(If x is an integer multiple of 2π, the answer has the form $\dfrac{0}{0}$, but the formula is still valid if we resolve it by L' Hôpital's rule.)

(g) Prove that for any integer $n \geq 0$ and any x real

$$\sum_{k=0}^{n} \cos(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \cos\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(nx + \frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

(If x is an integer multiple of 2π, the answer has the form $\dfrac{0}{0}$, but the formula is still valid if we resolve it by L' Hôpital's rule.)

We also obtain, $\quad \dfrac{1}{2} + \displaystyle\sum_{k=1}^{n} \cos(kx) = \dfrac{\sin\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$

(See also **Problem 1.2.27** for generalizations.)

1.2.16 (a) Use the **complex definition of cosine** and the **Binomial Theorem** to prove: If $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, then

$$\cos^{2n+1}(z) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos\{[2(n-k)+1]z\}.$$

(b) Use the **complex definition of cosine** and the **Binomial Theorem** to prove: If $n \in \mathbb{N}$ and $z \in \mathbb{C}$, then

$$\cos^{2n}(z) = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[2(n-k)z] + \frac{\binom{2n}{n}}{2^{2n}}.$$

(c) Plug $z = 0$ in the equations in **(a)** and **(b)**, simplify and derive two identities with the combinatorial numbers involved.

(d) Differentiate the equations in **(a)** and **(b)**, simplify and derive two new identities. Plug $z = 0$ to derive two identities with the combinatorial numbers involved.

(We can also take second, third, etc., derivatives to derive more identities and formulae.)

[See also **Problem I 2.2.28, (a)** and **(e)** for the powers of sine.]

1.2.17 Consider the two real **hyperbolic functions**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \forall\ x \in \mathbb{R}.$$

(a) Use the real power series of e^x to show that for all $x \in \mathbb{R}$,

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!},$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

(b) Make the natural extensions of the above definitions and the power series to any complex number $z = a + ib$ in $\mathbb{C}$, by replacing x with z. Prove that the exponential expressions and the power series that you obtain coincide.

[The **definitions** of the remaining four **complex hyperbolic functions** for any complex number $z = a + ib$ in terms of $\sinh(z)$ and $\cosh(z)$ or exponentials are given in the text, **Section 1.2**.]

1.2.18 Use the properties and the Maclaurin series of the functions e^z, $\sin(z)$, $\cos(z)$, $\sinh(z)$ and $\cosh(z)$ to find their Taylor series with center any point $a \in \mathbb{C}$.

1.2.19 Show the useful identities for any complex number $z = x + iy$:
(See also the **next four Problems**.)

$$\begin{aligned}
\sin(-z) &= -\sin(z), & \cos(-z) &= \cos(z),\\
\tan(-z) &= -\tan(z), & \cot(-z) &= -\cot(z),\\
\sec(-z) &= \sec(z), & \csc(-z) &= -\csc(z),\\
\sinh(-z) &= -\sinh(z), & \cosh(-z) &= \cosh(z),\\
\tanh(-z) &= -\tanh(z), & \coth(-z) &= -\coth(z),\\
\mathrm{sech}(-z) &= \mathrm{sech}(z), & \mathrm{csch}(-z) &= -\mathrm{csch}(z),
\end{aligned}$$

$$\begin{aligned}
\cos^2(z) + \sin^2(z) &= 1, & \cosh^2(z) - \sinh^2(z) &= 1,\\
\sec^2(z) - \tan^2(z) &= 1, & \mathrm{sech}^2(z) + \tanh^2(z) &= 1,\\
\csc^2(z) - \cot^2(z) &= 1, & -\mathrm{csch}^2(z) + \coth^2(z) &= 1,
\end{aligned}$$

$$\cosh(z) + \sinh(z) = e^z, \qquad \cosh(z) - \sinh(z) = e^{-z},$$

$$\begin{aligned}
\sin(z) &= -i\ \sinh(iz), & \cos(z) &= \cosh(iz),\\
\sinh(z) &= -i\ \sin(iz), & \cosh(z) &= \cos(iz),\\
\sin(iz) &= i\ \sinh(z), & \cos(iz) &= \cosh(z),\\
\sinh(iz) &= i\ \sin(z), & \cosh(iz) &= \cos(z),
\end{aligned}$$

$$\begin{aligned}
\sin(z) &= \sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y),\\
\cos(z) &= \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y),\\
\sinh(z) &= \sinh(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y),\\
\cosh(z) &= \cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y),
\end{aligned}$$

$$\begin{aligned}
|\sin(z)|^2 &= \sin^2(x) + \sinh^2(y), & |\cos(z)|^2 &= \cos^2(x) + \sinh^2(y),\\
|\sinh(z)|^2 &= \sin^2(y) + \sinh^2(x), & |\cosh(z)|^2 &= \cos^2(y) + \sinh^2(x).
\end{aligned}$$

1.2.20 Show the useful identities for any complex number $z = x + iy$:

$$\begin{aligned}
\tan(iz) &= i\ \tanh(z), & \cot(iz) &= -i\ \coth(z),\\
\tanh(iz) &= i\ \tan(z), & \coth(iz) &= -i\ \cot(z),\\
\sec(iz) &= \mathrm{sech}(z), & \csc(iz) &= -i\,\mathrm{csch}(z),\\
\mathrm{sech}(iz) &= \sec(z), & \mathrm{csch}(iz) &= -i\ \csc(z).
\end{aligned}$$

1.2.21 The hyperbolic functions exhibit many identities analogous to trigonometric identities that we find in a book of trigonometry (up to + or − signs). (We have written a few of them in some of the problems and the text.) Search the bibliography and find a collection of such identities. Take a look at them and verify a few of them for practice, or produce some of these identities yourselves! For instance prove the following:

$$(a)\quad \sinh(0) = 0, \quad \cosh(0) = 1, \quad \tanh(0) = 0, \quad \lim_{z\to\infty}[\tanh(z)] = 1, \text{ etc.}$$

$$\text{(b)}\qquad \sinh^2\left(\frac{z}{2}\right) = \frac{\cosh(z) - 1}{2}, \qquad \sinh\left(\frac{z}{2}\right) = \pm\sqrt{\frac{\cosh(z) - 1}{2}},$$

$$\text{(c)}\qquad \cosh^2\left(\frac{z}{2}\right) = \frac{\cosh(z) + 1}{2}, \qquad \cosh\left(\frac{z}{2}\right) = \sqrt{\frac{\cosh(z) + 1}{2}},$$

$$\text{(d)}\qquad \tanh\left(\frac{z}{2}\right) = \frac{\cosh(z) - 1}{\sinh(z)} = \frac{\sinh(z)}{\cosh(z) + 1} = \pm\sqrt{\frac{\cosh(z) - 1}{\cosh(z) + 1}}.$$

1.2.22 For a complex number $z = x + iy$ show

$$\tan(z) = \tan(x + iy) =$$
$$\frac{\sin(x)\cos(x) + i\sinh(y)\cosh(y)}{\cos^2(x) + \sinh^2(y)} = \frac{\sin(2x) + i\sinh(2y)}{\cos(2x) + \cosh(2y)}$$

and find the analogous formulae for $\cot(z)$, $\sec(z)$, $\csc(z)$, $\tanh(z)$, $\coth(z)$, $\operatorname{sech}(z)$ and $\operatorname{csch}(z)$.

1.2.23 Prove that $e^{\bar{z}} = \overline{e^z}$ and then the following properties of conjugation with the trigonometric and hyperbolic functions:

$$\begin{aligned}
\sin(\bar{z}) &= \overline{\sin(z)}, & \cos(\bar{z}) &= \overline{\cos(z)},\\
\tan(\bar{z}) &= \overline{\tan(z)}, & \cot(\bar{z}) &= \overline{\cot(z)},\\
\sec(\bar{z}) &= \overline{\sec(z)}, & \csc(\bar{z}) &= \overline{\csc(z)},
\end{aligned}$$

$$\begin{aligned}
\sinh(\bar{z}) &= \overline{\sinh(z)}, & \cosh(\bar{z}) &= \overline{\cosh(z)},\\
\tanh(\bar{z}) &= \overline{\tanh(z)}, & \coth(\bar{z}) &= \overline{\coth(z)},\\
\operatorname{sech}(\bar{z}) &= \overline{\operatorname{sech}(z)}, & \operatorname{csch}(\bar{z}) &= \overline{\operatorname{csch}(z)}.
\end{aligned}$$

1.2.24 For $z,\ w \in \mathbb{C}$ prove the following identities:

$$\text{(a)}\quad \cosh(z)+\sinh(z)=e^{z}, \qquad \text{(b)}\quad \cosh(z)-\sinh(z)=e^{-z},$$
$$\text{(c)}\quad \sinh(z+w)=\sinh(z)\cosh(w)+\cosh(z)\sinh(w),$$
$$\text{(d)}\quad \sinh(z-w)=\sinh(z)\cosh(w)-\cosh(z)\sinh(w),$$
$$\text{(e)}\quad \cosh(z+w)=\cosh(z)\cosh(w)+\sinh(z)\sinh(w),$$
$$\text{(f)}\quad \cosh(z-w)=\cosh(z)\cosh(w)-\sinh(z)\sinh(w),$$
$$\text{(g)}\quad \tanh(z+w)=\frac{\tanh(z)+\tanh(w)}{1+\tanh(z)\tanh(w)}, \quad \tanh(z-w)=\ldots?,$$
$$\text{(h)}\quad \sinh(z+w)+\sinh(z-w)=2\sinh(z)\cosh(w),$$
$$\text{(i)}\quad \sinh(z+w)-\sinh(z-w)=2\cosh(z)\sinh(w),$$
$$\text{(j)}\quad \cosh(z+w)+\cosh(z-w)=2\cosh(z)\cosh(w),$$
$$\text{(k)}\quad \cosh(z+w)-\cosh(z-w)=2\sinh(z)\cosh(w),$$
$$\text{(l)}\quad e^{-z}=\frac{\operatorname{sech}(z)}{1+\tanh(z)}=\frac{1-\tanh(z)}{\operatorname{sech(z)}}, \qquad e^{z}=\ldots?,$$
$$\text{(m)}\quad \sinh(z)+\sinh(w)=2\sinh\left(\frac{z+w}{2}\right)\cosh\left(\frac{z-w}{2}\right),$$
$$\text{(n)}\quad \sinh(z)-\sinh(w)=2\cosh\left(\frac{z+w}{2}\right)\sinh\left(\frac{z-w}{2}\right),$$
$$\text{(o)}\quad \cosh(z)+\cosh(w)=2\cosh\left(\frac{z+w}{2}\right)\cosh\left(\frac{z-w}{2}\right),$$
$$\text{(p)}\quad \cosh(z)-\cosh(w)=2\sinh\left(\frac{z+w}{2}\right)\sinh\left(\frac{z-w}{2}\right).$$

1.2.25 For a real number x prove

$$\text{(a)}\qquad \operatorname{arcsinh}(x)=\ln\left(x+\sqrt{x^2+1}\right), \quad \forall\ x\in\mathbb{R},$$
$$\text{(b)}\ \operatorname{arccosh}^{\pm}(x)=\ln\left(x\pm\sqrt{x^2-1}\right)=\pm\ln\left(x+\sqrt{x^2-1}\right),\ \text{if}\quad x\geq 1,$$
$$\text{(c)}\qquad \operatorname{arctanh}(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad \text{if}\quad |x|<1.$$

After you study **Subsection 1.5.3**, return back to rewrite these formulae for $z \in \mathbb{C}$.

1.2.26 From **Example 1.2.1**, find the power series of $\ln(1-x)$ and then use it to find the power series for $\ln(1+x)$ and $\ln\left(\dfrac{1+x}{1-x}\right)$. Write all these results in the complex variable z.

1.2.27 (a) If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$, prove the formulae for the following two finite sums that can also be used to estimate infinite series:

$$\sum_{k=1}^{n} \sin[a+(k-1)d] = \frac{\sin\left[a+\frac{(n-1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\cos\left(a-\frac{d}{2}\right)-\cos\left(a+nd-\frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)},$$

$$\sum_{k=1}^{n} \cos[a+(k-1)d] = \frac{\cos\left[a+\frac{(n-1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\sin\left(a+nd-\frac{d}{2}\right)-\sin\left(a-\frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

(b) Replace a in both formulae in **(a)** by $a+d$ to prove that

$$\sum_{k=1}^{n} \sin(a+kd) = \frac{\sin\left[a+\frac{(n+1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\cos\left(a+\frac{d}{2}\right)-\cos\left(a+nd+\frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)},$$

$$\sum_{k=1}^{n} \cos(a+kd) = \frac{\cos\left[a+\frac{(n+1)d}{2}\right]\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\sin\left(a+nd+\frac{d}{2}\right)-\sin\left(a+\frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

(c) Apply both formulae in **(a)** for $d = a$ to prove that

$$\sum_{k=1}^{n} \sin(ka) = \frac{\sin\left[\frac{(n+1)a}{2}\right]\sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)} = \frac{\cos\left(\frac{a}{2}\right)-\cos\left(na+\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)},$$

$$\sum_{k=1}^{n} \cos(ka) = \frac{\cos\left[\frac{(n+1)a}{2}\right]\sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)} = \frac{\sin\left(na+\frac{a}{2}\right)-\sin\left(\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)}.$$

$$\text{So,} \qquad \frac{1}{2}+\sum_{k=1}^{n} \cos(ka) = \frac{\sin\left(na+\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)}.$$

For every $n \in \mathbb{N}_0$ and for every $x \in \mathbb{R}$, the last sum is denoted by $D_n(x)$ and is called the n^{th}-**Dirichlet kernel**. It plays an important role in Fourier Analysis. Prove that

$$D_n(x) := \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos(kx) = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)},$$

and $D_n(0) = 1 + 2n$.

Next differentiate the appropriate equalities and simplify to prove

$$\sum_{k=1}^{n} k\sin(kx) = \frac{(n+1)\sin(nx) - n\sin[(n+1)x]}{4\sin^2\left(\frac{x}{2}\right)},$$

and

$$\sum_{k=1}^{n} k\cos(kx) = \frac{(n+1)\cos(nx) - n\cos[(n+1)x] - 1}{4\sin^2\left(\frac{x}{2}\right)}.$$

(d) Use both formulae in **(a)** with $d = 2a$ to prove that

$$\sum_{k=1}^{n} \sin[(2k-1)a] = \frac{\sin^2(na)}{\sin(a)}, \quad \text{and} \qquad \sum_{k=1}^{n} \cos[(2k-1)a] = \frac{\sin(2na)}{2\sin(a)}.$$

(e) Replace a with $2a$ in the first two equations of **(c)** to prove

$$\sum_{k=1}^{n} \sin(2ka) = \frac{\sin[(n+1)a]\sin(na)}{\sin(a)} = \frac{\cos(a) - \cos[(2n+1)a]}{2\sin(a)},$$

$$\sum_{k=1}^{n} \cos(2ka) = \frac{\cos[(n+1)a]\sin(na)}{\sin(a)} = \frac{\sin[(2n+1)a] - \sin(a)}{2\sin(a)}.$$

$$\text{So,} \quad \frac{1}{2} + \sum_{k=1}^{n} \cos(2ka) = \frac{\sin[(2n+1)a]}{2\sin(a)}.$$

[See also, **Problem 1.2.15, (f)** and **(g)**.]

(f) Prove that following two equalities

$$\cos\left(\frac{\pi}{17}\right) + \cos\left(\frac{3\pi}{17}\right) + \ldots + \cos\left(\frac{15\pi}{17}\right) = \frac{1}{2},$$

$$\cos\left(\frac{\pi}{23}\right) + \cos\left(\frac{3\pi}{23}\right) + \ldots + \cos\left(\frac{21\pi}{23}\right) = \frac{1}{2}.$$

[Hint: Use the complex expressions of sine and cosine as in the text or in **Problem 1.2.15** and the formulae for geometric sums.]

1.2.28 (a) If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$, prove

$$\sum_{k=1}^{n} \sin^2[a+(k-1)d] = \frac{n}{2} - \frac{\cos\left[2a+(n-1)d\right]\sin(nd)}{2\sin(d)} = \frac{n}{2} - \frac{\sin\left[2a+(2n-1)d\right]+\sin(2a-d)}{4\sin(d)}.$$

(b) Apply the formula in **(a)** when $d = a$ or $d = 2a$ to prove that

$$\sum_{k=1}^{n} \sin^2(ka) = \frac{n}{2} - \frac{\cos[(n+1)a]\sin(na)}{2\sin(a)} = \frac{n}{2} - \frac{\sin[(2n+1)a]+\sin(a)}{4\sin(a)},$$

$$\sum_{k=1}^{n} \sin^2[(2k-1)a] = \frac{n}{2} - \frac{\cos(2na)\sin(2na)}{2\sin(2a)} = \frac{n}{2} - \frac{\sin(4na)}{4\sin(2a)}.$$

(c) Apply this formula with $2d$ in the places of both a and d [or simply put $2d$ for a in the first formula of (b)] to prove

$$\sum_{k=1}^{n} \sin^2(2kd) = \frac{n}{2} - \frac{\cos[2(n+1)d]\sin(2nd)}{2\sin(2d)} = \frac{n}{2} - \frac{\sin[2(2n+1)d]+\sin(d)}{4\sin(2d)}.$$

(d) Find the analogous formulae for the sums

$$\sum_{k=1}^{n} \cos^2[a+(k-1)d], \qquad \sum_{k=1}^{n} \cos^2(ka),$$

$$\sum_{k=1}^{n} \cos^2[(2k-1)a], \qquad \sum_{k=1}^{n} \cos^2(2ka).$$

[Hint: Use the complex expressions of sine and cosine as in the text or in **Problem 1.2.15** and the formulae for geometric sums.]

1.2.29 Use the geometric series to show that if $-\frac{\pi}{6} < x < \frac{\pi}{6}$ (real) then

$$\sum_{n=1}^{\infty} 2^{n-1}\sin^n(x) = \frac{\sin(x)}{1-2\sin(x)}.$$

1.2.30 (a) Use the appropriate **telescopic series partial sum**

$$\sum_{k=1}^{n} [f(k) - f(k+1)] = f(1) - f(n+1)$$

to show

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^{n+2}}\right)\cos\left(\frac{3\pi}{2^{n+2}}\right) = \frac{1}{2}.$$

(b) Now for $z \in \mathbb{C}$, prove that $\tan(z) = \cot(z) - 2\cot(2z)$ and then

$$\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\left(\frac{z}{2^n}\right) = \frac{1}{z} - \cot(z).$$

At $z = m\pi$, with $m \in \mathbb{Z}$, both sides become infinite.
At $z = 0^{\pm}$, we have $0 = \pm\infty \mp \infty$.

[Hint: In **(b)** you may use an appropriate **telescopic series partial sum** of the form $\sum_{k=1}^{n}[f(k+1) - f(k)] = f(n+1) - f(1)$.]

1.2.31 (a) For any $x \geq 0$ and $y \geq 0$ real numbers prove that

$$\arctan\left(\frac{x-y}{1+xy}\right) = \arctan(x) - \arctan(y).$$

(b) Use **(a)** to prove that for any $x \in \mathbb{R}$ and any $n \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \arctan\left[\frac{x}{1+n(n+1)x^2}\right] =$$

$$\begin{cases} \dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right), & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -\dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right), & \text{if} \quad x < 0. \end{cases}$$

Justify the discontinuity at $x = 0$.

(c) Prove

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \frac{\pi}{4}.$$

[Hint: $\dfrac{1}{2n^2} = \dfrac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$.]

1.2.32 (a) For $n \in \mathbb{N}$, state the exponential forms of the n n$^{\text{th}}$ roots of unity $\omega_0,\ \omega_1,\ \ldots,\ \omega_{n-1}$, and of the n n$^{\text{th}}$ roots of any complex number $z \neq 0$.

(b) For $n \in \mathbb{N}$ and $\omega_l = e^{i\frac{2l\pi}{n}}$, where $l = 0,\ 1,\ 2,\ \ldots,\ n-1$, use the finite geometric sum to prove

$$\omega_0^j + \omega_1^j + \ldots + \omega_{n-1}^j = \begin{cases} 0, & \text{if} \quad j \in \{1,\ 2,\ \ldots\ n-1\} + n\,\mathbb{Z}, \\ \\ n, & \text{if} \quad j \in n\,\mathbb{Z}. \end{cases}$$

(c) Generalize **(b)** to the roots of any $z \in \mathbb{C}$.

(d) For $n \in \mathbb{N}$ and $\omega \neq 1$ an n$^{\text{th}}$ root of unity (i.e., of 1), prove that

$$\sum_{l=0}^{n-1} \omega^l = 1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

1.2.33 (a) Let $n \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots,\ \omega_{n-1}$ be the n n$^{\text{th}}$ roots of unity (i.e., of 1). Prove

$$\begin{gathered} \omega_0 + \omega_1 + \ldots + \omega_{n-1} = 0, \\ \omega_0\omega_1 + \omega_0\omega_2 + \ldots + \omega_{n-2}\omega_{n-1} = 0, \\ \omega_0\omega_1\omega_2 + \ldots + \omega_{n-3}\omega_{n-2}\omega_{n-1} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \omega_0\omega_1\ldots\omega_{n-2} + \ldots + \omega_1\omega_2\ldots\omega_{n-1} = 0, \\ \omega_0\omega_1\ldots\omega_{n-1} = (-1)^{n+1}. \end{gathered}$$

(b) Generalize these equations to the roots of any $z \in \mathbb{C}$.

[Hint: Use Vieta's formulae for polynomials.]

1.2.34 Consider any $n \in \mathbb{N}$ and the first complex n$^{\text{th}}$ root of unity (i.e., of 1)

$$\omega = e^{\frac{2i\pi}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right). \quad (\omega_1 := \omega,\ \ \omega_0 = 1,\ \ \text{etc.})$$

(a) Prove that all the n n$^{\text{th}}$ roots of unity are given by $\omega_k = \omega^k$, for $k = 0,\ 1,\ 2,\ 3,\ \ldots,\ n-1$.

(b) Prove that

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \ldots + z^2 + z + 1) = (z-1)\prod_{k=1}^{n-1}(z - \omega^k)$$

$$\text{and then,} \qquad \prod_{k=1}^{n-1}(1-\omega^k) = \prod_{k=1}^{n-1}(1-\omega_k) = n.$$

(c) Let $m = \left[\left[\frac{n}{2}\right]\right]$ be the integer part of $\frac{n}{2}$. Prove

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}, \qquad \text{and} \qquad \prod_{k=1}^{m} \sin\left(\frac{k\pi}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}.$$

(See also **Problem I 2.6.55.**.)

[Hint: First prove $|1-\omega^k| = 2\left|\sin\left(\frac{k\pi}{n}\right)\right| = 2 \cdot \sin\left(\frac{k\pi}{n}\right)$. Then use **(b)**, the symmetries of the values of sine, $\sin\left(\frac{k\pi}{n}\right) > 0$ for $k = 1,\ 2,\ \ldots,\ n-1$, etc.]

(d) Use **(c)** and limits of Riemann sums to prove

$$\int_0^{\pi} \ln[\sin(x)]\,dx = 2\int_0^{\frac{\pi}{2}} \ln[\sin(x)]\,dx = -\pi\ln(2).$$

(See also **Problems I 2.1.19, 1.7.132** and **Example 1.7.50**.)

(e) Write explicitly the two complex symmetric matrices

$F_n = [f_{ij}]_{n\times n}$ and $G_n = \frac{1}{n} \cdot [g_{ij}]_{n\times n}$, where

$f_{ij} = \omega^{\,(i-1)(j-1)}$ and $g_{ij} = \omega^{\,-(i-1)(j-1)}$, with $i,\ j = 1,\ 2,\ \ldots,\ n$, and prove that they are inverse of each other. Also, find their determinants and check that these matrices are not Hermitian for $n \geq 3$, but for $n \geq 1$ the matrices $\frac{1}{\sqrt{n}} \cdot F_n$ and $\sqrt{n} \cdot G_n$ are unitary. (The matrices F_n and G_n are called **Fourier matrices**.)

[Hint: Use **(a), (b)** and the previous two problems. For the determinants you may use induction, or row reduction, or find and compare with Vandermonde's[10] determinant, or another way.]

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[10] Vandermonde Alexandre-Theophile, French mathematician, chemist and musician, 1735-1796.

(f) Consider any complex numbers $z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-1}$. For $k = 0,\ 1,\ 2,\ \ldots,\ n-1$ put

$$A_k = z_0 + z_1\omega^k + z_2\omega^{2k} + \ldots + z_{n-1}\omega^{(n-1)k}.$$

$$\text{Then prove that} \quad \sum_{k=0}^{n-1} |A_k|^2 = n \sum_{k=0}^{n-1} |z_k|^2.$$

[Hint: One way is to use **(e)** above with the unitary matrix.]

1.2.35 For $n \in \mathbb{N}$ and $z \in \mathbb{C}$ find the formulae for the three finite sums

$$\text{(a)} \quad \sum_{k=0}^{n} \left(ze^{i\theta}\right)^k, \qquad \text{(b)} \quad \sum_{k=0}^{n} z^k \cos(k\theta), \qquad \text{(c)} \quad \sum_{k=1}^{n} z^k \sin(k\theta).$$

1.2.36 For $n \in \mathbb{N}$ write the binomial expansion of $\left(1 + e^{i\theta}\right)^n$ and also rewrite it using the half angle formulae for sine and cosine. Then, find the formulae for the sums

$$\text{(a)} \qquad \sum_{k=0}^{n} \binom{n}{k} \cos(k\theta), \qquad\qquad \text{(b)} \qquad \sum_{k=1}^{n} \binom{n}{k} \sin(k\theta).$$

1.2.37 (a) The **integral sine function** is given by

$$\mathrm{Si}(x) := \int_0^x \frac{\sin(t)}{t}\, dt, \quad \text{for all} \quad x \in \mathbb{R}.$$

Find the power series with center $c = 0$ for the integrand and then integrate term by term to get a power series for $\mathrm{Si}(x)$. What is its radius of convergence?

(See also **Example I 1.1.21**.)

(b) The **integral cosine function** is given by

$$\mathrm{Ci}(x) := \int_x^\infty \frac{\cos(t)}{t}\, dt, \quad \text{for all} \quad x > 0.$$

Why $\mathrm{Ci}(0)$ is not defined?

Find the first five terms of the power series of $\mathrm{Ci}(x)$ with center $c = \pi$. [The $a_0 = \mathrm{Ci}(\pi)$ remains as integral. It cannot be computed explicitly.] What is the radius of convergence of this power series?

[See also **Problem I 2.2.33, (j)**.]

1.2.38 Prove that the series expansion of the Gamma function in

Problem I 2.6.62 converges for all complex numbers $z \in \mathbb{C} - \{0,\ -1,\ -2, \dots\}$.

[See also **property** $(\mathbf{\Gamma}, \mathbf{8})$ and **Problem 1.6.22**.]

1.3 Limits, Continuity and Derivatives

In this exposition, we do not plan to present all the topological aspects of the complex plane and the implied results on the complex functions, sequences and series. These important results, if not known by now, can be studied from a book of complex analysis. (See bibliography.)

As far as the limits are concerned, we mention what we need for the goals of this text. So, we have:

$$z = x+iy \to z_0 = x_0+iy_0 \ \Leftrightarrow\ (x,y) \to (x_0,y_0) \ \Leftrightarrow\ x \to x_0 \text{ and } y \to y_0.$$

This follows from the inequality

$$\max\{|x-x_0|, |y-y_0|\} \le \sqrt{(x-x_0)^2+(y-y_0)^2} \le |x-x_0|+|y-y_0|,$$

since $|z-z_0| = \sqrt{(x-x_0)^2+(y-y_0)^2}$, for all complex numbers $z = x+iy$ and $z_0 = x_0+iy_0$.

Similarly, by the same inequality, we have

$$\begin{aligned} g(z) = u(x,y) + iv(x,y) &\longrightarrow a+ib \in \mathbb{C}, \\ \text{as} \quad z = x+iy &\longrightarrow z_0 = x_0+iy_0 \in \mathbb{C}, \end{aligned}$$

if and only if

$$\begin{aligned} u(x,y) &\longrightarrow a, \quad \text{as} \quad (x,y) \longrightarrow (x_0,y_0), \quad \text{and} \\ v(x,y) &\longrightarrow b, \quad \text{as} \quad (x,y) \longrightarrow (x_0,y_0). \end{aligned}$$

We understand that if

$$\lim_{z\to z_0} g(z) = \lim_{x+iy\to x_0+iy_0} [u(x,y)+iv(x,y)] = a+ib \in \mathbb{C},$$

then the answer $a+ib$, to this limit, is always the same no matter in what way $z = x+iy$ approaches the constant $z_0 = x_0+iy_0$ in the complex plane $\mathbb{C}$. There are, of course, infinitely many ways and paths that $z = x+iy$ can follow to reach $z_0 = x_0+iy_0$. We can express this by the following general and mathematically rigorous definition:

$$\lim_{z\to z_0} f(z) = L \in \mathbb{C} \overset{def}{\Longleftrightarrow} \forall\, \epsilon > 0,\ \exists\, \delta > 0 \ :\ |z-z_0| < \delta \Rightarrow |f(z)-L| < \epsilon.$$

So: **If in such a limit two different ways of approaching the number (constant, point) $z_0 = x_0 + iy_0$ yield two different answers, then this limit does not exist.**

In the real line $\mathbb{R}$, we distinguish two infinities, the $+\infty$ and $-\infty$. They are answers to certain real limits. (Suppose $a > 0$ real constant, then $\frac{a}{0^{\pm}} = \pm\infty$ and $\frac{-a}{0^{\pm}} = \mp\infty$. The limit form $\frac{0}{0}$ is indeterminate, that is, any answer, real number, $\pm\infty$, or does not exist, is possible. The same is true for the indeterminate limit $\frac{\pm\infty}{\pm\infty}$.)

In the complex plane $\mathbb{C}$, some limits have answers $a \pm i\infty$, where $a \in \mathbb{R}$, or $\pm\infty + ib$, where $b \in \mathbb{R}$, and/or $\pm\infty \pm i\infty$. In the complex plane, all of these unbounded answers are considered to be one infinity, the complex infinity (∞). So, in the complex plane, we do not distinguish between various infinities unless we need to declare its specific form out of these **eight** stated choices. For any $c \in \mathbb{C}$, we have $\frac{c}{\infty} = 0$, but $\frac{\infty}{\infty}$ is indeterminate.

A complex function has limit a complex number at the complex infinity if by definition

$$\lim_{z\to\infty} f(z) = L \in \mathbb{C} \overset{def}{\Longleftrightarrow} \forall\ \epsilon > 0,\ \exists\ R > 0\ :\ |z| > R \Longrightarrow |f(z) - L| < \epsilon.$$

(In this definition, R depends on ϵ.)

Therefore, if $\lim_{z\to\infty} f(z) = L \in \mathbb{C}$ and for any $R > 0$ we define the truncated function

$$f_R(z) = \begin{cases} f(z), & \text{if} \quad |z| > R, \\ \\ L, & \text{if} \quad |z| \le R, \end{cases}$$

then the convergence $\lim_{R\to\infty} f_R(z) = L$ is uniform in z. We use this fact very often, but a lot of times tacitly, when we perform complex integration, especially when we switch limits and integrals over finite ranges. So, in the sequel, we must keep in mind this fact, especially when we emphasize that we have a limit as z approaches the complex infinity.

The definition and the properties of continuous functions from $\mathbb{C}$ to $\mathbb{C}$ transfer the same as with the real functions of a real variable. So:

Definition 1.3.1 *A complex function $f : \mathcal{D} \longrightarrow \mathbb{C}$ of one complex variable z, where the domain $\mathcal{D} \subseteq \mathbb{C}$, is **continuous at** $z_0 \in \mathcal{D}$ if*

$$\lim_{z\to z_0} f(z) = f(z_0)\ [= f(\lim_{z\to z_0} z)].$$

*It is called **continuous in** $\mathcal{D}$ if it is continuous at every point of $\mathcal{D}$.*

The sum, difference, multiplication, division and composition of continuous functions are continuous. For the division, we exclude the points at which the divisor function (denominator) is zero.

Example 1.3.1 Consider the complex function $f(z) = \dfrac{z^2}{z+1}$.

If $z_0 \neq -1$, by continuity we get

$$\lim_{z \to z_0} \frac{z^2}{z+1} = \frac{z_0^2}{z_0+1}, \quad \text{otherwise,} \quad \lim_{z \to -1} \frac{z^2}{z+1} = \infty.$$

▲

Example 1.3.2 Consider $z = x + iy$ and the complex function

$$f(z) = \frac{\mathrm{Re}(z^2)}{z\bar{z}} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Then:

$$\lim_{z=x+i0 \to 0+i0} f(z) = \lim_{z=x+i0 \to 0+i0} \frac{x^2}{x^2} = 1,$$

and

$$\lim_{z=0+iy \to 0+i0} f(z) = \lim_{z=0+iy \to 0+i0} \frac{-y^2}{y^2} = -1.$$

Therefore, $\lim\limits_{z=x+iy \to 0+i0} f(z)$ does not exist.

(See also **Examples 1.6.6** and **1.6.8**.)

▲

Example 1.3.3 One way to prove that $f(z) = z^2 = (x + iy)^2$ is continuous at every point $z_0 = x_0 + iy_0 \in \mathbb{C}$ is the following:

$$0 \leq |z^2 - z_0^2| = |z + z_0| \cdot |z - z_0| \leq$$
$$(|x + x_0| + |y + y_0|) \cdot (|x - x_0| + |y - y_0|) \to (2x_0 + 2y_0) \cdot (0 + 0) = 0,$$

as $z = x + iy \to z_0 = x_0 + iy_0$, or as ($x \to x_0$ and $y \to y_0$), or as $(x, y) \to (x_0, y_0)$.

Therefore, $z^2 \to z_0^2$ as $z \to z_0$, i.e., the function $f(z) = z^2$ is continuous at every point $z_0 \in \mathbb{C}$. (Prove this in a different way, too.)

▲

We continue with the following:

Definition 1.3.2 *A complex-valued function f of one complex variable z is said to be* **analytic** *or* **complex analytic** *at a point $c \in \mathbb{C}$ if we can write*

$$f(z) = \sum_{n=0}^{\infty} b_n (z - c)^n,$$

and this complex power series has (maximum) radius of convergence some $0 < R \leq \infty$, i.e., the power series converges for all z such that $|z - z_0| < R$ and diverges for any z such that $|z - z_0| > R$. The complex number c is called the ***center of the power series****.*

We denote the **open disc** with center c and radius R by $D(c, R) = \{z \in \mathbb{C} \mid |z - c| < R\}$ and the **closed disc** with center c and radius R by $\overline{D(c, R)} = \{z \in \mathbb{C} \mid |z - c| \leq R\}$. Both of them have boundary the **circle** with center c and radius R, $C(c, R) = \{z \in \mathbb{C} \mid |z - c| = R\}$.

So, in the above **definition**, the complex power series of a complex function $f(z)$ analytic at c, i.e., $f(z) = \sum_{n=0}^{\infty} b_n (z - c)^n$, converges for any $z \in D(c, R)$ and diverges for any $z \in \mathbb{C} - \overline{D(c, R)}$ (the complement of the closed disc). At a point of their boundary $C(c, R)$ anything can happen and must be checked individually. By the Absolute Root or Ratio Tests **the convergence is absolute in** $D(c, R)$. If at a point of $C(c, R)$ the power series converges, the convergence may or may not be absolute.

Such a function is differentiable at any $z \in D(c, R)$ for $R > 0$. As in calculus, we can prove here that its derivative is

$$f'(z) = \sum_{n=0}^{\infty} n b_n (z - c)^{n-1} = \sum_{n=1}^{\infty} n b_n (z - c)^{n-1}.$$

Since $\lim_{n \to \infty} \sqrt[n]{n} = 1$ or $\lim_{n \to \infty} \frac{n+1}{n} = 1$, the radius of convergence of this new complex power series is also R, that is, the same as the radius of convergence of the complex power series we started with. [See **Problem 1.3.1, (a)**.] Then to prove this result, we can use the complex analog of **Theorem I 2.3.15**, applied point-wise to $S_n(z) = \sum_{k=0}^{n} b_k (z - c)^k$ and $S'_n(z) = \sum_{k=1}^{n} k\, b_k (z - c)^{k-1}$ for any z in the common disc of convergence $D(c, R)$, with $R > 0$.

Inductively, the k^{th} order derivative of $f(z)$, where $k \in \mathbb{N}$, is

$$f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1) b_n (z - c)^{n-k} =$$

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1) b_n (z - c)^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, b_n (z - c)^{n-k}$$

and the radius of convergence of this series is again $R > 0$. So, such a function is infinitely differentiable in $D(c, R)$.

In fact, such a function $f(z)$ is analytic at any $z_0 \in D(c, R)$. To show this, we consider the power series

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

with center z_0 and radius of convergence $r = R - |z_0 - c| > 0$, so that $D(z_0, r) \subseteq D(c, R)$ and coefficients a_k's, given by Taylor's formula,

$$\forall \quad k = 0,\ 1,\ 2,\ 3,\ \ldots, \qquad a_k = \frac{g^{(k)}(z_0)}{k!} = \frac{f^{(k)}(z_0)}{k!} =$$
$$\frac{1}{k!} \sum_{n=k}^{\infty} n(n-1) \ldots (n-k+1) b_n (z_0 - c)^{n-k}.$$

By the convergence of the initial power series in $D(c, R)$, all of these coefficients are finite numbers. The argument that this power series $g(z)$ converges and is equal to $f(z)$ in $D(z_0, r)$, and so its radius of convergence ρ is at least r (i.e., $\rho \geq r$), is accessible at this level but not that trivial. So, we skip it here for the sake of brevity. The interested reader may consult Cartan 1973, 37-39, in the bibliography. In the proof, the absolute convergence of power series and the Binomial Theorem is used. An example in which $\rho > r$ is also provided.

In general, for any complex function $w = f(z)$, we define the complex derivative of $f(z)$ by:

Definition 1.3.3 *The **complex derivative** or simply the **derivative** or the **first derivative of a complex function** $f(z)$ at a point z is defined (in the same way as we have seen for real functions) to be*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

In this definition, we recognize that h is a complex number that tends to $0 = 0 + i0$. Equivalently, $h = s + it \to 0$, or $|h| \to 0$, or $\sqrt{s^2 + t^2} \to 0$, or $(s, t) \to (0, 0)$, that is, h is in a disc of radius r around 0, and $r \to 0$. If this limit does not exist at some point, then we say that $f(z)$ does not have a (complex) derivative at that point.

Definition 1.3.4 *A complex function $f : \mathcal{D} \longrightarrow \mathbb{C}$, where $\mathcal{D} \subseteq \mathbb{C}$ is open, that possesses complex derivative at every point $z \in \mathcal{D}$ is called a **holomorphic function** in its domain $\mathcal{D}$.*

It turns out that the concepts "**complex analytic,**" i.e., complex power series, and "**holomorphic**" in a domain $\mathcal{D}$ are equivalent. We will discuss this fact more and use it in what follows.

We now write $z = x + iy$ and $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where the functions u and v are real functions of two real variables. Since the limit, in the definition of the (complex) derivative, exists if and only if it yields the same value no matter how $h = s + it$ approaches $0 = 0 + i0$, i.e., $(s, t) \longrightarrow (0, 0)$, we have the following:

(I) First, we choose $h = s + i0$ in the limit of the derivative of $f(z)$ to obtain

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{s\to 0} \frac{f(x+s+iy) - f(x+iy)}{s} =$$
$$\lim_{s\to 0} \frac{u(x+s,y) + iv(x+s,y) - u(x,y) - iv(x,y)}{s} =$$
$$\lim_{s\to 0} \frac{u(x+s,y) - u(x,y)}{s} + \lim_{s\to 0} i \frac{v(x+s,y) - v(x,y)}{s} =$$
$$\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y).$$

(II) Next, we choose $h = 0 + it$ in the limit of the derivative of $f(z)$ to obtain

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{t\to 0} \frac{f[x+i(y+t)] - f(x+iy)}{t} =$$
$$\lim_{t\to 0} \frac{u(x,y+t) + iv(x,y+t) - u(x,y) - iv(x,y)}{it} =$$
$$\lim_{t\to 0} \frac{u(x,y+t) - u(x,y)}{it} + \lim_{t\to 0} i \frac{v(x,y+t) - v(x,y)}{it} =$$
$$\frac{1}{i}\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y).$$

From the computations **(I)** and **(II)**, we see that:

(1) If the complex derivative of $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ exists, then the partial derivatives of $u(x, y)$ and $v(x, y)$ do exist.
(2) If the derivative exists, then the two limits in **(I)** and **(II)** must be equal, and thus we obtain the following **two formulae for the complex derivative**

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y). \tag{1.4}$$

We can also write this as

$$f'(z) = f_x(z) = u_x(x,y) + iv_x(x,y) = -if_y(z) = -i[u_y(x,y) + iv_y(x,y)].$$

(3) The equality of the two limits in **(I)** and **(II)** forces the validity of

the following **two necessary conditions**

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y).$$

These two first-order partial differential equations are called the **Cauchy-Riemann equations or conditions in Cartesian coordinates** x **and** y (even though they were known to and used by D' Alembert and Euler earlier). Both together form a system of two first-order partial differential equations.

The existence of the partial derivatives of $u(x,y)$ and $v(x,y)$ and the two equations that must satisfy are only **necessary conditions** for the complex derivative to exist. On their account, we also obtained two additional formulae for the derivative of $f(z)$, i.e.,

$$f'(z) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) + i\frac{\partial v}{\partial x}(x,y). \tag{1.5}$$

In conclusion, the **general necessary and sufficient conditions for the existence of the complex derivative of a complex function** are the following:

"*A complex function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, with $z = x + iy \in \mathcal{D} \subseteq \mathbb{C}$ open, has complex derivative at a point $z_0 = x_0 + iy_0 \in \mathcal{D}$ if and only if (1) the two real functions $u(x,y)$ and $v(x,y)$ are differentiable*[11] *at $z_0 = (x_0, y_0)$ and (2) they satisfy the Cauchy-Riemann conditions at $z_0 = (x_0, y_0)$.*"

(This result is usually proven in a course of complex analysis. It says that differentiability and the Cauchy-Riemann conditions imply the existence of complex derivative and vice-versa. Even though its proof is not hard, we omit it here for the sake of brevity but make a note of this result. Also, see and compare with **Problem 1.3.14**.)

For keeping the exposition simpler, we assume for sufficiency that $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann conditions and are of class $\mathfrak{C}^1$ in $\mathcal{D}$. The $\mathfrak{C}^1$ condition is a bit stronger than the simple differentiability, but we do not lose much under this assumption.

[11] A function $g : \mathcal{R} \longrightarrow \mathbb{R}^m$, where $\mathcal{R} \subseteq \mathbb{R}^n$ open, $m \in \mathbb{N}$ and $n \in \mathbb{N}$ is **differentiable at** $\vec{x}_0 \in \mathcal{R}$ if by definition there exists a matrix of real numbers $A_{m\times n} = (a_{ij})$ such that $\lim_{\vec{x} \to \vec{x}_0} \frac{g(\vec{x}) - g(\vec{x}_0) - A(\vec{x} - \vec{x}_0)}{\|\vec{x} - \vec{x}_0\|} = 0$. In such a case, the i^{th} row of $A_{m\times n}$ consists of the partial derivatives of the i^{th} component of g. However, it can happen that the partial derivatives of g at a point $\vec{x}_0$ exist but g is not differentiable at $\vec{x}_0$. But, if all the partial derivatives, except possibly one, are continuous, then g is differentiable. (Review this material again!)

Notice that for any holomorphic function $f(z)$ in $\mathcal{D} \subseteq \mathbb{C}$ we have

$$\left|\frac{df(z)}{dz}\right| = u_x^2 + v_x^2 = u_y^2 + v_y^2 = \text{etc.} = u_x v_y - u_y v_x = \frac{\partial(u,v)}{\partial(x,y)} \geq 0$$

and so the Jacobian determinant of the transformation $T : D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$: $(x,y) \longrightarrow T(x,y) := [u(x,y), v(x,y)]$ is non-negative. ($D \simeq \mathcal{D}$.)

Even though the complex derivative is defined in a way analogous to the derivative of a real function of a real variable, the Cauchy-Riemann necessary conditions impose a strong restriction for the complex derivative to exist.[12] The complex derivative exists if and only if u and v are differentiable and satisfy the Cauchy-Riemann conditions.

The **Cauchy-Riemann equations or conditions expressed in polar coordinates**

$$(r,\theta) \longrightarrow z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$$

are very useful in many situations. We have:

If a complex function, with real and imaginary part expressed in polar coordinates,

$$f(z) = f\left(re^{i\theta}\right) = f\left\{r[\cos(\theta) + i\sin(\theta)]\right\} = u(r,\theta) + iv(r,\theta)$$

has complex derivative $f'(z)$, then

$$r\,u_r(r,\theta) = \ v_\theta(r,\theta) \quad \text{ and } \quad r\,v_r(r,\theta) = -u_\theta(r,\theta).$$

Also, **the derivative in polar coordinates** at a point

$$z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)$$

is computed to be

$$f'(z) = f'\left(re^{i\theta}\right) = e^{-i\theta}[u_r(r,\theta) + iv_r(r,\theta)] =$$
$$[\cos(\theta) - i\sin(\theta)] \cdot [u_r(r,\theta) + iv_r(r,\theta)] =$$
$$u_r(r,\theta)\cos(\theta) + v_r(r,\theta)\sin(\theta) + i[v_r(r,\theta)\cos(\theta) - u_r(r,\theta)\sin(\theta)], \ \text{ etc.}$$

[See and solve **Problem 1.3.8**. The polar form of the Cauchy-Riemann equations and the complex derivative is naturally useful when we work in **polar coordinates** (r,θ).]

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[12] This is because that in the complex field there are two linearly independent directions, the real and the imaginary, whereas in the real field there is only the real one. Then all linear combinations of 1 and i with real coefficients form the set of complex numbers, which turns out to be an algebraic field.

Most rules of complex derivatives are analogous to those we learn in calculus. For instance, using either the definition of the derivative $f'(z)$ as a limit, or expression **(1.4)** or **(1.5)**, or the expression in polar coordinates found above, or differentiating power series, we can directly prove the following rules:

1. If a function $w = f(z)$ has derivative at z_0, then it is continuous at z_0.
2. For any constant $c \in \mathbb{C}$, $\dfrac{d(c)}{dz} = 0$.
3. For any integer $n \in \mathbb{Z}$, $\dfrac{d(z^n)}{dz} = nz^{n-1}$.
4. For any complex $z \in \mathbb{C}$, $\dfrac{d(e^z)}{dz} = e^z$.
5. **Sum and/or difference rule**: Provided the derivatives involved exist, we have
$$\frac{d}{dz}[f(z) \pm g(z)] = \frac{d}{dz}[f(z)] \pm \frac{d}{dz}[g(z)] = f'(z) \pm g'(z).$$
6. **Factoring a multiplicative constant rule**: Provided the derivatives involved exist, we have for any complex constant c
$$\frac{d}{dz}[c \cdot f(z)] = c \cdot \frac{d}{dz}[f(z)] = c \cdot f'(z).$$
7. **Product rule**: Provided the derivatives involved exist, we have
$$\frac{d}{dz}[f(z) \cdot g(z)] =$$
$$\frac{d}{dz}[f(z)] \cdot g(z) + f(z) \cdot \frac{d}{dz}[g(z)] = f'(z) \cdot g(z) + f(z) \cdot g'(z).$$
8. **Reciprocal rule**: Provided the derivatives involved exist, we have
$$\frac{d}{dz}\left[\frac{1}{g(z)}\right] = \frac{-\dfrac{d}{dz}[g(z)]}{g^2(z)} = \frac{-g'(z)}{g^2(z)}.$$

9. **Quotient rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] =$$
$$\frac{\frac{d}{dz}[f(z)] \cdot g(z) - f(z) \cdot \frac{d}{dz}[g(z)]}{g^2(z)} = \frac{f'(z) \cdot g(z) - f(z) \cdot g'(z)}{g^2(z)}.$$

10. **Chain rule**: Provided the derivatives involved exist, and if we set $w = g(z)$ we have

$$\frac{d}{dz}[f \circ g](z) = \frac{d}{dw}[f(w)] \cdot \frac{d}{dz}[g(z)] = [f' \circ g](z) \cdot g'(z) = f'[g(z)] \cdot g'(z).$$

11. **Inverse function rule**: Provided the derivatives involved exist, we have

$$\frac{d}{dz}[f^{-1}(z)] = \frac{1}{\frac{df}{dz}[f^{-1}(z)]} = \frac{1}{[f' \circ f^{-1}](z)} = \frac{1}{f'[f^{-1}(z)]}.$$

12.
$$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z).$$

13.
$$\frac{d}{dz}\sinh(z) = \cosh(z) \quad \text{and} \quad \frac{d}{dz}\cosh(z) = \sinh(z).$$

14. For the derivatives of all six **complex trigonometric** and all six **complex hyperbolic functions** see **Problems 1.3.15** and **1.3.16**.

15. For the derivatives of the **complex logarithm**, the **complex power** and the **inverses** of the **complex trigonometric** and **complex hyperbolic functions**, we must study first **Subsections 1.5.3, 1.5.5** and **1.5.6** and then find the inverses and compute their derivatives.

Remember that if $f'(c)$ exists, then $f(z)$ is continuous at $c \in \mathbb{C}$. The proof is analogous to the one in calculus. (Have a look at it and write one for this case.) The converse is not true.

Example 1.3.4 Consider the function $f(z) = \overline{z}$. This is continuous at every point of the complex plane $\mathbb{C}$. (Prove this as an exercise.) But,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h\to 0} \frac{\overline{h}}{h} = \text{does not exist.}$$

(E.g., when $h = s + 0i$ this limit is 1, and when $h = 0 + ti$ it is -1.)

Therefore, the simple complex continuous function $f(z) = \overline{z}$ does not have complex derivative.

We easily observe that the Cauchy-Riemann necessary conditions are not satisfied. (Check this!)

So, by the chain rule stated above, all complex functions that involve $\overline{z}$ in a non-trivial way do not have complex derivatives.

▲

Problems

1.3.1 (a) If a complex power series $\sum_{n=0}^{\infty} b_n(z-c)^n$, with center $c \in \mathbb{C}$, has radius of convergence $R > 0$, then prove that its derivative

$$\sum_{n=0}^{\infty} nb_n(z-c)^{n-1} = \sum_{n=1}^{\infty} nb_n(z-c)^{n-1}$$

and its indefinite integral $C + \sum_{n=0}^{\infty} b_n \frac{(z-c)^{n+1}}{n+1}$ have also radius of convergence R.

(b) What is the radius of convergence of the second derivative

$$\sum_{n=0}^{\infty} n(n-1)b_n(z-c)^{n-2} = \sum_{n=2}^{\infty} n(n-1)b_n(z-c)^{n-2}, \text{ etc.,}$$

of the complex power series in **(a)**?

1.3.2 Consider $z = x + iy$ and the complex function

$$f(z) = \frac{\operatorname{Im}(z^2)}{z\overline{z}}.$$

Prove that $\lim_{z=x+iy \to 0+i0} f(z)$ does not exist.

1.3.3 Prove that the limits $\lim_{z=x+iy \to 0+i0} e^{\frac{1}{z}}$ and $\lim_{z \to \infty} e^z$ do not exist.

1.3.4 Consider the exponential function of the variable $z = x + iy$

$$f(z) = e^z = e^x \cos(y) + ie^x \sin(y).$$

(a) What function is defined by f when $y = 0$, that is, $z = x + i0 = x$ is real?

(b) Prove $f'(z) = e^z$.

[Hint: Use the formula of $f'(z)$ found in equation **(1.4)** or **(1.5)**.]

1.3.5 Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Prove that $\mathrm{Re}[f(z)]$ or $\mathrm{Im}[f(z)]$ is constant if and only if $f(z)$ is constant.

1.3.6 If for a complex function $f(z)$ the $f'(c)$ exists at $c \in \mathbb{C}$, prove that $f(z)$ is continuous at $c \in \mathbb{C}$.

The converse is not true. Give a counterexample!

1.3.7 (a) Prove that $f'(z) = 0$ in an open connected or path connected (see **Definition 1.4.2** in the next section) region[13] $\mathcal{R} \subseteq \mathbb{C}$ if and only if $f(z) \equiv c$ in $\mathcal{R}$, for some constant $c \in \mathbb{C}$.

(b) Give an example of a complex function $f(z)$ such that $f'(z) = 0$ in an open region which is not connected (e.g., it consists of at least two open disjoint subsets) and $f(z)$ is not identically constant.

[Hint: In **(a)** use the formula of $f'(z)$ found in equation **(1.4)** or **(1.5)** and multi-variable calculus to show that the functions $u(x, y)$ and $v(x, y)$ are constant in $\mathcal{R}$. Or, use the chain rule with paths. See the beginning of the next section.]

1.3.8 (a) Prove that **in polar coordinates**

$$(r, \theta) \longrightarrow z = r\,[\cos(\theta) + i\sin(\theta)] = re^{i\theta}$$

the Cauchy-Riemann conditions or equations for a complex function written as

$$f(z) = f\left(re^{i\theta}\right) = f\left\{r\,[\cos(\theta) + i\sin(\theta)]\right\} = u(r, \theta) + i\,v(r, \theta)$$

for which $f'(z)$ exists become

$$ru_r = v_\theta \quad \text{and} \quad r\,v_r = -u_\theta.$$

[Hint: Use the relations

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right) + (\text{appropriate constant})$$

[13] In order to avoid too much topology and too many topological definitions in topological and/or metric spaces, you may think of a **connected set** intuitively as a set in the plane which consists of one piece. More accurately and for our purposes in this text, we can consider it to be a set in $\mathbb{C}$ in which we can join any two points in it by a continuous path lying entirely in the set. We also call such a set a **path connected set**. See **Definition 1.4.2** in the next section.

(valid locally). Then prove and use the differential operators

$$\frac{\partial}{\partial x} = \frac{x}{r} \cdot \frac{\partial}{\partial r} - \frac{y}{r^2} \cdot \frac{\partial}{\partial \theta} = \cos(\theta) \cdot \frac{\partial}{\partial r} - \frac{\sin(\theta)}{r} \cdot \frac{\partial}{\partial \theta},$$
$$\frac{\partial}{\partial y} = \frac{y}{r} \cdot \frac{\partial}{\partial r} + \frac{x}{r^2} \cdot \frac{\partial}{\partial \theta} = \sin(\theta) \cdot \frac{\partial}{\partial r} + \frac{\cos(\theta)}{r} \cdot \frac{\partial}{\partial \theta}.$$

(Simplify your computations at each step.)

Or by taking partial derivatives of $f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + i\,v(r,\theta)$ with respect to r and θ, prove

$$f'(z) = \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) e^{-i\theta} = \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)\left(-ie^{-i\theta}\right),$$

and then derive the Cauchy-Riemann conditions in polar coordinates.]

(b) Prove that at a point $z = re^{i\theta} = r[\cos(\theta) + i\sin(\theta)]$, the complex derivative of a complex function $f(z)$ in polar coordinates is given by

$$f'(z) = f'\left(re^{i\theta}\right) = e^{-i\theta}[u_r(r,\theta) + iv_r(r,\theta)] =$$
$$[\cos(\theta) - i\sin(\theta)] \cdot [u_r(r,\theta) + iv_r(r,\theta)] =$$
$$u_r(r,\theta)\cos(\theta) + v_r(r,\theta)\sin(\theta) + i[v_r(r,\theta)\cos(\theta) - u_r(r,\theta)\sin(\theta)], \text{ etc.}$$

(c) Verify the Cauchy-Riemann conditions in polar coordinates for the exponential function

$$e^z = e^{re^{i\theta}} = e^{r[\cos(\theta)+i\sin(\theta)]} = e^{r\cos(\theta)}\{\cos[r\sin(\theta)] + i\sin[r\sin(\theta)]\}.$$

(d) Apply the rule **(b)** of derivatives in polar coordinates to show $\left(z^2\right)' = 2z$ and $\left(\dfrac{1}{z}\right)' = \dfrac{-1}{z^2}$.

1.3.9 Consider the function $z^2 = x^2 - y^2 + i2xy$. Show that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ satisfy the Cauchy-Riemann equations. Check the same thing for z^3.

1.3.10 The function $f(z) = \mathrm{Re}(z) = x$ is a real analytic and therefore continuous function. Show that it is not a holomorphic function.

Show the same thing for the following functions:
$g(z) = z\bar{z}$, $h(z) = \bar{z}$, $p(z) = |z|$, and $q(z) = \mathrm{Im}(z) = y$.

[Hint: Check the Cauchy-Riemann conditions.]

1.3.11 (a) Use the Cauchy-Riemann equations to show that if $z = x+iy$ and $f(z) = u(x,y) + iv(x,y)$ is holomorphic in an open set $\mathcal{D} \subseteq \mathbb{C}$, then the two functions $u(x,y)$ and $v(x,y)$ are **harmonic** in $\mathcal{D}$, or else each of them satisfies the **Laplace equation**

$$\Delta w := \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

in $\mathcal{D}$, provided that the functions u and v are of class $\mathfrak{C}^2$, i.e., they are twice continuously differentiable.

(b) Next, under the same conditions, prove that the function $g(x,y) := |f(z)|^2 = u^2(x,y) + v^2(x,y)$ satisfies the relation

$$\Delta g = 2\left|\vec{\nabla}(u)\right|^2 + 2\left|\vec{\nabla}(v)\right|^2 = 4\left|\vec{\nabla}(u)\right|^2 = 4\left|\vec{\nabla}(v)\right|^2 \geq 0,$$

where $\vec{\nabla}(u)$ is the gradient of u, etc.

(c) Then prove:

$$\Delta g \equiv 0 \quad \text{in} \quad \mathcal{D} \quad \Longleftrightarrow \quad f \quad \text{is constant in} \quad D.$$

1.3.12 Prove that the Laplacian of a real sufficiently differentiable function $w = w(x,y)$ of two variables x and y, written in polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, has the form

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

[Hint: Use the relations and the operators in the hint in **Problem 1.3.8**, along with the chain rule on the function $w(x,y)$. Simplify completely at each step.]

1.3.13 Let $\mathcal{R} \subseteq \mathbb{C}$ be a region and $f(z) = u(x,y) + iv(x,y) \,:\, \mathcal{R} \to \mathbb{C}$ holomorphic ($z = x + iy$). Prove that $\forall\ r \in \mathbb{R}$:

$$\text{(a)} \qquad \frac{\partial^2\left[|f(z)|^r\right]}{\partial x^2} + \frac{\partial^2\left[|f(z)|^r\right]}{\partial y^2} = r^2|f(z)|^{r-2}|f'(z)|^2.$$

$$\text{(b)} \qquad \frac{\partial^2\left[|u(x,y)|^r\right]}{\partial x^2} + \frac{\partial^2\left[|u(x,y)|^r\right]}{\partial y^2} = r(r-1)|u(x,y)|^{r-2}|f'(z)|^2.$$

(c) For $v(x,y)$ write and prove the equation as the one in **(b)**.

1.3.14 Consider the complex function

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & \text{if} \quad z \neq 0, \\[2ex] 0, & \text{if} \quad z = 0. \end{cases}$$

Then check the following nine statements:

(a) With $z = x + iy$, write $f(z)$ explicitly in the form

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

(b) Prove directly that for every $z \in \mathbb{C}$, $f(z)$ is continuous but does not have complex derivative at z.

(c) Prove that both $u(x, y)$ and $v(x, y)$ have partial derivatives at every $(x, y) \in \mathbb{R}^2$.

(d) Prove that at the point $(x_0, y_0) = (0, 0)$ the partial derivatives of $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann conditions.

(e) Prove that both partial derivatives of both $u(x, y)$ and $v(x, y)$ are discontinuous at $(x_0, y_0) = (0, 0)$.

(f) Prove directly that both functions $u(x, y)$ and $v(x, y)$ are not differentiable at $(x_0, y_0) = (0, 0)$.

(g) Prove directly that both functions $u(x, y)$ and $v(x, y)$ do not satisfy the Cauchy-Riemann conditions at every point $(x, y) \neq (0, 0)$.

(h) Prove directly that both functions $u(x, y)$ and $v(x, y)$ have continuous partial derivatives at every point $(x, y) \neq (0, 0)$, and therefore they are differentiable at such a point.

1.3.15 Prove the derivatives of the complex trigonometric functions

$$\frac{d\,[\sin(z)]}{dz} = \cos(z), \qquad \frac{d\,[\cos(z)]}{dz} = -\sin(z),$$

$$\frac{d\,[\tan(z)]}{dz} = \sec^2(z), \qquad \frac{d\,[\cot(z)]}{dz} = -\csc^2(z),$$

$$\frac{d\,[\sec(z)]}{dz} = \sec(z)\tan(z), \qquad \frac{d\,[\csc(z)]}{dz} = -\csc(z)\cot(z).$$

1.3.16 Prove the derivatives of the complex hyperbolic functions

$$\frac{d\,[\sinh(z)]}{dz} = \cosh(z), \qquad \frac{d\,[\cosh(z)]}{dz} = \sinh(z),$$

$$\frac{d\,[\tanh(z)]}{dz} = \mathrm{sech}^2(z), \qquad \frac{d\,[\coth(z)]}{dz} = -\mathrm{csch}^2(z),$$

$$\frac{d\,[\mathrm{sech}(z)]}{dz} = -\mathrm{sech}(z)\tanh(z), \qquad \frac{d\,[\mathrm{csch}(z)]}{dz} = -\mathrm{csch}(z)\coth(z).$$

1.3.17 Prove that the function

$$f(z) = \begin{cases} e^{\frac{-1}{z^4}}, & \text{if } z \neq 0, \\ \\ 0, & \text{if } z = 0, \end{cases}$$

satisfies the Cauchy-Riemann equations at every $z \in \mathbb{C}$ (even at $z = 0$), but does not have complex derivative or differential at $z = 0$.

1.4 Line Integrals in the Complex Plane

Definition 1.4.1 *A **complex path** (or continuous complex path) in the complex plane $\mathbb{C}$ is a function $C : [a, b] \to R \subseteq \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a bounded closed interval of the real numbers, defined by*

$$C(t) = z(t) = x(t) + iy(t),$$

such that $z(t)$ is continuous, i.e., both $x(t)$ and $y(t)$ are continuous, at every $t \in [a, b]$. (At a and b, we consider the appropriate side continuities.)

If the derivative

$$C'(t) = z'(t) = x'(t) + iy'(t)$$

*exists for all $t \in [a, b]$, we say that $C(t)$ is a **differentiable complex path** on $[a, b]$. (At a and b, we consider the appropriate side derivatives.)*

*If $C'(t)$ is continuous, i.e., $z'(t)$ is continuous, i.e., both $x'(t)$ and $y'(t)$ are continuous, then we say that $C(t)$ is a **continuously differentiable complex path**.*

Definition 1.4.2 *A non-empty set $\emptyset \neq \mathcal{U} \subseteq \mathbb{C}$ is called **path connected** if for any two points p and q in $\mathcal{U}$ there is a continuous path $C : [a, b] \to \mathcal{U} \subseteq \mathbb{C}$ such that $C(a) = p$, $C(b) = q$ and $C([a, b]) \subseteq \mathcal{U}$, that is, the whole path lies in the set $\mathcal{U}$.*

Important remark: In this definition, if the set $\mathcal{U}$ is open and path connected, then we can prove that the path $C(t)$ can be chosen to be a continuously differentiable complex path. This involves some standard arguments of general topology and analysis, which we skip here. Otherwise, we would have to write a whole chapter on the topological facts of the complex plane. In fact, any open and connected subset of the complex plane is path connected, and the paths joining any two points of it can be chosen to be not just continuous but also continuously differentiable. We can use this important fact freely in the sequel.

Now we consider a complex function

$$f : \mathcal{R} \longrightarrow \mathbb{C} \;\; : \;\; z = x + iy \in \mathcal{R} \longrightarrow w = f(z) = u(x, y) + iv(x, y)$$

of the complex variable $z = x + iy$ from a region $\mathcal{R} \subseteq \mathbb{C}$ into $\mathbb{C}$ and a complex path in $\mathcal{R}$

$$C : [a, b] \longrightarrow \mathcal{R} \subseteq \mathbb{C} \;\; : \;\; t \in [a, b] \longrightarrow C(t) = x(t) + iy(t) = z(t).$$

Therefore, the composition $(f \circ C)(t) = f[C(t)]$ is possible for all $t \in [a, b]$.

Next, we assume that $C'(t) = x'(t) + iy'(t) = z'(t)$ is continuous, i.e., C is a **continuously differentiable path**. Then, along this path we define the **complex line integral** as follows:

$$\int_C f(z)\, dz := \int_a^b f[C(t)]d[C(t)] =$$
$$\int_a^b f[x(t) + iy(t)] \cdot\, d[x(t) + iy(t)] =$$
$$\int_a^b f[x(t) + iy(t)] \cdot [x'(t) + iy'(t)]\, dt.$$

So, if $f(z) = u(x, y) + iv(x, y)$ is continuous over C and $C(t)$ is continuously differentiable, then by multiplying out we find

$$\int_C f(z)\, dz =$$
$$\int_a^b \{u[x(t), y(t)] + iv[x(t), y(t)]\}[x'(t) + iy'(t)]\, dt =$$
$$\int_a^b \{u[x(t), y(t)]x'(t) - v[x(t), y(t)]y'(t)\}\, dt +$$
$$i \int_a^b \{u[x(t), y(t)]y'(t) + v[x(t), y(t)]x'(t)\}\, dt.$$

(By the continuity of all functions involved here, both final integrals exist.)

If $C(t)$ is continuous but does not possess derivatives at finitely many values $t_i \in [a,b]$ for $i = 1, 2, 3, \ldots, n$, (usually due to corners), but $C'(t)$ is continuous at all other points t of $[a,b]$, we say that $C(t)$ is a **continuous and piecewise continuously differentiable path**. Suppose, without loss of generality, that $a = t_0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_n \leq b = t_{n+1}$. Then the complex line integral is naturally defined to be the sum of the integrals over the individual smooth pieces. That is,

$$\int\limits_C f(z)\,dz = \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} f[x(t) + iy(t)] \cdot [x'(t) + iy'(t)]\,dt.$$

If a path C is a **union of continuously differentiable paths** C_i for $i = 1, 2, 3, \ldots, n$ (successive or disjoint or both), i.e., $C = \bigcup\limits_{i=1}^{n} C_i$, then we simply call it a **piecewise continuously differentiable path**. In such a case, for any complex function $f(z)$ we have again

$$\int\limits_C f(z)dz = \sum_{i=1}^{n} \int\limits_{C_i} f(z)dz.$$

We evaluate this integral by parameterizing each C_i, etc.

This definition suggests to use the $+$ instead of the $\cup$, in this context. That is, we prefer to write $C = \sum_{i=1}^{n} C_i = C_1 + C_2 + \ldots + C_n$ instead of $C = \bigcup\limits_{i=1}^{n} C_i$. We are going to use this notation from now on.

Important Properties of the Complex Line Integral:

Here we state five properties of the complex line integrals that we are going to use in the material that follows. The proofs of the first four are like the proofs of the corresponding properties of line integrals, found in any book of multi-variable calculus, with some straightforward adjustments to our context. The fifth one is harder, and we prove it here.

(1) The rule of u-substitution with definite integrals

$$\int_a^b f[u(x)]u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

verifies that the value of $\int\limits_C f(z)\,dz$ **is independent of the parametrization** of the path C.

(2) The line integral switches sign if we switch direction of traversing the path. If two paths have exactly the same range in $\mathbb{C}$ but are traversed in opposite directions, then we call them **opposite paths**. If we denote one of them by C, then we denote its opposite by $-C$. Then

$$\int\limits_{-C} f(z)\,dz = -\int\limits_{C} f(z)\,dz$$

So, the direction of the motion along the path is significant for the sign of the value of the line integral, but not for its absolute value.

(3) For any complex path C and any complex functions $f(z)$ and $g(z)$, we have the usual **linearity property of the integral**

$$\int\limits_{C} (f+g)(z)\,dz = \int\limits_{C} f(z)\,dz + \int\limits_{C} g(z)\,dz.$$

One way to prove this is to perform the computations in both sides separately and see that they yield the same result.

(4) For any complex path C, any complex function $f(z)$ and any complex number $c = a + ib$, we have the usual **linearity property of the integral**

$$\int\limits_{C} (cf)(z)\,dz = c\int\limits_{C} f(z)\,dz.$$

Again, one way to prove this is to perform the computations in both sides separately and see that they yield the same result.

(5) For any complex path C and any complex function $f(z)$, we have the **important inequality**

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz|.$$

The **properties (1)-(4)** are straightforward and follow from the definitions and carrying out the computations. Also, **property (5)** is fairly easy in the real case, as it has been shown in the calculus courses. But, proving **property (5)** in the complex case becomes tricky. So, we present:

Proof of (5) It is sufficient to prove this for C a continuously differentiable path. That is,

$$C : [a,b] \longrightarrow R \subseteq \mathbb{C} \quad : \quad t \in [a,b] \longrightarrow C(t) = x(t) + iy(t) = z(t),$$

such that

$$C'(t) = x'(t) + iy'(t) = z'(t)$$

exists for all $t \in [a,b]$ and it is continuous. Then

$$\int_C f(z)\,dz = \int_a^b f[z(t)] \cdot\ d[z(t)] = \int_a^b f[z(t)] \cdot z'(t)\,dt.$$

We let

$$I = \int_C f(z)\,dz = \int_a^b f[z(t)] \cdot z'(t)\,dt$$

and

$$J = \int_C |f(z)|\ |dz| = \int_a^b |f[z(t)] \cdot z'(t)|\ dt,$$

where

$$|z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

and

$$d\,[z(t)] = z'(t)\,dt = [x'(t) + iy'(t)]\,dt.$$

Obviously, $0 \le |I| < \infty$, $0 \le J < \infty$ and we must prove $|I| \le J$. If $I = 0$, this is trivially valid.

If $I \neq 0$, we write $I = |I|e^{i\theta}$. [I.e., $\theta = \mathrm{Arg}(I)$.] We let $c = \dfrac{|I|}{I} = e^{-i\theta}$, a complex constant with $|c| = 1$. Then:

$$0 < \left| \int_a^b f[z(t)] \cdot z'(t)\,dt \right| = |I| =$$

$$cI = c \int_a^b f[z(t)] \cdot z'(t)\,dt = \int_a^b c \cdot f[z(t)] \cdot z'(t)\,dt.$$

We write

$$c \cdot f[z(t)] \cdot z'(t) = p(t) + iq(t)\,(= \text{real part} + i \cdot \text{imaginary part})\,.$$

Then, $0 < |I| = cI$ implies

$$0 < \left| \int_a^b f[z(t)] \cdot z'(t)\,dt \right| = \int_a^b cf[z(t)] \cdot z'(t)\,dt = \int_a^b p(t)\,dt + i \int_a^b q(t)\,dt.$$

Since this is positive real, we conclude

$$\int_a^b q(t)\,dt = 0.$$

Also, by the properties of the absolute value, we get

$$p(t) \leq |c \cdot f[z(t)] \cdot z'(t)| =$$
$$|c|\,|f[z(t)]|\,|z'(t)| = 1 \cdot |f[z(t)]|\,|z'(t)| = |f[z(t)]|\,|z'(t)|.$$

By the previous two observations and the inequality properties of real integrals, we eventually find

$$0 < |I| = \left| \int_a^b f[z(t)] \cdot z'(t)\,dt \right| = \int_a^b p(t)\,dt \leq$$
$$\int_a^b |f[z(t)]|\,|z'(t)|\,dt = \int\limits_C |f(z)|\,|dz| = J.$$

If $C = C_1 + C_2 + \ldots + C_n$ is a path with each piece C_i continuously differentiable, this inequality is valid on each C_i. Then we get

$$\left| \int_C f(z)dz \right| = \left| \sum_{i=1}^n \int_{C_i} f(z)dz \right| \leq \sum_{i=1}^n \left| \int_{C_i} f(z)dz \right| \leq$$
$$\sum_{i=1}^n \int_{C_i} |f(z)|\,|dz| = \int_C |f(z)|\,|dz|.$$

Examples

Example 1.4.1 Let

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

and C be the unit circle $\{z \in \mathbb{C} : \ |z| = 1\}$ traversed in the positive direction (counterclockwise). We indicate this by writing C^+. Then, a parametrization of C^+ is

$$z(t) = \cos(t) + i\sin(t) \quad \text{with} \quad 0 \leq t \leq 2\pi.$$

Hence,

$$dz(t) = de^{it} = d[\cos(t) + i\sin(t)] =$$
$$[-\sin(t) + i\cos(t)]\,dt = i[\cos(t) + i\sin(t)]\,dt = iz(t) = ie^{it}$$

and

$$f[z(t)] = \cos^2(t) - \sin^2(t) + i2\cos(t)\sin(t).$$

So,

$$\int_{C^+} f(z)\,dz = \int_0^{2\pi} \{[\cos^2(t) - \sin^2(t)][-\sin(t)] - 2\cos(t)\sin(t)\cos(t)\}\,dt +$$
$$i\int_0^{2\pi} \{[\cos^2(t) - \sin^2(t)]\cos(t) + 2\cos(t)\sin(t)[-\sin(t)]\}\,dt =$$
$$\int_0^{2\pi} [-\cos(2t)\sin(t) - \sin(2t)\cos(t)]\,dt +$$
$$i\int_0^{2\pi} [\cos(2t)\cos(t) - \sin(2t)\sin(t)]\,dt =$$
$$\int_0^{2\pi} [-\sin(2t+t)]\,dt + i\int_0^{2\pi} \cos(2t+t)\,dt =$$
$$\int_0^{2\pi} [-\sin(3t)]\,dt + i\int_0^{2\pi} \cos(3t)\,dt = 0 + i0 = 0.$$

Note: In general, we may have to reside in this lengthy way in terms of $x(t)$ and $y(t)$ in order to compute complex line integrals. This is especially the case when $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is not given as an explicit function of z and the path C is not given as a complex expression of the parameter t. Instead, either one is given as some function of x and y.

In this example, however, we observe that the contour is

$$C^+ = \left\{ z = z(t) = e^{it} \;\mid\; 0 \le t \le 2\pi \right\}.$$

Therefore, on the contour C^+ we have $f(z) = z^2 = \left(e^{it}\right)^2 = e^{2it}$. That is, both f *and* C^+ *have been explicitly expressed by the parameter* t. *Then the line integral can be computed faster*, as follows:

$$\int_{C^+} f(z)\,dz = \int_0^{2\pi} \left(e^{it}\right)^2\, de^{it} = \int_0^{2\pi} e^{2it} i e^{it}\,dt = i\int_0^{2\pi} e^{3it}\,dt =$$
$$\frac{i}{3i}\left[e^{3it}\right]_0^{2\pi} = \frac{1}{3}\left(e^{6\pi i} - e^0\right) = \frac{1}{3}(1-1) = 0.$$

From now on, any time we can do so, we will employ this shorter complex computation method to evaluate the complex line integral. The initial lengthier way will be used in all other cases.

▲

Example 1.4.2 Let C^+ be as before the positive unit circle with center the origin, and

$$\text{(a)} \quad f(z) = \frac{1}{z}, \qquad\qquad \text{(b)} \quad g(z) = \frac{1}{z^2}.$$

Then:

$$\text{(a)} \quad \int\limits_{C^+} f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{it}}\,de^{it} = \int_0^{2\pi} e^{-it} i e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

If we travel the path of the unit circle in the negative direction, indicated by the opposite path of C^+, i.e., $C^- = -C^+$, then we get

$$\int\limits_{C^-} f(z)\,dz = \int\limits_{C^-} \frac{1}{z}\,dz = -2\pi i.$$

$$\text{(b)} \qquad \int\limits_{C^+} g(z)\,dz = \int_0^{2\pi} e^{-2it} i e^{it}\,dt = i \int_0^{2\pi} e^{-it}\,dt =$$

$$\frac{i}{-i}\left[e^{-it}\right]_0^{2\pi} = -1\left(e^{-2\pi i} - e^0\right) = -(1-1) = 0.$$

Note: *Every complex line integral can be separated into real and imaginary parts to yield two real line integrals.* This strategy is very effective when computing real integrals. For instance, from these two examples, we get:

$$\text{(a)} \qquad \int\limits_{C^+} \frac{1}{z}\,dz = \int\limits_{C^+} \frac{xdx + ydy}{x^2 + y^2} + i \int\limits_{C^+} \frac{-ydx + xdy}{x^2 + y^2} = 0 + 2\pi i.$$

So, we find the following two real line integrals

$$\int\limits_{C^+} \frac{xdx + ydy}{x^2 + y^2} = 0 \qquad \text{and} \qquad \int\limits_{C^+} \frac{-ydx + xdy}{x^2 + y^2} = 2\pi.$$

Also,

$$\int\limits_{C^-} \frac{xdx + ydy}{x^2 + y^2} = -0 = 0 \qquad \text{and} \qquad \int\limits_{C^-} \frac{-ydx + xdy}{x^2 + y^2} = -2\pi.$$

Similarly,

$$\text{(b)} \int\limits_{C^+} \frac{1}{z^2}\,dz = \int\limits_{C^+} \frac{(x^2 - y^2)dx + 2xydy}{(x^2 + y^2)^2} + i \int\limits_{C^+} \frac{-2xydx + (x^2 - y^2)dy}{(x^2 + y^2)^2} = 0.$$

Then both real and imaginary parts are equal to zero, i.e.,

$$\int\limits_{C^\pm} \frac{(x^2 - y^2)dx + 2xydy}{(x^2 + y^2)^2} = 0 \qquad \text{and} \qquad \int\limits_{C^\pm} \frac{-2xydx + (x^2 - y^2)dy}{(x^2 + y^2)^2} = 0.$$

We can, of course, verify these results directly with the methods of calculus. However, this efficient technique to evaluate complicated real line integrals gives answers easily and quickly.

▲

Example 1.4.3 We consider the points $P = (-1, 1)$ and $Q = (1, 2)$ and the complex function $f(z) = f(x + iy) = x + y - ix^2$. We would like to compute

$$\int_F f(z)dz = \int_F (x + y - ix^2)\, d(x + iy) = \int_F (x + y - ix^2)\, (dx + idy)$$

where F is the straight segment PQ, moving from P to Q.

A straight segment like this, in complex analysis, is denoted by $[z_1, z_2]$, where $z_1 = -1+i$ is the complex number representing the initial point P and $z_2 = 1+2i$ is the complex number representing the terminal point Q.

Here, we have no choice but to integrate using the long method. The usual parametrization of a segment like F is

$$\begin{gathered} F = [z_1, z_2] = \{z = (1 - t)z_1 + tz_2 \mid 0 \le t \le 1\} = \\ \{z = (1 - t)(-1 + i) + t(1 + 2i) \mid 0 \le t \le 1\} = \\ \{z = (-1 + 2t) + i(1 + t) \mid 0 \le t \le 1\}. \end{gathered}$$

Then, along F, $x = -1 + 2t$, $y = 1 + t$, $x + y = 3t$, $x^2 = 1 - 4t + 4t^2$, $dz = (2 + i)dt$, and we find

$$\int_F f(z)\, dz = \int_F (x + y - ix^2)\, d(x + iy) =$$

$$\int_0^1 [3t - i(1 - 4t + 4t^2)](2 + i)dt = (2 + i)\left[\frac{3t^2}{2} - i\left(t - 2t^2 + \frac{4t^3}{3}\right)\right]_0^1 =$$

$$(2 + i)\left[\frac{3}{2} - i\left(1 - 2 + \frac{4}{3}\right) - (0 + i0)\right] = (2 + i)\left(\frac{3}{2} - \frac{i}{3}\right) = \frac{10}{3} + i\frac{5}{6}.$$

Now we take a different path L that joins P and Q and compute the path integral of the same function. We let $O = (0, 0)$ the origin, and we pick L to be the union of the two straight segments PO and OQ. We write

$$L = [z_1, 0 + i0] \cup [0 + i0 z_2] = [-1 + i, 0 + i0] \cup [0 + i0, 1 + 2i].$$

As we have already said, in this context we prefer to use the $+$ instead of the $\cup$. So, from now on, we are going to use the notation with the

+. That is, we prefer $L = [-1+i, 0+i0] + [0+i0, 1+2i]$ instead of $L = [-1+i, 0+i0] \cup [0+i0, 1+2i]$.

A parametrization of this new path L is

$$L = \{z = (t-1)(-1+i) \mid 0 \le t \le 1\} + \{z = (1+i2)t \mid 0 \le t \le 1\} = \{z = -t+1+i(t-1) \mid 0 \le t \le 1\} + \{z = t+i2t \mid 0 \le t \le 1\}.$$

Along this new path, we compute the integral

$$\int_L (x+y-ix^2)\,(dx+idy) =$$

$$\int_0^1 [0 - i(-t+1)^2]\,(-1+i)\,dt + \int_0^1 (t+2t) - it^2)\,(1+i2)\,dt =$$

$$(-1+i)(-i)\int_0^1 (t^2-2t+1)\,dt + (1+i2)\int_0^1 (3t-it^2)\,dt =$$

$$(1+i)\left[\frac{t^3}{3} - t^2 + t\right]_0^1 + (1+i2)\left[\frac{3t^2}{2} - i\frac{t^3}{3}\right]_0^1 =$$

$$= (1+i)\left[\frac{1}{3} - 1 + 1 - 0\right] + (1+i2)\left[\frac{3}{2} - \frac{i}{3} - 0\right] = \frac{5}{2} + i3.$$

We see that the two path integrals of the given function $f(z)$ are different, even though both paths start at P and terminate at Q.

Along the path opposite to F, written as $-F$ from Q to P, the line integral has the opposite value, i.e.,

$$\int_{-F} (x+y-ix^2)\,(dx+idy) = -\left(\frac{10}{3} + i\frac{5}{6}\right),$$

and similarly for $-L$.

▲

Example 1.4.4 Compute $\displaystyle\int_{C^+} \frac{1}{z}\,dz$, where C^+ is the circle of center -2 and radius 1, travelled in the positive direction.

A parametrization of the path C^+ is

$$z(t) = -2 + e^{it} \qquad \text{with} \qquad 0 \le t \le 2\pi.$$

So,

$$\int_{C^+} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{-2+e^{it}}\,d(-2+e^{it}) = i\int_0^{2\pi} \frac{e^{it}dt}{-2+e^{it}}.$$

To make the denominator of $\dfrac{e^{it}}{e^{it}-2}$ real, we multiply both numerator and denominator by the conjugate of the denominator, which is

$$\overline{e^{it}-2} = \overline{e^{it}} - \overline{2} = \overline{e^{it}} - 2 = e^{-it} - 2.$$

Then

$$\frac{e^{it}}{e^{it}-2} = \frac{1-2e^{it}}{1+4-2(e^{it}+e^{-it})} = \frac{1-2\cos(t)-i2\sin(t)}{5-4\cos(t)}.$$

So,

$$\int\limits_{C^+} \frac{1}{z}\,dz = 2\int_0^{2\pi} \frac{\sin(t)}{5-4\cos(t)}dt + i\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}dt.$$

We now compute both integrals by elementary methods. In the first integral, we use the substitution $u = \cos(t)$ to find

$$\int_0^{2\pi} \frac{\sin(t)}{5-4\cos(t)}dt = -\int_1^1 \frac{du}{5-4u} = 0.$$

The second integral can be written as

$$\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}\,dt = \frac{1}{2}\int_0^{2\pi} \frac{5-4\cos(t)-3}{5-4\cos(t)}\,dt =$$
$$\frac{1}{2}\int_0^{2\pi} \left[1 - \frac{3}{5-4\cos(t)}\right]\,dt = \pi - \frac{3}{2}\int_0^{2\pi} \frac{dt}{5-4\cos(t)}.$$

Now, in order to use the substitution $w = \tan\left(\dfrac{u}{2}\right)$, tangent of half angle, in this rational expression of $\cos(t)$ [and $\sin(t)$], we need the dummy variable to vary in $[-\pi, \pi]$, because tangent is undefined (discontinuous) at $\pm\dfrac{\pi}{2}$. (See **Example I 1.1.2**.) So, we let first $u = t - \pi$, and then using the substitution the last integral becomes

$$-\frac{3}{2}\int_{-\pi}^{\pi} \frac{du}{5+4\cos(u)} = -\frac{3}{2}\cdot\frac{2}{3}\left[\arctan\left[\frac{1}{3}\tan\left(\frac{u}{2}\right)\right]\right]_{-\pi}^{\pi} =$$
$$-\arctan(\infty) + \arctan(-\infty) = \frac{-\pi}{2} + \frac{-\pi}{2} = -\pi.$$

Therefore,

$$\int_0^{2\pi} \frac{1-2\cos(t)}{5-4\cos(t)}dt = \pi - \pi = 0 \quad \text{and finally} \quad \int\limits_{C^+} \frac{1}{z}\,dz = 0.$$

▲

Problems

1.4.1 Compute $I = \int\limits_C z\overline{z}\,dz$ along the piece of the parabola C: $y = x^2$, for $-2 \leq x \leq 4$.

1.4.2 Compute the four integrals:

$$I_1 = \int\limits_{[-i,i]} z\,dz, \quad I_2 = \int\limits_{[-i,i]} |z|\,dz, \quad I_3 = \int\limits_{[1+i,-2+3i]} \frac{1}{z}\,dz, \quad I_4 = \int\limits_{[1+i,-2+3i]} \frac{1}{\overline{z}}\,dz.$$

1.4.3 In many situations, we make a holomorphic change of the variable $z = x + iy$ as $z = z(w) = x(u, v) + iy(u, v)$, where $w = u + iv$. Prove that

$$dz = dx + idy = z'(w)(du + idv) = z'(w)dw.$$

1.4.4 With C^+ the circle of center 0 and radius 2 travelled in the positive direction, evaluate the following two integrals:

$$I_1 = \oint\limits_{C^+} z\overline{z}\,dz, \qquad I_2 = \oint\limits_{C^+} \frac{1}{\overline{z}}\,dz.$$

How much are these integrals if we travel the above circle in the negative direction?

1.5 Cauchy-Goursat Theorem and Consequences

1.5.1 Complex Preliminaries and Notation

The result 0 that we found in **Examples 1.4.1** and **1.4.4** were not accidents. We must notice and remember the following three conditions.

(1) The functions $w = z^2$ and $w = \dfrac{1}{z}$ were both defined at every point of the respective paths and at every point inside the regions enclosed by these paths.

(2) The paths were **closed**, that is, they had the same initial and terminal points and they were **simple**, that is, they had no self-intersections. So, each of them **enclosed a region** of the complex plane without ambiguity.

We call the region of the complex plane enclosed by a simple closed path **the set of its interior points** or **the inside of the simple closed path**. We call the unbounded region of the plane not enclosed by such a path the **set of its exterior points** or **the outside of the simple closed path**.

(3) We must observe that the functions $w = z^2$ and $w = \dfrac{1}{z}$ have complex derivatives $\dfrac{dw}{dz} = 2z$ and $\dfrac{dw}{dz} = \dfrac{-1}{z^2}$, respectively, which exist **at every point** of the corresponding paths and the enclosed regions. (As we see, for $w = \dfrac{1}{z}$ the troublesome point $z = 0$ is not in that region.) So, $w = z^2$ and $w = \dfrac{1}{z}$ are holomorphic. So, according to the most important fact of Complex Analysis that we are going to state right after, they are complex analytic, i.e., they are locally represented as convergent complex power series. Namely, for the **two examples**, we have:

(a) In $\{z \in \mathbb{C} \mid \ |z| \leq 1\}$,

$$f(z) = z^2 = 0 + 0z + 1z^2 + 0z^3 + \dots$$

is already in power series form.

(b) In $\{z \in \mathbb{C} \mid \ |z+2| \leq 1\}$, with the help of the geometric series, we have that

$$g(z) = \frac{1}{z} = \frac{1}{(z+2)-2} = \frac{-1}{2}\frac{1}{1-\left(\dfrac{z+2}{2}\right)} =$$

$$\frac{-1}{2}\sum_{n=0}^{\infty}\left(\frac{z+2}{2}\right)^n = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}(z+2)^n = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}[z-(-2)]^n.$$

In fact, this power series is convergent for all z's such that $\left|\dfrac{z+2}{2}\right| < 1$, that is, all z's in the open disc $D(-2,2) := \{z \ : \ |z+2| < 2\}$.

Also, $D(-2,2)$ purely contains the region of the closed disc $\overline{D(-2,1)} = \{z \ : \ |z+2| \leq 1\}$ enclosed by the given path.

Notice that $z = 0$ does not belong to the open disc $D(-2,2)$.

Now we state:

Theorem 1.5.1 ("Most Important Fact" of Complex Analysis) *Every complex function $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$ which* *is defined and holomorphic in an open region $\mathcal{R} \subseteq \mathbb{C}$, that is, the complex derivative $\dfrac{dw}{dz} = f'(z)$ exists at every point $z \in \mathcal{R}$, is locally expressed as a power series in the region $\mathcal{R}$. I.e., for every point $z_0 \in \mathcal{R}$ and every $r > 0$ such that the open disc $D(z_0, r) \subseteq \mathcal{R}$, there are complex coefficients a_0, a_1, a_2, a_3, $\ldots$ such that*

$$w = f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall\, z \in D(z_0, r) \subseteq \mathcal{R}.$$

Therefore, this power series with center z_0 and radius $r > 0$ converges, and it is equal to $f(z)$ at every point $z \in D(z_0, r) \subseteq \mathcal{R}$. (In particular, $f(z_0) = a_0$.)

Proof We can just use this well-known fact of Complex Analysis without proving it. Here, we simply provide a proof-sketch under the extra assumption that $u(x, y)$ and $v(x, y)$ are of class $\mathfrak{C}^2$, that is, twice continuously differentiable. (For details, see **Appendix 1.5.9** at the end of this section.)

We have seen (**Problem 1.3.11**) that if $w = f(z) = u(x, y) + iv(x, y)$ is holomorphic in $\mathcal{R}$, i.e., $f'(z)$ exists at every point of the region $\mathcal{R}$, then by the Cauchy-Riemann conditions, and under the assumption that $u(x, y)$ and $v(x, y)$ are twice continuously differentiable, both $u(x, y)$ and $v(x, y)$ are harmonic in $\mathcal{R}$. I.e.,

$$u_{xx} + u_{yy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = 0.$$

The solutions of these Laplace equations in discs can be obtained by the **Poisson integral formula (1.14)** (obtained in **Appendix 1.5.9** at the end of this section).

This formula proves that both $u(x, y)$ and $v(x, y)$ are analytic functions, that is, power series in x and y, in discs (proven in **Theorem 1.5.8**).

But the harmonic functions u and v are closely related by means of the Cauchy-Riemann conditions in such a way that the two power series that represent them in any given disc, $D(z_0, r)$ with $r > 0$, can be added together according to the rule $u(x, y) + iv(x, y)$ and then rewritten so that their sum is a power series in the complex variable z. This power series is equal to the complex holomorphic function

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

(proven in **Theorem 1.5.10**).

Hence, $w = f(z)$ is a complex power series in any disc $D(z_0, r) \subseteq \mathcal{R}$ and therefore complex analytic in $\mathcal{R}$.

■

Remark 1: We have just seen that this fundamental result of Complex Analysis can be established through the **solutions of the Laplace equation in discs** by the **Poisson integral formula**. Proving first the so called **Cauchy integral formula**, which we will see soon in **Subsection 1.5.8**, from the existence of $f'(z)$ and then obtaining the analyticity of $f(z)$ as a result, is the standard way followed in most advanced books of complex analysis. In this way, we avoid the extra condition that requires $u(x, y)$ and $v(x, y)$ to be twice continuously differentiable. In such a complex analysis book, this hypothesis is immediately obtained from the analyticity of $f(z)$.

This approach needs several preliminaries that we cannot develop at this level and thus establish the Cauchy integral formula from the existence of the $f'(z)$ before the analyticity of $f(z)$. At this level, we are content with this extra condition, for it is not very restrictive to applications and simplifies the approach to the subject matter and proofs involved a lot.

In this course for beginners, and our brief and oriented toward applications exposition, we follow Weierstraßś approach on this subject. Hence, from the **power series representation** we derive the **Cauchy integral formula** in **Subsection 1.5.8**. In more advanced courses of complex analysis, the classic approach is to establish the Cauchy integral formula first and then the power series representation. But, both approaches are essentially equivalent.

Our goal here is not to develop a rigorous and advanced complex analysis course. This would need many more mathematical prerequisites. As we have said, we want to expose to the novices the most important tools that complex analysis can provide in computing various proper and improper Riemann integrals.

Remark 2: For a power series, we observe

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = 0, \quad \forall\ z \in D(z_0, r) \iff$$
$$a_n = 0,\ \forall\ n \in \mathbb{N}_0.$$

Because the power series, claimed by **Theorem 1.5.1**, can be found by the Taylor series formula in the complex variable z, in exactly the same way as we have used it in calculus, we also called them **Taylor series**. That is,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad \forall\ z \in D(z_0, r),\ (r > 0).$$

Hence, the previous coefficients a_0, a_1, a_2, a_3, ... satisfy the relation

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \ \forall \ n = 0, 1, 2, 3, \dots \ .$$

The radius of convergence is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}},$$

provided that these limits exist.

[When the first limit exists, prove as an exercise that both exist and are equal. (See **Problem 1.2.4**.) Also, do not forget that in the real line we have the limit rules $\frac{1}{0^+} = +\infty$ and $\frac{1}{+\infty} = 0^+$.]

The power series converges for every z inside the disc $D(z_0, R)$. In fact, it converges absolutely. But, for z on the boundary of the disc,

$$\partial D(z_0, R) := \{z \mid z \in \mathbb{C} \ : \ |z - z_0| = R\},$$

the power series may converge or may diverge. In general, a boundary point must be checked individually.

Notice that for complex analytic or holomorphic functions, we do not need to check if Taylor's Remainder converges to zero as $n \to \infty$. We can straightly compute the power series, and then we must find its radius of convergence.

Many times there are various shortcuts for finding these complex power series. These depend on the case and already known results. For instance, to find the power series of the functions $\cos(z)$, $\sin(z)$, $\cosh(z)$, $\sinh(z)$, we can more easily use their definitions by means of the exponential function

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Another way is to use the power series of real functions and then replace the real variable with the complex variable, as we have seen in **Section 1.2**.

In the following list, we cite a few basic examples of power series along with their domain of convergence, representing some frequently used complex analytic functions. These power series have center $z_0 = 0$. When the center of the power series is $z_0 = 0$, we call the power series a **Maclaurin series**. This is a special case of the Taylor series. (Convergence at boundary points is checked point by point, in general!)

1. $\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots, \quad \forall\, z: \ |z| < 1.$

2. $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots + \dfrac{z^n}{n!} + \ldots, \qquad \forall\, z \in \mathbb{C}.$

3. $\cos(z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n}}{(2n)!} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \ldots + (-1)^n \dfrac{z^{2n}}{(2n)!} + \ldots,$
   $\forall\, z \in \mathbb{C}.$

4. $\sin(z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{(2n+1)!} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \ldots +$
   $(-1)^n \dfrac{z^{2n+1}}{(2n+1)!} + \ldots, \qquad \forall\, z \in \mathbb{C}.$

5. $\cosh(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!} = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \ldots + \dfrac{z^{2n}}{(2n)!} + \ldots, \quad \forall\, z \in \mathbb{C}.$

6. $\sinh(z) = \sum_{n=0}^{\infty} \dfrac{z^{2n+1}}{(2n+1)!} = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \ldots + \dfrac{z^{2n+1}}{(2n+1)!} + \ldots,$
   $\forall\, z \in \mathbb{C}.$

7. $\arccos(z) =$
   $\dfrac{\pi}{2} - \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \dfrac{z^{2n+1}}{2n+1} = \dfrac{\pi}{2} - \sum_{n=0}^{\infty} \binom{n+\frac{1}{2}}{n} \dfrac{z^{2n+1}}{2n+1} =$
   $\dfrac{\pi}{2} - \dfrac{z}{1} - \dfrac{1}{2} \cdot \dfrac{z^3}{3} - \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{z^5}{5} - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \dfrac{z^7}{7} - \ldots, \quad \forall\, z: \ |z| < 1.$

8. $\arcsin(z) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \dfrac{z^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \binom{n+\frac{1}{2}}{n} \dfrac{z^{2n+1}}{2n+1} =$
   $\dfrac{z}{1} + \dfrac{1}{2} \cdot \dfrac{z^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4} \cdot \dfrac{z^5}{5} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \dfrac{z^7}{7} + \ldots, \quad \forall\, z: \ |z| < 1.$

9. $\arctan(z) = \sum_{n=0}^{\infty} (-1)^n \dfrac{z^{2n+1}}{2n+1} = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \ldots,$
   $\forall\, z: \ |z| < 1.$

(10. For the very useful power series of **complex logarithms** and **complex binomial power series**, see, respectively, **Sections 1.5.3** and **1.5.5**, where we define the complex logarithms and the complex

powers.)

Now, we return to the previous discussion. In **Example 1.4.2, (a)**, the function $f(z) = \dfrac{1}{z}$ was integrated over the unit circle with center the origin, $C(0,1) = \{z \ : \ |z| = 1\}$, in the positive direction. The result of this line integral was found to be

$$\int_{C^+(0,1)} \frac{1}{z}\, dz = 2\pi i.$$

This path is simple, and closed, and encloses the open unit disc: $D(0,1) = \{z \ : \ |z| < 1\}$, which is inside the region $\mathcal{R} = D(0,2)$. But, at the origin $z = 0$, which is a point of this region $\mathcal{R}$, the function $f(z) = \dfrac{1}{z}$ is not defined, let alone having derivative. There is no power series representation of this function with center $z_0 = 0$.

Similarly, given any $r > 0$ and any $z_0 \in \mathbb{C}$, we let $C(z_0, r) = \{z \ : \ |z - z_0| = r\}$ be the circle of center z_0 and radius r. This path encloses the open disc $D(z_0, r) = \{z \ : \ |z - z_0| < r\}$, with center z_0 and radius r, which is inside the region $\mathcal{R} = D(0, 2r)$. Then, moving along this path in the positive direction, we obtain

$$\int_{C^+(z_0,r)} \frac{1}{z - z_0}\, dz = 2\pi i.$$

(Check this in an analogous way as before.)

But, in **Example 1.4.2, (b)**, we found that

$$\int_{C(0,1)} \frac{1}{z^2}\, dz = 0.$$

even though the function $g(z) = \dfrac{1}{z^2}$ is also not defined at the origin $z = 0$. Similarly, for any $r > 0$, we have

$$\int_{C^\pm(z_0,r)} \frac{1}{(z - z_0)^2}\, dz = 0,$$

We must examine carefully what is going on here. But before we do that, we need to introduce the following **notation**, which we are going to use from now on. Given any $r \geq 0, \ 0 \leq r_1 < r_2 \leq \infty$, and $z_0, \ z_1, \ z_2, \in \mathbb{C}$, we denote:

1. **Closed directed straight segment** in the direction from $z_1 \in \mathbb{C}$ to $z_2 \in \mathbb{C}$:
   $[z_1, z_2] = \{z \; : \; z = (1-t)z_1 + tz_2, \; 0 \le t \le 1\}$.
2. **Circle** with center z_0 and radius r:
   $C(z_0, r) = \{z \; : \; |z - z_0| = r\}$.
3. **Open disc** with center z_0 and radius $r > 0$:
   $D(z_0, r) = \{z \; : \; |z - z_0| < r\}$.
4. **Closed disc** with center z_0 and radius r:
   $\overline{D(z_0, r)} = \{z \; : \; |z - z_0| \le r\} = D(z_0, r) \cup C(z_0, r)$.
5. **Open center-punctured disc** with center z_0 and radius $r > 0$:
   $D^o(z_0, r) = D(z_0, r) - \{z_0\} = \{z \; : \; 0 < |z - z_0| < r\}$.
6. **Closed center-punctured disc** with center z_0 and radius $r > 0$:
   $\overline{D(z_0, r)}^{\,o} = \overline{D(z_0, r)} - \{z_0\} = \{z \; : \; 0 < |z - z_0| \le r\} =$
   $D^o(z_0, r) \cup C(z_0, r)$.
7. **Open annulus** with center z_0 and radii $r_1 < r_2$:
   $A(z_0, r_1, r_2) = \{z \; : \; r_1 < |z - z_0| < r_2\}$.
8. **Closed annulus** with center z_0 and radii $r_1 < r_2$:
   $\overline{A(z_0, r_1, r_2)} = \{z \; : \; r_1 \le |z - z_0| \le r_2\}$.
9. **Circle** with center z_0 and radius $r > 0$ travelled in the **positive direction**:
   $C^+(z_0, r) = \left\{z = z_0 + re^{it}, \; 0 \le t \le 2\pi\right\}$.
10. **Circle** with center z_0 and radius $r > 0$ travelled in the **negative direction**:
    $C^-(z_0, r) = \left\{z = z_0 + re^{-it}, \; 0 \le t \le 2\pi\right\}$.
11. **Complex line integral** of a complex function $w = f(z)$ along a **closed path** C, i.e., a path with initial and terminal points the same, in the positive or negative direction: $\oint\limits_{C^\pm} f(z)\, dz$.
    (For any path C, the opposite of C^+ is simply $C^- = -C^+$.)

1.5.2 Cauchy-Goursat Theorem

In the **previous part of this section**, we discussed the **Taylor and Maclaurin series**. We also saw **Theorem 1.5.1**, which we called the *"Most Important Fact" of Complex Analysis*. According to that, if $f(z)$

is defined and holomorphic in an open region $\mathcal{R} \subseteq \mathbb{C}$ and $D(z_0, r) \subseteq \mathcal{R}$, for some $r > 0$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ \forall\, z \in D(z_0, r),$$

i.e., $f(z)$ is a Taylor series in the open disc $D(z_0, r)$.

Hence, an **antiderivative** of $f(z)$ is

$$F(z) = c + \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n+1}, \ \ \forall\, z \in D(z_0, r),$$

where $c \in \mathbb{C}$ is a constant, that is, $F'(z) = f(z)$, since we can differentiate power series term by term in the disc of convergence.

Similarly, if in the same region we consider a function given by the series

$$g(z) = \sum_{n=2}^{\infty} b_n (z - z_0)^{-n}, \ \ \forall\, z \in D^o(z_0, r),$$

(we begin summation with $n = 2$ and not 1), then $g(z)$ has antiderivative

$$G(z) = c + \sum_{n=2}^{\infty} b_n \frac{(z - z_0)^{-n+1}}{-n+1}, \ \ \forall\, z \in D^o(z_0, r),$$

where $c \in \mathbb{C}$ is constant.

The function $F(z)$ is defined in every disc $D(z_0, r) \subseteq \mathcal{R}$ in which $f(z)$ is defined, and it is continuous in this disc. Similarly, $G(z)$ is defined in $D^o(z_0, r)$ and is continuous. So, along any circle $C(z_0, \rho)$ with $0 < \rho < r$ or in any open set $\mathcal{A}$ inside the $D^o(z_0, r)$ containing $C(z_0, \rho)$ [i.e., $C(z_0, \rho) \subset \mathcal{A}$ (open) $\subset D^o(z_0, r)$], the functions $F(z)$ and $G(z)$ are defined and continuous.

The existence of a continuous antiderivative is the **crucial fact** that applies in **Example 1.4.2, (b)**, in order to get $\displaystyle\int_{C(0,1)} \frac{1}{z^2}\, dz = 0$. That is, the function $g(z) = \dfrac{1}{z^2}$ has antiderivative the function $G(z) = \dfrac{-1}{z}$ defined and continuous in $D^o(0, 1)$.

In **Example 1.4.2, (a)**, however, we have that

$$\int_{C(0,1)^{\pm}} \frac{1}{z}\, dz = \pm 2\pi i.$$

The non-zero answer happened because, as we shall shortly see in **Subsection 1.5.3** about complex logarithms, the function $f(z) = \dfrac{1}{z}$ does not have antiderivative which is both defined and continuous everywhere in $D^o(0,1)$. In general, we have the following **important Theorem**:

Theorem 1.5.2 *Suppose $w = f(z)$ has a continuous antiderivative $F(z)$ in an open set $\mathcal{R} \subseteq \mathbb{C}$, and C is a continuous and piecewise differentiable path (simple or not simple) contained in $\mathcal{R}$ with a parametrization $z = z(t) : [a,b] \longrightarrow \mathcal{R}$ [and so joining the complex numbers (points) $z_a = z(a) \in \mathcal{R}$ and $z_b = z(b) \in \mathcal{R}$].*

Then,

$$\int_C f(z)\,dz = F(z_b) - F(z_a),$$

and so the path integral is independent of the path.

If the path C is closed, i.e., $z(a) = z(b)$, this integral is zero.

Proof By the stated hypotheses we have the following:
$F(z_a) = F[z(a)] = \lim_{t\to a^+} F[z(t)]$ and $\lim_{t\to b^-} F[z(t)] = F[z(b)] = F(z_b)$. Without loss of generality, we assume that C is continuous and differentiable, and so by the chain rule we have:

$$\forall\ t:\ a < t < b,\ \ \frac{d}{dt}\left\{F[z(t)]\right\} = \frac{d}{dz}\left\{F[z(t)]\right\}\frac{d}{dt}[z(t)] = f[z(t)]\,\frac{d}{dt}[z(t)].$$

So,

$$\int_C f(z)\,dz = \int_{a^+}^{b^-} f[z(t)]\,\frac{d}{dt}[z(t)]\,dt = \int_{a^+}^{b^-}\frac{d}{dt}F[z(t)]\,dt = [\,F[z(t)]\,]_{a^+}^{b^-} =$$

$$F[z(b^-)] - F[z(a^+)] = F[z(b)] - F[z(a)] = F(z_b) - F(z_a).$$

If C is closed, then $z_a = z(a) = z(b) = z_b$, and so this integral is $F(z_b) - F(z_a) = 0$.

■

A special case of **this general Theorem** is the following:

Corollary 1.5.1 *For any complex function of the form*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=2}^{\infty} b_n(z-z_0)^{-n},\ \ \forall\ z \in D^o(z_0,r)$$

(notice that the power with exponent $n = -1$ is missing) and any closed continuous and piecewise continuously differentiable path C inside $D^o(z_0,r)$, we have that

$$\int_C f(z)\,dz = 0.$$

Proof In the discussion before the **above Theorem**, we explained why such a function has a continuous antiderivative in the center-punctured disc $D^o(z_0, r)$, and so the result follows from the Theorem.

■

Remark: In **this Corollary**, if $b_n = 0,\ \forall\ n = 2,\ 3,\ 4,\ \dots$, then the function $f(z)$ is complex analytic (or holomorphic) in the whole disc, and we can replace the center-punctured disc $D^o(z_0, r)$ with the whole disc $D(z_0, r)$.

Example 1.5.1 The function $f(z) = z^2$ has as continuous antiderivatives the functions $F(z) = \dfrac{z^3}{3} + c$, where c is any constant, in the whole $\mathbb{C}$. So, for any path C joining the numbers $-3i$ and 9, we have

$$\int_C z^2\,dz = \int_{-3i}^{9} z^2\,dz = \left[\frac{z^3}{3}\right]_{-3i}^{9} = \frac{9^3}{3} - \frac{(-3i)^3}{3} = 243 - 9i.$$

Similarly, along any path that does not contain 0, i.e., in the open set $\mathbb{C} - \{0 + i0\}$, and joins $-3i$ and 9, we find

$$\int_{-3i}^{9} \frac{1}{z^2}\,dz = \left[-\frac{1}{z}\right]_{-3i}^{9} = \frac{-1}{9} - \frac{-1}{-3i} = \frac{-1+3i}{9}.$$

Also, along any path joining $-1 + 3i$ and $9 - 2i$, we find

$$\int_{-1+3i}^{9-2i} \sin(z)\,dz = [\cos(z)]_{-1+3i}^{9-2i} = \cos(9-2i) - \cos(-1+3i) =$$
$$\frac{e^{2+9i} + e^{-2-9i} - e^{-3-i} - e^{3+i}}{2}.$$

(See also **Example 1.5.4**.)

▲

Now we present a version of the fundamental Cauchy-Goursat Theorem that we will use in this text:

Theorem 1.5.3 (Cauchy-Goursat Theorem) [14] *Let $\mathcal{R} \subseteq \mathbb{C}$ be an open region and let $f : \mathcal{R} \longrightarrow \mathbb{C}$ be a complex* ***holomorphic*** *function [i.e., $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at every point*

[14] There are more general versions of this Theorem, sometimes called the Global Cauchy Theorem, in which the paths are not simple but homotopic to a point, or the domains have holes and the paths may contain holes in their interior, etc., but we do not need these generalizations in our goals here. The interested reader can consult the bibliography in complex analysis.

$z \in \mathcal{R}$]. Then for any simple closed path C in $\mathcal{R}$ for which all of its interior points are in $\mathcal{R}$, we have

$$\int_C f(z)\,dz = 0.$$

Proof We observe that if $\mathcal{R} = D(z_0, r)$ is an open disc for some center $z_0 \in \mathbb{C}$ and radius $r > 0$, then we have the power series representation of the holomorphic function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for some appropriate complex numbers $a_n,\ n = 0, 1, 2, \dots$. This has a continuous antiderivative in $\mathcal{R} = D(z_0, r)$ the power series

$$F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z - z_0)^{n+1},$$

and the result follows from the **previous general Theorem** or its **Corollary**.

So, we assume that $\mathcal{R} \neq D(z_0, r)$ (refer to **Figure 1.2**). In this general case and at this level, we can prove the Theorem under the assumption that $u(x,y)$ and $v(x,y)$ are of class $\mathfrak{C}^1$ (i.e., their first order partial derivatives exist and are continuous). This extra assumption is not restrictive for the usual applications.

We let $\mathcal{D}$ be the interior of C in $\mathcal{R}$. Then

$$\oint_{C^{\pm}} f(z)dz = \oint_{C^{\pm}} [u(x,y) + iv(x,y)](dx + idy) =$$

$$\oint_{C^{\pm}} (udx - vdy) + i \oint_{C^{\pm}} (vdx + udy).$$

Under all assumptions imposed here, we can apply the analytic form of Green's[15] Theorem in the region $\mathcal{D}$, as we see it in calculus.[16] So, we

---

[15] George Green, English mathematician, 1793-1841.

[16] **Green's Theorem:** Let $P(x,y)$ and $Q(x,y)$ be two real functions of two variables which are continuously differentiable at every point of a bounded region $\mathcal{G}$ and every point of its boundary $\partial\mathcal{G}$, in the plane $\mathbb{R}^2$. We also assume that the boundary $\partial\mathcal{G}$ is piecewise differentiable. Then:

$$\oint_{\partial\mathcal{G}^{\pm}} P\,dx + Q\,dy = \pm \int\!\!\int_{\mathcal{G}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

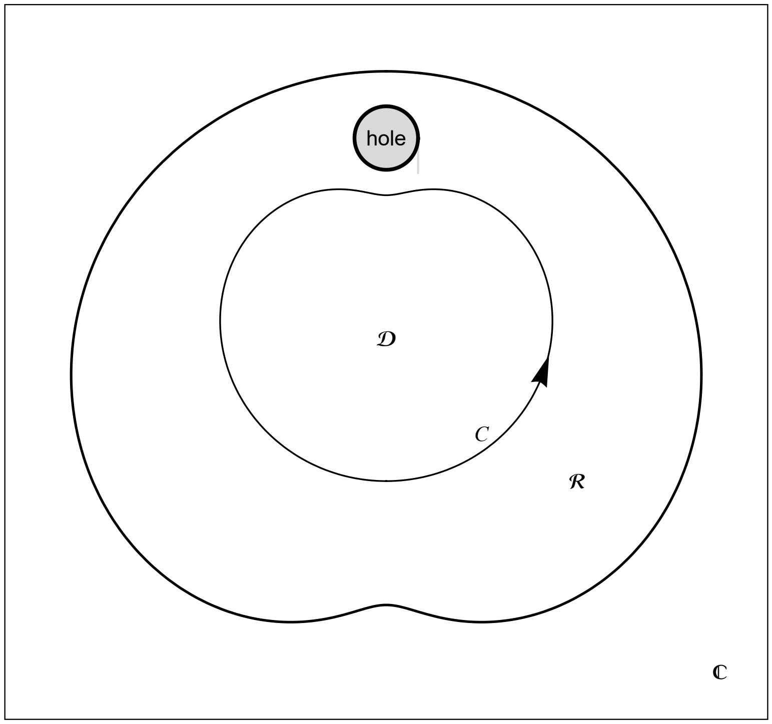

FIGURE 1.2: Region $\mathcal{R} \subseteq \mathbb{C}$, Contour C in $\mathcal{R}$ with interior $\mathcal{D}$

have

$$\oint_{C^{\pm}} f(z)dz = \pm \left[\int\int_{\mathcal{D}} (-v_x - u_y)dxdy + i\int\int_{\mathcal{D}} (u_x - v_y)dxdy \right].$$

But, the **Cauchy-Riemann conditions**, $u_x = v_y$ and $u_y = -v_x$, hold in $\mathcal{D} \cup C$ since $f'(z)$ exists at every $z \in \mathcal{D} \cup C$. Then,

$$\oint_{C^{\pm}} f(z)dz = \pm \left[\int\int_{\mathcal{D}} 0\, dxdy + i\int\int_{\mathcal{D}} 0\, dxdy \right] = 0 + i0 = 0,$$

and the proof is finished!

■

Remark 1: This result was known first to Gauß but was published first by Cauchy. A bit later, Goursat[17] relaxed some hypotheses and

[17] Edouard Jean-Baptiste Goursat, French mathematician, 1858-1936.

improved the proof that Cauchy had provided. His proof is somewhat involved but not difficult, and it is more general because it is based on the existence of $f'(z)$ only. Nothing is mentioned about $u(x,y)$ and $v(x,y)$ let alone to be of class $\mathfrak{C}^1$.

Remark 2: The **Cauchy-Goursat Theorem** claims that, under its hypotheses, the non-existence of holes in the interior of the simple closed path "contour" C is a sufficient condition to draw its conclusion.

However, this condition is not necessary because we have seen the example

$$\int_{C(0,1)} \frac{1}{z^2}\,dz = 0,$$

and in this example the origin forms a hole inside the unit circle $C(0,1)$, since the function $f(z) = \dfrac{1}{z^2}$ is not defined at $z = 0$.

In this example, we apply **Theorem 1.5.2** because

$$F(z) = \frac{-1}{z}$$

is a continuous antiderivative of $f(z) = \dfrac{1}{z^2}$ in $\overline{D}^{\,o}(0,1)$.

But, as we will see in the following subsection, we cannot claim and do the same things for the function $g(z) = \dfrac{1}{z}$ in $\overline{D}^{\,o}(0,1)$ for which we have seen that

$$\int_{C^+(0,1)} \frac{1}{z}\,dz = 2\pi i.$$

On the one hand, the **Cauchy-Goursat Theorem** does not apply here, and on the other hand, the function $g(z) = \dfrac{1}{z}$ does not have a continuous antiderivative in $\overline{D}^{\,o}(0,1)$, as we explain in the subsection that follows.

Notice that one condition of the **Cauchy-Goursat Theorem** was that all interior points of the simple closed path C belonged to the open region $\mathcal{R}$. This motivates the following definition:

Definition 1.5.1 *A region $\mathcal{R} \subseteq \mathbb{C}$ is called* ***simply connected*** *if for every simple closed path C lying in $\mathcal{R}$ the inside of C is a subset of $\mathcal{R}$.*

In view of this definition, we state an important consequence of the Cauchy-Goursat Theorem. We will not prove it, since under our approach in this text we will not use it. Its proof is not hard, and the interested reader can easily locate it in the bibliography.

Theorem 1.5.4 *Let $f : \mathcal{R} \longrightarrow \mathbb{C}$ be a holomorphic function in an open, connected and simply connected region $\mathcal{R} \subseteq \mathbb{C}$. We pick any $z_0 \in \mathcal{R}$, and for any $z \in \mathcal{R}$ we define*

$$F(z) = \int_C f(z)dz, \tag{1.6}$$

where C is any piecewise differentiable path lying in $\mathcal{R}$ and joining z_0 and z. Then $F(z)$ is a well-defined function, i.e., independent of the chosen path, and it is a continuous antiderivative of $f(z)$, i.e., $F'(z) = f(z)$.

Under these hypotheses, if C is any closed path inside the region $\mathcal{R}$, then

$$\int_C f(z)\,dz = 0.$$

The converse of the Cauchy-Goursat Theorem is also true. It is called Morera's Theorem, and we state it without its proof, which is not hard. It can be found with its proof in any good book of complex analysis. Many books choose for C's only triangles for a bit more generality.

Theorem 1.5.5 (Morera's Theorem) *Let $\mathcal{R} \subseteq \mathbb{C}$ be an open region in the complex plane and $f : \mathcal{R} \longrightarrow \mathbb{C}$ a complex continuous function defined on the open region $\mathcal{R}$, $[f(z) = f(x+iy) = u(x,y) + iv(x,y)]$.*

If *for any simple closed piecewise differentiable path C contained in $\mathcal{R}$, $(C \subset \mathcal{R})$, it holds that*

$$\int_C f(z)\,dz = 0,$$

then *the function $f(z)$ is* ***holomorphic in*** *$\mathcal{R}$. I.e., $f'(z)$ exists at every point $z \in \mathcal{R}$.*

[In fact, $f(z) = F'(z)$ and so $f'(z) = F''(z)$, where $F(z)$ is given by ***(1.6)*** *above. Since $F(z)$ is holomorphic, then any derivative of $F(z)$ is also holomorphic. Also, Morera's Theorem is usually stated with simple closed piecewise differentiable paths C being closed triangles only. But, the two statements are equivalent.]*

1.5.3 Complex Logarithm

Someone motivated by the fact $\frac{d}{dx}\ln(x) = \frac{1}{x}$ in the realm of the real functions may be tempted to claim that the function $f(z) = \frac{1}{z}$ has as a continuous antiderivative in any $D^o(0,r)$ with $r > 0$, the function

$$F(z) = \log(z)$$

that we will define in what follows, whereas we have seen in **Example 1.4.2, (a)**

$$\int_{C(0,1)} \frac{1}{z}\, dz = 2\pi i \neq 0.$$

Is this a contradiction to **Theorems 1.5.2** and **1.5.3**? The answer is no! In the sequel, we define and understand what we mean by

$$F(z) = \log(z)$$

in the complex plane and along any continuous closed path that encloses the origin in its interior. We shall see that this function is defined continuously only locally. That is, we cannot define the complex $\log(z)$ in any continuous manner everywhere in any $D^o(0, r)$ with $r > 0$ or in $\mathbb{C} - \{0\}$ for the same matter.

So, we must understand the definition of the **complex logarithmic function**. We remember that the trigonometric representation of complex numbers is $z = re^{i\theta}$, where $r = |z|$ and the angle θ is an argument of z. As we have seen in **Subsection 1.1.2, Parts (2)** and **(3)**, when $z \neq 0 \Leftrightarrow r > 0$, this angle is determined (mod 2π). To define and evaluate $\log(z)$ for any $z \neq 0$, we use the exponential form

$$z = e^{\ln r + i\theta},$$

where θ is determined (mod 2π).

We begin as follows: Given a $z \in \mathbb{C} - \{0\}$, we have $r = |z| > 0$. Then, using an argument θ and **Problem 1.2.12**, we find

$$z = re^{i\theta} = e^{\ln(r)} e^{i\theta} \cdot 1 = e^{\ln(r)} e^{i\theta} e^{i2k\pi} =$$
$$e^{\ln(r) + i\theta + i2k\pi} = e^{\ln(r) + i(\theta + 2k\pi)},$$

where $k \in \mathbb{Z}$. So, there are infinitely many values

$$\{w_k = \ln(r) + i(\theta + 2k\pi) \mid k \in \mathbb{Z}\},$$

such that $e^{w_k} = z$. In fact, these are precisely all the possible complex numbers $w = w_k$ for which $e^{w_k} = z$ (prove this claim!). So, here we give the definition:

Definition 1.5.2 *For any $z \in \mathbb{C} - \{0\}$* ***the complex logarithm*** $\log(z)$ *of z is defined to be* ***the set*** *of all complex values w such that $e^w = z$.* ***A complex logarithm*** *or* ***a branch of logarithm*** *of z is any particular value w_k taken out of the set of all possible values.*

Remarks on this definition:

1. For any $z \in \mathbb{C} - \{0\}$, we pick a principal argument $\theta = \mathrm{Arg}(z)$ in a fixed initial interval and we have

   $$w = \log(z) = \{w_k = \ln(r) + i(\theta + 2k\pi) \mid k = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots\},$$

   that is, the $\log(z)$ is the set of the complex numbers

   $$\log(z) =$$
   $$\ln(r) + i(\theta + 2\pi\mathbb{Z}) = \ln(r) + i\ \arg(z) = \ln(r) + i\,\mathrm{Arg}(z) + i\,2\pi\mathbb{Z}.$$

   In many concrete applications, we must designate and fix an initial interval for the principal arguments $\theta = \mathrm{Arg}(z)$.

   I.e., the complex $\log(z)$ has infinitely many values obtained by replacing $\theta[= \mathrm{Arg}(z)]$ with $\theta + 2k\pi$ with k an integer. This happens because e^z is not one-to-one in $\mathbb{C}$. We have seen that $e^{2k\pi i} = 1$ if and only if $k \in \mathbb{Z}$, and so for each $k \in \mathbb{Z}$ we get a different value in the set $\arg(z)$, i.e., $\arg(z)$ contains countably infinite values.

   We say that $w = \log(z)$ is a "**multi-value function**" with values the infinitely many complex numbers found above.

   In local coordinates, if $z = x + iy$, we can write

   $$\log(z) = \ln\left(\sqrt{x^2 + y^2}\right) + i\left[\arctan\left(\frac{y}{x}\right) + c + 2k\pi\right], \qquad (1.7)$$

   where $k \in \mathbb{Z}$ and c is a corrective constant to obtain the correct principal argument (as in **Example 1.1.3**).

2. Obviously, $\log(z)$ cannot be defined at the origin $z = 0$ because $e^z \neq 0$, $\forall\ z \in \mathbb{C}$. [Also, there is no $\ln(0)$.]

3. For any $z \in \mathbb{C} - \{0\}$, we have that

   $$e^{\log(z)} = e^w = \{e^{w_k} \mid k \in \mathbb{Z}\} = \{z\}.$$

   Hence, we simply write $e^{\log(z)} = z$.

4. For any $w \in \mathbb{C}$, we have that

   $$\log(e^w) = w + i2\pi\mathbb{Z} \supset \{w\}.$$

5. If we designate an initial interval $[a, a + 2\pi)$ or $(a, a + 2\pi]$, where $a \in \mathbb{R}$, for the principal argument $\mathrm{Arg}_0(z)$ [see **Subsection 1.1.2, Parts (2)** and **(3)**], then $\forall\ k \in \mathbb{Z}$, we write

   $$\log_{(k)}(z) = w_k = \ln(r) + i[\mathrm{Arg}_0(z) + 2k\pi]$$

[(k) is an index, in parentheses, so that it is not confused with a logarithmic base].

As in equation **(1.7)** above, in local coordinates $z = x + iy$, we write

$$\log_{(k)}(z) = \ln\left(\sqrt{x^2+y^2}\right) + i\left[\arctan\left(\frac{y}{x}\right) + c + 2k\pi\right], \ \forall \ k \in \mathbb{Z},$$

where c is an appropriate corrective constant.

So,

$$e^{\log_{(k)}(z)} = z, \ \forall \ k \in \mathbb{Z}.$$

We call all the different values $w_k = \log_{(k)}(z)$ of the complex $\log(z)$ the **branches of the complex logarithm**. Here, k is simply an index written as a subscript and should not be confused with some logarithmic base, which for the complex logarithm is always e.

Hence, when we work with the complex logarithm we must keep in mind whether we use all of its values as a set of numbers or one value of a particular branch.

Example 1.5.2 In calculus and real analysis, we learn $\ln(e) = 1$. That is, when we write $\ln(e)$, by definition, we mean the unique real answer such that $e^{\ln(e)} = e$.

However, in the general complex analysis $\log(e)$ is the following set of complex numbers:

$$\log(e) = \{\, \ln(e) + i(0 + 2k\pi) = 1 + i2k\pi \ | \ k \in \mathbb{Z}\,\} = 1 + i\,2\pi\mathbb{Z},$$

where we have designated as an initial interval of principal arguments the $[0, 2\pi)$, let us say.

For each $k \in \mathbb{Z}$, we have a branch of logarithm. For instance, if $k = 0$, we have $\log_{(0)}(e) = 1$. This is the real value that we learn in a calculus course. If $k = -10$, then $\log_{(-10)}(e) = 1 - i\,20\pi$, which is a complex number. Similarly, $\log_{(10)}(e) = 1 + i\,20\pi$.

▲

About continuity and discontinuity of the argument and the complex logarithm

The angles $\theta = \arg(z)$ are measured from the positive x-axis. Let us here designate the interval $(-\pi, \pi]$ to be the interval in which the principal argument $\mathrm{Arg}_0(z)$ varies. With these values of $\theta = \mathrm{Arg}_0(z)$, we can write any complex number $z \neq 0$ in trigonometric or exponential form uniquely. We observe that for $\theta = \pi$, we achieve all non-positive numbers in the complex plane $\mathbb{C}$.

Now, any negative number may be approached in many ways, but here we consider two: first by a sequence of complex numbers of the upper half plane, and second by a sequence of complex numbers of the lower half plane.

So, we consider any $a < 0$ and $z_n = r_n e^{i\theta_n}$ $n = 1, 2, 3, \ldots$ a sequence of complex numbers in the upper half plane such that $0 < \theta_n < \pi$, $n = 1, 2, 3, \ldots$ and $z_n \longrightarrow a$, as $n \longrightarrow \infty$. Then $r_n \longrightarrow |a|$ and $\theta_n \longrightarrow \pi$, as $n \longrightarrow \infty$. So, with an initial interval of $\mathrm{Arg}_0(z)$ the interval $(-\pi, \pi]$, we have

$$\log(z_n) = \ln(r_n) + i\theta_n \longrightarrow \ln(|a|) + i\pi, \text{ as } n \longrightarrow \infty.$$

We now consider $\zeta_n = \rho_n e^{i\phi_n}$ $n = 1, 2, 3, \ldots$ a sequence of complex numbers in the lower half plane such that $-\pi < \phi_n < 0$, $n = 1, 2, 3, \ldots$ and $\zeta_n \longrightarrow a$, as $n \longrightarrow \infty$. Then $\rho_n \longrightarrow |a|$ and $\phi_n \longrightarrow -\pi$, as $n \longrightarrow \infty$. So, with an initial interval of $\mathrm{Arg}_0(z)$ the interval $(-\pi, \pi]$, we have

$$\log(\zeta_n) = \ln(\rho_n) + i\phi_n \longrightarrow \ln(|a|) - i\pi, \text{ as } n \longrightarrow \infty.$$

Thus, even though

$$\lim_{n\to\infty} z_n = a \text{ and } \lim_{n\to\infty} \zeta_n = a,$$

i.e., z_n and ζ_n have the same limit as $n \longrightarrow \infty$, we have that

$$\lim_{n\to\infty} \log(z_n) = \ln(|a|) + i\pi \neq \ln(|a|) - i\pi = \lim_{n\to\infty} \log(\zeta_n).$$

Since the two limits are different, the $\lim_{z\to a} \log(z)$ does not exist. Therefore, the $\log(z)$ we have considered here cannot be extended continuously to any number $a \leq 0$. In other words, $\log(z)$ is not a continuous function in $\mathbb{C}$ and, in fact, is not continuous in any punctured disc $D^o(0, r)$, $r > 0$.

We see that apart from the origin at which $\log(z)$ cannot be defined in any way, $w = \log(z)$ has a jump, equal to $2\pi i$, at every point of the negative x-axis. This discontinuity is due to the discontinuities of the $\arg(z)$ along the non-positive x-axis, if we take the interval $(-\pi, \pi]$ to be the interval of the principal argument $\mathrm{Arg}_0(z)$.

So, $\arg(z)$ and hence $\log(z)$ are continuously defined in the region $\mathbb{C} - \{a \mid a \leq 0\} = \mathbb{C} - \{\text{non-positive } x\text{-axis}\}$ and not in the whole $\mathbb{C}$! This means that in any situation in which we need to use continuous $\arg(z)$ and $\log(z)$, we must exclude the non-positive x-axis from $\mathbb{C}$, and then the principal argument must vary in the interval $(-\pi, \pi)$ and not in $(-\pi, \pi]$. That is, a maximal domain in $\mathbb{C}$ in which the principal argument and the logarithm are continuous is not the whole $\mathbb{C}$ but the $\mathbb{C} - \{\text{non-positive } x\text{-axis}\}$.

Instead of the non-positive x-axis, we could have excluded from $\mathbb{C}$ any closed half-line starting at the origin

$$\{z = re^{i\phi} \mid r \geq 0 \text{ and } \phi \in \mathbb{R} \text{ constant}\}.$$

For instance, we can take away the non-negative x-axis by choosing the initial interval of continuous $\mathrm{Arg}_0(z)$ to be the interval $(0, 2\pi)$. (If we take the initial interval to be $[0, 2\pi)$, then we take into account the whole $\mathbb{C}$, and discontinuities occur along the non-negative x-axis.) Advanced theorems of complex analysis prove that we can also exclude other curves that start at the origin as well. But the half lines starting at the origin are good enough for the scope of this book and most applications.

With $\{a \mid a \leq 0\} = \{\text{non-positive } x\text{-axis}\}$ excluded from $\mathbb{C}$, we can define the function $w = \log(z)$ continuously in $\mathbb{C} - \{a \mid a \leq 0\} = \mathbb{C} - \{\text{non-positive } x\text{-axis}\}$, if $-\pi < \mathrm{Arg}_0(z) < \pi$. In this way, we achieve a continuous piece of the multi-value function $w = \log(z)$ out of the infinitely many pieces of the complex log. We call such a piece a **continuous branch of** $w = \log(z)$. As we have already said, we achieve continuity at the expense of losing the whole $\mathbb{C}$. For this continuous branch of the complex logarithm, the non-positive x-axis is called the **branch cut**. The origin is the common point of all these infinitely many branch cuts, and it is called **branch point** or **branching point** of the complex logarithm.

A chosen fixed initial branch of the logarithm is also called **principal branch of the complex logarithm**. For instance the continuous branch of $w = \log(z)$ in the domain $\mathbb{C} - \{a \mid a \leq 0\} = \mathbb{C} - \{\text{non-positive } x\text{-axis}\}$, can be chosen to be the principal branch, if we like so, but this is not necessary. (As we have said, we may chose a different branch cut, e.g., the non-negative x-axis, or any ray starting at the origin.) For this choice, $-\pi < \mathrm{Arg}_0(z) < \pi$ is the **principal argument of** $w = \log(z)$. (With other choices, we have other principal arguments.) At times we may need to close the (principal) argument in one of the end points only. Thus, we may consider $-\pi < \mathrm{Arg}_0(z) \leq \pi$, or $-\pi \leq \mathrm{Arg}_0(z) < \pi$.

Now, **the derivative of any continuous branch** (continuity is a necessary condition for the existence of derivative) of the complex function $w = \log(z)$ can be found by the chain rule and using the relations $e^{\log_{(k)}(z)} = z$, for any $k \in \mathbb{Z}$, and $\dfrac{d}{dz}(e^z) = e^z$. [Or, we can directly use the derivative rule for inverse functions. Or, we can locally use equations **(1.7)** and **(1.4)** or **(1.5)**.] With $\log(z)$ representing all the continuous branches $\{\log_{(k)}(z) \mid k \in \mathbb{Z}\}$, we find

$$\frac{d}{dz}[\log(z)] = \frac{1}{z}, \quad z \in \mathbb{C} - \{\text{the branch cut of } \log(z)\},$$

and so (by the chain rule)

$$\frac{d}{dz}\{\log[f(z)]\} = \frac{f'(z)}{f(z)}, \quad z \in \mathbb{C} - \{\text{the branch cut of } \log[f(z)]\}\,.$$

Next, by computing the derivatives of all orders of $f(z) = \log(1+z)$ at $z = 0$, we derive the following Maclaurin power series for the continuous branch of $f(z) = \log(1+z)$ for which $\log(1) = 0$:

$$\log(1+z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots, \ \ \forall\, z: \ |z| < 1.$$

This is the power series of the branch $w_0 = \log_{(0)}(1+z)$ (i.e., $k = 0$) if the $\mathrm{Arg}_0(z)$ varies in $(-\pi, \pi)$.

To obtain the power series of any branch $w_k = \log_{(k)}(1+z)$, $k \in \mathbb{Z}$, we must add the constant $2k\pi i$ to this power series.

The basic **properties of the complex logarithms** have been listed together with the related properties of the complex power functions in **Section 1.5.5**, in which we also explain the **adjustment of the argument** for both. Next, we present the following:

Example 1.5.3 Find all complex values of the complex logarithms $\log(1)$, $\log(5)$, $\log(-5)$ and $\log(5-2i)$, with initial arguments in $(-\pi, \pi]$.

Since [by **Part (a)** of **Problem 1.2.12**]

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) = 1 \iff \theta = 2k\pi \text{ with } k \text{ integer},$$

then we have that

$$\log(1) = \{2k\pi i \ | \ k \in \mathbb{Z}\}.$$

So, for $k = 0$, we find $\log_{(0)}(1) = \ln(1) = 0$ (as we know it from calculus).

Next, we have $\quad 5 = 5e^{i0}, \quad -5 = 5e^{i\pi}, \quad$ and

$$5 - 2i = \sqrt{5^2 + (-2)^2}\, e^{i\phi} = \sqrt{29}\, e^{i\phi},$$

with $\phi \in (-\pi, \pi]$ the unique IV quadrant angle determined by the two relations $\cos(\phi) = \dfrac{5}{\sqrt{29}}$ and $\sin(\phi) = \dfrac{-2}{\sqrt{29}}$. Therefore, $\phi = \arcsin\left(\dfrac{-2}{\sqrt{29}}\right) \in (-\pi, \pi]$.

So, we find

$$\log(5) = \{\ln(5) + i(2k\pi + 0) = \ln(5) + i\,2k\pi \ | \ k \in \mathbb{Z}\},$$

$$\log(-5) = \{\ln(5) + i(2k\pi + \pi) = \ln(5) + i(2k+1)\pi \ | \ k \in \mathbb{Z}\},$$

$$\log(5-2i) = \{\ln(\sqrt{29}) + i(2k\pi + \phi) = \frac{\ln(29)}{2} + i(2k\pi + \phi) \ | \ k \in \mathbb{Z}\}.$$

If all initial arguments were measured in $[0, 2\pi)$, then the answers would be:

$$\log(1) = \{2k\pi i \mid k \in \mathbb{Z}\},$$

$$\log(5) = \{\ln(5) + i(2k\pi + 0) = \ln(5) + i\,2k\pi \mid k \in \mathbb{Z}\},$$

$$\log(-5) = \{\ln(5) + i(2k\pi + \pi) = \ln(5) + i(2k+1)\pi \mid k \in \mathbb{Z}\},$$

$$\log(5-2i) = \{\ln(\sqrt{29}) + i(2k\pi + \psi) = \frac{\ln(29)}{2} + i(2k\pi + \psi) \mid k \in \mathbb{Z}\},$$

with $\psi \in [0, 2\pi)$ the unique IV quadrant angle determined by the two relations $\cos(\psi) = \dfrac{5}{\sqrt{29}}$ and $\sin(\psi) = \dfrac{-2}{\sqrt{29}}$. Therefore, $\psi = 2\pi + \arcsin\left(\dfrac{-2}{\sqrt{29}}\right) \in [0, 2\pi)$ in this case. (That is, the first three answers of both cases coincide, but the fourth one is different.)

▲

Example 1.5.4 In the **open right half plane**, i.e., $\mathrm{Re}(z) > 0$, the function

$$f(z) = \frac{z^3 + 5z^2 - 3z + 6}{z^2}$$

has as a continuous antiderivative the function

$$F(z) = \frac{z^2}{2} + 5z - 3\log_{(0)}(z) - \frac{6}{z},$$

where we have picked the principal branch of the complex logarithm $\log_{(0)}(z)$ with principal argument in $[-\pi, \pi)$.

So, for any path C lying in the open right half plane $[\mathrm{Re}(z) > 0]$ and joining the numbers $1+i$ and $2-5i$, by **Theorem 1.5.2**, we find

$$\int_C f(z)\,dz =$$

$$\int_{1+i}^{2-5i} \frac{z^3 + 5z^2 - 3z + 6}{z^2}\,dz =$$

$$\left[\frac{z^2}{2} + 5z - 3\log_{(0)}(z) - \frac{6}{z}\right]_{1+i}^{2-5i} =$$

$$\left[\frac{(2-5i)^2}{2} + 5(2-5i) - 3\log_{(0)}(2-5i) - \frac{6}{2-5i}\right] -$$

$$\left[\frac{(1+i)^2}{2} + 5(1+i) - 3\log_{(0)}(1+i) - \frac{6}{1+i}\right] =$$

$$= \left[\frac{-21-20i}{2} + 10 - 25i - 3\ln\left(\sqrt{29}\right) - 3\arcsin\left(\frac{-5}{\sqrt{29}}\right)i - \frac{12+30i}{29}\right] -$$

$$- \left[i + 5 + 5i - 3\ln\left(\sqrt{2}\right) - 3\frac{\pi}{4}\,i - 3 + 3i\right] =$$

$$= -\frac{169}{58} - \frac{3}{2}\ln\left(\frac{29}{2}\right) + i\left[\frac{3\pi}{4} + 3\arcsin\left(\frac{5\sqrt{29}}{29}\right) - \frac{1306}{29}\right].$$

But on the any circle $C = C^+(0,r)$, $r > 0$, positively oriented, we get

$$\int_{C^+(0,\,r)} f(z)\,dz = (-3)\cdot 2\pi i = -6\pi i.$$

▲

1.5.4 Series Deduced from the Logarithmic Series

Using the **principal argument Arg**, the principal branch of the complex logarithm of $1+z$ is given by

$$\mathrm{Log}(1+z) = \ln|1+z| + i\mathrm{Arg}(1+z)$$

and, as we have seen, it has power series expansion

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \ldots \quad \forall\ z \in \mathbb{C},\ : |z| < 1.$$

Then with

$$z = r[\cos(\theta) + i\sin(\theta)] = re^{i\theta},$$

we have

$$\mathrm{Log}(1+z) = \ln\sqrt{1 + 2r\cos(\theta) + r^2} + i\mathrm{Arg}(1+z),$$

where $-\pi < \mathrm{Arg}(1+z) < \pi$ with possibly either end point included. This $\mathrm{Arg}(1+z)$ is found by adjusting appropriately the

$$\arctan\left[\frac{r\sin(\theta)}{1+r\cos(\theta)}\right].$$

So, for $0 \le r < 1$ and $-\pi \le \theta \le \pi$, we have

$$\frac{1}{2}\ln\left[1 + 2r\cos(\theta) + r^2\right] + i\mathrm{Arg}(1+z) =$$
$$\sum_{n=1}^{\infty}(-1)^{n-1}r^n\frac{\cos(n\theta) + i\sin(n\theta)}{n},$$

and thus we obtain the following two infinite series:

$$\text{(I)} \qquad \frac{1}{2}\ln\left[1+2r\cos(\theta)+r^2\right] = \sum_{n=1}^{\infty}(-1)^{n-1}r^n\frac{\cos(n\theta)}{n},$$

and

$$\text{(II)} \qquad \operatorname{Arg}(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}r^n\frac{\sin(n\theta)}{n}.$$

Since $0 \le r < 1$ the convergence of both series is absolute and uniform by the **Weierstraß M-Test, I 2.3.3**. In both equations, both sides, as functions of θ, are 2π-periodic, and so these formulae are valid for all values of θ.

By convergence of the series and **Abel's Lemma**, these two formulae hold even if $r = 1$, provided that $\theta \neq \pm\pi \pmod{2\pi}$. Therefore, we obtain the new series:

$$\text{(I}'\text{)} \qquad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(n\theta)}{n} = \frac{1}{2}\ln\left[2+2\cos(\theta)\right] =$$

$$\ln\left[2\cos\left(\frac{\theta}{2}\right)\right] = \ln(2) + \ln\left[\cos\left(\frac{\theta}{2}\right)\right]$$

and

$$\text{(II}'\text{)} \qquad \operatorname{Arg}(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n\theta)}{n} =$$

$$\arctan\left[\frac{\sin(\theta)}{1+\cos(\theta)}\right] + k\pi = \arctan\left[\tan\left(\frac{\theta}{2}\right)\right] + k\pi = \frac{\theta}{2} + k\pi,$$

where $k \in \mathbb{Z}$ must be chosen so that $-\frac{\pi}{2} \le \frac{\theta}{2} + k\pi \le \frac{\pi}{2}$.

Putting $\theta = 0$ in the first series (I$'$), we find the well known **Brouncker series**

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots.$$

(See also **Example I 1.3.12**.)

Also, if $k = 0$ and $-\pi \le \theta \le \pi$, the second series (II$'$) yields the useful series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n\theta)}{n} = \frac{\theta}{2}.$$

Then, plugging $\theta = \frac{\pi}{2}$, we obtain the well known **Gregory-Leibniz**

formula for π[18]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots =$$
$$2\left(\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \ldots\right).$$

[See also **Problem 1.2.7, (b)** and **Example 1.7.24**.]

Next, we replace θ by 2θ in the cosine series (II$'$) above and integrate it term by term to

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{\sin(2n\theta)}{2n^2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\ln(2) + \int_0^{\frac{\pi}{2}} \ln[\cos(\theta)]d\theta.$$

(The term by term integration can be justified by, e.g., the **Beppo-Levi Theorem, I 2.3.14**.) Since the first side gives zero, we find the result of **Problem I 2.1.19,, (a), (b), (d)**.

$$\int_0^{\frac{\pi}{2}} \ln[\cos(\theta)]d\theta = -\frac{\pi}{2}\ln(2) = \int_0^{\frac{\pi}{2}} \ln[\sin(\theta)]d\theta.$$

Also, since $\cos(2n\theta) = 1 - 2\sin^2(n\theta) = 2\cos^2(n\theta) - 1$, then from the first series (I$'$), we find

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin^2(n\theta)}{n} = -\frac{1}{2}\ln[\cos(\theta)]$$

and

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2(n\theta)}{n} = \ln(2) + \frac{1}{2}\ln[\cos(\theta)].$$

(See also **Problems I 2.2.46** and **1.5.22**.)

If we now integrate formula (I) on $[0,\pi]$, we get

$$\int_0^{\pi} \ln\left[1 + 2r\cos(\theta) + r^2\right]d\theta =$$
$$2\sum_{n=1}^{\infty}(-1)^{n-1}r^n\int_0^{\pi}\frac{\cos(n\theta)}{n}d\theta = 2\sum_{n=1}^{\infty}0 = 2\cdot 0 = 0$$

and so obtain the result of **Problem I 2.2.46**, when $0 \le r \le 1$, and of **Example 1.7.50**.

[18]This formula was also found independently a few years before Leibniz by the Scottish mathematician James Gregory (also spelled, Gregorie), 1638-1675.

In general we have: If $0 \leq r \leq 1$

$$\int_a^b \ln\left[1 + 2r\cos(\theta) + r^2\right] d\theta = \; 2\sum_{n=1}^{\infty}(-1)^{n-1}r^n\left[\frac{\sin(n\theta)}{n^2}\right]_a^b =$$

$$2\sum_{n=1}^{\infty}(-1)^{n-1}r^n\left[\frac{\sin(nb)-\sin(na)}{n^2}\right].$$

Example 1.5.5 We multiply the series (I′)

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2n\theta)}{n} = \ln(2) + \ln[\cos(\theta)]$$

by θ and integrate term by term on $\left[0, \frac{\pi}{2}\right]$. (The term by term integration can be justified by, e.g., the **Beppo-Levi Theorem, I 2.3.14.**.) Then,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^{\frac{\pi}{2}}\theta\cdot\frac{\cos(2n\theta)}{n}\,d\theta = \int_0^{\frac{\pi}{2}}\ln(2)\,\theta\,d\theta + \int_0^{\frac{\pi}{2}}\theta\ln[\cos(\theta)]\,d\theta.$$

By elementary integration, we find

$$\sum_{k=1}^{\infty}\frac{-1}{2(2k-1)^3} = \frac{\pi^2}{8}\ln(2) + \int_0^{\frac{\pi}{2}}\theta\ln[\cos(\theta)]\,d\theta.$$

Then, using the result of **Problem 1.7.68**, we obtain

$$\int_0^{\frac{\pi}{2}}\theta\ln[\cos(\theta)]\,d\theta = -\frac{\pi^2}{8}\ln(2) - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} =$$

$$-\frac{\pi^2}{8}\ln(2) - \frac{7}{16}\sum_{n=1}^{\infty}\frac{1}{n^3} = -\frac{\pi^2}{8}\ln(2) - \frac{7}{16}\zeta(3).$$

Letting $\theta = \frac{\pi}{2} - u$ in this integral and using the fact that

$$\int_0^{\frac{\pi}{2}}\ln[\cos(\theta)]d\theta = -\frac{\pi}{2}\ln(2) = \int_0^{\frac{\pi}{2}}\ln[\sin(\theta)]d\theta,$$

[found before this example and in **Problem I 2.1.19, (a), (b), (d)**], we obtain the integral

$$\int_0^{\frac{\pi}{2}}u\ln[\sin(u)]\,du = -\frac{\pi^2}{8}\ln(2) + \frac{7}{16}\zeta(3).$$

(Compare also with **Problems I 2.1.21** and **1.5.22**. These results are used in the final result of **Example I 2.5.3**.)

▲

1.5.5 Complex Power Functions

We need to say a few words about the **multi-value complex power function**. Besides its own interest, we are going to use it in several cases in the sequel. This multi-value complex function is:

$$f(z) = z^{\alpha},$$

where, in general, $\alpha \in \mathbb{C}$ is constant and $z \in \mathbb{C}$ is the complex variable. This function is defined by means of the complex $\log(z)$ and the exponential function e^z in the following way:

If $z \neq 0$, we define

$$z^{\alpha} = e^{\alpha \log(z)} \overset{def}{=} e^{\alpha \ln(|z|) + i\alpha \arg(z)},$$

where, as we have already seen,

$$\arg(z) = \{\mathrm{Arg}_0(z) + 2k\pi \mid k \in \mathbb{Z}\} = \{\mathrm{Arg}_0(z)\} + 2\pi\mathbb{Z}$$

is a countable set of numbers. So, this function is **a multi-value complex function**, i.e., its output is **a set of numbers**, in general. We can choose particular values from this set as the continuous branches of the power function. In fact, we pick continuous branches of the complex logarithm, and then we find the corresponding continuous branches of the power.

In general, we may consider any domain $\mathbb{C} - \{\text{a ray from the origin}\}$, and so the principal argument $\mathrm{Arg}_0(z)$ is in an open interval $(\phi, 2\pi + \phi)$, where ϕ is some angle.

Example 1.5.6 In calculus and real analysis, we learn that $e^{\sqrt{2}}$ is a unique real value. However, in the general complex analysis, $e^{\sqrt{2}}$ viewed as a complex power, is the following set of numbers:

$$e^{\sqrt{2}} = \{\ e^{[\log(e)]\sqrt{2}}\ \} =$$
$$\{\, e^{[1+i(0+2k\pi)]\sqrt{2}} \ \mid\ k \in \mathbb{Z}\,\} = \{\, e^{(1+i2k\pi)\sqrt{2}} \ \mid\ k \in \mathbb{Z}\,\}.$$

For each $k \in \mathbb{Z}$, we have a branch of this complex power. For instance, if $k = 0$, we find the unique real answer $e^{\sqrt{2}}$, as we know it in calculus or real analysis. But, if, for instance, $k = -10$, then we find the complex number $e^{\sqrt{2}(1-i20\pi)} = e^{\sqrt{2}}[\,\cos(20\pi\sqrt{2}) - i\sin(20\pi\sqrt{2})\,]$ and so on.

Note: It would be appropriate to introduce notations indicating if the expression $e^{\sqrt{2}}$, and any expression a^b with a and b real numbers, is considered as a real power with a unique real number as answer, if such a real exists, or as a complex power with infinitely many answers, out of which we can pick particular branches which we can indicate with indices $k \in \mathbb{Z}$. But, in order to avoid too crowded notations, we are not going to introduce and use such notations. We must simply keep in mind when expanding complex powers, what we mean with the complex $\log(z)$ or $\log_{(k)}(z)$ with $k \in \mathbb{Z}$.

▲

It does not always hold that each branch of the complex logarithm will yield a different branch of the complex power. This depends on the exponent. For instance: As we have seen in **Section 3.1**, if $\alpha = n \in \mathbb{Z}$ (i.e., a **is an integer**), then $f(z) = z^n \left(= z^{\frac{n}{1}}\right)$ is a unique-value function. Indeed:

If $z = 0$ and $n \in \mathbb{N}$, then we have $0^n = 0 \cdot 0 \cdots 0 = 0$, by the original definition of a power with a natural exponent. Also, we continuously consider $0^0 = 1$, and therefore holomorphically, since $z^0 = 1$ for all $z \neq 0$. (For $n = -1,\ -2,\ \ldots,$ the power 0^n is not defined or set to be the complex ∞.)

If $z \neq 0$ and $n \in \mathbb{Z}$, then we obtain

$$
\begin{gathered}
f(z) = z^n = e^{n\log(z)} = e^{n[\ln(|z|)+i\arg(z)]} = e^{n\left[\ln(|z|)+i\mathrm{Arg}_0(z)+2k\pi i\right]} = \\
e^{n\ln(|z|)+i\,n\mathrm{Arg}_0(z)+2kn\pi i} = e^{n\ln(|z|)} \cdot e^{in\mathrm{Arg}_0(z)} \cdot e^{2kn\pi i} = \\
|z|^n \cdot \{\cos[n\mathrm{Arg}_0(z)] + i\sin[n\mathrm{Arg}_0(z)]\} \cdot 1 = \\
\{|z| \cdot \{\cos[\mathrm{Arg}_0(z)] + i\sin[\mathrm{Arg}_0(z)]\}\}^n = \\
\begin{cases}
z^n = z \cdot z \cdot z \cdots z, & \text{if} \quad n > 0, \\[2ex]
1, & \text{if} \quad n = 0, \\[2ex]
z^n = \frac{1}{z} \cdot \frac{1}{z} \cdot \frac{1}{z} \cdots \frac{1}{z}, & \text{if} \quad n < 0,
\end{cases}
\end{gathered}
$$

where the last two steps follow from the **De Moivre formula** [**Subsection 1.1.2**, **Part (5)**]. This answer coincides with the original definition of the power function with integer exponent $n \in \mathbb{Z}$ and base $z \neq 0$.

In general, if the exponent α **is not an integer**, then the function $f(z) = z^\alpha$ is a multi-value function. Indeed:

If $\alpha = \dfrac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\}$, i.e., α is **rational**, then $f(z) = z^\alpha$ has at most $|q|$ values. The $|q|$ $|q|^{\text{th}}$ roots of $z^{\mathrm{sign}(q)\cdot p}$ repeat as k runs $\mathbb{Z}$ in the expression $2k\,\pi\,i$. If p and q are relatively prime, then there are exactly $|q|$ different values. (See **Problem 1.1.27**.)

Example 1.5.7 (a) For any integer $n = \dfrac{n}{1}$, we have just seen that $f(z) = z^n$ has exactly one value.

(b) The n^{th} order roots of complex numbers, as we have studied them in **Subsection 1.1.2, Part (6)**, can be viewed in the context of the complex power function. For every $n \in \mathbb{N}$ and any $z \neq 0$ complex number, the power $z^{\frac{1}{n}}$ represents n different complex numbers, namely the n n^{th} roots of z.

(c) When $\alpha = \dfrac{1}{2}$, we have seen in **Subsection 1.1.3** that the complex square root function has two values. Here we find the two continuous branches of $\sqrt{z} = z^{\frac{1}{2}}$.

We have that for $z \neq 0$

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\ln|z| + \frac{i}{2}\arg(z)} = \sqrt{|z|}e^{\frac{i}{2}\arg(z)}.$$

We consider as continuous $\text{Arg}_0(z)$ the one in the initial interval $(0, 2\pi)$. That is, we work in the open domain

$$\mathbb{C} - \{\text{non-negative } x\text{-axis}\} \subset \mathbb{C}.$$

Thus, we obtain two distinct branches: one for $0 < \arg(z) < 2\pi$ and another for $2\pi < \arg(z) < 4\pi$. For all the other values of the $\arg(z)$, the branches of $\sqrt{z} = z^{\frac{1}{2}}$ repeat, and any one of them is equal to one of these two. Indeed:

If we call $\text{Arg}_0(z)$ the argument for the first branch, then we have the following two distinct branches of $\sqrt{z} = z^{\frac{1}{2}}$:

$$(1) \quad \sqrt{z} = z^{\frac{1}{2}} = \sqrt{|z|}\, e^{i\dfrac{\text{Arg}_0(z)}{2}}, \quad \text{when} \quad 0 < \arg(z) := \text{Arg}_0(z) < 2\pi.$$

$$(2) \quad \sqrt{z} = z^{\frac{1}{2}} = \sqrt{|z|}\, e^{i\dfrac{\text{Arg}_0(z) + 2\pi}{2}}, \qquad \text{when}$$

$$2\pi < \arg(z) = \arg_1(z) = \text{Arg}_0(z) + 2\pi < 4\pi.$$

E.g., let $z = -2 = 2e^{i\arg(-2)}$. Then $\text{Arg}_0(-2) = \pi$, and the **first** branch gives $\sqrt{-2} = \sqrt{2}e^{i\frac{\pi}{2}} = +\sqrt{2}\,i$, whereas the **second** branch gives $\sqrt{-2} = \sqrt{2}e^{i\frac{\pi+2\pi}{2}} = \sqrt{2}e^{i\frac{3\pi}{2}} = -\sqrt{2}\,i$. Both values are the known values of $\sqrt{-2} = \left(\pm\sqrt{2}\right) i$.

Remark: In this example, the exponent $\alpha = \dfrac{1}{2}$ is positive. It makes sense to define $\sqrt{0} = 0$. This extension holds for positive exponents only.

That is, if $\alpha > 0$, then $0^\alpha = 0$. We observe that all the continuous branches of powers with positive exponents meet at $z = 0$. If the exponents are negative, we do not make any such extension unless we set all answers to be ∞.

▲

As we have seen in **Example 1.5.6**, the power $e^{\sqrt{2}}$ represents infinitely many different values in the complex domain. In fact, this is the case with any complex power $f(z) = z^\alpha$ with exponent α an **irrational real**. This follows from the fact that whenever α is irrational the expression $e^{i\alpha\, 2k\pi}$ yields a different value for each different $k \in \mathbb{Z}$. Indeed: If $e^{i\alpha\, 2k\pi} = e^{i\alpha\, 2l\pi}$ with k and l integers, then $e^{i\alpha\, 2(k-l)\pi} = 1$. Therefore, $\alpha\, 2(k-l)\pi = 2m\pi$ for some integer m [see **Part (a)** of **Problem 1.2.12**]. Then, $\alpha(k-l) = m$. Since α is irrational, this is possible only if $k = l$ and $m = 0$. (Otherwise, $k \neq l$ and then $\alpha = \dfrac{m}{k-l}$ would be a rational number, which is a contradiction.)

Example 1.5.8 Find the continuous branches of

$$z^{\sqrt{2}} = e^{\sqrt{2}[\ln|z| + i\arg(z)]} = |z|^{\sqrt{2}} e^{i\sqrt{2}\arg(z)} = |z|^{\sqrt{2}} e^{i\sqrt{2}\,[\,\mathrm{Arg}_0(z) + 2k\pi]}.$$

Since $\sqrt{2}$ is irrational, there are countably infinitely many continuous branches of this function obtained by the countably infinitely many branches of the $\arg(z)$. E.g., we consider $\mathrm{Arg}_0(z)$ in $(0, 2\pi)$, and then for every integer $k = 0,\ \pm 1,\ \pm 2,\ \pm 3, \ldots$ we get a continuous complex argument such that $2k\pi < \arg(z) < 2(k+1)\pi$.

▲

If α is a **complex number with Im**$(\alpha) \neq \mathbf{0}$, then, again, $f(z) = z^\alpha$ has infinitely many answers. The continuous branches of this function are obtained by using the continuous branches of the complex $\log(z)$. We illustrate this in the following:

Example 1.5.9 Find the value of

$$(1-i)^{\,(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln\left(\sqrt{2}\right) + i\arg(1-i)\right]},$$

corresponding to $\mathrm{Arg}_0(1-i) = \dfrac{7\pi}{4} \in [0, 2\pi)$.

We have

$$(1-i)^{\,(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln\left(\sqrt{2}\right) + i\frac{7\pi}{4}\right]} =$$

$$e^{\left[2\ln\left(\sqrt{2}\right) - 3\cdot\frac{7\pi}{4}\right] + i\left[3\ln\left(\sqrt{2}\right) + 2\cdot\frac{7\pi}{4}\right]} = e^{\left[\ln(2) - \frac{21\pi}{4}\right] + i\left[\frac{3}{2}\ln(2) + \frac{14\pi}{4}\right]} =$$

$$e^{\left[\ln(2) - \frac{21\pi}{4}\right]}\left\{\cos\left[\frac{3}{2}\ln(2) + \frac{14\pi}{4}\right] + i\sin\left[\frac{3}{2}\ln(2) + \frac{14\pi}{4}\right]\right\} =$$

$$= e^{\left[\ln(2) - \frac{21\pi}{4}\right]} \left\{ \cos\left[\frac{3}{2}\ln(2) + \frac{3\pi}{2}\right] + i \sin\left[\frac{3}{2}\ln(2) + \frac{3\pi}{2}\right] \right\} =$$

$$e^{\left[\ln(2) - \frac{21\pi}{4}\right]} \left\{ \sin\left[\frac{3}{2}\ln(2)\right] - i \cos\left[\frac{3}{2}\ln(2)\right] \right\} =$$

$$\frac{2}{e^{\frac{21\pi}{4}}} \left\{ \sin\left[\frac{3}{2}\ln(2)\right] - i \cos\left[\frac{3}{2}\ln(2)\right] \right\}.$$

All the infinitely many values of this complex power are

$$(1-i)^{(2+3i)} = e^{(2+3i)\log(1-i)} = e^{(2+3i)\left[\ln\left(\sqrt{2}\right) + i\left(\frac{7\pi}{4} + 2k\pi\right)\right]} =$$

$$e^{\left\{2\ln\left(\sqrt{2}\right) - 3\left(\frac{7}{4} + 2k\right)\pi + i\left[3\ln\left(\sqrt{2}\right) + \frac{7\pi}{2} + 2k\pi\right]\right\}} =$$

$$\ldots = \frac{2}{e^{3\left(\frac{7}{4} + 2k\right)\pi}} \left\{ \sin\left[\frac{3}{2}\ln(2)\right] - i \cos\left[\frac{3}{2}\ln(2)\right] \right\}$$

for $k = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots$.

▲

Using the chain rule, the derivative rules for exponentials and the complex logarithm, we obtain the following two **derivative rules**:

(1) For any constant $\alpha \in \mathbb{C}$, the derivative of a continuous branch of $f(z) = z^{\alpha}$ is

$$\frac{d}{dz}(z^{\alpha}) = \alpha z^{\alpha - 1}, \quad z \in \mathbb{C} - \{\text{the branch cut of } \log(z)\}, \text{ in general,}$$

and so

$$\frac{d}{d\alpha}[g^{\alpha}(z)] = \alpha g^{\alpha - 1}(z) g'(z), \quad z \in \mathbb{C} - \{\text{the branch cut of } \log[g(z)]\},$$

in general.

(2) If we take the derivative with respect to α (considering now α to be the variable), we have that the derivative of a continuous branch of $f(z) = z^{\alpha}$ is

$$\frac{d}{d\alpha}(z^{\alpha}) = z^{\alpha}\log(z), \quad z \in \mathbb{C} - \{\text{the branch cut of } \log(z)\}, \text{ in general,}$$

and

$$\frac{d}{d\alpha}[g^{\alpha}(z)] = g^{\alpha}(z)\log[g(z)], \quad z \in \mathbb{C} - \{\text{the branch cut of } \log[g(z)]\},$$

in general.

In the usual way, we also obtain the **complex binomial series** with complex exponents

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad \alpha \in \mathbb{C}, \quad \forall\, z:\ |z|<1,$$

where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}$, $\forall\, n \in \mathbb{N}$.
Whenever $(1+z)^\alpha$ is a multi-value function, this power series represents the continuous branch which is equal to 1 at $z=0$.

1.5.6 Properties of Complex Logarithms and Powers

Since a complex logarithm and a complex power are a set of complex numbers in general, we would like to accurately state and understand the **basic properties of complex logarithms and complex powers** as sets of numbers.

Let z, z_1 and z_2 be in $\mathbb{C}-\{0\}$. With $\log(z)$, $\log(z_1)$ and $\log(z_2)$, we mean the sets of all values of their **complex logarithms**. Similarly, the **complex powers** z^a and z^b with a and b in $\mathbb{C}$ mean the sets of their values.

Using the properties of the argument, the operations among subsets of $\mathbb{C}$ that we have seen in **Subsection 1.1.2, Parts (2), (3)** and **(7)**, and the definitions and remarks of **Subsection 1.5.3**, we can directly prove that **the corresponding sets** of these complex numbers satisfy the following properties:

1.
$$\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2).$$

2.
$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2).$$

3.
$$\log\left(z^a\right) = a \cdot \log(z) + i\,2\pi\mathbb{Z} \supseteq a \cdot \log(z).$$

4.
$$z^a \cdot z^b \supseteq z^{a+b}.$$

   Equality may fail. For instance, if $z \neq 0$, $\mathrm{Im}(a) \neq 0$, and $b=-a+n$, with $n \in \mathbb{Z}$. Then $z^{a+b} = z^n$ has one value, but may $z^a \cdot z^b$ have countably infinitely many values.

5.
$$(z_1 \cdot z_2)^a = z_1^a \cdot z_2^a.$$

6.
$$\left(\frac{z_1}{z_2}\right)^a = z_1^a \cdot z_2^{-a}.$$

7.
$$\left(z^a\right)^b = z^{ab} \cdot e^{ib2\pi\mathbb{Z}} \supseteq z^{ab}.$$

Notice: If $z \neq 0$, then, in general,

$$\left(z^a\right)^b = z^{ab} \cdot e^{ib2\pi\mathbb{Z}} \neq \left(z^b\right)^a = z^{ab} \cdot e^{ia2\pi\mathbb{Z}},$$

unless a and b are especially chosen. Besides the choices $a = b \in \mathbb{C}$, or both a and b are integers, not necessarily equal, find some other choices of $a \neq b$ for which we obtain equality. (See **Problem 1.5.13**.)

Also, we must notice that when we designate a common initial interval $(r, r + 2\pi)$, $r \in \mathbb{R}$, [e.g., $(-\pi, \pi]$, $[-\pi, \pi)$, $[0, 2\pi]$, etc.] for the continuous principal arguments of z_1 and z_2, then $\operatorname{Arg}_0(z_1) \pm \operatorname{Arg}_0(z_2)$, etc., may not be in this interval. We may need to add or subtract an appropriate constant to these results in order to return into $(r, r + 2\pi)$. (See the **next Example**.)

Examples

Example 1.5.10 Let $(a, a + 2\pi) = (-\pi, \pi)$, i.e., $a = -\pi$.

Suppose $\operatorname{Arg}_0(z_1) = \dfrac{3\pi}{4}$ and $\operatorname{Arg}_0(z_2) = \dfrac{2\pi}{3}$. Then $\operatorname{Arg}_0(z_1) + \operatorname{Arg}_0(z_2) = \dfrac{3\pi}{4} + \dfrac{2\pi}{3} = \dfrac{17\pi}{12}$. This number is outside $(-\pi, \pi)$. But, if we subtract 2π from it, we find $\dfrac{17\pi}{12} - 2\pi = -\dfrac{7\pi}{12}$, which is in $(-\pi, \pi)$.

Similarly, if $\operatorname{Arg}_0(z_1) = -\dfrac{3\pi}{4}$ and $\operatorname{Arg}_0(z_2) = \dfrac{5\pi}{6}$, then $\operatorname{Arg}_0(z_1) - \operatorname{Arg}_0(z_2) = -\dfrac{3\pi}{4} - \dfrac{5\pi}{6} = -\dfrac{19\pi}{12}$. This number is outside $(-\pi, \pi)$. But, if we add 2π to it, we find $-\dfrac{19\pi}{12} + 2\pi = \dfrac{7\pi}{12}$, which is in $(-\pi, \pi)$.

In the same way, we make similar adjustments in situations similar to this!

▲

Example 1.5.11 Find the continuous branches of $\dfrac{1}{\sqrt[3]{z}} = \dfrac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}}$.

We have that for $z \neq 0$

$$\frac{1}{\sqrt[3]{z}} = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = e^{\frac{-1}{3}\ln|z| - \frac{i}{3}\arg(z)} = \frac{1}{\sqrt[3]{|z|}}\, e^{-\frac{i}{3}\arg(z)}.$$

We consider as continuous $\text{Arg}_0(z)$ the one in the initial interval $(0, 2\pi)$. That is, we work in the open domain

$$\mathbb{C} - \{\text{non-negative } x\text{-axis}\} \subset \mathbb{C}.$$

Thus, we obtain three distinct branches: one for $0 < \arg(z) < 2\pi$, another for $2\pi < \arg(z) < 4\pi$ and another for $4\pi < \arg(z) < 6\pi$. For all other values of the $\arg(z)$, the branches of $\dfrac{1}{\sqrt[3]{z}} = \dfrac{1}{z^{\frac{-1}{3}}}$ repeat, and any one of them is equal to one of these three. Indeed:

For the first branch, we use as argument $\arg(z)$ the principal argument $\text{Arg}_0(z)$ and then find the following three branches of $z^{\frac{-1}{3}} = \dfrac{1}{z^3}$:

$$w_1 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\text{Arg}_0(z)}{3}}, \quad \text{when} \quad 0 < \text{Arg}_0(z) < 2\pi,$$

$$w_2 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\text{Arg}_0(z)+2\pi}{3}}, \quad \text{when} \quad 2\pi < \arg_1(z) < 4\pi,$$

$$w_3 = \frac{1}{z^{\frac{1}{3}}} = z^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{|z|}}\, e^{-i\frac{\text{Arg}_0(z)+4\pi}{3}}, \quad \text{when} \quad 4\pi < \arg_2(z) < 6\pi.$$

For instance, let $z = -8 = 8e^{i\arg(-8)}$ with $\text{Arg}_0(-8) = \pi$, and we would like to evaluate the three values of $(-8)^{\frac{-1}{3}} = \dfrac{1}{\sqrt[3]{-8}}$. We find that the value of each branch is:

$$w_1 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{2}e^{-i\frac{\pi}{3}} = \frac{1}{2}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] =$$
$$\frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)\right] = \frac{1}{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{4}(1 - i\sqrt{3}).$$

$$w_2 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{\sqrt[3]{|-8|}}e^{-i\frac{\pi+2\pi}{3}} = \frac{1}{2}[\cos(\pi) + i\sin(\pi)] = -\frac{1}{2}.$$

$$w_3 = (-8)^{\frac{-1}{3}} = \frac{1}{\sqrt[3]{-8}} = \frac{1}{2}e^{-i\frac{\pi+4\pi}{3}} = \frac{1}{2}\left[\cos\left(-\frac{5\pi}{3}\right) + i\sin\left(-\frac{5\pi}{3}\right)\right] =$$
$$\frac{1}{2}\left[\cos\left(\frac{5\pi}{3}\right) - i\sin\left(\frac{5\pi}{3}\right)\right] = \frac{1}{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{4}(1 + i\sqrt{3}).$$

▲

Example 1.5.12 Find all the values of

$$i^{\,i} = e^{i\log(i)} = e^{i[\ln|1| + i\arg(i)]} = e^{-\arg(i)}.$$

Since all possible values of $\arg(i)$ are $\dfrac{\pi}{2} + 2k\pi$, we find that all the values of the complex power $i^{\,i}$ are the real numbers $e^{-\left(\frac{\pi}{2}+2k\pi\right)}$, with $k = 0,\ \pm 1,\ \pm 2,\ \pm 3, \dots$.

▲

Example 1.5.13 Find the infinitely many values of the complex power

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)\log(-2+i)}.$$

We let $\mathrm{Arg}_0(-2+i) = \theta$ be the unique II-quadrant angle $\theta \in [0, 2\pi)$ determined by $\cos(\theta) = \dfrac{-2}{\sqrt{5}}$ and $\sin(\theta) = \dfrac{1}{\sqrt{5}}$ and measured in radians. (We can approximate this θ to several decimal digits.) Then

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)[\ln(\sqrt{5})+i(\theta+2k\pi)]} =$$
$$e^{\frac{3\ln(5)}{2}+2\theta+4k\pi}\,\{\cos[-\ln(5)+3\theta+6k\pi] + i\sin[-\ln(5)+3\theta+6k\pi]\} =$$
$$e^{\frac{3\ln(5)}{2}+2\theta+4k\pi}\,\{\cos[-\ln(5)]\cos(3\theta) - \sin[-\ln(5)]\sin(3\theta) +$$
$$i\,\{\sin[-\ln(5)]\cos(3\theta) + \cos[-\ln(5)]\sin(3\theta)\}\}, \quad \text{with} \quad k \in \mathbb{Z}.$$

But, by triple angle formulae, we have

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) = 4\cdot\left(\frac{-2}{\sqrt{5}}\right)^3 - 3\cdot\left(\frac{-2}{\sqrt{5}}\right) =$$
$$4\cdot\left(\frac{-8}{5\sqrt{5}}\right) - 3\cdot\left(\frac{-2}{\sqrt{5}}\right) = \frac{-2\sqrt{5}}{25},$$
$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta) = 3\cdot\frac{1}{\sqrt{5}} - 4\cdot\left(\frac{1}{\sqrt{5}}\right)^3 =$$
$$\frac{3}{\sqrt{5}} - \frac{4}{5\sqrt{5}} = \frac{11\sqrt{5}}{25}.$$

Substituting these values and simplifying, we find

$$(-2+i)^{\,(3-2i)} = e^{(3-2i)\log(-2+i)} =$$
$$e^{2\theta+4k\pi}\cdot\{-2\cos[\ln(5)] + 11\sin[\ln(5)] + i\,\{2\sin[\ln(5)] + 11\cos[\ln(5)]\}\},$$

where $k \in \mathbb{Z}$.

▲

Example 1.5.14 We want to find all (complex) solutions of the (complex) equation $\sin(z) = -2$. By the definition of the complex sine, we must solve the complex equation:

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i} = -2 \quad \Longleftrightarrow \quad \left(e^{iz}\right)^2 + 4ie^{iz} - 1 = 0.$$

Then, by the quadratic formula, we get

$$e^{iz} = \left(-2 \pm \sqrt{3}\right) i \quad \Longleftrightarrow \quad iz = \log\left[\left(-2 \pm \sqrt{3}\right) i\right].$$

Since both $-2 \pm \sqrt{3}$ are negative, we can choose $\mathrm{Arg}_0[(-2 \pm \sqrt{3})i] = \dfrac{3\pi}{2} \in [0, 2\pi)$. (If these numbers have different arguments, we must consider them apart in two different formulae.)

Then we get

$$z = \frac{1}{i}\log\left[\left(-2 \pm \sqrt{3}\right) i\right] = -i\log\left[\left(-2 \pm \sqrt{3}\ \right) i\right] =$$
$$= -i\left[\ln\left|-2 \pm \sqrt{3}\right| + \left(2k + \frac{3}{2}\right)\pi i\right] = \left(2k + \frac{3}{2}\right)\pi - i\ln\left(2 \mp \sqrt{3}\right),$$

with $k \in \mathbb{Z}$.

If we observe that $2 - \sqrt{3} = \dfrac{1}{2 + \sqrt{3}}$, then we can write

$$z := z_k^{\pm} = \left(2k + \frac{3}{2}\right)\pi \pm i\ln\left(2 + \sqrt{3}\right), \quad \forall\ k \in \mathbb{Z}.$$

So, the above equation has infinitely many complex solutions, two for each integer $k = 0,\ \pm 1,\ \pm 2, \ldots$. The solutions were expected to be complex since we have set the sine function equal to $-2 \notin [-1, 1]$ (the range of the real sine). If we plug any of these numbers back into $\sin(z)$, we easily verify that the value we get is -2.

In this example, we could have found first the multi-value function **inverse of sine**, in general, and then plug in the number -2. That is, by the definition of sine,

$$w = \sin^{-1}(z) \iff z = \sin(w).$$

Thus,

$$z = \frac{e^{iw} - e^{-iw}}{2i} \ \text{ or equivalently } \ \left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 = 0.$$

So, by the quadratic formula, we find $e^{iw} = iz \pm \left(1 - z^2\right)^{\frac{1}{2}}$ and then, the **inverse of sine** is the infinitely-value function

$$w^{\pm} = \sin^{-1}(z) = -i\log\left[iz \pm \left(1 - z^2\right)^{\frac{1}{2}}\right], \quad \forall\ z \in \mathbb{C}.$$

This function has infinitely many values produced by the $\pm$ choice and the infinitely many values of the complex logarithm.

If by $\left(1-z^2\right)^{\frac{1}{2}}$ we mean the two-value function of the power with exponent $\frac{1}{2}$, then we can replace the $\pm$ with simply $+$ in the above formula. Also notice $iz-\left(1-z^2\right)^{\frac{1}{2}} = \frac{-1}{iz+(1-z^2)^{\frac{1}{2}}}$. But when working with the $\sin^{-1}(z)$, we may need to introduce convenient **branch cuts**: E.g., $(-\infty,-1)$ and $(1,\infty)$ for $\sqrt{1-z^2}=\sqrt{1+z}\cdot\sqrt{1-z}$, with $0 \le \arg(1-z) < 2\pi$ and $-\pi < \arg(1+z) \le \pi$, respectively. We also consider the $+$ only to obtain the **single value function**

$$w = \sin^{-1}(z) = -i\,\mathrm{Log}\left[iz+\left(1-z^2\right)^{\frac{1}{2}}\right], \quad \forall\ z \in \mathbb{C}-\{\text{the branch cuts}\},$$

where $\mathrm{Log}(z)$ is the single value **complex logarithm with branch cut** $(-\infty,0]$ and $-\pi < \arg(z) = \mathrm{Arg}_0(z) \le \pi$.

The derivatives of the two formulae respectively are

$$\frac{d\left(w^{\pm}\right)}{dz} = \frac{d}{dz}\left[\sin^{-1}(z)\right] = \frac{\pm 1}{\sqrt{1-z^2}}, \quad \forall\ z \in \mathbb{C}-\{\text{the branch cuts}\},$$

as we can directly check. Without loss of generality, **we use the** $+$.

Now, for $z=-2$, we find

$$\sin^{-1}(-2) = -i\log\left[-i2 \pm (-3)^{\frac{1}{2}}\right] = -i\log\left[i\left(-2\pm\sqrt{3}\right)\right]$$

which are the values we have found above, etc.

If we plug $z=1$, a value within the real range of sine, we find

$$\sin^{-1}(1) = -i\log(i) = -i\left[\ln(1) + i\left(\frac{\pi}{2}+2k\pi\right)\right] = \\ -ii\left(\frac{\pi}{2}+2k\pi\right) = \frac{\pi}{2}+2k\pi, \quad \text{with} \quad k \in \mathbb{Z}.$$

Similarly, $\sin^{-1}(0) = l\pi$, with $l \in \mathbb{Z}$, and

$$\sin^{-1}\left(-\frac{1}{2}\right) = \begin{cases} \dfrac{5\pi}{6}+2k\pi, & \text{with} \quad k \in \mathbb{Z}, \\ \\ \dfrac{7\pi}{6}+2k\pi, & \text{with} \quad k \in \mathbb{Z}. \end{cases}$$

In these last three examples, all the values found are real because 1, 0, and $\frac{1}{2}$ are in the range of the real sine. These are exactly the values that we know from elementary trigonometry.

Remark: In the same way, as in this example, we can find the general formulae of the **inverse functions of the trigonometric and hyperbolic functions (multi-value functions)**. Then, given any complex number, we can plug it into these formulae to compute the corresponding infinitely many answers. For example:

$$\cos^{-1}(z) = -i\log\left[z \pm \left(z^2-1\right)^{\frac{1}{2}}\right] = -i\log\left[z \pm i\left(1-z^2\right)^{\frac{1}{2}}\right], \ \forall \ z \in \mathbb{C},$$

and

$$\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{1-iz}{1+iz}\right), \quad \forall \ z \in \mathbb{C} - \{\pm i\}.$$

The complex cosine assumes all values in $\mathbb{C}$. But when working with the $\cos^{-1}(z)$, we may need to introduce convenient branch cuts. E.g., $(-\infty,-1)$ and $(1,\infty)$ for $\sqrt{1-z^2} = \sqrt{1+z}\cdot\sqrt{1-z}$, with $0 \le \arg(1-z) < 2\pi$ and $-\pi < \arg(1+z) \le \pi$, respectively.
Then, we rewrite (check) and use

$$\cos^{-1}(z) = \frac{\pi}{2} + i\,\mathrm{Log}\left(iz + \sqrt{1-z^2}\right), \ \forall \ z \in \mathbb{C} - \{\text{the branch cuts}\},$$

where $\mathrm{Log}(z)$ is the single value **complex logarithm with branch cut** $(-\infty,0]$ and $-\pi < \arg(z) = \mathrm{Arg}_0(z) \le \pi$.

We can check directly that the complex function $z = \tan(w)$ does not assume the complex values $z = \pm i$ only. [See also **Problem 1.5.15, (b)**.] But when working with the $\tan^{-1}(z)$, we may need to introduce convenient branch cuts, e.g., $(-i\infty,-i]$ for $\log(1-iz)$ and $[i,i\infty)$ for $\log(1+iz)$. Then we use

$$\tan^{-1}(z) = \frac{1}{2}\,i\left[\log(1-iz) - \log(1+iz)\right], \ \forall \ z \in \mathbb{C} - \{\text{the branch cuts}\}.$$

In $\mathbb{C} - \{$the branch cuts of $\sqrt{1-z^2}\}$, we have the derivative

$$\frac{d}{dz}\left[\cos^{-1}(z)\right] = \frac{\mp i}{\sqrt{z^2-1}} = \frac{\mp 1}{\sqrt{1-z^2}}, \quad \forall \ z \in \mathbb{C} - \{\text{the branch cuts}\}.$$

In $\mathbb{C} - \{\pm i\}$, we have the derivative

$$\frac{d}{dz}\left[\tan^{-1}(z)\right] = \frac{1}{1+z^2}, \quad \forall \ z \in \mathbb{C} - \{\pm i\}.$$

We do similar work and have analogous conclusions for the **inverse of the complex sine and complex cotangent**. Solve and study **Problem 1.5.15**.

▲

1.5.7 Consequence.

Going back to $f(z) = \dfrac{1}{z}$ in the $D^o(0, r)$, with $r > 1$, and $C = C(0, 1)$, we now see that this function does not have a continuous antiderivative along C. In fact, at $a = -1 \in \mathbb{C}$, the $\log(z)$ has a jump equal to $+2\pi i$ when we cross $a = -1$ as we travel along C. This jump is precisely equal to

$$\oint_{C^+(0,1)} \frac{1}{z}\, dz = +2\pi i.$$

Similarly, we have that for any $r > 0$ and any $z_0 \in \mathbb{C}$

$$\oint_{C^\pm(z_0,r)} \frac{1}{z - z_0}\, dz = \pm 2\pi i.$$

Therefore, for any $r > 0$ and any complex number c fixed, we have

$$\oint_{C^\pm(0,r)} \frac{c}{z}\, dz = \pm 2c\pi i,$$

and similarly, if $z_0 \in \mathbb{C}$ fixed

$$\oint_{C^\pm(z_0,r)} \frac{c}{z - z_0}\, dz = \pm 2c\pi i.$$

So, now we observe that when we integrate a function written as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \;\; \forall\ z \in D^o(z_0, r),$$

along a closed path C inside a punctured-disc $D^o(z_0, r)$, with $r > 0$, only the term $\dfrac{b_1}{z - z_0}$ may contribute a non-zero result to the integral, but all the other additive terms of the series contribute zero.

In fact, for the scope of this book, we use the following (partial) **Theorem**:

Theorem 1.5.6 *Let C be a simple, closed, continuous, piecewise continuously differentiable path in $\mathbb{C}$ and $f(z)$ be a function given by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \;\; \forall\ z \in \mathbb{C} - \{z_0\},$$

where $z_0 \in \mathbb{C}$ *fixed* ***but not on the path*** C. *Then:*

(a) If z_0 *lies inside the region that* C *encloses, then*

$$\oint_{C^{\pm}} f(z)\,dz = \pm 2b_1\pi i.$$

(b) If z_0 *lies outside the region that* C *encloses, then*

$$\oint_{C^{\pm}} f(z)\,dz = 0.$$

Proof (Refer to **Figure 1.3**, below.) By the previous discussion, it is enough to prove **this Theorem** for the function

$$f(z) = \frac{b_1}{z - z_0}.$$

We let $\mathcal{D}$ be the interior of the simple closed path C, that is, the open region that C encloses.

(a) Assume z_0 lies inside the region $\mathcal{D}$. Then this function $f(z)$ does not have a continuous antiderivative in $\mathcal{D} - \{z_0\}$.

We take $r > 0$ small enough so that C and $C(z_0, r)$ do not intersect or touch. Call $\mathcal{R} = \mathcal{D} - \overline{D(z_0, r)}$. This is the part of $\mathcal{D}$ lying between C and the circle $C(z_0, r)$.

As we have already seen, for the circle $C(z_0, r)$, we get

$$\oint_{C^{\pm}(z_0,r)} \frac{b_1}{z - z_0}\,dz = \pm\int_0^{2\pi} b_1 \frac{1}{z_0 + re^{i\theta} - z_0}\,d(re^{i\theta}) =$$
$$\pm b_1 \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}}\,d\theta = \pm b_1 i \int_0^{2\pi} d\theta = \pm 2b_1\pi i,$$

and so

$$\oint_{C^{\mp}(z_0,r)} \frac{b_1}{z - z_0}\,dz = \mp 2b_1\pi i.$$

Next, in order to apply Green's Theorem in $\mathcal{R}$, we must be careful with the relative orientation of the two pieces of the boundary of $\mathcal{R}$, and we need to write our function as

$$\frac{1}{z - z_0} = u(x,y) + iv(x,y) =$$
$$\frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2} + i\frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}$$

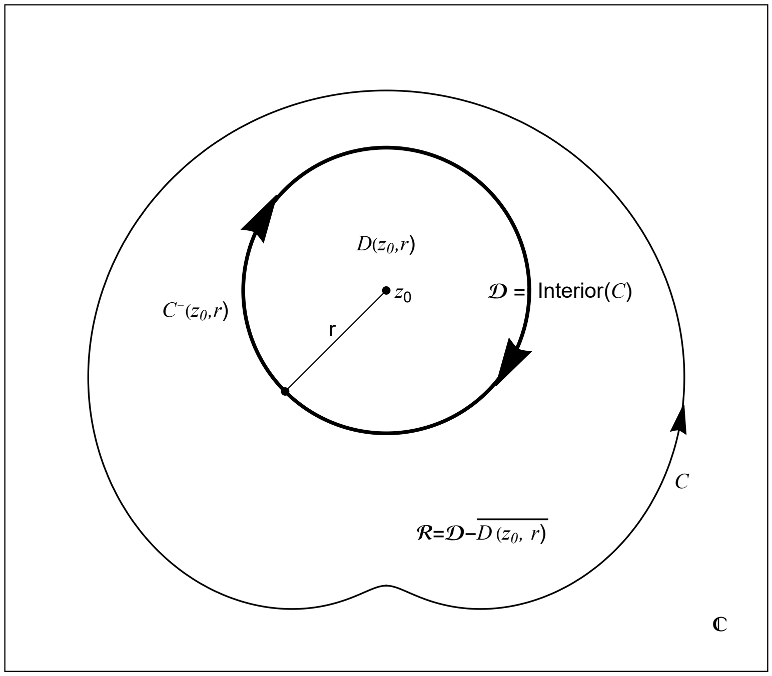

FIGURE 1.3: Contour C in $\mathbb{C}$ with point z_0 in its interior $\mathcal{D}$

where $z = x + iy$ and $z_0 = x_0 + iy_0$. Then we have:

$$\oint_{\partial\mathcal{R}} \frac{b_1}{z - z_0}\,dz =$$

$$\oint_{C^{\pm}} \frac{b_1}{z - z_0}\,dz + \oint_{C^{\mp}(z_0,r)} \frac{b_1}{z - z_0}\,dz =$$

$$b_1 \oint_{C^{\pm}} [u(x,y) + iv(x,y)](dx + idy) +$$

$$b_1 \oint_{C^{\mp}(z_0,r)} [u(x,y) + iv(x,y)](dx + idy) =$$

$$b_1 \left[\oint_{C^{\pm}} (udx - vdy) + i \oint_{C^{\pm}} (vdx + udy) \right] +$$

$$b_1 \left[\oint_{C^{\mp}(z_0,r)} (udx - vdy) + i \oint_{C^{\mp}(z_0,r)} (vdx + udy) \right].$$

Therefore,

$$\oint_{\partial\mathcal{R}} \frac{b_1}{z-z_0}\,dz =$$

$$b_1\left[\oint_{C^{\pm}} + \oint_{C^{\mp}(z_0,r)}\right](udx - vdy) + b_1 i\left[\oint_{C^{\pm}} + \oint_{C^{\mp}(z_0,r)}\right](vdx + udy).$$

Then by Green's Theorem in the region $\mathcal{R}$ (as we have seen it in calculus), we get

$$\oint_{\partial\mathcal{R}} \frac{b_1}{z-z_0}\,dz = \pm b_1\left[\iint_{\mathcal{R}}(-v_x - u_y)dxdy + i\iint_{\mathcal{R}}(u_x - v_y)dxdy\right].$$

But, in $\mathcal{R}$, the function $\dfrac{1}{z-z_0}$ has complex derivative at every point, equal to $\dfrac{-1}{(z-z_0)^2}$. (Notice that $z_0 \notin \mathcal{R}$.) Therefore, it is holomorphic in $\mathcal{R}$, and so the **Cauchy-Riemann conditions** hold for this function in $\mathcal{R}$. That is,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Therefore, both double integrals above are equal to zero, and so

$$\oint_{\partial R} \frac{b_1}{z-z_0} = \oint_{C^{\pm}} \frac{b_1}{z-z_0}\,dz + \oint_{C^{\mp}(z_0,r)} \frac{b_1}{z-z_0}\,dz = 0.$$

Since

$$\oint_{C^{\pm}(z_0,r)} \frac{b_1}{z-z_0}\,dz = \mp 2b_1\pi i,$$

we finally find

$$\oint_{C^{\pm}} \frac{b_1}{z-z_0}\,dz = -(\mp 2b_1\pi i) = \pm 2b_1\pi i.$$

(b) Assume $z_0 \notin \mathcal{D}\cup C$, that is, it lies outside the closed region $\mathcal{D}\cup C$ that C encloses. Since $f'(z) = \dfrac{-1}{(z-z_0)^2}$ is defined at every point of $\mathcal{D}\cup C$, then by the **Cauchy-Goursat Theorem, 1.5.3,** we immediately get

$$\oint_{C^{\pm}} f(z)\,dz = 0.$$

[In this case, we could also apply **Theorem 1.5.2** because in the bounded region $\mathcal{D} \cup C$ the function $f(z)$ has a continuous antiderivative which is any continuous branch of the function $\log(z - z_0)$.]

■

Remark: If in the **previous Theorem** $z_0 \in C$, then the $\oint\limits_{C^\pm} f(z)\,dz$ is dealt individually. Its existence, non-existence and evaluation depend on the given $f(z)$ and C.

Examples

Example 1.5.15 Let C be the boundary of the square $\{z = x + iy \ : \ |x| \leq 2,\ |y| \leq 2\}$. Then, due to **Theorem 1.5.6, Part (a)**, all the following integrals give the same value:

$$\oint\limits_{C^\pm} \frac{1}{z}\,dz = \oint\limits_{C^\pm} \frac{1}{z-1}\,dz = \oint\limits_{C^\pm} \frac{1}{z-i}\,dz = \oint\limits_{C^\pm} \frac{1}{z+1}\,dz =$$
$$\oint\limits_{C^\pm} \frac{1}{z+i}\,dz = \oint\limits_{C^\pm} \frac{1}{z-(1-i)}\,dz = \pm 2\pi i.$$

Similarly, by the **same Theorem, Part (b)**, we have:

$$\oint\limits_{C^\pm} \frac{1}{z-3}\,dz = \oint\limits_{C^\pm} \frac{1}{z-3i}\,dz = \oint\limits_{C^\pm} \frac{1}{z+3}\,dz =$$
$$\oint\limits_{C^\pm} \frac{1}{z+3i}\,dz = \oint\limits_{C^\pm} \frac{1}{z-(4-5i)}\,dz = 0.$$

▲

Example 1.5.16 For any $a > 0$ and $b > 0$ constants, let C be the ellipse $\{z = x + iy \ : \ x = a\cos(\theta),\ y = b\sin(\theta),\ 0 \leq \theta \leq 2\pi\}$. Then, due to **Theorem 1.5.6**, we have that

$$\oint\limits_{C^+} \frac{1}{z}\,dz = 2\pi i.$$

From this, we get

$$\int_0^{2\pi} \frac{-a\sin(\theta) + ib\cos(\theta)}{a\cos(\theta) + ib\sin(\theta)}\,d\theta = \int_0^{2\pi} \frac{(b^2 - a^2)\sin(\theta)\cos(\theta) + iab}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\,d\theta =$$
$$2\pi i = 0 + 2\pi i.$$

Separating real and imaginary parts, we get the following two real integrals:

$$\int_0^{2\pi} \frac{(b^2 - a^2)\sin(\theta)\cos(\theta)}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 0,$$

and

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 2\pi.$$

Therefore, we also obtain the following two real integrals:

$$\int_0^{2\pi} \frac{\sin(\theta)\cos(\theta)}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = 0,$$

and

$$\int_0^{2\pi} \frac{1}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = \frac{2\pi}{ab}.$$

So,

$$\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2(\theta) + b^2\sin^2(\theta)}\, d\theta = \frac{\pi}{2ab}.$$

(Compare this with **Examples I 2.2.3**, **I 2.6.21** and **1.8.3**. Also, for comparison, try to evaluate the second integral by pure calculus methods!)

▲

From the two **previous Examples**, we get a sense of the power of the **Theorem 1.5.6**.

1.5.8 Cauchy Integral Formula

Let $w = f(z)$ be a holomorphic function in an open region $\mathcal{R} \subseteq \mathbb{C}$ and C be a simple, closed, continuous and piecewise continuously differentiable path in $\mathcal{R}$. Let z_0 be located inside the region $\mathcal{D} \subset \mathcal{R} \subseteq \mathbb{C}$ that C encloses and let $0 < r < p$, where $p > 0$ and such that $D(z_0, p) \subseteq \mathcal{D}$. Then from **Theorem 1.5.1**, we know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \ \forall\ z \in D(z_0, p),$$

where $a_n = \dfrac{f^{(n)}(z_0)}{n!} \in \mathbb{C}, \ \ \forall\ n = 0,\ 1,\ 2,\ \dots$.

So, $f(z_0) = a_0$ and

$$\frac{f(z)}{z - z_0} = \frac{a_0}{z - z_0} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n-1}, \ \ \forall\ z \in D(z_0, p).$$

Hence

$$\oint_{C^+(z_0,r)} \frac{f(z)}{z-z_0}\,dz = a_0 2\pi i = 2\pi i f(z_0).$$

As in the proof of **Theorem 1.5.6**, we can use Green's Theorem to get that for any simple, closed, continuous and piecewise continuously differentiable path C around the point z_0, we have that

$$\oint_{C^+} \frac{f(z)}{z-z_0}\,dz = \oint_{C^+(z_0,r)} \frac{f(z)}{z-z_0}\,dz.$$

Therefore,

$$\oint_{C^+} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

[See also **Problem 1.5.19, (a)**.] This is the powerful and useful result:

Theorem 1.5.7 (Cauchy integral formula) *Let $w = f(z)$ be a holomorphic function in an open region $\mathcal{G} \subseteq \mathbb{C}$ and C be any simple, closed, continuous and piecewise continuously differentiable path in $\mathcal{G}$. We let $\mathcal{D} :=$ the inside of C, and we assume $\mathcal{D} \subseteq \mathcal{G}$. In particular, this is true when $\mathcal{G}$ is simply connected. Then for any $z_0 \in \mathcal{D}$ we have that*

$$\oint_{C^+} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) \iff f(z_0) = \frac{1}{2\pi i}\oint_{C^+} \frac{f(z)}{z-z_0}\,dz. \tag{1.8}$$

This Theorem says that all the values of a holomorphic function obtained at the interior points of the path C, as in the Theorem, can be completely determined by the values of the function at just the points of C, via the stated formula. (For a generalization, see **Problem 1.5.29**.)

Under the conditions of this Theorem, if $z_0 \in \mathcal{G} - (\mathcal{D} \cup C)$, then, by the **Cauchy-Goursat Theorem, 1.5.3**, the above integral is equal to zero, because with $z_0 \in \mathcal{G} - (\mathcal{D} \cup C)$, $\dfrac{f(z)}{z-z_0}$ is holomorphic in $\mathcal{D} \cup C$.

If C^+ is a positively oriented circle with center z_0 and radius $R > 0$ such that $\overline{D(z_0, R)} \subseteq \mathcal{G}$, then we have that along C^+ $z = z_0 + Re^{it}$ with $0 \le t \le 2\pi$ and $dz = Rie^{it}dt$. In this case the **Cauchy integral formula** gives

$$\oint_{C^+} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} f\left(z_0 + Re^{it}\right) i\,dt = 2\pi i f(z_0) \iff$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + Re^{it}\right) dt. \tag{1.9}$$

This last equation, for obvious reasons, is called **the Mean Value Property of Holomorphic Functions**. The value of a holomorphic function at the center of a disc is the average or mean value of its values along the boundary of the disc.

Note: In our exposition, we use the fact that a holomorphic function is complex analytic (i.e., complex power series) and then we derive the Cauchy integral formula. Most books of complex analysis prove the Cauchy integral formula for holomorphic functions first by basic principles and then working in direction opposite to our approach (in which they make use of the geometric power series) establish the complex analyticity. There are several proofs of this very fundamental result. For another proof, see **Problem 1.5.19, (a)**.

Examples

Example 1.5.17 If $C = C(0,1)$, then by **(1.8)** we find

$$\oint_{C^+} \frac{\cos(z)}{z}\, dz = 2\pi i \cos(0) = 2\pi i.$$

Also, by **(1.9)** we find

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\left(e^{it}\right) dt = \cos(0) = 1.$$

Similarly,

$$\oint_{C^+} \frac{\sin(z)}{z}\, dz = 2\pi i \sin(0) = 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(e^{it}\right) dt = \sin(0) = 0.$$

Also,

$$\oint_{C^+} \frac{z^2}{z-\frac{1}{2}}\, dz = 2\pi i \left(\frac{1}{2}\right)^2 = \frac{\pi i}{2},$$

since $\dfrac{1}{2} \in D(0,1)$, but we do not apply **(1.9)** to this example because $\dfrac{1}{2}$ is not the center of the disc $D(0,1)$.

Now, for any $R > 0$ and $C = C(2,R)$, by **(1.8)** and **(1.9)** we respectively obtain

$$\oint_{C^+} \frac{\sin\left(e^z\right)}{z-2}\, dz = 2\pi i \sin\left(e^2\right) \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} \sin\left(e^{2+Re^{it}}\right) dt = \sin\left(e^2\right).$$

So,

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left\{e^{2+R\cos(t)}\left\{\cos[\sin(t)] + i\sin[\sin(t)]\right\}\right\}\,dt = \sin\left(e^2\right).$$

We can split this complex integral into two real integrals, one equal to $\sin\left(e^2\right)$ and the other to zero.

▲

Example 1.5.18 If $C = C(0,2)$, then by the **Cauchy integral formula, 1.8**, we get:

(a) $$\oint_{C^+} \frac{dz}{z^2-1} = \oint_{C^+} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz = \frac{1}{2}(2\pi i - 2\pi i) = 0.$$

(b) $$\oint_{C^+} \frac{dz}{z^2-8} = 0,$$

since $\pm\sqrt{8} = \pm 2\sqrt{2}$ lie outside the $D(0,2)$ and so $\dfrac{1}{z^2-8}$ is holomorphic in $D(0,2\sqrt{2}) \supset D(0,2)\cup C(0,2) = \overline{D(0,2)}$.

Along the same path, we also have

$$\oint_{C^+} \frac{dz}{z^2+2z-3} = \oint_{C^+} \frac{dz}{(z+3)(z-1)} = \oint_{C^+} \frac{\frac{1}{z+3}}{z-1}\,dz = 2\pi i \cdot \frac{1}{1+3} = \frac{\pi i}{2}.$$

▲

Under the same conditions on $f(z)$, by considering z_0 as parameter in the **Cauchy integral formula**, we can differentiate k times with respect to z_0 and under the integral sign to obtain the more **general formula**:

$$\oint_{C^+} \frac{f(z)}{(z-z_0)^{k+1}}\,dz = \frac{2\pi i f^{(k)}(z_0)}{k!}, \quad \forall\ k = 0,\ 1,\ 2,\ 3,\ \ldots, \qquad (1.10)$$

where C and z_0 are as in **Theorem 1.5.7**, above. That is, we start with the Cauchy integral formula ($k = 0$) and then we take derivative k times with respect to z_0, where $k = 1,\ 2,\ 3,\ \ldots$, and we find:

$$\oint_{C^+} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0),$$

$$\oint_{C^+} \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0),$$

$$\oint_{C^+} \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i f''(z_0)}{2} = \pi i f''(z_0),$$

$$\oint_{C^+} \frac{f(z)}{(z-z_0)^4}\,dz = \frac{2\pi i f'''(z_0)}{6} = \frac{\pi i f'''(z_0)}{3},$$

and so on for all higher order derivatives, as the above general formula states.

If $C = C^+(z_0, R)$ under the same conditions as in **equation (1.9)**, then **equation (1.10)** becomes

$$\frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + Re^{it}\right) e^{-ikt}\,dt = \frac{R^k f^{(k)}(z_0)}{k!},\ \forall\ k = 0,\,1,\,2,\,\ldots. \quad (1.11)$$

Examples

Example 1.5.19 If $C = C(0,1)$, then by **(1.10)** we find

$$\oint_{C^+} \frac{\sin(z)}{z^2}\,dz = 2\pi i \sin'(z)|_{z=0} = 2\pi i \cos(0) = 2\pi i,$$

and

$$\oint_{C^+} \frac{e^z}{z^2}\,dz = 2\pi i (e^z)'|_{z=0} = 2\pi i e^0 = 2\pi i \cdot 1 = 2\pi i.$$

Then, by **(1.11)**, we find the two integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(e^{it}\right) e^{-it}\,dt = \frac{\cos(0)}{1!} = 1,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{e^{it}-it}\,dt = \frac{e^0}{1!} = 1.$$

Similarly, with $C = C(2,3)$ we find the integrals

$$\oint_{C^+} \frac{\sin(z)}{(z-2)^4}\,dz = \frac{2\pi i \sin^{(3)}(2)}{3!} = \frac{-2\pi i \sin(2)}{6} = \frac{-\pi i \sin(2)}{3},$$

$$\oint_{C^+} \frac{1}{(z-2)^3}\,dz = \frac{2\pi i (1'')|_{z=2}}{2!} = \pi i \cdot 0 = 0,$$

$$\oint_{C^+} \frac{\cos(z)}{z^2}\,dz = 2\pi i \cos'(z)|_{z=2} = -2\pi i \sin(2),$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\left(2+3e^{it}\right)e^{-3it}\,dt = \frac{3^3\sin^{(3)}(2)}{3!} = \frac{-9\pi i\sin(2)}{2},$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-2it}\,dt = 0,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\left(2+3e^{it}\right)e^{-it}\,dt = \frac{3\cos'(2)}{2!} = \frac{3}{2}\sin(2).$$

▲

Example 1.5.20 If $C = C(0,2)$, then

$$\oint_{C^+} \frac{e^z}{(z-1)^3}\,dz = \frac{2\pi i\left(e^z\right)''\Big|_{z=1}}{2!} = \frac{2\pi ie}{2} = \pi ei,$$

$$\oint_{C^+} \frac{z^3}{(z-i)^2}\,dz = 2\pi i(z^3)'|_{z=i} = 2\pi i3i^2 = -6\pi i,$$

$$\oint_{C^+} \frac{z^3}{(z+i)^5}\,dz = \frac{2\pi i(z^3)^{(4)}|_{z=-i}}{4!} = \frac{0}{24} = 0,$$

and

$$\oint_{C^+} \frac{(z+1)^7}{[z-(1+i)]^{10}}\,dz = \frac{2\pi i[(z+1)^7]^{(9)}\Big|_{z=1+i}}{9!} = 0.$$

▲

Problems

1.5.1 With $C = C(0,2)$ justify the use of the **Cauchy-Goursat Theorem, 1.5.3**, and show that the values of the following four integrals are all zero.

$$I_1 = \oint_{C^+} \frac{dz}{z^2+z+3}, \qquad I_2 = \oint_{C^+} \frac{|z|e^z}{(z-3)^2}\,dz,$$

$$I_3 = \oint_{C^+} \frac{dz}{z^2+16}, \qquad I_4 = \oint_{C^+} \frac{z^2-1}{z^2+10}\,dz.$$

1.5.2 Consider the function

$$f(z) = e^{z^2} \int_0^z e^{-\zeta^2} d\zeta, \quad \forall \ z \in \mathbb{C}.$$

Prove that $f(0) = 0$ and $f'(z) = 1 + 2zf(z)$. Then, use these relations to prove that the Taylor series of $f(z)$ with center $z_0 = 0$ is

$$f(z) = \sum_{n=1}^{\infty} \frac{n!\, 2^{2n-1}}{(2n)!} z^{2n-1}, \quad \forall \ z \in \mathbb{C}.$$

1.5.3 Find all complex values of $\log(5i)$, $\log(-5i)$ and $\log(-3+5i)$.

1.5.4 Solve the exponential equations
$$(2+3i)^z = 5, \qquad (-5)^z = 3-2i, \qquad (2-3i)^z = 3+2i.$$

1.5.5 Consider as the initial interval of $\mathrm{Arg}_0(z)$ the $[0, 2\pi)$ and prove the discontinuity of $\arg(z)$ and $\log(z)$ along the non-negative x-axis.

[Hint: Imitate what we did in the text with initial interval $[-\pi, \pi)$.]

1.5.6 Pick any principal argument $a < \mathrm{Arg}(z) \leq b$ or $a \leq \mathrm{Arg}(z) < b$ such that $b - a = 2\pi$ (a, b are real constants). Prove that for each $k \in \mathbb{Z}$ the corresponding branch of the complex logarithm maps the set $\mathbb{C} - \{0\}$ in one-to-one manner onto the horizontal strip

$$\{z \mid z = x + i(y + 2k\pi) \ \text{ and } \ -\infty < x < \infty \ \text{ and } \ a < y \leq b\},$$

or

$$\{z \mid z = x + i(y + 2k\pi) \ \text{ and } \ -\infty < x < \infty \ \text{ and } \ a \leq y < b\},$$

respectively.

1.5.7 Verify the Cauchy-Riemann conditions for the complex logarithm $f(z) = \log(z)$, $z \neq 0$, in both Cartesian and polar coordinates. Then, using the expressions of the complex derivative of $f(z)$ in either coordinate system, derive $\dfrac{d}{dz}[\log(z)] = \dfrac{1}{z}$.

1.5.8 Find all complex values of $(-3)^{\frac{1}{6}}$, $(-5i)^{\pi}$ and $(-3+5i)^{2-i}$.

1.5.9 Find all complex continuous branches of the functions $z^{\frac{1}{4}}$, $z^{-\frac{1}{4}}$, z^{π} and z^{i}.

1.5.10 (a) For any $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ and any complex number z, prove the set-relations (with non-positive exponents assume $z \neq 0$)

$$z^{\frac{p}{q}} = \left(z^{\frac{1}{q}}\right)^p \subseteq (z^p)^{\frac{1}{q}}.$$

(b) Compute all complex answers to $(-5)^{\frac{3}{5}}$, $(2-6i)^{-\frac{4}{5}}$, $(1+i)^{\frac{4}{-5}}$ and $i^{\frac{-4}{5}}$.

1.5.11 We start with the true relation $e^{2\pi i} = 1$ and continue as follows:

$$\left(e^{2\pi i}\right)^{\frac{1}{6}} = 1^{\frac{1}{6}} \quad \Longrightarrow \quad e^{\frac{\pi}{3}i} = 1 \quad \Longrightarrow$$
$$\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = 1 \quad \Longrightarrow \quad \frac{1}{2} + i\frac{\sqrt{3}}{2} = 1\,??$$

Find the mistake!

1.5.12 (a) Locate and explain the error made in

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = -1.$$

(See also **Problem 1.1.52**.)

(b) Conclude when the elementary algebra rule about square roots $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ is valid if a and b are complex numbers.

1.5.13 Prove that for a, b in $\mathbb{C}$, $e^{ib2\pi\mathbb{Z}} = e^{ia2\pi\mathbb{Z}}$ if and only if the following two conditions hold:

(1) $\forall\ m \in \mathbb{Z},\ \ \exists\ n \in \mathbb{Z}\ \ :\ \ am - bn \in \mathbb{Z}.$

(2) $\forall\ k \in \mathbb{Z},\ \ \exists\ l \in \mathbb{Z}\ \ :\ \ bk - al \in \mathbb{Z}.$

Besides the choices $a = b \in \mathbb{C}$, or both a and b integers (in $\mathbb{Z}$) not necessarily equal, find some other choices of $a \neq b$ for which we obtain equality.

1.5.14 Set each of the functions $\cos(z), \sin(z), \cosh(z), \sinh(z)$ equal to 1 and find all solutions of each equation. Then, do the same thing by setting them equal to -1 and 0.

[Hint: Imitate **Example 1.5.14**. For 0, the work is easier.]

1.5.15 (a) Imitate **Example 1.5.14** to find the **inverse functions (multi-value functions) of the complex trigonometric and hyperbolic functions** and state explicitly some convenient **branch cuts**. E.g.: $\sinh^{-1}(z) = \log\left(z \pm \sqrt{z^2+1}\right)$, $\cosh^{-1}(z) = \log\left(z \pm \sqrt{z^2-1}\right)$, ... (You may find pertinent bibliography, e.g., Spiegel 1964, etc. and/or papers posted in the internet, etc.)

(b) Prove that the **range** of the **complex** $\sin(z)$, $\cos(z)$, $\sinh(z)$ and

$\cosh(z)$ is the whole complex plane $\mathbb{C}$, the **range** of e^z is $\mathbb{C} - \{0\}$ and the **range** of the **complex** $\tan(z)$ and $\cot(z)$ is the set $\mathbb{C} - \{\pm i\}$. Next, find the ranges of the **complex** $\tanh(z)$, $\coth(z)$, $\sec(z)$, $\text{sech}(z)$, $\csc(z)$, and $\text{csch}(z)$.

(c) Find the derivatives of the inverses you have found in **(a)** and their domains.

(d) Use **(a)** or imitate **Example 1.5.14** to find all solutions of the following twelve equations:

$$\begin{array}{llll} \cos(z) = 2, & \sin(z) = -3, & \cosh(z) = -4, & \sinh(z) = 5, \\ \tan(z) = 2, & \cot(z) = -3, & \tanh(z) = -4, & \coth(z) = 5, \\ \sec(z) = 2, & \csc(z) = -3, & \text{sech}(z) = -4, & \text{csch}(z) = 5. \end{array}$$

(e) Find in bibliography the real trigonometric and hyperbolic functions (e.g., Thomas 1968, etc.) and compare the analogous results with those found in **(a), (b)** and **(c)** for the complex trigonometric and hyperbolic functions. Observe and state the similarities and the differences.

1.5.16 Find the Taylor series of $\dfrac{1}{a-z}$ with center $c \neq a$ $(a,\ c \in \mathbb{C})$ and its radius of convergence. Do the same for e^z.
[Hint: $\dfrac{1}{a-z} = \dfrac{1}{a-c} \cdot \dfrac{1}{1 - \dfrac{z-c}{a-c}}$ and $e^z = e^c e^{z-c}$.]

1.5.17 (a) Find the Maclaurin series of $h(z) = \log(9+z)$ and its radius of convergence.

(b) Find the Taylor series of $f(z) = \log(z)$ and $g(z) = \dfrac{1}{\sqrt{z}}$ with center $c = 9$ and their radii of convergence.
[Hint: (a) $9 + z = 9\left(1 + \dfrac{z}{9}\right)$.
(b) $z = 9\left(1 + \dfrac{z-9}{9}\right)$ and $\dfrac{1}{\sqrt{z}} = \dfrac{1}{3}\left(1 + \dfrac{z-9}{9}\right)^{\frac{-1}{2}}$.]

1.5.18 Prove that the **complex Riemann Zeta function**

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

converges for all z with $\text{Re}(z) > 1$ and the convergence is uniform on the every subset of the complex plane where $\text{Re}(z) \geq c > 1$, with c constant greater than 1. Also, $\zeta(x) = \infty$ for all real $x \leq 1$, and it converges pointwise for all $z = 1 + iy$ with $y \neq 0$, but the convergence is not uniform in the subset $\{z = x + iy \mid x \geq 1\} - \{1\}$.

[Hint: You may use some **Theorems** from **Section I 2.3** adjusted to the complex plane. See also **Problem I 2.3.26**.]

1.5.19 (a) If $f(z)$ is holomorphic in an open region $\mathcal{R} \subseteq \mathbb{C}$ and for some $r > 0$ the circle $C(z_0, r)$ is inside $\mathcal{R}$, use just the continuity of $f(z)$ to prove that

$$\lim_{r \to 0} \int_{C^-(z_0,r)} \frac{f(z)}{z - z_0}\, dz = -2\pi i f(z_0). \quad \text{(Notice the negative orientation.)}$$

Then, for any C^+ simple closed contour in $\mathcal{R}$ with z_0 in its interior, use Green's Theorem on the path $C^+ + C^-(z_0, r)$ and its interior, for $r > 0$ small enough, and the Cauchy-Riemann conditions for the holomorphic function $g(z) = \dfrac{f(z)}{z - z_0} := U(x, y) + iV(x, y)$, for $z \neq z_0$, to prove the **Cauchy integral formula** stated in **Theorem 1.5.7**.

(b) Use the Cauchy integral formula and its derivatives with respect to z_0 to derive the **Cauchy estimates of the derivatives of** $f(z)$ **at** z_0

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \max_{|z - z_0| = r} |f(z)|, \quad \forall\ n = 0,\ 1,\ 2,\ \ldots$$

and for every $r > 0$ such that the circle $C(z_0, r)$ lies inside the open region $\mathcal{R}$.

Then, if $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$ in a disc $D(z_0, R) \subseteq \mathcal{R}$, with $R > 0$, prove that for every r such that $0 < r < R$, we have:

$$|a_n| \leq \frac{\displaystyle\max_{|z - z_0| = r} |f(z)|}{r^n}, \quad \forall\ n = 0,\ 1,\ 2,\ \ldots .$$

(c) For $n = 0$, we obtain

$$| f(z_0)| \leq \max_{|z - z_0| = r} |f(z)|.$$

Use this to justify the claim: If $f(z)$ is holomorphic in an open region $\mathcal{R} \subseteq \mathbb{C}$ and $\overline{D(z_0, r)} \subset \mathcal{R}$, then the maximum of $|f(z)|$ over $\overline{D(z_0, r)}$ is obtained on the boundary $C(z_0, r)$ of the disc, and so $|f(z)|$ cannot have a strict interior maximum in $\mathcal{R}$.

(This property of holomorphic functions is called **Maximum Modulus Principle**. This principle can be refined to the fact that $|f(z)|$

cannot have any interior maximum, strict or non-strict, in $\mathcal{R}$, unless $|f(z)|$ is identically constant in $\mathcal{R}$. The proof of this claim is more complicated and is relegated to a more advanced course of analysis.)

(d) A holomorphic function over the entire $\mathbb{C}$ is called an **entire function**. [E.g., $f(z) = e^z$.] Use the Cauchy estimates to prove that if $f(z)$ is an entire **bounded** function, i.e., there exists a $B > 0$ such that $\forall\ z \in \mathbb{C}\ |f(z)| \leq B$, then $f(z)$ is identically constant.

(e) Obviously, every polynomial $P(z)$ with complex coefficients is an entire function. Prove that if degree$[P(z)] \geq 1$, then $P(z)$ has a root in $\mathbb{C}$, i.e., there is $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$, thus proving the **Fundamental Theorem of Algebra**.
[Hint: If not, then $f(z) = \dfrac{1}{P(z)}$ would be an entire bounded function in $\mathbb{C}$, i.e., there is a constant $B \geq 0$ such that $|f(z)| \leq B$ for every $z \in \mathbb{C}$. Then $P(z)$ would be identically constant, which is a contradiction!]

(f) If $f(z)$ is an entire function such that

$$\lim_{z\to\infty} \frac{f(z)}{z^n} = 0, \quad \text{for an} \quad n \in \mathbb{N},$$

then $f^{(n)}(z) = 0$ at every $z \in \mathbb{C}$, and so $f(z)$ is a polynomial of degree at most $n - 1$.

(g) If for some $R > 0$, $f(z)$ is a holomorphic function in the region $\mathcal{R} := \mathbb{C} - \overline{D(0,R)} = \{z \in \mathbb{C} \mid |z| > R\} \subset \mathbb{C}$ and $\lim\limits_{z\to\infty} f(z) = c \in \mathbb{C}$ (exists), then for every $n \in \mathbb{N}$, $\lim\limits_{z\to\infty} f^{(n)}(z) = 0$. In this case, the function $f(z)$ need not be constant, but then, it must have singularities in $\overline{D(0,R)}$.

If a function $f\ :\ (r,\infty) \longrightarrow \mathbb{R}$, where $r \in \mathbb{R}$, is differentiable and both limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} f'(z)$ exist, then the second limit must be zero. In this case, give examples in which one of the limits exists and the other does not, or both limits do not exist.

1.5.20 Suppose $f : D(0,1) \longrightarrow \mathbb{C}$ is holomorphic and satisfies the condition $|f(z)| \leq \dfrac{1}{1-|z|}$, $\forall\ z \in D(0,1)$. Use the **Cauchy estimates of the previous Problem** to prove that the coefficients of the Taylor series $f(z) = \sum\limits_{n=0}^{\infty} c_n z^n$ of $f(z)$ in $D(0,1)$ satisfy

$$|c_n| \leq (n+1)\left(1 + \frac{1}{n}\right)^n < e(n+1), \quad \forall\ n \in \mathbb{N}_0.$$

1.5.21 The following five steps lead to a contradiction. Find where the mistake occurred.

(1) $$\forall\ z \in \mathbb{C},\ \ z - \sqrt{z^2 - 1} \neq 0. \text{ (Check!)}$$

So,

$$f(z) = \log\left(z - \sqrt{z^2 - 1}\right)$$

is defined for all $z \in \mathbb{C}$.

(2) $$\left(z - \sqrt{z^2 - 1}\right) \cdot \left(z + \sqrt{z^2 - 1}\right) = 1. \text{ (Check!)}$$

So, changing the branch of the square root $\sqrt{z^2 - 1}$ to the other one, $-\sqrt{z^2 - 1}$, in $f(z)$, changes $f(z)$ to $-f(z)$.

(3) If we now let $g(z) = f^2(z)$, $g(z)$ is an entire function, since it is single-valued and hence the singularities at $z = \pm 1$ (the branch points of the above square roots) are removable.

(4) $$0 \leq \left|\frac{g(z)}{z}\right| \leq \frac{\ln^2(2|z| + 1)}{|z|} \longrightarrow 0, \quad \text{as} \quad |z| \longrightarrow \infty.$$

(5) By the **previous Problem, (f)**, $g(z)$ is constant. (Contradiction!)

1.5.22 Rework all **Subsection 1.5.4** with the function $\mathrm{Log}(1 - z)$.

1.5.23 Evaluate the integral

$$\oint_C \frac{z^2 - 5z + 7}{(z - 1)^3}\, dz$$

along any simple, closed, continuous and piecewise continuously differentiable path C, that has $z_0 = 1$ in the inside, by using:

(a) The Cauchy integral formula.

(b) Partial fraction decomposition.

1.5.24 Prove that

$$\int_0^1 (-1)^x dx = \frac{2i}{(2n + 1)\pi},$$

where $n \in \mathbb{Z}$. (That is, this integral has infinity many values.)

1.5.25 (a) Evaluate the following ten integrals by using **equation (1.8)** or **equation (1.10)**:

$$I_1 = \oint_{C^+(0,1)} \frac{e^z}{z^n}\,dz, \ \ n \in \mathbb{Z}, \qquad I_2 = \oint_{C^+(2i,1)} \frac{e^z}{z^n}\,dz, \ \ n \in \mathbb{Z},$$

$$I_3 = \oint_{C^+(0,1)} \frac{z}{z-2}\,dz, \qquad I_4 = \oint_{C^+(2,1)} \frac{z}{z-2}\,dz,$$

$$I_5 = \oint_{C^+(0,3)} \frac{4e^z}{z-2}\,dz, \qquad I_6 = \oint_{C^+(0,3)} \frac{5z}{z+2}\,dz,$$

$$I_7 = \oint_{C^+(0,4)} \frac{\sin(z)}{(z+3)^3}\,dz, \qquad I_8 = \oint_{C^+(0,4)} \frac{\tan(z)}{(z+1)^3}\,dz,$$

$$I_9 = \oint_{C^+(0,1)} \frac{\sin(z)}{z^n}\,dz, \ \ n \in \mathbb{Z}, \qquad I_{10} = \oint_{C^+(0,1)} \frac{\cos(z)}{z^n}\,dz, \ \ n \in \mathbb{Z}.$$

(b) Now apply **equation (1.9)** or **equation (1.11)** to those integrals in **(a)** on which either of these equations can apply and find the corresponding integrals.

1.5.26 With $C = C(0,2)$ evaluate the following six integrals:

$$I_1 = \oint_{C^+} \frac{dz}{z^2+z+1}, \qquad I_2 = \oint_{C^+} \frac{|z|e^z}{z^2}\,dz, \qquad I_3 = \oint_{C^+} \frac{dz}{z^2(z^2+16)},$$

$$I_4 = \oint_{C^+} \frac{z^2-1}{z^2+1}\,dz, \qquad I_5 = \oint_{C^+} 3z\overline{4z}\,dz, \qquad I_6 = \oint_{C^+} \frac{3}{\overline{4z}}\,dz.$$

1.5.27 Show that:

$$\text{(a)} \quad g(w) = \int_{C^{\pm}(0,3)} \frac{z^3-z}{(z-w)^3}\,dz = \begin{cases} \pm 6\pi i w, & \text{if} \quad |w| < 3, \\ 0, & \text{if} \quad |w| > 3. \end{cases}$$

$$\text{(b)} \qquad \int_{C^{\pm}(0,1)} \frac{dz}{z^2+2z+2}\,dz = 0.$$

1.5.28 Evaluate the following ten line integrals, if C is the boundary of the square $\{z = x + iy \ : \ |x| \leq 2, \ |y| \leq 2\}$:

$$I_1 = \oint_{C^+} \frac{\cos(z)}{z}\, dz, \qquad I_2 = \oint_{C^+} \frac{\cos(z)}{z^4}\, dz, \qquad I_3 = \oint_{C^+} \frac{z^2}{2z+3}\, dz,$$

$$I_4 = \oint_{C^+} \frac{\cosh^2(z)}{z^2}\, dz, \qquad I_5 = \oint_{C^+} \frac{dz}{z^2+2}, \qquad I_6 = \oint_{C^+} \frac{dz}{(z^2+2)^2},$$

$$I_7 = \oint_{C^+} \frac{\cos(z)}{z-\frac{\pi}{6}}\, dz, \qquad I_8 = \oint_{C^+} \frac{e^z}{z(z+1)}\, dz,$$

$$I_9 = \oint_{C^+} \frac{z}{z(z-i)}\, dz, \qquad I_{10} = \oint_{C^+} \frac{z}{z(z+i)}\, dz.$$

[Hint: You may use the **Cauchy integral formulae**, but in some of them you may need partial fractions first.]

1.5.29 (A **generalization of the Cauchy integral formula**.) Under the conditions of **Theorem 1.5.7**, consider n points $z_1, \ z_2, \ \ldots \ z_n$ in $\mathcal{D}$. Prove that

$$\frac{1}{2\pi i} \oint_{C^+} \frac{f(z)dz}{(z-z_1)(z-z_2)\ldots(z-z_n)} = \sum_{i=1}^{n} \left[f(z_i) \prod_{\substack{j=1 \\ j\neq i}}^{n} (z_i - z_j)^{-1} \right].$$

Next, take derivatives of this with respect to z_i to derive new formulae.

[Hint: Use partial fractions and **Theorem 1.5.7**.]

1.5.30 Notice that the integrals

$$\oint_{C^+(0,2)} \frac{z}{z-2}\, dz \qquad \text{and} \qquad \oint_{C^+(1,1)} \frac{z}{z-2}\, dz$$

are improper and cannot be found by the Cauchy integral formula, since $z = 2$ is on the paths of integration.

Use convenient parameterizations of the paths to evaluate these integrals or to show that they do not exist.

Do the same for their principal values.

1.5.31 If $C = [\infty - \pi i, 0 - \pi i] + [-\pi i, \pi i] + [0 + \pi i, \pi i + \infty]$, then prove

$$\int_C e^{e^z}\, dz = 2\pi i.$$

1.5.32 Let $a > 0$ and C any path joining the numbers of the x-axis $-a$ and a, all other points of which are in the upper half plane. Prove

$$\int_C z^i\,dz = \frac{a\,(1+e^{-\pi})\,\{\cos[\ln(a)] + i\sin[\ln(a)]\}}{2} \cdot (1-i).$$

[Hint: Take as branch cut the non-positive y-axis and principal argument in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Then use

$$F(z) = \frac{z^{i+1}}{i+1} = \frac{z^{i+1}}{2}(1-i),$$

as an antiderivative of $f(z) = z^i$.]

1.5.33 (a) If $r > 0$ constant and $a = c + id \in \mathbb{C}$ such that $|a| = \sqrt{c^2+d^2} \neq r$, prove

$$\int_{C^+(0,r)} \frac{|dz|}{|z-a|^4} = 2\pi r \frac{|a|^2 + r^2}{\left||a|^2 - r^2\right|^3}.$$

Then show

$$\int_0^{2\pi} \frac{d\theta}{\{c^2+d^2+r^2-2r[c\cos(\theta)+d\sin(\theta)]\}^2} = 2\pi \frac{c^2+d^2+r^2}{\left|c^2+d^2-r^2\right|^3}.$$

E.g.,

$$\int_0^{2\pi} \frac{d\theta}{\{66 - 10[\pm 4\cos(\theta) \pm 5\sin(\theta)]\}^2} = \frac{33\pi}{1024},$$

with four combinations of $+$ and $-$.

(b) Also, find the integral $\displaystyle\int_{C^+(0,r)} \frac{|dz|}{|z-a|^2}$.

[Hint: Consider the three cases $a = 0$, $0 < |a| < r$ and $|a| > r$. The first case is easy.

In the last two cases prove and use that on $C^+(0,r)$ we have the following general formulae that are used in all similar situations:

$$z = re^{i\theta}, \quad \text{so} \quad dz = ire^{i\theta}d\theta, \quad \text{and notice} \quad |dz| = -ir\frac{dz}{z}.$$

$$\text{Also,} \qquad z|z-a|^2 = -\bar{a}(z-a)\left(z - \frac{r^2}{|a|^2}a\right).$$

Then, the given integral takes the form

$$\int\limits_{C^+(0,r)} \frac{|dz|}{|z-a|^4} = -\frac{ir}{\bar{a}^2} \int\limits_{C^+(0,r)} \frac{zdz}{(z-a)^2(z-b)^2}, \quad \text{where} \quad b = \frac{r^2}{|a|^2}a.$$

Now, use partial fractions and compute.]

1.5.34 Let $\mathcal{E}$ be the ellipse given in the $(z = x+iy)$-plane by the equation

$$F(x,y) = 0, \quad \text{where} \quad F(x,y) = x^2 - xy + y^2 + x + y.$$

Prove

$$\int\limits_{\mathcal{E}} \frac{dz}{z^4+1}\, dz = -\frac{\pi\sqrt{2}}{4}(1-i).$$

[Hint: In general, a point (u,v) of the $(z = x+iy)$-plane is inside, on, and outside an ellipse $\mathcal{E}$ given by the equation $F(x,y) = 0$ in which the coefficients of x^2 and y^2 are positive, if $F(u,v) < 0$, $F(u,v) = 0$, and $F(u,v) > 0$, respectively.

So, find the roots of $z^4 + 1 = 0$ and check which ones lie inside the ellipse and none on it, use partial fractions and compute. Concretely, find that $e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ is inside the ellipse and the other three

$$e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \qquad e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}, \qquad e^{i\frac{7\pi}{4}} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

are outside.

Note: In this example, the corresponding rotation of the axes, by $45°$, easily determines the normal equation of the ellipse and its position in relation to the unit circle $C(0,1)$, which contains the four fourth roots of -1. Thus, from the geometry of this problem, we can easily see which of the fourth roots of -1 are inside or outside $\mathcal{E}$. (No such root is on $\mathcal{E}$.)

That is, the change of coordinates is

$$x = \frac{\sqrt{2}}{2}(\eta - \xi), \qquad y = \frac{\sqrt{2}}{2}(\eta + \xi),$$

and the normal equation of the ellipse is

$$\frac{(\eta + a)^2}{a^2} + \frac{\xi^2}{b^2} = 1, \quad \text{where} \quad a = \sqrt{2} \quad \text{and} \quad b = \sqrt{\frac{2}{3}}.$$

Now draw the figure of $\mathcal{E}$ together with the unit circle $C(0,1)$ and see where the four fourth roots of -1 are located in relation to the ellipse $\mathcal{E}$.

Notice that, this geometric method is not convenient for most ellipses. The analytic method of the previous two paragraphs is always convenient and easy to use. For example, compute the above integral over the ellipse $5x^2 + 4xy + 2y^2 - 24x - 12y + 18 = 0.$]

1.5.35 The **Rodrigues formula**[19] for the **Legendre polynomials** is usually written as

$$P_n(z) = \frac{1}{2^n\, n!}\, \frac{d^n}{dz^n} \left[\left(z^2 - 1 \right)^n \right], \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \dots .$$

(a) If C is a smooth simple closed path around a point $z \in \mathbb{C}$, use the **Cauchy integral formula** to prove

$$P_n(z) = \frac{1}{2\pi i}\, \frac{1}{2^n} \int_{C^+} \frac{(w^2-1)^n}{(w-z)^{n+1}}\, dw, \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \dots .$$

(b) Let C^+ be the circle of center z and radius $\sqrt{|z^2-1|}$. Show that

$$P_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \left[z + \sqrt{z^2-1}\, \cos(\theta) \right]^n d\theta, \quad \forall\ n = 0,\ 1,\ 2,\ 3,\ \dots .$$

[Hint: Use **Part (a)**, the convenient parametrization of the circle, the $z^2 - 1 = |z^2-1| e^{i\phi}$ for some fixed $0 \le \phi < 2\pi$, the periodicity of the integrand as a function of θ and **Item (7.)** of **Problem I 1.1.8.**.

Another way is to work backwards and use **Example 1.6.15**, **Problem 1.6.20** and **Lemma 1.8.1**. So, redo this problem in this way when you have studied these topics.]

1.5.36 If $C = C(0,2)$, then by **Theorem 1.5.6**, we have that

$$\oint_{C^+} \frac{dz}{z-1} = 2\pi i.$$

Show that by making the substitution $u(t) = 2e^{it}$ where $0 \le t \le 2\pi$, we obtain

$$\oint_{C^+} \frac{dz}{z-1} = \int_0^{2\pi} \frac{2ie^{it}dt}{2e^{it}-1}.$$

Now let $v = e^{it}$, so that $v(0) = 1,\ v(2\pi) = 1$ and $dv = ie^{it}dt$. Then

$$\oint_{C^+} \frac{dz}{z-1} = \int_0^{2\pi} \frac{2ie^{it}dt}{2e^{it}-1} = \int_1^1 \frac{2dv}{2v-1} = 0 \neq 2\pi i.$$

Find and explain the mistake!

[19] Benjamin Olinde Rodrigues, French mathematician, 1795-1851.

1.5.9 Appendix

The Potential or Laplace Equation in a Disc and the Poisson Integral Formula

We have seen in **Problem 1.3.12** that the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

of a function $u = u(x,y)$ of two variables x and y, written in polar coordinates r, θ such that $x = r\cos(\theta)$ and $y = r\sin(\theta)$, has the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

This form of Δu is convenient when solving the Laplace equation in a disc with prescribed boundary data. Here, without loss of generality, we consider the disc $D(0,a) = \left\{(x,y) \mid x^2 + y^2 < a^2\right\}$ with radius $a > 0$ and center the origin. If the center is not the origin, then we apply a shift transformation of the coordinates so that the center is the new origin. That is, without loss of generality, we consider the problem

$$\begin{cases} \Delta u(x,y) = 0, & x^2 + y^2 < a^2, \\ \\ u(x,y) = h(x,y), & x^2 + y^2 = a^2, \end{cases} \tag{1.12}$$

which in polar coordinates takes the form

$$\begin{cases} \Delta v(r,\theta) = 0, & 0 \le r < a, \quad -\pi < \theta \le \pi, \\ \\ v(a,\theta) = f(\theta), & -\pi < \theta \le \pi, \end{cases} \tag{1.13}$$

where $v(r,\theta) = u[r\cos(\theta), r\sin(\theta)]$ and $f(\theta) = h[a\cos(\theta), a\sin(\theta)]$.

Here, we consider a harmonic $\mathfrak{C}^2$ function[20] $u(x,y) = v(r,\theta)$ defined everywhere inside the disc $0 \le r < a$. Therefore, as continuous, it is bounded on any smaller closed disc $0 \le r \le b < a$ around the origin.

Then we have

$$\begin{cases} \dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & 0 \le r < a, \quad -\pi < \theta \le \pi, \\ \\ v(a,\theta) = f(\theta), & -\pi < \theta \le \pi. \end{cases}$$

[20] A function is of class $\mathfrak{C}^2$ if its second partial derivatives exist and are continuous.

To solve this boundary value problem, we use the method of separation of variables. That is, we write $v(r,\theta) = R(r)\Theta(\theta)$ and plug it into the equation. Then for $v(r,\theta) \neq 0$, we find

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

Hence,

$$-\left(r^2\frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = \mu,$$

where μ is a separation constant. So, we must consider three cases: μ positive, zero and negative.

Case $\mu = \lambda^2 > 0$:
In this case, we have $\Theta'' - \lambda^2\Theta = 0$, which has the general solution

$$\Theta(\theta) = Ae^{\lambda\theta} + Be^{-\lambda\theta}.$$

Since $\Theta(\theta)$ must be 2π periodic, in order to have continuous solutions defined at every point in the disc $r < a$, we immediately find that we must have $A = 0 = B$, or $\Theta \equiv 0$. This results in $v \equiv 0$, and so this case does not produce any non-trivial solutions.

Case $\mu = 0$:
In this case, $\Theta'' = 0$, which has the general solution

$$\Theta(\theta) = A\theta + B.$$

This solution is 2π periodic only if $A = 0$, so that $\Theta(\theta) = B = B \cdot 1$. Therefore, we consider the constant function

$$\Theta_0(\theta) = 1.$$

In this case, the differential equation for $R(r)$ has the form

$$r^2\frac{R''}{R} + r\frac{R'}{R} = 0 \quad \text{or} \quad rR'' + R' = 0.$$

Two linearly independent solutions of this equation are

$$R(r) = 1 \quad \text{and} \quad R(r) = \ln(r).$$

Since $\lim_{r\to 0+} \ln(r) = \infty$, this solution does not qualify to be a part of the solution of the problem posed here. We accept only the solution $R(r) = 1$. Therefore, we obtain only one basic solution for the problem, namely:

$$v_0(r,\theta) = 1 \cdot 1 = 1.$$

Case $\mu = -\lambda^2 < 0$:
In this case, $\Theta'' + \lambda^2\Theta = 0$, which has two independent solutions

$$\Theta(\theta) = \cos(\lambda\theta) \quad \text{and} \quad \Theta(\theta) = \sin(\lambda\theta).$$

Since we work in a disc, these solutions along with their first and second derivatives must be 2π-periodic. This happens if and only if

$$\cos(\lambda 2\pi) = \cos(\lambda\, 0) = \cos(0) = 1 \iff$$
$$2\lambda\pi = 2n\pi, \quad n \in \mathbb{Z} \iff \lambda = n \in \mathbb{Z}.$$

Also, $\lambda = n \in \mathbb{Z}$ satisfies the 2π-periodicity of $\sin(\lambda\theta)$, and we do not obtain any other λ's from it either. So, all λ's are precisely the integer numbers.

We had already found the basic solution $v_0(r, \theta) = 1$, which is also obtained by $\cos(\lambda\theta)$ when $\lambda = n = 0$ and $R(r) = 1$ and since sine is odd and cosine is even, we do not lose anything if we consider

$$0 < \lambda = n = 1,\ 2,\ 3, \ldots$$

only. So, we consider only:

$$\Theta_n(\theta) = \cos(n\theta) \ \text{ and } \ \Theta_n(\theta) = \sin(n\theta) \ \text{ with } \ \lambda = n = 1,\ 2,\ 3, \ldots .$$

Then, $R(r)$ satisfies the second-order ordinary differential equation

$$r^2R'' + rR' - n^2R = 0.$$

[For $n = 0$, this reduces to $rR'' + R' = 0$, which was studied above and gave the solution $R(r) = 1$.]

This linear homogeneous ordinary differential equation is of **Euler type**. To solve it, we try solutions of the form $R(r) = r^\alpha$, where α is some constant exponent to be determined. Then we obtain:

$$r^2\alpha(\alpha - 1)r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2r^\alpha = 0.$$

Hence,

$$\left[\alpha(\alpha - 1) + \alpha - n^2\right] r^\alpha = 0,$$

from which we find

$$\alpha^2 - n^2 = 0, \quad \text{and so} \quad \alpha = \pm\, n.$$

So, for $n = 1,\ 2,\ 3, \ldots$ we obtain the two independent solutions for $R(r)$

$$R_n(r) = r^n \quad \text{and} \quad R_n(r) = r^{-n}.$$

But, for $n = 1,\ 2,\ 3,\ \ldots$, we have $\lim\limits_{r\to 0+} r^{-n} = \infty$. Therefore, these solutions do not qualify for being a part of the final solution.

Hence, this case has produced the following basic solutions for **Problem (1.13)**:

$$v_n(r,\theta) = r^n \cos(n\theta) \ \text{ and } \ v_n(r,\theta) = r^n \sin(n\theta), \ \text{ for } \ n = 1,\ 2,\ 3,\ \ldots.$$

Putting the last two cases together, we observe that all the basic solutions of this problem are

$$r^n \cos(n\theta), \ \text{ for } \ n = 0,\ 1,\ 2, \ldots \ \text{ and } \ r^n \sin(n\theta), \ \text{ for } \ n = 1,\ 2,\ 3, \ldots.$$

Then, by the linearity of the Laplacian Δ, the general solution of $\Delta v(r,\theta) = 0$ in the disc $0 \le r < a$ is a linear combination of these basic solutions. That is,

$$v(r,\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right],$$

with a_n and b_n real constants.

To satisfy the boundary condition $v(r,\theta) = f(\theta),\ -\pi < \theta \le \pi$, we must have

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n a^n \cos(n\theta) + b_n a^n \sin(n\theta)\right].$$

This means that $a_0,\ a_n a^n,\ b_n a^n,\ n = 1,\ 2,\ 3, \ldots$ are the **Fourier coefficients** of the 2π periodic function $f(\theta)$. So, for all $n = 1,\ 2,\ 3, \ldots$ we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi, & a_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,d\phi, \\ b_0 &= 0, & b_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,d\phi. \end{aligned}$$

The **Fourier series theory** proves that if $f(\theta)$ is continuous and 2π-periodic, then this Fourier series solution $u(r,\theta)$ is indeed a solution of the boundary value problem **(1.12)** or **(1.13)**. Also, by the theory of harmonic functions, we prove that the solution is unique. (These two statements need proofs, which we must skip as going far beyond the material in this text.) By these two facts, we are certain that we have completely and uniquely found the solution of the boundary value **Problem (1.13)** and therefore the solution of **Problem (1.12)**.

As far as finding the solution of **Problem (1.12)** or **(1.13)**, we could stop at this point. However, we continue because the Fourier series solution for $v(r,\theta)$ found above can be put in a very interesting compact integral formula, the so-called **Poisson integral formula**. To achieve this, we first replace the integral representations of the Fourier coefficients into the series for $v(r,\theta)$, and we find

$$v(r,\theta) =$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\frac{1}{\pi a^n}\left[\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,d\phi\right] r^n\cos(n\theta) +$$
$$\sum_{n=1}^{\infty}\frac{1}{\pi a^n}\left[\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,d\phi\right] r^n\sin(n\theta).$$

We factor out the fraction $\dfrac{1}{2\pi}$ and write

$$v(r,\theta) =$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[1+\sum_{n=1}^{\infty}\frac{2}{a^n}r^n\left[\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)\right]\right]\,d\phi =$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[-1+2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)]\right]\,d\phi.$$

Now, we use **Euler's formula** for the cosine and the summation of the geometric series to obtain

$$2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos[n(\theta-\phi)] = 2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\frac{e^{in(\theta-\phi)}+e^{-in(\theta-\phi)}}{2} =$$
$$\frac{1}{1-\dfrac{r}{a}\,e^{i(\theta-\phi)}}+\frac{1}{1-\dfrac{r}{a}\,e^{-i(\theta-\phi)}} = \cdots = \frac{2\left[a^2-ar\cos(\theta-\phi)\right]}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

Therefore,

$$v(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\left[-1+\frac{2\left[a^2-ar\cos(\theta-\phi)\right]}{a^2+r^2-2ar\cos(\theta-\phi)}\right]\,d\phi,$$

which eventually becomes

$$v(r,\theta) = \frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)}{a^2+r^2-2ar\cos(\theta-\phi)}\,d\phi. \tag{1.14}$$

This formula for the solution $v(r,\theta)$ is called the **Poisson integral formula** for the solution of the Laplace equation in the disc. (See also **Problem 1.5.45**.)

In the Poisson integral formula, the boundary value function $f(\phi)$ may in general be just absolutely integrable, that is, $\int_{-\pi}^{\pi} |f(\phi)|\, d\phi < \infty$, and not necessarily 2π-periodic and/or continuous as we required earlier.

Definition 1.5.3 *The expression*

$$P_{(a,\phi)}(r,\theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} =$$
$$-1 + 2\sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta - \phi)] = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta - \phi)],$$

where $0 \le r < a$ and $\theta,\ \phi$ are real constants, derived above as a function of (r,θ) for any given (a,ϕ), is called the ***Poisson kernel*** *in the disc $D(0,a)$.*

Remark: We observe that as a real function of two variables $0 \le r < a$ and $0 \le \theta \le 2\pi$ the $P_{(a,\phi)}(r,\theta)$ is a real analytic function since it is a composition of finitely many real analytic functions.

We observe that

$$\lim_{0<r\to a^-} P_{(a,\phi)}(r,\phi) = +\infty.$$

Also,

$$\frac{a-r}{a+r} \le P_{(a,\phi)}(r,\theta) \le \frac{a+r}{a-r}.$$

So, for given $r \ge 0$ we have

$$\lim_{a\to\infty} P_{(a,\phi)}(r,\theta) = 1.$$

(Check these results.)

Theorem 1.5.8 *The harmonic function which is the solution of the boundary value problem*

$$\begin{aligned} \Delta u(x,y) &= 0, \qquad & x^2 + y^2 < a^2, \\ u(x,y) &= h(x,y), & x^2 + y^2 = a^2, \end{aligned}$$

or in polar coordinates

$$\begin{aligned} \Delta v(r,\theta) &= 0, \qquad & 0 \le r < a, \quad -\pi < \theta \le \pi, \\ v(a,\theta) &= f(\theta), & -\pi < \theta \le \pi, \end{aligned}$$

where $v(r,\theta) = u[r\cos(\theta), r\sin(\theta)]$ with $f(\theta) = h[a\cos(\theta), a\sin(\theta)]$ and $f(\theta)$ absolutely integrable, is real analytic inside the disc $r < a$. Therefore, all harmonic functions $u(x,y)$ are analytic inside their domain of definition.

Proof The claim of this Theorem follows from the Poisson integral formula, the analyticity of the Poisson kernel and the fact that we can switch the integral $\int_{-\pi}^{\pi}$ with the infinite summation when we write the Poisson kernel as an infinite series in two variables. For instance, all hypotheses of **Theorem I 2.3.11** are fulfilled. (See also **Example I 2.3.8**.)

■

In this context we also state (without proof) the following Theorem[21] because it is very important and relevant.

Theorem 1.5.9 *At all points* $0 \le \phi \le 2\pi$ *at which* $f(\theta)$ *is continuous we have*

$$\lim_{(r,\theta)\to(a,\phi)} v(r,\theta) = f(\phi).$$

So, if $f(\theta)$ *is continuous for all* $0 \le \theta \le 2\pi$ *(and so it must be* 2π*-periodic) then the solution* $v(r,\theta)$ *of* ***Problem (1.12)*** *or* ***(1.13)*** *obtained as a Fourier series or by the Poisson integral formula extends continuously to all points of the boundary of the disc* $D(0,a)$ *with values equal to the boundary data, i.e.,* $v(a,\theta) = f(\theta)$ *for all* $0 \le \theta \le 2\pi$*.*

Remark: The two unbounded solutions

$$\ln(r), \quad r^{-n}, \quad n = 1,\ 2,\ 3,\ \dots,$$

found for $R(r)$, must also be used when we solve the Laplace equation in an annulus $A(0, r_1, r_2)$, since we stay away from $r = 0$. The same is valid for any domain that does not contain a neighborhood of the origin. (See **Problem 1.5.44**.)

The Mean Value Property

By the Poisson integral formula, we find that

$$u(0,0) = v(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\, d\phi.$$

Similarly, for any $0 < \rho \le a$ fixed, we get that

$$u(0,0) = v(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\phi)\, d\phi.$$

That is, the value of u (or v) at the center of the disc is equal to the mean (average) value of u (or v) along the circumference of any circle with center the center of the disc and radius $0 < \rho \le a$.

[21] For the **proof**, see the bibliography, e.g., Weinberger 1965, sections 24 and 25.

In general, we consider a harmonic function $u : \mathcal{U} \longrightarrow \mathbb{R}$, *where* $\mathcal{U} \subseteq \mathbb{R}^2$ *open. Then, for any point* $P = (x_0, y_0) \in \mathcal{U}$, *any* $r > 0$ *such that* $D(P, r) \subset \mathcal{U}$ *and any* $0 \leq \rho \leq r$, *we let*

$$v(\rho, \phi) := u[x_0 + \rho\cos(\phi), y_0 + \rho\sin(\phi)]$$

and we analogously have

$$\forall \;\; 0 < \rho \leq r, \quad u(P) = u(x_0, y_0) = v(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho, \phi)\, d\phi.$$

This property of harmonic functions is called the **mean value property of harmonic functions**.

Example 1.5.21 We can easily check that $u(x, y) = \ln(x^2 + y^2)$ is a harmonic function in $\mathbb{R}^2 - \{(0, 0)\}$. So, by the mean value property at the point $(x_0, y_0) = (2, 3)$ and with $0 < \rho = 1 < r = \sqrt{13}$, we get

$$\int_{-\pi}^{\pi} \ln\left\{[2 + \cos(\phi)]^2 + [3 + \sin(\phi)]^2\right\} \stackrel{\rho=1}{=}$$
$$\int_{-\pi}^{\pi} \ln[14 + 4\cos(\phi) + 6\sin(\phi)]\, d\phi = 2\pi\ln(13) = 16.11605212\ldots \stackrel{\rho=2}{=}$$
$$\int_{-\pi}^{\pi} \ln[17 + 8\cos(\phi) + 12\sin(\phi)]\, d\phi, \quad \text{etc.}$$

▲

From the above mean value property we obtain an **equivalent mean value property**. Integrating the equation of the above mean value property, we obtain

$$\int_0^r v(0, \theta)\rho\, d\rho = \frac{1}{2\pi}\int_0^r \int_{-\pi}^{\pi} v(\rho, \phi)\, d\phi\, \rho\, d\rho, \implies$$
$$u(P) = v(0, \theta) = \frac{1}{\pi r^2}\int\int_{D(P, r)} u(x, y)\, dA,$$

where $dA = dxdy = \rho\, d\rho\, d\phi$ is the area element in $\mathbb{R}^2$.

We obtain the converse of the equivalence of these two mean value properties of the harmonic functions by differentiating with respect to r the equation

$$\pi r^2 u(x_0, y_0) = \int_0^r \int_{-\pi}^{\pi} u[x_0 + \rho\cos(\phi), y_0 + \rho\sin(\phi)]\, d\phi\, \rho\, d\rho. \quad \text{(Check.)}$$

That is, we have obtained the following **equivalent mean value property** of the harmonic functions:

Let $u : \mathcal{U} \longrightarrow \mathbb{R}$, where $\mathcal{U} \subseteq \mathbb{R}^2$ open is harmonic. Then, for any point $P = (x_0, y_0)$ and $r > 0$ such that $D(P, r) \subset \mathcal{U}$, we have

$$u(x_0, y_0) = \frac{1}{\text{Area}[D(P, r)]} \iint\limits_{D(P,r)} u(x, y)\, dA.$$

Application: Suppose $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a square integrable harmonic function. Then, for any $P = (x_0, y_0) \in \mathbb{R}^2$ and $r > 0$, by the mean value property and **Hölder's inequality in** $\mathbb{R}^2$ [e.g., see **project Problem I 2.6.65, Item (1.)**], we have

$$|u(x_0, y_0)| = \frac{1}{\text{Area}[D(P, r)]} \int\int_{D(P,r)} |u(x, y)| \cdot 1\; dA \leq$$

$$\frac{1}{\text{Area}[D(P, r)]} \left[\int\int_{D(P,r)} u^2(x, y)\; dA\right]^{\frac{1}{2}} \left[\int\int_{D(P,r)} 1^2\; dA\right]^{\frac{1}{2}}.$$

Now, let $C = \int_{\mathbb{R}^2} u^2(x, y)\; dA \geq 0$, a finite constant by hypothesis. Then, as $r \longrightarrow \infty$, we get

$$|u(x_0, y_0)| \leq \frac{1}{\text{Area}[D(P, r)]} \sqrt{C}\, \{\text{Area}[D(P, r)]\}^{\frac{1}{2}} = \sqrt{\frac{C}{\pi r}} \longrightarrow 0.$$

Therefore, $u(x_0, y_0) = 0$ at any $P = (x_0, y_0) \in \mathbb{R}^2$. So, the only harmonic function defined at every point of $\mathbb{R}^2$ which is square integrable is $u \equiv 0$.

Problems

1.5.37 Check that the functions $R(r) = 1$ and $R(r) = \ln(r)$ are linearly independent solutions to the Laplace equation in polar coordinates.

1.5.38 Find all solutions $v(r)$ to the Laplace equation $\Delta v = 0$ in polar coordinates that are independent of θ, and all solutions $v(\theta)$ that are independent of r.

1.5.39 Prove that the Poisson kernel, **Definition 1.5.3**, is harmonic in the disc $r < a$.

1.5.40 Find the function $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ to which the series of functions in **Example I 2.3.8** converges uniformly on $[-\pi, \pi]$.

1.5.41 Let $\mathcal{R} \subseteq \mathbb{C}$ be a region and $w = f(z) : \mathcal{R} \longrightarrow \mathbb{C} - \{0\}$ holomorphic. Prove $\text{Log}\,[\,|f(z)|\,]$ is harmonic.

1.5.42 Use **Definition 1.5.3**, and prove that for any $m \in \mathbb{N}_0$

$$\int_{-\pi}^{\pi} P_{(a,\phi)}(r,\theta)\cos(m\theta)\,d\theta = \int_{0}^{2\pi} P_{(a,\phi)}(r,\theta)\cos(m\theta)\,d\theta = 2\pi\left(\frac{r}{a}\right)^m.$$

Compare also with **Example 1.8.2** and **Problem 1.5.46, (c)**.

[Hint: You may use the **Weierstraß M-Test, Theorem I 2.3.3,**, to justify the term-by-term integration and use the theory of **Section I 2.3**.]

1.5.43 (a) Prove

$$\int_{-\pi}^{\pi} \arctan\left[\frac{1+\sqrt{2}\cos(\phi)}{\sqrt{3}+\sqrt{2}\sin(\phi)}\right] d\phi = \frac{\pi^2}{3}.$$

(b) Prove for every $\rho > 0$,

$$\int_{-\pi}^{\pi} e^{\pm\rho\cos(\phi)}\cos[\rho\sin(\phi)]\,d\phi = 2\pi$$

and

$$\int_{-\pi}^{\pi} e^{\pm\rho\cos(\phi)}\sin[\rho\sin(\phi)]\,d\phi = 0.$$

1.5.44 Solve the Laplace equation $\Delta u = 0$ in the annulus $A(0, 0 < R_1 < R_2 < \infty)$ with boundary conditions:
(a) $u(R_1,\theta) = 0$ and $u(R_2,\theta) = f(\theta)$.
(b) $u(R_1,\theta) = g(\theta)$ and $u(R_2,\theta) = 0$.
(c) $u(R_1,\theta) = g(\theta)$ and $u(R_2,\theta) = f(\theta)$.
(See the **Remark** after **Theorem 1.5.9**.)

1.5.45 Let $w = Re^{i\phi}$ and $z = re^{i\theta}$ with $0 \le r < R$.
(a) Prove

$$|w-z|^2 = (w-z)\overline{(w-z)} = (w-z)(\overline{w}-\overline{z}) = R^2 - 2Rr\cos(\theta-\phi) + r^2.$$

(b) Prove

$$\mathrm{Re}\left(\frac{w+z}{w-z}\right) = \frac{1}{2}\left[\frac{w+z}{w-z} + \overline{\left(\frac{w+z}{w-z}\right)}\right] = \frac{1}{2}\left(\frac{w+z}{w-z} + \frac{\overline{w}+\overline{z}}{\overline{w}-\overline{z}}\right) =$$

$$\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}, \quad \text{the Poisson kernel.}$$

(c) Find the $\quad \operatorname{Im}\left(\dfrac{w+z}{w-z}\right)$.

(d) Prove

$$\operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{w}{w-z} - \frac{w}{w-\frac{R^2}{\overline{z}}}.$$

(e) Prove that $dw = iw\,d\phi$ and that the number $\dfrac{R^2}{\overline{z}}$ lies outside the closed disc $\overline{D(0,R)}$.

(f) Use **(d)**, **(e)**, the **Cauchy integral formula** (**Theorem 1.5.7**) and **(b)** above, to prove that if $f(z)$ is a holomorphic function in an open region $\mathcal{R}$ that contains the closed disc $\overline{D(0,R)}$ and $z = re^{i\theta}$, where $0 \le r \le R$, then

$$f(z) = f\left(re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f\left(Re^{i\phi}\right)}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, d\phi.$$

(g) Under the same assumptions as in **(f)** and for a closed disc such that $\overline{D(a,R)} \subset \mathcal{R} \subseteq \mathbb{C}$, the formula in **(f)** becomes

$$f(a+z) = f\left(a + re^{i\theta}\right) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f\left(a + Re^{i\phi}\right)}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, d\phi.$$

(h) Use **equation (1.14)**, **(b)**, **(f)** and **(g)** to prove that any real harmonic function $u(x,y)$ in an open region $\mathcal{R} \subseteq \mathbb{C}$ containing the closed disc $\overline{D(0,R)}$, and if $z = x + iy = re^{i\theta}$, satisfies

$$u(x,y) = u(z) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) u\left(Re^{i\phi}\right)\, d\phi =$$
$$\operatorname{Re}\left[\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{Re^{i\phi}+z}{Re^{i\phi}-z}\right) u\left(Re^{i\phi}\right)\, d\phi\right].$$

[Analogously for any $D(a,R) \subseteq \mathcal{R} \subseteq \mathbb{C}$.] So now prove that any real harmonic function $u(x,y)$ in an open region $\mathcal{R}$ is the real part of a holomorphic function.

1.5.46

(a) Use $u \equiv 1$ in the **formula 3.14** to prove that for any $r < a$ and any θ

$$\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}\, d\phi = 2\pi.$$

(See also **Example 1.8.2** and **Problem 1.8.3**.)

(b) For any $0 \leq r < a$ and any θ: Use $u(x, y) = x$ and limits to prove that

$$\int_0^{2\pi} \frac{(a^2 - r^2)\cos(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi = \frac{2\pi r}{a}\cos(\theta).$$

Now use $u(x, y) = y$ and limits to prove that

$$\int_0^{2\pi} \frac{(a^2 - r^2)\sin(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi = \frac{2\pi r}{a}\sin(\theta).$$

[Compare with **Problems 1.8.4** and **1.8.11, (d)**.]

(c) Differentiate

$$P_{(a,\phi)}(r, \theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} =$$

$$-1 + 2\sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta - \phi)] = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos[n(\theta - \phi)],$$

where $0 \leq r < a$ and θ, ϕ are reals, with respect to each variable to derive new series expressions. Make sure that the operations are legitimate.

Do similar work with integration.

(See also **Problem 1.5.42**.)

Application to Holomorphic Functions

Suppose we have the solutions $u(r, \theta)$ and $v(r, \theta)$ of two boundary value problems of the type **(1.12)** or **(1.13)** that we solved before, given in polar variables (r, θ) by the series

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right],$$

and

$$v(r, \theta) = c_0 + \sum_{n=1}^{\infty} \left[c_n r^n \cos(n\theta) + d_n r^n \sin(n\theta)\right],$$

where the coefficients a_n's, b_n's, c_n's, d_n's are real constants. So, these two functions, $u(r, \theta)$ and $v(r, \theta)$, are harmonic.

We moreover assume that $u(r, \theta)$ and $v(r, \theta)$ satisfy the Cauchy-Riemann conditions (written here in polar coordinate form)

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad u_\theta = -r v_r.$$

Then, developing the Cauchy-Riemann conditions for these functions (given in the above series forms), we find that:

$$\forall\ n = 1,\ 2,\ 3,\ \dots,\quad a_n = d_n \quad \text{and} \quad b_n = -c_n,$$

(check this easily). So,

$$u(r,\theta) + iv(r,\theta) =$$

$$a_0 + ic_0 + \sum_{n=1}^{\infty}\{a_n r^n[\cos(n\theta) + i\sin(n\theta)] + b_n r^n[\sin(n\theta) - i\cos(n\theta)]\} =$$

$$a_0 + ic_0 + \sum_{n=1}^{\infty}\{a_n r^n[\cos(n\theta) + i\sin(n\theta)] - ib_n r^n[\cos(n\theta) + i\sin(n\theta)]\} =$$

$$a_0 + ic_0 + \sum_{n=1}^{\infty}\{(a_n - ib_n) r^n[\cos(n\theta) + i\sin(n\theta)]\} = C_0 + \sum_{n=1}^{\infty} C_n z^n,$$

where $C_0 = a_0 + ic_0$, $C_n = a_n - ib_n$ and by the **De Moivre formula** we have:

$$r^n[\cos(n\theta) + i\sin(n\theta)] = z^n = (x+iy)^n.$$

Therefore, the function

$$f(z) = f\left(r[\cos(\theta) + i\sin(\theta)]\right) = u(r,\theta) + iv(r,\theta) = C_0 + \sum_{n=1}^{\infty} C_n z^n,$$

$$\text{or} \qquad f(z) = f(x+iy) = C_0 + \sum_{n=1}^{\infty} C_n z^n =$$

$$C_0 + \sum_{n=1}^{\infty} C_n (x+iy)^n = U(x,y) + iV(x,y),$$

is a complex power series of the complex variable z, and therefore it is holomorphic. So, we have:

Theorem 1.5.10 *If the functions $u(x,y)$ and $v(x,y)$ are harmonic in a disc $D(0,a)$, ($a > 0$ constant) and they satisfy the Cauchy-Riemann conditions, then the function $u(x,y)+iv(x,y)$ is a holomorphic function $f(z)$ in the complex variable $z = x+iy$.*

Problem 1.5.47 Suppose the function $f\ :\ \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic.

(a) Prove that $g(z) = \overline{f(\bar{z})}$ is also holomorphic.
(b) Under what condition(s) $g = f$?
(c) Give an example such that $g \neq f$.

End of Appendix 1.5.9.

1.6 Roots, Singularities, Residues

1.6.1 Definitions, Laurent Expansion and Examples

We consider an open region $\mathcal{R} \subseteq \mathbb{C}$ and a holomorphic function $f : \mathcal{R} \longrightarrow \mathbb{C}$, that is, $f'(z)$ exists $\forall\, z \in \mathcal{R}$. Since $\mathcal{R}$ is open, for any $z \in \mathcal{R}$ there is a number $r = r(z) > 0$ such that $D(z, r) \subseteq \mathcal{R}$. That is, all points of $\mathcal{R}$ are interior points. Also, by **Theorem 1.5.1**, $f(z)$ is a power series, $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, in any disc $D(z_0, r) \subseteq \mathcal{R}$, with $r > 0$.

Definition 1.6.1 *We call a* $z_0 \in \mathcal{R}$ ***root of order or multiplicity*** $m = 1,\ 2,\ 3, \ldots$ *of* $f(z)$*, if there is* $r > 0$ *such that in the open disc* $D(z_0, r) \subseteq \mathcal{R}$ *we can write*

$$f(z) = (z - z_0)^m g(z), \quad \forall\, z \in D(z_0, r),$$

where $g(z)$ *is a holomorphic function in* $D(z_0, r)$ *and* $g(z_0) \neq 0$.

A root of order (or multiplicity) one is also called ***a simple root****.*

This definition implies immediately that: $z_0 \in \mathcal{R}$ **is a root of order or multiplicity** $m \geq 1$ of the holomorphic function $f(z)$ if and only if $f(z_0) = f'(z_0) = f''(z_0) = \ldots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$. This is a convenient condition to detect the order m of a root. It is easily seen by differentiating the $f(z) = (z - z_0)^m g(z)$ m times and using $g(z_0) \neq 0$.

(Many times, we need to find the order of a root. Usually this condition is convenient to use. Give a complete proof of this "if and only if" claim, using the analyticity of the functions $f(z)$ and $g(z)$. See also **Example 1.6.13** for a special case.)

Example 1.6.1 $z_0 = 0$ is a root of order $m = 3$ of the holomorphic function $f(z) = z^2 \sin z$, which is defined in all $\mathbb{C}$.

Indeed: $f(0) = 0^2 \sin(0) = 0 \cdot 0 = 0$,
$f'(0) = 2z \sin(z) + z^2 \cos(z)|_{z=0} = 0 + 0 = 0$,
$f''(0) = 2 \sin(z) + 4z \cos(z) - z^2 \sin(z)|_{z=0} = 0 + 0 - 0 = 0$, and
$f^{(3)}(0) = 6 \cos(z) - 6z \sin(z) - z^2 \cos(z)|_{z=0} = 6 - 0 - 0 = 6 \neq 0$.

All the other roots of $f(z)$ are the real numbers $z_k = k\pi$, with $k \neq 0$ integer, which are the other roots of $\sin(z)$. All of them have order $m = 1$.

Indeed: $f(z_k) = z_k^2 \sin(z_k) = k^2\pi^2 \sin(k\pi) = k^2\pi^2 \cdot 0 = 0$ and
$f'(z_k) = 2z \sin(z) + z^2 \cos(z)|_{z_k} = 0 + k^2\pi^2(-1)^k = k^2\pi^2(-1)^k \neq 0$.

▲

Next we consider an open region $\mathcal{R} \subseteq \mathbb{C}$ and z_0 (an interior) point of $\mathcal{R}$. We define:

Definition 1.6.2 *Suppose that $z_0 \in \mathcal{R}$ and $f : \mathcal{R} - \{z_0\} \longrightarrow \mathbb{C}$ is a holomorphic function [that is, $f(z)$ is defined at every point of $\mathcal{R}$ except z_0]. Then we call the point z_0 an* ***isolated singularity*** *of $f(z)$.*

The region $\mathcal{R}$ in this definition could for convenience be just an open disc $D(z_0, r)$, for some $r > 0$. I.e., $f(z)$ is defined in the punctured disc $\mathcal{R} = D^{\circ}(z_0, r)$.

Definition 1.6.3 *An isolated singularity is called* ***a removable singularity*** *or* ***non-essential singularity*** *if we can extend the function $f(z)$ to the point z_0 (i.e., we can define $f(z)$ at z_0) so that the extended function is holomorphic in the entire $\mathcal{R}$ (including z_0).*

If we cannot do this, then this singularity is called a ***non-removable*** *or* ***essential singularity****.*

So, if z_0 is a removable singularity of $f(z)$, then at $z = z_0$ we can assign an appropriate value $f(z_0)$ of $f(z)$ so that the complex derivative $f'(z_0)$ exists at $z = z_0$, too. If z_0 is a non-removable singularity, it is impossible to achieve such a thing.

Definition 1.6.4 *A singularity z_0 of $f(z)$ which is not isolated is called* ***a non-isolated singularity.***

So, if z_0 is a non-isolated singularity of $f(z)$, then $\forall\ r > 0$ such that $D(z_0, r) \subseteq \mathcal{R}$ there exists $z_r \in D(z_0, r)$ such that $z_r \neq z_0$ and z_r is a singularity of $f(z)$. Therefore, there exists a sequence (z_n), $n \in \mathbb{N}$, of singularities of $f(z)$, converging to z_0, as $n \to \infty$.

Now we state a partial version of the Laurent Series Expansion Theorem around an isolated singularity, which is sufficient for the scope of this text.

Theorem 1.6.1 (Laurent Series Expansion, a partial version) *We consider an open region $\mathcal{R} \subseteq \mathbb{C}$ and $z_0 \in \mathcal{R}$ an isolated singularity of a holomorphic function $f : \mathcal{R} - \{z_0\} \longrightarrow \mathbb{C}$. Then we can find $0 < r \leq \infty$ such that $D(z_0, r) = \{ z \in \mathbb{C} \ : \ |z - z_0| < r\} \subseteq \mathcal{R}$, and the following expansion of $f(z)$*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \qquad (1.15)$$

$$\forall\ z \in D^{o}(z_0, r) = D(z_0, r) - \{z_0\}$$

holds for appropriate complex coefficients a_n's and b_n's, and it is unique.

This series expansion is called **the Laurent**[22] **series expansion of** $f(z)$ **with center** z_0**.** (This expansion was also known to Weierstraß.)

For the sake of brevity, we omit the proof even though we have developed enough tools to provide it. (Consult any pertinent bibliography, in which you can find the most general statement of this Theorem and its proof.)

From the form of this expansion, we easily observe (for instance, using **Theorem 1.5.6** or the **Cauchy integral formula 1.5.8**, etc.) that the **coefficients of the Laurent expansion** can be found by

$$a_n = \frac{1}{2\pi i} \oint\limits_{C^+(z_0,\,\rho)} f(z)(z - z_0)^{-(n+1)}\, dz, \quad \forall\ n = 0,\ 1,\ 2,\ 3, \dots\,,$$

and

$$b_n = \frac{1}{2\pi i} \oint\limits_{C^+(z_0,\,\rho)} f(z)(z - z_0)^{(n-1)}\, dz, \quad \forall\ n = 1,\ 2,\ 3,\ 4, \dots,$$

where $0 < \rho < r$. These coefficients, evaluated by these formulae, do not depend on the choice of ρ, $0 < \rho < r$, and so the Laurent series expansion of a function about a given center z_0 is unique.

The Laurent series expansion **(1.15)** reduces to the Taylor series expansion of $f(z)$ with center $z = z_0$ when $f(z)$ has no singularity in $D(z_0, r)$, i.e., $f(z)$ is holomorphic at z_0 and/or z_0 is a removable singularity of $f(z)$. In such a case, $b_n = 0$, $\forall\ n \in \mathbb{N}$ and $a_n = \dfrac{f^{(n)}(z_0)}{n!}$, $\forall\ n \in \mathbb{N}_0$. $f(z)$ is a power series with center z_0 in the whole disc $D(z_0, r)$ and $f(z_0) = a_0$.

In general, some or all of the a_n's and/or some or all b_n's may be zero. If all the b_n's are zero, then $f(z)$ has a removable singularity at z_0. In such a case, $f(z_0) = a_0$ and $f(z)$ is holomorphic at $z = z_0$, also.

This version of **this Theorem** is not the most general one, but it is sufficient for the needs we face in this text. The interested reader can find the theorem in various intermediate and/or advanced level books of complex analysis. In the general version, $f(z)$ is holomorphic in the domain

$$\begin{gathered} \mathbb{C} - \{\text{a set of isolated singularities}\}, \quad \text{i.e.,} \\ \mathcal{R} = \mathbb{C} - \{\text{isolated singularities of } f(z)\}, \end{gathered}$$

and the center z_0 may be any point, regular or singular for $f(z)$. Then, the Laurent expansion may be found in any annulus

[22] Pierre Alphonse Laurent, French mathematician, 1813-1854.

$A = \{ z \in \mathbb{C} \ : \ \rho < |z| < r \} \subseteq \mathbb{C}$ that does not contain any singularity, and $0 \leq \rho < r \leq \infty$ are appropriate constants.

The maximum $r > 0$ for which expansion **(1.15)** holds is the distance of z_0 to the nearest singularity of $f(z)$ other than z_0. It is strictly positive because the singularity z_0 is isolated. I.e., in this case, the radius of convergence is the distance of the center z_0 of the Laurent series to the nearest isolated essential singularity. If there is no singularity in $\mathbb{C}$ other than z_0, then $r = \infty$. Also, if $f(z)$ has no singularity in the whole $\mathbb{C}$, then its Taylor series with any center has radius of convergence ∞.

If z_0 is a non-isolated singularity, then we cannot obviously have Laurent series expansion of $f(z)$ with center z_0. That is, we do not have Laurent series expansions with centers non-isolated essential singularities.

Definition 1.6.5 *In the Laurent expansion of $f(z)$ with center the isolated singularity z_0, the coefficient b_1 is special and is called* ***the residue of*** *$f(z)$* ***at*** *z_0.*

Notation: We write

$$b_1 = \operatorname*{Res}_{z=z_0} f(z), \quad \text{or} \quad b_1 = \operatorname*{Res}_{z=z_0} [f(z)].$$

Important Remark: We have seen in **Subsection 1.5.7**, and we shall see it again in the sequel, that the residues are very important in complex integration. We must also keep in mind that we do not have residues for non-isolated essential singularities. We cannot define them in such a case, because there is no Laurent series expansion around a non-isolated singularity. Also, at an isolated removable singularity, the Laurent series becomes a Taylor series, and so the residue at an isolated removable singularity is equal to zero.

Question: Why have we singled out b_1 and given it a special name?
Answer: This is because we have already seen that b_1 plays the most important role in the line integral along any simple, closed, continuous and piecewise continuously differentiable path C that encloses z_0. Namely, we have seen several times in previous sections that

$$\oint_{C^\pm} f(z)\, dz = \pm 2\pi i b_1 = \pm 2\pi i \operatorname*{Res}_{z=z_0} f(z).$$

So, finding the isolated singularities and their corresponding residues is the most important piece of information in computing complex line integrals and in turn two real line integrals.

Example 1.6.2 (a) Consider the function

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

It has two isolated singularities at $z_1 = 1$ and $z_2 = 1$.

(1) In the semi-open annulus $A(0, r_1 = 0, r_2 < 1)$ which is the open disc $D(0,1)$, we have:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{-1}{2\left(1-\frac{z}{2}\right)} + \frac{1}{1-z} =$$
$$-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n,$$

which is the Taylor series of $f(z)$ in $D(0,1)$.

(2) Similarly in the open annulus $A(0, r_1 = 1, r_2 = 2)$, we find

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \sum_{n=0}^{\infty}\frac{-1}{2^{n+1}} z^n + \sum_{n=1}^{\infty}\frac{-1}{z^n},$$

in which both parts of the Laurent expansion appear.

(3) Finally in the unbounded open annulus $A(0, r_1 = 2, r_2 = \infty)$, we find

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \sum_{n=2}^{\infty}\frac{2^{n-1}-1}{z^n}.$$

The Laurent series of $f(z)$ with center $z_0 = 1$ in the open annulus $A(1, r_1 = 0, r_2 = 1)$ is

$$f(z) = \frac{-1}{1-(z-1)} - \frac{1}{z-1} = \sum_{n=0}^{\infty}(-1)(z-1)^n - \frac{1}{z-1}.$$

Similarly the Laurent series of $f(z)$ with center $z_0 = 2$ in the open annulus $A(2, r_1 = 0, r_2 = 1)$ is

$$f(z) = \frac{1}{z-2} - \frac{1}{(z-2)+1} = \sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n + \frac{1}{z-2}.$$

We see that $\operatorname*{Res}_{z=1}[f(z)] = -1$ and $\operatorname*{Res}_{z=2}[f(z)] = 1$. So,

$$\oint_{C^{\pm}(1,\ 0.5)} f(z)\, dz = \mp 2\pi i,$$

and

$$\oint_{C^{\pm}(2,\ 0.5)} f(z)\,dz = \pm 2\pi i.$$

With the help of the geometric series, we find the following Laurent series expansions with center $z_0 = 0$:

(b) In the open annulus $A(0, r_1 = 0, r_2 = \infty) = \mathbb{C} - \{0\}$, we have

$$g(z) := e^z + e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 2 + \sum_{n=1}^{\infty} \frac{z^n}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}.$$

At the isolated singularity $z_0 = 0$, we see that $\operatorname*{Res}_{z=0}[g(z)] = 1$. So,

$$\oint_{C^{\pm}(0,\rho)} g(z)\,dz = \pm 2\pi i, \quad \forall \ \rho > 0.$$

(c) In the open annulus $A(0, r_1 = 1, r_2 = \infty)$, after some work, we have

$$h(z) := \sin\left(\frac{1}{z-1}\right) = \frac{1}{z} + \frac{1}{z^2} + \frac{5}{6} \cdot \frac{1}{z^3} + \frac{1}{2} \cdot \frac{1}{z^4} + \dots .$$

The function here has an isolated singularity at $z = 1$.

In the open disc $D(0, r_1 = 1)$, the function $h(z) := \sin\left(\frac{1}{z-1}\right)$ is holomorphic and so we find its Taylor series

$$h(z) := \sin\left(\frac{1}{z-1}\right) = -\sin(1) - \frac{\cos(1)}{1!}\, z - \frac{\sin(1) + 2\cos(1)}{2!}\, z^2 + \dots .$$

But, in the open annulus $A(z_0 = 1, r_1 = 0, r_2 = \infty)$, by the power series expansion of $\sin(z)$, we obtain the Laurent series

$$h(z) := \sin\left(\frac{1}{z-1}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{1}{(z-1)^{2k+1}}.$$

(See also **Example 1.6.6**.)

From this, we see that $\operatorname*{Res}_{z=1}[h(z)] = 1$. So,

$$\oint_{C^{\pm}(1,\rho)} h(z)\,dz = \pm 2\pi i, \quad \forall \ \rho > 0.$$

▲

A **Theorem** characterizing removable (isolated) singularities is the following:

Theorem 1.6.2 (Riemann) *Let $\mathcal{R} \subseteq \mathbb{C}$ be an open region, z_0 be an interior point of $\mathcal{R}$ and $f : \mathcal{R} - \{z_0\} \longrightarrow \mathbb{C}$ be a holomorphic function. Then the following three statements are equivalent.*

(a) The isolated singularity z_0 of $f(z)$ is removable.

(b) The $\lim_{z \to z_0} f(z)$ exists as a complex number L.

(c) There is some $r > 0$ such that $\overline{D(z_0, r)} \subset \mathcal{R}$ and $f(z)$ is bounded on the closed punctured disc $\overline{D(z_0, r)}^{\,o}$.

Proof (a) $\Longrightarrow$ (b) Since we assume that z_0 is a removable singularity of $f(z)$, by definition we can define $f(z)$ at z_0 such that $f(z_0) = L$ and $f'(z_0)$ exists. So, $f(z)$ is continuous in all $\mathcal{R}$, and therefore $\lim_{z \to z_0} f(z) = L$ exists.

(b) $\Longrightarrow$ (c) Suppose $\lim_{z \to z_0} f(z) = L \in \mathbb{C}$ exists as a complex number L. Then at $z = z_0$, we define $f(z)$ by $f(z_0) = L$. This assignment makes $f(z)$ continuous in all $\mathcal{R}$.

Since $\mathcal{R}$ is an open set, there is $\rho > 0$ such that $D(z_0, \rho) \subseteq \mathcal{R}$. Then for any $0 < r < \rho$, we have $\overline{D(z_0, r)} \subseteq \mathcal{R}$. Then $f(z)$ is continuous on the closed and bounded set $\overline{D(z_0, r)}$ and so, by the Extreme Value Theorem, is bounded on $\overline{D(z_0, r)}$.

Another way to show this boundedness is by the definition of limit. We have that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - L| < \epsilon$. So, for any $0 < r < \delta$ we get that if $|z - z_0| < r$, then $|f(z)| < |L| + \epsilon$. Therefore, $f(z)$ is bounded on $\overline{D(z_0, r)}$ by $|L| + \epsilon$.

(c) $\Longrightarrow$ (a) We suppose that there is some $r > 0$ such that $\overline{D(z_0, r)} \subset \mathcal{R}$ and $f(z)$ is bounded on closed punctured disc $\overline{D(z_0, r)}^{\,o}$. Then in $\mathcal{R}$, we define

$$g(z) = \begin{cases} (z - z_0)^2 f(z), & \text{if} \quad z \neq z_0, \\ \\ 0, & \text{if} \quad z = z_0. \end{cases}$$

Obviously, $g'(z)$ exists for all $z \neq z_0$. At $z = z_0$, we observe that if $z \in \overline{D(z_0, r)}^{\,o}$, then we have

$$\lim_{z_0 \neq z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} (z - z_0) f(z) = 0,$$

since $\lim_{z \to z_0} (z - z_0) = 0$ and $f(z)$ is bounded on $\overline{D(z_0, r)}^{\,o}$. Hence, $g'(z_0) = 0$ also exists. That is, $g(z)$ is holomorphic in all $\mathcal{R}$.

Then $g(z)$ is a power series in $D(z_0, r)$. I.e.,

$$g(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots\,, \quad \forall \; z \in D(z_0, r).$$

Since $g(z_0) = 0$ and $g'(z_0) = 0$, we have $c_0 = 0$ and $c_1 = 0$. So, $g(z) = c_2(z-z_0)^2 + c_3(z-z_0)^3 + \ldots = (z-z_0)^2 f(z), \quad \forall\ z \in D(z_0, r)$.

Hence, $f(z) = c_2 + c_3(z-z_0) + c_4(z-z_0)^2 + \ldots, \quad \forall\ z \in D(z_0, r)^o$. That is, $f(z)$ is a power series with center z_0 in $D(z_0, r)^o$ and therefore in $D(z_0, r)$, if we set $f(z_0) = c_2$. Thus, $f(z)$ extends to z_0 holomorphically, and so z_0 is a removable singularity. [Finally, $f(z)$ is holomorphic in the whole $\mathcal{R}$.]

■

Remark: The proof of the **Part** [$\underline{\textbf{(c)} \Longrightarrow}$**(a)**] carries through if we replace the boundedness of $f(z)$ on $\overline{D(z_0, r)}^{\,o}$ with the more general condition:

$\exists\ M > 0$ *and* $\alpha < 1$ *real constants, such that*

$$|f(z)| \leq \frac{M}{|z-z_0|^\alpha}, \ \ \forall\ z \in \overline{D(z_0, r)}^{\,o}.$$

From this Theorem, we immediately have the following:

Corollary 1.6.1 *Let $\mathcal{R} \subseteq \mathbb{C}$ be open, $E \subset \mathcal{R}$ be a finite subset of $\mathcal{R}$. We assume that the function $f : \mathcal{R} \longrightarrow \mathbb{C}$ is continuous in the whole $\mathcal{R}$ and holomorphic in $\mathcal{R} - E$. Then $f(z)$ is holomorphic in the whole $\mathcal{R}$ (i.e., it has complex derivative even at the exceptional points of E).*

Important Remark: Another way, besides invoking the **above Theorem**, to **prove this Corollary** is achieved by invoking **Morera's Theorem, 1.5.5,** and check that it applies. Moreover, **Morera's Theorem** proves **this Corollary** even when the exceptional set E contains not only finitely many isolated points of $\mathcal{R}$ but even segments of paths in $\mathcal{R}$ on which the function $f(z)$ is continuous, but we know nothing about the existence of complex derivatives on them. Eventually, $f(z)$ has complex derivative even on those exceptional segments.

Definition 1.6.6 *An isolated singularity z_0 of an otherwise holomorphic function $f(z)$ in a disc $D(z_0, r)$ $(r > 0)$ is called a* ***pole of order*** *$m \geq 1$ of $f(z)$, if in the* ***Laurent expansion (1.15)***

$$b_n = 0, \ \ \forall\ n \geq m+1, \quad but \quad b_m \neq 0.$$

The expression

$$\sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n} = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \ldots + \frac{b_1}{z-z_0}$$

is called the ***principal part of the pole*** *z_0.*

A ***pole of order one*** *is also called a* ***simple pole****.*

We observe that a pole is an essential (non-removable) singularity. If in the Laurent series infinitely many b_n's are non-zero, then the singularity z_0 is essential but not a pole.

In analogy with the definition of a root of order m, we easily observe that ***this definition of a pole is equivalent to the condition***

$$(z-z_0)^m f(z) = g(z) \Longleftrightarrow f(z) = (z-z_0)^{-m} g(z),$$

where $g(z)$ is a holomorphic function in a disc $D(z_0, r)$, for some $r > 0$, and $(b_m =) g(z_0) \neq 0$.

(Many times, we need to find the order of a pole. Usually this condition is convenient to use. Give a complete proof of this claim. See also **Example 1.6.13** for a special case.)

A **Theorem** characterizing a pole is the following:

Theorem 1.6.3 *Let $\mathcal{D} \subseteq \mathbb{C}$ be open, $z_0 \in \mathcal{D}$ and $f : \mathcal{D} - \{z_0\} \longrightarrow \mathbb{C}$ be a holomorphic function. Then z_0 is a pole of $f(z)$ if and only if $\lim_{z \to z_0} f(z) = \infty$.*

Proof ($\Longrightarrow$) This is immediately true since for any function the principal part of a pole approaches ∞, as $z \longrightarrow z_0$, and also the power series part of the **Laurent expansion (1.15)** is equal to the constant a_0 at z_0.

($\Longleftarrow$) If $\lim_{z \to z_0} f(z) = \infty$, then $f(z)$ cannot have any roots in a disc $D(z_0, \rho) \subseteq \mathcal{D}$ for some $\rho > 0$ and also $\lim_{z \to z_0} \dfrac{1}{f(z)} = \dfrac{1}{\infty} = 0$.

Therefore, for any $0 < r < \rho$, the function $g(z) := \dfrac{1}{f(z)}$ is bounded on $\overline{D(z_0, r)}$, and it is holomorphic in $\overline{D(z_0, r)} - \{z_0\}$.

So, $g(z_0) = 0$, and by the **previous Theorem** $g(z)$ is holomorphic in $\overline{D(z_0, r)}$. Then we let $k \geq 1$ be the order of this root of $g(z)$, and by definition there is a holomorphic function $h(z)$ in $D(z_0, r)$ and

$$g(z) = (z-z_0)^k h(z), \ \ \forall \ z \in D(z_0, r) \ \text{ and with } \ h(z_0) \neq 0.$$

Then for sufficiently small $r > 0$, we have $h(z) \neq 0$, $\forall \ z \in D(z_0, r)$, and

$$f(z) = (z-z_0)^{-k} \frac{1}{h(z)}.$$

Since $\dfrac{1}{h(z)}$ is holomorphic in $D(z_0, r)$, it is a power series

$$\frac{1}{h(z)} = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad \forall \ z \in D(z_0, r), \quad \text{with} \quad c_0 \neq 0.$$

Hence, we have: $c_0 \neq 0$ and $\forall\ z \in D(z_0, r)$

$$f(z) = (z-z_0)^{-k} \sum_{n=0}^{\infty} c_n (z-z_0)^n =$$

$$\frac{c_0}{(z-z_0)^k} + \ldots + \frac{c_{k-1}}{z-z_0} + c_k + c_{k+1}(z-z_0) + c_{k+2}(z-z_0)^2 + \ldots\,,$$

thus proving that z_0 is a pole of $f(z)$ with order k.

$\blacksquare$

Because of this result, we say that any polynomial of degree ≥ 1 has a pole at the complex infinity. For any such polynomial $p(z)$, we obviously have $\lim\limits_{z\to\infty} p(z) = \infty$. (See also **Corollary 1.7.1**.)

In view of the **previous two Theorems**, we conclude that z_0 is an essential singularity of $f(z)$ which is not a pole if and only if the $\lim\limits_{z\to z_0} f(z)$ oscillates among the complex numbers. Hence, we have shown that if $\mathcal{D} \subseteq \mathbb{C}$ is an open set and $z_0 \in \mathcal{D}$ is an isolated singularity of a holomorphic function $f(z)$ in $\mathcal{D} - \{z_0\}$, then:

1. z_0 is a removable singularity of $f(z) \iff \lim\limits_{z\to z_0} f(z) := L \in \mathbb{C}$.

2. z_0 is a pole of $f(z) \iff \lim\limits_{z\to z_0} f(z) = \infty$ (complex infinity).

3. z_0 is an essential singularity but not a pole of $f(z) \iff \lim\limits_{z\to z_0} f(z)$ oscillates in $\mathbb{C}$.

Also, in view of the **Laurent expansion (1.15)**, the **previous two Theorems** and comments, we conclude that:

1. z_0 is a removable singularity of $f(z) \iff$ in **(1.15)** $b_n = 0,\ \forall\ n \in \mathbb{N}$.

2. z_0 is a pole of $f(z) \iff$ in **(1.15)** $b_n \neq 0$, for finitely many n's. The order of the pole is the maximum n for which $b_n \neq 0$.

3. z_0 is an essential singularity but not a pole of $f(z) \iff$ in **(1.15)** $b_n \neq 0$, for infinitely many n's.

Examples

Example 1.6.3 The function $f(z) = e^z$ is holomorphic in the entire $\mathbb{C}$. (We know $f'(z) = e^z,\ \ \forall\ z \in \mathbb{C}$). Therefore, $f(z) = e^z$ has no singularity, and at any point $z_0 \in \mathbb{C}$ the residue of $f(z) = e^z$ is zero ($b_1 = 0$).

$\blacktriangle$

Example 1.6.4 The function $f(z) = \dfrac{z^4 - 1}{z - 1}$ has an isolated singularity at $z = 1$. This isolated singularity is removable because

$$\lim_{1 \neq z \to 1} f(z) = \lim_{z \to 1} \left(z^3 + z^2 + z + 1\right) = 4.$$

So, the extended function

$$g(z) = \begin{cases} f(z), & \text{if} \quad z \neq 1, \\ \\ 4, & \text{if} \quad z = 1, \end{cases}$$

is not only continuous but also holomorphic in the whole $\mathbb{C}$. In fact,

$$\frac{dg(z)}{dz}\Big|_{z=1} = (3z^2 + 2z + 1)|_{z=1} = 6.$$

The residue of $f(z)$ at $z = 1$ is zero (i.e., $b_1 = 0$).

▲

Example 1.6.5 The function $f(z) = \dfrac{e^z - z - 1}{z^2}$ has an isolated singularity at $z = 0$. This isolated singularity is removable because

$$\lim_{\substack{z \to 0 \\ z \neq 0}} f(z) = \lim_{z \to 0} \left(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots\right) = \frac{1}{2}.$$

So, the extended function

$$g(z) = \begin{cases} f(z), & \text{if} \quad z \neq 0, \\ \\ \dfrac{1}{2}, & \text{if} \quad z = 0, \end{cases}$$

is not only continuous but also holomorphic in $\mathbb{C}$. In fact,

$$\frac{dg(z)}{dz}\Big|_{z=0} = \frac{1}{3!} = \frac{1}{6}.$$

The residue of $f(z)$ at $z = 0$ is zero ($b_1 = 0$).

▲

Example 1.6.6 Since

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right),$$

we have that $z_0 = 0$ is a root of $\sin(z)$ of order one. The $\sin(z)$ is holomorphic in the entire $\mathbb{C}$. Therefore, at any $z_0 \in \mathbb{C}$, the residue of $\sin(z)$ is $b_1 = 0$.

Also, the function

$$g(z) = \frac{\sin(z)}{z}, \quad \forall \ z \neq 0,$$

has an isolated singularity at $z = 0$. This singularity is removable. Indeed, we observe that

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots, \quad \forall \ z \neq 0,$$

and so we can extend $g(z)$ at $z = 0$ by defining $g(0) = 1$. The extended $g(z)$ is a power series which is defined and converges for all $z \in \mathbb{C}$. Therefore, it is holomorphic in the entire $\mathbb{C}$. The residue of $g(z)$ at $z_0 = 0$ is $b_1 = 0$.

But,

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots, \quad \forall \ z \neq 0.$$

Therefore, $\sin\left(\frac{1}{z}\right)$ has an isolated singularity at $z_0 = 0$, which is not removable and it is not a pole. The corresponding residue is $b_1 = 1$.

Also,

$$\lim_{z \to 0} \sin\left(\frac{1}{z}\right) = \text{does not exist. (This limit oscillates!)}$$

▲

Example 1.6.7 The function

$$f(z) = \frac{(z-1)^{10}}{z-3}$$

has a root of order 10 at $z_0 = 1$ and a singularity at $z_0 = 3$. To find the corresponding residue, we must observe that

$$f(z) = \frac{(z-1)^{10}}{z-3} = \frac{[(z-3)+2]^{10}}{z-3} =$$

$$\frac{\sum_{k=0}^{10} \binom{10}{k} (z-3)^{10-k} 2^k}{z-3} = \left[\sum_{k=0}^{9} 2^k \binom{10}{k} (z-3)^{10-k-1}\right] + \frac{2^{10}}{z-3} =$$

$$\left[\sum_{k=0}^{9} 2^k \binom{10}{k} (z-3)^{9-k}\right] + \frac{2^{10}}{z-3}.$$

This shows that $z_0 = 3$ is a pole of order one with corresponding residue $b_1 = 2^{10} = 1024$. [Observe that, $\lim_{z \to 3} f(z) = \infty$.]

▲

Example 1.6.8 We know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots$$

is defined everywhere in $\mathbb{C}$, and it has no root and no singularity. But,

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} - \dots,$$

is not defined at $z_0 = 0$ and cannot be extended to it as a holomorphic function. Therefore, $z_0 = 0$ is an essential singularity, which is not a pole. The corresponding residue is $b_1 = 1$.

Also,

$$\lim_{z \to 0} e^{\frac{1}{z}} = \text{does not exist.}$$

This limit oscillates! For instance: If $x \in \mathbb{R}$, then

$$\lim_{x \to 0^+} e^{\frac{1}{x}} = e^{\infty} = \infty,$$

and

$$\lim_{x \to 0^-} e^{\frac{1}{x}} = e^{-\infty} = 0.$$

Similarly, if $y \in \mathbb{R}$, then

$$\lim_{y \to 0^\pm} e^{\frac{1}{iy}} = \lim_{y \to 0^\pm} e^{\frac{-i}{y}} = \lim_{y \to 0^\pm} \left[\cos\left(\frac{1}{y}\right) - i \sin\left(\frac{1}{y}\right)\right] = \text{does not exist.}$$

▲

Example 1.6.9 We have seen that the continuous branch of $\log(z)$

$$f(z) = \log(z), \quad z \in \mathbb{C} - \{a \mid a \leq 0\}$$

is holomorphic, since

$$\frac{d}{dz} \log(z) = \frac{1}{z}, \quad z \in \mathbb{C} - \{a \mid a \leq 0\}.$$

Here, we consider $-\pi < \operatorname{Arg}(z) < \pi$.

So, for this branch of the complex logarithm, all the points of the non-positive x-axis, or the whole branch cut, are singularities. This function is discontinuous at each of these points, and therefore it is impossible to be extended holomorphically at any one of them. (Continuity is a necessary condition for holomorphicity.) These singularities form a continuous closed half line starting at the origin, and therefore they are not isolated. If z_0 is one of these singularities, then for any $r > 0$ the disc $D(z_0, r)$ obviously contains the segment $(z_0 - r, z_0 + r)$ which consists of singularities. Keep in mind: **There are no residues for non-isolated singularities**.

▲

Example 1.6.10 The function

$$f(z) = \frac{1}{1 - e^{\frac{1}{z}}}$$

has countably infinitely many singularities, the points $z_0 = 0$ and $z_n = \dfrac{1}{2n\pi i}$, $n = \pm 1,\ \pm 2,\ \pm 3, \ldots$.

$z_0 = 0$ is a non-isolated singularity because $z_n = \dfrac{1}{2n\pi i} \to z_0 = 0$, as $n \to \pm\infty$. So, we cannot find the residue of $f(z)$ at $z_0 = 0$. All the other singularities (z_n), $n \in \mathbb{Z} - \{0\}$, are isolated, and we can compute the corresponding residues.

▲

Example 1.6.11 Let $f(z)$ be a holomorphic function in some region $\mathcal{R}$ which contains z_0 as interior point and $f(z_0) \neq 0$. Then the function

$$g(z) = \frac{f(z)}{(z - z_0)^m}, \quad z \in R - \{z_0\}$$

has a pole of order m at z_0. This follows from the fact that there is $r > 0$ such that $D(z_0, r) \subseteq \mathcal{R}$ and

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad z \in D(z_0, r),$$

where $c_n = \dfrac{f^{(n)}(z_0)}{n!}$ are the **Taylor coefficients** of $f(z)$ at z_0 and $n = 0,\ 1,\ 2,\ 3, \ldots$. So,

$$g(z) = \frac{c_0}{(z - z_0)^m} + \frac{c_1}{(z - z_0)^{m-1}} + \ldots + \frac{c_{m-2}}{(z - z_0)^2} + \frac{c_{m-1}}{z - z_0} + \\ c_m + c_{m+1}(z - z_0) + c_{m+2}(z - z_0)^2 + \ldots .$$

Since $c_0 = f(z_0) \neq 0$, by definition, z_0 is a pole of order m for the function $g(z)$. Its residue at $z = z_0$ is

$$\operatorname*{Res}_{z=z_0} g(z) = c_{m-1} = \frac{f^{(m-1)}(z_0)}{(m-1)!}.$$

Also,

$$\lim_{z \to z_0} g(z) = \infty.$$

▲

Example 1.6.12 The polynomial $p(z) = z^2 + z + 1$ has the two roots

$$z_1 = \frac{-1 + i\sqrt{3}}{2}, \quad z_1 = \frac{-1 - i\sqrt{3}}{2}.$$

So, the function

$$f(z) = \frac{1}{z^2 + z + 1} = \frac{1}{(z - z_1)(z - z_2)} = \frac{-i\sqrt{3}}{2}\left(\frac{1}{z - z_1} - \frac{1}{z - z_2}\right).$$

has two singularities at the points $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$ and $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$.

Using these partial fractions, we find that the residue of $f(z)$ at $z_1 = \dfrac{-1 + i\sqrt{3}}{2}$ is

$$\operatorname*{Res}_{z=z_1} f(z) = \operatorname*{Res}_{z=z_1} \frac{1}{z^2 + z + 1} = \frac{-i\sqrt{3}}{3}$$

and the residue at $z_2 = \dfrac{-1 - i\sqrt{3}}{2}$ is

$$\operatorname*{Res}_{z=z_2} f(z) = \operatorname*{Res}_{z=z_2} \frac{1}{z^2 + z + 1} = \frac{i\sqrt{3}}{3}.$$

▲

Example 1.6.13 Let $f(z)$ have a root of order k and $g(z)$ have a root of order l at z_0, where $f(z)$ and $g(z)$ are both holomorphic functions in some region which contains z_0 as an interior point. Then, there is some $r > 0$ such that

$$f(z) = (z - z_0)^k \, p(z), \quad \forall \;\; z \in D(z_0, r),$$

with $p(z)$ holomorphic in $D(z_0, r)$ and $p(z_0) \neq 0$. Similarly, there is some $\rho > 0$ such that

$$g(z) = (z - z_0)^l \, q(z), \quad \forall \;\; z \in D(z_0, \rho),$$

with $q(z)$ holomorphic in $D(z_0, \rho)$ and $q(z_0) \neq 0$.

Then we have the **rule**: At $z = z_0$, the function

$$\frac{f(z)}{g(z)} \quad \text{has} \quad \begin{cases} \text{Regular point (not a root)}, & \text{if } k = l. \\ \text{Root of order } k - l, & \text{if } k > l. \\ \text{Isolated singularity, pole of order } m = l - k, & \text{if } k < l. \end{cases}$$

In the latter case, for $\sigma = \min\{r, \rho\}$ we have

$$\frac{f(z)}{g(z)} = \frac{p(z)}{(z - z_0)^{l-k} q(z)}, \quad \forall \ z \in D^o(z_0, \sigma).$$

If $l - k = 1$, that is, z_0 is a simple pole, then

$$b_1 = \operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{p(z_0)}{q(z_0)}.$$

If $l - k > 1$, then by **Example 1.6.11**, the residue is

$$b_1 = \operatorname*{Res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{\left[\frac{p(z)}{q(z)}\right]^{(l-k-1)}\Big|_{z=z_0}}{(l-k-1)!}.$$

In the next section, we develop several ways to compute the residues of functions and of various fractions $\dfrac{f(z)}{g(z)}$ in particular.

▲

1.6.2 Five Ways to Evaluate Residues

Suppose that $w = f(z)$ is a holomorphic function in $D^o(z_0, r)$ for some $r > 0$. Then z_0 is an isolated singularity of $f(z)$. We want to have some convenient ways to compute its residue at z_0, denoted by $\operatorname*{Res}_{z=z_0} f(z)$.

Way 1. We have already seen that if C is any simple, closed, continuous and piecewise continuously differentiable path in $D^o(z_0, r)$ that encloses z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_{C^+} f(z)\, dz.$$

In particular, we can choose C to be any convenient circumference of a circle $C(z_0, \rho)$ with $\rho < r$.

Way 2. If in any way we manage to find the Laurent Series of $f(z)$ with center z_0

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} b_n(z-z_0)^{-n}, \ \ \forall\ z \in D^o(z_0, r),$$

then we readily have

$$\operatorname*{Res}_{z=z_0} f(z) = b_1.$$

Way 3. (**Example 1.6.11** revisited.) Suppose that

$$f(z) = \frac{h(z)}{(z-z_0)^m}, \ \ \forall\ z \in D^o(z_0, r),$$

with $h(z)$ holomorphic in all $D(z_0, r)$, $h(z_0) \neq 0$ and m some positive integer (fixed). In this situation, z_0 is a pole of order m. Then we can use the formula:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[h(z)\right]\Big|_{z=z_0} =$$
$$\frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}.$$

This follows from the fact that $h(z)$ can be written as a power series

$$h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n, \ \ \forall\ z \in D^o(z_0, r),$$

with

$$c_n = \frac{h^{(n)}(z_0)}{n!}.$$

So, we have that

$$\operatorname*{Res}_{z=z_0} f(z) = c_{m-1} = \frac{h^{(m-1)}(z_0)}{(m-1)!} = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}.$$

Remark: There is no such formula for an isolated essential singularity that it is not a pole! (This is another substantial difference between a pole and an isolated essential singularity which is not a pole!)

Way 4. Sometimes the function $h(z)$ in **Way 3** is not expressed in a convenient closed form, and so the computation of its derivatives is not efficient. Then, to find the $\operatorname*{Res}_{z=z_0} f(z)$ we use a manipulation similar to the one explained in the **following example**. In fact, with this method we compute the whole principal part of a pole $z = z_0$. Many times, in complex analysis, we need the whole principal part and not just the residue of a pole!

Example 1.6.14 We consider the function

$$f(z) = \frac{\cot(z)}{z^2} = \frac{\cos(z)}{z^2 \sin(z)} =$$
$$\frac{\cos(z)}{z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}} = \frac{\cos(z)}{z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}}.$$

Since $\cos(0) = 1$, $f(z)$ has singularity at $z_0 = 0$.

If we set

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n},$$

this power series is convergent $\forall\ z \in \mathbb{C}$ (use, for instance, the **Ratio Test** to prove this), and $g(0) = 1$. So, there is some $\epsilon_1 > 0$ such that $g(z) \neq 0, \forall\ z \in D(0, \epsilon_1)$. Now, to compare with **Way 3**, we let

$$h(z) = \frac{\cos(z)}{g(z)}, \ \ \forall\ z \in D(0, \epsilon_1).$$

We have that $h(0) = \frac{1}{1} = 1 \neq 0$. So, there is $0 < \epsilon \leq \epsilon_1$ such that $h(z) \neq 0$, and it is holomorphic $\forall\ z \in D(0, \epsilon)$. Then in $D(0, \epsilon)$, we can write $h(z)$ as power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \neq 0, \ \ \forall\ z \in D(0, \epsilon),$$

(with $c_0 = h(0) = 1$). Since $f(z) = \frac{h(z)}{z^3}$, we get

$$f(z) = \frac{c_0}{z^3} + \frac{c_1}{z^2} + \frac{c_2}{z} + c_3 + c_4 z + c_5 z^2 + c_6 z^3 + \dots .$$

We rewrite this as

$$f(z) = \frac{b_3}{z^3} + \frac{b_2}{z^2} + \frac{b_1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots .$$

We now see that $z_0 = 0$ is a pole of $f(z)$ of order three, and we must compute the residue b_1. For this, we work as follows. First, we see that

$$b_3 = z^3 f(z)|_{z=0} = \left.\frac{z \cos(z)}{\sin(z)}\right|_{z=0} = \frac{0}{0} =?.$$

At this point, we use the **complex L' Hôpital's rule** which is applied in exactly the same way as in the real case of an **indeterminate limit of the form** $\frac{0}{0}$, which we know from calculus. **(Remember to simplify the expressions to the fullest every time you apply L' Hôpital's rule).** So,

$$b_3 = \frac{[z\cos(z)]'}{[\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z)}{\cos(z)}\Big|_{z=0} = \frac{1-0}{1} = 1.$$

Then,

$$b_2 = z^2\left[f(z) - \frac{1}{z^3}\right]\Big|_{z=0} =$$
$$\left(\frac{\cos(z)}{\sin(z)} - \frac{1}{z}\right)\Big|_{z=0} = \frac{z\cos(z) - \sin(z)}{z\sin(z)}\Big|_{z=0} =$$
$$\frac{[z\cos(z) - \sin(z)]'}{[z\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z) - \cos(z)}{\sin(z) + z\cos(z)}\Big|_{z=0} =$$
$$\frac{[-z\sin(z)]'}{[\sin(z) + z\cos(z)]'}\Big|_{z=0} = \frac{-\sin(z) - z\cos(z)}{\cos(z) + \cos(z) - z\sin(z)}\Big|_{z=0} = \frac{0}{2} = 0.$$

Finally,

$$b_1 = z\left[f(z) - \frac{1}{z^3} - \frac{0}{z^2}\right]\Big|_{z=0} =$$
$$\left(\frac{\cos(z)}{z\sin(z)} - \frac{1}{z^2}\right)\Big|_{z=0} = \frac{z\cos(z) - \sin(z)}{z^2\sin(z)}\Big|_{z=0} =$$
$$\frac{[z\cos(z) - \sin(z)]'}{[z^2\sin(z)]'}\Big|_{z=0} = \frac{\cos(z) - z\sin(z) - \cos(z)}{2z\sin(z) + z^2\cos(z)}\Big|_{z=0} =$$
$$\frac{[-\sin(z)]'}{[2\sin(z) + z\cos(z)]'}\Big|_{z=0} = \frac{-\cos(z)}{2\cos(z) + \cos(z) - z\sin(z)}\Big|_{z=0} = \frac{-1}{3}.$$

So,

$$\operatorname*{Res}_{z=0} f(z) = \frac{-1}{3}.$$

By analyzing the method of this rather complicated example, we could write a general algorithm for finding the residues in such cases.

▲

Way 5. Here we redo the **previous Example** in a way that combines the initial part of **Way 4** and the formula of **Way 3**. This way is also convenient in complicated cases.

So, we reconsider the **previous Example**. There, we came up with

$$f(z) = \frac{\cot(z)}{z^2} = \frac{1}{z^3}\frac{\psi(z)}{g(z)},$$

where $\psi(z) = \cos(z)$ and, as before,

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = 1 - \frac{1}{6} z^2 + \frac{1}{120} z^4 - \dots .$$

Then $\psi(0) = \cos(0) = 1,\ \ \psi'(0) = -\sin(0) = 0,\ \ \psi''(0) = -\cos(0) = -1$, etc., and $g(0) = 1,\ \ g'(0) = 0,\ \ g''(0) = \dfrac{-2}{6} = \dfrac{-1}{3}$, etc.

So, we conclude that $z = 0$ is a pole of order three and then, according to the formula in **Way 3**, we get:

$$\begin{gathered}
\operatorname*{Res}_{z=0} f(z) = \\
\frac{1}{(3-1)!} \left[z^3 f(z) \right]'' \Big|_{z=0} = \\
\frac{1}{2!} \left[\frac{\psi(z)}{g(z)} \right]'' \Big|_{z=0} = \frac{1}{2} \left[\psi(z) g^{-1}(z) \right]'' \Big|_{z=0} = \\
\frac{1}{2} \left[\psi''(z) g^{-1}(z) - 2\psi'(z) g^{-2}(z) g'(z) + \right. \\
\left. 2\psi(z) g^{-3}(z) (g')^2(z) - \ \psi(z) g^{-2}(z) g''(z) \right] \big|_{z=0} = \\
\frac{1}{2} \left[\psi''(0) g^{-1}(0) - 2\psi'(0) g^{-2}(0) g'(0) + \right. \\
\left. 2\psi(0) g^{-3}(0) (g')^2(0) - \psi(0) g^{-2}(0) g''(0) \right] = \\
\frac{1}{2} \left[-1 \cdot 1 - 2 \cdot 0 \cdot 1 \cdot 0 + 2 \cdot 1 \cdot 1 \cdot 0 - 1 \cdot 1 \cdot \left(\frac{-1}{3} \right) \right] = \\
\frac{1}{2} \left(-1 + \frac{1}{3} \right) = -\frac{1}{3}.
\end{gathered}$$

Long Division of Power Series and Residues

The **example** of computing the residue of

$$\begin{gathered}
f(z) = \frac{\cot(z)}{z^2} = \frac{\cos(z)}{z^2 \sin(z)} = \\
\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{\displaystyle z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+3}}
\end{gathered}$$

at the pole $z = 0$, treated in **Ways 4** and **5** before, can be dealt with faster with the long division of power series.

We perform a few steps of the long division of the two Maclaurin series written in increasing order of the powers of z

$$f(z) = \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots}{z^3 - \frac{z^5}{6} + \frac{z^7}{120} - \dots},$$

using the usual scheme that we use in a calculus or analysis course and in applications. For this particular example, we need three quick steps, and we find that the initial five terms of the quotient are

$$z^{-3} + 0z^{-2} - \frac{1}{3}z^{-1} + 0 - \frac{1}{45}z \dots .$$

(The terms that do not appear have coefficients equal to zero. Check this for yourselves.)

Faster than in **Ways 4** and **5** above, we find again that the whole **principal part of the pole** $z = 0$ is

$$z^{-3} + 0z^{-2} - \frac{1}{3}z^{-1}$$

and then

$$\operatorname*{Res}_{z=0} f(z) = \frac{-1}{3}.$$

Leibniz Rule for Higher Derivatives of Products

To compute higher derivatives of products, something that appears often while computing residues, we find it convenient to use the **Leibniz Rule for higher derivatives of products**. This is stated as follows:

For any two differentiable functions $f(z)$ and $g(z)$ with high order derivatives, the n$^{\text{th}}$ derivative of their product is

$$\frac{d^n}{dz^n}[f(z) \cdot g(z)] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}[f(z)] \cdot \frac{d^k}{dz^k}[g(z)], \tag{1.16}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. We remember that $0! = 1$ and so $\binom{n}{0} = 1$. For $k \in \mathbb{N}$, we simplify

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k(k-1)\dots 3 \cdot 2 \cdot 1}.$$

This rule is easily proven by mathematical induction.

Examples

Example 1.6.15 For $a,\ b,\ c,\ d \in \mathbb{C}$ constants and $n,\ l,\ m, \in \mathbb{N}$, by the Leibniz rule for higher derivatives of products, we have

$$\frac{d^n}{dz^n}[(az+b)^l \cdot (cz+d)^m] =$$

$$\sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}[(az+b)^l] \cdot \frac{d^k}{dz^k}[(cz+d)^m] =$$

$$\sum_{k=0}^{n} \binom{n}{k} l\,(l-1)\dots(l-n+k+1)a^{n-k}(az+b)^{l-n+k} \ \times$$

$$m(m-1)\dots(m-k+1)c^k(cz+d)^{m-k}.$$

Notice that if p and q are in $\mathbb{N}$, then the product $p\,(p-1)\dots(p-q+1)$ will be zero for all $q \geq p+1$.

▲

Example 1.6.16 For $f(z) = \dfrac{z+1}{z-2}$, find the $\operatorname*{Res}_{z=2} f(z)$.

(a) Let $C = C^+(2,r)$ be the positively oriented circle of center $z_0 = 2$ and radius some $r > 0$. Then

$$b_1 = \frac{1}{2\pi i}\oint_C \frac{z+1}{z-2}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{2+re^{i\theta}+1}{2+re^{i\theta}-2} rie^{i\theta}\,d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (3+re^{i\theta})\,d\theta = \frac{1}{2\pi}(3 \cdot 2\pi + 0) = 3.$$

(b) We have $f(z) = \dfrac{z+1}{z-2} = \dfrac{z-2+3}{z-2} = 1 + \dfrac{3}{z-2}$. So,

$$\operatorname*{Res}_{z=2} f(z) = b_1 = 3.$$

(c) By **Example 1.6.13**, $z = 2$ is a pole of order one, and so we can apply the formula in **Way 3** to find

$$b_1 = (z-2)f(z)|_{z=2} = (z+1)|_{z=2} = 2+1 = 3.$$

▲

Example 1.6.17 Find the $\operatorname*{Res}_{z=2}\left[\dfrac{z^2-2z+5}{(z-2)^3}\right]$.

We see that $z = 2$ is an isolated singularity for the given rational function (fraction of two polynomials), since at $z = 2$ the numerator is

$2^2 - 2 \cdot 2 + 5 = 5 \neq 0$. We can use any of the ways of the **previous Example** to find that

$$\operatorname*{Res}_{z=2}\left[\frac{z^2-2z+5}{(z-2)^3}\right] = b_1 = 1.$$

For instance,

$$f(z) = \frac{z^2-2z+5}{(z-2)^3} = \frac{z^2-4z+4+2z-4+5}{(z-2)^3} =$$
$$\frac{(z-2)^2+2(z-2)+5}{(z-2)^3} = \frac{1}{z-2} + \frac{2}{(z-2)^2} + \frac{5}{(z-2)^3}.$$

So, $\quad b_1 = 1$.

Or, by **Example 1.6.13**, $z = 2$ is a pole of order three, and so we can apply the formula in **Way 3** to find

$$b_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-2)^3 f(z)\right]\Big|_{z=2} = \frac{1}{2}\frac{d^2}{dz^2}(z^2-2z+5)|_{z=2} = \frac{1}{2}\cdot(2) = 1.$$

▲

Example 1.6.18 Find the $\operatorname*{Res}_{z=1}\left[\dfrac{e^z}{(z-1)^2}\right]$.

Since $e^1 = e \neq 0$, by **Example 1.6.13**, $z = 1$ is a pole of order two, and so we can apply the formula in **Way 3** to find the residue

$$\operatorname*{Res}_{z=1}\left[\frac{e^z}{(z-1)^2}\right] = \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2\frac{e^z}{(z-1)^2}\right]\Big|_{z=1} = 1\cdot(e^z)'\Big|_{z=1} = e.$$

Also, using the power series expansion of e^z about $z = 1$, we get

$$e^z = e \cdot e^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}.$$

Therefore,

$$f(z) = e\sum_{n=0}^{\infty}\frac{(z-1)^{n-2}}{n!} = e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \sum_{n=2}^{\infty}\frac{(z-1)^{n-2}}{n!}\right].$$

So,

$$\operatorname*{Res}_{z=1}\left[\frac{e^z}{(z-1)^2}\right] = e \cdot 1 = e.$$

▲

Example 1.6.19 Find the residue of

$$f(z) = \exp\left(z + \frac{1}{z}\right) \quad \text{at} \quad z = 0.$$

We have that

$$\exp\left(z+\frac{1}{z}\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(z+\frac{1}{z}\right)^{n}=$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left[\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,z^{k}\,\frac{1}{z^{n-k}}\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\frac{1}{k!(n-k)!}\,z^{2k-n}\right].$$

We observe that $z=0$ is an essential singularity which is not a pole, since there are infinitely many negative exponents. In this example, the sought residue is the total coefficient of $z^{-1}=\dfrac{1}{z}$.

A non-zero coefficient of $\dfrac{1}{z}$ is achieved whenever $2k-n=-1$, or $n=2k+1$, i.e., whenever n is odd. If $n=2k+1$, for $k=0,\ 1,\ 2,\ \ldots,$ the corresponding summand of the coefficient of $\dfrac{1}{z}$ is written as

$$\frac{1}{k!(n-k)!}=\frac{1}{k!(2k+1-k)!}=\frac{1}{k!(k+1)!}=\frac{1}{(k!)^{2}(k+1)}.$$

By adding all of these summands for $k=0,\ 1,\ 2,\ldots$, we find

$$\operatorname*{Res}_{z=0}\ \exp\left(z+\frac{1}{z}\right)=\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}=\sum_{k=0}^{\infty}\frac{1}{(k!)^{2}(k+1)}.$$

▲

Example 1.6.20 For any $a\in\mathcal{R}$ constant, find the residues of

$$f(z)=\sin\left(\frac{1}{z}\right)\cdot e^{az}\quad\text{at}\quad z=0.$$

We have that for $z\neq 0$

$$f(z)=\left(\frac{1}{z}-\frac{1}{3!}\cdot\frac{1}{z^{3}}+\frac{1}{5!}\cdot\frac{1}{z^{5}}-\ldots\right)\cdot$$
$$\left[1+az+\frac{1}{2!}(az)^{2}+\frac{1}{3!}(az)^{3}+\frac{1}{4!}(az)^{4}+\ldots\right].$$

So, we observe that we can get $\dfrac{1}{z}$ in infinitely many ways which give as its coefficient

$$\operatorname*{Res}_{z=0}f(z)=1-\frac{a^{2}}{2!\cdot 3!}+\frac{a^{4}}{4!\cdot 5!}-\frac{a^{6}}{6!\cdot 7!}+\ldots=\sum_{n=0}^{\infty}\frac{(-1)^{n}\,a^{2n}}{(2n)!\cdot(2n+1)!}.$$

▲

Example 1.6.21 Find the residues of

$$f(z) = \frac{z-2}{z^2(z-1)} \quad \text{at} \quad z=0 \quad \text{and} \quad z=1.$$

By **Example 1.6.13**, $z = 0$ is a pole of order two and $z = 1$ is a pole of order one. So, we can apply the formula in **Way 3** to find

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\frac{d}{dz}\left[z^2 f(z)\right]\Big|_{z=0} = \frac{d}{dz}\left(\frac{z-2}{z-1}\right)\Big|_{z=0} =$$
$$\frac{(z-1)-(z-2)}{(z-1)^2}\Big|_{z=0} = \frac{1}{1} = 1.$$

Similarly,

$$\operatorname*{Res}_{z=1} f(z) = \left[(z-1)f(z)\right]\Big|_{z=1} = \frac{z-2}{z^2}\Big|_{z=1} = \frac{-1}{1} = -1.$$

▲

Example 1.6.22 Find the residues of $f(z) = \dfrac{1}{(1+z^2)^{n+1}}$ at $z = -i$ and $z = +i$, for $n = 0,\ 1,\ 2,\ 3,\ \ldots$.

Since

$$f(z) = \frac{1}{(1+z^2)^{n+1}} = \frac{1}{(z-i)^{n+1}(z+i)^{n+1}},$$

we have that $z = -i$ and $z = +i$ are both poles of order $n + 1$. Then,

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left[(z-i)^{n+1}f(z)\right]\Big|_{z=i} = \frac{1}{n!}\frac{d^n}{dz^n}\left[\frac{1}{(z+i)^{n+1}}\right]\Big|_{z=i} =$$
$$\left[\frac{(-1)^n(n+1)(n+2)\ldots(2n)}{n!}\cdot\frac{1}{(z+i)^{2n+1}}\right]\Big|_{z=i} =$$
$$\frac{(-1)^n(n+1)(n+2)\ldots(2n)}{n!}\cdot\frac{1}{(2i)^{2n+1}} =$$
$$\frac{(-1)^n 1\cdot 2\ldots n(n+1)(n+2)\ldots(2n)}{(n!)^2\, 2^{2n+1}\, i^{2n}\, i} = \frac{-i(2n)!}{(n!)^2\, 2^{2n+1}} = \frac{-i}{2^{2n+1}}\binom{2n}{n}.$$

Similarly,

$$\operatorname*{Res}_{z=-i} f(z) = \frac{1}{n!}\frac{d^n}{dz^n}\left[(z+i)^{n+1}f(z)\right]\Big|_{z=-i} =$$
$$\frac{(-1)^n(2n)!}{(n!)^2\, 2^{2n+1}\,(-i)^{2n}(-i)} = \frac{-(2n)!}{(n!)^2\, 2^{2n+1}\, i} = \frac{i(2n)!}{(n!)^2\, 2^{2n+1}} = \frac{i}{2^{2n+1}}\binom{2n}{n}.$$

▲

Example 1.6.23 Find the residues of

$$f(z) = \frac{z^{2m}}{1 + z^{2n}}$$

at the roots of the denominator. Here $m,\ n = 1,\ 2,\ 3,\ \dots$.

The roots of the denominator are the complex numbers that satisfy the equation

$$z^{2n} = -1 = e^{\pi i}.$$

So, these roots are the $2n$ $(2\mathrm{n})^{\mathrm{th}}$ roots of -1

$$z_k = e^{i\frac{(2k+1)\pi}{2n}}, \quad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ \dots,\ 2n-1.$$

These roots are **simple**, that is, of order one, and so they are simple poles for the given function $f(z)$. Thus, for $k = 0,\ 1,\ 2,\ 3, \dots,\ 2n-1$ we have

$$\operatorname*{Res}_{z=z_k} f(z) = [(z - z_k) f(z)]\Big|_{z=z_k} = \left[\frac{(z - z_k) z^{2m}}{1 + z^{2n}}\right]\Big|_{z=z_k} =$$

$$(\text{use L' Hôpital's rule}) \ \ \frac{z^{2m} + (z - z_k) 2m z^{2m-1}}{2n z^{2n-1}}\Big|_{z=z_k} =$$

$$\frac{z_k^{2m}}{2n z_k^{2n-1}} = \big(\text{remember } z_k^{2n} = -1\big) \ \frac{-1}{2n} z_k^{2m+1} = \frac{-1}{2n} e^{i\frac{(2k+1)(2m+1)}{2n}\pi}.$$

(Notice how we use the **complex L' Hôpital's rule** in this example and imitate it in similar situations.)

▲

Problems

1.6.1 (a) Find the roots of $\sin(z)$ and their orders.

(b) Explain why $\sin(z)$ cannot be equal to $f^2(z)$ for some holomorphic function $f\ :\ \mathbb{C} \longrightarrow \mathbb{C}$.

(c) Do **(a)** and **(b)** for $\cos(z)$.

1.6.2 What kind of point is $z = 0$ for each function?

$$f(z) = \frac{\sin^5(z)}{z^2}, \qquad g(z) = \frac{\sin^2(z)}{z^5}, \qquad h(z) = \frac{\sin^5(z)}{z^5}.$$

[Hint: Find $\lim\limits_{z \to 0} \dfrac{\sin^5(z)}{z^5}$.]

1.6.3 Use mathematical induction to prove the **Leibniz rule for higher derivatives of products (1.16)**.

In **problems 1.6.4-1.6.16** that follow **compute the residues** of the given functions at the given **singular points**. First, justify why the given points are **isolated singularities** of the given functions. Otherwise, justify why they are **non-isolated singularities**.

1.6.4

$$f_1(z) = \frac{z+1}{(z-1)^3(z-4)}, \quad \text{at} \quad z_0 = 1 \quad \text{and} \quad z_0 = 4.$$

1.6.5

$$f_2(z) = \frac{1}{z^2+2z+2},$$

at the two roots of the denominator. (Find these roots first.)

1.6.6

$$f_4(z) = \frac{1}{e^z - 1},$$

at the infinitely many roots of the denominator. Find these roots and their orders first, then justify why they are isolated singularities and compute the corresponding residues.

1.6.7

$$f_3(z) = \frac{1}{\sin\left(\frac{1}{z^2}\right)}, \quad \text{at} \quad z_0 = 0.$$

1.6.8

$$f_5(z) = \frac{z^2}{1+z^4}, \quad \text{at all roots of the denominator.}$$

(Find these roots first. In exponential form, they are more convenient.)

1.6.9

$$f_6(z) = \frac{1}{1+z^5}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

1.6.10

$$f_7(z) = \frac{1}{1+z^{10}}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

1.6.11

$$f_8(z) = \frac{1}{z^2\sin(z)}, \quad \text{at} \quad z_0 = 0.$$

1.6.12

$$f_9(z) = \frac{e^z - 1}{\sin^3(z)}, \quad \text{at} \quad z_0 = 0.$$

1.6.13

$$f_{10}(z) = \frac{1}{z^5 - 1}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

1.6.14

$$f_{11}(z) = \frac{z^2}{z^4 - 1}, \quad \text{at all roots of the denominator.}$$

(Find these roots in exponential form.)

1.6.15

$$f_{12}(z) = \frac{\cot(z)}{z} = \frac{\cos(z)}{z\sin(z)}, \quad \text{at all roots of the denominator.}$$

1.6.16

$$f_{13}(z) = \frac{\cot(z)}{z^3} = \frac{\cos(z)}{z^3\sin(z)}, \quad \text{at all roots of the denominator.}$$

1.6.17 Prove:
(a) $\lim\limits_{z\to\infty} e^{-z} = \lim\limits_{z\to\infty} \dfrac{1}{e^z}$ does not exist. (b) $\forall\ a \in \mathbb{C}$, $\lim\limits_{z\to\infty} \dfrac{z}{z^2 \pm a} = 0$.

1.6.18 Consider the function $f(z) = \dfrac{1}{1 - e^{\frac{1}{z}}}$ in $\mathbb{C}$.

(a) Show that $f(z)$ has singularities at the points $z_0 = 0$ and $z_n = \dfrac{1}{2n\pi i}$ with $n = \pm 1,\ \pm 2,\ \pm 3,\ \dots$.

(b) Show that $z_0 = 0$ is a non-isolated singularity and all the other singularities are isolated.

(c) Compute the residues at the isolated singularities.

1.6.19 Study the example treated in "**Way 4**" of evaluating residues and write a general algorithm that always computes the residues when the singularity is a pole of order m, in the way described there.

1.6.20 (a) Prove the equalities

$$\left[z+\sqrt{z^2-1}\;\frac{w+\frac{1}{w}}{2}\right]^n \frac{1}{iw} = \frac{\left(\sqrt{z^2-1}\;w^2+2zw+\sqrt{z^2-1}\;\right)^n}{i2^n w^{n+1}} =$$

$$\begin{cases} \dfrac{\left[\sqrt{z^2-1}\;\left(w+\dfrac{z+1}{\sqrt{z^2-1}}\right)\left(w+\dfrac{z-1}{\sqrt{z^2-1}}\right)\right]^n}{i\,2^n\,w^{n+1}}, & \text{if} \quad z \neq \pm 1, \\ \dfrac{1}{iw}, & \text{if} \quad z = 1, \\ \dfrac{(-1)^n}{iw}, & \text{if} \quad z = -1. \end{cases}$$

(b) Consider z as a constant and w as a variable in the expression in **(a)**. Find the order of the pole $w = 0$ and compute the corresponding residue in each of the three cases.

1.6.21 Let Log(z) (with capital L) mean the holomorphic branch of the complex $\log(z)$ defined in the domain

$$D := \mathbb{C} - \{x \mid x \leq 0\} = \{\,(r,\,\theta) \mid r \geq 0, \text{ and } -\pi < \theta < \pi\,\}.$$

(a) Prove that the composite complex logarithmic function

$$h(z) := \mathrm{Log}(1-z) = \mathrm{Log}(1-x-iy)$$

is defined and is holomorphic in the domain

$$D := \mathbb{C} - \{x \mid 1-x \leq 0\} = \mathbb{C} - \{x \mid x \geq 1\}.$$

(b) Show that in this domain $\mathcal{D}$

$$\mathrm{Re}\,h(z) = \ln|1-z| \quad \text{and} \quad -\pi < \mathrm{Im}\,h(z) < \pi.$$

(c) Now consider the function $g(z) := \dfrac{h(z)}{z}$ in this domain $\mathcal{D}$ and prove that it has a removable singularity at $z = 0$ and non-isolated singularities at all real $x \geq 1$.

(d) What can you say about the residues of $g(z)$ at these singularities?

1.6.22 As we have proved in **property** $(\boldsymbol{\Gamma}, \mathbf{8})$,

$$\forall\; z \in \mathbb{C} - \{0,\ -1,\ -2,\ \ldots\},\ \ \Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\frac{1}{z+n} + \int_1^{\infty} x^{p-1}e^{-z}dz.$$

Prove that any $n \in \{0, -1, -2, \dots\}$ is a simple pole (order 1) with residue $\dfrac{(-1)^n}{n!}$.

[See also **Problems I 2.6.62**, and **1.2.38**.]

1.6.23 On the basis of what you know, prove the following result:

Let $r > 0$, $c \in \mathbb{C}$ and $f : D(c,r) \longrightarrow \mathbb{C}$ be a holomorphic function. There is an infinite sequence of complex numbers (z_n), $n \in \mathbb{N}$, in $D(c,r)$, such that $f(z_n) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = c$ if and only if $f(z) = 0$ for all $z \in D(c,r)$.

1.7 Contour Integration and Integrals

In this section, we will integrate holomorphic (complex analytic) functions along simple, closed, continuous and piecewise continuously differentiable paths. From now on, we shall call such paths **contours**.

1.7.1 Residue Theorem and Examples

The main tool of integration in the sequel is the following version of a generalization of **Theorem 1.5.6**. (In more advanced literature, we can find more general settings of this Theorem. For our level and purposes, the one presented here is enough.)

Theorem 1.7.1 (Residue Theorem) *Let $\mathcal{R} \subseteq \mathbb{C}$ be an open region and $w = f(z)$ be a complex holomorphic function in $\mathcal{R} - \{z_1, z_2, z_3, \dots, z_n, \dots, z_l\}$, where $1 \le n \le l$ and each point of the finite exceptional set $\{z_1, z_2, z_3, \dots, z_l\}$ is an isolated singularity of $f(z)$.*

Then, for any C simple, closed and piecewise continuously differentiable contour such that no singularity of $f(z)$ is on C and C encloses the isolated singularities $z_1, z_2, z_3, \dots, z_n$ but no other singularity, then for the line integral of $f(z)$ along C^+ or C^- the following integral equality holds, respectively:

$$\oint_{C^+} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathop{Res}_{z=z_k} f(z), \quad \text{or} \quad \oint_{C^-} f(z)\,dz = -2\pi i \sum_{k=1}^{n} \mathop{Res}_{z=z_k} f(z).$$

(If $l > n$, the singularities $\{z_{n+1}, \dots, z_l\}$ lie outside C and play no role in this result!)

Remark: If $n = 0$, then the empty summation is equal to zero, and we are in the case of the **Cauchy-Goursat Theorem, 1.5.3.**

In advanced literature, we find more general settings of this important Theorem, in which the contour may not be simple, and apart from the singularities the domain may have other holes within the contour, etc. We refer the interested reader to advanced bibliography on complex analysis. For our level and purposes, the version of this Theorem presented here is enough.

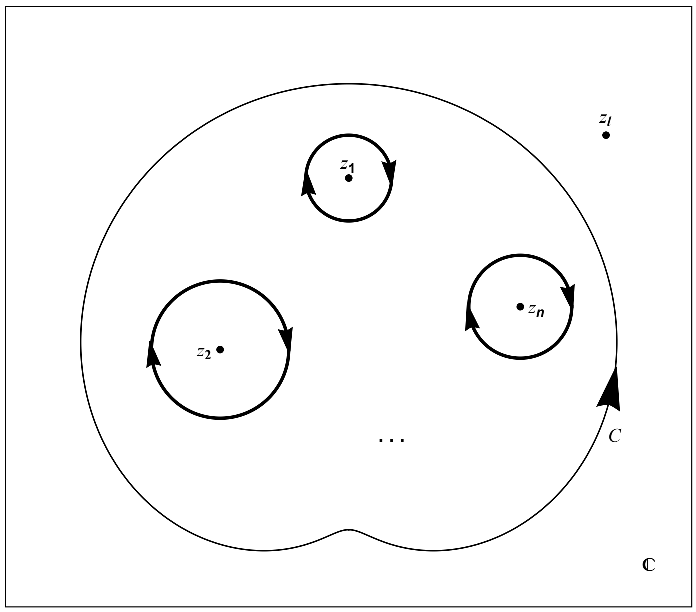

FIGURE 1.4: Contour and discs

Proof (Refer to **Figure 1.4**.) We consider open discs $D(z_k, r_k)$, where $r_k > 0$, for $k = 1,\ 2,\ 3,\ \dots,\ n$ small enough to lie in the inside of C and to be pairwise disjoint. Then, we apply Green's Theorem to the function $f(z)$ in the closed domain

$$\left\{ [C \cup (\text{inside of } C)] - \bigcup_{k=1}^{n} D(z_k, r_k) \right\} \subset \mathcal{R} \subseteq \mathbb{C}$$

in the same way as we did in **Theorem 1.5.6**. That Theorem dealt with only one singularity in the whole $\mathbb{C}$. Now, there may be any finite number of singularities in the region $\mathcal{R} \subseteq \mathbb{C}$. These generalizations do not change the course of the proof, and the result follows in the same way as it did then. (Study that proof one more time.)

■

We continue with the following definition:

Definition 1.7.1 *Let $w = f(z)$ be a complex function of $z \in \mathbb{C}$ holomorphic in $\mathbb{C} - \overline{D(0,R)} = \{z \in C \quad | \quad |z| > R\}$ [the complement of the closed disc $\overline{D(0,R)}$], for some $R > 0$. We call **residue at infinity** of $w = f(z)$ the quantity $\underset{z=\infty}{Res}\, f(z) := -\underset{z=0}{Res}\left[\frac{1}{z^2}\, f\left(\frac{1}{z}\right)\right]$.*

The following **Theorem** is interesting but of little use to applications:

Theorem 1.7.2 *Let $f(z)$ be a holomorphic function in $\mathbb{C}-\{a_1,\ a_2,\ \ldots,\ a_n\}$, where $n \geq 0$ integer, with singularities (isolated) at the points $a_1,\ a_2, \ldots, a_n$. Then,*

$$\underset{z=a_1}{Res}\, f(z) + \underset{z=a_2}{Res}\, f(z) + \ldots + \underset{z=a_n}{Res}\, f(z) + \underset{z=\infty}{Res}\, f(z) = 0.$$

Proof We consider any $R > 0$ for which $|a_i| < R$, for all $i = 1,\ 2, \ldots, n$ and $|a_i| > \dfrac{1}{R}$, for all $a_i \neq 0$. Then, by **Definition 1.6.5** or **Subsection 1.6.2, Way 1**, we have

$$\underset{z=\infty}{Res}\, f(z) := -\underset{z=0}{Res}\left[\frac{1}{z^2}\, f\left(\frac{1}{z}\right)\right] = -\frac{1}{2\pi i}\int_{C^+\left(0,\frac{1}{R}\right)} \frac{1}{z^2}\, f\left(\frac{1}{z}\right) dz.$$

We use the change of variables $w = \dfrac{1}{z}$, and we find

$$\underset{z=\infty}{Res}\, f(z) = \frac{1}{2\pi i}\int_{C^-(0,R)} f(w)dw = -\frac{1}{2\pi i}\int_{C^+(0,R)} f(w)\, dw =$$
$$-[\underset{z=a_1}{Res}\, f(z) + \underset{z=a_2}{Res}\, f(z) + \ldots + \underset{z=a_n}{Res}\, f(z)],$$

and the result follows.

■

Corollary 1.7.1 *The residue at infinity of an entire function, in particular of a polynomial, is zero.*

(See also **Problem 1.7.7**.)

Examples.

Example 1.7.1 Find the integral $\displaystyle\oint_{C^+} \frac{5z-3}{z(z-2)}\, dz$, if:

(a) $C := C(0,1)$, (b) $C := C(0,3)$, and (c) $C := C(4,1)$.

First, we compute the residues of the function $f(z) = \dfrac{5z-3}{z(z-2)}$ at the isolated singularities $z_0 = 0$ and $z_0 = 2$. At all other points of $\mathbb{C}$, this function is defined and holomorphic. We evaluate

$$\operatorname*{Res}_{z=0} f(z) = [zf(z)]|_{z=0} = \frac{5z-3}{z-2}\Big|_{z=0} = \frac{-3}{-2} = \frac{3}{2},$$

$$\operatorname*{Res}_{z=2} f(z) = [(z-2)f(z)]|_{z=2} = \frac{5z-3}{z}\Big|_{z=2} = \frac{7}{2}.$$

So, by the **Residue Theorem, 1.7.1**, we have:

(a) Since $z = 0$ is the only singularity inside $C := C(0,1)$, then

$$\oint\limits_{C^+(0,1)} \frac{5z-3}{z(z-2)}\, dz = 2\pi i \cdot \frac{3}{2} = 3\pi i.$$

(b) Now, both singularities are inside $C := C(0,3)$, and so

$$\oint\limits_{C^+(0,3)} \frac{5z-3}{z(z-2)}\, dz = 2\pi i \left(\frac{3}{2} + \frac{7}{2}\right) = 10\pi i.$$

(c) Finally, inside $C := C(4,1)$ there are no singularities of $f(z)$. Therefore, the line integral is zero, i.e.,

$$\oint\limits_{C^+(4,1)} \frac{5z-3}{z(z-2)}\, dz = 0.$$

Remark: If we parameterize the contours (circles) of the above line integrals and then separate the real and imaginary parts of each complex line integral, we obtain two real integrals in each case.

Notice that $\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=2} f(z) = \dfrac{3}{2} + \dfrac{7}{2} = 5$ and

$$\operatorname*{Res}_{z=\infty} f(z) := -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\operatorname*{Res}_{z=0}\left[\frac{5-3z}{z(1-2z)}\right] = -\frac{5-3z}{1-2z}\Big|_{z=0} = -5,$$

and $5 + (-5) = 0$, thus checking the previous theorem.

▲

Example 1.7.2 Evaluate the integral $\displaystyle\oint\limits_{C^+} \frac{dz}{e^z - 1}$ if $C = C(0, 3\pi)$. Then, compute the two real integrals resulting from it by using the parametrization of $C = C(0, 3\pi)$ given by $z = 3\pi e^{i\theta}$ with $0 \le \theta \le 2\pi$.

Inside $C = C(0, 3\pi)$, the function $f(z) = \dfrac{dz}{e^z - 1}$ has three singularities, namely $-2\pi i$, 0 and $2\pi i$, which are simple roots of the denominator.

Using one of the methods, that we have seen, we find that each of the three residues is equal to 1. E.g.,

$$\operatorname*{Res}_{z=-2\pi i} f(z) = \{[z-(-2\pi i)]f(z)\}\Big|_{z=-2\pi i} = \frac{z+2\pi i}{e^z-1}\Big|_{z=-2\pi i} =$$
$$\text{(use L' Hôpital's rule)} \quad \frac{1}{e^z}\Big|_{z=-2\pi i} = \frac{1}{e^{-2\pi i}} = \frac{1}{1} = 1.$$

So,

$$\oint_{C^+} \frac{dz}{e^z-1} = 2\pi i(1+1+1) = 6\pi i.$$

From this integral, we can now evaluate two real ones, if we use the parametrization $z = 3\pi e^{i\theta}$, with $0 \leq \theta \leq 2\pi$, of $C = C(0, 3\pi)$ and then separate the real and imaginary parts. So,

$$\int_0^{2\pi} \frac{3\pi i e^{i\theta} d\theta}{e^{3\pi e^{i\theta}} - 1} = 6\pi i, \text{ or } \int_0^{2\pi} \frac{e^{i\theta} d\theta}{e^{3\pi e^{i\theta}} - 1} = 2.$$

Now we replace $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and separate real and imaginary parts to get

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\cos[\theta - 3\pi\sin(\theta)] - \cos(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos[3\pi\sin(\theta)] + e^{6\pi\cos(\theta)}}\, d\theta = 2$$

and

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\sin[\theta - 3\pi\sin(\theta)] - \sin(\theta)}{1 - 2e^{3\pi\cos(\theta)}\cos[3\pi\sin(\theta)] + e^{6\pi\cos(\theta)}}\, d\theta = 0.$$

(Who could have imagined these real integrals?)

▲

Example 1.7.3 Any polynomial $P(z) = a_n z^n + a_{n-1} z^1 + \ldots + a_1 z + a_0$, with $n \geq 0$ integer, has $\operatorname*{Res}_{z=\infty} P(z) = 0$.

This follows from **Theorem 1.7.2**, since any polynomial has no singularities. But, we can also see it directly from **Definitions 1.6.5** and **1.7.1**, since

$$\operatorname*{Res}_{z=\infty} P(z) := -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\, P\left(\frac{1}{z}\right)\right] =$$
$$-\operatorname*{Res}_{z=0} \frac{1}{z^2}\left(a_0 + \frac{a_1}{z} + \ldots + \frac{a_{n-1}}{z^{n-1}} + \frac{a_n}{z^n}\right) =$$
$$\operatorname*{Res}_{z=0}\left(\frac{0}{z} - \frac{a_0}{z^2} - \frac{a_1}{z^3} - \ldots - \frac{a_{n-1}}{z^{n+1}} - \frac{a_n}{z^{n+2}}\right) = 0.$$

▲

Example 1.7.4 Show that for $m \geq 1$ integer and $a \in \mathbb{C}$ constant

$$\operatorname*{Res}_{z=\infty}\left[\frac{1}{(z-a)^m}\right] = \begin{cases} -1, & \text{if} \quad m = 1, \\ \\ 0, & \text{if} \quad m \geq 2. \end{cases}$$

This follows by direct computation of the residue of

$$\frac{-1}{z^2}\frac{1}{\left(\frac{1}{z}-a\right)^m} = \frac{-z^{m-2}}{(1-az)^m}$$

at $z = 0$. This is -1 when $m = 1$ and 0 when $m \geq 2$. (Check this. This result follows also from **Theorem 1.7.2**.)

▲

Problems

1.7.1 Evaluate

$$\oint_{C^+} \frac{dz}{z^2+2z+2}$$

if: (a) $C = C(0,1)$, (b) $C = C(0,5)$ and (c) $C = C(100,1)$.

1.7.2 Evaluate

$$\oint_{C^+} \frac{dz}{(z-1)^3(z-4)}, \quad \text{and} \quad \oint_{C^+} \frac{z-2}{z^2(z-1)}\, dz$$

if the simple closed contour C is: (a) $C = C(0,2)$, (b) $C = C(0,5)$ and (c) $C = C\left(3, \dfrac{3}{2}\right)$.

1.7.3 Evaluate

$$\oint_{C^+} \frac{e^z}{z}\, dz, \quad \text{and} \quad \oint_{C^+} \frac{e^z}{z-1}\, dz$$

where $C = C(0,2)$. Then, obtain two real integrals from each of these complex line integrals.

1.7.4 If $C = C(0,3)$, evaluate $\displaystyle\oint_{C^+} \frac{e^z}{(z-2)(z+4)},\, dz$ and $\displaystyle\oint_{C^+} \frac{\sin(z)}{z^2}\, dz$.

1.7.5 Evaluate $\displaystyle\oint_{C^+} \frac{e^{z^2}}{z-2}\, dz$, and $\displaystyle\oint_{C^+} \frac{e^z}{z-2}\, dz$,

where C is the boundary of the rectangle
$\{z = x + iy \mid 0 \leq x \leq 4,\ -2 \leq y \leq 2\}$.
Then, find the two corresponding real integrals.

1.7.6 If $C = C(0, 4)$, evaluate

$$\oint_{C^+} \frac{dz}{\sin(z)}, \quad \text{and} \quad \oint_{C^+} \frac{dz}{\sinh(z)}.$$

1.7.7 Suppose $\lim\limits_{z\to\infty} [-zf(z)] = l \in \mathbb{C}$ (i.e., the limit exists). Prove that $\underset{z=\infty}{Res}\, f(z) = l$.

1.7.2 Contour Integration and Improper Real Integrals

We are going to use integrations of complex functions along appropriately chosen contours to evaluate improper real integrals and integrals of Fourier type. This method is very powerful, for it computes very difficult integrals and at the same time proves their existence. Choosing the correct contour(s) and then applying the **Residue Theorem, 1.7.1**, is an art that takes some experience. Then we take the appropriate limits to obtain the real improper integral. We are going to analyze the most important cases of such integral techniques which are sufficient for the needs of an undergraduate student. We also present in detail several examples in order to practice and to get a feeling of what is going on. Not all possible cases, encountered in the bibliography, can be presented in this exposition. Some of them are above the level of this text and they require a higher level course of complex analysis. Several useful **Lemmata**, depending on the particular cases, will be proven and used in order to speed up the various methods of the evaluations of the respective integrals.

Example 1.7.5 The integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan(\infty) - \arctan(-\infty) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$$

has been computed elementarily. For easy practice, we will use contour integration to establish this result.

To this end, we consider the complex function $f(z) = \dfrac{1}{1+z^2}$ in $\mathbb{C}$. The denominator has two simple roots, the $+i$ and the $-i$, which are isolated singularities, poles of order one. So,

$$\operatorname*{Res}_{z=i} f(z) = [(z-i)f(z)]\big|_{z=i} = (z-i)\frac{1}{(z-i)(z+i)}\big|_{z=i} = \frac{1}{2i}.$$

We consider any $R > 1$ and the contour $C = [-R, R] + S_R^+$ consisting of two parts: (1) the straight segment of the x-axis $[-R, R]$ from $-R$ to R, and (2) the positively oriented upper half of $C(0, R)$, denoted by S_R^+. See **Figure 1.5**. These parts are respectively parameterized by:

(1) $\{z = x + 0i \mid -R \le x \le R\}$ and (2) $\{z = Re^{i\theta} \mid 0 \le \theta \le \pi\}$.
(From now on, we will use these notations in similar situations.)

We have chosen the contour C in this way, so that at least one of the singularities, namely the $z = +i$, is enclosed in it. Then, we apply **the Residue Theorem, 1.7.1**, to find

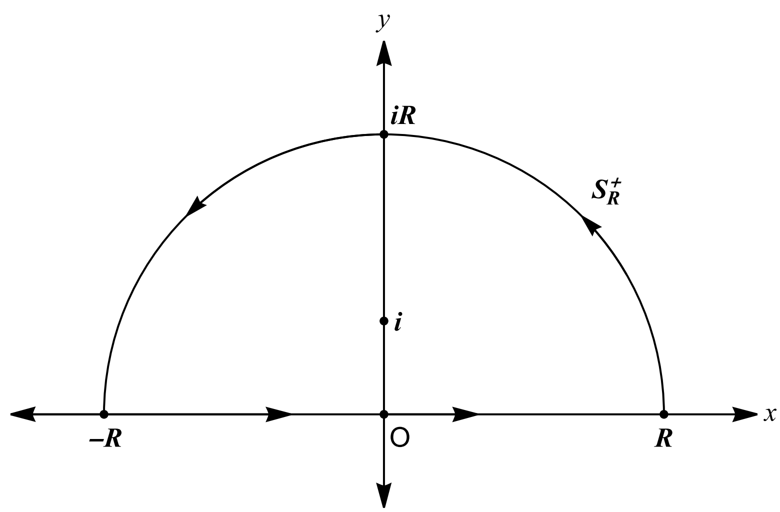

FIGURE 1.5: Contour 1 for Example 1.7.5

$$\oint_{C^+} \frac{dz}{z^2+1} = 2\pi i \frac{1}{2i} = \pi.$$

But, then

$$\pi = \oint_{C^+} \frac{dz}{z^2+1} = \oint_{[-R,R]} \frac{dz}{z^2+1} + \oint_{S_R^+} \frac{dz}{z^2+1} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_0^{\pi} \frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}.$$

This equality is valid for all $R > 1$.

Then, we let $R \to \infty$. In this limit process, the constant π does not change, and the two partial integrals become:

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^2+1} = \text{P.V.}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} = \int_{-\infty}^{\infty}\frac{dx}{x^2+1}$$

and

$$\lim_{R\to\infty}\int_{0}^{\pi}\frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} = 0.$$

The latter limit needs a proof of course. Such a proof is the following. The inequality

$$0 \leq \left|\int_C f(z)dz\right| \leq \int_C |f(z)|\,|dz|$$

for the complex variable $z \in \mathbb{C}$ and C any path in $\mathbb{C}$ is always valid. (See **property (5)** of the complex line integral in **Section 1.4**.) Using this property and the properties of the absolute value of complex numbers (see **Subsection 1.1.1**, **Problems 1.1.6, 1.1.7**, **1.1.8**, etc.), we get:

$$\begin{gathered}
0 \leq \lim_{R\to\infty}\left|\int_{0}^{\pi}\frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}\right| \leq \lim_{R\to\infty}\int_{0}^{\pi}\left|\frac{Rie^{i\theta}}{1+R^2e^{i2\theta}}\right|d\theta = \\
= \lim_{R\to\infty}\int_{0}^{\pi}\frac{\left|Rie^{i\theta}\right|}{|1+R^2e^{i2\theta}|}\,d\theta = \lim_{R\to\infty}\int_{0}^{\pi}\frac{R}{|1+R^2e^{i2\theta}|}\,d\theta \leq \\
\lim_{R\to\infty}\int_{0}^{\pi}\frac{R}{||1|-|R^2e^{i2\theta}||}\,d\theta = \lim_{R\to\infty}\int_{0}^{\pi}\frac{R}{|1-R^2|}\,d\theta = \\
\lim_{1<R\to\infty}\int_{0}^{\pi}\frac{R}{R^2-1}\,d\theta = \lim_{1<R\to\infty}\frac{R}{R^2-1}\pi = 0.
\end{gathered}$$

So, by the Squeeze Theorem for limits, we obtain

$$\lim_{R\to\infty}\left|\int_{0}^{\pi}\frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}}\right| = 0.$$

Then, by the property of limits

$$\lim_{t\to a}|A_t| = 0 \iff \lim_{t\to a}A_t = 0,$$

we have

$$\lim_{R\to\infty}\int_{0}^{\pi}\frac{Rie^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} = 0.$$

Thus, we have finally proven

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \pi.$$

We observe that by having taken the **symmetric limit** as $R \to \infty$, we have obtained the evaluation of the **principal value** of this real improper integral. Since we can easily prove (using real methods) that this integral exists, its principal value is equal to its value.

Remark: Since $f(x) = \dfrac{1}{1+x^2}$ is an even, function in $\mathbb{R}$ we also get:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2},$$

or

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_{-\infty}^{0} \frac{dx}{1+x^2} = 2\int_{0}^{\infty} \frac{dx}{1+x^2}.$$

▲

Example 1.7.6 Similarly, we find

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = 2\int_{-\infty}^{0} \frac{x^2}{1+x^4}\,dx = 2\int_{0}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} = \frac{\pi\sqrt{2}}{2},$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = 2\int_{-\infty}^{0} \frac{dx}{1+x^6} = 2\int_{0}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3},$$

$$\int_{0}^{\infty} \frac{dx}{1+x^n} = \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}, \quad n = 2,\, 3,\, 4, \dots .$$

All these integrals are special cases of the general integral

$$\int_{0}^{\infty} \frac{x^j}{1+x^l}\,dx = \frac{\pi}{l\sin\left[\frac{(j+1)\pi}{l}\right]},$$

where $j = 0,\ 1,\ 2, \dots$ and $l > j+1$ integer.

We are going to prove this general integral formula in the sequel. But before we do that, we need to present two **Lemmata**, which we are going to use in the proof in order to facilitate the involved computations. These lemmata are very useful in many similar situations.

▲

In most of the results and examples that follow, we end up with limits of integrals on closed subintervals of $[0, 2\pi]$. These limits are uniform, and thus we apply **Theorem I 2.3.9**. The uniform convergence in $\theta \in [a,b] \subseteq [0,2\pi]$ is a byproduct of the existence of a certain limit as z approaches the complex infinity.

Lemma 1.7.1 *Let $w = f(z)$ be a complex function defined and continuous in an open subset of $\mathbb{C}$ of the type*

$$\mathbb{C} - \{\textit{finitely many points of } \mathbb{C}\}$$

and satisfying the condition

$$\lim_{z\to\infty} z f(z) = 0,$$

where ∞ is the complex infinity.

For any $0 \le \theta_1 \le \theta_2 \le 2\pi$ and $R > 0$, we consider the closed or open arc of the $C(0,R)$ from θ_1 to θ_2, denoted by A_R. This arc, as a path, can be parameterized by

$$A_R = \left\{z = Re^{i\theta}, \quad \theta_1 \le \theta \le \theta_2\right\}$$

when it is closed, and by

$$A_R = \left\{z = Re^{i\theta}, \quad \theta_1 < \theta < \theta_2\right\}$$

when it is open. Then, in either case we have

$$\lim_{R\to\infty} \int_{A_R} f(z)\,dz = \lim_{R\to\infty} \int_{\theta_1}^{\theta_2} f(Re^{i\theta}) R\, i\, e^{i\theta} d\theta = 0,$$

and the convergence in this limit is uniform.

Proof We take $R > 0$ large enough so that all finitely many points of $\mathbb{C}$ at which $f(z)$ is not defined are located inside the circle $C(0,R)$.

We let

$$M(R) = \operatorname*{Maximum}_{z\in A_R} |f(z)|.$$

Then, by the condition $\lim\limits_{z\to\infty} z\, f(z) = 0$, we get $\lim\limits_{R\to\infty} R\, M(R) = 0$. Consequently,

$$\left|\int_{A_R} f(z)\,dz\right| = \left|\int_{\theta_1}^{\theta_2} f(Re^{i\theta}) R\, i\, e^{i\theta} d\theta\right| \le \int_{\theta_1}^{\theta_2} \left|f(Re^{i\theta})\right| R \cdot 1 \cdot\, d\theta \le$$

$$R\, M(R)(\theta_2 - \theta_1) \to 0, \quad \text{as} \quad R \to \infty.$$

Hence,

$$\lim_{R\to\infty} \int_{A_R} f(z)\,dz = \lim_{R\to\infty} \int_{\theta_1}^{\theta_2} f(Re^{i\theta}) R\, i\, e^{i\theta} d\theta = 0.$$

The convergence is uniform by the **Weierstraß M-Test, Theorem I 2.3.3**, for instance.

■

Lemma 1.7.2 *Let $w = f(z)$ be a complex function defined and holomorphic in the open subset of $\mathbb{C}$*

$$\mathbb{C} - \{z_1,\, z_2,\, z_3,\, \ldots,\, z_k\}$$

and satisfying the condition

$$\lim_{z\to\infty} zf(z) = 0,$$

where ∞ is the complex infinity. We assume that ***each*** *of the k exceptional points $z_1, z_2, z_3, \ldots, z_k$ is an isolated singularity of $f(z)$ and* ***not on the real axis****.*

(a) If ***all*** *the exceptional points located in the upper half plane are the $z_1, z_2, z_3, \ldots, z_l$, where $1 \le l \le k$, then*

$$P.V. \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{j=1}^{l} \underset{z=z_j}{Res} f(z).$$

(b) If $0 \le l < k$, that is, the $z_{l+1}, z_{l+2}, \ldots, z_k$ are ***precisely*** *the exceptional points located in the lower half plane, then we also have*

$$P.V. \int_{-\infty}^{\infty} f(x)\, dx = -2\pi i \sum_{j=l+1}^{k} \underset{z=z_j}{Res} f(z).$$

Proof (a) As in **Example 1.7.5** of this **section**, we let $C = [-R, R] + S_R^+$, where S_R^+ is the upper half of the circle $C(0, R)$ positively oriented and $R > 0$ is large enough so that C encloses all $z_1, z_2, z_3, \ldots, z_l$. Then, by the **Residue Theorem, 1.7.1**, we get

$$\oint_{C^+} f(z)\, dz = 2\pi i \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j} f(z),$$

or

$$\int_{-R}^{R} f(x)\, dx + \int_{S_R^+} f(z)\, dz = 2\pi i \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j} f(z).$$

Now apply **Lemma 1.7.1** with $A_R = S_R^+$ and take the (symmetric) limit as $R \to \infty$. The result

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j} f(z)$$

follows immediately.

(b) The second result is obtained in a similar way by considering the contour $C = [-R, R] + T_R^-$, with $R > 0$ large enough and T_R^- the lower half of the circle $C(0, R)$ negatively oriented. The minus $(-)$ sign in this second equality is necessary because, in this case, we traverse C in the negative (clockwise) direction.

■

In the **next two examples**, we examine and evaluate three cases of integrals, even though the first two can be drawn as byproducts of the third one. We do this in order to practice with and emphasize the choice of the contour from case to case and to also observe the similarities and the differences of these three cases.

Example 1.7.7 We use the **previous Lemma, 1.7.2,** to prove the following result:

For all integers l ***and*** j ***such that:*** $j = 0,\ 1,\ 2,\ 3,\ 4, \ldots$ ***and*** $l > j + 1$***, we have:***

$$\int_0^{\infty} \frac{x^j}{1 + x^l}\, dx = \frac{\pi}{l\, \sin\left[\frac{(j+1)\pi}{l}\right]}.$$

(Compare this result with (Compare this result with **Examples I 2.2.5** and **I 2.2.6** and **Problems I 2.2.6, I 2.2.47** and **I 2.4.18**.).)

The function $\dfrac{z^j}{1 + z^l}$ has finitely many (isolated) singularities, namely the roots of the denominator $1 + z^l$, which are all simple.

If l is odd, then the root $z = -1$ is in the real axis, but this does not matter because we integrate over $[0, \infty)$.

Instead of taking into account all of these singularities, in this case, we consider only one, namely

$$z_0 = e^{\frac{\pi i}{l}}.$$

We enclose it by the three-piece positively oriented contour

$$C^+ = [0, R] + A_R^+ + [Re^{\frac{2\pi i}{l}}, 0],$$

where $[0, R]$ and $[Re^{\frac{2\pi i}{l}}, 0]$ are straight segments and

$$A_R^+(\theta) = \left\{ z = Re^{i\theta} \quad | \quad 0 \leq \theta \leq \frac{2\pi}{l} \right\}$$

is an arc. (See **Figure 1.6**, below.)

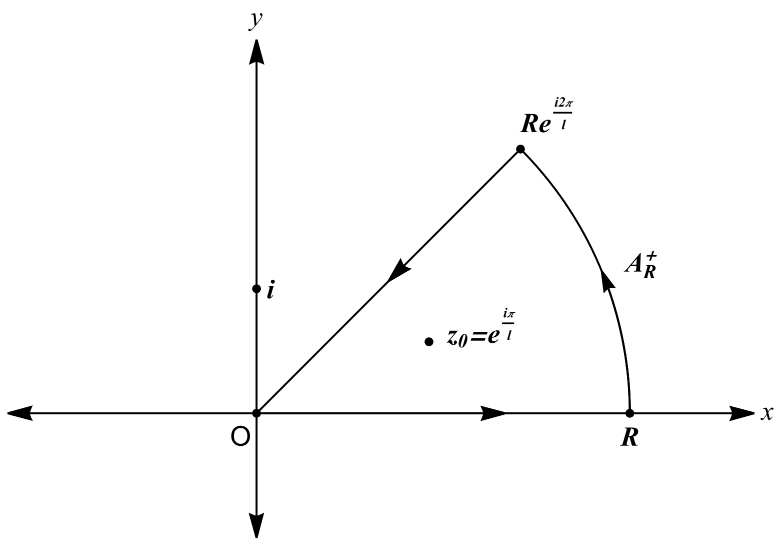

FIGURE 1.6: Contour 2 for Example 1.7.5

We have that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{(z-z_0)z^j}{1+z^l}\Big|_{z=z_0} \left(= \frac{0}{0}\right) = \frac{z^j + j(z-z_0)z^{j-1}}{lz^{l-1}}\Big|_{z=z_0} =$$

$$\frac{z_0^j}{lz_0^{l-1}} = \frac{z_0^{j+1}}{l(-1)} = \frac{-e^{i\frac{(j+1)\pi}{l}}}{l}.$$

So,

$$\oint\limits_{C^+} \frac{z^j dz}{1+z^l} = \int\limits_{[0,R]} \frac{z^j dz}{1+z^l} + \int\limits_{A_R^+} \frac{z^j dz}{1+z^l} + \int\limits_{[Re^{\frac{2\pi i}{l}},0]} \frac{z^j dz}{1+z^l} = -2\pi i \frac{e^{i\frac{(j+1)\pi}{l}}}{l}.$$

First, since $\lim\limits_{z\to\infty} \dfrac{z\cdot z^j}{1+z^l} = 0$, by **Lemma 1.7.1**, we have that

$$\oint\limits_{A_R^+} \frac{z^j dz}{1+z^l} \to 0, \text{ as } R\to\infty.$$

Second,

$$\lim_{R\to\infty} \int\limits_{[0,R]} f(z)\,dz = \int_0^\infty \frac{x^j dx}{1+x^l}.$$

Third,

$$\int_{[0,Re^{\frac{2\pi i}{l}}]} f(z)\,dz =$$

$$\int_0^R \frac{x^j e^{i\frac{2\pi j}{l}} e^{\frac{2\pi i}{l}} dx}{1+x^l e^{2\pi i}} = \int_0^R \frac{x^j e^{i\frac{2\pi(j+1)}{l}} dx}{1+x^l} = e^{i\frac{2\pi(j+1)}{l}} \int_0^R \frac{x^j\,dx}{1+x^l}.$$

Taking limit as $R \to \infty$, we get

$$\lim_{R\to\infty} \int_{[Re^{\frac{2\pi i}{l}},0]} f(z)\,dz = -e^{i\frac{2\pi(j+1)}{l}} \int_0^\infty \frac{x^j\,dx}{1+x^l}.$$

Therefore, by using the **Residue Theorem, 1.7.1**, and the three-part-limit (as $R \to \infty$) for $\oint\limits_{C^+} \frac{z^j dz}{1+z^l}$, we get

$$\int_0^\infty \frac{x^j\,dx}{1+x^l} - e^{i\frac{2\pi(j+1)}{l}} \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$
$$\left[1 - e^{i\frac{2\pi(j+1)}{l}}\right] \int_0^\infty \frac{x^j\,dx}{1+x^l} dx = -2\pi i \frac{e^{i\frac{(j+1)\pi}{l}}}{l}.$$

So, by Euler's formula (see **Section 1.2**)

$$\left\{1 - \cos\left[\frac{2(j+1)\pi}{l}\right] - i\sin\left[\frac{2(j+1)\pi}{l}\right]\right\} \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$
$$\frac{-2\pi i}{l}\left\{\cos\left[\frac{(j+1)\pi}{l}\right] + i\sin\left[\frac{(j+1)\pi}{l}\right]\right\}.$$

Using the double angle formulae in trigonometry, we find

$$\left\{2\sin^2\left[\frac{(j+1)\pi}{l}\right] - 2i\sin\left[\frac{(j+1)\pi}{l}\right]\cos\left[\frac{(j+1)\pi}{l}\right]\right\} \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$
$$2\sin\left[\frac{(j+1)\pi}{l}\right] \cdot \left\{\sin\left[\frac{(j+1)\pi}{l}\right] - i\,\cos\left[\frac{(j+1)\pi}{l}\right]\right\} \int_0^\infty \frac{x^j\,dx}{1+x^l} =$$
$$\frac{2\pi}{l}\left\{\sin\left[\frac{(j+1)\pi}{l}\right] - i\cos\left[\frac{(j+1)\pi}{l}\right]\right\}.$$

Separating the real and imaginary parts, or simply dividing by the bracket in front of the integral, we find the important result:

Result: For all integers l and j such that $l > j+1$ and $j = 0,\ 1,\ 2,\ 3,\ 4, \ldots$ we have:

$$\int_0^\infty \frac{x^j}{1+x^l}\,dx = \frac{\pi}{l\,\sin\left[\frac{(j+1)\pi}{l}\right]}.$$

Remark 1: Here, the contour integration method proves the existence of the integral and evaluates it at the same time.

Remark 2: If $l = j+1$ and $j = 0,\ 1,\ 2,\ 3,\ 4, \ldots$, then the result extends as

$$\int_0^\infty \frac{x^j}{1+x^{j+1}}\,dx = \left[\frac{1}{j+1}\ln\left|1+x^{j+1}\right|\right]_0^\infty = \infty = \frac{\pi}{(j+1)\sin(\pi^+)} = \frac{\pi}{0^+}.$$

Remark 3: If $j = 2m \geq 0$ and $l = 2n$ are **both even integers**, then the integrand is an even function, and by this result we also get

$$\int_{-\infty}^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{2\pi}{2n\,\sin\left[\frac{(2m+1)\pi}{2n}\right]} = \frac{\pi}{n\,\sin\left[\frac{(2m+1)\pi}{2n}\right]},$$

where n is integer such that $2n > 2m+1$ and $m = 0,\ 1,\ 2,\ 3,\ 4, \ldots$.

Remark 4: If the integer $j = 2k+1$ is odd and the integer $l = 2n$ is even, then the integrand is an odd function, and so

$$\text{P.V.} \int_{-\infty}^\infty \frac{x^{2k+1}}{1+x^{2n}}\,dx = 0.$$

Remark 5: If the integer $l = 2n+1$ is odd, then the integral

$$\int_{-\infty}^\infty \frac{x^j}{1+x^l}\,dx$$

is also improper at $x = -1$, which is a simple pole on the coordinate x-axis. This integral does not exist. (Prove!) But, we can compute its principal value.

If $j = 2n$, then $\dfrac{x^{2n}}{1+x^{2n+1}}$ has antiderivative $\dfrac{\ln\left|x^{2n+1}\right|}{2n+1}$ and the principal value of the integral is zero. (Verify!)

If $j \leq 2n-1 = l-2$, then the principal value is evaluated by **Theorem 1.7.4**.

▲

Example 1.7.8 The result in the **previous Example** generalizes to the following important **result**:

For $l = 1,\ 2,\ 3,\ 4, \ldots$ **integer and** $\alpha \in \mathbb{R} - \mathbb{Z}$ **such that** $0 < \alpha + 1 < l$ **(or** $-1 < \alpha < l - 1$**), we have:**

$$\int_0^\infty \frac{x^\alpha}{1 + x^l}\, dx = \frac{\pi}{l \sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

For example, if $\alpha = \dfrac{1}{2}$ and $l = 2$, we find:

$$\int_0^\infty \frac{\sqrt{x}}{1 + x^2}\, dx = \frac{\pi}{2 \sin\left[\frac{(\frac{1}{2}+1)\pi}{2}\right]} = \frac{\pi}{2 \cdot \frac{\sqrt{2}}{2}} = \frac{\pi\sqrt{2}}{2}.$$

(Compare this result with **Examples I 2.2.5, I 2.6.24** and **Problem I 2.6.21**.)

Remark: The case $\alpha \in \mathbb{Z}$ was studied in the **previous Example**. This general integral formula is also useful in the study of the Gamma and Beta functions, studied in **Section I 2.6**.

The **proof** of this result for α **not an integer** is more complicated than the proof in the **previous Example**. The reason is that in such a case the power function is defined by:

$$z^\alpha = e^{\alpha \log(z)}.$$

To obtain continuous branches of $\log(z)$, we must introduce appropriate branch cuts. So, the function

$$f(z) = \frac{z^\alpha}{1 + z^l}$$

has as isolated singularities all the roots of the denominator and as non-isolated singularities a whole branch cut, which can be taken to be a closed half line starting at the origin.

Under the conditions $0 < \alpha + 1 < l$ imposed on l and α, this function satisfies the condition of **Lemma 1.7.2**:

$$\lim_{z \to \infty} z f(z) = 0.$$

When $\alpha \in \mathbb{R} - \mathbb{Z}$ this integral is dealt with in the following two **cases**: **(a)** $l = 2, 3, 4, \ldots$, and **(b)** $l = 1$.

Case (a): $l = 2, 3, 4, \ldots$. In this case, besides the roots of the denominator, the contour must also avoid the non-isolated singularities of the branch cut and the origin. We achieve this by considering the branch cut to be the closed lower y-semi-axis $\{z = 0 + iy \mid y \leq 0\}$.

Then, we choose the following contour C, this time consisting of four parts (in order to avoid the origin and the branch cut). We consider any $0 < r < 1 < R < \infty$, and we define

$$C = [r, R] + A_R^+ + [Re^{\frac{2\pi i}{l}}, re^{\frac{2\pi i}{l}}] + A_r^-,$$

where we have the straight segments $[r, R]$, $[Re^{\frac{2\pi i}{l}}, re^{\frac{2\pi i}{l}}]$ and the arcs

$$A_R^+(\theta) = \left\{Re^{i\theta} \quad | \quad 0 \leq \theta \leq \frac{2\pi}{l}\right\},$$

and

$$A_r^-(\theta) = \left\{re^{i\theta} \quad | \quad \frac{2\pi}{l} \geq \theta \geq 0\right\},$$

as in **Figure 1.7**.

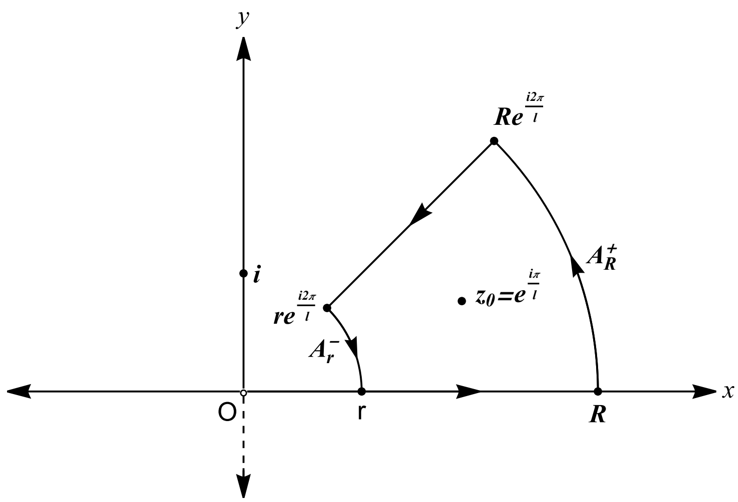

FIGURE 1.7: Contour 3 for Example 1.7.8, Case (a)

We now follow the same method as in the **previous Example**, but we take two limits: one as $R \to \infty$ and the other as $r \to 0^+$.

Again, by **Lemma 1.7.1**, we have that

$$\lim_{R\to\infty} \int\limits_{A_R^+} \frac{z^\alpha}{1 + z^l}\, dz = 0.$$

Under the conditions $0 < r < 1$ and $\alpha > -1$, we have that

$$\lim_{r\to 0^+} \int\limits_{A_r^-} \frac{z^\alpha}{1+z^l}\,dz = 0.$$

This follows easily from

$$\left| \int\limits_{A_r^-} \frac{z^\alpha}{1+z^l}\,dz \right| = \left| \int_{\frac{2\pi}{l}}^{0} \frac{\left(re^{i\theta}\right)^\alpha ire^{i\theta}}{1+\left(re^{i\theta}\right)^l}\,d\theta \right| \le \int_0^{\frac{2\pi}{l}} \left| \frac{r^\alpha e^{i\alpha\theta} ire^{i\theta}}{1+r^l e^{il\theta}} \right| d\theta \le$$

$$\int_0^{\frac{2\pi}{l}} \frac{r^{\alpha+1}}{1-r^l}\,d\theta = \frac{2\pi}{l}\frac{r^{\alpha+1}}{1-r^l} \longrightarrow \frac{0}{l} = 0, \quad \text{as} \quad 0 < r \longrightarrow 0^+.$$

Having done this, the remaining computation is exactly the same as in the **previous Example**. (Finish it!) So, we have obtained the following:

Result:

$$\forall\ l = 2,\ 3,\ 4, \ldots, \quad \text{and} \quad \forall\ \alpha \in \mathbb{R}\ :\ -1 < \alpha < l-1$$
$$\int_0^\infty \frac{x^\alpha}{1+x^l}\,dx = \frac{\pi}{l\,\sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

Case (b) $l = 1$. This case is substantially different from **Case (a)** because, as we shall see, we cannot stay away from the branch cut.

Here, we must prove that for all $-1 < \alpha < 0$ real, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \frac{\pi}{\sin[(\alpha+1)\pi]} = \frac{-\pi}{\sin(\alpha\pi)} = \frac{\pi}{\sin(-\alpha\pi)}.$$

Notice that the only root of the denominator is $z = -1$. We take as the branch cut the closed half line of the non-negative real semi-axis $\{z = x + 0i \mid x \ge 0\}$. In **Figure 1.8** this is indicated by the **dashed line.**[23] That is, we have chosen the positive continuous argument $0 < \arg(z) < 2\pi$. So, we write $z = -1 = e^{i\pi}$, and then

$$\operatorname*{Res}_{z=-1}\left(\frac{z^\alpha}{1+z}\right) = (-1)^\alpha = \left(e^{i\pi}\right)^\alpha = e^{i\alpha\pi}.$$

We now pick numbers r and R such that $0 < r < 1 < R < \infty$ and as appropriate contour

$$C = [r,R] + A_R^+ + [R,r] + A_r^-,$$

[23]**Note**: In this text, branch cuts in contours are indicated by dashed lines. The same is true for arcs that are depicted but not used.

where

$$A_R^+(\theta) = Re^{i\theta}, \quad 0 < \theta < 2\pi \quad \text{and} \quad A_r^-(\theta) = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

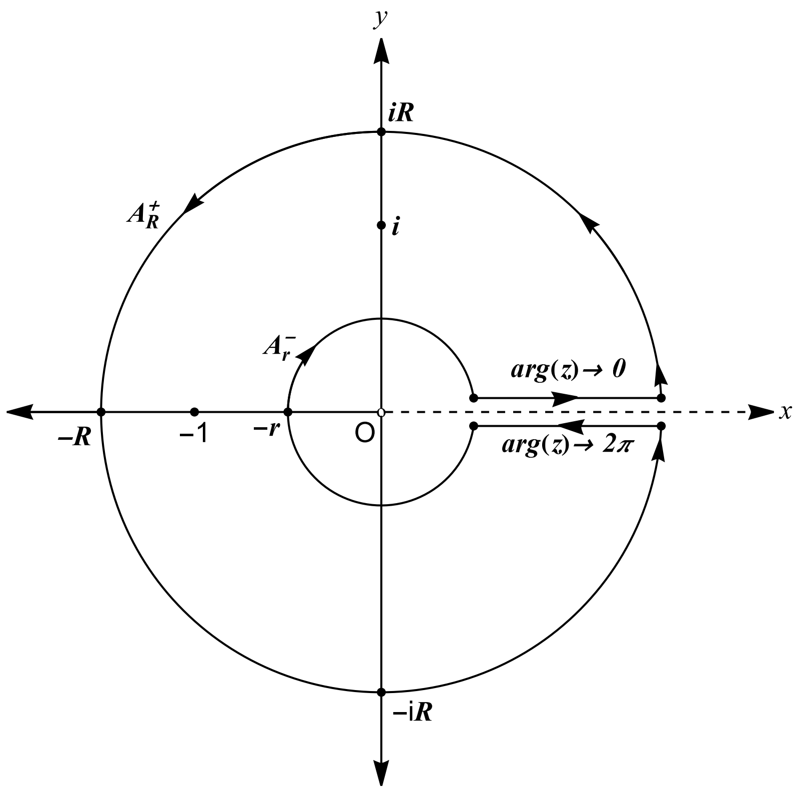

FIGURE 1.8: Contour 4 for Example 1.7.8, Case (b)

Notice the strict inequalities in the interval of θ, and pay attention to the direction of each part of the contour.

By the **Residue Theorem, 1.7.1**, we obtain

$$\oint_{C^+} \frac{z^\alpha}{1+z}\, dz =$$

$$\int_{[r,R]} \frac{z^\alpha dz}{1+z} + \int_{A_R^+} \frac{z^\alpha dz}{1+z} + \int_{[R,r]} \frac{z^\alpha dz}{1+z} + \int_{A_r^-} \frac{z^\alpha dz}{1+z} = 2\pi i e^{i\alpha\pi}, \qquad (1.17)$$

and we must take limits as $R \to \infty$ and as $r \to 0^+$.

By **Lemma 1.7.1**, we get

$$(1) \qquad \lim_{R\to\infty} \int\limits_{A_R^+} f(z)\,dz = 0.$$

Again, under the condition $-1 < \alpha < 0$ and using the parametrization of A_r^- as in **Case (a)**, in exactly the same way, we prove that

$$(2) \qquad \lim_{r\to 0^+} \int\limits_{A_r^-} \frac{z^\alpha}{1+z}\,dz = 0,$$

Next, we must compute the two partial integrals along the branch cut, that is, over the intervals $[r, R]$ and $[R, r]$. Someone may think that the two integrals cancel because we integrate over opposite intervals. However, this is not so, because the function is discontinuous along the branch cut. Its limits are different when we approach the branch cut from above (in the upper half plane) vs approaching it from below (in the lower half plane). Notice also the indication of the two different arguments in **Figure 1.8**.

In the case of the (positive) segment $[r, R]$, as we travel along the contour the arc A_r^- indicates[24] that we approach the branch cut from above, and so the limit of the $\arg(z)$ is 0. Hence, along $[r, R]$ we compute the real integral $\displaystyle\int_r^R \frac{\left(xe^{0i}\right)^\alpha}{1+x}\,dx = \int_r^R \frac{x^\alpha}{1+x}\,dx$. Taking limits, we get

$$(3) \qquad \lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R \frac{x^\alpha}{1+x}\,dx = \int_0^\infty \frac{x^\alpha}{1+x}\,dx.$$

But, in the case of the (negative) segment $[R, r]$, we approach the branch cut from below, as the arc A_R^+ indicates, and so the limit of the $\arg(z)$ is 2π. Hence, along $[R, r]$ we must compute the integral $\displaystyle\int_R^r \frac{(xe^{2\pi i})^\alpha}{1+x}\,dx$. The integrand now is different from the one along $[r, R]$ above, and this happens because $-1 < \alpha < 0$ is not an integer and so $e^{2\pi\alpha i} \neq 1$. Taking limits, we get

$$(4) \qquad \lim_{\substack{r\to 0\\ R\to\infty}} \int_R^r \frac{(xe^{2\pi i})^\alpha}{1+x}\,dx = -e^{2\pi i\alpha} \int_0^\infty \frac{x^\alpha}{1+x}\,dx.$$

[24]**Note**: We emphasize all these details because **in this Figure and all the figures with branch cuts that follow**, in order to simplify the drawing of contours and writing of too many limits with a lot of ϵ's and δ's, we give the figure of the limiting contour of the contours that other books draw and use. Make a note of this, and also have a look at other books to see this difference. Also in this book, be aware as to how the contours are made, and be careful how you approach the branch cuts as you travel along the contours in the direction indicated.

Now, by taking the indicated limits in **(1.17)** by the four computed pieces **(1), (2), (3)** and **(4)**, we find

$$\int_0^\infty \frac{x^\alpha dx}{1+x} - e^{2\pi i\alpha}\int_0^\infty \frac{x^\alpha dx}{1+x} = \left(1-e^{2\pi i\alpha}\right)\int_0^\infty \frac{x^\alpha}{1+x}\,dx = 2\pi i e^{i\alpha\pi}.$$

So we have the important result: For any $-1<\alpha<0$ real,[25]

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \frac{2\pi i e^{i\alpha\pi}}{1-e^{2\pi i\alpha}} = \frac{\pi}{\frac{e^{-i\alpha\pi}-e^{i\alpha\pi}}{2i}} = \frac{\pi}{-\frac{e^{i\alpha\pi}-e^{-i\alpha\pi}}{2i}} =$$
$$\frac{\pi}{-\sin(\alpha\pi)} = \frac{\pi}{\sin(-\alpha\pi)} = \frac{\pi}{\sin[(\alpha+1)\pi]}.$$

Remark 1: If $\alpha\ge 0$ or $\alpha\le -1$, then $\displaystyle\int_0^\infty \frac{x^\alpha}{1+x}\,dx=\infty$.

Indeed, if $\alpha\ge 0$, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx \ge \int_0^1 \frac{x^\alpha}{1+x}\,dx + \int_1^\infty \frac{1}{1+x}\,dx = \int_0^1 \frac{x^\alpha}{1+x}\,dx + \infty = \infty,$$

and if $\alpha\le -1$, we have

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx \ge \int_0^1 \frac{x^\alpha}{2}\,dx + \int_1^\infty \frac{x^\alpha}{1+x}\,dx = \infty + \int_1^\infty \frac{x^\alpha}{1+x}\,dx = \infty.$$

Remark 2: For the **real case** of the integral $\displaystyle\int_{-\infty}^0 \frac{x^\alpha}{1+x}\,dx$ or its principal value, we need x^α to be defined as real. So, we need α to be a reduced rational with an odd denominator.

We notice that this integral is improper at $-\infty$ and $x=-1$. Essentially, each case needs a separate investigation. For example, if $\alpha=0$ then

$$\text{P.V.}\int_{-\infty}^0 \frac{1}{1+x}\,dx \quad \text{does not exist},$$

and therefore the integral itself does not exist.

Remark 3: The integrals of **Examples 1.7.7** and **1.7.8, Case**

[25] The formula is also valid for $\alpha=z\in\mathbb{C}$, with $-1<\mathrm{Re}(z)<0$. I.e.,

$$\int_0^\infty \frac{x^z}{1+x}\,dx = \frac{\pi}{\sin[(z+1)\pi]}, \quad \text{or equivalently} \quad \int_0^\infty \frac{x^{w-1}}{1+x}\,dx = \frac{\pi}{\sin(w\pi)},$$

with $0<\mathrm{Re}(w)<1$. Here, we use the complex sine. See Titchmarsh 1939, p. 105. This offers a taste of complex integral formulae, but they are out of the scope of this book.

(a), can be drawn from this integral by making the u-substitution: $u = x^l \Longleftrightarrow x = u^{\frac{1}{l}}$ and then $dx = \dfrac{1}{l} u^{\frac{1}{l}-1} du$. (Work it out.)

Remark 4: If $-1 < \alpha < 0$ and $b > 0$, then using the change of variables $x = bu$ and so $dx = b\,du$, we find

$$\int_0^\infty \frac{x^\alpha}{b+x}\,dx = b^\alpha \int_0^\infty \frac{u^\alpha}{1+u}\,du = \frac{b^\alpha \pi}{\sin(-\alpha\pi)}.$$

Remark 5: Compare the integrals found and discussed in this **Example** and **Remarks** with the integrals found and discuss in **properties (B, 8)** and **(B, 5)** of the Beta function in **Subsection I 2.6.2**. There, we allow powers of the whole denominator with exponents other than just 1. (See also the pertinent **Examples** and **Problems** of **Subsection I 2.6.2** and **Problem 1.7.13**, especially I_{11} and I_{12}.)

Remark 6: We must keep in mind that in a contour integration along a branch cut, if the argument for the integrand function is ϕ along one of its sides, then along the other side the argument is $\phi + 2\pi$, **if** we move around the branch cut in the positive direction in order to reach the other side, and it is $\phi - 2\pi$, **if** we move in the negative direction.

▲

Example 1.7.9

$$\int_0^\infty \frac{x^{\frac{-1}{3}}}{10+x}\,dx = \frac{10^{\frac{-1}{3}}\pi}{\sin\frac{\pi}{3}} = \frac{2\pi\sqrt[3]{100}\sqrt{3}}{30}.$$

▲

Example 1.7.10 To find $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,dx$, we perform the u-substitution

$$u = x^{\sqrt{2}} \Longleftrightarrow x = u^{\frac{1}{\sqrt{2}}} \qquad \text{and} \qquad dx = \frac{1}{\sqrt{2}} u^{\frac{1}{\sqrt{2}}-1} du.$$

Since $\alpha = \dfrac{3}{2\sqrt{2}} - 1 > 0$, we find

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,dx = \frac{1}{\sqrt{2}} \int_0^\infty \frac{u^{\frac{3}{2\sqrt{2}}-1}}{1+u}\,du = \infty.$$

▲

Example 1.7.11 To find $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{3}}}\,dx$, we perform the u-substitution

$$u = x^{\sqrt{3}} \Longleftrightarrow x = u^{\frac{1}{\sqrt{3}}} \qquad \text{and} \qquad dx = \frac{1}{\sqrt{3}} u^{\frac{1}{\sqrt{3}}-1} du.$$

Since $-1 < \alpha = \dfrac{3}{2\sqrt{3}} - 1 < 0$, we find

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{3}}}\,dx = \frac{1}{\sqrt{3}}\int_0^\infty \frac{u^{\frac{3}{2\sqrt{3}}-1}}{1+u}\,du = \frac{1}{\sqrt{3}}\frac{\pi}{\sin\left(\frac{\sqrt{3}}{2}\pi\right)}.$$

▲

Example 1.7.12 To find $\displaystyle\int_0^\infty \frac{x^8}{1+x^{4\pi}}\,dx$, we perform the u-substitution

$$u = x^{4\pi} \Longleftrightarrow x = u^{\frac{1}{4\pi}} \qquad \text{and} \qquad dx = \frac{1}{4\pi}u^{\frac{1}{4\pi}-1}du.$$

Since $-1 < \alpha = \dfrac{9}{4\pi} - 1 < 0$, we find

$$\int_0^\infty \frac{x^8}{1+x^{4\pi}}\,dx = \frac{1}{4\pi}\int_0^\infty \frac{u^{\frac{9}{4\pi}-1}}{1+u}\,du = \frac{1}{4\pi}\frac{\pi}{\sin\left[\left(1-\frac{9}{4\pi}\right)\pi\right]} = \frac{1}{4\sin\left(\frac{9}{4}\right)}.$$

▲

Example 1.7.13 Now, we present an interesting **mathematical application** of contour integration. We are going to be sketchy, but we provide enough information so that the interested reader can provide the missing details.

For $n \in \mathbb{N}$, we define

$$I_n = \int_0^\infty f_n(x)\,dx, \quad \text{where} \quad f_n(x) = \begin{cases} \dfrac{x^n - 2x + 1}{x^{2n} - 1}, & \text{if} \quad x \neq 1, \\[2ex] \dfrac{n-2}{2n}, & \text{if} \quad x = 1\,. \end{cases}$$

(a) For $n = 1$, we simplify and easily find that $I_1 = -\infty$.

(b) For $n \geq 2$, we find that $I_n = 0$.[26] To prove this, we notice that $z = 1$ is a removable singularity of $f_n(z)$, where $z = x + iy$ the complex variable. Then, for $n \geq 2$, we choose $R > 1$ and the contour

$$C^+ = [0, R] + A_R^+(\theta) + \left[Re^{\frac{3\pi i}{2n}}, 0\right],$$

where $A_R^+(\theta) = \left\{z = Re^{i\theta} \;\mid\; 0 \leq \theta \leq \dfrac{3\pi}{2n}\right\}$.

[26] Mathematics Magazine, Problem 1912, Vol. 86, Number 1, February 2013.

We observe that $f_n(z)$ has a unique pole of order one in the interior of this contour, which is the $2n^{\text{th}}$ root of unity, $z_* = e^{\frac{\pi i}{n}}$. The corresponding residue is $\operatorname*{Res}_{z=z_*} f_n(z) = -\dfrac{e^{\frac{2\pi i}{n}}}{n}$.

Then, we apply the **Residue Theorem, 1.7.1,** and take the limit as $R \longrightarrow \infty$ to obtain the result. (Fill in the details.)

(c) Also, we easily find that

$$f(x) \overset{pw}{:=} \lim_{n\to\infty} f_n(x) = \begin{cases} 2x-1, & \text{if} \quad 0 \le x < 1, \\ \dfrac{1}{2}, & \text{if} \quad x = 1, \\ 0, & \text{if} \quad x > 1. \end{cases}$$

Therefore, the convergence is not uniform and $\displaystyle\int_0^\infty f(x)\,dx = 0$.

(d) Using the result in **Example 1.7.8** and making the necessary adjustments, we find that $I_\alpha = 0$ for all real numbers $\alpha > 1$.

▲

Example 1.7.14 Here, we examine and evaluate another kind of integral, in which we use branch cuts and the contours have some segments parallel to the branch cuts.

For all $a > 1$ real constant, we want to evaluate the integral

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)}.$$

With $a > 1$, we work with the complex function

$$f(z) = \frac{1}{(z+a)\sqrt{z^2-1}}.$$

This has an isolated singularity at $z_0 = -a$, which is a simple pole, and non-isolated singularities along the branch cut(s) that we are going to choose for making $\sqrt{z^2-1}$ a continuous function in the open domain $\mathbb{C} - \{\text{the union of the branch cut(s)}\}$.

We have

$$\sqrt{z^2-1} = (z^2-1)^{\frac{1}{2}} = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}.$$

In view of these two factors, we choose convenient branch cuts that shift $z-1$ and $z+1$ to the origin. We choose

$$\mathbb{C} - [+1, +\infty) \quad \text{for} \quad \sqrt{z-1} \qquad \text{and} \qquad \mathbb{C} - [-1, +\infty) \quad \text{for} \quad \sqrt{z+1}.$$

The intersection of these branch cuts is $[+1, +\infty)$ and their union is $[-1, +\infty)$, both intervals of the real axis.

The corresponding restrictions on the arguments are

$$0 < \arg(z-1) < 2\pi \quad \text{and} \quad 0 < \arg(z+1) < 2\pi.$$

Then, by the definition of the non-integer powers through complex logarithms, we have

$$\sqrt{z^2-1} = |z^2-1|^{\frac{1}{2}} e^{i\frac{\arg(z^2-1)}{2}} = \sqrt{|z^2-1|}e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}.$$

Now, for any $0 < r < \min\{1,\ a-1\}$ and any $(1 <)a < R$, we consider the contour

$$\begin{aligned} C^+ = [R, 1+r]^- + T_r^- + [1-r, -1+r]^- + A_r^- + \\ [-1+r, 1-r]^+ + S_r^- + [1+r, R]^+ + A_R^+, \end{aligned}$$

consisting of two positive segments, two negative segments, three negative arcs and one positive arc, as in **Figure 1.9**. The arcs are given analytically by:

$$\begin{aligned} T_r^- &= \left\{ z = 1 + re^{i\theta} \mid 2\pi > \theta > \pi \right\}, \\ A_r^- &= \left\{ z = -1 + re^{i\theta} \mid 2\pi > \theta > 0 \right\}, \\ S_r^- &= \left\{ z = 1 + re^{i\theta} \mid \pi > \theta > 0 \right\}, \\ A_R^+ &= \left\{ z = Re^{i\theta} \mid 0 < \theta < 2\pi \right\}. \end{aligned}$$

We must understand that **the positive segments are approached by staying in the upper half plane, whereas the negative segments are approached by staying in the lower half plane**. Then, the arguments of $\arg(z-1)$ and $\arg(z+1)$ along these segments are:

Along $[R, 1+r]^-$:	$\arg(z-1) = \arg(z+1) = 2\pi.$
Along $[1-r, -1+r]^-$:	$\arg(z-1) = \pi, \quad \arg(z+1) = 2\pi.$
Along $[-1+r, 1-r]^+$:	$\arg(z-1) = \pi, \quad \arg(z+1) = 0.$
Along $[1+r, R]^+$:	$\arg(z-1) = \arg(z+1) = 0.$

Next, the residue at $z_0 = -a$ is

$$\operatorname*{Res}_{z_0=-a} f(z) = f(z)(z+a)|_{z_0=-a} = \frac{1}{\sqrt{z^2-1}}\Big|_{z_0=-a} =$$

$$\frac{1}{\sqrt{|z^2-1|}\, e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}}\Big|_{z_0=-a} = \frac{1}{\sqrt{|(-a)^2-1|}\, e^{i\frac{\arg(-a-1)+\arg(-a+1)}{2}}} =$$

$$\frac{1}{\sqrt{|a^2-1|}\, e^{i\frac{\pi+\pi}{2}}} = \frac{1}{\sqrt{|a^2-1|}\, e^{i\pi}} = \frac{1}{-\sqrt{|a^2-1|}}.$$

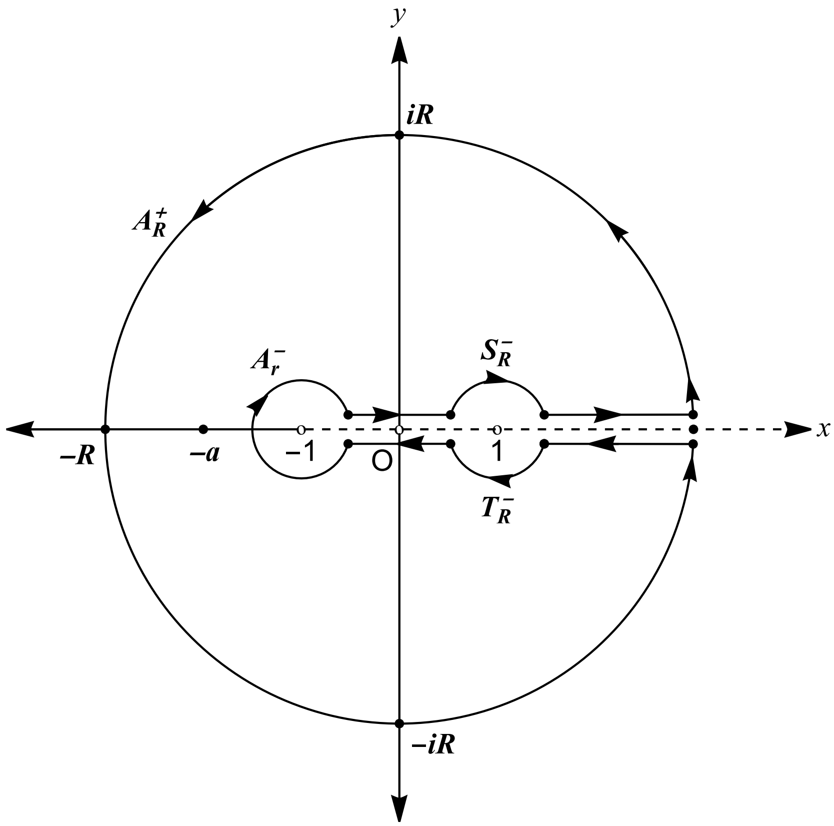

FIGURE 1.9: Contour 5 for Example 1.7.14

Now, we integrate $f(z)$ along C^+ and then we take limit as $R \to \infty$ and $r \to 0^+$. Thus, by **the Residue Theorem, 1.7.1**, the final result is equal to

$$2\pi i \cdot \frac{1}{-\sqrt{|a^2-1|}} = \frac{-2\pi i}{\sqrt{|a^2-1|}} = \frac{-2\pi i}{\sqrt{a^2-1}}, \quad \text{when} \quad a > 1 \quad \text{constant.}$$

Since $\lim\limits_{z\to\infty} zf(z) = 0$, by **Lemma 1.7.1**, we have $\lim\limits_{R\to\infty} \int\limits_{A_R^+} f(z)\,dz = 0.$

Also, we easily prove that

$$\lim_{r\to 0^+} \int\limits_{T_r^-} f(z)\,dz = \lim_{r\to 0^+} \int\limits_{A_r^-} f(z)\,dz = \lim_{r\to 0^+} \int\limits_{S_r^-} f(z)\,dz = 0.$$

Now, on $[R, 1+r]^-$ we have

$$f(z) = f(x) = \frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \tag{1.18}$$

$$\frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\frac{2\pi+2\pi}{2}}} = \frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\,2\pi}} = \frac{1}{(x+a)\sqrt{x^2-1}}.$$

Similarly, on $[1+r, R]^+$ we have

$$f(z) = f(x) = \frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \tag{1.19}$$

$$\frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\frac{0+0}{2}}} = \frac{1}{(x+a)\sqrt{x^2-1}\, e^{i\,0}} = \frac{1}{(x+a)\sqrt{x^2-1}}.$$

Hence, on these two opposite segments the function $f(z)$ remains the same, and so the two integrals over these opposite segments cancel each other.

[**Note**: We expect this phenomenon to happen along the common part of two branch cuts each of which corresponds to a factor of a function under consideration. Along the common part of the branch cuts, the function extends continuously, since crossing the common part changes both factors in a way that their product remains unchanged.

We add that in such a case the function remains also holomorphic along the common part of the two branch cuts. This is a consequence of **Morera's Theorem, 1.5.5**, since, by a simple limit argument, the holomorphicity of $f(z)$ in $\mathbb{C} - \{\text{union of the branch cuts}\}$ and its continuity in $\{\mathbb{C} - \{\text{union of the branch cuts}\}\} \cup \{\text{intersection of the branch cuts}\}$ imply

$$\int_C f(z)dz = 0$$

for any simple closed contour C such that C and its interior are inside the open region

$\{\mathbb{C} - \{\text{union of the branch cuts}\}\} \cup \{\text{intersection of the branch cuts}\}$.]

Next, we need to work out the two remaining integrals over the opposite segments $[1-r, -1+r]^-$ and $[-1+r, 1-r]^+$ only. On $[1-r, -1+r]^-$ we have:

$$f(z) = f(x) =$$

$$\frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi+2\pi}{2}}} =$$

$$\frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{3\pi}{2}}} = \frac{1}{-i(x+a)\sqrt{1-x^2}}.$$

Similarly, on $[-1+r, 1-r]^+$ we have:

$$f(z) = f(x) =$$
$$\frac{1}{(x+a)\sqrt{|x^2-1|}\, e^{i\frac{\arg(x-1)+\arg(x+1)}{2}}} = \frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi+0}{2}}} =$$
$$\frac{1}{(x+a)\sqrt{1-x^2}\, e^{i\frac{\pi}{2}}} = \frac{1}{i(x+a)\sqrt{1-x^2}}.$$

Finally, we find

$$\int_1^{-1} \frac{1}{-i(x+a)\sqrt{1-x^2}}\, dx + \int_{-1}^{1} \frac{1}{i(x+a)\sqrt{1-x^2}}\, dx = \frac{-2\pi i}{\sqrt{|a^2-1|}},$$

or

$$\int_{-1}^{1} \frac{1}{(x+a)\sqrt{1-x^2}} dx + \int_{-1}^{1} \frac{1}{(x+a)\sqrt{1-x^2}} dx =$$
$$i\frac{-2\pi i}{\sqrt{|a^2-1|}} = \frac{2\pi}{\sqrt{|a^2-1|}}.$$

So, we have eventually found: For all $a > 1$ real constant

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)} = \frac{\pi}{\sqrt{a^2-1}}.$$

Remark 1: When $|a| \leq 1$, this integral is improper at $x = a$ and does not exist. E.g., for $a = 0$, we get

$$\int_{-1}^{0} \frac{dx}{x\sqrt{1-x^2}} \leq \int_{-1}^{0} \frac{dx}{x} = -\infty \quad \text{and} \quad \int_{0}^{1} \frac{dx}{x\sqrt{1-x^2}} \geq \int_{0}^{1} \frac{dx}{x} = \infty,$$

and so

$$\int_{-1}^{1} \frac{dx}{x\sqrt{1-x^2}} = \text{does not exist}.$$

But,

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{x\sqrt{1-x^2}} = 0,$$

since we integrate an odd function over an interval symmetrical about the origin.

For $a = -\frac{1}{2}$, we have

$$\int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} = \int_{-\frac{3}{2}}^{\frac{1}{2}} \frac{du}{u\sqrt{1-\left(u+\frac{1}{2}\right)^2}} = \text{does not exist}$$

and similarly

$$\int_{-1}^{1} \frac{dx}{(1+x)\sqrt{1-x^2}} = +\infty \quad \text{and} \quad \int_{-1}^{1} \frac{dx}{(-1+x)\sqrt{1-x^2}} = -\infty,$$

etc. We examine the principal value in the **next Example**.

Remark 2: The integral evaluated in this **Example** may be considered as an integral of a rational function of $\sin(\theta)$ and $\cos(\theta)$, which in a calculus course is treated with the change of variables

$$u = \tan\left(\frac{\theta}{2}\right) \quad \Longleftrightarrow \quad \theta = 2\arctan(u),$$

called **tangent of half angle substitution**. From this, we find:

$$d\theta = \frac{2}{1+u^2}\,du,$$
$$\sin(\theta) = \sin[2\arctan(u)] = \frac{2u}{1+u^2},$$
$$\cos(\theta) = \cos[2\arctan(u)] = \frac{1-u^2}{1+u^2}.$$

These results change an (indefinite) integral of a rational function of $\sin(\theta)$ and $\cos(\theta)$ into an (indefinite) integral of a rational function of u, which is usually computed by partial fraction decomposition, etc. Then, from this indefinite integral, we compute the definite.

For example, we find that

$$\int \frac{d\theta}{-1+\sin(\theta)} = \frac{2}{\tan\left(\frac{\theta}{2}\right)-1} + C.$$

So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{-1+\sin(\theta)} =$$
$$\frac{2}{\tan\left(\frac{\pi}{4}\right)-1} - \frac{2}{\tan\left(\frac{-\pi}{4}\right)-1} = \frac{2}{0^-} - \frac{2}{-2} = -\infty + 1 = -\infty.$$

Also,

$$\int \frac{d\theta}{1+\sin(\theta)} = \frac{-2}{\tan\left(\frac{\theta}{2}\right)+1} + C,$$

and so

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{1+\sin(\theta)} =$$
$$\frac{-2}{\tan\left(\frac{\pi}{4}\right)+1} - \frac{-2}{\tan\left(\frac{-\pi}{4}\right)+1} = \frac{-2}{2} - \frac{-2}{0^+} = -1 - (-\infty) = \infty.$$

Using this method, we can compute proper and improper integrals and principal values of the rational functions of $\sin(\theta)$ and $\cos(\theta)$.

Remark 3: See **Section 1.8.1** for a different treatment of some integrals of this kind along with an easier uniform treatment of integrals of rational functions of sines and cosines on the interval $[0, 2\pi]$ or more generally over an interval of length 2π.

▲

We would like to examine the principal value of the integral in the **previous Example** when $|a| < 1$. But first, in case of isolated simple poles, we need the following convenient **Lemma**, which we will use in the sequel. As we see, in this example a is a simple isolated pole.

Lemma 1.7.3 *Let the complex function $f(z)$ have a **simple pole** at $z_0 \in \mathbb{C}$ and let $A_\theta(z_0, \epsilon) = \{z = z_0 + \epsilon e^{i\phi} \mid \theta_0 \leq \phi \leq \theta_0 + \theta\}$ be a circular arc centered at z_0, of radius $\epsilon > 0$ and angle θ, where θ_0 is a fixed angle. (See **Figure 1.10**.) Then,*

$$\lim_{\epsilon \to 0} \int_{A_\theta(z_0,\epsilon)} f(z)dz = i\theta \mathop{Res}_{z=z_0} f(z).$$

Proof Let $\operatorname*{Res}_{z=z_0} f(z) = b_1$. Since $z = z_0$ is a simple pole for $f(z)$, there is $r > 0$ such that in the closed disc $\overline{D(z_0, r)}$ we have

$$f(z) = \frac{b_1}{z - z_0} + g(z)$$

with $g(z)$ a holomorphic function and therefore bounded. So, there is $M > 0$ such that

$$|g(z)| \leq M, \ \ \forall \ \ z \in \overline{D(z_0, r)}.$$

Thus, $$\int_{A_\theta(z_0,\epsilon)} f(z)\, dz = \int_{A_\theta(z_0,\epsilon)} \frac{b_1}{z - z_0}\, dz + \int_{A_\theta(z_0,\epsilon)} g(z)\, dz.$$

But,

$$\int_{A_\theta(z_0,\epsilon)} \frac{b_1}{z - z_0} dz = b_1 \int_{\theta_0}^{\theta_0+\theta} \frac{\epsilon i e^{i\phi}}{\epsilon e^{i\phi}} d\phi = b_1 \theta\, i,$$

and for $0 < \epsilon < r$

$$\left| \int_{A_\theta(z_0,\epsilon)} g(z)dz \right| \leq \int_{A_\theta(z_0,\epsilon)} |g(z)||dz| \leq M\epsilon\,\theta \longrightarrow 0, \ \text{ as } \ \epsilon \longrightarrow 0.$$

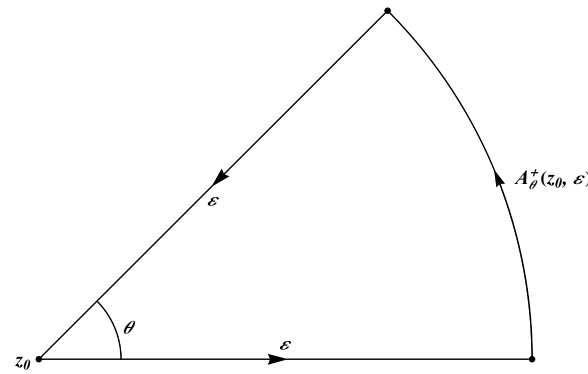

FIGURE 1.10: Contour 6 for Lemma 1.7.3

Finally,

$$\lim_{\epsilon \to 0} \int_{A_\theta(z_0,\epsilon)} f(z)dz = b_1 \theta\, i + 0 = \theta i \operatorname*{Res}_{z=z_0} f(z),$$

and the proof is finished!

■

Example 1.7.15 We will show that if $|b| < 1$

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{(b+x)\sqrt{1-x^2}} = \text{P.V.} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{b + \sin(\theta)} = 0.$$

Let us work with $b = -\dfrac{1}{2}$, and the same work proves the result for every b such that $|b| < 1$. We observe

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} =$$

$$\lim_{\delta \to 0^+} \left[\int_{-1}^{\frac{1}{2}-\delta} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} + \int_{\frac{1}{2}+\delta}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} \right] =$$

$$\lim_{\delta \to 0^+} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}-\delta} \frac{d\theta}{-\frac{1}{2}+\sin(\theta)} + \int_{\frac{\pi}{6}+\delta}^{\frac{\pi}{2}} \frac{d\theta}{-\frac{1}{2}+\sin(\theta)} \right].$$

We consider again the **previous Example**, and we observe that $z = \dfrac{1}{2}$ is on the branch cut and behaves like a "simple pole" for either functions, similar to **(1.18)** and **(1.19)**,

$$f^{\pm}(z) = \frac{1}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}},$$

where the $+$ and $-$ indicate that $f(z)$ is in the upper half plane and in the lower half plane, respectively, as we cannot cross the branch cut on the segment $-1 < x < 1$.

To find the principal value we use a contour, that is the one we use in the **previous Example** with two additional semicircles around $x = \dfrac{1}{2}$ and the consequent modification of the segments in between. Now we choose $0 < r < \dfrac{1}{4}$ and the contour

$$C_*^+ = [R, 1+r]^- + T_r^- + [1-r, \frac{1}{2}+r]^- + T_{*r}^- + [\frac{1}{2}-r, -1+r]^- + A_r^-$$
$$+[-1+r, \frac{1}{2}-r]^+ + S_{*r}^- + [\frac{1}{2}+r, 1-r]^+ + S_r^- + [1+r, R]^+ + A_R^+.$$

The two new arcs around $x = \dfrac{1}{2}$ are: $T_{*r}^- = \left\{z = \frac{1}{2} + re^{i\theta} \mid 2\pi > \theta > \pi\right\}$, and $S_{*r}^- = \left\{z = \frac{1}{2} + re^{i\theta} \mid \pi > \theta > 0\right\}$. The other arcs are the same as before.

We also observe that there are no singularities inside this contour and so no residues to compute. Therefore,

$$\int_{C_*^+} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = 0.$$

The question is what the newly introduced arcs contribute to the integral. We examine this now. **Lemma 1.7.3** can apply here, and we find that in the lower half plane

$$\int_{T_{*r}^-} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = (\pi - 2\pi)i \frac{1}{\sqrt{\left|\left(\frac{1}{2}\right)^2 - 1\right|}} e^{i\frac{\arg\left(\frac{1}{2}-1\right)+\arg\left(\frac{1}{2}+1\right)}{2}} =$$
$$-\pi i \sqrt{\frac{4}{3}}\, e^{i\frac{\pi+2\pi}{2}} = -\pi i \sqrt{\frac{4}{3}}\,(-i) = -\frac{2\pi}{\sqrt{3}} = -\frac{2\pi\sqrt{3}}{3}.$$

In the upper half plane, we get

$$\int_{S_{*r}^-} \frac{dz}{\left(-\frac{1}{2}+z\right)\sqrt{z^2-1}} = (0 - \pi)i \frac{1}{\sqrt{\left|\left(\frac{1}{2}\right)^2 - 1\right|}} e^{i\frac{\arg\left(\frac{1}{2}-1\right)+\arg\left(\frac{1}{2}+1\right)}{2}} =$$
$$-\pi i \sqrt{\frac{4}{3}}\, e^{i\frac{\pi+0}{2}} = -\pi i \sqrt{\frac{4}{3}}\, i = \frac{2\pi}{\sqrt{3}} = \frac{2\pi\sqrt{3}}{3}.$$

So, the net contribution of the new arcs is $-\dfrac{2\pi}{\sqrt{3}} + \dfrac{2\pi}{\sqrt{3}} = 0$. Therefore,

$$\text{P.V.} \int_{-1}^{1} \frac{dx}{\left(-\frac{1}{2}+x\right)\sqrt{1-x^2}} = 0.$$

The same result is true if we replace $-\dfrac{1}{2}$ with any b such that $|b| < 1$.

Remark: This result can also be obtained by elementary integration if we use the indefinite integral formula that can be derived by the half angle substitution. (See **Problem 1.7.8**.) If $a \neq 0$, b and c real constants, then

$$\int \frac{dx}{b + c\sin(ax)} =$$

$$\begin{cases} \dfrac{-2}{a\sqrt{b^2-c^2}} \cdot \arctan\left[\sqrt{\dfrac{b-c}{b+c}} \tan\left(\dfrac{\pi}{4} - \dfrac{ax}{2}\right)\right] + C, & \text{if } b^2 > c^2, \\[2ex] \dfrac{-1}{a\sqrt{c^2-b^2}} \cdot \ln\left[\left|\dfrac{c + b\sin(ax) + \sqrt{c^2-b^2}\cos(ax)}{b + c\sin(ax)}\right|\right] + C, & \text{if } b^2 < c^2, \\[2ex] \dfrac{\mp 1}{ab} \cdot \tan\left(\dfrac{\pi}{4} \mp \dfrac{ax}{2}\right) + C, & \text{if } b = \pm c \neq 0. \end{cases}$$

▲

We continue with a technical and convenient **Theorem** for integrals of the special type $\int_0^{\infty} x^r f(x)dx$, where r is a real constant but not an integer. Some integrals of this type can also be evaluated by other methods, but sometimes **this Theorem** is convenient and efficient.

Theorem 1.7.3 *We consider a complex function $f(z)$ and a real number r that satisfy the following hypotheses:*

1. *$f(z)$ is analytic in $\mathbb{C} - \{z_1, z_2, \ldots, z_n\}$, $n \geq 0$ integer, where $z_1, z_2, \ldots, z_n$ are (isolated) singularities of $f(z)$.*
2. *All $z_1, z_2, \ldots, z_n$ are not on the (strictly) positive x-axis.*
3. *$r > -1$ is not an integer.*
4. *There are constants $M_1 > 0$, $R_1 > 0$, $b > r+1$, such that if $|z| \geq R_1$, then $|f(z)| \leq \dfrac{M_1}{|z|^b}$.*
5. *There are constants $M_2 > 0$, $R_2 > 0$, $d < r+1$, such that if $0 < |z| \leq R_2$, then $|f(z)| \leq \dfrac{M_2}{|z|^d}$.*

Then, using in $z^r = e^{r\log(z)}$ the branch $0 < Arg(z) < 2\pi$, we have

$$\int_0^\infty x^r f(x)dx = \frac{-\pi e^{-r\pi i}}{\sin(r\pi)} \sum_{i=1}^{n} \underset{z=z_i}{Res}[z^r f(z)],$$

and the integral converges absolutely.

These hypotheses were properly imposed so that the proof works out. We omit the proof here, but the interested reader can find it in the bibliography, e.g: Marsden and Hoffman 1987, 304-307.

We can directly check that the hypotheses of this Theorem are satisfied when $f(z) = \dfrac{P(z)}{Q(z)}$ is a rational function, where $P(z)$ and $Q(z)$ are polynomials of degrees $p \geq 0$ and $q \geq 0$, respectively, and satisfy the following hypotheses:

1. We assume that $P(z)$ and $Q(z)$ have no common factors, or else $f(z)$ is completely simplified.
2. $Q(z)$ has no zero on the positive x-axis.
3. The number r is not an integer and satisfies $-1 < r < q - p - 1$.
4. If $Q(0) = 0$ and, as root of $Q(z)$, $z = 0$ has order $m \geq 1$, then we also need $m < r + 1$.

We illustrate this result with the following example:

Example 1.7.16 We would like to check if the integral

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx$$

satisfies the conditions of **Theorem 1.7.3** and evaluate it.

(1) The function $f(z) = \dfrac{1}{z^2+z+1}$ is analytic in $\mathbb{C} - \{e^{\frac{2\pi}{3}i},\ e^{\frac{i4\pi}{3}i}\}$, and the numbers $z_1 = e^{\frac{2\pi}{3}i},\ z_2 = e^{\frac{i4\pi}{3}i}$ are isolated singularities (poles of order 1).

(2) The numbers $z_1 = e^{\frac{2\pi}{3}i},\ z_2 = e^{\frac{4\pi}{3}i}$ are not on the positive x-axis.

(3) $r = \dfrac{1}{3} > -1$ is not an integer.

(4) Since

$$\left|\frac{f(z)}{\frac{1}{z^2}}\right| = \left|\frac{z^2}{z^2+z+1}\right| \longrightarrow 1, \quad \text{as} \quad z \longrightarrow \infty,$$

we conclude that there is $R_1 > 0$ such that if $|z| \geq R_1$, then $\left|\frac{f(z)}{\frac{1}{z^2}}\right| \leq 2$ or $|f(z)| \leq \frac{2}{|z|^2}$. So, we can pick $M_1 = 2$ and $b = 2$.

(5) Since

$$\left|\frac{f(z)}{\frac{1}{z}}\right| = \left|\frac{z}{z^2+z+1}\right| \longrightarrow 0, \quad \text{as} \quad z \longrightarrow 0,$$

we conclude that there is $R_2 > 0$ such that if $0 < |z| \leq R_2$, then $\left|\frac{f(z)}{\frac{1}{z}}\right| \leq 1$ or $|f(z)| \leq \frac{1}{|z|}$. So, we can pick $d = 1 < \frac{1}{3} + 1$ and $M_2 = 1$. So, all the hypotheses of **Theorem 1.7.3** are met.

Hence,

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx = \frac{-\pi e^{-\frac{\pi}{3}i}}{\sin\left(\frac{\pi}{3}\right)}\left\{\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] + \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right]\right\}.$$

Now, since

$$f(z) = \frac{1}{(z-z_1)(z-z_2)}$$

we get that

$$\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] = \frac{e^{\frac{2\pi}{9}i}}{z_1 - z_2} \quad \text{and} \quad \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right] = \frac{e^{\frac{4\pi}{9}i}}{z_2 - z_1}$$

within the branch $0 < \operatorname{Arg}(z) < 2\pi$.

Then,

$$\operatorname*{Res}_{z=z_1}\left[z^{\frac{1}{3}}f(z)\right] + \operatorname*{Res}_{z=z_2}\left[z^{\frac{1}{3}}f(z)\right] = \frac{e^{\frac{2\pi}{9}i} - e^{\frac{4\pi}{9}i}}{e^{\frac{2\pi}{3}i} - e^{\frac{4\pi}{3}i}}.$$

Putting these together, we find

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx =$$

$$\frac{-\pi e^{-\frac{\pi}{3}i}}{\sin\left(\frac{\pi}{3}\right)} \cdot \frac{e^{\frac{2\pi}{9}i} - e^{\frac{4\pi}{9}i}}{e^{\frac{2\pi}{3}i} - e^{\frac{4\pi}{3}i}} = \frac{\pi\left(-e^{\frac{-\pi}{9}i} + e^{\frac{\pi}{9}i}\right)}{\sin\left(\frac{\pi}{3}\right) 2i\frac{\sqrt{3}}{2}} = \frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right).$$

(See also **Problem 1.7.147** for another method.)

▲

The integrals in **Theorem 1.7.3** are immediately related to the Mellin[27] transform of a function.

[27] Robert Hjalmar Mellin, Finnish mathematician, 1854-1933.

Definition 1.7.2 *The* ***Mellin transform*** *of a real function* $y = f(x)$ *with* $0 < x < \infty$ *or* $0 \leq x < \infty$ *[we consider* $f(x) = 0$ *for* $x < 0$*] is defined by*

$$\mathcal{M}\{f(x)\}(s) := \phi(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

for all those s's for which this integral exists.

This transform has a lot of applications in mathematics, engineering and computer science. The inverse transform is

$$\mathcal{M}^{-1}\{\phi(s)\}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \phi(s)\, ds$$

for an appropriate constant c. We state that the Mellin transform exists if $x^k f(x)$ is absolutely integrable on $(0, \infty)$ for some $k > 0$. Then, the inverse transform also exists for $c > k$.

Several integrals that we have evaluated so far and involve the product of a function with some power x^α may be viewed as Mellin transforms. For example, the $\Gamma(p)$ is the Mellin transform of $f(x) = e^{-x}$, with $x \in (0, \infty)$. Also, in **Example 1.7.8**, we have proved

For $l = 1,\ 2,\ 3,\ 4, \ldots$ integer and $\alpha \in \mathbb{R}$ such that $l > \alpha + 1 > 0$

(or $l - 1 > \alpha > -1$) we have: $\displaystyle\int_0^\infty \frac{x^\alpha}{1 + x^l}\, dx = \frac{\pi}{l\, \sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$

Then, under these conditions, this integral can be viewed as

$$\mathcal{M}\left\{\frac{1}{1 + x^l}\right\}(\alpha + 1) = \frac{\pi}{l\, \sin\left[\frac{(\alpha+1)\pi}{l}\right]}.$$

Replacing $\alpha + 1$ with s, we obtain the Mellin transform

$$\mathcal{M}\left\{\frac{1}{1 + x^l}\right\}(s) = \frac{\pi}{l\, \sin\left(\frac{s\pi}{l}\right)}.$$

Many other such integrals in the examples and the problems of this text may be translated as Mellin transforms in an analogous way.

Now, we are going to evaluate the **Fresnel integrals** of **Examples I 1.3.20, I 2.4.2** and **Problem I 2.6.11** by using contour integration. To do this, we first need to prove the so-called **Jordan's**[28] **Lemma**. **This Lemma** is also very useful in the computation of Fourier transforms, as we shall see soon.

[28] Marie Ennemond Camille Jordan, French mathematician, 1838-1922.

Lemma 1.7.4 (Jordan's Lemma) *If* $0 \leq \theta_1 < \theta_2 \leq \pi$ *and* $\mu > 0$, *then*

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt < \frac{\pi}{\mu}.$$

Proof Since $e^{-\mu\sin(t)} > 0$, we obtain

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \leq \int_0^{\pi} e^{-\mu\sin(t)}dt = \int_0^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt + \int_{\frac{\pi}{2}}^{\pi} e^{-\mu\sin(t)}dt.$$

Using $u = \pi - t$, we find

$$\int_{\frac{\pi}{2}}^{\pi} e^{-\mu\sin(t)}dt = \int_0^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt.$$

So,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \leq 2\int_0^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt.$$

But, for $0 \leq t \leq \dfrac{\pi}{2}$, we have that $\dfrac{2}{\pi}t \leq \sin(t)$. This inequality is seen graphically, since $\sin(t)$ is a concave function in the interval $\left[0, \dfrac{\pi}{2}\right]$, because $\sin''(t) = -\sin(t) \leq 0$ for $0 \leq t \leq \dfrac{\pi}{2}$. Therefore, $y = \sin(t)$ is greater than or equal to the straight segment function $y = \dfrac{2}{\pi}t$ in $\left[0, \dfrac{\pi}{2}\right]$.

So, $e^{-\mu\sin(t)} \leq e^{-\frac{2\mu t}{\pi}}$ for all $t \in \left[0, \dfrac{\pi}{2}\right]$. Hence,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}\,dt \leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2\mu t}{\pi}}\,dt = 2\left(\frac{-\pi}{2\mu}\right)\left[e^{-\frac{2\mu t}{\pi}}\right]_0^{\frac{\pi}{2}} =$$
$$\frac{-\pi}{\mu}\left(e^{-\mu} - 1\right) = \frac{\pi}{\mu}\left(1 - e^{-\mu}\right) < \frac{\pi}{\mu}.$$

■

Remark: Jordan's Lemma also implies the following inequalities:
(1)If $\pi \leq \theta_1 < \theta_2 \leq 2\pi$ and $\lambda > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{\lambda\sin(t)}dt < \frac{\pi}{\lambda}.$$

(2) If $-\dfrac{\pi}{2} \leq \theta_1 < \theta_2 \leq \dfrac{\pi}{2}$ and $\sigma > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{-\sigma\cos(t)}dt < \frac{\pi}{\sigma}.$$

(3) If $\dfrac{\pi}{2} \leq \theta_1 < \theta_2 \leq \dfrac{3\pi}{2}$ and $\tau > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{\tau\cos(t)} dt < \frac{\pi}{\tau}.$$

Example 1.7.17 Now we can compute the **Fresnel integrals** by using contour integration techniques.

$$\int_0^\infty \cos\left(x^2\right) dx = \int_0^\infty \sin\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4}.$$

(Compare also with **Example I 2.4.2** and **Problem I 2.6.11**.)

We consider $f(z) = e^{-z^2}$, which is holomorphic in $\mathbb{C}$. This function has no singularities in $\mathbb{C}$ and so, by **the Cauchy-Goursat Theorem, 1.5.3**, its integral along any closed, continuous and piecewise continuously differentiable path is zero.

Then, for any $R > 0$, we consider the contour

$$C = [0, R] + A_R^+ + [Re^{\frac{i\pi}{4}}, 0],$$

where A_R^+ is the positively oriented arc parameterized by $z = Re^{i\theta}$ with $0 \leq \theta \leq \dfrac{\pi}{4}$. Then,

$$\oint\limits_{C^+} e^{-z^2} dz = \int\limits_{[0,\,R]} e^{-z^2} dz + \int\limits_{A_R^+} e^{-z^2} dz + \int\limits_{[Re^{\frac{i\pi}{4}},\,0]} e^{-z^2} dz = 0.$$

Now,

$$\int\limits_{[0,R]} e^{-z^2} dz = \int_0^R e^{-x^2} dx,$$

and, as we have found in **Section I 2.1, (2.1)**,

$$\lim_{R\to\infty} \int_0^R e^{-x^2} dx = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Next, since $e^{\frac{i\pi}{2}} = i$ and along $[Re^{\frac{i\pi}{4}}, 0]$ we have $z = xe^{\frac{i\pi}{4}}$ with $R > x > 0$, we get

$$\int\limits_{[Re^{\frac{i\pi}{4}},\,0]} e^{-z^2} dz = \int_R^0 e^{-x^2 e^{\frac{i\pi}{2}}} e^{\frac{i\pi}{4}} dx = -e^{\frac{i\pi}{4}} \int_0^R e^{-x^2 i} dx.$$

Lastly, on A_R^+ we get

$$\int\limits_{A_R^+} e^{-z^2} dz = iR \int_0^{\frac{\pi}{4}} e^{-R^2 e^{2ix}} e^{ix} dx.$$

We observe that

$$e^{-R^2 e^{2ix}} = e^{-R^2[\cos(2x)+i\sin(2x)]} = e^{-R^2\cos(2x)} e^{-iR^2\sin(2x)}.$$

Therefore,

$$\left| e^{-R^2 e^{2ix}} e^{ix} \right| = e^{-R^2\cos(2x)}.$$

So, we have

$$\left| \int\limits_{A_R^+} e^{-z^2} dz \right| \leq \int\limits_{A_R^+} \left| e^{-z^2} \right| |\, dz\, | \leq \int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\phi)} R\, d\phi \overset{\left(u=-\phi+\frac{\pi}{4}\right)}{=}$$

$$R \int_0^{\frac{\pi}{4}} e^{-R^2\sin(2u)} du \overset{(v=2u)}{=} \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2\sin(v)} dv <$$

$$\text{(by } \textbf{Jordan's Lemma 1.7.4}\text{)} \quad \frac{R}{2} \cdot \frac{\pi}{R^2} = \frac{\pi}{2R}.$$

Since $\lim\limits_{R\to\infty} \dfrac{\pi}{2R} = 0$, we get

$$\lim_{R\to\infty} \int\limits_{A_R^+} e^{-z^2} dz = 0.$$

So, as $R \to \infty$, we finally get

$$\frac{\sqrt{\pi}}{2} - e^{\frac{i\pi}{4}} \int_0^\infty e^{-ix^2} dx = 0.$$

Then,

$$\int_0^\infty \left[\cos\left(x^2\right) - i\sin\left(x^2\right)\right] dx =$$

$$\frac{\sqrt{\pi}}{2} e^{\frac{-i\pi}{4}} = \frac{\sqrt{\pi}}{2} \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2\pi}}{4} (1 - i).$$

Now, we separate the real and imaginary parts of this equality and obtain the final result

$$\int_0^\infty \cos\left(x^2\right) dx = \frac{\sqrt{2\pi}}{4} = \int_0^\infty \sin\left(x^2\right) dx.$$

Remark 1: This computation was carried out even though we knew nothing about the convergence of these integrals. So, the contour integration evaluates and proves the existence of the integral at the same time.

Remark 2: In our approach here, to compute the Fresnel integrals, we use the integral

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

that was computed earlier with the help of double integration in **Section I 2.1**.

There is also the reverse approach. By choosing an appropriate contour, different from the one in this example, we can compute the Fresnel integrals first, and then from them we can compute $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.[29]

▲

Example 1.7.18 In this example, we show

$$\int_{-\infty}^{\infty} e^{-x^2} \cos(4x) dx = e^{-4}\sqrt{\pi},$$

which is a special example of the integral of Laplace in **Example I 2.2.14** and **Problems I 2.3.16,, 1.7.32, (a)**.

(This is generalized in **Problem 1.7.32, (a)**, which is essentially the same with **Example I 2.2.14**, but the methods used are different.)

For any $R > 0$, we consider the rectangle contour in **Figure 1.11**

$$C^+ = [-R, R] + [R, R + 2i] + [R + 2i, -R + 2i] + [-R + 2i, -R].$$

Again, as in the **previous Example**, we have

$$\oint_{C^+} e^{-z^2} dz = 0,$$

and so

$$\int_{-R}^{R} e^{-x^2} dx + \int_0^2 e^{-(R+iy)^2} i\, dy +$$
$$\int_R^{-R} e^{-(x+2i)^2} dx + \int_2^0 e^{-(-R+iy)^2} i\, dy = 0.$$

[29] E.g., see the bibliography: Marsden and Hoffman 1987 and 1993.
In these two books and others, such as Copson 1948, Markushevich 1977, Brown and Churchill 2008, etc., one can find many interesting and complicated examples and/or exercises of contour integration.

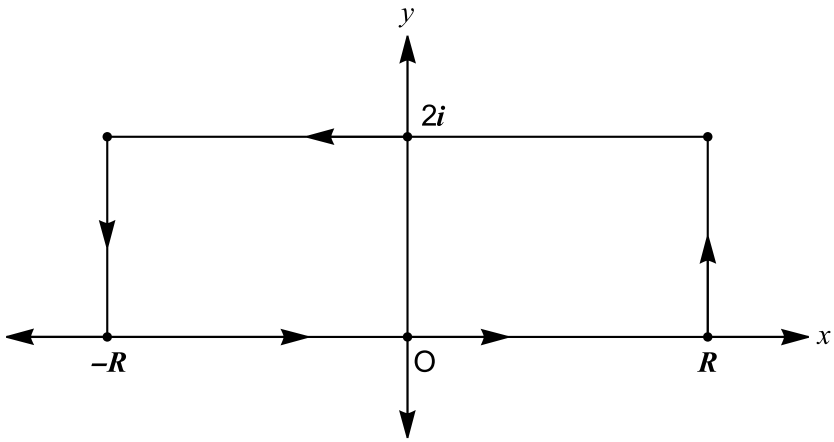

FIGURE 1.11: Contour 7 for Example 1.7.18

Then, we have

$$\left|\int_0^2 e^{-(R+iy)^2} i\,dy\right| \le \int_0^2 \left|e^{-R^2}e^{-2Riy}e^{y^2}\right| dy =$$
$$\int_0^2 e^{-R^2}e^{y^2}dy \le 2e^{-R^2}e^4 \to 0, \quad \text{as} \quad R \to \infty.$$

Similarly,

$$\left|\int_2^0 e^{-(-R+iy)^2} idy\right| \le 2e^{-R^2}e^4 \to 0, \quad \text{as} \quad R \to \infty.$$

Hence, taking limit as $R \to \infty$, we get

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-(x+2i)^2}dx.$$

Thus, by the equality of the real parts and the **Integral (I 2.1)** in **Section I 2.1**, we have

$$\text{Re}\left[\int_{-\infty}^{\infty} e^{-(x+2i)^2}dx\right] = \text{Re}\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right] = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

Developing this, we get

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(4x)\,dx = e^{-4}\sqrt{\pi}.$$

Remark 1: Since $f(x) = e^{-x^2}\cos(4x)$ is even, we get

$$\int_{-\infty}^{0} e^{-x^2}\cos(4x)\,dx = \int_{0}^{\infty} e^{-x^2}\cos(4x)\,dx =$$
$$\frac{e^{-4}\sqrt{\pi}}{2} \simeq 0.01623181234006586\ldots.$$

Remark 2: The equality of the imaginary parts gives

$$\int_{-\infty}^{\infty} e^{-x^2}\sin(4x)\,dx = 0,$$

a fact already known, since the function $f(x) = e^{-x^2}\sin(4x)$ is odd and the integral exists.

Remark 3: For any $R > 0$, if we integrate e^{-z^2} over the contour

$$C = [0, R] + [R, R + 2i] + [R + 2i, 2i] + [2i, 0](= \text{rectangle})$$

and then take the limit as $R \to \infty$, using the **Integral (2.1)** and the **Error Function (2.2)** (**Section 2.1**) and the first remark above, we find

$$\int_{0}^{\infty} e^{-x^2}\sin(4x)\,dx = e^{-4}\int_{0}^{2} e^{y^2}dy \simeq 0.3013403889237924\ldots.$$

(Check this. Keep in mind that the indefinite integral $\int e^{y^2}dy$ cannot be found in closed form by means of elementary functions. So, we compute $\int_{0}^{2} e^{y^2}dy$ numerically.)

▲

Problems

1.7.8 Prove the general indefinite integral formula of the **Remark** of **Example 1.7.15** and also derived the analogous formula if we replace sine by cosine.

1.7.9 Consider $f(z) = \dfrac{e^{-az}}{z^2+1}$ with $a \in \mathbb{R}$ constant and $C(0, R)$ the circumference of the circle with center the origin and radius $R > 0$.

(a) If $a \neq 0$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)} |zf(z)|\right] = +\infty.$$

(b) If $a = 0$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)}|zf(z)|\right] = 0.$$

(c) If $a > 0$ real constant and $A_R = \{z = Re^{i\theta} \mid 0 \le \theta \le \frac{\pi}{2}\}$, the arc of the circle $C(0,R)$ between the angles 0 and $\frac{\pi}{2}$, then prove

$$\lim_{R\to\infty}\left[\max_{z\in A_R}|zf(z)|\right] = 0.$$

1.7.10 Conclude the integrals of **Example 1.7.7** and **Case (a) of Example 1.7.8** from the integral of **Case (b) of Example 1.7.8**.

1.7.11 Prove that the integral $\int_0^\infty \frac{3x^2-2}{x^4+5x^2+6}\,dx$ exists, and then use contour integration to evaluate it.

1.7.12 (a) Using contour integration, evaluate $\int_{-\infty}^\infty \frac{dx}{(x-i)(x-2i)}\,dx.$

(b) Separate the real and imaginary parts to find the two corresponding real integrals.

1.7.13 Compute the following twelve integrals by whichever methods you prefer. You may use direct appropriate contour integrations or any other correct methods and integral formulae already used or proved in **previous Examples**, etc.

$$I_1 = \int_{-\infty}^\infty \frac{x^2}{1+x^4}\,dx, \qquad I_2 = \int_{-\infty}^\infty \frac{1}{1+x^6}\,dx,$$

$$I_3 = \int_0^\infty \frac{dx}{1+x^n}, \quad \forall\, n = 2,3,\ldots, \qquad I_4 = \int_0^\infty \frac{\sqrt[3]{x^2}}{1+x^2}\,dx,$$

$$I_5 = \int_0^\infty \frac{\sqrt[3]{x^5}}{1+x^3}\,dx, \qquad I_6 = \int_0^\infty \frac{dx}{\sqrt[3]{x}(x+1)},$$

$$I_7 = \int_0^\infty \frac{x^{\frac{-1}{2}}}{100+x}\,dx, \qquad I_8 = \int_0^\infty \frac{x^{\frac{5}{2}}}{100+x^4}\,dx,$$

$$I_9 = \int_{-1}^1 \frac{dx}{(x+10)\sqrt{1-x^2}}, \qquad I_{10} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{20+\sin(\theta)},$$

$$I_{11} = \int_0^\infty \frac{\sqrt[3]{x}}{(1+x)^3}\,dx, \qquad I_{12} = \int_0^\infty \frac{x^{\frac{-2}{3}}}{(1+x)^4}\,dx.$$

1.7.14 Prove

$$\text{(a)} \qquad \forall \quad k \in \mathbb{N}_0, \quad \int_0^1 \frac{x^k}{x^{k+2}+1}\,dx = \int_1^\infty \frac{1}{u^{k+2}+1}\,du.$$

$$\text{(b)} \qquad \forall \quad k \in \mathbb{N}_0, \quad \int_0^1 \frac{1}{x^{k+2}+1}\,dx = \int_1^\infty \frac{u^k}{u^{k+2}+1}\,du.$$

$$\text{(c)} \qquad \forall \quad k \in \mathbb{N}_0, \quad \int_0^1 \frac{x^k+1}{x^{k+2}+1}\,dx = \int_1^\infty \frac{u^k+1}{u^{k+2}+1}\,du =$$
$$\int_0^\infty \frac{1}{x^{k+2}+1}\,dx = \int_0^\infty \frac{x^k}{x^{k+2}+1}\,dx = \frac{\pi}{(k+2)\cdot\sin\left(\frac{\pi}{k+2}\right)}.$$

[Hint: Use $u = \dfrac{1}{x}$ and **Example 1.7.7** (or **1.7.8**).]

[See also **Problem I 1.1.4, (d)**. Also, compare with **Example I 2.6.22**. In both use $k = 4$.]

(d) If $\quad k \in \mathbb{N}_0, \quad$ then

$$\pi = (k+2)\cdot\sin\left(\frac{\pi}{k+2}\right)\sum_{n=0}^\infty \frac{(-1)^n[2(k+2)n+k+2]}{[(k+2)n+1][(k+2)n+k+1]}.$$

(e) For $k \in \mathbb{N}_0$ and $\rho = 3,\ 4,\ 5,\ \ldots$, examine equalities analogous to those in **(a)** and **(b)** with integrals of the type

$$\int_0^1 \frac{x^k}{x^{k+\rho}+1}\,dx,$$

etc. (Use $x = u^{\frac{-1}{\rho-1}}$ or any change of variables that you may find better.)

1.7.15 Use partial fractions and arc-tangent, or contour integration, to show that for any $a > 0$ and $b > 0$

$$\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} =$$
$$2\int_{-\infty}^0 \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

(Consider two cases: $a \neq b$ and $a = b$.)

1.7.16 (1) Prove:

$$\text{(a)} \qquad \forall\ a \in \mathbb{R}, \quad \int_{-\infty}^{\infty} \frac{x^2\,dx}{\left(x^2+a^2\right)^3} =$$
$$2\int_0^{\infty} \frac{x^2\,dx}{\left(x^2+a^2\right)^3} = 2\int_{-\infty}^{0} \frac{x^2\,dx}{\left(x^2+a^2\right)^3} = \frac{\pi}{8|a|^3}.$$

$$\text{(b)} \qquad \int_{-\infty}^{\infty} \frac{x^2\,dx}{1+x^2+x^4} =$$
$$2\int_0^{\infty} \frac{x^2\,dx}{1+x^2+x^4} = 2\int_{-\infty}^{0} \frac{x^2\,dx}{1+x^2+x^4} = \frac{\pi\sqrt{3}}{3}.$$

$$\text{(c)} \qquad \int_{-\infty}^{\infty} \frac{x^6\,dx}{\left(1+x^4\right)^2} =$$
$$2\int_0^{\infty} \frac{x^6\,dx}{\left(1+x^4\right)^2} = 2\int_{-\infty}^{0} \frac{x^6\,dx}{\left(1+x^4\right)^2} = \frac{3\pi\sqrt{2}}{8}.$$

(2) Now, for the integrals in **(a)**, **(b)**, and **(c)** in which $0 < x < \infty$, let $u = x^2$ and derive the three corresponding integrals in u.

1.7.17 (a) Use contour integration and **Example 1.6.22** to show that for all $n = 0,\ 1,\ 2,\ \dots,$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{\left(x^2+1\right)^{n+1}} = 2\int_0^{\infty} \frac{dx}{\left(x^2+1\right)^{n+1}} = \frac{\pi(2n)!}{2^{2n}(n!)^2} = \frac{\pi}{2^{2n}}\binom{2n}{n}.$$

(b) Now prove that for any $a > 0$, $b > 0$ and for any $n = 0,\ 1,\ 2,\ \dots,$ we have

$$\int_{-\infty}^{\infty} \frac{dx}{\left(ax^2+b\right)^{n+1}} = 2\int_0^{\infty} \frac{dx}{\left(ax^2+b\right)^{n+1}} =$$
$$\frac{1}{b^n\sqrt{ab}}\frac{\pi(2n)!}{2^{2n}(n!)^2} = \frac{1}{b^n\sqrt{ab}}\frac{\pi}{2^{2n}}\binom{2n}{n}.$$

(c) Let $x = \cot(u)$ in **(a)** to derive that for any $n = 0,\ 1,\ 2,\ \dots,$

$$\int_0^{\pi} \sin^{2n}(u)\,du = 2\int_0^{\frac{\pi}{2}} \sin^{2n}(u)\,du = \frac{\pi(2n)!}{2^{2n}(n!)^2} = \frac{\pi}{2^{2n}}\binom{2n}{n}.$$

Also, let $x = \tan(u)$ in **(a)** to derive for any $n = 0,\ 1,\ 2,\ \dots,$

$$\int_0^{\pi} \cos^{2n}(u)\,du = 2\int_0^{\frac{\pi}{2}} \cos^{2n}(u)\,du = \frac{\pi(2n)!}{2^{2n}(n!)^2} = \frac{\pi}{2^{2n}}\binom{2n}{n}.$$

(These two facts were already known from the Gamma and Beta functions. See, also **Problems I 2.6.25,, 1.8.12, 1.8.17** and **Examples I 2.6.19, 1.8.4, 1.8.5, 1.8.6**.)

1.7.18 Prove that $\displaystyle\int_0^1 \frac{dz}{z^2+1} = \frac{\pi}{4} + k\pi$, with $k \in \mathbb{Z}$, if this integral is taken along any path from 0 to 1 that does not contain i and $-i$. (The $k \in \mathbb{Z}$ depends on the chosen path.)

1.7.19 Use contour integration to prove:

(a) If $\beta \in \mathbb{R}$ and $c > 0$ constants

$$\int_{-\infty}^{\infty} e^{-cx^2}\sin(\beta x^2)\,dx =$$
$$2\int_0^{\infty} e^{-cx^2}\sin(\beta x^2)\,dx = \int_0^{\infty} e^{-cu}\,\frac{\sin(\beta u)}{\sqrt{u}}\,du =$$
$$\frac{2}{\sqrt{\pi}}\int_0^{\infty} \frac{\beta\,dv}{\beta^2 + \left(c+v^2\right)^2} = \operatorname{sign}(\beta)\sqrt{\frac{\pi}{2}}\sqrt{\frac{-c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

(See also **Example I 2.4.2**, **Remark 3**.)

(b) If $\beta \in \mathbb{R}$ and $c > 0$ constants

$$\int_{-\infty}^{\infty} e^{-cx^2}\cos(\beta x^2)\,dx =$$
$$2\int_0^{\infty} e^{-cx^2}\cos(\beta x^2)\,dx = \int_0^{\infty} e^{-cu}\,\frac{\cos(\beta u)}{\sqrt{u}}\,du =$$
$$\frac{2}{\sqrt{\pi}}\int_0^{\infty} \frac{\left(c+v^2\right)\,dv}{\beta^2 + \left(c+v^2\right)^2} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

1.7.20 Solve **Problem I 2.6.56** by integrating the function

$$f(z) = \frac{e^{iz}}{z^{1-m}}, \quad \text{with} \quad 0 < m < 1,$$

along the contour

$$C^+ = [r, R] + A^+ + [Ri, ri] + A^-,$$

where $0 < r < R$ real constants, $A^+(\theta) = \{z = Re^{i\theta} \mid 0 \le \theta \le \frac{\pi}{2}\}$ and $A^-(\theta) = \{z = re^{i\theta} \mid \frac{\pi}{2} \ge \theta \ge 0\}$, and then take limits as $r \longrightarrow 0$ and $R \longrightarrow \infty$.

(See also **Example I 2.6.7**.)

1.7.21 Prove that the two integrals

$$\int_0^\infty \frac{\sqrt{x}}{x^2 \pm x + 1}\, dx$$

exist, and then use contour integration to evaluate them.

[Hint: Notice that $x^3 \pm 1 = (x \pm 1)(x^2 \mp x + 1)$. So, for easier computations, you may use exponential representations of the isolated singularities which are the cubic roots of ± 1. Use the **contour of Figure 1.8** of **Example 1.7.8, Case (b)**, and work analogously. Or, use **Theorem 1.7.3**.]

1.7.22 (a) Prove: $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^3 + x^2 + x + 1}\, dx = \frac{\pi}{2}\,(\sqrt{2} - 1).$

(b) Evaluate: $\displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{x^3 + x^2 + x + 1}\, dx.$

[Hint:Use $f(z) = \dfrac{1}{z^3 + z^2 + z + 1} = \dfrac{1}{(z - i)(z + i)(z + 1)}$ and **Theorem 1.7.3**.]

1.7.23 Prove

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2 + 4x + 8}\, dx = \frac{\pi\sqrt{2}}{2} \cdot \frac{\sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\pi\sqrt{6}}{6}\sqrt{2 - \sqrt{3}}.$$

1.7.24 (a) If $-1 < a < 1$ and $b > 0$, prove

$$\int_0^\infty \frac{x^a}{x^2 + b^2}\, dx = \frac{\pi\, b^{a-1}}{2\cos\left(\frac{\pi a}{2}\right)}.$$

(b) If $a \ne 0$ and $a \ne 1$ and $-1 < a < 2$, prove

$$\int_0^\infty \frac{x^a}{x^3 + 1}\, dx = \frac{-\pi\left\{1 + 2\cos\left[\frac{2\pi(a+1)}{3}\right]\right\}}{3\sin(\pi a)}.$$

Determine that the answers when $a = 0$ and/or $a = 1$ are the same and equal to $\dfrac{2\pi\sqrt{3}}{9}$. (Use L' Hôpital's rule or partial fractions, etc. See also **Problem I 1.3.2.**.)

1.7.25 (a) Prove that if $-1 < p < 1$ and $a > b > 0$,

$$\int_0^\infty \frac{x^p}{(x+a)(x+b)}\,dx = \frac{\pi}{\sin(p\pi)} \cdot \frac{a^p - b^p}{a-b}.$$

(b) If $p = 0$ in **(a)**, then

$$\int_0^\infty \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{a-b} \cdot \ln\left(\frac{a}{b}\right).$$

(c) Let in **(a)** $a \longrightarrow b$ and keep b constant. Use L' Hôpital's rule to show

$$\int_0^\infty \frac{x^p}{(x+b)^2}\,dx = \frac{p\,\pi b^{p-1}}{\sin(p\pi)}.$$

E.g., if $-1 < p < 1$ and $b > 0$,

$$\int_0^\infty \frac{x^p}{(x+1)^2}\,dx = \frac{p\,\pi}{\sin(p\pi)}, \qquad \text{and} \qquad \int_0^\infty \frac{1}{(x+b)^2}\,dx = \frac{1}{b}.$$

[Hint: In **(a)**, use the **contour of Figure 1.8** of **Example 1.7.8, Case (b)**. See also **properties (B, 5)** and **(B, 8)** of the Beta function.]

1.7.26 (a) Prove that if $-1 < a < 1$ and $0 < \theta < \pi$,

$$\int_0^\infty \frac{x^a}{x^2 + 2x\cos(\theta) + 1}\,dx = \frac{\pi\sin(a\theta)}{\sin(a\pi)\sin\theta}.$$

[Hint: Use the **contour of Figure 1.8** of **Example 1.7.8, Case (b)**.]

(b) Show that the answer in **(a)** is $\dfrac{\theta}{\sin(\theta)}$ when $a = 0$, by using L' Hôpital's rule.

(c) Show that **(a)** gives the correct answer even when $\theta = 0$. I.e.,

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,dx = \frac{a\pi}{\sin(a\pi)}.$$

[See also **properties (B, 5)** and **(B, 8)** of the Beta function.]

(d) When $\theta = \pi$, show that $x = 1$ is a singularity and both sides of the formula in **(a)** are $+\infty$.

1.7.27 For $-1 < \alpha < 3$, prove

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)^2}\,dx = \frac{\pi}{4}\frac{1-\alpha}{\cos\left(\frac{\alpha\pi}{2}\right)} = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2}, \frac{3-\alpha}{2}\right).$$

(At the end points $\alpha = -1$ or 3, the equality becomes $\infty = \infty$.)

1.7.28 If $0 < a \le b$ and $0 < c < 2$, prove

$$\int_a^b \left(\frac{b-x}{x-a}\right)^{c-1}\frac{dx}{x} = \frac{\pi}{\sin(c\pi)}\left[1 - \left(\frac{b}{a}\right)^{c-1}\right].$$

1.7.29 For any $a > 1$ constant, prove the integral formula

$$\int_{-1}^1 \frac{dx}{(a-x)\sqrt{1-x^2}} \overset{x=\sin(\theta)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a-\sin(\theta)}.$$

(See also **Example 1.8.1** and its **Remarks**.)

1.7.30 (Practice problem!) Prove that

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2}$$

in the following two ways:

(1) By letting $x = \sec(u)$ (elementary way).

(2) By considering the complex function $f(z) = \dfrac{1}{z\sqrt{z^2-1}}$ and do work analogous to **Example 1.7.14**.

In this way, notice that, in general,

$$\frac{1}{\sqrt{z^2-1}} = \frac{1}{\sqrt{z-1}}\cdot\frac{1}{\sqrt{z+1}}.$$

Then, take branch cuts $[1,\infty)$ and $(-\infty,-1]$ for the first and the second factor, respectively. So, we have

$$\frac{1}{\sqrt{z-1}} = \frac{1}{\sqrt{|z-1|}}e^{i\frac{\arg(z-1)}{2}} \quad \text{with} \quad 0 < \arg(z-1) < 2\pi,$$

and

$$\frac{1}{\sqrt{z+1}} = \frac{1}{\sqrt{|z+1|}}e^{i\frac{\arg(z+1)}{2}} \quad \text{with} \quad -\pi < \arg(z+1) < \pi.$$

Notice that $z = 0$ is a simple pole of $f(z)$ with residue $-i$ (prove this).

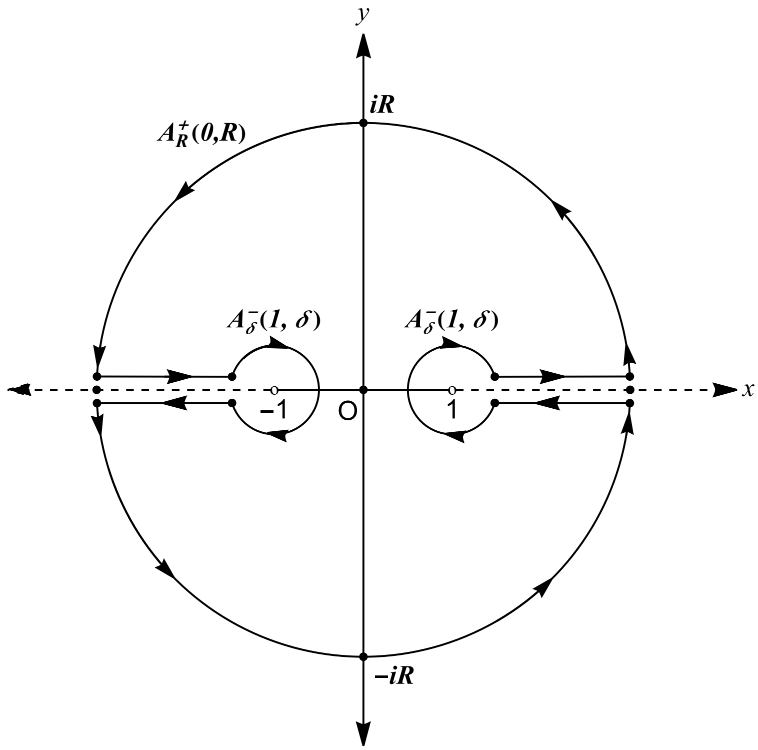

FIGURE 1.12: Contour 8, for practice Problem 1.7.29

As a **contour**, use the one provided in **Figure 1.12** for appropriate $r > 0$ and $R > 0$. Find the correct arguments of the complex numbers involved along the segments of the branch cuts, as you approach them staying in the upper half plane and/or staying in the lower half plane.

Then, use the **Residue Theorem, 1.7.1**, and take the appropriate limits. The limits along the three arcs are zero. The limits along the four straight segments parallel to the branch cuts are the same. Finally, we get

$$4\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}} = 2\pi,$$

and so the result follows.

[This problem, even though easy by means of the substitution in **Way (1)**, is suggested in order to practice with contour integration, branch cuts and correct arguments!]

1.7.31 Prove that $\displaystyle\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)\sqrt{1-x^2}}$ does not exist, but

$$\text{P.V.}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)\sqrt{1-x^2}} = \frac{2\ln(2+\sqrt{3})}{\sqrt{3}}.$$

1.7.32 (a) For $a > 0$ and $b > 0$ constants, prove

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\cos(bx)\,dx =$$

$$2\int_{-\infty}^{0} e^{-a^2x^2}\cos(bx)\,dx = 2\int_{0}^{\infty} e^{-a^2x^2}\cos(bx)\,dx = \frac{\sqrt{\pi}}{a}e^{\frac{-b^2}{4a^2}}.$$

(This is an **integral of Laplace**. Compare this with **Examples I 2.2.14, I 3.7.18**, and **Problem I 2.3.16**.)
[Hint: First, notice $e^{-a^2x^2}\cos(bx) = e^{-\frac{b^2}{4a^2}}\,\mathrm{Re}\left[e^{-\left(ax+\frac{b}{2a}i\right)^2}\right]$, and then for any $R > 0$ choose as an appropriate contour the rectangle

$$C = [-R,R] + [R, R+\frac{b}{2a}i] + [R+\frac{b}{2a}i, -R+\frac{b}{2a}i] + [-R+\frac{b}{2a}i, -R]$$

and consider the integral $\displaystyle\oint_{C^+} e^{-z^2}\,dz$.]

(b) Now prove that $\forall\ c \in \mathbb{R}$, $\displaystyle\int_{-\infty}^{\infty} e^{-(x+ic)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.
(See also **Problem I 2.2.43**.)

1.7.33 Work as in the **previous Problem** or **Example 1.7.18** and its **Remark 3** to prove that for $a \neq 0$ and $b \in \mathbb{R}$ constants

$$\int_0^{\infty} e^{-a^2x^2}\sin(bx)dx = \frac{1}{2a^2}e^{-\frac{b^2}{4a^2}}\int_0^b e^{\frac{t^2}{4a^2}}dt.$$

(Compare with **Problem I 2.2.43**.)

1.7.34 Use **Problem 1.7.32, (a)**, to prove that for every $a \neq 0$ and τ real constants

$$\int_{-\infty+i\tau}^{\infty+i\tau} e^{-(az)^2}dz = \int_{-\infty}^{\infty} e^{-a^2(\pm\sigma+i\tau)^2}d\sigma = \frac{\sqrt{\pi}}{a}e^{\tau^2\left(a^2-\frac{1}{a^2}\right)}$$

$$\text{and so}\quad \int_{-\infty}^{\infty} e^{-a^2(\pm\sigma+i\tau)^2}d\sigma = \frac{\sqrt{\pi}}{a}e^{\tau^2\left(a^2-\frac{1}{a^2}\right)}.$$

[Compare with **Problem 1.7.32, (b)**.]

1.7.35 If $-1 < \lambda < 1$, prove

$$\int_{-\infty}^{\infty} e^{\lambda x}\operatorname{sech}(x)\,dx = \int_{-\infty}^{\infty} e^{\lambda x}\frac{1}{\cosh(x)}\,dx = \frac{\pi}{\cos\left(\frac{\lambda\pi}{2}\right)}.$$

[See also **Problem 1.7.39, (a)**.]

1.7.36 Use contour integration to prove

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sz}}{\sqrt{z+1}}\,dz = 2i\sqrt{\pi}\,\frac{e^{-s}}{\sqrt{s}},$$

where $s > 0$ and $\gamma > 0$ constants. So,

$$\int_{-\infty}^{\infty} \frac{e^{s(\gamma+it)}}{\sqrt{\gamma+1+it}}\,dt = 2\sqrt{\pi}\,\frac{e^{-s}}{\sqrt{s}},$$

[Hint: $(\gamma-i\infty, \gamma+i\infty)$ is the infinite vertical straight line $x = \gamma$. Choose as the branch cut the interval $(-\infty, -1]$ and an appropriate contour.]

1.7.37 Let $p > 0$, $q > 0$ and $n \in \mathbb{N}$. Prove

(a) $$\int_0^{\infty} x^{n-1}e^{-px}\cos(qx)\,dx = \frac{(n-1)!\ \mathrm{Re}[(p+iq)^n]}{(p^2+q^2)^n},$$

(b) $$\int_0^{\infty} x^{n-1}e^{-px}\sin(qx)\,dx = \frac{(n-1)!\ \mathrm{Im}[(p+iq)^n]}{(p^2+q^2)^n}.$$

[Hint: Integrate the function $f(z) = z^{n-1}e^z$ along a contour similar to the contour in **Figure 1.6**, but choosing the angle at the origin O appropriately. Also, $\Gamma(n) = (n-1)!$.]

1.7.38 Integrate the function $\dfrac{1-e^{-z}}{z}$ on the two simple closed contours

$C_1 = [0,1] + \left\{e^{i\theta}, 0 \le \theta \le \dfrac{\pi}{2}\right\} + [i, 0]$ and

$C_2 = [1,0] + [0,-i] + \{e^{i\theta}, \dfrac{3\pi}{2} \le \theta \le 2\pi\}$.

For $R > 1$, integrate the function $\dfrac{e^{-z}}{z}$ on the two simple closed contours

$C_3 = [1,R] + \left\{Re^{i\theta}, 0 \le \theta \le \dfrac{\pi}{2}\right\} + [Ri, i] + \left\{e^{i\theta}, \dfrac{\pi}{2} \ge \theta \ge 0\right\}$ and

$C_4 = [R,1] + \left\{e^{i\theta}, 0 \ge \theta \ge -\dfrac{\pi}{2}\right\} + [-i, -Ri] + \left\{Re^{i\theta}, -\dfrac{\pi}{2} \le \theta \le 0\right\}$

and take limits as $R \longrightarrow \infty$.

Combine appropriately the four results found with the result of **Problem I 2.3.21** to find

$$\gamma = \int_0^1 [1-\cos(y)]\frac{dy}{y} - \int_1^{\infty} \cos(y)\frac{dy}{y}.$$

1.7.3 Infinite Isolated Singularities and Integrals

Suppose we need to evaluate

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

by means of residues, where $f(z)$ is holomorphic in the upper closed half plane $\mathrm{Im}(z) \geq 0$, or in the lower closed half plane $\mathrm{Im}(z) \leq 0$, except at a set of (countably) infinite isolated singularities $A = \{z_n \mid n \in \mathbb{N}\}$.

Let us assume that we work in the **upper closed half plane** [$\mathrm{Im}(z) \geq 0$] under the following hypotheses:

1. We assume that there are no singularities of $f(z)$ on the x-axis. That is, $\mathrm{Im}(z_n) \neq 0,\ \forall\ n \in \mathbb{N}$.

2. Since the singularities are isolated, there is no convergent subsequence of $(z_n)_{n\in\mathbb{N}}$, and they can be ordered so that $|z_1| \leq |z_2| \leq |z_3| \leq \dots$. Then we have: $|z_n| \to \infty$, as $n \to \infty$.

3. For any positive integer l, we can find a number $R_l > 0$ and a simple closed contour C_l with interior $\mathcal{D}_l$ in the upper closed half plane such that:

   (a) $R_l \uparrow \infty$, as $l \to \infty$ (strictly increasing to ∞).

   (b) $\forall\ l \in \mathbb{N},\ C_l \cap A = \emptyset$, that is, no singularity lies on any of the contours.

   (c) Since the singularities are isolated, any interior $\mathcal{D}_l$ contains finitely many of them. Assume $|A \cap \mathcal{D}_l| = k_l > 0$ integer and let $A \cap \mathcal{D}_l = \{z_1, z_2, \dots, z_{k_l}\}$.

   (d) $C_l = [-R_l, R_l] \cup P_l$ and $P_l \cap \{x-\text{axis}\} = \{-R_l, R_l\}$ (set of two elements). That is, the contour C_l consists of two parts: the closed segment $[-R_l, R_l]$ of the x-axis (and C_l has no other points of the x-axis) and a curve P_l in the upper half plane.

   (e) For any $k \neq l$, we have $P_k \cap P_l = \emptyset$, that is, the parts of the contours not on the x-axis are pairwise disjoint.

   (f) If $k < l$, then $\mathcal{D}_k \subsetneq \mathcal{D}_l$, that is, the interiors of the contours are in strict increasing order.

   (g) $A \subset \bigcup\limits_{l\in\mathbb{N}} \mathcal{D}_l$, that is, all singularities are in the infinite union of the interiors of the contours.

4. Finally, we assume

$$\lim_{l\to\infty} \int\limits_{P_l} f(z)\,dz = 0.$$

Then, under the above hypotheses (conditions), by the **Residue Theorem, 1.7.1**, we get

$$\oint_{C_l^+} f(z)\,dz = 2\pi i \sum_{k=1}^{k_l} \operatorname*{Res}_{z=z_k} f(z),$$

and so

$$\int_{-R_l}^{R_l} f(x)\,dx = 2\pi i \sum_{k=1}^{k_l} \operatorname*{Res}_{z=z_k} f(z) - \int_{P_l} f(z)\,dz.$$

Taking limits as $l \longrightarrow \infty$, by the last hypothesis, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z),$$

provided that the integral exists and the series converges.

More generally, we have computed the principal value of the integral, if this principal value exists, since the limit is taken over the symmetrical intervals $[-R_l, R_l] \longrightarrow (-\infty, \infty)$, as $l \longrightarrow \infty$. I.e.,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z),$$

provided that the principal value of the integral exists and the series converges.

This infinite series of residues may or may not converge even if the integral exists. There may be cases in which the integral exists but the series oscillates. For example, see **Problem 1.7.39**. **But, if the integral exists or its principal value exists and the series converges, the above equalities are valid, and thus we have evaluated the integral or its principal value by this series of numbers.**

With analogous hypotheses and work in the **lower closed half plane** $\mathrm{Im}(z) \leq 0$, we find: **Either**

$$\int_{-\infty}^{\infty} f(x)\,dx = -2\pi i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z), \text{ where } \mathrm{Re}(z_n) < 0, \ \forall\, n \in \mathbb{N}, \quad \textbf{or}$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = -2\pi i \sum_{n=1}^{\infty} \operatorname*{Res}_{z=z_n} f(z), \text{ where } \mathrm{Re}(z_n) < 0, \ \forall\, n \in \mathbb{N},$$

provided that the integral or its principal value exists and the series converges.

Example 1.7.19 We easily observe that the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{sech}(x)}{1+x^2}\,dx$$

exists. We will try to evaluate it as a convergent infinite series.

The corresponding complex function

$$f(z) = \frac{1}{(1+z^2)\cosh(z)}$$

has singularities at the solutions of $(1+z^2)\cosh(z) = 0$. These are

$$z = \pm i \quad \text{and} \quad z = \frac{2k+1}{2}\pi\, i \quad \text{with} \quad k \in \mathbb{Z}\,.$$

No singularity lies on the x-axis, and there are infinitely many **isolated** singularities in either of the half planes.

We choose to work in the upper closed half plane in which the singularities are:

$$i,\ \ \frac{1}{2}\pi\, i,\ \ \frac{3}{2}\pi\, i,\ \ \frac{5}{2}\pi\, i,\ \ \ldots\,.$$

These are simple poles and their **absolute values tend to** ∞.

For $l = 1,\ 2,\ 3,\ \ldots$, we pick numbers $0 < R_l = l\pi \uparrow \infty$ and simple closed contours $C_l^+ = [-R_l, R_l] \cup S_{R_l}^+$, where
$P_l := S_{R_l}^+ = \{z = R_l e^{i\theta} \mid 0 \le \theta \le \pi\}$ is the upper positively oriented semicircle with center the origin and radius $R_l = l\pi$.

The numbers $l\pi$ and contours C_l^+ satisfy the conditions stated at the beginning of this subsection. In fact, the first three are obvious, but we need to verify the fourth one. So, in some way we need to verify

$$\lim_{l\to\infty} \int\limits_{P_l = S_{R_l}^+} f(z)\,dz = 0.$$

Here, we cannot apply **Lemma 1.7.1** because the condition $\lim\limits_{z\to\infty} z f(z) = 0$ does not hold due to the existence of infinitely many poles that tend to ∞. So, we must verify **hypothesis (4)** for our particular choice of the contours!

For these particular paths $P_l = S_{R_l}^+$, we can prove that

$$\lim_{l\to\infty} \left[R_l \max_{z\in S_{R_l}^+} |f(z)| \right] = 0.$$

Then, as we prove **Lemma 1.7.1**, in the same way we obtain **hypothesis (4)**. So, we have:

$$R_l \max_{z \in S^+_{R_l}} |f(z)| = R_l \max_{z \in S^+_{R_l}} \frac{1}{|1+z^2||\cosh(z)|} \leq$$
$$l\pi \max_{z \in S^+_{R_l}} \left[\frac{1}{|z|^2 - 1} \cdot \frac{1}{|\cosh(z)|} \right] = \frac{l\pi}{l^2\pi^2 - 1} \max_{z \in S^+_{R_l}} \frac{1}{|\cosh(z)|} =$$
$$\frac{l\pi}{l^2\pi^2 - 1} \frac{1}{\min\limits_{z \in S^+_{R_l}} |\cosh(z)|}.$$

Since for any $z = x + yi$ we have $|\cosh(z)|^2 = \cos^2(y) + \sinh^2(x)$ (see **Problem 1.2.19**), and $x,\ y$ satisfy $x^2 + y^2 = l^2\pi^2$, with $0 \leq y \leq l\pi$, we might think of using the Lagrange[30] multipliers method to find the minimum of $|\cosh(z)|^2$. Unfortunately, the derived system of three equations in three unknowns cannot be solved easily. So, we must find a way to estimate a lower bound of $|\cosh(z)|^2$ (from which we get a lower bound of $|\cosh(z)|$), which, hopefully, is greater than (not equal to) zero. This is done as follows:

If $l\pi - \dfrac{\pi}{4} \leq y \leq l\pi$, then $\dfrac{1}{2} \leq \cos^2(y) \leq 1.$

If $0 \leq y < l\pi - \dfrac{\pi}{4}$, then for $l \geq 1$ we have

$$|x| \geq \sqrt{l^2\pi^2 - \left(l - \frac{1}{4}\right)\pi^2} = \pi\sqrt{\frac{l}{2} - \frac{1}{16}} = \pi\sqrt{\frac{8l-1}{16}} =$$
$$\frac{\pi}{4}\sqrt{8l-1} \geq \frac{\pi}{4}\sqrt{7}.$$

Since for $x \geq 0$, $\sinh(x) \geq 0$ and increasing, we have

$$\sinh^2(x) = \frac{e^{2x} + e^{-2x} - 2}{4} \geq \frac{e^{\frac{\pi}{2}\sqrt{7}} + e^{-\frac{\pi}{2}\sqrt{7}} - 2}{4} > 15.45\ldots\ .$$

Therefore, in either case, we have that

$$|\cosh(z)|^2 = \cos^2(y) + \sinh^2(x) > \frac{1}{2}, \quad \text{and so} \quad \frac{1}{|\cosh(z)|} < \sqrt{2}.$$

We then conclude

$$0 < R_l \max_{z \in S^+_{R_l}} |f(z)| < \frac{l\pi\sqrt{2}}{l^2\pi^2 - 1} \longrightarrow 0, \text{ as } l \longrightarrow \infty.$$

[30] Joseph-Louis Lagrange, considered to be a French mathematician, Italian born and named Giuseppe Luigi Lagrancia or Lagrangia, 1736-1813.

Hence, for the chosen contours, we obtain

$$\lim_{l\to\infty} \int\limits_{P_l=S^+_{R_l}} f(z)\,dz = 0\cdot\pi = 0.$$

We have observed that the singularities are simple poles. Now, we must calculate their residues.

$$\operatorname*{Res}_{z=i} f(z) =$$
$$\frac{z-i}{(z^2+1)\cosh(z)}|_{z=i} = \frac{1}{(z+i)\cosh(z)}|_{z=i} = \frac{1}{2i\,\cosh(i)} = \frac{1}{2i\,\cos(1)}.$$

For any $n \geq 1$ integer, we have:

$$\operatorname*{Res}_{z=\frac{2n-1}{2}\pi\, i} f(z) =$$
$$\frac{z-\frac{2n-1}{2}\pi\, i}{(z^2+1)\cosh(z)}|_{z=\frac{2n-1}{2}\pi\, i} = \frac{1}{(z^2+1)\sinh(z)}|_{z=\frac{2n-1}{2}\pi\, i} =$$
$$= \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]\sinh\left(\frac{2n-1}{2}\pi\, i\right)} = \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]\, i\,\sin\left(\frac{2n-1}{2}\pi\right)} =$$
$$\frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right](-1)^{n-1}\, i} = \frac{(-1)^n}{\left[\left(\frac{2n-1}{2}\right)^2\pi^2-1\right]\, i}.$$

So, finally,

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)\cosh(x)} = 2\pi\, i\left\{\frac{1}{2i\cos(1)} + \sum_{n=1}^{\infty}\frac{(-1)^n}{i\left[\left(n-\frac{1}{2}\right)^2\pi^2-1\right]\, i}\right\} =$$
$$\frac{\pi}{\cos(1)} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2-1},$$

and we see that this series converges absolutely.

Since $f(x) = \dfrac{1}{(1+x^2)\cosh(x)}$ is an even function, we also get

$$\int_{-\infty}^{0}\frac{dx}{(1+x^2)\cosh(x)} = \int_{0}^{\infty}\frac{dx}{(1+x^2)\cosh(x)} =$$
$$\frac{\pi}{2\cos(1)} + \pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2-1}.$$

▲

Example 1.7.20 Notice

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = 0,$$

since the function

$$f(x) = \frac{1}{(1+x^2)\sinh(x)}$$

is odd. But,

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\, dx$$

does not exist.

[Prove this using $\sinh(0^+) = 0^+$ and $\sinh(0^-) = 0^-$, which imply $f(0^+) = +\infty$ and $f(0^-) = -\infty$. Next, $\lim_{x\to 0} \frac{\sinh(x)}{x} = 1$, as $x \to 0$, and $\int_{-\infty}^{\infty} \frac{dx}{x}$ does not exist. Now, use the **p-Test, Example I 1.3.5.**.]

Here, the corresponding complex function has a singularity on the x-axis at $z = 0$.

(Continue in **Example 1.7.22**.)

▲

Isolated Singularities on Coordinate Axis and Cauchy Principal Value

We begin by proving again the **result**

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2 \pi^2 - 1},$$

using **another method**, which is especially convenient with problems in which $f(z)$ has finitely or infinitely many **isolated simple poles**. In such a case, **Lemma 1.7.3** is a useful and convenient tool. In this new proof, we use the contour C_k in **Figure 1.13**.

(With **isolated poles of order greater than one** and other types of **isolated singularities**, we work case by case.)

We have already seen that in the closed upper half plane the function

$$f(z) = \frac{1}{(1+z^2)\cosh(z)}$$

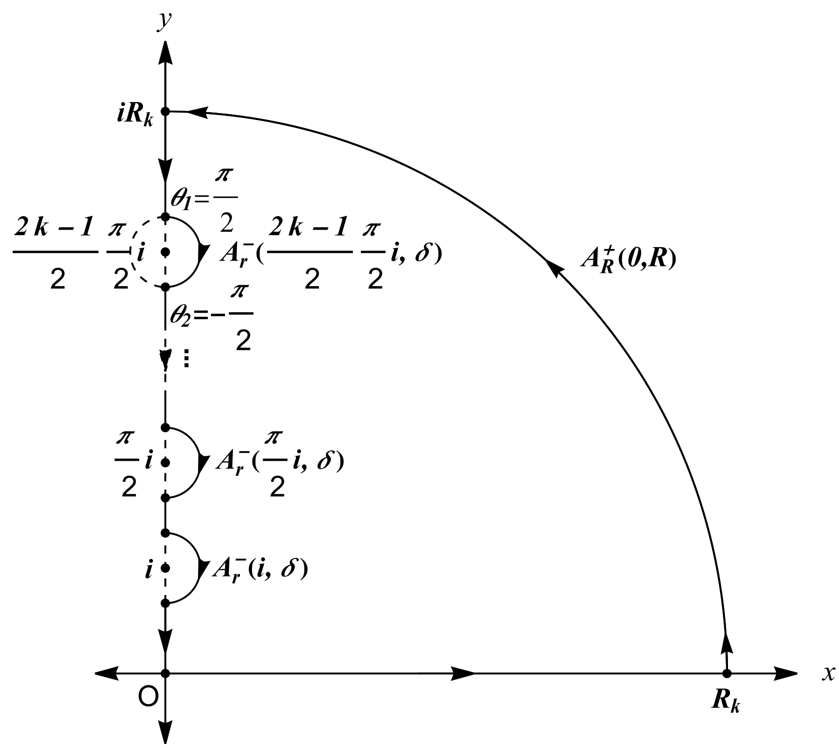

FIGURE 1.13: Contour 9 for the result examined here

has simple poles at the numbers

$$i, \ \frac{1}{2}\pi\, i, \ \frac{3}{2}\pi\, i, \ \frac{5}{2}\pi\, i, \ \ldots \ ,$$

with corresponding residues:

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2i\,\cos(1)}$$

and for any $k \geq 1$ integer

$$\operatorname*{Res}_{z=\frac{2k-1}{2}\pi\, i} f(z) = \frac{(-1)^k}{\left[\left(\frac{2k-1}{2}\right)^2 \pi^2 - 1\right]\, i}.$$

We have also seen if $R_k = k\pi$ and

$$A_k^+(0, R_k) = \left\{ z = R_k e^{i\alpha} \ \mid \ 0 \leq \alpha \leq \frac{\pi}{2} \right\}.$$

Then
$$\lim_{k\to\infty} R_k \max_{z\in A_k^+(0,R_k)} |f(z)| = 0.$$

We take $0 < \delta < \dfrac{1}{2}\left(\dfrac{\pi}{2} - 1\right) \simeq 0.2853982\ldots$ small enough so that all the small semicircles of C_k^+, **Figure 1.13**, with centers the poles of $f(z)$ and radius δ, are pairwise disjoint. Given that there are no singularities of $f(z)$ in the interior of this contour, by the **Residue Theorem, 1.7.1**, we have:
$$\int_{C_k^+} f(z)dz = 0.$$

We must observe:

1. Along any negatively oriented semicircle, as above, the angle difference is
$$\theta_2 - \theta_1 = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi.$$

2. For any interval $[ui,\ vi]$ of the imaginary axis that does not contain any pole, the integral
$$\int_u^v f(z)dz =$$
$$\int_u^v \frac{d(iy)}{[1+(iy)^2]\cosh(iy)} = i\int_u^v \frac{dy}{(1-y^2)\cos(y)}$$
is pure imaginary, and so it does not affect the real integral
$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)}.$$

Then, by **Lemma 1.7.3** and taking the limit as $k\to\infty$, we find
$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} +$$
$$(-\pi i)\left\{\frac{1}{2\cos(1)i} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2\pi^2 - 1\right]i}\right\} = 0,$$

and so we obtain the **result**

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2 - 1}.$$

Important Remarks

1. Here, we first prove that

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^\infty \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2 \pi^2 - 1},$$

and then we use the evenness of the integrand function to conclude

$$\int_{-\infty}^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{\cos(1)} + 2\pi \sum_{n=1}^\infty \frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2 \pi^2 - 1}.$$

2. If there were isolated singularities w's of $f(z)$ in the interior of the contours C_k^+'s, then we should add the $2\pi i \sum_w \operatorname*{Res}_{z=w} f(z)$ in the second side of the equality.

3. Since by construction of the contours C_k^+'s we avoid a symmetrical interval $(p-\delta i, p+\delta i)$ around each pole p of $f(z)$ located in the upper closed half plane, by letting $\delta \to 0$ we obtain:

$$\text{P.V.} \int_0^\infty f(iy)\, d(iy) = \text{P.V.} \int_0^\infty \frac{d(iy)}{[1+(iy)^2]\cosh(iy)} =$$

$$\text{P.V.} \ \ i \int_0^\infty \frac{dy}{(1-y^2)\cos(y)} = 0,$$

and so

$$\text{P.V.} \int_0^\infty \frac{dy}{(1-y^2)\cos(y)} = 0.$$

I.e., the **Cauchy principal value** of this integral is equal to 0.

4. The integrand function has infinitely many isolated simple poles on the negative imaginary half axis. Then, in the same way, we get

$$\text{P.V.} \int_{-\infty}^0 \frac{dy}{(1-y^2)\cos(y)} = 0.$$

This with the evenness of $f(y) = \dfrac{1}{(1-y^2)\cos(y)}$ implies

$$\text{P.V.} \int_{-\infty}^\infty \frac{dy}{(1-y^2)\cos(y)} = 0.$$

5. If the integral exists, then its equal to its **Cauchy principal value**. In this example, the integral of $f(z)$ along the upper closed half imaginary axis, $[0i, \infty i)$, does not exist. (Prove this!)

6. Let

$$A = [0i, (1-\delta)i] \bigcup \left[(1+\delta)i, \left(\frac{\pi}{2} - \delta\right) i\right] \bigcup$$
$$\left\{ \bigcup_{l=1}^{\infty} \left[\left(\frac{2l-1}{2}\pi + \delta\right) i, \left(\frac{2l+1}{2}\pi - \delta\right) i \right] \right\}.$$

Then, we have:

$$(a) \qquad \int_A f(z)dz = 0,$$
$$(b) \qquad \lim_{0 \le R \to \infty} \int_{A \cap [0i, Ri]} f(z)dz = 0,$$
$$(c) \qquad \text{P.V.} \int_0^{\infty} f(z)dz = 0.$$

7. Analogous remarks and computations can be made when there are **finitely or infinitely many isolated singularities** on the x-axis or the y-axis.

We end this subsection by adding a result about the Cauchy principal value. In view of **Lemmata 1.7.1, 1.7.2 and 1.7.3**, we write and can directly prove the **Theorem** that follows on **evaluating the Cauchy principal value of a certain type of integrals**.

If a function $f(x)$ has finitely many isolated singularities (x_i), with $i = 1,\ 2,\ \ldots,\ n \in \mathbb{N}$, on the x-axis, we pick small $\epsilon > 0$ and big $R > 0$ and then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx =$$

$$\lim_{\substack{\epsilon \to 0^+ \\ R \to \infty}} \left[\int_{-R}^{x_1 - \epsilon} f(x)dx + \sum_{i=1}^{n} \int_{x_i - \epsilon}^{x_i + \epsilon} f(x)dx + \int_{x_n + \epsilon}^{R} f(x)dx \right].$$

Now, we state:

Theorem 1.7.4 *Suppose a complex function $f(z)$ satisfies the conditions of **Lemma 1.7.1** or **Lemma 1.7.2** with the exception that it is allowed to have finitely many **simple** (isolated) **poles** on the x-axis. Then we have:*

$$P.V. \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum [\textit{residues of } f(z) \textit{ in the upper half plane}] +$$
$$\pi i \sum [\textit{residues of } f(z) \textit{ on the real axis}]$$

or

$$P.V. \int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum[\textit{residues of } f(z) \textit{ in the lower half plane}] -$$

$$\pi i \sum[\textit{residues of } f(z) \textit{ on the real axis}].$$

So, by adding these two equations and dividing by 2, we also have

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \pi i \left\{ \sum[\textit{residues of } f(z) \textit{ in the upper half plane}] - \right.$$

$$\left. \sum[\textit{residues of } f(z) \textit{ in the lower half plane}] \right\}.$$

(The **proof** follows from **Lemmata 1.7.1, 1.7.2 and 1.7.3**.)

Example 1.7.21 Find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 - 1}\, dx.$$

The complex function

$$f(z) = \frac{1}{z^4 - 1} = \frac{1}{(z-1)(z+1)(z-i)(z+i)}$$

has two simple poles on the x-axis, the numbers $z = 1$ and $z = -1$, one simple pole, $z = i$, in the upper half plane and satisfies the conditions of **Theorem 1.7.4**. The corresponding residues are:

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{2(1-i)(1+i)} = \frac{1}{4},$$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{-2(-1-i)(-1+i)} = \frac{-1}{4},$$

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(i-1)(i+1)2i} = \frac{-1}{4i}.$$

Therefore, by **Theorem 1.7.4**, we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 - 1}\, dx = 2\pi i \left(\frac{-1}{4i} \right) + \pi i \left(\frac{1}{4} - \frac{1}{4} \right) = \frac{-\pi}{2}.$$

▲

Example 1.7.22 We continue with **Example 1.7.20**. There we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\, dx = 0,$$

but

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \int_{-\infty}^{\infty} \frac{\operatorname{csch}(x)}{1+x^2}\, dx \quad \text{does not exist.}$$

The singularities of $f(z) = \dfrac{1}{(1+z^2)\sinh(z)}$ are all the roots of the denominator $g(z) := (1+z^2)\sinh(z)$, $z = \pm i$ and $z = k\pi i$ with $k \in \mathbb{Z}$. These roots of $g(z)$ are simple, and so they are simple poles of $f(z)$.

For $k = 0$, we get the pole $z = 0$, which is on the x-axis, and

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

We work in the upper half plane, and so

$$\operatorname*{Res}_{z=i} f(z) = \frac{-1}{2\sin(1)} = \frac{-1}{2}\csc(1).$$

At $z = k\pi i$, with $k \in \mathbb{N}$, we find

$$\operatorname*{Res}_{z=k\pi i} f(z) = \frac{1}{(1-k^2\pi^2)(-1)^k} = \frac{(-1)^{k-1}}{\pi^2\left[k^2 - \left(\frac{1}{\pi}\right)^2\right]}.$$

Therefore, by **Theorem 1.7.4**, and since the above principal value is zero, it must be

$$\pi i \cdot 1 + 2\pi i \cdot \left(\frac{-1}{2}\right)\csc(1) + 2\pi i \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi^2\left[k^2 - \left(\frac{1}{\pi}\right)^2\right]} =$$

$$\pi i \left\{1 - \csc(1) + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 - \left(\frac{1}{\pi}\right)^2}\right\} = 0.$$

From this equality, we find the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2 - \left(\frac{1}{\pi}\right)^2} = \frac{\pi^2[\csc(1) - 1]}{2}$$

[This sum also follows by **Problem 1.7.59, (b)**.]

Thinking in the reverse way, we can use **Theorem 1.7.4** and the obvious fact that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)\sinh(\pi x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\operatorname{csch}(\pi x)}{a^2+x^2}\, dx = 0$$

and derive the result of **Problem 1.7.59, (b)** for any $a \notin \mathbb{Z}$.

▲

Application: Here, we use the results of **Problem 1.7.51** below in a statistical application. If we assume that some measurements $X = x$ have normal distribution (see **Subsection 2.1.1, Application 1**)

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty,$$

and we want to find the average or mean value of $Y = \dfrac{1}{x}$, then by **Problem 1.7.51, (a)** this mean value does not exist. So, if we experiment with many such measurements in an irregular or a random way, we can get any awkward results, in general.

If, however, we make many measurements of X in a way that its values x's near zero are evenly or almost evenly distributed about the singular value $X = 0$, or we have strong natural reasons to suppose so, then using principal values, by **Problem 1.7.51, (d)** with $a = \sqrt{2}\,\sigma$, we find

$$E(Y) = \frac{1}{\sigma\sqrt{2\pi}}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{x}\, dx \ =$$

$$\frac{e}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{x^2+2\sigma^2} \left\{ x\, \cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right] - \sqrt{2}\,\sigma \sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\} dx.$$

For example, if $\mu = 2$ and $\sigma = \dfrac{1}{\sqrt{2}} \simeq 0.70711\ldots$ and we make a large number of measurements that satisfy the above condition, 3000 of them let us say, then, by **Problem 1.7.51, (e)**, we find

$$E(Y) \simeq \frac{1}{\sigma\sqrt{2\pi}} \cdot 1.06822\ldots = \frac{1}{\sqrt{\pi}} \cdot 1.06822\ldots \simeq 0.60268\ldots\ .$$

If $\mu = 0$, the principal value of the expected value of $Y = \dfrac{1}{x}$ of many measurements symmetrical about zero is zero, as expected.

Problems

1.7.39 (a) Prove directly that

$$\int_{-\infty}^{\infty} \frac{1}{\cosh(x)}\,dx = \int_{-\infty}^{\infty} \operatorname{sech}(x)\,dx = 2\int_{0}^{\infty} \frac{1}{\cosh(x)}\,dx =$$
$$2\int_{0}^{\infty} \operatorname{sech}(x)\,dx = 2\int_{-\infty}^{0} \frac{1}{\cosh(x)}\,dx = 2\int_{-\infty}^{0} \operatorname{sech}(x)\,dx = \pi,$$

but this integral cannot be computed by the infinite series of the respected residues.

(See also **Problem 1.7.35**.)

(b) Prove that the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x}{\cosh(x)}\,dx$ exists absolutely and so is equal to zero as an integral of an odd function. But, explain why this integral cannot be computed by the infinite series of the respected residues, as we have seen for **Example 1.7.19**.

[See also **Problem 1.7.139, (a),** g_3**,** for the value of this integral on $[0,\infty)$.]

1.7.40 Since $\displaystyle\frac{z}{\sinh(z)}\Big|_{z=0} = 1$, the singularity of $f(z) = \dfrac{z}{(1+z^4)\sinh(z)}$ at $z=0$ is removable. Then, prove

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^4)\sinh(x)}\,dx = \int_{-\infty}^{\infty} \frac{x}{1+x^4}\,\operatorname{csch}(x)\,dx =$$
$$\frac{\pi\,\sinh\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right)+\sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4\pi^4+1}.$$

1.7.41 For $f(x) = \dfrac{x}{\sinh(x)}$, prove that its integral over $\mathbb{R}$ exists and

$$0 < \int_{-\infty}^{\infty} \frac{x}{\sinh(x)}\,dx = \int_{-\infty}^{\infty} x\operatorname{csch}(x)\,dx < 3+\sqrt{e},$$

but it cannot be evaluated by the infinite series of residues.

1.7.42 Let $a>0$ and $b\in\mathbb{R}$ constants. Prove

$$\int_{0}^{\infty} \frac{\cos(bx)}{\cosh(x)+\cosh(a)}\,dx = \frac{\pi\sin(ab)}{\sinh(a)\sinh(\pi b)}.$$

[Hint: Integrate the function $f(z) = \dfrac{e^{ibz}}{\cosh(z)+\cosh(a)}$ along the rectangle with vertices $(\pm R, \pm R + 2\pi i)$.]

1.7.43 (a) For $-\pi < a < \pi$, prove that

$$\int_{-\infty}^{\infty} \frac{\sinh(ax)}{\sinh(\pi x)}\,dx = 2\int_{0}^{\infty} \frac{\sinh(ax)}{\sinh(\pi x)}\,dx = 2\int_{-\infty}^{0} \frac{\sinh(ax)}{\sinh(\pi x)}\,dx = \tan\Big(\frac{a}{2}\Big).$$

(b) In general, if $b > 0$ and $-b < a < b$,

$$\int_{-\infty}^{\infty} \frac{\sinh(ax)}{\sinh(bx)}\,dx = 2\int_{0}^{\infty} \frac{\sinh(ax)}{\sinh(bx)}\,dx = 2\int_{-\infty}^{0} \frac{\sinh(ax)}{\sinh(bx)}\,dx = \frac{\pi}{b}\tan\Big(\frac{a\pi}{2b}\Big).$$

(See also **Problem I 2.3.22, (c), (d), 1.7.80**, **Example 1.7.25**, **Corollary 1.7.5**.)

(c) Use **Problem I 2.3.22, (b)**, to show that if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, then

$$\tan(x) = 8x\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 - 4x^2}.$$

1.7.44 By means of infinite series, evaluate the following two integrals

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)[\cos(x)+4]}$ and (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)[\sin(x)+2]}$.

1.7.45 (a) Use complex analysis to prove that if the real quadratic polynomial ax^2+bx+c ($a \neq 0$) has two real distinct roots, and so $\mathbf{b^2 > 4ac}$, then

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = 0.$$

Prove that without the principal value, the integral does not exist.

(b) Achieve the result in **(a)** with calculus techniques.

(c) If the polynomial has one real double root, and so $\mathbf{b^2 = 4ac}$, then prove that the integral in **(a)** (without the principal value) is equal to

$$\text{sign}(\mathbf{a})\cdot\infty.$$

(d) If the polynomial has two complex conjugate roots and so $\mathbf{b^2 < 4ac}$, then the integral has finite value and converges absolutely. Using complex analysis or calculus techniques prove that this value is

$$\frac{\mathbf{2}\pi}{\sqrt{\mathbf{4ac - b^2}}}.$$

(See also **Problem I 1.2.21**.)

1.7.46 Consider a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real coefficients, simple real roots r_i, with $i = 0,\ 1,\ \ldots,\ k \leq n$, and simple complex roots z_j, with $j = 0,\ 1,\ \ldots,\ 2m \leq n$. (So, $k + 2m = n$.) Find the formula for $\text{P.V.}\displaystyle\int_{-\infty}^{\infty} \frac{1}{P(x)}\,dx$.

Investigate the same question if the complex roots are not simple.

1.7.47 Find the following five principal values:

$$\text{(a)}\quad \text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{x^2-1} (= 0), \qquad \text{(b)}\quad \text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{x^3-1} \left(= \frac{-\pi}{\sqrt{3}}\right),$$

$$\text{(c)}\ \ \text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{x^3+1}, \quad \text{(d)}\ \ \text{P.V.}\int_{-\infty}^{\infty} \frac{xdx}{x^3-1}, \quad \text{(e)}\ \ \text{P.V.}\int_{-\infty}^{\infty} \frac{xdx}{x^3+1}.$$

1.7.48 (a) If a and $b > 0$ are real constants, the

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x\left[(x-a)^2 + b^{\,2}\right]}\,dx,$$

exists. Find it by using: (1) Real calculus. (2) Complex analysis.

(b) Show that as an integral, not principal value, the integral in **(a)** does not exist.

(c) Now prove

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{(x-2)^2}\,dx = +\infty, \qquad \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{(x-2)^2(x-3)}\,dx = -\infty,$$

$$\text{and} \qquad \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{(x-2)^2(x-3)(x-4)^2}\,dx = \text{does not exist.}$$

1.7.49 (A result of **Legendre A.- M.**. See also **Example 1.7.25** and **Problem 1.7.60**.)

(a) Integrate the complex functions

$$f(z) = \frac{e^{aiz}}{e^{2\pi z}-1} \quad \text{and} \quad g(z) = \frac{e^{-aiz}}{e^{2\pi z}-1},$$

where $a \in \mathbb{C}$ such that $|\text{Im}(a)| < 2\pi$, over the positively oriented **contour** provided in **Figure 1.14**, use **Lemma 1.7.3**, let $\epsilon \to 0^+$ and $R \to \infty$ and then subtract the two results to prove:

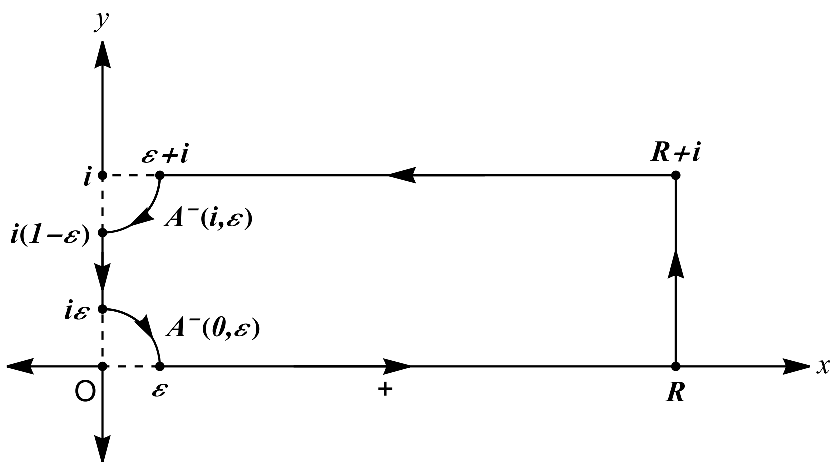

FIGURE 1.14: Contour 10 for Problem 1.7.49, (a)

$$\int_0^\infty \frac{\sin(ax)}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}.$$

(The condition $|\mathrm{Im}(a)| < 2\pi$ is used in showing that the limit of the integral of either function over the interval of constant length $[R+0i, R+1i] = [R, R+i]$ is zero, as $R \to \infty$.)

(b) Now, manipulate the constants α and β in $\mathbb{C}$ such that $\mathrm{Re}(\beta) > 0$ and $|\mathrm{Im}(\alpha)| < \mathrm{Re}(\beta)$, to prove

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x}-1}\,dx = \frac{\pi}{2\beta}\coth\left(\frac{\alpha\pi}{\beta}\right) - \frac{1}{2a}.$$

(c) From the processes in **(a)** and **(b)**, show the byproduct

$$\lim_{\epsilon\to 0^+}\left\{\lim_{r\to\infty}\left[2\,\sinh\left(\frac{2\pi\alpha}{\beta}\right)\int_\epsilon^r \frac{\cos(\alpha u)}{e^{\beta u}-1}\,du\right] + \int_\epsilon^{\frac{2\pi}{\beta}-\epsilon} \sinh(\alpha u)\cot\left(\frac{\beta u}{2}\right)\,du\right\} = 0.$$

(d) Prove that under the same conditions on the constants α and β, we have

$$\int_0^\infty \frac{\sin(\alpha x)}{e^{\beta x}+1}\,dx = -\frac{\pi}{2}\,\mathrm{csch}\left(\frac{\alpha\pi}{\beta}\right) + \frac{1}{2\alpha}.$$

1.7.50 (See also **Example 1.7.25** and **Problem 1.7.60**.) Imitate or use the **previous Problem** and some identities in **Problems 1.2.19** and **1.2.20** to prove that for constants a and a in $\mathbb{C}$ such that $\mathrm{Re}(b) > 0$ and $|\mathrm{Re}(a)| < \mathrm{Re}(b)$, we have

$$\text{(a)} \quad \int_0^\infty \frac{\sinh(ax)}{e^{bx} - 1}\, dx = -\frac{\pi}{2b} \cot\left(\frac{a\pi}{b}\right) + \frac{1}{2a},$$

and

$$\text{(b)} \quad \int_0^\infty \frac{\sinh(ax)}{e^{bx} + 1}\, dx = \frac{\pi}{2b} \csc\left(\frac{a\pi}{b}\right) - \frac{1}{2a}.$$

1.7.51 The following results, besides being interesting for their sake, can be useful to problems in probability, statistics and expected values with the normal distribution, as we have seen in the application above.

(a) For any $\mu \in \mathbb{R}$ and any $a > 0$, prove

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx \quad \text{does not exist.}$$

(b) Consider any $b \in \mathbb{R} - \{0\}$, any $a > 0$ and $0 < \epsilon < R$ constants. Integrate the function $f(z) = \dfrac{e^{-\frac{(z-\mu)^2}{a^2}}}{z}$ along the contour

$$C = [-R, -\epsilon] + S_\epsilon^- + [\epsilon, R] + [R, R+bi] + [R+bi, -R+bi] + [-R+bi, -R],$$

take limits as $R \to \infty$ and $\epsilon \to 0^+$ and use **Lemma 1.7.3** [$z = 0$ is a simple pole for $f(z)$] to prove

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx =$$

$$e^{\frac{b^2}{a^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2 + b^2} \left\{ x \cos\left[\frac{2b(x-\mu)}{a^2}\right] - b \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx,$$

(the same answer for all b's) and

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2 + b^2} \left\{ b \cos\left[\frac{2b(x-\mu)}{a^2}\right] + x \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx =$$

$$\mathrm{sign}(b)\, \pi e^{\frac{-(\mu^2+b^2)}{a^2}}.$$

(c) What do you observe happening in

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2} \left\{ b \cos\left[\frac{2b(x-\mu)}{a^2}\right] + x \sin\left[\frac{2b(x-\mu)}{a^2}\right] \right\} dx =$$

$$\operatorname{sign}(b)\, \pi e^{\frac{-(\mu^2+b^2)}{a^2}}, \quad \text{as} \quad b \longrightarrow 0.$$

(d) Replace b with ab or a (a and b as before) to get the following two simpler forms of

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx \ =$$

$$e^{b^2} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+(ab)^2} \left\{ x \cos\left[\frac{2b(x-\mu)}{a}\right] - ab \sin\left[\frac{2b(x-\mu)}{a}\right] \right\} dx,$$

(the same answer for all b's) and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\, dx \ =$$

$$e \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+a^2} \left\{ x \cos\left[\frac{2(x-\mu)}{a}\right] - a \sin\left[\frac{2(x-\mu)}{a}\right] \right\} dx.$$

(e) Let $a = 1$ and $\mu = 2$. Estimate that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-(x-2)^2}}{x}\, dx \ = 1.06822\ldots.$$

From this, guess the estimate of $\ \text{P.V.} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-(x+2)^2}}{x}\, dx.$

(f) With $a > 0$, $b \neq 0$ and μ as before, integrate

$$e^{\frac{b^2}{a^2}} \cdot \frac{e^{-\frac{(z-\mu)^2}{a^2}}}{z^2+b^2} \left\{ z \cos\left[\frac{2b(z-\mu)}{a^2}\right] - b \sin\left[\frac{2b(z-\mu)}{a^2}\right] \right\}$$

along the rectangular contour

$$C = [-R, R] + [R, R+ci] + [R+ci, -R+ci] + [-R+ci, -R],$$

where $c > |b| > 0$ constant, and observe that the imaginary part of this integral is $\pi e^{-\frac{\mu^2}{a^2}}$ (and so independent of b).

[Hint: You do not need to carry out all the computations. Use the

Residue Theorem, 1.7.1.]

(g) With $a > 0$, $b \neq 0$ and μ as before, integrate

$$\frac{e^{-\frac{(z-\mu)^2}{a^2}}}{z^2+b^2}\left\{b\,\cos\left[\frac{2b(z-\mu)}{a^2}\right] + z\,\sin\left[\frac{2b(z-\mu)}{a^2}\right]\right\}$$

along the rectangular contour

$$C = [-R, R] + [R, R+ci] + [R+ci, -R+ci] + [-R+ci, -R],$$

where $c > |b| > 0$ constant, and observe that the real part of this integral is

$$\mathrm{sign}(b)\,\pi e^{\frac{-(\mu^2+b^2)}{a^2}}.$$

[Hint: You do not need to carry out all the computations. Use the **Residue Theorem, 1.7.1**.]

(h) For any $\mu \in \mathbb{R}$, any $a > 0$ and any n **even positive integer**, prove directly that

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^n}\,dx = \infty = \mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^n}\,dx.$$

(i) Use the appropriate integration by parts and some of the previous results to prove that

$$\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^3}\,dx = \begin{cases} 0, & \text{if } \mu = 0, \\ \\ \mathrm{sign}(\mu)\cdot\infty, & \text{if } \mu \neq 0. \end{cases}$$

Generalize this result if the denominator is x^n, with $n \geq 3$ odd integer.

[(j) For a method to compute the integrals

$$\int_{-\infty}^{\infty} x^k e^{-\frac{(x-\mu)^2}{a^2}}\,dx, \quad \text{for} \quad k = 0,\ 1,\ 2,\ 3,\ \ldots,$$

refer to **Application 1** of **Subsection I 2.1.1**, or **Examples I 2.2.16** and **I 2.3.24**, and/or **Problem I 2.6.6**. These integrals exist!]

1.7.4 Infinite Isolated Singularities and Series

We consider a holomorphic function $f(z)$ in $\mathbb{C} - A$, where $A \subset \mathbb{C}$ is a countable set of isolated singularities of $f(z)$ in the complex plane $\mathbb{C}$.

Suppose we can find simple closed contours (circles, squares, parallelograms, etc.) C_l with interior $\mathcal{D}_l$, such that:

$$\text{(a)} \quad \mathbb{C} = \bigcup_{l=1}^{\infty} \mathcal{D}_l, \quad \text{and} \quad \text{(b)} \quad \lim_{l \to \infty} \oint_{C_l} f(z)\, dz = 0.$$

Then, under these two conditions, we immediately get

$$\sum_{w \in A} \operatorname*{Res}_{z=w} f(z) = 0.$$

This innocent observation allows us to evaluate certain infinite series, at times complicated, as we do in the following:

Example 1.7.23 For any $a \in \mathbb{C}$ such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer), prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a),$$

from which we immediately obtain the **result**:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a}\left[\pi \coth(\pi a) - \frac{1}{a}\right].$$

We consider the function

$$f(z) = \frac{1}{z^2 + a^2} \cot(\pi z),$$

which has simple poles at $z = \pm a\,i$ and $z = n$ for all $n \in \mathbb{Z}$.

Convenient contours, in this case, are the (positively oriented) squares with vertices $\pm(n + \frac{1}{2}) \pm (n + \frac{1}{2})i$, for $n = 1, 2, 3, \ldots$. (See **Figure 1.15**.)

The residues are:

$$\operatorname*{Res}_{z=a\,i} f(z) = \frac{1}{2a\,i} \cot(\pi a\,i) = \frac{-1}{2a} \coth(\pi a),$$

$$\operatorname*{Res}_{z=-a\,i} f(z) = \frac{1}{-2a\,i} \cot(-\pi a\,i) = \frac{-1}{2a} \coth(\pi a),$$

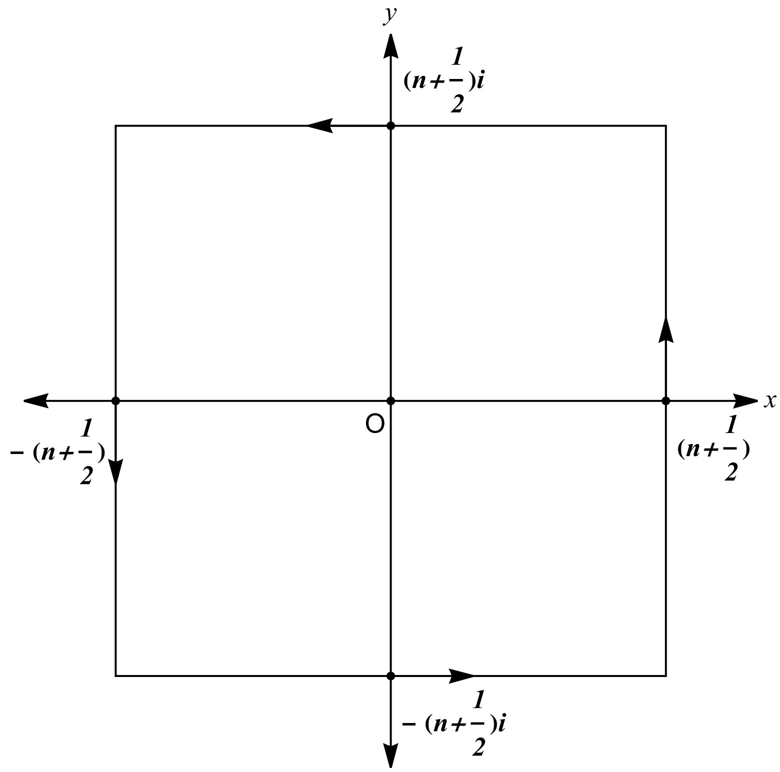

FIGURE 1.15: Contour 11 for Example 1.7.23

and $\forall\ n \in \mathbb{Z}$, we have

$$\operatorname*{Res}_{z=n} f(z) = \frac{(z-n)\cos(\pi z)}{(z^2+a^2)\sin(\pi z)}|_{z=n} =$$
$$\frac{(-1)^n}{n^2+a^2}\frac{1}{\pi\ \cos(\pi z)}|_{z=n} = \frac{1}{\pi\,(n^2+a^2)}.$$

Now, we must check if

$$\lim_{n\to\infty} \oint\limits_{C_n^+} f(z)dz = 0, \quad (\text{where} \quad n \in \mathbb{N}).$$

Obviously,

$$\oint\limits_{C_n^+} f(z)dz = I_1(n) + I_2(n) + I_3(n) + I_4(n),$$

with

$$I_1(n) = \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left[x - \left(n+\frac{1}{2}\right) i\right]^2 + a^2} \cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\} dx,$$

$$I_2(n) = \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left[\left(n+\frac{1}{2}\right) + y\, i\right]^2 + a^2} \cot\left\{\pi\left[\left(n+\frac{1}{2}\right) + y\, i\right]\right\} dy,$$

$$I_3(n) = \int_{\left(n+\frac{1}{2}\right)}^{-\left(n+\frac{1}{2}\right)} \frac{1}{\left[x + \left(n+\frac{1}{2}\right) i\right]^2 + a^2} \cot\left\{\pi\left[x + \left(n+\frac{1}{2}\right) i\right]\right\} dx,$$

and

$$I_4(n) =$$

$$\int_{\left(n+\frac{1}{2}\right)}^{-\left(n+\frac{1}{2}\right)} \frac{1}{\left[-\left(n+\frac{1}{2}\right) + y\, i\right]^2 + a^2} \cot\left\{\pi\left[-\left(n+\frac{1}{2}\right) + y\, i\right]\right\} dy.$$

We will show that for $j = 1,\ 2,\ 3,\ 4$

$$\lim_{n\to\infty} I_j(n) = 0.$$

We will do this for $j = 1$ and 2, and the work is similar with $j = 3$ and 4. We consider $n \geq a$ and we have:

$$|I_1(n)| \leq$$

$$\int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{\left|x - \left(n+\frac{1}{2}\right) i\right|^2 - a^2} \left|\cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right| dx =$$

$$\int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{x^2 + \left(n+\frac{1}{2}\right)^2 - a^2} \left|\cot\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right| dx.$$

But (see **Problem 1.2.19**), we have

$$\left|\cos\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right|^2 = \cos^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]$$

and

$$\left|\sin\left\{\pi\left[x - \left(n+\frac{1}{2}\right) i\right]\right\}\right|^2 = \sin^2(\pi x) + \sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right].$$

Therefore,

$$\left|\cot\left\{\pi\left[x-\left(n+\frac{1}{2}\right)i\right]\right\}\right| =$$

$$\sqrt{\frac{\cos^2(\pi x)+\sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]}{\sin^2(\pi x)+\sinh^2\left[\left(n+\frac{1}{2}\right)\pi\right]}} \longrightarrow \sqrt{1}=1, \quad \text{as} \quad n\longrightarrow\infty.$$

So, for n large $\left|\cot\left\{\pi\left[x-\left(n+\frac{1}{2}\right)i\right]\right\}\right| < 2.$

Hence, for n large, we get

$$|I_1(n)| \le \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{2}{x^2+\left(n+\frac{1}{2}\right)^2-a^2}\,dx =$$

$$\left[\frac{2}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\arctan\left[\frac{x}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]\right]_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} =$$

$$\frac{2}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\cdot\left\{\arctan\left[\frac{n+\frac{1}{2}}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]-\right.$$

$$\left.\arctan\left[-\frac{n+\frac{1}{2}}{\sqrt{\left(n+\frac{1}{2}\right)^2-a^2}}\right]\right\} \longrightarrow \frac{2}{\infty}\left[\arctan(1)-\arctan(-1)\right] =$$

$$\frac{2}{\infty}\left(\frac{\pi}{4}-\frac{-\pi}{4}\right)=\frac{\pi}{\infty}=0, \quad \text{as} \quad n\longrightarrow\infty.$$

Similarly, for n large, we have

$$|I_2(n)| \le \int_{-\left(n+\frac{1}{2}\right)}^{\left(n+\frac{1}{2}\right)} \frac{1}{y^2+\left(n+\frac{1}{2}\right)^2-a^2}\,dy \longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty$$

and

$$I_3(n)\longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty \qquad \text{and} \qquad I_4(n)\longrightarrow 0, \quad \text{as} \quad n\longrightarrow\infty.$$

Therefore,

$$\lim_{n\to\infty}\oint_{C_n} f(z)dz = 0.$$

Then,

$$\sum_{w\in A}\operatorname*{Res}_{z=w} f(z)=0 \quad \text{and so,} \quad 2\cdot\frac{-1}{2a}\cdot\coth(\pi a)+\sum_{n=-\infty}^{\infty}\frac{1}{\pi(n^2+a^2)}=0.$$

Hence, we obtain the following **Result:**

$$\forall\ a \in \mathbb{C}, \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a), \qquad \textit{and so}$$

$$\forall\ a \in \mathbb{C}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a}\left[\pi\coth(\pi a) - \frac{1}{a}\right].$$

(When $a = ni$ with $n \in \mathbb{Z}$ the two equations take the form $\infty = \infty$.)

By this result, we also obtain the following series expansion of $\coth(z)$*:*

$$\forall\ z \in \mathbb{C}, \quad \coth(z) = \frac{1}{z} + 2z\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2+z^2}.$$

Corollary 1.7.2 *Notice that this final sum can be rewritten as*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi a\ \cosh(\pi a) - \sinh(\pi a)}{2a^2\sinh(\pi a)}.$$

Corollary 1.7.3 *By letting a be real in the* ***above Corollary*** *and applying L' Hôpital's rule as $a \longrightarrow 0^+$, we find* ***Euler's sum***[31]

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(For a direct proof of **Euler's sum** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ above, see **Example I 2.4.3** and **Problem 1.7.52**.)

---

[31] For even exponents, we can prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ (e.g., see **Problem 1.7.75**), $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$, etc. These and other sums can also be found by the **Fourier series theory**, the related **Parseval's equation** and several other related results. For instance, the **Poisson summation formula** applied to $f(x) = e^{-|x|}$, yields the **Result** above and the result of **Corollary 1.7.2**. The interested reader can consult pertinent bibliography, since here, we do not want to roam into the Fourier series theory.

(Marc-Antoine Parseval, French mathematician, 1755-1836.)

(Consult pertinent books and articles. E.g., Apostol 1974, chapter 11, Rudin 1976, chapter 8, Weinberger 1965, chapter 4, etc.)

The $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is not known yet. It still remains a famous open problem. In 1979, the Greek-French mathematician Roger Apéry, 1916-1994, proved that this number is irrational. It still remains open whether it is algebraic or transcendental. These sums are related to the **Riemann Zeta function** and the **Bernoulli numbers**, after Jakob Bernoulli.

Corollary 1.7.4 *Replacing a with $ai \in \mathbb{C}$ in the above **result**, we find the sum*

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a}\left[\frac{1}{a} - \pi\cot(\pi a)\right].$$

By this result, we also obtain the following series expansion of $\cot(z)$*:*

$$\forall\ z \in \mathbb{C}, \quad \cot(z) = \frac{1}{z} - 2z\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}.$$

Also

$$\forall\ z \in \mathbb{C}, \quad \pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Differentiating this term by term, we obtain

$$\forall\ z \in \mathbb{C}, \quad \left[\frac{\pi}{\sin(\pi z)}\right]^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

So with $z = \dfrac{1}{2}$*, we obtain* $\quad \pi^2 = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{1}{2} - n\right)^2}, \quad$ *etc.*

(See also **Problems 1.7.53** and **1.7.83**.)

▲

Example 1.7.24 We suppose that $a \notin \mathbb{Z}$ and rewrite the result of the **previous Corollary** as

$$\pi\cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a}{n(a-n)}.$$

(The third expression yields the second one by associating the terms of the opposite indices.)

We are going to manipulate this triple equation to derive some interesting results. (See also **Problem 1.7.83**.)

We notice that

$$\frac{a}{n(a-n)} = \frac{1}{a-n} + \frac{1}{n}.$$

So, by considering the symmetric summation about $n = 0$ and associating the terms with opposite indices, the fractions $\dfrac{1}{n}$'s cancel, and we find

$$\pi\cot(\pi a) - \frac{1}{a} = \lim_{m\to\infty} \sum_{\substack{n=-m \\ n\neq 0}}^{m} \frac{1}{a-n},$$

or

$$\pi \cot(\pi a) = \lim_{m \to \infty} \sum_{n=-m}^{m} \frac{1}{a-n}. \tag{1.20}$$

Now, we work with

$$\lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{a-n}.$$

We associate the even and odd indices separately and we obtain

$$\sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{a-n} = \sum_{n=-k}^{k} \frac{1}{a-2n} - \sum_{n=-(k+1)}^{k+1} \frac{1}{a-1-2n} + \frac{1}{a-2k-3} =$$
$$= \frac{1}{2} \sum_{n=-k}^{k} \frac{1}{\frac{a}{2}-n} - \frac{1}{2} \sum_{n=-(k+1)}^{k+1} \frac{1}{\frac{a-1}{2}-n} + \frac{1}{a-2k-3}.$$

We take the limit as $k \longrightarrow \infty$, use **equation (1.20)** above and find

$$\lim_{k \to \infty} \sum_{n=-(2k+1)}^{2k+1} \frac{(-1)^n}{a-n} = \frac{\pi}{2} \cot\left(\frac{\pi a}{2}\right) - \frac{\pi}{2} \cot\left[\frac{\pi(a-1)}{2}\right] = \dots =$$
$$\frac{\pi}{\sin(\pi a)} = \pi \csc(\pi a).$$

Hence, we have obtained the important formula

$$\frac{\pi}{\sin(\pi a)} = \pi \csc(\pi a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-n}$$

(since this alternating series converges).

Now, if we let $a = \dfrac{1}{2}$, we find the well known **Gregory-Leibniz formula for** π.

$$\pi = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{1}{2}-n} = 2\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} + \sum_{n=-1}^{-\infty} \frac{(-1)^{n+1}}{2n-1}\right] =$$
$$4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right).$$

[See also **Problem 1.2.7, (b)**, and **Subsection 1.5.4**.]

With different bracketing, we can also write it as

$$\pi = 2 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{4}-n^2} = 2 + 2\left(\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \dots\right).$$

[This formula can also be derived from the power series expansion of $\arctan(z)$, if we let $z = 1$ and use **Leibniz's alternating series Test** (see **Example I 1.3.12** and its **footnote**) and **Abel's Lemma** (see **Theorem 1.2.1** and its **footnote**).]

▲

Example 1.7.25 By means of series summation, we will evaluate the integral

$$\int_0^\infty \frac{\sin(ax)}{e^{bx} + c}\, dx,$$

where a, b and c are appropriate complex constants. In the process, we will discover the conditions that a, b, and c must satisfy.

First, we rewrite the $\sin(ax)$ by its exponential form, and then we expand the integrand $\dfrac{\sin(ax)}{e^{bx} + c}$ as an infinite series by means of the geometric series, as follows:

$$\frac{\sin(ax)}{e^{bx} + c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right) e^{-bx}\, \frac{1}{1 + ce^{-bx}}.$$

To apply the geometric series expansion to the last fraction, we need minimum condition $\left|ce^{-bx}\right| < 1$ for all $x \in (0, \infty)$. Therefore, we must stipulate the two conditions:

$$(1) \quad \mathrm{Re}(b) > 0, \qquad\qquad (2) \quad |c| \leq 1.$$

Then, we have:

$$\frac{\sin(ax)}{e^{bx} + c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right) e^{-bx} \sum_{n=0}^{\infty} \left(-ce^{-bx}\right)^n =$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} (-c)^n \left(e^{[ia - b(n+1)]x} - e^{-[ia + b(n+1)]x}\right). \tag{1.21}$$

To integrate the final sum on $[0, \infty)$ term by term, we need the condition:

$$(3) \quad |\mathrm{Im}(a)| < (0 + 1)\mathrm{Re}(b) = \mathrm{Re}(b).$$

Under this condition, we can apply **Theorem I 2.3.11**. We observe that the function

$$f(x) = \frac{\sin(ax)}{e^{bx} - 1}$$

is continuous for $x > 0$ and approaches the value $\dfrac{a}{b}$, as $x \longrightarrow 0$. (Use L' Hôpital's rule. If we put $c \neq 1$ in the place of -1, then the value of the new function is directly 0.) Therefore, $B := \max_{0\le x\le 1}|f(x)|$ is finite. So, we can use as dominating function the following function

$$g(x) = \begin{cases} B, & \text{if} \quad 0 \le x \le 1, \\ \dfrac{e^{|Im(a)|x}}{e^{Re(b)x}-1}, & \text{if} \quad 1 < x < \infty. \end{cases}$$

Now, we integrate the above series in **(1.21)**, term by term and use the fact that for any real numbers u, v, α and β the formula

$$\int_u^v e^{(\alpha+i\beta)x}dx = \frac{e^{v(\alpha+i\beta)} - e^{u(\alpha+i\beta)}}{\alpha+i\beta}$$

is valid. (See **Problem 1.7.58**).

Then, after a few straightforward computations, we arrive at the following useful result:

Result: *If a, b and c are complex constants such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| \le 1$, then*

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx =$$
$$a\sum_{n=0}^\infty (-c)^n \frac{1}{a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2}. \tag{1.22}$$

(See also **Problem I 2.3.22.**.)

Corollary 1.7.5 *This formula implies the following nine* ***corollaries:***

(A) *For $c=-1$, $Re(b)>0$ and $|Im(a)| < Re(b)$, by using the result of* ***Example 1.7.23****, we find*

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}-1}\,dx = \frac{a}{b^2}\sum_{n=1}^\infty \frac{1}{n^2+\left(\frac{a}{b}\right)^2} =$$
$$\frac{a}{b^2}\left[\frac{\pi}{2\frac{a}{b}}\coth\left(\frac{\pi a}{b}\right) - \frac{1}{2\left(\frac{a}{b}\right)^2}\right] = \frac{\pi}{2b}\coth\left(\frac{\pi a}{b}\right) - \frac{1}{2a},$$

which agrees with the result in ***Problem 1.7.49, (b)****.*

(B) *Similarly, for $c=1$, by using* ***Problem 1.7.59, (a)****, we obtain the result in* ***Problem 1.7.49, (d)****, i.e:*

$$\textit{If } |Im(a)| < Re(b), \textit{ then: } \int_0^\infty \frac{\sin(ax)}{e^{bx}+1}\,dx = -\frac{\pi}{2b}\operatorname{csch}\left(\frac{\pi a}{b}\right) + \frac{1}{2a}.$$

(C) *For $c = 0$, only the first term of the infinite sum survives, under the rule $(-c)^0 = 1$, and we find:*

$$\textit{If } |Im(a)| < Re(b), \textit{ then: } \int_0^\infty \sin(ax)\, e^{-bx}\, dx = \frac{a}{a^2 + b^2},$$

which agrees with the result in ***Problem I 1.2.13.****.*

(D) *With a, b and c real constants such that $b > 0$, and $|c| < 1$, we obtain the values of other real integrals by the corresponding infinite sums.*

(E) *We assume $a \neq 0$. Then, we divide both sides of the formula by a and take the limit as $a \longrightarrow 0$. (Notice that switching limit and integral and limit and sum is legitimate here.) Then, since $\lim_{\alpha \to 0} \frac{\sin(\alpha x)}{\alpha} = x$, we obtain the* ***result****:*

If b and c are complex constants such that $Re(b) > 0$ and $|c| \leq 1$, then

$$\int_0^\infty \frac{x}{e^{bx} + c}\, dx = \frac{1}{b^2} \sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2}. \tag{1.23}$$

For $Re(b) > 0$ and $c = -1$, by the result of ***Corollary 1.7.3 of Example 1.7.23****, we obtain*

$$\int_0^\infty \frac{x}{e^{bx} - 1}\, dx = \frac{1}{b^2} \sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{b^2} \frac{\pi^2}{6}. \tag{1.24}$$

(See and compare with ***Problem I 2.6.7.****.)*

For $Re(b) > 0$ and $c = 1$ and the result of ***Problem 1.7.59, (c)****, we obtain the* ***result***

$$\int_0^\infty \frac{x}{e^{bx} + 1}\, dx = \frac{1}{b^2} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2} = \frac{1}{b^2} \frac{\pi^2}{12}. \tag{1.25}$$

[See also and compare with ***Problem 1.7.86, (5)****.]*

(F) *If in the equations* ***(1.23), (1.24) and (1.25)*** *above we let $u = e^x$ or $x = \ln(u)$, then we find, respectively:*

If b and c are complex constants such that $Re(b) > 0$, and $|c| \leq 1$, then

$$\int_1^\infty \frac{\ln(u)}{(u^b + c)u}\, du = \frac{1}{b^2} \sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2},$$

$$\int_1^\infty \frac{\ln(u)}{(u^b - 1)u}\, du = \frac{1}{b^2} \frac{\pi^2}{6}, \qquad \textit{and} \qquad \int_1^\infty \frac{\ln(u)}{(u^b + 1)u}\, du = \frac{1}{b^2} \frac{\pi^2}{12}.$$

(G) *If in **(F)** we let* $u = \dfrac{1}{x}$*, then we find, respectively:*

If b *and* c *are complex constants such that* $Re(b) > 0$*, and* $|c| \leq 1$*, then*

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{c+x^b}\,dx = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1-x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{6}, \qquad \textit{and} \qquad \int_0^1 \frac{-x^{b-1}\ln(x)}{1+x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12}.$$

(H) *If* a*,* b*,* c *and* d *are complex constants such that* $Re(b) > 0$*,* $|Im(a)| < Re(b)$*,* $d \neq 0$ *and* $\left|\dfrac{c}{d}\right| \leq 1$*, then the initial integral is dealt by*

$$\int_0^\infty \frac{\sin(ax)}{de^{bx}+c}\,dx = \frac{1}{d}\int_0^\infty \frac{\sin(ax)}{e^{bx}+\frac{c}{d}}\,dx,$$

*and we use **equation (1.22)** with* $\dfrac{c}{d}$ *in the place of* c*.*

*We work analogously for a similar situation with **equation (1.23)** in **(E)** above.*

(I) *If in **equation (1.22)** we replace* a *with* ia *and use the identity* $\sin(iz) = i\sinh(z)$*, we find:*

If a*,* b *and* c *are complex constants such that* $Re(b) > 0$*,* $|Re(a)| < Re(b)$ *and* $|c| \leq 1$*, then*

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx}+c}\,dx = a\sum_{n=0}^{\infty}(-c)^n\frac{1}{-a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2-\left(\frac{a}{b}\right)^2}.$$

*From this, we can obtain again **equations (1.23), (1.24)** and **(1.25)**. (See also **Problem 1.7.78**.)*

▲

Example 1.7.26 Working as in the **previous Example, 1.7.25**, with cosine in the place of sine, we find:

If a*,* b *and* c *are complex constants such that* $Re(b) > 0$*,* $|Im(a)| < Re(b)$ *and* $|c| < 1$*, then*

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}, \tag{1.26}$$

and we have analogous remarks as in ***(H)*** of the **previous Corollary**.

Notice that here we have $|c| < 1$, in general. If $|c| = 1$, we need to check the formula for the individual c. For instance, with $c = 1$ we get

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2},$$

which converges by the **alternating series Test**. For example, we have

$$\int_0^\infty \frac{\cos(x)}{e^x+1}\,dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2+1}, \quad \text{etc.}$$

Also, for $a = 0$, $c = 1$ and b such that $\mathrm{Re}(b) > 0$, we find

$$\int_0^\infty \frac{1}{e^{bx}+1}\,dx \overset{x=\ln(u)}{=} \int_1^\infty \frac{1}{u(u^b+1)}\,du = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} = \frac{\ln(2)}{b},$$

which, when $b > 0$ real, can also be verified by the substitutions $u = e^x$ and then $v = u^b$.

More generally, for $a = 0$, b such that $\mathrm{Re}(b) > 0$ and $-1 < c \leq 1$ real, using the power series for $\ln(1+x)$, we find

$$\int_0^\infty \frac{1}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n} = \frac{\ln(1+c)}{bc}.$$

(See also **Problem I 1.2.20**.)

But, if $c = -1$,

$$\sum_{n=1}^\infty \frac{n}{n^2+\left(\frac{a}{b}\right)^2}$$

diverges. In this case, besides the infinite interval of integration, the integral becomes also improper at $x = 0$.

If we take the derivative of **equation (1.26)** with respect to a, we find: *If a, b and c are complex constants such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| < 1$, then*

$$\int_0^\infty \frac{x\sin(ax)}{e^{bx}+c}\,dx = \frac{2a}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{\left[n^2+\left(\frac{a}{b}\right)^2\right]^2}.$$

If now we divide this equation by a and the take limit as $a \longrightarrow 0$, we find: *Under the conditions $Re(b) > 0$ and $|c| < 1$, we have*

$$\int_0^\infty \frac{x^2}{e^{bx}+c}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^3},$$

and so

$$\int_0^\infty \frac{x^2}{e^{bx}+1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3}, \quad \text{and} \quad \int_0^\infty \frac{x^2}{e^{bx}-1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

Putting $x = \ln(u)$ in the last two equalities, with $b > 0$ real, we get

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b+1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3}, \text{ and } \int_1^\infty \frac{\ln^2(u)}{u(u^b-1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

[Notice that

$$\text{if } b \geq 0, \int_0^1 \frac{\ln^2(u)}{u(u^b+1)}\,du = +\infty, \text{ and if } b > 0, \int_0^1 \frac{\ln^2(u)}{u(u^b-1)}\,du = -\infty.]$$

If now we let $u = \dfrac{1}{x}$ in the above two integrals, with $b > 0$ real, we obtain

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1+x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

and

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1-x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

If in **equation (1.26)** we replace a with ia and use the identity $\cos(iz) = \cosh(z)$, we find: *If a, b and c are complex constants such that $Re(b) > 0$, $|Re(a)| < Re(b)$ and $|c| < 1$, then*

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2 - \left(\frac{a}{b}\right)^2}.$$

▲

Example 1.7.27 We consider the integral

$$I := \int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx,$$

where $a \in \mathbb{R}$, $b > 0$ and $c \geq -1$. Under these conditions, the integral converges absolutely. (The proof is easy!) Unless $a = 0$ in which case $I = 0$ for any b and c, for $c < -1$ and $a \neq 0$ the integral does not exist, due to the singularity at $x = \dfrac{\ln(-c)}{b} > 0$.

When $-1 \leq c \leq 1$, we have found an infinite sum formula for it in **Example 1.7.25**, **Formula (1.22)**. Here, we will find a formula for $c > 1$. To this end, we work as follows:

We consider $c > 1$ and write

$$I := \int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx}+c}\,dx + \int_{\frac{\ln(c)}{b}}^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx.$$

Then, we work with each part separately. So, we have:

$$\int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx}+c}\,dx = \frac{1}{c}\int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{1+\frac{e^{bx}}{c}}\,dx.$$

Since $\dfrac{e^{bx}}{c} < 1$ when $0 \le x < \dfrac{\ln(c)}{b}$, we have the geometric series

$$\frac{1}{1+\frac{e^{bx}}{c}} = \sum_{n=0}^{\infty}\left(\frac{-e^{bx}}{c}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{c^n}\,e^{nbx}.$$

The convergence is uniform on every interval $[0,r]$, with $0 < r < \dfrac{\ln(c)}{b}$, by the **Weierstraß M-Test, Theorem I 2.3.3**, since $\dfrac{e^{br}}{c} < 1$ and so the positive (geometric) series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{e^{br}}{c}\right)^n$ has a finite sum.

So, by **Corollary I 2.3.2 of Theorem 2.3.9**, we can integrate term by term below, and with the help of the well-known formulae in **Problem 1.7.58, (d)**, we find:

$$\int_0^{\frac{\ln(c)}{b}} \frac{\sin(ax)}{e^{bx}+c}\,dx = \frac{1}{c}\sum_{n=0}^{\infty}\frac{(-1)^n}{c^n}\int_0^{\frac{\ln(c)}{b}} \sin(ax)e^{nbx}\,dx =$$

$$\frac{a}{c}\sum_{n=0}^{\infty}\frac{(-1)^n}{(a^2+b^2n^2)\,c^n} + \frac{b}{c}\sin\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^n n}{a^2+b^2n^2} -$$

$$\frac{a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^n}{a^2+b^2n^2}.$$

For the second part of the integral I, we use the substitution $x = u + \dfrac{\ln(c)}{b}$, and we find

$$\int_{\frac{\ln(c)}{b}}^{\infty} \frac{\sin(ax)}{e^{bx}+c}\,dx =$$

$$\frac{1}{c}\cos\left[\frac{a\ln(c)}{b}\right]\int_0^{\infty}\frac{\sin(au)}{e^{bu}+1}\,du + \frac{1}{c}\sin\left[\frac{a\ln(c)}{b}\right]\int_0^{\infty}\frac{\cos(au)}{e^{bu}+1}\,du.$$

Then, using **formulae (1.22)** and **(1.26)** with $c = 1$, we find

$$\int_{\frac{\ln(c)}{b}}^{\infty} \frac{\sin(ax)}{e^{bx}+c}\,dx =$$

$$\frac{a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{a^2+b^2n^2} + \frac{b}{c}\sin\left[\frac{a\ln(c)}{b}\right]\sum_{n=0}^{\infty}\frac{(-1)^{n-1}n}{a^2+b^2n^2}.$$

Adding the two partial results found, we finally obtain the following:
Result: *For* $a \in \mathbb{R}$, $b > 0$ *and* $c > 1$, *we have*

$$I := \int_0^\infty \frac{\sin(ax)}{e^{bx} + c}\, dx = \frac{1 - \cos\left[\frac{a\ln(c)}{b}\right]}{ac} +$$
$$\frac{a}{c}\sum_{n=1}^{\infty} \frac{(-1)^n}{(a^2 + b^2n^2)c^n} + \frac{2a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{a^2 + b^2n^2},$$

which is also valid for $c = 1$, *as in* ***Corollary 1.7.5, (B), of Example 1.7.25***.

If now we divide by a in the above result and take the limit as $a \longrightarrow 0$, we find the following:
Result: *For* $b > 0$ *and* $c > 1$, *we have*

$$\int_0^\infty \frac{x}{e^{bx} + c}\, dx = \frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} =$$
$$\frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\frac{\pi^2}{12},$$

which is also valid for $c = 1$, *as in* ***equality (1.25)***.

▲

Remark 1: As we have already seen in **Examples 1.7.19** (treated twice), **1.7.25, 1.7.26, 1.7.27** and in several **problems**, we can compute improper integrals or their principal values as infinite sums and/or infinite sums as improper integrals or their principal values. To obtain these kinds of formulae we use partial fractions, the geometric series, **Problem 1.7.58, (b)**, **Theorem I 2.3.11**, the **Residue Theorem, 1.7.1**, etc. (See also **Problems I 2.3.28**, **1.7.72** and its **footnote**.)

Remark 2: All of the examples studied so far and those that will be studied in the sequel show the power of contour integration in obtaining formulae for difficult improper integrals and infinite sums very strongly. They also show that the choice of contour is a very important piece of art that needs a great deal of experience.

Problems

1.7.52 Find **Euler's sum** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ directly by considering the complex function $f(z) = \dfrac{\cot(z)}{z^2}$, contours analogous to the ones used in **Example 1.7.23**, computing $\operatorname*{Res}_{z=0}[f(z)] = \dfrac{-1}{3}$ (see **Example 1.6.14**) and

$\operatorname*{Res}_{z=n\pi}[f(z)] = \dfrac{1}{n^2\pi^2}, \ \forall \ n \in \mathbb{Z}.$

(See also **Example I 2.4.3** and **Corollary 1.7.3 of Example 1.7.23**.)

1.7.53 Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a}\left[\frac{1}{a} - \pi\cot(\pi a)\right]$$

by considering the complex function $f(z) = \dfrac{\coth(\pi z)}{z^2 + a^2}$ and working in a way similar to the one in **Example 1.7.23**.

(See also **Corollary 1.7.4 of Example 1.7.23**.)

1.7.54 Prove that, if $n \in \mathbb{N}$, then

$$\text{(a)} \qquad \sum_{k=1}^{\infty} \frac{1}{(2k)^{2n}} = \frac{1}{2^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}},$$

$$\text{(b)} \qquad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2k}} = \left(1 - \frac{1}{2^{2n}}\right) \sum_{k=1}^{\infty} \frac{1}{k^{2n}},$$

$$\text{(c)} \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} = \left(\frac{1}{2^{2n-1}} - 1\right) \sum_{k=1}^{\infty} \frac{1}{k^{2n}}.$$

1.7.55 Use **Euler's sum** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ to prove the sums

$$\text{(a)} \ \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}, \quad \text{(b)} \ \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}, \quad \text{(c)} \ \sum_{n=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12},$$

$$\text{(d)} \quad \sum_{k=1}^{\infty} \frac{1}{\left(k - \frac{1}{2}\right)^2} = \frac{\pi^2}{2}, \qquad \text{(e)} \quad \sum_{k=0}^{\infty} \frac{2k+1}{k^2(k+1)} = \frac{\pi^2}{6} + 1.$$

(See also **Example I 2.3.23.**.)

1.7.56 Prove

$$\sum_{k=0}^{\infty} \left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 - \left(\frac{1}{2}\right)^2} = \frac{\pi}{4}.$$

[Hint: Use some already proven results or the power series of $\arctan(x)$ with **Abel's Lemma**. (See **footnote** of **Theorem 1.2.1**.)]

1.7.57 Prove that

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\left(\frac{1}{N+in}+\frac{1}{N-in}\right)=2\int_{-1}^{1}\frac{dt}{t^2+1}=\pi.$$

1.7.58 (a) Prove that for any complex number $\alpha+i\beta\neq 0+i0$ (α and β real) fixed, we have

$$\int e^{(\alpha+i\beta)x}dx=\frac{e^{(\alpha+i\beta)x}}{\alpha+i\beta}+w,\ \text{ where } w=c+id \text{ is a complex constant.}$$

(b) Use **(a)** to prove that for any $u,v\in\mathbb{R}$ and for any complex number $\alpha+i\beta\neq 0+i0$ (α and β are real) fixed, we have

$$\int_u^v e^{(\alpha+i\beta)x}dx=\frac{e^{v(\alpha+i\beta)}-e^{u(\alpha+i\beta)}}{\alpha+i\beta}.$$

(c) Prove that when $\alpha+i\beta=0+i0$, the answers in **(a)** and **(b)** are, respectively, $x+w$ and $v-u$.

(d) Use **(a)** to prove that when $\alpha+i\beta\neq 0+i0$,

$$\int e^{\alpha x}\cos(\beta x)\,dx=\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\cos(\beta x)+\beta\sin(\beta x)\right]+C_1,$$

and

$$\int e^{\alpha x}\sin(\beta x)\,dx=\frac{e^{\alpha x}}{\alpha^2+\beta^2}\left[\alpha\sin(\beta x)-\beta\cos(\beta x)\right]+C_2,$$

where α and β are fixed real numbers and C_1, C_2 are the real constants of integration.

(In this way, you avoid integration by parts in these types of indefinite integrals! See **Problems I 1.2.13** and **I 1.2.15**.)

1.7.59 (a) For any $a\in\mathbb{C}$ such that $a\neq ni$, with $n\in\mathbb{Z}$, use the function $f(z)=\dfrac{\csc(\pi z)}{z^2+a^2}$ to find

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{a}\operatorname{csch}(\pi a),$$

from which we get

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}=\frac{\pi}{2a}\operatorname{csch}(\pi a)-\frac{1}{2a^2}.$$

We also obtain the following series expansion of csch(z):

$$\operatorname{csch}(z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 + z^2}.$$

(b) Now, replace a with $ai \in \mathbb{C}$, where $a \notin \mathbb{Z}$, to prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 - a^2} = \frac{-\pi}{a} \csc(\pi a),$$

from which we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a} \csc(\pi a).$$

We also obtain the following series expansion of csc(z):

$$\csc(z) = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2}.$$

(See also **Example 1.7.22**.)

(c) Now, prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$

1.7.60 Use the four summation results of (1) **Example 1.7.23** (2) its **Corollary 1.7.4** (3) **Problem 1.7.59** and (4) the integral summation result of **Example 1.7.25**, to derive the four general integrals claimed in **Problems 1.7.49** and **1.7.50**.

1.7.61 Find the integrals: (a) $\displaystyle\int_0^{\infty} \frac{\sin(2x)}{7e^{3x} \pm 5}\, dx$, (b) $\displaystyle\int_0^{\infty} \frac{x}{7e^{3x} \pm 5}\, dx.$

1.7.62 Use the results of **Corollary 1.7.5, (E)** or **(F)** of **Example 1.7.25** to verify the following five integrals:

$$I_1 = \int_1^{\infty} \frac{\ln(u)}{(u+1)u}\, du = \frac{\pi^2}{12}, \qquad I_2 = \int_1^{\infty} \frac{\ln(u)}{(u-1)u}\, du = \frac{\pi^2}{6},$$

$$I_3 = \int_1^{\infty} \frac{\ln(u)}{(u^2+1)u}\, du = \frac{\pi^2}{48}, \qquad I_4 = \int_1^{\infty} \frac{\ln(u)}{(u^2-1)u}\, du = \frac{\pi^2}{24},$$

$$I_5 = \int_1^{\infty} \frac{\ln(u)}{u^2 - 1}\, du = \frac{\pi^2}{8}.$$

[See also and compare with **Examples I 2.3.23, I 3.7.47, I 2.4.5** and **Problem 1.7.86, (5)**.]

1.7.63 Use the **previous Problem** to verify the following integrals:

$$I_1 = \int_0^1 \frac{\ln(v)}{1+v}\,dv = \frac{-\pi^2}{12}, \qquad I_2 = \int_0^1 \frac{\ln(v)}{1-v}\,dv = \frac{-\pi^2}{6},$$
$$I_3 = \int_0^1 \frac{v\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{24}, \qquad I_4 = \int_0^1 \frac{\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{8}.$$

[Hint: In some of them you may also use the geometric series and the power series of $\ln(1-x)$, for $-1 < x < 1$ and integration. See also **Examples I 2.4.4** and **I 2.4.5**.]

1.7.64 Use the **previous Problem** (and some elementary integration rules and techniques, if necessary) to verify the following six integrals:

$$I_1 = \int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}, \qquad I_2 = \int_0^1 \frac{\ln(1-x)}{x}\,dx = \frac{-\pi^2}{6},$$
$$I_3 = \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \int_1^\infty \ln\left(\frac{x+1}{x-1}\right)\frac{dx}{x} =$$
$$\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)dx = \int_0^\infty \ln\left[\coth\left(\frac{x}{2}\right)\right]dx = \frac{\pi^2}{4},$$
$$I_4 = \int_0^1 \frac{x\ln(x)}{1-x}\,dx = 1 - \frac{\pi^2}{6}, \qquad I_5 = \int_0^1 \ln(x)\ln(1-x)\,dx = 2 - \frac{\pi^2}{6},$$
$$I_6 = \int_0^1 \ln(x)\ln(1+x)\,dx = 2 - 2\ln(2) - \frac{\pi^2}{12}.$$

(See also **Problem I 2.6.5**.)

[Hint: In some of them you may also use the geometric series and the power series of $\ln(1-x)$, for $-1 < x < 1$ and integration. See also **Examples I 2.4.4** and **I 2.4.5**.]

1.7.65 Use $f(z) = \dfrac{1}{z^3\cos(\pi z)}$ to find

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^3} = \frac{1}{8}\sum_{n=0}^\infty \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{32}, \quad \text{and so} \quad \sum_{n=0}^\infty \frac{(-1)^n}{\left(n+\frac{1}{2}\right)^3} = \frac{\pi^3}{4}.$$

1.7.66 Use the geometric series, **Problem I 2.1.17** (or **I 2.6.8**) and the **previous Problem** to prove

$$\int_0^1 \frac{x^2\ln^2(x)}{1+x^2}\,dx = 2\left(1 - \frac{\pi^3}{32}\right)$$

1.7.67 Explain why the method of **Example 1.7.23** (or any other method presented in this **section**) cannot be used to figure out the following sums:

$$\text{(a)} \quad \sum_{n=-\infty}^{\infty} \frac{1}{n^3+a^3}, \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{n^3+a^3}, \quad \text{where} \quad a \in \mathbb{C}-\mathbb{Z}.$$

$$\text{(b)} \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^3+a^3}, \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+a^3}, \quad \text{where} \quad a \in \mathbb{C}-\mathbb{Z}.$$

$$\text{(c)} \qquad \sum_{n=1}^{\infty} \frac{1}{n^3}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.$$

1.7.68 (a) If $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} = \alpha$, then prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{8}{7}\,\alpha$.

(b) If $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k)^3} = \beta$, then prove $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = 8\beta$. (So, $\alpha = 7\beta$.)

1.7.69 (a) For any $a \in \mathbb{C}-\mathbb{Z}$, use $f(z) = \dfrac{\cot(\pi z)}{(z+a)^2}$ to find

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \pi^2 \csc^2(\pi a) = \frac{\pi^2}{\sin^2(\pi a)}.$$

(b) Now, for any $a \in \mathbb{C}-\mathbb{Z}$, use $f(z) = \dfrac{\csc(\pi z)}{(z+a)^2}$ to find

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \csc(\pi a)\cot(\pi a) = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

1.7.70 (a) For $a \in \mathbb{C}-\mathbb{Z}$ and $b \in \mathbb{C}-\mathbb{Z}$, use $f(z) = \dfrac{\cot(\pi z)}{(z+a)(z+b)}$ to prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\cot(\pi a) - \cot(\pi b)\right].$$

Also, use this result to solve **Problems 1.7.56** and **1.7.69, (a)**.

(b) Then, prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2+a^2} = \frac{\pi}{2ai}\left\{\cot[\pi(z-ai)] - \cot[\pi(z+ai)]\right\},$$
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2-a^2} = \frac{\pi}{2a}\left\{\cot[\pi(z-a)] - \cot[\pi(z+a)]\right\}.$$

(c) Now, find the eight summations

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 \pm n + 1}, \qquad \sum_{n=-\infty}^{\infty} \frac{1}{n^2 \pm n - 1},$$
$$\sum_{n=0}^{\infty} \frac{1}{n^2 \pm n + 1}, \qquad \sum_{n=0}^{\infty} \frac{1}{n^2 \pm n - 1}.$$

[See also **Problem 1.7.74, (c), (d)**.]

(d) Prove that for $0 < a < 1$

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1-a)} = \frac{\pi}{1-2a}\cot(\pi a)$$

(If $a = \frac{1}{2}$ the answer is $\frac{\pi^2}{2}$, by L' Hôpital's rule.)

(e) For $a \in \mathbb{C} - \mathbb{Z}$, prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+a} = \pi\cot(\pi a),$$

whereas $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+a} = \infty$, and $\displaystyle\sum_{n=-\infty}^{-1} \frac{1}{n+a} = -\infty$, for all $a \in \mathbb{R}$. If $a \in \mathbb{C}$, then these two infinities are the complex infinity.

(f) For $a \in \mathbb{C} - \mathbb{Z}$, $b \in \mathbb{C} - \mathbb{Z}$ and $c \in \mathbb{C} - \mathbb{Z}$ prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)(n+c)} =$$
$$\pi\left[\frac{\cot(\pi a)}{(b-a)(c-a)} + \frac{\cot(\pi b)}{(a-b)(c-b)} + \frac{\cot(\pi c)}{(a-c)(b-c)}\right].$$

[(g) We can use **Results (a) and (b)** to derive several known summations. Spot and justify some of these summations in this way.]

1.7.71 (a) For $a \in \mathbb{C} - \mathbb{Z}$ and $b \in \mathbb{C} - \mathbb{Z}$, use $f(z) = \dfrac{\csc(\pi z)}{(z+a)(z+b)}$ to prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\csc(\pi a) - \csc(\pi b)\right].$$

Use this result to solve **Problem 1.7.69, (b)**.

(b) Then, prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2 + a^2} = \frac{\pi}{2ai}\left\{\csc[\pi(z-ai)] - \csc[\pi(z+ai)]\right\},$$
$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2 - a^2} = \frac{\pi}{2a}\left\{\csc[\pi(z-a)] - \csc[\pi(z+a)]\right\}.$$

(c) Now, find the eight summations

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 \pm n + 1}, \qquad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 \pm n - 1},$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 \pm n + 1}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 \pm n - 1}.$$

[See also **Problem 1.7.74, (c)**.]

(d) For $a \in \mathbb{C} - \mathbb{Z}$ prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+a} = \pi \csc(\pi a),$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

[See also **Problem I 2.3.7, (c)**.]

(e) $b \in \mathbb{C} - \mathbb{Z}$ and $c \in \mathbb{C} - \mathbb{Z}$ prove

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)(n+c)} =$$
$$\pi\left[\frac{\csc(\pi a)}{(b-a)(c-a)} + \frac{\csc(\pi b)}{(a-b)(c-b)} + \frac{\csc(\pi c)}{(a-c)(b-c)}\right].$$

[(f) We can use **Results (a) and (b)** to derive several known summations. Spot and justify some of these summations in this way.]

1.7.72 Use **Problem 1.7.70, (a)**, to prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{3}.$$

[Compare this with **Problem I 2.3.33, (5)**.][32]

1.7.73 (a) Use $f(z) = \dfrac{\cot(\pi z)}{1+z^4}$ to find $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{1+n^4}$. Then, prove that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^4} = \frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)+\sinh\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right)+\sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

How much is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{1+n^4}$? Generalize to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a^4+n^4}$, with $a > 0$.

[Hint: See **Subsection 4.2, List items 114 and 115**, at the back.]

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[32]Exact answers to series are complicated or impossible, in general. Also, most exact answers are found by making an involved combination of computations and facts.

E.g., the sum $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{12} + \frac{3\ln(3)}{4}$ is complicated. [See **Problems I 2.3.28** and **I 2.3.32, (c)**.] But, the sum done in **Problem I 2.3.7, (d)**, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{6}$ is easier. We show below three more summations:

(1) By subtracting the two sums and simplifying, we find

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)} = \frac{3\ln(3)}{2} - \frac{\sqrt{3}\,\pi}{6}.$$

(2) By adding these two sums and simplifying, we find

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(6k+1)} = \frac{\sqrt{3}\,\pi}{8} + \frac{3\ln(3)}{8}.$$

(3) Combining this last sum with **Problem 1.7.70, (a)** with $a = \dfrac{1}{2}$, $b = \dfrac{1}{6}$ and simplifying, we find

$$\sum_{k=1}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)\left(k-\frac{1}{6}\right)} = \frac{3\sqrt{3}\,\pi}{2} - \frac{9\ln(3)}{2},$$

and so on.

(b) Use $f(z) = \dfrac{\csc(\pi z)}{1+z^4}$ to find $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^4}$. Then, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^4} = \frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)+\sinh\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right)+\sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right].$$

How much is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{1+n^4}$? Generalize to $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{a^4+n^4}$, with $a > 0$.

1.7.74 (a) Check the two relations

$$\frac{z^2}{z^3+1} = \frac{1}{3}\left(\frac{1}{z+1} + \frac{2z-1}{z^2-z+1}\right) \qquad \text{and} \qquad \ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

and use them to prove

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n^2}{n^3+1} = \frac{1}{3}\left[1 - \ln(2) + \pi\,\mathrm{sech}\left(\pi\frac{\sqrt{3}}{2}\right)\right].$$

[Hint: You may work as in **Problem I 2.3.7** with partial fractions and with the third roots of -1.

Or, use $f(z) = \dfrac{(2z-1)\csc(\pi z)}{z^2-z+1}$ and notice

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1}.]$$

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = \pi\,\mathrm{sech}\left(\pi\frac{\sqrt{3}}{2}\right).$$

(You can now form other combinations of summations using this result.)

(c) Now use $f(z) = \dfrac{\cot(\pi z)}{z^2-z+1}$ to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2-n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2-n+1} = \frac{\pi\sqrt{3}}{3}\tanh\left(\pi\frac{\sqrt{3}}{2}\right).$$

[See also **Problem 1.7.70, (c)**.]

(d) Find the sums $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-n+1}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2-n+1}$.
[See also **Problem 1.7.71, (c)**.]

1.7.75 For any $a \in \mathbb{C} - i\mathbb{Z}$ prove

$$\sum_{n=1}^{\infty}\frac{1}{\left(n^2+a^2\right)^2}=\frac{-2+a\pi\coth(a\pi)+a^2\pi^2\operatorname{csch}\left(a^2\pi^2\right)}{4a^4}.$$

Then, prove that if $a=0$, the sum is $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

1.7.76 Prove that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2+n^2}=\infty$$

1.7.77 Find the conditions on the complex constants a, b and c so that you can compute the integral

$$\int_0^{\infty}\frac{\sin^2(ax)}{e^{bx}+c}\,dx$$

as a series. Find this series and the byproducts of the obtained result.

[Hint: Follow the steps of **Example 1.7.25** and some parts in **Corollary 1.7.5**. See also **Problem I 2.3.22.**.]

1.7.78 (a) Imitate the process of **Example 1.7.25** (or **Problem I 2.3.22.**) to find the conditions that the complex constants a, b and c must satisfy in order to evaluate the integral

$$\int_0^{\infty}\frac{\sinh(ax)}{e^{bx}+c}\,dx$$

as an infinite summation. In the process, find this infinite summation.

(b) Obtain the same result by using the results of **Example 1.7.25** and the identity $\sinh(z)=-i\sin(iz)$.

(c) Then, check your results against the results of **Problem 1.7.50**.

(d) Replace $\sin(x)$ in **Example 1.7.25** or $\sinh(x)$ here with $\cos(x)$ or $\cosh(x)$ to obtain analogous results in any possible way. (See also **Example 1.7.26**.)

1.7.79 For a any real constant, $b>0$ and $c\geq 1$, imitate the process of

Example 1.7.27 to find the integral

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx$$

as an infinite summation.

1.7.80 Prove the following series expansions:

$$\operatorname{sech}(z) = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 + 4z^2}.$$
$$\sec(z) = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 - 4z^2}.$$
$$\tanh(z) = 8z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 + 4z^2}.$$
$$\tan(z) = 8z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 - 4z^2}.$$

[See also **Problem 1.7.43, (c)** for a partial result.]

1.7.81 Prove that if $-\pi < a < \pi$, and $t \notin \mathbb{Z}$ (not an integer),

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(na)}{t^2-n^2} = \frac{\pi}{2t}\cdot\frac{\cos(at)}{\sin(\pi t)} - \frac{1}{2t^2}$$
$$\text{and} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n\sin(na)}{t^2-n^2} = \frac{\pi}{2}\cdot\frac{\sin(at)}{\sin(\pi t)}.$$

[Hint: Consider $\lim_{n\to\infty} \sum_{k=-n}^{n} (-1)^k \frac{e^{iak}}{t-k}$.]

1.7.82 (a) If $a > 0$ and $b > 0$ constants, prove:

$$\sum_{n=1}^{\infty} \frac{1}{a+bn^2} = \frac{1}{2}\left[\frac{\pi}{\sqrt{ab}}\cdot\coth\left(\pi\sqrt{\frac{a}{b}}\right) - \frac{1}{a}\right],$$
$$\sum_{n=1}^{\infty} \frac{n^2}{a^4+n^4} = \frac{\pi}{2\sqrt{2}a}\cdot\frac{\sinh\left(\pi a\sqrt{2}\right) - \sin\left(\pi a\sqrt{2}\right)}{\cosh\left(\pi a\sqrt{2}\right) - \cos\left(\pi a\sqrt{2}\right)}.$$

(b) If $a > 0$ and $b > 0$ constants, find the sums

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{a+bn^2} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{a^4+n^4}.$$

1.7.83 (a) For any $a \in \mathbb{C} - \mathbb{Z}$, compute all the residues of the function

$$f(z) := \frac{\pi \cot(\pi z)}{z(z-a)},$$

integrate it along the square contours C_n^+, $n \in \mathbb{N}$, of **Example 1.7.23** and take the limit as $n \longrightarrow \infty$ to re-derive the result of **Corollary 1.7.4 of Example 1.7.23** and **Example 1.7.24**

$$\pi \cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}.$$

(b) Let

$$g(a) := \pi \cot(\pi a) - \frac{1}{a},$$

with $a \in \mathbb{C}$. Prove that $\lim_{a \to 0} g(a) = 0$. So, by **Riemann's Theorem, 1.6.2,** and/or its **Corollary 1.6.1**, $g(a)$, with $a \in \mathbb{C}$, is also analytic at $a = 0$.

Then, integrate $g(a)$ (with respect to a) from 0 to any $z \in \mathbb{C}$ by means of a continuous antiderivative and using

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z),$$

with $\text{Arg}(z) \in [-\pi, \pi)$, to obtain

$$\text{Log}\left[\frac{\sin(\pi z)}{z}\right] = \log(\pi) + \sum_{n=1}^{\infty} \text{Log}\left(1 - \frac{z^2}{n^2}\right),$$

where

$$\log(\pi) = \text{Log}(\pi) + 2k\pi i$$

is a complex logarithm of π, for some $k \in \mathbb{Z}$, not necessarily the $\text{Log}(\pi)$, i.e., k may not be zero.

[Hint: To establish this relation, we need a combination of justifications. We will sketchily provide them here, and the reader supplies the missing details.

First, $\pi \cot(\pi a)$ and $\dfrac{1}{a}$ have continuous antiderivatives in $\mathbb{C} - \{x\text{-non-positive axis}\}$ the $\text{Log}\,[\sin(\pi a)]$ and $\text{Log}(a)$, respectively. Then,

$$\text{Log}\,[\sin(\pi a)] - \text{Log}(a) = \text{Log}\left[\frac{\sin(\pi a)}{a}\right] + 2k\pi i,$$

where $k = 0$, or 1, or -1, since the principal argument $\text{Arg}(z)$ is in $[-\pi, \pi)$.

Then, the suggested integration from 0 to z [knowing that at 0 we get $\lim_{a\to 0}\frac{\sin(\pi a)}{a} = \pi > 0$ and since $\mathrm{Log}(\pi) = \ln(\pi) + 0i$], gives

$$\mathrm{Log}\left[\frac{\sin(\pi z)}{z}\right] - \mathrm{Log}(\pi).$$

Next, the sum

$$\sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$$

converges absolutely and uniformly on any closed disc $\overline{D(0,R)}$, for any $R > 0$. This is so because, given an $R > 0$, we pick the integer $m = [\![R]\!] + 1$, and then for any $a \in \overline{D(0,R)}$ we have

$$\sum_{n=m}^{\infty} \left|\frac{2a}{a^2 - n^2}\right| \leq \sum_{n=m}^{\infty} \frac{2|a|}{n^2 - |a|^2} \leq \sum_{n=m}^{\infty} \frac{2|R|}{n^2 - |R|^2} < \infty.$$

So, the series converges absolutely. The uniform convergence follows from the **Weierstraß M-Test, Theorem I 2.3.3**.

So, switching integration and summation below is legitimate by the uniform convergence, and we get

$$\int_0^z \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}\, da = \sum_{n=1}^{\infty} \int_0^z \frac{2a}{a^2 - n^2}\, da =$$

$$\lim_{m\to\infty} \sum_{n=1}^{m} \int_0^z \frac{2a}{a^2 - n^2}\, da = \lim_{m\to\infty} \sum_{n=1}^{m} \int_0^z d\left[\mathrm{Log}\left(1 - \frac{a^2}{n^2}\right)\right] =$$

$$\lim_{m\to\infty} \left\{ \sum_{n=1}^{m} \left[\mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) - \mathrm{Log}(1)\right] + h_m 2\pi i \right\} =$$

$$\lim_{m\to\infty} \left\{ \sum_{n=1}^{m} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) + h_m 2\pi i \right\}$$

for some integer h_m. But, this limit exists, and therefore after m large enough we must have $h_m = h$ constant integer. [By the way, notice that for all $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) = \mathrm{Log}(1) = 0 + 0i.]$$

So, the second side takes the form

$$\sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) + h2\pi i.$$

Then, both sides yield the equality

$$\mathrm{Log}\left[\frac{\sin(\pi z)}{z}\right] - \mathrm{Log}(\pi) = \sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) + h2\pi i,$$

or, if we set

$$\log(\pi) = \mathrm{Log}(\pi) + h2\pi i,$$

we eventually find

$$\mathrm{Log}\left[\frac{\sin(\pi z)}{z}\right] = \mathrm{Log}(\pi) + h2\pi i + \sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) =$$

$$\log(\pi) + \sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right).]$$

(c) Conclude that

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Let $z = \frac{1}{2}$ to find $\frac{1}{\pi}$ and then π as infinite products! (**Wallis**[33] **product**.)

Use this to find Euler's sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Change z to iz to obtain:

$$\sinh(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right).$$

Now, put $\frac{z}{\pi}$ in the place of z in both formulae above to find the two corresponding formulae for $\sin(z)$ and $\sinh(z)$ as infinite products.[34]

Now prove

$$e^z - 1 = e^{\frac{z}{2}} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right), \quad \forall\ z \in \mathbb{C}.$$

[33] John Wallis, English mathematician, 1616-1703.

[34] Similar work proves:

$$\forall\ z \in \mathbb{C},\ \ \cos(z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{\pi^2(2n-1)^2}\right], \ \text{ and } \ \cosh(z) = \prod_{n=1}^{\infty} \left[1 + \frac{4z^2}{\pi^2(2n-1)^2}\right].$$

1.7.84 (a) Write explicitly the **Wallis product** in item **(c)** of the **previous problem** and find[35]

$$\pi = 2 \cdot \lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right).$$

(b) Justify why

$$\pi = 2 \cdot \lim_{n\to\infty} \left[\frac{2^2 \cdot 4^2 \dots (2n-2)^2}{3^2 \cdot 5^2 \dots (2n-1)^2} \cdot 2n \right].$$

(c) Use **(b)** to prove

$$\sqrt{\pi} = \lim_{n\to\infty} \left[\frac{2^{2n} \cdot (n!)^2}{(2n)! \cdot \sqrt{n}} \right].$$

[Compare with **Problem I 2.6.66, (b)**.]

1.7.85 Let $f(z)$ be holomorphic in $\mathbb{C}$ except at the simple poles $a_1,\ a_2,\ a_3\ \dots$. Suppose $0 < |a_1| \le |a_2| \le |a_3| \le \dots$ and the corresponding residues are $R_1,\ R_2,\ R_3,\ \dots$. Then, for any simple contour C_n with no poles on it and containing the first n poles inside of it, prove

$$\oint_{C_n} f(w) \left(\frac{1}{w-z} - \frac{1}{w} \right) dw = 2\pi i \left[f(z) - f(0) + \sum_{k=1}^{n} R_n \left(\frac{1}{a_n - z} - \frac{1}{a_n} \right) \right].$$

When these poles are infinitely many and the contours are appropriately chosen, we can take limits as $n \to \infty$ and if the respected limits exist, we derive an expansion of the function $f(z)$. Provide examples or refer to some examples already treated in the text or other problems!

1.7.86 Project: Examine the following work and provide the missing details and the omitted parts of the proofs provided.

(1) For $a \in \mathbb{C}$ and $k \in \mathbb{N}$, we set

$$z^k - a^k = (z - \omega_0 a)(z - \omega_1 a) \dots (z - \omega_{k-1} a),$$

where $\omega_\rho = \cos\left(\frac{2\rho\pi}{k}\right) + i \sin\left(\frac{2\rho\pi}{k}\right) = e^{\frac{i2\rho\pi}{k}}$, $\rho = 0,\ 1,\ 2,\ \dots,\ k-1$, are the k k$^{\text{th}}$ roots of unity$(=1)$.

[35] This result can be also proven by basic calculus. Study such a proof from a calculus book that contains it.

(2) If $a \neq 0$, using partial fractions we have

$$\frac{1}{z^k - a^k} = \sum_{\rho=0}^{k-1} \frac{A_\rho}{z - \omega_\rho a}, \quad \text{and so} \quad 1 = \sum_{\rho-0}^{k-1} A_\rho \frac{z^k - a^k}{z - \omega_\rho a}.$$

Then, for $z = \omega_\sigma a$ ($\sigma = 0,\ 1,\ 2,\ \dots,\ k-1$), we obtain

$$1 = A_\sigma \frac{kz^{k-1}}{1}|_{z=\omega_\sigma a} = A_\sigma k \omega_\sigma^{k-1} a^{k-1} = A_\sigma \frac{ka^{k-1}}{\omega_\sigma}, \quad (\omega_\sigma^k = 1).$$

So,

$$A_\sigma = \frac{\omega_\sigma}{ka^{k-1}} \quad \text{for} \quad \sigma = 0,\ 1,\ 2, \dots,\ k-1.$$

Hence,

$$\frac{1}{z^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \frac{\omega_\rho}{z - \omega_\rho a}.$$

(3) Then, if we pick z and a such that $\mathrm{Re}(z - |a|) = \mathrm{Re}(z) - |a| > 0$, we have

$$\frac{1}{z^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty e^{-(z-\omega_\rho a)x} dx.$$

(4) We choose $z = n \in \mathbb{N}$ and $0 < |a| < 1$. Then,
$\mathrm{Re}(n - |a|) > n - 1 \geq 0$, $\forall\, n \in \mathbb{N}$ and

$$\sum_{n=1}^\infty \frac{1}{n^k - a^k} = \frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty \sum_{n=1}^\infty e^{-(n-\omega_\rho a)x} dx =$$

$$\frac{1}{ka^{k-1}} \sum_{\rho=0}^{k-1} \omega_\rho \int_0^\infty \frac{e^{-(1-\omega_\rho a)x}}{1 - e^{-x}}\, dx = \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{ka^{k-1}\,(e^x - 1)}\, dx.$$

Since

$$\sum_{\rho=0}^{k-1} \omega_\rho^m = 0, \quad \text{for} \quad m = 1,\ 2,\ \dots,\ k-1 \quad \text{and} \quad \sum_{\rho=0}^{k-1} \omega_\rho^k = k,$$

by using L' Hôpital's rule $(k-1)$ times in a row, we find

$$\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k} = \lim_{a \to 0} \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{ka^{k-1}\,(e^x - 1)}\, dx = \int_0^\infty \frac{x^{k-1}}{(k-1)!\,(e^x - 1)}\, dx.$$

(Switching of integral and sum is legitimate. For the **Zeta function**, see **Problem I 2.3.26**. For $k = 1$, we obtain $\infty = \infty$.)

Then, by switching the integral and the summation (which is legitimate here), we obtain

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \int_0^{\infty} \frac{x^{k-1}}{(k-1)!} \sum_{n=1}^{\infty} e^{-nx} dx = \frac{1}{\Gamma(k)} \sum_{n=1}^{\infty} \int_0^{\infty} x^{k-1} e^{-nx} dx.$$

(5) The **previous equation** is a result that follows from the following more general result.

Use the **property of the Gamma function**, $(\mathbf{\Gamma}, \mathbf{11})$,

$$\forall \ x > 0, \ \forall \ p > 0, \ \frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} u^{p-1} e^{-xu} du$$

and observe that switching of integral and sum is legitimate and thus we obtain: $\forall \ p > 1$,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \sum_{n=1}^{\infty} \int_0^{\infty} x^{p-1} e^{-nx} dx = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{x^{p-1}}{e^x - 1} \, dx.$$

By the way, the change of variables $x = \ln(u)$ transforms this formula to

$$\forall \ p > 1, \quad \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_1^{\infty} \frac{[\ln(u)]^{p-1}}{u(u-1)} \, du,$$

and then the change of variables $u = \dfrac{1}{v}$ transforms this formula to

$$\forall \ p > 1, \quad \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{1} \frac{[-\ln(v)]^{p-1}}{1-v} \, dv.$$

(6) Now, for any $m \in \mathbb{N}$ and any $a \in \mathbb{C}$ such that $0 < |m-a| < 1$, we similarly obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \sum_{n=1}^{m} \frac{1}{n^k - a^k} + \int_0^{\infty} \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\omega_\rho a)x}}{ka^{k-1}(e^x - 1)} \, dx.$$

(7) Next, if we pick any kth root of -1, ψ let us say, and we replace a with $-a$, we find

$$\forall \ 0 < |a| < 1, \quad \sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{\psi\omega_\rho a x}}{ka^{k-1}(e^x - 1)} \, dx,$$

and $\forall \ m \in \mathbb{N}, \ \forall \ a \in \mathbb{C} : \quad 0 < |m-a| < 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \sum_{n=1}^{m} \frac{1}{n^k + a^k} + \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\psi\omega_\rho a)x}}{ka^{k-1}(e^x - 1)} \, dx.$$

We observe that, in this case, the last two formulae are also valid for $a = m \in \mathbb{N}$.

1.7.5 Fourier Type Integrals

In this section, we deal with the **Fourier type integrals** and computation of Fourier transforms. The complex Fourier type integrals considered here are mainly computed by using the following two **Lemmata**.

Lemma 1.7.5 *Let $w = f(z)$ be a complex function defined and continuous in an open subset of $\mathbb{C}$ of the type*

$$\mathbb{C} - \{\text{finitely many points of } \mathbb{C}\}$$

and satisfying the condition

$$\lim_{z \to \infty} f(z) = 0,$$

where ∞ is the complex infinity. Then:

(a) For every, $\mu > 0$, $R > 0$ and S_R the upper half of $C(0, R)$, we have

$$\lim_{R \to \infty} \int_{S_R} f(z) e^{i\mu z} dz = 0.$$

(b) For every, $\lambda < 0$, $R > 0$ and T_R the lower half of $C(0, R)$, we have

$$\lim_{R \to \infty} \int_{T_R} f(z) e^{i\lambda z} dz = 0.$$

Proof We prove **(a)**. [**(b)** is done similarly with the necessary adjustments.]

We take $R > 0$ greater than the lengths of all the finitely many exceptional points at which $f(z)$ is not defined. We let

$$M(R) = \underset{z \in S_R}{\text{Maximum}} |f(z)| .$$

Then, by the hypothesis $\lim\limits_{z \to \infty} f(z) = 0$, we get $\lim\limits_{R \to \infty} M(R) = 0$.

Hence,

$$\left| \int_{S_R} f(z) e^{i\mu z} dz \right| = \left| \int_0^{\pi} f(Re^{i\theta}) e^{i\mu R[\cos(\theta) + i\sin(\theta)]} Rie^{i\theta} d\theta \right| \leq$$

$$\int_0^{\pi} \left| f(Re^{i\theta}) \right| R\, e^{-\mu R \sin(\theta)} d\theta < \quad \text{(by } \textbf{Jordan's Lemma 1.7.4}\text{)}$$

$$M(R) R \frac{\pi}{\mu R} = M(R) \frac{\pi}{\mu} \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty,$$

and this implies the result. (The convergence is uniform.)

■

Lemma 1.7.6 *Let $w = f(z)$ be a complex function defined and holomorphic in the open subset of $\mathbb{C}$*

$$\mathbb{C} - \{z_1,\, z_2,\, z_3,\, \ldots,\, z_k\} \quad \textit{with} \quad k \geq 1$$

and satisfying the condition

$$\lim_{z \to \infty} f(z) = 0,$$

where ∞ is the complex infinity. We assume that ***each*** *of the k exceptional points $z_1,\, z_2,\, z_3,\, \ldots,\, z_k$ is an isolated singularity of $f(z)$ and is not on the real axis.*

(a) If ***all*** *the exceptional points located in the upper half plane are the $z_1,\, z_2,\, z_3,\, \ldots,\, z_l$ with $1 \leq l \leq k$, then for any $\mu > 0$ we have*

$$P.V. \int_{-\infty}^{\infty} f(x) e^{i\mu x}\, dx = 2\pi i \sum_{j=1}^{l} \underset{z=z_j}{Res} \left[f(z) e^{i\mu z} \right].$$

(b) If the $z_{l+1},\, z_{l+2},\, \ldots,\, z_k$ are ***all*** *the exceptional points located in the lower half plane with $0 \leq l < k$, then for any $\lambda < 0$ we have*

$$P.V. \int_{-\infty}^{\infty} f(x) e^{i\lambda x}\, dx = -2\pi i \sum_{j=l+1}^{k} \underset{z=z_j}{Res} \left[f(z) e^{i\lambda z} \right].$$

Proof We observe that $f(z)$ and $f(z)e^{i\mu z}$ [or $f(z)e^{i\lambda z}$] have exactly the same singularities. Then, the results follow from **the Residue Theorem, 1.7.1**, and **Lemma 1.7.5** applied to $f(z)e^{i\mu z}$ [or $f(z)e^{i\lambda z}$].

■

Example 1.7.28 If $a > 0$ and $b \neq 0$ constants, compute

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2}\, dx = \int_{-\infty}^{\infty} \frac{\cos(ax) + i \sin(ax)}{x^2 + b^2}\, dx.$$

The parameter b appears in the square, and so, without loss of generality, we assume that $b > 0$. Otherwise, we use $|b| > 0$.

First of all, for $b > 0$, this integral exists because

$$\int_{-\infty}^{\infty} \left| \frac{e^{iax}}{x^2 + b^2} \right| dx = \int_{-\infty}^{\infty} \frac{1}{x^2 + b^2}\, dx = \frac{1}{b}\pi,$$

i.e., it converges absolutely. So, its value and its principal value coincide.

If $b = 0$, this integral does not exist because $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2}\, dx = \infty$, since near $x = 0$ the integrand behaves like $\dfrac{1}{x^2}$. Also, we have that for

$a > 0$ (or $\neq 0$) the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2}\,dx = \infty - \infty$ does not exist, since near $x = 0$ the integrand behaves like $\dfrac{\pm a}{2x}$.

The function $f(z) = \dfrac{1}{z^2 + b^2}$ satisfies the condition $\lim\limits_{z\to\infty} f(z) = 0$ and has two singularities $z_1 = bi$ and $z_2 = -bi$ not on the real axis. So, this function is holomorphic in $\mathbb{C} - \{z_1,\ z_2\}$.

We consider $z_1 = bi$ the singularity located in the upper half plane $(b > 0)$. Then,

$$\operatorname*{Res}_{z=bi}\left[f(z)e^{iaz}\right] = \frac{(z-bi)e^{iaz}}{(z-bi)(z+bi)}\Big|_{z=bi} = \frac{e^{iabi}}{2bi} = \frac{e^{-ab}}{2ib}.$$

So, by **Lemma 1.7.6, (a)**, we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\,dx = 2\pi i\frac{e^{-ab}}{2ib} = \frac{\pi}{b}\,e^{-ab} = \frac{\pi}{be^{ab}}.$$

Remark: From this integral, we get the following two real integrals: For $a > 0$ and $b > 0$

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{be^{ab}} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx = 0.$$

In the first integral, the integrand function is even, and so

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{0}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}.$$

This formula also gives the correct answer for $a = 0$. I.e., this integral is continuous in a at $a = 0$. Moreover, by the evenness of the integrand, the integral does not change if we replace a with $-a$, and therefore we must use $|a|$ in the second side. Finally, we have that for any $b > 0$ and $a \in \mathbb{R}$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{0}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-|a|b}.$$

In the second integral, the integrand function is odd and the integral exists.[36] So, we could á-priori say that its value is immediately zero. But,

[36] In general, we can prove in the same way that $\forall\ n = 1,\ 2,\ 3,\ \dots$

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^{2n}+1}\,dx = \frac{\pi}{n}\sum_{k=1}^{n} e^{-\sin(t_k)}\cdot\sin[t_k + \cos(t_k)], \quad \text{where} \quad t_k = \frac{\pi}{2n}(2k+1),$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^{2n}+1}\,dx = 0.$$

With a and/or b other than 1, we perform the appropriate transformations.

the integral

$$\int_0^\infty \frac{\sin(ax)}{x^2+b^2}\,dx,$$

where $a > 0$ and $b > 0$ constants, cannot be found in a nice closed form or by basic means, even though it converges absolutely.

(See also **Example 1.7.33** in which the initial integral here is viewed as a Fourier transform.)

▲

Example 1.7.29 We consider the integral

$$F(a) := \int_{-\infty}^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx,$$

where $a \geq 0$ and $b > 0$.

We observe that the function $f(x) := \dfrac{\sin(ax)}{x(x^2+b^2)}$ is even in $\mathbb{R}$, [i.e., $f(-x) = f(x),\ \forall\ x \in \mathbb{R}$] and absolutely bounded by the function $g(x) := \dfrac{a}{x^2+b^2} \geq 0$ [i.e., $|f(x)| \leq g(x),\ \forall\ x \in \mathbb{R}$]. Since $\displaystyle\int_{-\infty}^\infty g(x)\,dx = \frac{a\pi}{b}$ is finite, $F(a)$ converges absolutely for all $a \geq 0$. For $a = 0$, we have $F(0) = 0$.

Now, $\forall\ a \in \mathbb{R}$

$$\left|\frac{\partial}{\partial a}\left[\frac{\sin(ax)}{x(x^2+b^2)}\right]\right| = \left|\frac{\cos(ax)}{x^2+b^2}\right| \leq \frac{1}{x^2+b^2}$$

with

$$\int_{-\infty}^\infty \frac{1}{x^2+b^2}\,dx = \frac{\pi}{b},$$

finite and independent of a.

Then, by the differentiation **Part II of Theorem I 2.2.1** and the **previous Example**, we get

$$\frac{d\,F(a)}{da} = F'(a) = \int_{-\infty}^\infty \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}e^{-ab}.$$

Therefore,

$$F(a) = \frac{\pi}{b}\cdot\frac{e^{-ab}}{-b} + C.$$

Since $F(0) = 0$, we have that $C = \dfrac{\pi}{b^2}$, and so if $a >$ and $b > 0$, we have

$$F(a) := \int_{-\infty}^\infty \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \frac{\pi}{b^2}\cdot\left(1 - e^{-ab}\right).$$

Notice that $F(-a) = -F(a)$. So,

$$\forall\ a \in \mathbb{R},\ \ F(a) := \int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\, dx = \operatorname{sign}(a)\frac{\pi}{b^2} \cdot \left(1 - e^{-|a|b}\right).$$

As we said above, $f(x)$ is even in $\mathbb{R}$. So,

$$\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\, dx = \int_{-\infty}^{0} \frac{\sin(ax)}{x(x^2+b^2)}\, dx = \frac{\pi}{2b^2} \cdot \left(1 - e^{-ab}\right).$$

Remark: We have computed the above integral $F(a)$ by combining real and complex analyses. We can also compute it quickly by pure complex analysis if we use the **Theorem** that follows. [Try the next theorem on this example, with $f(z) = \dfrac{e^{iaz}}{z(z^2+b^2)}$!]

Pure real analysis can compute this integral but with extensive arguments. Using **Theorem I 2.2.1**, as we have used it above, and **Theorem I 2.3.15**, in order to prove that $F''(a) = \displaystyle\int_{-\infty}^{\infty} \frac{-x\sin(ax)}{x^2+b^2}\, dx$ as a uniform limit of derivatives, we can show that for $a \geq 0$, $F(a)$ satisfies the initial value-problem:

$$\begin{cases} \dfrac{1}{b^2}\ F''(a) - F(a) = \dfrac{-\pi}{b^2}, \\ \\ F(0) = 0, \\ \\ F'(0) = \dfrac{\pi}{b}, \end{cases}$$

whose solution is easily found to be $F(a) = \dfrac{\pi}{b^2} \cdot \left(1 - e^{-ab}\right)$. (Etc.)

By this example, we see how powerful the methods of complex analysis are in comparison to real analysis ones.

▲

Example 1.7.30 If $a \geq 0$ and $b > 0$, the differentiation **Part II of Theorem I 2.2.1** applies immediately if we differentiate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx = \frac{\pi}{b} e^{-ab}$$

with respect to b. Then, for $a \geq 0$ and $b > 0$, we find:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{\left(x^2+b^2\right)^2}\, dx = \frac{\pi}{2b^3} e^{-ab}(1+ab).$$

▲

In view of **Lemmata 1.7.3 and 1.7.6** we write and can directly prove the **Theorem** that follows, analogous to **Theorem 1.7.4**, on **evaluating the Cauchy principal value of certain types of integrals**.

Theorem 1.7.5 *Suppose a complex function $f(z)$ satisfies the conditions of **Lemma 1.7.6** with the exception that it is allowed to have finitely many (isolated) **simple poles** on the x-axis.*

Then, we have: If $a > 0$ constant

$$P.V. \int_{-\infty}^{\infty} e^{iaz} f(x)dx =$$

$$2\pi i \sum [\text{residues of } e^{iaz} f(z) \text{ in the upper half plane}] +$$

$$\pi i \sum [\text{residues of } e^{iaz} f(z) \text{ on the real axis}].$$

If $a < 0$ constant

$$P.V. \int_{-\infty}^{\infty} e^{iaz} f(x)dx =$$

$$-2\pi i \sum [\text{residues of } e^{iaz} f(z) \text{ in the lower half plane}] -$$

$$\pi i \sum [\text{residues of } e^{iaz} f(z) \text{ on the real axis}].$$

(The proof follows from **Lemmata 1.7.3 and 1.7.6**.)

Example 1.7.31 Find the

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x^3 - 1}\, dx.$$

The complex function

$$f(z) = \frac{1}{z^3 - 1} = \frac{1}{(z-1)\left(z - e^{\frac{2\pi i}{3}}\right)\left(z - e^{\frac{4\pi i}{3}}\right)}$$

has one simple pole on the x-axis, the number $z = 1$, one simple pole at $z = e^{\frac{2\pi i}{3}}$, in the upper half plane, and satisfies the conditions of **Theorem 1.7.5**. The corresponding residues are

$$\operatorname*{Res}_{z=1} f(z) = \frac{e^i}{\left(1 + \frac{1}{2} - \frac{\sqrt{3}}{2} i\right)\left(1 + \frac{1}{2} + \frac{\sqrt{3}}{2} i\right)} = \frac{e^i}{3},$$

$$\operatorname*{Res}_{z=e^{\frac{2\pi i}{3}}} f(z) = \frac{e^{ie^{i\frac{2\pi}{3}}}}{\left(e^{\frac{2\pi i}{3}} - 1\right)\left(e^{\frac{2\pi i}{3}} - e^{\frac{4\pi i}{3}}\right)} = \dots$$

$$= \frac{-e^{\frac{-\sqrt{3}}{2}}}{6}\left\{\cos\left(\frac{1}{2}\right) - \sqrt{3}\sin\left(\frac{1}{2}\right) - i\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right)\right]\right\}.$$

Therefore, by **Theorem 1.7.5**, we find

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x^3-1}\,dx = \text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)+i\sin(x)}{x^3-1}\,dx =$$

$$2\pi i\frac{-e^{\frac{-\sqrt{3}}{2}}}{6}\left\{\cos\left(\frac{1}{2}\right) - \sqrt{3}\sin\left(\frac{1}{2}\right) - i\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right)\right]\right\} +$$

$$\pi i\left(\frac{e^i}{3}\right) = -\frac{\pi}{3}\,e^{\frac{-\sqrt{3}}{2}}\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) - \sin(1)\right] -$$

$$i\frac{\pi}{3}\left[\cos\left(\frac{1}{2}\right)\sqrt{3}\sin\left(\frac{1}{2}\right) + \cos(1)\right].$$

Separating the real and imaginary parts, we obtain the two real principal values:

$$(1)\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)}{x^3-1}\,dx = -\frac{\pi}{3}\,e^{\frac{-\sqrt{3}}{2}}\left[\sqrt{3}\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) - \sin(1)\right],$$

$$(2)\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{\sin(x)}{x^3-1}\,dx = -\frac{\pi}{3}\left[\cos\left(\frac{1}{2}\right)\sqrt{3}\sin\left(\frac{1}{2}\right) + \cos(1)\right].$$

▲

Example 1.7.32 For $b \in \mathbb{R}$ find the

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}\,dx = \text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(ax)+i\sin(ax)}{x^2-b^2}\,dx,$$

and consequently two real principal values.

The complex function

$$f(z) = \frac{e^{iaz}}{z^2-b^2} = \frac{e^{iaz}}{(z-b)(z+b)}$$

has two simple poles of the real axis and satisfies the conditions of **Theorem 1.7.5**.

The residues are easily computed to be

$$\operatorname*{Res}_{z=b} f(z) = \frac{e^{abi}}{2b} = \frac{\cos(ab)+i\sin(ab)}{2b}$$

$$\operatorname*{Res}_{z=-b} f(z) = \frac{e^{-abi}}{-2b} = \frac{-\cos(ab)+i\sin(ab)}{2b}$$

So, by the Theorem we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 - b^2}\, dx = \begin{cases} -\pi \dfrac{\sin(ab)}{b}, & \text{if } a > 0, \\ \\ \pi \dfrac{\sin(ab)}{b}, & \text{if } a < 0. \end{cases}$$

This formula is also correct as $0 = 0$, when $a = 0$.

Therefore

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 - b^2}\, dx = \begin{cases} -\text{sign}(a)\, \pi\, \dfrac{\sin(ab)}{b}, & \text{if } \ a \neq 0, \\ \\ 0, & \text{if } \ a = 0, \end{cases}$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2 - b^2}\, dx = 0.$$

(These two principal values can also be found with elementary methods by decomposing $f(z)$ into partial fractions and using the Dirichlet integral of **Example I 2.2.8**.)

Now in view of the results in **Example 1.7.28** and its **Remark**, we also get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4 - b^4}\, dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2 - b^2)(x^2 + b^2)}\, dx =$$

$$\frac{1}{2b^2}\, \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 - b^2}\, dx - \frac{1}{2b^2}\, \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + b^2}\, dx =$$

$$\begin{cases} -\dfrac{\pi}{2b^3} \left[\text{sign}(a) \dfrac{\sin(ab)}{b} + e^{-|a|b} \right], & \text{if } a \neq 0, \\ \\ -\dfrac{\pi}{2b^3}, & \text{if } a = 0. \end{cases}$$

▲

Now, we want to state the definition of the **Fourier transform** for some classes of functions. We consider an **absolutely integrable** function $f : \mathbb{R} \longrightarrow \mathbb{R}$, that is, $\displaystyle\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Then, for any $t \in \mathbb{R}$, the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x) e^{itx} dx$ exists. This follows immediately by the identity $\left| e^{itx} \right| = 1$, and so the integral converges absolutely by the absolute integrability of $f(x)$. Obviously: $\displaystyle\int_{-\infty}^{\infty} \left|f(x)e^{itx}\right| dx \le \int_{-\infty}^{\infty} |f(x)|dx < \infty$.

So, for the class of the absolutely integrable functions, we define:

Definition 1.7.3 *For any real absolutely integrable function $y = f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define the* ***Fourier transform*** *of $y = f(x)$, written as $\hat{f} := \mathcal{F}[f(x)] : \mathbb{R} \longrightarrow \mathbb{C}$, to be the improper integral*

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx, \quad \forall\ t \in \mathbb{R} \tag{1.27}$$

(which exists for all $t \in \mathbb{R}$).

For a **square integrable** function $f : \mathbb{R} \longrightarrow \mathbb{R}$, i.e., $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$, the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)e^{itx}dx$ may exist for some t's and may not exist for others or not at all. In this case, we use the principal value to define the Fourier transform of $y = f(x)$ by:

Definition 1.7.4 *For any real square integrable function $y = f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define the* ***Fourier transform*** *of $y = f(x)$, written as $\hat{f} := \mathcal{F}[f(x)] : \mathbb{R} \longrightarrow \mathbb{C}$, to be the principal value*

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = P.V. \int_{-\infty}^{\infty} f(x)e^{itx}dx =$$
$$\lim_{0<M\to\infty} \int_{-M}^{M} f(x)e^{itx}dx, \quad \forall\ t \in \mathbb{R}. \tag{1.28}$$

If the improper integral in **(1.28)** exists, then it is equal to its principal value. So, **Definition 1.7.4** implies, and therefore extends or includes, **Definition 1.7.3**.

In **Lemma 1.7.6**, we have seen a class of functions for which the principal value in **(1.28)** exists. So, for the functions that satisfy the conditions of **this Lemma**, we define their Fourier transform as in **Definition 1.7.4**.

In this book, we mostly concentrate on the evaluation of this integral and some of its basic properties. We will need some results that we will readily take from analysis, and we will present a few applications. The large number of properties and applications of this integral belong to a big chapter of analysis, titled **"Fourier transform."** This is a very important chapter to mathematics, engineering, science, and application.

In many books, **Definitions 1.7.3** and **1.7.4** include the multiplicative constant $\dfrac{1}{\sqrt{2\pi}}$ in front of the integral. That is, by many authors, the **Fourier transform** is defined by:

$$\tilde{f}(t) = \mathfrak{F}[f(x)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}}\hat{f}(t), \quad \forall\ t \in \mathbb{R}, \tag{1.29}$$

or

$$\tilde{f}(t) = \mathfrak{F}[f(x)](t) = \frac{1}{\sqrt{2\pi}}\ \text{P.V.} \int_{-\infty}^{\infty} f(x)e^{itx}dx = \frac{1}{\sqrt{2\pi}}\hat{f}(t), \quad \forall\ t \in \mathbb{R}. \tag{1.30}$$

Here, we use the symbols $\tilde{f}$ and $\mathfrak{F}$ instead of $\hat{f}$ and $\mathcal{F}$ in the **Definitions 1.7.3** and **1.7.4**. There are some reasons for this constant, but we do not obviously need it for the computation of the integral.

We also notice that

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx = \int_{-\infty}^{\infty} f(x)\left[\cos(tx) + i\sin(tx)\right]dx = \int_{-\infty}^{\infty} f(x)\cos(tx)\,dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)\,dx.$$

So, to compute $\hat{f}(t)$, we may compute the last two integrals separately in any possible way, or using already computed integrals, or using the evenness and/or the oddity of certain functions, etc. We call the first integral the **cosine transform of** $f(x)$ and the second the **sine transform of** $f(x)$. Sometimes one may exist but not the other or its principal value may, or may not exist, etc. Obviously, **the cosine transform of an odd function and the sine transform of an even function are zero**.

{From now on, we must be careful to notice and distinguish between $\hat{f}$ and $\tilde{f}$, and between $\mathcal{F}[f]$ and $\mathfrak{F}[f]$.}

We observe that on the set of functions that the **Fourier, cosine** and **sine transforms** are well defined (exist) these transforms are **linear**. I.e., assuming that the Fourier transforms involved exist and $a \in \mathbb{R}$, or $a \in \mathbb{C}$, we have:

$$(1) \quad \mathcal{F}[f(x) + g(x)](t) = \mathcal{F}[f(x)](t) + \mathcal{F}[g(x)](t),$$
$$(2) \qquad \mathcal{F}[af(x)](t) = a\mathcal{F}[f(x)](t),$$

and similarly for the cosine and the sine transforms.

It can fairly easily be proved that if $y = f(x)$ is a nice real function on $\mathbb{R}$, and its Fourier, or sine, or cosine, transform exists, then it is unique. (For this, see the **Inversion Theorem, 1.7.6**, below.)

In our exposition, we will need the following **three important results** from advanced analysis. The first two results deal with the **behavior** of the Fourier transform at $\pm\infty$ and the third with its **invertibility**.

Result 1: The first result concerning the behavior is the following fundamental Lemma:

Lemma 1.7.7 (Riemann-Lebesgue Lemma) *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an **absolutely integrable** function, then its Fourier transform $\hat{f}$, given by **(1.27)** or **(1.29)**, is uniformly continuous on $\mathbb{R}$ and $\lim_{t\to\pm\infty} \hat{f}(t) = 0$.*

$$\left[\textit{So, } \lim_{t\to\pm\infty} \int_{-\infty}^{\infty} f(x)\cos(tx)\,dx = 0, \ \textit{and} \lim_{t\to\pm\infty} \int_{-\infty}^{\infty} f(x)\sin(tx)\,dx = 0. \right]$$

Also, $\forall\ t \in \mathbb{R}$, $|\hat{f}(t)| \le \|f\|_1$, that is, $|\hat{f}|$ is bounded by $\int_{-\infty}^{\infty} |f(x)|dx$. We write: $\|f\|_\infty \le \|f\|_1$.

[A **proof** of this key result in Fourier analysis can be found in Rudin 1987, Theorem 9.6, 182-183; Weinberger 1965, 316, etc.

We also remind the reader that a function $f : \mathcal{D} \longrightarrow \mathbb{R}$, with $\mathcal{D} \subseteq \mathbb{R}$, is called **uniformly continuous on its domain** $\mathcal{D}$ if by definition:

$$\forall\ \ \epsilon > 0,\ \ \exists\ \ \delta := \delta(\epsilon) > 0, \quad \text{such that} \quad \forall\ u \in \mathcal{D} \quad \text{and} \quad \forall\ v \in \mathcal{D},$$
$$|u - v| < \delta \Longrightarrow |f(u) - f(v)| < \epsilon.$$

That is, the choice of $\delta > 0$ depends only on the à-priori choice of $\epsilon > 0$ (not on the points $u \in \mathcal{D}$ and $v \in \mathcal{D}$), i.e., $\delta = \delta(\epsilon)$.

The **simple continuity** on $\mathcal{D}$ is defined point-wise and so for the simple continuity at a point $w \in \mathcal{D}$, the positive number δ depends on both à-priori choices of $\epsilon > 0$ and $w \in \mathcal{D}$, i.e., $\delta := \delta(\epsilon, w) > 0$.

We notice that the uniform continuity depends on both the function f and its domain $\mathcal{D}$ and it implies the simple continuity. (Simple continuity does not necessarily imply uniform continuity.) Also, it is obvious from the definition that if f is uniformly continuous on $\mathcal{D}$, then it is uniformly continuous on any subset of $\mathcal{D}$.]

Result 2: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is **square integrable**, then $\hat{f}$, given by **(1.28)** or **(1.30)**, exists and is square integrable but may not be continuous and we have (the two forms of) the so-called **Parseval equation**.

$$\int_{-\infty}^{\infty} |\hat{f}(t)|^2\,dt = 2\pi \int_{-\infty}^{\infty} |f(x)|^2\,dx (< \infty), \tag{1.31}$$

and

$$\int_{-\infty}^{\infty} |\tilde{f}(t)|^2\,dt = \int_{-\infty}^{\infty} |f(x)|^2\,dx (< \infty). \tag{1.32}$$

So, concerning the behavior of Fourier transform of square integrable functions at $\pm\infty$, by **(1.31)** and/or **(1.32)** we have that, if the limits $\lim_{t\to\pm\infty} \hat{f}(t) = \lim_{t\to\pm\infty} \sqrt{2\pi}\tilde{f}(t)$ exist, they must be zero.

[The above **Equations (1.31) and (1.32)** are consequences of the so called **Plancherel**[37] **Theorem**, a key result in Fourier analysis. See, e.g., Rudin 1987, 185-187.]

Remark: **Equation (1.32)** shows that the Fourier transform $\tilde{f}(t) = \mathfrak{F}[f(x)](t)$ is an isometry in the square integrable functions under the $\|\cdot\|_2$ norm. [See **Problem I 2.6.65, Item (16.)**]. This is the main reason why we define we put the factor $\frac{1}{\sqrt{2\pi}}$ in the definition of $\tilde{f}(t) = \frac{1}{\sqrt{2\pi}}\hat{f}(t)$.

For any complex function $h(z)$, we have $|h(z)|^2 = h(z)\overline{h(z)}$. If, now, f and g are two square integrable real or complex functions, then $\pm f \pm g$ and $\pm if \pm ig$, by **Minkowski's inequality** [see **project Problem I 2.6.65, Item (17.)**] are also square integrable. We apply **(1.31)** and/or **(1.32)** to $f \pm g$ and $f \pm ig$ along with the linearity of the Fourier transform to obtain the four respective Parseval equations. Combining these four equations we respectively obtain (work this out) the so-called **general Parseval equation**

$$\int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\,dt = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx, \tag{1.33}$$

or

$$\int_{-\infty}^{\infty} \tilde{f}(t)\overline{\tilde{g}(t)}\,dt = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx. \tag{1.34}$$

Here, we need to use the complex conjugation, even if f and g are real functions, since the Fourier transform is a complex integral, and so $|\hat{f}(t)|^2 = \hat{f}(t)\overline{\hat{f}(t)}$, etc. Also, the functions f and g may be complex, in general.

The **Parseval equation** and the **general Parseval equation** can be used to compute the improper integral of one side by the other side, when the other side is easier. (See **Examples 1.7.38, 1.7.41, 1.7.42, Problem 1.7.107**, etc.)

Result 3: The importance and usefulness of the Fourier transform depends on the following result. On the classes of functions for which the Fourier transform exists, **the Fourier transform is a one-to-one mapping or operator**. So, under some mild conditions, we can

[37] Michel Plancherel, Swiss mathematician, 1885-1967.

retrieve a function from its Fourier transform. Results of this type are called **inversion Theorems** for the Fourier transform. Depending on the conditions, there are several **inversion theorems**. Here, we state an **inversion Theorem** which we use in some examples and applications in the sequel. This is not the most general **inversion Theorem** but one adjusted to the level of this book. It is quite useful and applicable.

Theorem 1.7.6 (Inversion Theorem) *(a) Suppose that a function $f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, is absolutely integrable or square integrable and piecewise differentiable with $f'(x)$ absolutely integrable. Then,*

$$f(x) = \frac{1}{2\pi}\ P.V. \int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(t)e^{-itx}dt,$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}}\ P.V. \int_{-\infty}^{\infty} \tilde{f}(t)e^{-itx}dt = \frac{1}{\sqrt{2\pi}} \lim_{M\to\infty} \int_{-M}^{M} \tilde{f}(t)e^{-itx}dt.$$

If such an $f(x)$ has a jump discontinuity at a point $x = \alpha$, then we get

$$\frac{f(\alpha^+) + f(\alpha^-)}{2} =$$
$$\frac{1}{2\pi}\ P.V. \int_{-\infty}^{\infty} \hat{f}(t)e^{-it\alpha}dt = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(t)e^{-it\alpha}dt,$$

or

$$\frac{f(\alpha^+) + f(\alpha^-)}{2} =$$
$$\frac{1}{\sqrt{2\pi}}\ P.V. \int_{-\infty}^{\infty} \tilde{f}(t)e^{-it\alpha}dt = \frac{1}{\sqrt{2\pi}} \lim_{M\to\infty} \int_{-M}^{M} \tilde{f}(t)e^{-it\alpha}dt.$$

(b) If both f and $\hat{f}$ are absolutely integrable (then $\tilde{f}$ is so), we can drop the differentiability or the absolute integrability of the derivative and the principal value in the formulae. So, in this case we simply have:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt, \qquad or \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(t)e^{-itx}dt.$$

In this case we also have the following three necessary conditions analogous to the conclusions in the ***Riemann-Lebesgue Lemma, 1.7.7****:*
(1) $\lim_{x\to\pm\infty} f(x) = 0$. *(2) $f(x)$ is uniformly continuous in $\mathbb{R}$.*
(3) $\forall\ x \in \mathbb{R}$, $|f(x)| \le \frac{1}{2\pi}\|\hat{f}\|_1 = \frac{1}{\sqrt{2\pi}}\|\tilde{f}\|_1$. *We write:*

$$\|f\|_\infty \le \frac{1}{2\pi}\|\hat{f}\|_1 = \frac{1}{\sqrt{2\pi}}\|\tilde{f}\|_1.$$

Corollary 1.7.6 *If $f : \mathbb{R} \longrightarrow \mathbb{R}$, is absolutely integrable and at least one of the three necessary conditions in **(b)** of the **Inversion Theorem** fails, then $\hat{f}$ (or $\tilde{f}$) is not absolutely integrable.*

[E.g., see **Problem 1.7.116, (d)**.]

Hence, under the conditions of this **inversion Theorem**, the **inverse Fourier transform** to either Fourier transform defined by **(1.27)** and **(1.29)** is respectively defined to be

$$\mathcal{F}^{-1}[\hat{f}(t)](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{it(-x)}dt = \frac{1}{2\pi}\mathcal{F}[\hat{f}(t)](-x), \tag{1.35}$$

and so

$$f(-x) = \frac{1}{2\pi}\mathcal{F}\left[\mathcal{F}[f(x)](t)\right](x) \quad \Longleftrightarrow \quad \mathcal{F}\left[\mathcal{F}[f(x)](t)\right](x) = 2\pi f(-x), \tag{1.36}$$

and

$$\mathfrak{F}^{-1}[\tilde{f}(t)](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(t)e^{-itx}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(t)e^{it(-x)}dt = \mathfrak{F}[\tilde{f}(t)](-x), \tag{1.37}$$

and so

$$f(-x) = \mathfrak{F}\left[\mathfrak{F}[f(x)](t)\right](x). \tag{1.38}$$

So, if we apply these inverse Fourier transforms to the respective Fourier transforms $\hat{f}(t)$ and $\tilde{f}(t)$ of a function $f(x)$, under the conditions stated in the **inversion Theorem**, we retrieve the function $f(x)$. In particular, *under these conditions:*

The Fourier transform of $f(x)$ is zero if and only if the function $f(x) \equiv 0$, a.e.

We observe that the constant $\dfrac{1}{\sqrt{2\pi}}$ produces a symmetry in the expressions of the Fourier transform and its inverse. Besides some other reasons, this is one of the reasons as to why many authors prefer to put this constant in the definitions.

Examples

Example 1.7.33 Find the **Fourier transform** of

$$f(x) = \frac{1}{x^2+b^2}, \quad \text{where} \quad b>0 \quad \text{constant.}$$

In **Example 1.7.28**, we have found $\hat{f}(a) = \frac{\pi}{b}\, e^{-ba}$ for $a>0$. Now, for $a<0$, we find

$$\operatorname*{Res}_{z=-bi}\left[f(z)e^{iaz}\right] = \frac{(z+bi)e^{iaz}}{(z-bi)(z+bi)}\Big|_{z=-bi} = \frac{e^{ia(-bi)}}{-2bi} = \frac{e^{ab}}{-2bi}.$$

So, for $a<0$, by **Lemma 1.7.6, (b)**, we get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\, dx = -2\pi i \frac{e^{ab}}{-2ib} = \frac{\pi}{b} e^{ab}.$$

For $a=0$, we straightly compute

$$\int_{-\infty}^{\infty} \frac{1}{x^2+b^2}\, dx = \frac{\pi}{b}.$$

Putting the three cases $a>0$, $a=0$ and $a<0$ together in one formula, we find that the Fourier transform of the function

$$f(x) = \frac{1}{x^2+b^2}, \quad x \in \mathbb{R}$$

where $b>0$ constant, is

$$\hat{f}(a) = \frac{\pi}{b}\, e^{-b|a|}, \quad a \in \mathbb{R}\,.$$

Since the function $f(x) = \dfrac{1}{x^2+b^2}$ is absolutely integrable, we see that $\hat{f}(a) = \dfrac{\pi}{b}\, e^{-b|a|}$ is uniformly continuous in $\mathbb{R}$ and $\lim\limits_{a\to\pm\infty} f(a) = 0$, as claimed by the **Riemann-Lebesgue Lemma, 1.7.7**.

Notice that $\hat{f}(a)$ and $f(x)$ [and $f'(x) = \dfrac{-2x}{(x^2+b^2)^2}$] are absolutely integrable in $(-\infty,\infty)$. Then, if we apply the **inversion Theorem, 1.7.6, (b),** we will get the given function $f(x)$ back. Indeed,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi}{b}\, e^{-b|t|}e^{-itx}dt =$$
$$\frac{1}{2\pi}\int_{-\infty}^{0} \frac{\pi}{b}\, e^{bt}e^{-itx}dt + \frac{1}{2\pi}\int_{0}^{\infty} \frac{\pi}{b}\, e^{-bt}e^{-itx}dt =$$
$$\frac{1}{2\pi}\int_{-\infty}^{0} \frac{\pi}{b}\, e^{(b-ix)t}dt + \frac{1}{2\pi}\int_{0}^{\infty} \frac{\pi}{b}\, e^{-(b+ix)t}dt.$$

We use **Problem 1.7.58**, and, with $b > 0$, we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(t)e^{-itx}dt = \frac{1}{2\pi}\frac{\pi}{b}\left(\frac{1}{b-ix}+\frac{1}{b+ix}\right) = \frac{1}{x^2+b^2} = f(x).$$

Also notice: (1) $\lim\limits_{x\to\pm\infty} f(x) = 0$.

(2) $f(x)$ is uniformly continuous in $\mathbb{R}$. (Prove this!)

(3) $|f(x)| \leq \frac{1}{b^2}$, $\forall\ x \in \mathbb{R}$, and $\frac{1}{2\pi}\|\hat{f}\|_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi}{b}\,e^{-b|t|}dt = \frac{1}{b^2}$. So, $\|f\|_\infty \leq \frac{1}{2\pi}\|\hat{f}\|$, as claimed in the **inversion Theorem, 1.7.6, (b)**.

▲

Example 1.7.34 Compute

$$\int_{-\infty}^{\infty}\frac{xe^{iax}}{x^2+b^2}dx, \quad \text{where} \quad a>0 \quad \text{and} \quad b>0 \quad \text{constants.}$$

The function $f(z) = \frac{z}{z^2+b^2}$ satisfies the condition $\lim\limits_{z\to\infty} f(z) = 0$ and has two singularities $z_1 = bi$ and $z_2 = -bi$ not on the real axis. This function is holomorphic in $\mathbb{C} - \{z_1,\ z_2\}$.

We consider $z_1 = bi$ the singularity in the upper half plane. Then,

$$\operatorname*{Res}_{z=bi}\left[f(z)e^{iaz}\right] = \frac{z(z-bi)e^{iaz}}{(z-bi)(z+bi)}\Big|_{z=bi} = \frac{bie^{iabi}}{2bi} = \frac{1}{2e^{ab}}.$$

So, by the **Lemma 1.7.6, (a)**, we get that for $a > 0$

$$\int_{-\infty}^{\infty}\frac{xe^{iax}}{x^2+b^2}dx = 2\pi i\frac{1}{2e^{ab}} = \frac{\pi i}{e^{ab}} = i\,\pi\,e^{-ab}.$$

Remark 1: From this integral, we get the following two real integrals: If $a > 0$ and $b > 0$ constants, then:

$$\int_{-\infty}^{\infty}\frac{x\cos(ax)}{x^2+b^2}\,dx = 0 \text{ (odd function, whose improper integral exists).}$$

If $a = 0$ or $b = 0$ the integral is valid as principal value.

$$\int_{-\infty}^{\infty}\frac{x\sin(ax)}{x^2+b^2}\,dx = \frac{\pi}{e^{ab}} \text{ (even function, whose improper integral exists).}$$

Remark 2: Working as in the **previous example**, we find that the **Fourier transform** of $f(x) = \frac{x}{x^2+b^2}$, where $b > 0$ and $a \in \mathbb{R}$ constants, is

$$\hat{f}(a) = \begin{cases} 0, & \text{if} \quad a = 0, \\ \\ \operatorname{sign}(a)\,\pi i\,e^{-b|a|}, & \text{if} \quad a \neq 0. \end{cases}$$

We observe that this Fourier transform is discontinuous at $a = 0$. In fact, for $a = 0$, we must consider the principal value of the corresponding improper integral, because the improper integral does not exist in the general limiting sense. (Check this out.)

We can easily check that the function $f(x) = \dfrac{x}{x^2 + b^2}$ is not absolutely integrable but is square integrable. So, its Fourier transform $\hat{f}(a)$ is defined by means of principal value. It is discontinuous at $a = 0$ and its limits as $x \to \pm\infty$ exist and are equal to zero.

We can also check that the derivative of $f(x) = \dfrac{x}{x^2 + b^2}$ and its Fourier transform are absolutely integrable, as $b > 0$. So, if we apply the **inversion Theorem, 1.7.6, (a)**, we will get $f(x)$ back.

▲

Example 1.7.35 (Compare with **Examples I 1.3.18, I 2.2.8**, and **I 2.3.11**. See also **Problem 1.7.93**.)

$$\text{Verify that} \qquad \forall \ \ \alpha > 0 \qquad \text{constant,} \qquad \int_0^\infty \frac{\sin(\alpha x)}{x}\, dx = \frac{\pi}{2}.$$

Since $e^{i\alpha x} = \cos(\alpha x) + i \sin(\alpha x)$, we consider the complex function

$$g(z) = \frac{e^{i\alpha z}}{z}.$$

This function has only one singularity at $z = 0$, and so it is holomorphic in $\mathbb{C} - \{0\}$. Since the singularity is located on the real axis, we cannot apply **Lemma 1.7.6**.

This singularity is easily seen to be a simple pole with residue,

$$\operatorname*{Res}_{z=0} g(z) = zg(z)|_{z=0} = e^{i\alpha 0} = 1.$$

Also, since $\alpha > 0$, we must work in the upper half plane, according to **Lemma 1.7.5**. This is the point at which the positivity of α is used in this problem.

So, we must choose a contour in the upper half closed plane that avoids $z = 0$. Thus, we consider any $0 < r < R < \infty$ and then an appropriate choice of contour that itself and its interior does not contain $z = 0$ is

$$C = [r, R] + S_R^+ + [-R, -r] + S_r^-,$$

where S_R^+ is the upper half of $C(0, R)$, positively oriented, and S_r^- is the upper half of $C(0, r)$, negatively oriented. (See **Figure 1.16**.)

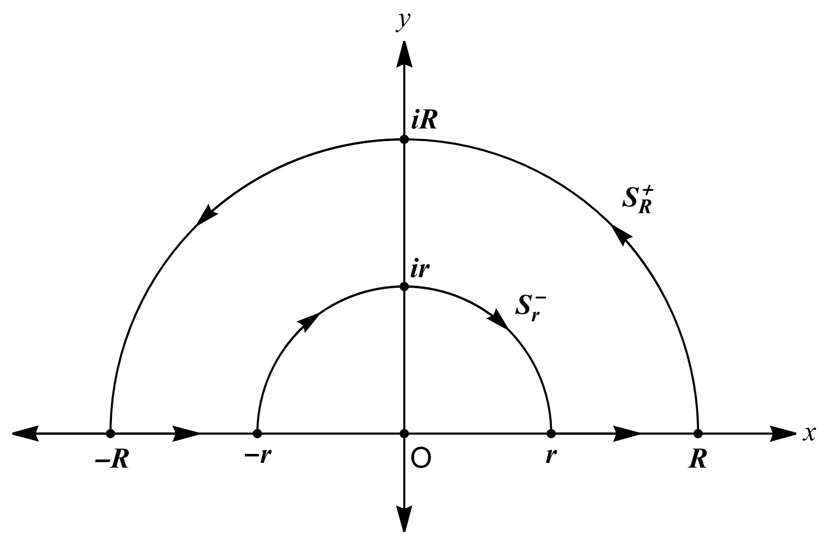

FIGURE 1.16: Contour 12 for Example 1.7.35

Then, by the **Cauchy-Goursat Theorem, 1.5.3**, we have

$$\oint\limits_{C^+} g(z)dz =$$

$$\int\limits_{[r,R]} g(z)dz + \int\limits_{S_R^+} g(z)dz + \int\limits_{[-R,-r]} g(z)dz + \int\limits_{S_r^-} g(z)dz = 0.$$

We see that

$$\lim_{\substack{r\to 0^+ \\ R\to\infty}} \left[\int\limits_{[r,R]} g(z)dz + \int\limits_{[-R,-r]} g(z)dz \right] = \lim_{\substack{r\to 0^+ \\ R\to\infty}} \int_{-R}^{-r} g(x)dx +$$

$$\lim_{\substack{r\to 0^+ \\ R\to\infty}} \int_r^R g(x)dx = \int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} \frac{\cos(\alpha x) + i\sin(\alpha x)}{x}\, dx.$$

Since for $f(z) = \dfrac{1}{z}$ the condition $\lim\limits_{z\to\infty} f(z) = 0$ is satisfied, by **Lemma 1.7.5** we have that

$$\lim_{R\to\infty} \int\limits_{S_R^+} g(z)dz = 0.$$

Next, for all $r > 0$, by **Lemma 1.7.3** we have

$$\int_{S_r^-} g(z)dz = -\pi i \operatorname*{Res}_{z=0} g(z) = -\pi i,$$

and so

$$\lim_{r\to 0^+} \int_{S_r^-} g(z)\,dz = -i\pi.$$

Putting these pieces together, we finally find

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x) + i\sin(\alpha x)}{x}\,dx - i\pi = 0$$

or

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x) + i\sin(\alpha x)}{x}\,dx = i\pi.$$

Separating real and imaginary parts, we find

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x}\,dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{x}\,dx = \pi.$$

Since $\dfrac{\sin(\alpha x)}{x}$ is an even function over $\mathbb{R}$, we obtain the **result**:

$$\forall\ \alpha > 0,\ \int_0^{\infty} \frac{\sin(\alpha x)}{x}\,dx = \frac{\pi}{2}.$$

Remark 1: If $\alpha < 0$, then we get $\displaystyle\int_0^{\infty} \frac{\sin(\alpha x)}{x}\,dx = -\frac{\pi}{2}$.

For $\alpha = 0$, we trivially get $\displaystyle\int_0^{\infty} \frac{\sin(\alpha x)}{x}\,dx = \int_0^{\infty} \frac{0}{x}\,dx = 0$.

Remark 2: We present some discussion on these results and the principal value. In **this example**, we, in fact, computed the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{x}\,dx = \pi, \quad \forall\ \alpha > 0,$$

with the help of the particularly chosen contour C.

However, because this integral exists, as proven in **Example I 1.3.18**, its principal value is equal to its value. Therefore, its value is equal to π.

But, the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x}\, dx$$

does not exist if all limiting processes are allowed. Whereas the limiting processes as $R \to \pm\infty$ do not matter, because $\lim\limits_{R\to\pm\infty} \dfrac{\cos(\alpha R)}{R} = 0$, problems arise when $r \to 0^{\pm}$ (from right and left). Notice for instance:

$$\int_{0^+}^{1} \frac{\cos(\alpha x)}{x}\, dx = \infty, \qquad \text{whereas} \qquad \int_{-1}^{0^-} \frac{\cos(\alpha x)}{x}\, dx = -\infty.$$

Finally, what we have computed by this integral is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{x}\, dx = 0,$$

which is naturally expected, since $\dfrac{\cos(\alpha x)}{x}$ is an odd function over $\mathbb{R}$.

Therefore, what we have accurately found is

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = i\pi, \ \ \forall \ \alpha > 0.$$

This computation extended to $\alpha = 0$ and $\alpha < 0$ is the Fourier transform of the function $f(x) = \dfrac{1}{x}$, $(x \neq 0)$, i.e.,

$$\hat{f}(\alpha) = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = \begin{cases} i\pi, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha = 0, \\ -i\pi, & \text{if } \alpha < 0. \end{cases}$$

Here, the function $f(x) = \dfrac{1}{x}$, $(x \neq 0)$, is neither absolutely nor square integrable in $\mathbb{R}$ but satisfies the conditions of **Lemma 1.7.6**. So, its Fourier transform is evaluated by the **Principal Value (1.28)** of **Definition 1.28**. We observe that this Fourier transform is not continuous at $\alpha = 0$ and its limit at $\pm\infty$ is not zero.

Note: The same discussion holds true for some **previous Examples**. Essentially, in some of these examples, we compute principal values of integrals. However, if the integrals we consider exist, then their values are equal to their principal values.

▲

Example 1.7.36 (Compare with **Example I 2.2.12**. See also **Problem 1.7.94**.) Verify that

$$\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{2}.$$

For this integral, we consider the function

$$g(z) = \frac{1 + 2iz - e^{2iz}}{z^2} = -\frac{(2i)^2}{2!} - \frac{(2i)^3}{3!}z - \dots$$

and we observe that $g(z)$ is holomorphic in $\mathbb{C}$. The origin $z = 0$ is not an essential singularity because we can extend $g(z)$ holomorphically at $z = 0$ by setting

$$g(0) = -\frac{(2i)^2}{2!} = 2.$$

Therefore, for any $R > 0$, we choose the positive contour $C^+ = [-R, R] + S_R^+$, with S_R^+ the upper half of $C(0, R)$ positively oriented. Then, by **the Cauchy-Goursat Theorem, 1.5.3**, we find

$$\oint_{C^+} g(z)\,dz = \int_{[-R,R]} g(z)\,dz + \int_{S_R^+} g(z)\,dz = 0.$$

By **Lemma 1.7.1** we get

$$\lim_{R\to\infty} \int_{S_R^+} \frac{1}{z^2}\,dz = 0.$$

Also, by **Lemma 1.7.3** we have

$$\lim_{R\to\infty} \int_{S_R^+} \frac{2iz}{z^2}\,dz = 2i \lim_{R\to\infty} \int_{S_R^+} \frac{1}{z}\,dz = 2i \cdot 1 \cdot i\pi = -2\pi,$$

and by **Lemma 1.7.5** we obtain

$$\lim_{R\to\infty} \int_{S_R^+} \frac{e^{2iz}}{z^2}\,dz = 0.$$

Therefore,

$$\lim_{R\to\infty} \int_{S_R^+} g(z)\,dz = -2\pi,$$

and so

$$\lim_{R\to\infty} \oint_{C^+} g(z)\,dz = \lim_{R\to\infty} \int_{[-R,R]} g(z)\,dz - 2\pi = 0.$$

Hence
$$\int_{-\infty}^{\infty} \frac{1 + 2ix - [\cos(2x) + i\sin(2x)]}{x^2}\, dx = 2\pi.$$

By separating real and imaginary parts, we obtain

$$\int_{-\infty}^{\infty} \frac{1 - \cos(2x)}{x^2}\, dx = 2\pi, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \frac{2x - \sin(2x)}{x^2}\, dx = 0.$$

(See also **Problems I 2.2.17** and **1.7.99**.)

In the first integral, we use the basic double angle trigonometric identity $\cos(2x) = 1 - 2\sin^2(x)$ to find $\int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \pi$. Since $\frac{\sin^2(x)}{x^2}$ is an even function over $\mathbb{R}$, we get $\int_0^{\infty} \frac{\sin^2(x)}{x^2}\, dx = \frac{\pi}{2}$.

Remark: The second integral is true only as a principal value because it does not exist in any general limiting process. That is,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{2x - \sin(2x)}{x^2}\, dx = 0.$$

▲

Example 1.7.37 The function $f(x) = \frac{\sin(x)}{x}$, $x \in \mathbb{R}$, is not absolutely integrable (**Example I 1.3.18, Problem I 1.3.16)**) but it is square integrable (**Example I 2.2.12**). Using the results of **Example I 2.2.8, Problem I 2.2.23 (a)** and basic trigonometric formulae, we find (fill in the details):

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} \frac{\sin(x)\cos(tx)}{x} + i0 = \begin{cases} 0, & \text{if} \quad |t| > 1, \\ \frac{\pi}{2}, & \text{if} \quad t = \pm 1, \\ \pi, & \text{if} \quad |t| < 1. \end{cases}$$

▲

Example 1.7.38 For any $a > 0$ constant, we let

$$f(x) = \begin{cases} 1, & \text{if} \quad |x| < a, \\ 0, & \text{if} \quad |x| \geq a. \end{cases}$$

Then, the Fourier transform of $f(x)$ is

$$\hat{f}(\xi) = \mathcal{F}[f(x)](\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx = \int_{-a}^{a} e^{ix\xi}dx =$$
$$\int_{-a}^{a} [\cos(\xi x) + i\sin(\xi x)]dx = \int_{-a}^{a} \cos(\xi x)dx + i\int_{-a}^{a} \sin(\xi x)]dx =$$
$$2\frac{\sin(a\xi)}{\xi} + i0 = 2\frac{\sin(a\xi)}{\xi}.$$

Also,

$$\tilde{f}(\xi) = \mathfrak{F}[f(x)](\xi) = \sqrt{\frac{1}{2\pi}}\ \mathcal{F}[f(x)](\xi) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(a\xi)}{\xi}.$$

We can use this result with the **Parseval equation (1.31)**. For $a > 0$, we get

$$\int_{-\infty}^{\infty} \left[2\frac{\sin(a\xi)}{\xi}\right]^2 d\xi = 2\pi\int_{-\infty}^{\infty} f^2(x)\,dx = 2\pi\int_{-a}^{a} 1^2\,dx = 2\pi \cdot 2a.$$

So, we find the already known **result** (see **Problem I 2.2.16**):

If $a > 0$, then

$$\int_{-\infty}^{\infty} \frac{\sin^2(a\xi)}{\xi^2}\,d\xi = \pi a.$$

[Similarly, if we had used **(1.32)**.]

If now for any $b > 0$ constant, we let

$$g(x) = \begin{cases} 1, & \text{if} \quad |x| < b, \\ \\ 0, & \text{if} \quad |x| \geq b, \end{cases} \qquad \text{then, as before,} \quad \hat{g}(\xi) = 2\frac{\sin(b\xi)}{\xi}.$$

Then, applying the **general Parseval equation, (1.33)**, we find

$$\int_{-\infty}^{\infty} \left[2\frac{\sin(a\xi)}{\xi}\right]\left[2\frac{\sin(b\xi)}{\xi}\right] d\xi = 2\pi\int_{-\infty}^{\infty} f(x)g(x)\,dx =$$
$$2\pi\int_{-\min\{a,\,b\}}^{\min\{a,\,b\}} 1 \cdot 1\,dx = 2\pi \cdot 2\min\{a,\,b\}.$$

So, we get

$$\int_{-\infty}^{\infty} \frac{\sin(a\xi)\sin(b\xi)}{\xi^2}\,d\xi = \pi \cdot \min\{a,\,b\}.$$

[Similar work if we had used **(1.34)**. See also **Problem I 2.2.39, (a)**.]

▲

1.7.6 Rules and Properties of the Fourier Transform

We have discussed the **linearity properties** of the Fourier transform, its **uniqueness** when it exists, that the Fourier transform is a **one-to-one operator** and an **inversion Theorem**. Here, we examine a few additional properties of the Fourier transform that follow from its definition and can be readily used as operational rules in applications.

(1) Suppose that the function $f(x)$ is continuous and piecewise continuously differentiable, $\mathcal{F}[f(x)](t)$ exists for all t and $\lim_{x\to\pm\infty} f(x) = 0$.

Then, for any $R_1 < 0 < R_2$, integrating by parts, we find

$$\int_{R_1}^{R_2} f'(x)e^{itx}dx = \left[f(x)e^{itx}\right]_{R_1}^{R_2} - it\int_{R_1}^{R_2} f(x)e^{itx}dx.$$

Letting $R_1 \to -\infty$ and $R_2 \to \infty$, by the hypotheses, we get

$$\mathcal{F}[f'(x)](t) = \int_{-\infty}^{\infty} f'(x)e^{itx}dx = 0 - it\int_{-\infty}^{\infty} f(x)e^{itx}dx = -it\mathcal{F}[f(x)](t).$$

So, under the stated conditions, we have the **rule for the Fourier transform of the derivative of a function**

$$\mathcal{F}[f'] = -it\mathcal{F}[f].$$

Remark: In view of the **previous Example**, **1.7.38**, the continuity of the function $f(x)$ in this **rule** is necessary.

There, $f'(x) = 0$, $\forall\, x \neq \pm a$, and so $\mathcal{F}[f'(x)](t) = 0$. But, $-it\mathcal{F}[f(x)](t) = -i\sqrt{\frac{2}{\pi}}\sin(at) \neq 0$.

In the above proof, the continuity of $f(x)$ is needed for computing the part $\left[f(x)e^{itx}\right]_{R_1}^{R_2}$, correctly, as we have explained in **Section I 1.1** and its problems. That is the problem with **Example 1.7.38**.

(2) If $f(x)$ is **real**, then $\hat{f}(-t) = \overline{\hat{f}(t)}$.

We have

$$\mathcal{F}[f(x)](t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx =$$

$$\int_{-\infty}^{\infty} f(x)\cos(tx)dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)du,$$

from which we get

$$\mathcal{F}[f(x)](-t) = \int_{-\infty}^{\infty} f(x)e^{-itx}dx =$$

$$= \int_{-\infty}^{\infty} f(x)\cos(tx)dx - i\int_{-\infty}^{\infty} f(x)\sin(tx)du,$$

and

$$\overline{\mathcal{F}[f(x)](t)} = \overline{\int_{-\infty}^{\infty} f(x)e^{itx}dx} =$$
$$\overline{\int_{-\infty}^{\infty} f(x)\cos(tx)dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)du} =$$
$$\int_{-\infty}^{\infty} f(x)\cos(tx)dx - i\int_{-\infty}^{\infty} f(x)\sin(tx)du,$$

that is, the last two results are equal.

We also observe:

$$\text{(a)} \quad \mathrm{Re}\{\mathcal{F}[f(x)](-t)\} = \mathrm{Re}\{\mathcal{F}[f(x)](t)\},$$

and

$$\text{(b)} \quad \mathrm{Im}\{\mathcal{F}[f(x)](-t)\} = -\mathrm{Im}\{\mathcal{F}[f(x)](t)\}.$$

Hence, when $f(x)$ is **real**, the **real part** of $\mathcal{F}[f(x)](t)$ is an **even** function of t and the **imaginary part** of it is an **odd** function of t.

(3) We can compute the derivative of the Fourier transform by differentiating under the integral sign, as long as the differentiability conditions of **Theorem I 2.2.1** are met. So,

$$\frac{d}{dt}\{\mathcal{F}[f(x)]\}(t) = \int_{-\infty}^{\infty} f(x)\frac{d}{dt}\left(e^{itx}\right)dx =$$
$$\int_{-\infty}^{\infty} ixf(x)e^{itx}dx = \mathcal{F}[ixf(x)](t),$$

provided that $xf(x)$ is absolutely integrable.

So, provided that $xf(x)$ is absolutely integrable, we have the **derivative of the Fourier transform rule**

$$\frac{d}{dt}\mathcal{F}[f(x)](t) = \mathcal{F}[ixf(x)](t) = i\mathcal{F}[xf(x)](t).$$

(4) The **scaling and shifting rule** is the following:

For any $a \neq 0$ and b real constants, we let $u = ax - b$ and get

$$\mathcal{F}[f(ax-b)](t) = \int_{-\infty}^{\infty} f(ax-b)e^{itx}dx = \frac{1}{a}\int_{-\mathrm{sign}(a)\infty}^{\mathrm{sign}(a)\infty} f(u)e^{\frac{it(u+b)}{a}}du =$$
$$\frac{1}{|a|}\,e^{\frac{ibt}{a}}\int_{-\infty}^{\infty} f(u)e^{\frac{itu}{a}}du = \frac{1}{|a|}\,e^{\frac{ibt}{a}}\,\mathcal{F}[f(x)]\left(\frac{t}{a}\right).$$

So, for any $a \neq 0$ and b real constants, we have the scaling and shifting rule

$$\mathcal{F}[f(ax-b)](t) = \frac{1}{|a|}\, e^{\frac{ibt}{a}}\, \mathcal{F}[f(x)]\left(\frac{t}{a}\right).$$

If $|a| = 1$, then we have only shifting, and if $b = 0$, we have only scaling.

(5) For any real constant c, we have

$$\mathcal{F}\left[e^{icx}f(x)\right](t) =$$
$$\int_{-\infty}^{\infty} e^{icx} f(x) e^{itx} dx = \int_{-\infty}^{\infty} f(x) e^{i(t+c)x} dx = \mathcal{F}[f(x)](t+c).$$

So, for any real constant c, we have the **rule of the multiplication by** e^{icx}

$$\mathcal{F}\left[e^{icx}f(x)\right](t) = \mathcal{F}[f(x)](t+c).$$

(6) Rule (5) and the **linearity** of the Fourier transform imply the following two **rules of the multiplication by** $\cos(cx)$ **or** $\sin(cx)$**, where** c **is any real constant**:

$$\mathcal{F}\left[\cos(cx)f(x)\right](t) = \frac{1}{2}\mathcal{F}\left[e^{icx}f(x)\right](t) + \frac{1}{2}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$
$$\frac{1}{2}\left\{\mathcal{F}[f(x)](t+c) + \mathcal{F}[f(x)](t-c)\right\},$$

and

$$\mathcal{F}\left[\sin(cx)f(x)\right](t) = \frac{1}{2i}\mathcal{F}\left[e^{icx}f(x)\right](t) - \frac{1}{2i}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$
$$\frac{-i}{2}\left\{\mathcal{F}[f(x)](t+c) - \mathcal{F}[f(x)](t-c)\right\}.$$

(7) In the context of the Fourier transform, we define as the **convolution of two real absolutely or square integrable functions** $f(x)$ **and** $g(x)$ **in** $\mathbb{R}$ the integral denoted by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-u)g(u)du.$$

So, the convolution $(f * g)(x)$, at $x \in \mathbb{R}$, is the integral over the whole $\mathbb{R}$ {not in an interval $[0, x]$, as we did in the Laplace transform} of the translate $f(x-u) = f[-(u-x)]$ of $f(-u)$ at x with respect to $g(u)du$.

(In some books, this integral is multiplied by the factor $\dfrac{1}{\sqrt{2\pi}}$. See also and compare with the definition of **convolution in the context of**

the Laplace transform, defined in **Example I 2.7.4**. See also **Problem 1.7.111**.)

(a) If $f(x)$ and $g(x)$ are real absolutely integrable functions in $\mathbb{R}$, then $(f * g)(x)$ is absolutely integrable and, using the notation introduced in **project Problem I 2.6.65**, we have:

If $\|f\|_1 < \infty$ **and** $\|g\|_1 < \infty$**, then** $\|(f * g)\|_1 \leq \|f\|_1 \cdot \|g\|_1 < \infty$.

This is proven as follows. We notice that for any absolutely integrable function $h(x)$ in $\mathbb{R}$

$$\int_{-\infty}^{\infty} h(x-u)\,du \overset{x-u=v-x}{=} \int_{-\infty}^{\infty} h(v-x)\,dv \overset{v-x=y}{=} \int_{-\infty}^{\infty} h(y)\,dy.$$

Now, for the positive function $|f(x-u)g(u)|$ in $\mathbb{R}$, we have that the iterated integral

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|f(x-u)g(u)|dx\right]du = \int_{-\infty}^{\infty}|g(u)|du\int_{-\infty}^{\infty}|f(x-u)|dx = \int_{-\infty}^{\infty}|g(u)|du\int_{-\infty}^{\infty}|f(x)|dx$$

is, by hypothesis, a product of two non-negative finite numbers and so non-negative finite.

Then, by the **Tonelli conditions** and **Fubini conditions** (see **Section I 2.4**), $f(x-u)g(u)$ is absolutely integrable in $\mathbb{R}^2$, and then we obtain the inequality

$$\int_{-\infty}^{\infty}|(f*g)(x)|\,dx = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(x-u)g(u)\,du\right|\,dx \leq \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}|f(x-u)g(u)|du\right]dx = \int_{-\infty}^{\infty}|f(x-u)|dx\int_{-\infty}^{\infty}|g(u)|du = \int_{-\infty}^{\infty}|f(x)|dx\int_{-\infty}^{\infty}|g(u)|du,$$

which proves what we have claimed.

(b) If $f(x)$ and $g(x)$ are real square integrable functions in $\mathbb{R}$, then $(f * g)(x)$ is bounded.

Indeed, by **Hölder's Inequality** (see **project Problem I 2.6.65**), we get

$$\forall\ x \in \mathbb{R}, \qquad |(f*g)(x)| \leq \int_{-\infty}^{\infty}|f(x-u)g(u)|du \leq$$

$$\leq \left[\int_{-\infty}^{\infty} |f(x-u)|^2 du\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |g(u)|^2 du\right]^{\frac{1}{2}} =$$

$$\left[\int_{-\infty}^{\infty} |f(u)|^2 du\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |g(u)|^2 du\right]^{\frac{1}{2}} = \text{finite constant},$$

as product of two finite constants, by hypothesis.

Remark: A phenomenon analogous to the one described in **Remark** following **Condition IV** of **Section I 2.4** may occur with the convolution of two absolutely integrable functions. I.e., their convolution may not be defined (exists) at every $x \in \mathbb{R}$. In **(a)** above, we prove that such a convolution is absolutely integrable itself, but not that is defined at every point. This is rather plausible, since the existence of the convolution of two absolutely integrable functions is proved by using **Condition IV** of **Section I 2.4**. Again, in advanced analysis, we can prove that the set of these exceptional points has Lebesgue measure zero. (See **Definition I 2.3.5**.)

For non-negative functions, this situation means that the convolution at these exceptional points is $+\infty$, and so is unbounded. But, this cannot happen if both functions are square integrable, as we prove in **(b)** above. We illustrate this in the example that follows and **Problem 1.7.116**. Also examine **Problems 1.7.114-1.7.125**.

Example 1.7.39 We let

$$f(x) = g(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}}, & \text{if } x \in [-1,0) \cup (0,1], \\ \\ 0, & \text{otherwise.} \end{cases}$$

Since these functions are non-negative and even, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = 2\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2\left[2x^{\frac{1}{2}}\right]_0^1 = 4$$

and so the functions are absolutely integrable. Therefore,

$$\int_{-\infty}^{\infty} |f*g|(x)\,dx \leq \int_{-\infty}^{\infty} |f(x)|\,dx \cdot \int_{-\infty}^{\infty} |g(x)|\,dx = 4 \times 4 = 16.$$

Since

$$\int_{-\infty}^{\infty} f^2(x)\,dx = \int_{-\infty}^{\infty} g^2(x)\,dx = 2\int_0^1 \frac{1}{x}\,dx = 2\,[\ln(x)]_0^1 = \infty,$$

the functions are not square integrable.

Now,

$$(f*g)(0)=\int_{-\infty}^{\infty} f(0-u)g(u)\,du=\int_{-1}^{1}\frac{1}{|u|}\,du=2\int_{0}^{1}\frac{1}{u}\,du=\infty,$$

and so $(f*g)(x)$ is not bounded and cannot be defined continuously at $x=0$. (See also **Problem 1.7.114**.)

Here both functions f and g are discontinuous in the whole $\mathbb{R}$. This phenomenon can happen even if both f and g are continuous. See **Problem 1.7.116**.

▲

(c) The operation of **convolution is commutative**, i.e., $f*g=g*f$. This is easily seen by making the change of variables $x-u=v$ to obtain

$$\int_{-\infty}^{\infty} f(x-u)g(u)du=\int_{-\infty}^{\infty} f(v)g(x-v)dv.$$

(d) It is straightforward that for any real constant a

$$a(f*g)=(af)*g=f*(ag).$$

(e) The convolution is **linear** with respect to each function position. I.e., for any real constants a and b, we have the two linearity relations

$$(af_1+bf_2)*g=a(f_1*g)+b(f_2*g),$$

and

$$f*(ag_1+bg_2)=a(f*g_1)+b(f*g_2).$$

(f) The convolution is **associative**, i.e., $(f*g)*h=f*(g*h)$. The proof is similar to the proof in **Example I 2.7.5**.

(g) Using either of the definitions of the Fourier transform **(1.27)** or **(1.28)**, we can prove in either case the **convolution Rule**, i.e.: **The Fourier transform of the convolution of two functions is equal to the product of their Fourier transforms**. I.e.,

$$\mathcal{F}[(f*g)(x)](t)=\mathcal{F}[f(x)](t)\cdot\mathcal{F}[g(x)](t).$$

The proof is analogous to the proof with the Laplace transform, **Example I 2.7.4**. It is even easier because switching of the order of integration, permissible here by the absolutely or square integrability, is more direct. That is,

$$\mathcal{F}[f(x)](t)\cdot\mathcal{F}[g(x)](t)=\int_{-\infty}^{\infty} e^{itx}f(x)\,dx\cdot\int_{-\infty}^{\infty} e^{ity}g(y)\,dy=$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(x+y)}f(x)g(y)\,dxdy,$$

either as an improper integral that converges or as a principal value. We let $x+y=w$, and we find

$$\mathcal{F}[f(x)](t)\cdot\mathcal{F}[g(x)](t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{itw}f(w-y)g(y)\,dwdy=$$
$$=\int_{-\infty}^{\infty}e^{itw}\left[\int_{-\infty}^{\infty}f(w-y)g(y)\,dy\right]dw=\int_{-\infty}^{\infty}e^{itw}(f*g)(w)\,dw=$$
$$\mathcal{F}[(f*g)(x)](t).$$

I.e.,

$$\mathcal{F}[(f*g)(x)](t)=\mathcal{F}[f(x)](t)\cdot\mathcal{F}[g(x)](t),$$

and so

$$(f*g)(x)=\mathcal{F}^{-1}\left\{\mathcal{F}[f(x)](t)\cdot\mathcal{F}[g(x)](t)\right\}(x).$$

The last equation constitutes a way of computing the convolution of two functions. We can use both equalities in applications in either direction, that is, $A=B$ and $B=A$.

(h) Comparison with the convolution in the context of the Laplace transform.

In the context of the Laplace transform, see **Example I 2.7.4**, we use functions $f\ :\ [0,\infty)\longrightarrow\mathbb{R}$ and $g\ :\ [0,\infty)\longrightarrow\mathbb{R}$.

We extend both functions to the whole $\mathbb{R}$ by setting $f\equiv 0$ and $g\equiv 0$ on $(-\infty,0)$. Then for such functions we get:

$$(f*g)(x):=\int_{-\infty}^{\infty}f(x-t)g(t)\,dt=\int_{0}^{x}f(x-t)g(t)\,dt,$$

which is the definition of convolution as stated in **Example I 2.7.4**.

This is so because:
(1) $g_{|(-\infty,0)}\equiv 0$ and thus, only if $t\geq 0$ there is contribution to $(f*g)(x)$.
(2) If $t>x$ or equivalently $x-t<0$, then $f(x-t)=0$. So, as a function of t, $f(x-t)\equiv 0$ on the interval (x,∞).

We must also notice that in the context of the Laplace transform, for $f\ :\ [0,\infty)\longrightarrow\mathbb{R}$ and $g\ :\ [0,\infty)\longrightarrow\mathbb{R}$ continuous functions, we have:

$$f*g=0\iff f\equiv 0\ \text{ or }\ g\equiv 0,$$

by the **Titchmarsh Theorem of convolution**. [See **Example I 2.7.5, (g)**.]

This famous result on the convolution in the context of Laplace transform is not true for real functions defined on the whole $\mathbb{R}$ and the convolution defined in the context of the Fourier transform in **item (7) of Subsection 1.7.6**, as the following example indicates.

We let

$$f(x) = \begin{cases} \sin(x), & \text{if } 0 \le x \le 2\pi, \\ \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) \equiv 1, \quad \forall\ x \in \mathbb{R}.$$

Then, f and g are continuous and

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-t)g(t)\,dt = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt =$$

$$\int_0^{2\pi} \sin(t)\,dt = [-\cos(x)]_0^{2\pi} = -1 + 1 = 0,$$

but neither f nor g is the zero function.

Compare this with **Example I 2.7.5, (e)**. The essential difference, in this context, is that the limits of the convolution integral here are fixed $\pm\infty$. Also, notice that the function f is continuous, absolutely and square integrable in $\mathbb{R}$, but the function g is a nonzero constant and so neither absolutely nor square integrable in $\mathbb{R}$. The Fourier transform of f exists, but the Fourier transform of g does not exist even as a principal value. (See also **Problem 1.7.115**.) However, the convolution of the two functions, defined on the whole $\mathbb{R}$, exists and it is zero. (See also **Problem 1.7.127**.)

If we now define

$$h(x) = \begin{cases} 1, & \text{if} \quad 0 \le x \le 2\pi, \\ \\ 0, & \quad\text{otherwise,} \end{cases}$$

which is both absolutely and square integrable, then we find by direct computation (check)

$$(f * h)(x) = \begin{cases} 1 - \cos(x), & \text{if} \quad 0 \le x \le 2\pi, \\ \\ 0, & \quad\text{otherwise.} \end{cases}$$

So, $f * h \ne 0$. Given that both functions are zero in $(-\infty, 0)$ this agrees with the Titchmarsh Theorem of convolution.

Examples

Example 1.7.40 By **Problem 1.7.32, (a)**, we have

$$\mathcal{F}\left[e^{-x^2}\right](t) = \int_{-\infty}^{\infty} e^{-x^2} e^{itx} dx = \int_{-\infty}^{\infty} e^{-x^2} \cos(tx) dx + i \cdot 0 = \sqrt{\pi}\, e^{-\frac{t^2}{4}}.$$

Then, by the scaling and shifting **Rule (4)**, we obtain that for any real constants $a > 0$ and b, we have

$$\mathcal{F}\left[e^{-a(x-b)^2}\right](t) = \mathcal{F}\left[e^{-[\sqrt{a}(x-b)]^2}\right](t) =$$
$$\frac{\sqrt{\pi}}{\sqrt{a}} e^{\frac{itb\sqrt{a}}{\sqrt{a}}}\, e^{-\frac{t^2}{4a}} = \sqrt{\frac{\pi}{a}}\, e^{itb} e^{-\frac{t^2}{4a}}.$$

Also, by **Rule (6)**, we find that for any real constant c

$$\mathcal{F}\left[e^{-x^2}\sin(cx)\right](t) = \frac{-i}{2}\left[\hat{f}(t+c) - \hat{f}(t-c)\right] =$$
$$\frac{-i}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} - \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right] = i\sqrt{\pi}\, e^{-\frac{(t^2+c^2)}{4}} \sinh\left(\frac{ct}{2}\right),$$

and similarly

$$\mathcal{F}\left[e^{-x^2}\cos(cx)\right](t) = \frac{1}{2}\left[\hat{f}(t+c) + \hat{f}(t-c)\right] =$$
$$\frac{1}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} + \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right],$$

i.e.,

$$\mathcal{F}\left[e^{-x^2}\cos(cx)\right](t) = \sqrt{\pi}\, e^{-\frac{(t^2+c^2)}{4}} \cosh\left(\frac{ct}{2}\right).$$

▲

Example 1.7.41 We compute

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \int_{-\infty}^{\infty} \frac{e^{-|x|}}{2}\, e^{itx} dx =$$
$$\frac{1}{2}\int_{-\infty}^{0} e^{x(1+it)} dx + \frac{1}{2}\int_{0}^{\infty} e^{x(-1+it)} dx =$$
$$\frac{1}{2}\left(\frac{1}{1+it} - \frac{1}{-1+it}\right) = \frac{1}{1+t^2}.$$

Then, with $f(x) = \dfrac{e^{-|x|}}{2}$, by the **convolution Rule, (g)**, we have

$$\left(\frac{1}{1+t^2}\right)^2 = \mathcal{F}[(f*f)(x)](t),$$

and so

$$g(x) := \mathcal{F}^{-1}\left[\left(\frac{1}{1+t^2}\right)^2\right](x) = (f*f)(x).$$

Therefore, to find the inverse transform $g(x)$, we need to compute the $(f * f)(x)$. For $x \le 0$, we have

$$(f * f)(x) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-u|-|u|}du =$$
$$\frac{1}{4}\left(\int_{-\infty}^{x} e^{2u-x}du + \int_{x}^{0} e^{x}du + \int_{-\infty}^{x} e^{-2u+x}du\right) = \frac{1}{4}(1-x)\,e^{x}.$$

Similarly, for $x \ge 0$, we find

$$(f * f)(x) = \frac{1}{4}\int_{-\infty}^{\infty} e^{-|x-u|-|u|}du =$$
$$\frac{1}{4}\left(\int_{-\infty}^{0} e^{2u-x}du + \int_{0}^{x} e^{-x}du + \int_{x}^{\infty} e^{-2u+x}du\right) = \frac{1}{4}(1+x)\,e^{-x}.$$

The two parts can be put together as one formula:

$$\forall\ \ x \in \mathbb{R}, \quad g(x) := \mathcal{F}^{-1}\left[\left(\frac{1}{1+t^2}\right)^2\right](x) = \frac{1}{4}(1+|x|)\,e^{-|x|}.$$

Now, we combine the result of the Fourier transforms here and of **Example 1.7.38**, the **inversion Theorem, 1.7.6**, and the **general Parseval equation (1.33)** [or **(1.34)**] to derive the known result in **Examples I 2.2.8, I 2.2.9, I 2.3.11,**, and **Problems I 2.4.11, I 2.5.23.**. We have

$$\mathcal{F}\left(\frac{e^{-|x|}}{2}\right)(t) = \frac{1}{1+t^2}.$$

If for any $a > 0$ constant, we let

$$f(x) = \begin{cases} 1, & \text{if} \quad |x| < a, \\ \\ 0, & \text{if} \quad |x| \ge a, \end{cases}$$

and then $\quad \hat{f}(t) = \mathcal{F}[f(x)](t) = 2\,\dfrac{\sin(at)}{t}, \quad$ not absolutely integrable.

Then, **inversion Theorem, 1.7.6, (a),** applies to get

$$\mathcal{F}[\hat{f}(x)](t) = \mathcal{F}\left[2\frac{\sin(ax)}{x}\right](t) = 2\pi f(t).$$

Also, by the **general Parseval equation (1.33)**, we obtain

$$2\pi\int_{-\infty}^{\infty}\frac{e^{-|x|}}{2}2\frac{\sin(ax)}{x}\,dx = \int_{-\infty}^{\infty}\frac{1}{1+t^2}2\pi f(t)\,dt = 2\pi\int_{-a}^{a}\frac{1}{1+t^2}\,dt$$
$$= 2\pi \cdot 2\arctan(a) = 4\pi \cdot \arctan(a).$$

Therefore, for any $a > 0$, we have

$$\int_{-\infty}^{\infty} e^{-|x|}\,\frac{\sin(ax)}{x}\,dx = 2\arctan(a),$$

and so

$$\int_{0}^{\infty} e^{-x}\,\frac{\sin(ax)}{x}\,dx = \arctan(a).$$

(See also **Examples I 2.2.8, I 2.2.9, I 2.3.11** and **Problems I 2.4.11, I 2.5.23.**.)

▲

Example 1.7.42 We want to find the Fourier transform of the continuous triangular function

$$g(x) = \begin{cases} 1 + x, & \text{if } -1 \le x \le 0, \\ 1 - x, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } |x| \le 1. \end{cases}$$

We avoid doing it directly (i.e.,by definition). We do it easier by using the derivative and **Rule (1)**, instead. So, we see that

$$g'(x) = \begin{cases} 1, & \text{if } -1 < x < 0, \\ -1, & \text{if } 0 < x < 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Then,

$$\mathcal{F}\left[g'(x)\right](t) = \int_{-\infty}^{\infty} g'(x)e^{itx}\,dx = \int_{-1}^{1} g'(x)e^{itx}\,dx =$$
$$\int_{-1}^{0} e^{itx}\,dx - \int_{0}^{1} e^{itx}\,dx = \frac{-2i[1-\cos(t)]}{t}.$$

Then, by **Rule (1)** we have

$$\frac{-2i[1-\cos(t)]}{t} = -it\mathcal{F}[g(x)](t)$$

and finally

$$\mathcal{F}[g(x)](t) = \frac{2[1-\cos(t)]}{t^2}.$$

Then, by **Parseval equation (1.31)** we get

$$\int_{-\infty}^{\infty} \left\{ \frac{2[1-\cos(t)]}{t^2} \right\}^2 dt = 2\pi \int_{-\infty}^{\infty} f^2(x)dx =$$
$$2\pi \left[\int_{-1}^{0} (1+x)^2\, dx + \int_{0}^{1} (1-x)^2\, dx \right] = 2\pi \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{4\pi}{3},$$

and so

$$\int_{-\infty}^{\infty} \frac{[1-\cos(t)]^2}{t^4}\, dt = 2 \int_{-\infty}^{0} \frac{[1-\cos(t)]^2}{t^4}\, dt =$$
$$2 \int_{0}^{\infty} \frac{[1-\cos(t)]^2}{t^4}\, dt = \frac{\pi}{3}.$$

Then, using $\cos(t) = 1 - 2\sin^2\left(\frac{t}{2}\right)$ and $x = \frac{t}{2}$, we find

$$\int_{0}^{\infty} \frac{\sin^4(x)}{x^4}\, dx = \int_{-\infty}^{0} \frac{\sin^4(x)}{x^4}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^4(x)}{x^4}\, dx = \frac{\pi}{3}.$$

(See also **Problem I 2.2.38** and its **hint**.)

▲

Example 1.7.43 By **Example 1.7.41**, we have

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \frac{1}{1+t^2}.$$

So, for the function

$$f(x) = \frac{1}{x^2+6x+10} = \frac{1}{(x+3)^2+1},$$

we apply the **inversion Theorem, 1.7.6, (b)**, or **Equations 1.35, 1.36**, and **Rule (4)**, with $a = 1$ and $b = -3$, [or use **Example 1.7.33** and **Rule (4)**] to find

$$\mathcal{F}[f(x)](t) = 2\,\pi e^{-3it} \frac{e^{-|t|}}{2} = \pi e^{-|t|-3it}.$$

If $a \neq 0$, $b \in \mathbb{R}$ and $f(x) = \frac{1}{(ax+b)^2+1}$, we find

$$\mathcal{F}[f(x)](t) = \frac{\pi}{|a|}\, e^{-\left|\frac{x}{a}\right| - \frac{bit}{a}}.$$

▲

Example 1.7.44 We let $f(x) = \displaystyle\int_{-\infty}^{x} e^{-(x-u)} g(u)du$, and we would like to find the relation of the Fourier transform of $f(x)$ with the Fourier transform of $g(x)$. We compute

$$\mathcal{F}\left[f(x)\right](t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{x} e^{-(x-u)} g(u)du\right] e^{itx} dx =$$
$$\int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{-(x-u)} g(u) e^{itx} dudx = \int_{-\infty}^{\infty} g(u) e^{u} \left[\int_{u}^{\infty} e^{(-1+it)} dx\right] du =$$
$$\int_{-\infty}^{\infty} g(u) e^{u} \left[\frac{e^{(-1+it)}}{-1+it}\right]_{u}^{\infty} du = \frac{1}{1-it} \int_{-\infty}^{\infty} g(u) e^{itu} du.$$

That is, we have the relation $\mathcal{F}[f(x)](t) = \dfrac{1}{1-it}\,\mathcal{F}[g(x)](t)$.

▲

Remark: As we did with the convolution in **Application 4 of Subsection I 2.7.2**, we can compute the convolution of two functions on $(-\infty, \infty)$ by using the Fourier and inverse Fourier transforms.

1.7.7 Applications

Application 1: Here, we give an application of the Fourier transform to an initial boundary value problem with the partial differential equation of the homogeneous wave equation. Namely, we want to solve the Cauchy problem for the wave equation:

$$\begin{cases} u_{yy}(x,y) - u_{xx}(x,y) = 0, & \text{for} \quad y > 0, \\[2ex] u(x,0) = 0, & \text{for} \quad -\infty < x < \infty, \\[2ex] u_y(x,0) = \begin{cases} 1, & \text{if} \quad |x| < a, \\[2ex] 0, & \text{if} \quad |x| \geq a. \end{cases} & \text{where } \ a > 0 \ \text{ constant,} \end{cases}$$

We consider the Fourier transform of $u(x,y)$ with respect to x. That is,

$$\tilde{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{ix\xi} dx.$$

Then, by taking the Fourier transform with respect to x of the given problem, we obtain the initial value problem (see also **Example 1.7.38**)

$$\begin{cases} \tilde{u}_{yy}(\xi,y)+\xi^2\tilde{u}(\xi,y)=0, \qquad \text{for} \quad y>0, \\ \tilde{u}(\xi,0)=0, \\ \tilde{u}_y(\xi,0)=\sqrt{\dfrac{2}{\pi}}\cdot\dfrac{\sin(a\xi)}{\xi}. \end{cases}$$

The general solution of the differential equation

$$\tilde{u}_{yy}(\xi,y)+\xi^2\tilde{u}(\xi,y)=0 \quad \text{is} \quad \tilde{u}(\xi,y)=A\cos(\xi y)+B\sin(\xi y),$$

where A and B are arbitrary constants. These constants are determined by the two initial conditions. From $\tilde{u}(\xi,0)=0$, we find $A=0$, and from $\tilde{u}_y(\xi,0)=\sqrt{\frac{2}{\pi}}\cdot\frac{\sin(a\xi)}{\xi}$, we find $B=\sqrt{\frac{2}{\pi}}\cdot\frac{\sin(a\xi)}{\xi^2}$. So, the solution of the differential equation that also satisfies the initial conditions is

$$\tilde{u}(\xi,y)=\sqrt{\frac{2}{\pi}}\cdot\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}.$$

The $\tilde{u}(\xi,y)$ is absolutely integrable in $(-\infty,\infty)$, and by the **inversion Theorem, 1.7.6**, we find

$$u(x,y)=$$
$$\frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}e^{-i\xi x}d\xi=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}e^{-i\xi x}d\xi=$$
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}[\cos(\xi x)-i\sin(\xi x)]\,d\xi.$$

This integral is real. Its imaginary part is zero as the integral of an absolutely integrable odd function in $(-\infty,\infty)$. Its real part is the integral of an absolutely integrable even function in $(-\infty,\infty)$. Then, we use some elementary trigonometric formulae and find

$$u(x,y)=$$
$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}\cos(\xi x)d\xi=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin(a\xi)\sin(\xi y)}{\xi^2}\cos(\xi x)\,d\xi=$$
$$\frac{1}{\pi}\int_0^{\infty}\left\{\frac{\cos(x\xi)\cos[(a-y)\xi]}{\xi^2}-\frac{\cos(x\xi)\cos[(a+y)\xi]}{\xi^2}\right\}d\xi=$$
$$\frac{1}{2\pi}\int_0^{\infty}\{\cos[(a+x-y)\xi]+\cos[(a-x-y)\xi]-$$
$$\cos[(a+x+y)\xi]-\cos[(a-x+y)\xi]\}\,\frac{d\xi}{\xi^2}.$$

We use integration by parts and the lower limit process to find

$$u(x,y) =$$

$$\frac{1}{2\pi}\lim_{\epsilon\to 0^+}\int_\epsilon^\infty \{\cos[(a+x-y)\xi]+\cos[(a-x-y)\xi]-$$

$$\cos[(a+x+y)\xi]-\cos[(a-x+y)\xi]\}\, d\left(\frac{-1}{\xi}\right) =$$

$$\frac{1}{2\pi}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\{\cos[(a+x-y)\epsilon]+\cos[(a-x-y)\epsilon]-$$

$$\cos[(a+x+y)\epsilon]-\cos[(a-x+y)\epsilon]\}+$$

$$\frac{1}{2\pi}\lim_{\epsilon\to 0^+}\left\{-(a+x-y)\int_\epsilon^\infty \frac{\sin[(a+x-y)\xi]}{\xi}d\xi-\right.$$

$$(a-x-y)\int_\epsilon^\infty \frac{\sin[(a-x-y)\xi]}{\xi}d\xi+$$

$$(a+x+y)\int_\epsilon^\infty \frac{\sin[(a+x+y)\xi]}{\xi}d\xi+$$

$$\left.(a-x+y)\int_\epsilon^\infty \frac{\sin[(a-x+y)\xi]}{\xi}d\xi\right\}.$$

The first limit is zero (by L' Hôpital's rule), and so we find

$$u(x,y) =$$

$$\frac{1}{2\pi}\left\{-(a+x-y)\int_0^\infty \frac{\sin[(a+x-y)\xi]}{\xi}d\xi-\right.$$

$$(a-x-y)\int_0^\infty \frac{\sin[(a-x-y)\xi]}{\xi}d\xi+$$

$$(a+x+y)\int_0^\infty \frac{\sin[(a+x+y)\xi]}{\xi}d\xi+$$

$$\left.(a-x+y)\int_0^\infty \frac{\sin[(a-x+y)\xi]}{\xi}d\xi\right\}.$$

But, from **Example I 2.2.10**, we have the **result**

$$\forall\ \beta\in\mathbb{R}, \qquad \frac{2}{\pi}\,\beta\int_0^\infty \frac{\sin(\beta\xi)}{\xi}\,dx = |\beta|.$$

Therefore, the final solution is written as

$$u(x,y) = \frac{1}{4}\left[\,|a+x+y|+|a-x+y|-|a+x-y|-|a-x-y|\,\right].$$

See **Figure 1.17** for the behavior of this solution in the upper half plane.

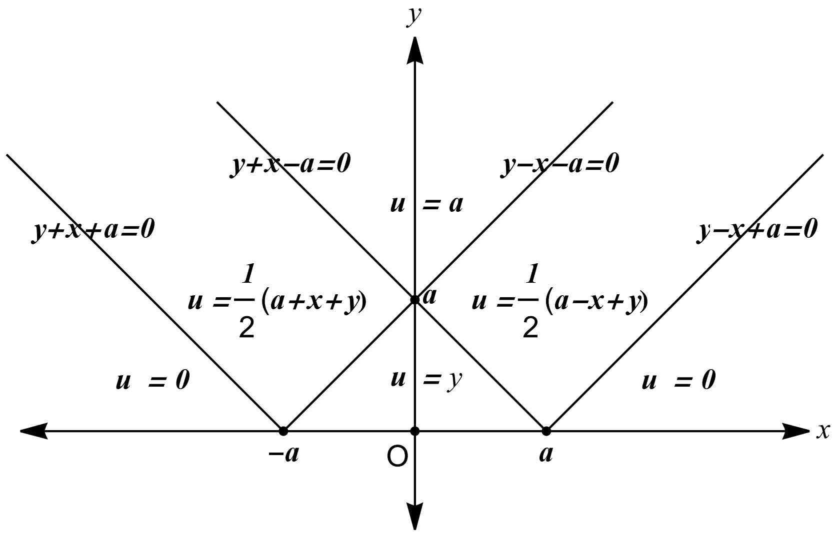

FIGURE 1.17: Sections of the solution in Application 1

Application 2: We consider the ordinary differential equation of the unknown function $u = u(x)$

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu = 0, \tag{1.39}$$

for which $x = 0$ is a regular singular point. This is **Bessel's differential equation with imaginary argument of order** 0.[38].

By taking the Fourier transform of the equation and using its linearity properties, we find

$$\mathcal{F}\left[x\frac{d^2u}{dx^2}\right](t) + \mathcal{F}\left[\frac{du}{dx}\right](t) - \mathcal{F}[x\,u(x)](t) = 0.$$

We assume that $u(x)$ is absolutely integrable, and for convenience we write $\mathcal{F}[u(x)](t) = \hat{u}(t)$. Then, we use the applicable properties from **Subsection 1.7.6** to obtain

$$\frac{1}{i}\frac{d}{dt}[-t^2\hat{u}(t)] - it\hat{u}(t) - \frac{1}{i}\frac{d}{dt}[\hat{u}(t)] = 0,$$

[38]**Bessel's differential equation with imaginary argument of order** $\nu \geq 0$ is $\dfrac{d^2u}{dz^2} + \dfrac{1}{z}\dfrac{du}{dz} - \left(1 + \dfrac{\nu^2}{z^2}\right)u = 0$. Its solutions are called **the Bessel functions with imaginary argument of order** ν. They are referred to by other names, too. With $\nu = 0$, we obtain the equation we have considered above.

from which we get the first-order, linear, homogeneous, ordinary, differential equation for $\hat{u}(t)$

$$(1+t^2)\hat{u}'(t) + t\hat{u}(t) = 0.$$

We solve this differential equation, and we find

$$\hat{u}(t) = c\frac{1}{\sqrt{t^2+1}},$$

where c is an arbitrary constant.

The $\hat{u}(t)$ found is square integrable, and by the **inversion Theorem, 1.7.6**, we get

$$u(x) = \frac{c}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-itx}}{\sqrt{t^2+1}}\, dt =$$
$$a \cdot \left[\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(tx)}{\sqrt{t^2+1}}\, dt - i\ \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(tx)}{\sqrt{t^2+1}}\, dt\right],$$

where a is an arbitrary constant.

The second principal value is zero since the integrand function is odd. Making the substitution $t = \sinh(\xi)$, we find

$$u(x) = a \cdot \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(tx)}{\sqrt{t^2+1}}\, dt = a \cdot \text{P.V.} \int_{-\infty}^{\infty} \cos[x\sinh(\xi)] d\xi.$$

This solution is in integral form. As $x = 0$ is a singular point for the differential equation, we notice that for the solution found $u(0) = \infty$. Apart from $x = 0$, the principal value found converges conditionally. (See **Problem I 1.3.17**.) The convergence is slow.

We can check that this solution satisfies the given differential equation. Indeed,

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu(x) = a \cdot\ \text{P.V.} \left\{\int_{-\infty}^{\infty} -x\cos[x\sinh(\xi)]\sinh^2(\xi)d\xi + \right.$$
$$\left. \int_{-\infty}^{\infty} -\sin[x\sinh(\xi)]\sinh(\xi)d\xi - \int_{-\infty}^{\infty} x\cos[x\sinh(\xi)]d\xi \right\} =$$
$$a \cdot \lim_{R\to\infty} \int_{-R}^{R} \frac{d}{d\xi}\left\{-\sin[x\sinh(\xi)]\cosh(\xi)\right\} d\xi =$$
$$a \lim_{R\to\infty} \left[-\sin[x\sinh(\xi)]\cosh(\xi)\,\right]_{-R}^{R} = a \lim_{R\to\infty} 0 = 0,$$
$$(\text{since } \sin[x\sinh(\xi)]\cosh(\xi) \text{ is an odd function in } \xi).$$

So, regardless of how many hypotheses were satisfied and how many were violated while taking and manipulating the Fourier transforms and applying an **inversion Theorem**, we reach an answer that in the end is a solution to the given differential equation. This tactic is frequently used when we solve applied problems of differential equations. We use the Fourier or Laplace transforms and the **inversion Theorems** without checking the hypotheses under which they apply, and in the end we check if the final answer solves the problem. If it does, we are finished.

Since $\cos[x\sinh(\xi)]$ is even, we can also write the solution as

$$u(x) = b\int_0^{\infty} \frac{\cos(tx)}{\sqrt{t^2+1}}\,dt = b\int_0^{\infty} \cos[x\sinh(\xi)]d\xi,$$

with b as an arbitrary constant. We also see that the solution $u(x)$ is an even function.

Application 3. The Cauchy problem for the one-dimensional **homogeneous** wave equation on an infinite line is solved by the D' Alembert formula. In an first course of partial differential equations, this formula is derived directly by making a change of variables and then using direct integration and the principle of superposition, etc. Here, we are going to derive it by demonstrating the power of the Fourier transform.

So, we consider the problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, & -\infty < x < \infty, \quad t > 0, \\ \\ u(x,0) = f(x), & -\infty < x < \infty, \\ \\ u_t(x,t) = g(x), & -\infty < x < \infty, \end{cases}$$

where $u = u(x,t)$ is of class $\mathfrak{C}^2\left[\mathbb{R}\times(0,\infty)\right]$ and continuous in $\mathbb{R}\times[0,\infty)$ and so the functions $f(x)$ and $g(x)$ must be of class $\mathfrak{C}^2\left(\mathbb{R}\right)$.

We temporarily assume that $u(x,t)$, $f(x)$ and $g(x)$ satisfy the conditions necessary for their Fourier transform to exist, the conditions of the **Inversion Theorem**, **1.7.6**, and their limits as $x \to \pm\infty$ are zero.

We consider the Fourier transform **(1.29)** or **(1.30)**, with respect to variable x,

$$\tilde{u}(\lambda,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{i\lambda x}dx.$$

Assuming that all the operations performed below are legitimate, using integration by parts, we compute

$$\tilde{u}_{tt}(\lambda,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_{tt}(x,t)e^{i\lambda x}dx =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c^2 u_{xx}(x,t)e^{i\lambda x}dx = \frac{c^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x}du_x(x,t) =$$
$$\frac{c^2}{\sqrt{2\pi}}\left\{\left[e^{i\lambda x}u_x(x,t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\lambda u_x(x,t)e^{i\lambda x}dx\right\} =$$
$$(0-0) - \frac{c^2}{\sqrt{2\pi}}(i\lambda)\int_{-\infty}^{\infty} u_x(x,t)e^{i\lambda x}dx = \ldots =$$
$$(0-0) + \frac{c^2}{\sqrt{2\pi}}(i\lambda)^2\int_{-\infty}^{\infty} u(x,t)e^{i\lambda x}dx = -c^2\lambda^2\tilde{u}(\lambda,t).$$

Therefore,

$$\tilde{u}_{tt}(\lambda,t) + c^2\lambda^2\tilde{u}(\lambda,t) = 0$$

and the initial conditions are translated as

$$\begin{cases} \tilde{u}(\lambda,0) = \tilde{f}(\lambda), & -\infty < \lambda < \infty, \\ \\ \tilde{u}_t(\lambda,t) = \tilde{g}(\lambda), & -\infty < \lambda < \infty. \end{cases}$$

Then, the solution of this initial-value-problem is of the form

$$\tilde{u}(\lambda,t) = A(\lambda)\cos(c\lambda t) + \mathrm{B}(\lambda)\sin(c\lambda t),$$

and so, $\tilde{u}(\lambda,0) = A(\lambda)$ and $\tilde{u}_t(\lambda,0) = c\lambda\mathrm{B}(\lambda)$.

Then, by the initial conditions, we straightly find that $A(\lambda) = \tilde{f}(\lambda)$ and $\mathrm{B}(\lambda) = \dfrac{\tilde{g}(\lambda)}{c\lambda}$. So, we obtain

$$\tilde{u}(\lambda,t) = \tilde{f}(\lambda)\cos(c\lambda t) + \frac{\tilde{g}(\lambda)}{c\lambda}\sin(c\lambda t).$$

Now, applying the **inversion Theorem**, **1.7.6**, we find

$$u(x,t) =$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\lambda)\cos(c\lambda t)e^{-i\lambda x}d\lambda + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{g}(\lambda)}{c\lambda}\sin(c\lambda t)e^{-i\lambda x}d\lambda =$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\lambda)\frac{e^{ic\lambda t}+e^{-ic\lambda t}}{2}e^{-i\lambda x}d\lambda +$$
$$\frac{1}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{g}(\lambda)}{\lambda}\frac{e^{ic\lambda t}-e^{-ic\lambda t}}{2i}e^{-i\lambda x}d\lambda =$$
$$\frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\lambda)\left[e^{i\lambda(x-ct)}+e^{i\lambda(x+ct)}\right]d\lambda +$$
$$\frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\lambda)\frac{e^{ic\lambda t}-e^{-ic\lambda t}}{\lambda i}e^{-i\lambda x}d\lambda =$$
$$\frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\lambda)\frac{e^{-i\lambda x}\left(e^{ic\lambda t}-e^{-ic\lambda t}\right)}{\lambda i}d\lambda.$$

We work with the last integral and use the **inversion Theorem**, **1.7.6**, to find

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\lambda)\frac{e^{-i\lambda x}\left(e^{ic\lambda t}-e^{-ic\lambda t}\right)}{\lambda i}\,d\lambda =$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\lambda)\left(\int_{x-ct}^{x+ct}e^{-i\lambda\bar{x}}d\bar{x}\right)d\lambda =$$
$$\int_{x-ct}^{x+ct}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\lambda)e^{-i\lambda\bar{x}}d\lambda\right]d\bar{x} = \int_{x-ct}^{x+ct}g(\bar{x})d\bar{x}.$$

Finally, the solution of the above problem is

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}g(\bar{x})d\bar{x},$$

which is exactly the **D' Alembert formula** of the solution.

For the **non-homogeneous** problem

$$\begin{cases} u_{tt}(x,t) - c^2u_{xx}(x,t) = F(x,t), & -\infty < x < \infty, \quad t > 0, \\ \\ u(x,0) = f(x), & -\infty < x < \infty, \\ \\ u_t(x,t) = g(x), & -\infty < x < \infty, \end{cases}$$

where $F(x,t)$ is a continuous function, the **D' Alembert formula** becomes

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}g(\bar{x})d\bar{x} +$$
$$\frac{1}{2c}\int_0^t\int_{x-c(t-\bar{t})}^{x+c(t-\bar{t})}F(\bar{x},\bar{t})d\bar{x}d\bar{t}.$$

Remark: Regardless of the necessary conditions for the Fourier transforms involved to exist and apply the inversion Theorem, we can directly check that the D' Alembert solutions of the above homogeneous and non-homogeneous problems are indeed the solutions of these problems, under the only assumptions that f is of class $\mathfrak{C}^2(\mathbb{R})$, $g(x)$ of class $\mathfrak{C}^1(\mathbb{R})$ (not $\mathfrak{C}^2$ now) and F is $\mathfrak{C}[R\times[0,\infty)]$. A lot of times in applied problems and differential equations, we use this tactic. We assume that everything is fine to take the Fourier transform and apply inversion in order to find a solution. Then, we check if this solution satisfies the initial problem under its relaxed minimum hypotheses. If this happens, we have solved the problem without worrying about how we have gotten its solution by using the Fourier transform. Hence, the Fourier transform is used as a special tool to get to the solution of the problem. If this goal is achieved, we prefer to check if the solution found with the help of the Fourier transform and inversion is good under the minimal assumptions of the given problem instead of stipulating the conditions under which we can compute Fourier transform and apply inversion.

Application 4. We want to solve the Cauchy problem for the **three-dimensional homogeneous wave equation** and derive the famous and multiply discussed **Kirchhoff's**[39] **formula**. This can be achieved by using the three-dimensional Fourier transform.

So, we consider the following initial value problem for the three dimensional homogeneous wave equation

$$\textbf{(I)}\begin{cases} u_{xx}+u_{yy}+u_{zz}-\dfrac{1}{c^2}u_{tt}=0, & (x,y,z)\in\mathbb{R}, \quad t>0, \\ u(x,y,z,0)=\phi(x,y,z), & (x,y,z)\in\mathbb{R}, \\ u_t(x,y,z,0)=\psi(x,y,z), & (x,y,z)\in\mathbb{R}, \end{cases}$$

where the function $u = u(x,y,z,t)$ is of class $\mathfrak{C}^2\left(\mathbb{R}^3\times\mathbb{R}^+\right)$ and $\mathfrak{C}\left(\mathbb{R}^3\times\mathbb{R}_0^+\right)$ and so the functions $\phi(x,y,z)$ and $\psi(x,y,z)$ must be of class $\mathfrak{C}^2\left(\mathbb{R}^3\right)$.

To solve this problem, we solve separately the following two problems:

$$\textbf{(II)}\begin{cases} u_{xx}+u_{yy}+u_{zz}-\dfrac{1}{c^2}u_{tt}=0, & (x,y,z)\in\mathbb{R}, \quad t>0, \\ u(x,y,z,0)=0, & (x,y,z)\in\mathbb{R}, \\ u_t(x,y,z,0)=\psi(x,y,z), & (x,y,z)\in\mathbb{R}, \end{cases}$$

and

$$\textbf{(III)}\begin{cases} u_{xx}+u_{yy}+u_{zz}-\dfrac{1}{c^2}u_{tt}=0, & (x,y,z)\in\mathbb{R}, \quad t>0, \\ u(x,y,z,0)=\phi(x,y,z), & (x,y,z)\in\mathbb{R}, \\ u_t(x,y,z,0)=0, & (x,y,z)\in\mathbb{R}. \end{cases}$$

[39]Gustav Robert Kirchhoff, German physicist and mathematician, 1824-1887.

By linearity and homogeneity the sum of the solutions of **Problems (II)** and **(III)** is the solution of **Problem (I)** (principle of superposition). In fact the solution of **Problem (II)** is given by the so called **Kirchhoff's formula**, even though this formula was first derived by **Poisson**. As we shall see, manipulating this solution we also find the solution of **Problem (III)**.

We consider the three-dimensional Fourier transform

$$\tilde{u}(\xi,\eta,\zeta,t) := \frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,z,t)e^{i(x\xi+y\eta+z\zeta)}dxdydz,$$

with corresponding inversion formula

$$u(x,y,z,t) = \frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{u}(\xi,\eta,\zeta,t)e^{-i(x\xi+y\eta+z\zeta)}d\xi d\eta d\zeta.$$

We apply the Fourier transform to **Problem (II)** and use its properties to find

$$\begin{cases} \rho^2\tilde{u}(\xi,\eta,\zeta,t) + \dfrac{1}{c^2}\tilde{u}_{tt}(\xi,\eta,\zeta,t) = 0, & (\xi,\eta,\zeta)\in\mathbb{R}^3, \quad t>0, \\ \tilde{u}(\xi,\eta,\zeta,0) = 0, & (\xi,\eta,\zeta)\in\mathbb{R}^3, \\ \tilde{u}_t(\xi,\eta,\zeta,0) = \tilde{\psi}(\xi,\eta,\zeta), & (\xi,\eta,\zeta)\in\mathbb{R}^3, \end{cases}$$

where $\rho^2 = \xi^2+\eta^2+\zeta^2$.

Then

$$\tilde{u}(\xi,\eta,\zeta,t) = A(\xi,\eta,\zeta)\cos(c\rho t) + \mathrm{B}(\xi,\eta,\zeta)\sin(c\rho t).$$

By the two initial conditions of the problem, we find that $A(\xi,\eta,\zeta)=0$ and

$$\mathrm{B}(\xi,\eta,\zeta) = \frac{\tilde{\psi}(\xi,\eta,\zeta)}{c\rho} =$$
$$\frac{1}{\left(\sqrt{2\pi}\right)^3}\frac{1}{c\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x',y',z')e^{i\left(x'\xi+y'\eta+z'\zeta\right)}dx'dy'dz'.$$

So,

$$\tilde{u}(\xi,\eta,\zeta,t) =$$
$$\frac{1}{\left(\sqrt{2\pi}\right)^3}\frac{\sin(c\rho t)}{c\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x',y',z')e^{i(x\xi'+y\eta'+z\zeta')}dx'dy'dz'.$$

Then, by the corresponding inversion formula we get

$$u(x,y,z,t) =$$
$$\frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(c\rho t)}{c\rho}\times$$
$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x',y',z')e^{i(x'\xi+y'\eta+z'\zeta)}dx'dy'dz'\right]\times$$
$$e^{-i(x\xi+y\eta+z\zeta)}d\xi d\eta d\zeta = \frac{1}{\left(\sqrt{2\pi}\right)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(c\rho t)}{c\rho}\times$$
$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x',y',z')e^{-i[(x-x')\xi+(y-y')\eta+(z-z')\zeta]}dx'dy'dz'\right]d\xi d\eta d\zeta.$$

We let

$$v(x,y,z,t) =$$
$$\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(c\rho t)}{\rho^2}\mathrm{B}(\xi,\eta,\zeta)e^{-i(x\xi+y\eta+z\zeta)}d\xi d\eta d\zeta =$$

$$= \frac{1}{c(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(c\rho t)}{\rho^3}\times$$
$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x',y',z')e^{-i[(x-x')\xi+(y-y')\eta+(z-z')\zeta]}dx'dy'dz'\right]d\xi d\eta d\zeta$$

and for given point (x,y,z), we let

$$I := I_{(x,y,z)}(x',y',z') =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(c\rho t)}{\rho^3}\,e^{-i[(x-x')\xi+(y-y')\eta+(z-z')\zeta]}d\xi d\eta d\zeta.$$

Hence,

$$v(x,y,z,t) = \frac{-1}{c(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\psi(x',y',z')\cdot I]\,dx'dy'dz'$$

and

$$u = \frac{1}{c^2}v_{tt}.$$

We put $r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, i.e., the distance of the points (x,y,z) and (x',y',z'). We also introduce spherical coordinates (r,θ,ϕ) in the (ξ,η,ζ)-space, with north pole in the direction of $(x-x',y-y',z-z')$. Since $\rho^2 = \xi^2+\eta^2+\zeta^2$ is rotation invariant, we have

$$\begin{cases} \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2}, \\ \xi = \rho \sin(\theta) \cos(\phi), \\ \eta = \rho \sin(\theta) \sin(\phi), \\ \zeta = \rho \cos(\theta), \\ (x - x')\xi + (y - y')\eta + (z - z')\zeta = \rho r \cos(\theta), \end{cases}$$

and

$$I = \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{\sin(c\rho t)}{\rho^3} e^{-i\rho r \cos(\theta)} \rho^2 \sin(\theta) d\theta d\phi d\rho =$$

$$2\pi \int_0^\infty \int_0^\pi \frac{\sin(c\rho t)}{\rho^3} e^{-i\rho r \cos(\theta)} \rho^2 \sin(\theta) d\theta d\rho.$$

But,

$$\int_0^\pi e^{-i\rho r \cos(\theta)} \sin(\theta) d\theta = -\frac{e^{i\rho r} - e^{-i\rho r}}{i\rho r} = -\frac{2}{\rho r} \sin(\rho r).$$

Thus,

$$I = \frac{-4\pi}{r} \int_0^\infty \frac{\sin(c\rho t)\sin(\rho r)}{\rho^2} d\rho =$$
$$\frac{2\pi}{r} \int_0^\infty \{\cos[\rho(ct - r)] - \cos[\rho(ct + r)]\}\, d\left(\frac{1}{\rho}\right) =$$
$$\frac{2\pi}{r} \lim_{\epsilon \to \infty} \left[\frac{\cos[\rho(ct - r)]}{\rho} - \frac{\cos[\rho(ct + r)]}{\rho}\right]_\epsilon^\infty +$$
$$\frac{2\pi}{r}(ct - r) \lim_{\epsilon \to \infty} \int_\epsilon^\infty \frac{\sin[\rho(ct - r)]}{\rho} d\rho -$$
$$\frac{2\pi}{r}(ct + r) \lim_{\epsilon \to \infty} \int_\epsilon^\infty \frac{\sin[\rho(ct + r)]}{\rho} d\rho =$$
$$[0, \text{ (the limit is zero by L' Hôpital's rule)}] +$$
$$\frac{2\pi}{r}(ct - r) \int_0^\infty \frac{\sin[\rho(ct - r)]}{\rho} d\rho - \frac{2\pi}{r}(ct + r) \int_0^\infty \frac{\sin[\rho(ct + r)]}{\rho} d\rho.$$

Then, by **Example I 2.2.10**, we obtain

$$I = -\frac{\pi^2}{r}\left(|ct+r| - |ct-r|\right) =$$

$$-\frac{\pi^2}{r}\left(ct+r-|ct-r|\right) = \begin{cases} -2\pi^2, & \text{if} \quad ct \geq r, \\ \\ -\dfrac{2\pi^2 ct}{r}, & \text{if} \quad ct \leq r. \end{cases}$$

Therefore,

$$v(x,y,z,t) =$$

$$\frac{-1}{4\pi c}\int\int\int\limits_{r\leq ct}\psi(x',y',z')dx'dy'dz' - \frac{t}{4\pi}\int\int\int\limits_{r\geq ct}\frac{\psi(x',y',z')}{r}dx'dy'dz'.$$

Now, we introduce spherical coordinates (r,θ,ϕ) in the (x',y',z')-space with origin at the point (x,y,z). So,

$$\begin{cases} r = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}, \\ \\ x'-x = r\sin(\theta)\cos(\phi), \\ \\ y'-y = r\sin(\theta)\sin(\phi), \\ \\ z'-z = r\cos(\theta). \end{cases}$$

Then, we have

$$v(x,y,z,t) =$$

$$\frac{-1}{4\pi c}\int_0^{ct}\int_0^{\pi}\int_0^{2\pi}\psi[x+r\sin(\theta)\cos(\phi), y+r\sin(\theta)\sin(\phi), z+r\cos(\theta)]\times$$
$$r^2\sin(\theta)d\phi d\theta d\rho -$$
$$\frac{t}{4\pi}\int_{ct}^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{\psi[x+r\sin(\theta)\cos(\phi), y+r\sin(\theta)\sin(\phi), z+r\cos(\theta)]}{r}\times$$
$$r^2\sin(\theta)d\phi d\theta d\rho.$$

Since $u = \frac{1}{c^2}v_{tt}$, after differentiating twice with respect to t (compute this), we find

$$u(x,y,z,t) =$$
$$\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\psi[x+ct\sin(\theta)\cos(\phi), y+ct\sin(\theta)\sin(\phi), z+ct\cos(\theta)]\times$$
$$\sin(\theta)d\phi d\theta =$$
$$\frac{t}{4\pi c^2t^2}\int_0^{\pi}\int_0^{2\pi}\psi[x+ct\sin(\theta)\cos(\phi), y+ct\sin(\theta)\sin(\phi), z+ct\cos(\theta)]\times$$
$$c^2t^2\sin(\theta)d\phi d\theta.$$

Let $d\Omega$ be the differential of the area of the unit sphere $S(0,1)$ and $d\omega$ the differential of the area of the sphere $S(0,ct)$. Then $d\omega = (ct)^2 d\Omega$. We also let α, β, and γ be the direction cosines of a non-zero vector through the origin, i.e., the cosines of the angles between the vector and the positive semi-axes of the Cartesian system of axes. Then,

$$\alpha = \sin(\theta)\cos(\phi), \quad \beta = \sin(\theta)\sin(\phi), \quad \gamma = \cos(\theta). \quad \text{(Check this!)}$$

Using these notations, we often encounter this formula written as

$$u(x,y,z,t) = \frac{t}{4\pi}\int\int_{S(0,1)}\psi(x+ct\alpha, y+ct\beta, z+ct\gamma)d\Omega =$$

$$\frac{t}{4\pi c^2t^2}\int\int_{S(0,ct)}\psi(x+ct\alpha, y+ct\beta, z+ct\gamma)d\omega.$$

But, 4π is the area of the unit sphere and $4\pi c^2t^2$ is the area of the sphere of radius $R = ct$. Therefore, the solution found $u(x,y,z,t)$ is the time t multiplied by the mean value of the function ψ over the surface of the sphere $S[(x,y,z),ct]$, whose center is the point $P = (x,y,z)$ and its radius is $R = ct$. I.e.,

$$u(x,y,z,t) = t\mathcal{M}_{S[(x,y,z),ct]}(\psi)$$

This formula of the solution of **Problem (II)** is called **Kirchhoff's formula**.

Now, we can forget about the conditions on Fourier transform and its inversion. As we have a formal formula of the solution of **Problem (II)** at hand, we can directly verify that this formula gives indeed the solution of the problem. (The interested reader may want to check that.)

Next, to looking at **Problem (II)** again, we observe that the function $w(x,y,z,t) = u_t(x,y,z,t)$ is the solution of the following problem

$$\begin{cases} w_{xx} + w_{yy} + w_{zz} = \dfrac{1}{c^2} w_{tt}, & (x,y,z) \in \mathbb{R}, \quad t > 0, \\ \\ w(x,y,z,0) = u_t(x,y,z,0) = \psi(x,y,z), & (x,y,z) \in \mathbb{R}, \\ \\ w_t(x,y,z,0) = u_{tt}(x,y,z,0) = 0, & (x,y,z) \in \mathbb{R}. \end{cases}$$

[The last equation is obtained from $u(x,y,z,0) = 0$ and the first equation, since $u_{tt}(x,y,z,0) = c^2[u_{xx} + u_{yy} + u_{zz}](x,y,z,0) = 0 + 0 + 0 = 0$.] Therefore, the solution of **Problem (III)** is

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left\{ t\mathcal{M}_{S[(x,y,z),ct]}(\phi) \right\}.$$

Finally, by linearity and the principle of superposition, the solution of the initial **Problem**, **(I)**, is

$$u(x,y,z,t) = t\mathcal{M}_{S[(x,y,z),ct]}(\psi) + \frac{\partial}{\partial t}\left\{ t\mathcal{M}_{S[(x,y,z),ct]}(\phi) \right\}.$$

[The interested reader may want to find pertinent bibliography to study the natural, and maybe philosophical, consequences of this law (Kirchhoff's formula and/or the final solution above), one of which is the famous Huygen's[40] principle.]

Application 5. We want to find the solution of the n**-dimensional homogeneous heat equation** by using the n-dimensional Fourier transform. The n**-dimensional Fourier transform** is a natural extension of the one-dimensional case to n dimensions. So, we consider the equation

$$u_{x_1x_1} + u_{x_2x_2} + \ldots + u_{x_nx_n} - u_t = 0,$$

where $u = u(x_1, x_2, \ldots, x_n, t)$ is a function defined in $\mathbb{R}^n \times \mathbb{R}^+$ that satisfies all the necessary conditions for the legitimacy of all the operations involved, in what follows. Since the equation is linear and homogeneous, any solution is determined up to an arbitrary multiplicative constant.

We consider the n-dimensional Fourier transform

$$\tilde{u}(\xi_1, \xi_2, \ldots, \xi_n, t) :=$$
$$\frac{1}{\left(\sqrt{2\pi}\right)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, x_2, \ldots x_n, t) e^{i(x_1\xi_1 + x_2\xi_2 + \ldots + \xi_n x_n)} dx_1 \ldots dx_n$$

[40] Christiaan Huygen, (Latin: Hugenius), Dutch mathematician and physicist, 1629-1695.

and apply it to the equation. Using the properties of the Fourier transform, we find

$$-\left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)\tilde{u}\left(\xi_1,\xi_2,\ldots,\xi_n,t\right)-\tilde{u}_t\left(\xi_1,\xi_2,\ldots\xi_n,t\right)=0.$$

Therefore,

$$\tilde{u}(\xi_1,\xi_2,\ldots,\xi_n,t)=C\left(\xi_1,\xi_2,\ldots,\xi_n\right)e^{-\left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)t},$$

where $C\left(\xi_1,\xi_2,\ldots,\xi_n\right)$ is an arbitrary $\mathfrak{C}^2$ function in the variables $\xi_1,\xi_2,\ldots,\xi_n$.

Then, by the analogous inversion formula, after letting $\vec{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$, we obtain the **general solution**

$$u(x_1,x_2,\ldots,x_n,t)=$$
$$\frac{1}{\left(\sqrt{2\pi}\right)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}C(\vec{\xi})e^{-\left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)t}e^{-i(x_1\xi_1+x_2\xi_2+\ldots+x_n\xi_n)}d\vec{\xi},$$

with $C(\vec{\xi})=C\left(\xi_1,\xi_2,\ldots,\xi_n\right)$ an arbitrary $\mathfrak{C}^2$ function in $\vec{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)$.

Next, we are going to elaborate this solution if

$$C(\vec{\xi})=C\left(\xi_1,\xi_2,\ldots,\xi_n\right)\equiv c$$

is an arbitrary constant. In this case, we have

$$u(x_1,x_2,\ldots,x_n,t)=$$
$$\frac{c}{\left(\sqrt{2\pi}\right)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{j=1}^{n}e^{-ix_j\xi_j-t\xi_j^2}\right)d\vec{\xi}=$$
$$\frac{c}{\left(\sqrt{2\pi}\right)^n}\left(\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{-ix_j\xi_j-t\xi_j^2}d\xi_j\right).$$

We observe that

$$-ix_j\xi_j-t\xi_j^2=-\frac{x_j^2}{4t}-\left(\sqrt{t}\xi_j+\frac{ix_j}{2\sqrt{t}}\right)^2$$

and we get

$$u(x_1,x_2,\ldots,x_n,t)=$$
$$\frac{c}{\left(\sqrt{2\pi}\right)^n}e^{-\frac{x_1^2+x_2^2+\ldots+x_n^2}{4t}}\left[\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{-\left(\sqrt{t}\xi_j+\frac{ix_j}{2\sqrt{t}}\right)^2}d\xi_j\right].$$

By **Problem 1.7.34**, we have

$$\int_{-\infty}^{\infty} e^{-\left(\sqrt{t}\xi_j + \frac{ix_j}{2\sqrt{t}}\right)^2} d\xi_j = \frac{\pi}{\sqrt{t}}$$

and so, in this case, we find the solution

$$u(x_1, x_2, \ldots x_n, t) = \frac{A}{\left(\sqrt{t}\right)^n} e^{-\frac{x_1^2+x_2^2+\ldots+x_n^2}{4t}},$$

where A is an arbitrary constant.

We can check directly that this solution satisfies the above heat equation regardless of the conditions for taking Fourier transform and applying inversion. This solution is not the only one, since other choices of $C(\vec{\xi}) = C(\xi_1, \xi_2, \ldots, \xi_n)$ will yield other solutions. Also, an initial condition $u(x_1, x_2, \ldots, x_n, 0) = f(x_1, x_2, \ldots, x_n)$ will determine the function C, as we have seen, for instance, in the previous **application, (4)**.

1.7.8 The Fourier Transform with Complex Argument

If in **Definition 1.7.3** [equation **(1.27)** or **(1.29)**] of the Fourier transform of a real function $y = f(x)$ on $\mathbb{R}$ we replace the e^{itx} with $e^{-sx}e^{itx} = e^{(it-s)x} = e^{i(t+is)x} = e^{iwx}$, where $w = t + is$, then we get what we call the **Fourier transform of** $y = f(x)$ **with complex argument the** $w = t + is$.

So, equations **(1.27)** and **(1.29)** with $w = t + is$ become respectively

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \int_{-\infty}^{\infty} f(x)e^{iwx}dx =$$

$$\int_{-\infty}^{\infty} f(x)e^{(-s+it)x}dx = \mathcal{F}[e^{-sx}f(x)](t), \quad \forall\ t \in \mathbb{R},$$

and

$$\tilde{f}(w) = \mathfrak{F}[f(x)](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{iwx}dx =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{(-s+it)x}dx = \mathfrak{F}[e^{-sx}f(x)](t), \quad \forall\ t \in \mathbb{R}.$$

We observe that $\mathcal{F}[f(x)](w) = \mathcal{F}[e^{-sx}f(x)](t)$, and so for the Fourier transform with complex argument $w = t + is$ to exist, we must require $e^{-sx}f(x)$ to be absolutely integrable. Depending on the value of $s = \mathrm{Im}(w)$, this condition may change, and so $\mathcal{F}[f(x)](w)$ may exist in some areas of the complex plane $\mathbb{C}$ but may not exist in others.

In the areas of the complex plane $\mathbb{C}$ in which the Fourier transforms with complex argument of the functions under consideration exist, all the properties and rules as they are referred to or analyzed in **Subsection 1.7.6** are valid in this new situation. All that we must do is simply replace the real variable t with the complex variable w, and any real constant is now considered to be a complex constant. Thus, we do not need to restate these rules. But, the reader may want to do so and prove them again. All proofs carry through analogously.

Before we study any example and application, we must study some **new properties of the Fourier transform with complex argument**.

(1) (a) Let us assume that $e^{-sx}f(x)$ is absolutely integrable for $s = a$ and $s = b$, with $a < b$. Then, $e^{-sx}f(x)$ is also absolutely integrable for any $s \in [a, b]$.

This claim follows immediately from the hypotheses and the obvious inequality

$$\int_{-\infty}^{\infty} \left| e^{-sx} f(x) \right| dx = \int_{-\infty}^{0} e^{-sx}|f(x)|dx + \int_{0}^{\infty} e^{-sx}|f(x)|dx \leq$$

$$\leq \int_{-\infty}^{0} e^{-bx}|f(x)|dx + \int_{0}^{\infty} e^{-ax}|f(x)|dx = \text{finite} + \text{finite} = \text{finite}\,(< \infty).$$

(b) If $e^{-sx}f(x)$ is square integrable for $s = a$ and $s = b$, with $a < b$, then $e^{-sx}f(x)$ is absolutely integrable for any $s \in (a, b)$.

This claim follows from the hypotheses, the **Cauchy-Schwarz inequality**. [See **project Problem I 2.6.65, Item (1.)**] and

$$\int_{-\infty}^{\infty} \left| e^{-sx} f(x) \right| dx = \int_{-\infty}^{0} e^{-sx}|f(x)|dx + \int_{0}^{\infty} e^{-sx}|f(x)|dx =$$

$$\int_{-\infty}^{0} e^{[-b+(b-s)]x}|f(x)|dx + \int_{0}^{\infty} e^{[-a-(s-a)]x}|f(x)|dx =$$

$$\int_{-\infty}^{0} \left[e^{(b-s)x}\right] \cdot \left[e^{-bx}|f(x)|\right] dx + \int_{0}^{\infty} \left[e^{-(s-a)x}\right] \cdot \left[e^{-ax}|f(x)|\right] dx \leq$$

$$\left[\int_{-\infty}^{0} e^{2(b-s)x} dx \cdot \int_{-\infty}^{0} e^{-2bx}|f(x)|^2 dx\right]^{\frac{1}{2}} +$$

$$\left[\int_{0}^{\infty} e^{-2(s-a)x} dx \cdot \int_{0}^{\infty} e^{-2ax}|f(x)|^2 dx\right]^{\frac{1}{2}} =$$

$$\left[\frac{1}{2(b-s)} \int_{-\infty}^{0} e^{-2bx}|f(x)|^2 dx\right]^{\frac{1}{2}} + \left[\frac{1}{2(s-a)} \int_{0}^{\infty} e^{-2ax}|f(x)|^2 dx\right]^{\frac{1}{2}} =$$

$$\text{finite} \ + \ \text{finite} \ = \ \text{finite} \ (< \infty).$$

(2) Under the **Condition (a)** or **(b)**, in **(1)**, the $\hat{f}(w) = \mathcal{F}[f(x)](w)$ exists for any $s \in [a, b]$ or any $s \in (a, b)$, respectively. Hence, it exists in the infinite horizontal strip of the complex plane $\mathbb{C}$ respectively defined by

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } a \leq s \leq b\}$$

or

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } a < s < b\}.$$

(3) In such a strip, the convergence is uniform in either s or t since $\left|e^{itx}\right| = 1$ and $e^{-sx}f(x)$ is absolutely integrable (see **Theorem I 2.3.11**, etc.). So, in such a strip, the $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is a continuous function in s and t.

(4) We then consider a maximal open horizontal strip

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } -\infty \leq p < s < q \leq \infty\} \subseteq \mathbb{C}$$

in which $\hat{f}(w) = \mathcal{F}[f(x)](w)$ exists. In such a strip, the Fourier transform $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is a holomorphic function.

This result follows by the **property (3)** above and **Morera's Theorem, 1.5.5**. We consider any simple closed contour $C \subset \mathbb{S}$, and we compute

$$\oint_C \hat{f}(w)dw = \oint_C \int_{-\infty}^{\infty} e^{iwx} f(x)\,dx\,dw =$$
$$\int_{-\infty}^{\infty} f(x) \oint_C e^{iwx}\,dw\,dx = \int_{-\infty}^{\infty} f(x) \cdot 0\,dx = 0.$$

(The switching of the order of the double integration follows from the **Tonelli conditions, Section I 2.4**, since $\left|e^{itx}\right| = 1$ and $e^{-sx}f(x)$ is absolutely integrable.)

So, with $e^{-sx}f(x)$ absolutely integrable in $\mathbb{R}$, by the **Riemann-Lebesgue Lemma, 1.7.7**, in a maximal open horizontal strip, in which $-\infty \leq p < s < q \leq \infty$, the limit of the Fourier transform $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is zero as $w \to \infty$, where ∞ here is the complex infinity within the strip, and so the convergence is uniform.

Example 1.7.45 We consider any $a > 0$, and we define

$$f(x) = \frac{1}{2a}\,e^{-a|x|} = \begin{cases} \dfrac{1}{2a}\,e^{-ax}, & \text{if } x > 0, \\[2ex] \dfrac{1}{2a}, & \text{if } x = 0, \\[2ex] \dfrac{1}{2a}\,e^{ax}, & \text{if } x < 0. \end{cases}$$

Notice that if $s \le -a$ or $s \ge a$, then the positive function $e^{-sx}f(x)$ is not absolutely integrable. (Check!) But, for $-a < s < a$, it is absolutely integrable. (Check!)

Then, in the open horizontal strip

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } -a < s < a\}$$

we have

$$\hat{f}(w) = \int_{-\infty}^{\infty} \frac{1}{2a}\, e^{iw} e^{-a|x|} dx =$$

$$\frac{1}{2a} \int_{-\infty}^{0} e^{(it-s+a)x} dx + \frac{1}{2a} \int_{0}^{\infty} e^{(it-s-a)x} dx =$$

$$\frac{1}{2a}\left(\frac{1}{a+it-s} - \frac{1}{-a+it-s}\right) = \frac{1}{2a} \cdot \frac{-2a}{-a^2-(t+is)^2} = \frac{1}{a^2+w^2}.$$

Hence, in the above maximal open horizontal strip $\mathbb{S}$, we have

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \frac{1}{a^2+w^2}.$$

We see that this Fourier transform with complex argument w is, as expected, holomorphic with respect to w in $\mathbb{S}$. Its limit at infinity is zero.

▲

(5) Since $e^{-sx}f(x)$ is absolutely integrable, the **inversion Theorem, 1.7.6**, can apply to $\hat{f}(w) = \mathcal{F}[f(x)](w) = \mathcal{F}[e^{-sx}f(x)](t)$. Then, we get

$$e^{-sx}f(x) =$$

$$\frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \hat{f}(w)e^{-itx} dt = \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(w)e^{-itx} dt.$$

So, with $w = t + is$, we have

$$f(x) = \frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \hat{f}(t)e^{sx}e^{-itx} dt =$$

$$\frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \hat{f}(w)e^{-i(t+is)x} dt = \frac{1}{2\pi}\ \text{P.V.} \int_{-\infty}^{\infty} \hat{f}(w)e^{-iwx} dt =$$

$$\frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} \hat{f}(w)e^{-iwx} dt. \tag{1.40}$$

Since the integration is performed only with respect to t, for any given $s = \mathrm{Im}(w)$, the integrals take place on the horizontal straight line $(-\infty + is, \infty + is)$. Then, if s varies in certain different subintervals of $(-\infty, \infty)$, we may find different $f(x)$'s as the inverse Fourier transforms of $\hat{f}(w)$.

Example 1.7.46 We will apply the **inversion Theorem**, that is, the **inversion formula (1.40)**, to the result of the **previous Example**.

For any $a > 0$ and

$$f(x) = \frac{1}{2a}\, e^{-a|x|},$$

we found (in the **previous Example**) that if $-a < s < a$, then

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \frac{1}{a^2 + w^2},$$

where $w = t + is$.

Since for any $-a < s < a$, $f(x)$ [or $e^{-sx} f(x)$] satisfies all conditions under which we can apply the **inversion formula (1.40)**, we expect to retrieve the function $f(x)$ if we apply the inversion formula to $\hat{f}(w) = \dfrac{1}{a^2 + w^2}$. But, we must expect this only for $-a < s < a$. If we apply the inversion formula for $a < s < \infty$ or $-\infty < s < -a$, then we must expect the possibility of finding different inverse transforms. So, in this example, we plan to explore this inversion.

Case $-a < s < a$**:** We let

$$f(x) = \frac{1}{2\pi} \int_{-\infty + si}^{\infty + si} \frac{e^{-iwx}}{a^2 + w^2}\, dt,$$

where $w = t + is$.

Subcase $x > 0$: To find this principal value, we use the negatively oriented contour

$$C^- = [-M + si, M + si] + A_R^-,$$

where $M > 0$ such that $M^2 + s^2 > a^2$, $R = \sqrt{M^2 + s^2}$ and A_R^- is the negative arc which is the part of the circle $C(0, R)$, with center the origin and radius R, and below the line $(-\infty + is, \infty + is)$. [So, along this arc $\theta = \mathrm{Arg}(w)$ satisfies $\arcsin\left(\dfrac{s}{M}\right) \geq \theta \geq -\pi - \arcsin\left(\dfrac{s}{M}\right)$.] This contour encloses the number $-ai$, which is a simple pole for the function $e^{-iwx} \hat{f}(w) = \dfrac{e^{-iwx}}{a^2 + w^2}$. (Make a figure of the contour.)

We have that

$$\operatorname*{Res}_{z=-ai} \left[e^{-iwx} \hat{f}(w) \right] = \frac{-e^{-ax}}{2ai},$$

and so by the **Residue Theorem, 1.7.1**, we find

$$\frac{1}{2\pi}\int_{C^-}\frac{e^{-iwx}}{a^2+w^2}\,dw = \frac{1}{2\pi}\left(\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw + \int_{A_R^-}\frac{e^{-iwx}}{a^2+w^2}\,dw\right) =$$
$$\frac{1}{2\pi}(-2\pi i)\frac{-e^{-ax}}{2ai} = \frac{e^{-ax}}{2a}. \tag{1.41}$$

Now, if $w = u + iv$ along A_R^-, then $v \le s$, and since $x > 0$, we have $|e^{-iwx}| = |e^{-iux}e^{vx}| \le e^{sx}$. Thus,

$$\left|\int_{A_R^-}\frac{e^{-iwx}}{a^2+w^2}\,dw\right| \le \int_{A_R^-}\frac{e^{sx}}{R^2-a^2}\,|dw| <$$
$$\frac{e^{sx}}{R^2-a^2}2\pi R \longrightarrow 0, \quad \text{as } R \longrightarrow \infty.$$

But $R = \sqrt{M^2+s^2}$, and so $R \longrightarrow \infty \Longleftrightarrow M \longrightarrow \infty$. Therefore, by taking the limit of the relation **(1.41)** above, as $M \longrightarrow \infty$, we find that

$$\text{for all } x > 0, \;\; f(x) = \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw = \frac{e^{-ax}}{2a}.$$

Subcase $x < 0$: This case is analogous to the previous subcase except we must use a different contour to obtain zero limit along the arc. The previous contour does not achieve this, as we can easily check, for the integrand approaches infinity in the limit. The contour this time is the positively oriented contour

$$C^+ = [-M+si, M+si] + A_R^+,$$

where M and R are the same as before but A_R^+ is the positive arc of the circle $C(0,R)$, with center the origin and radius R and above the line $(-\infty+is, \infty+is)$. This contour encloses the number ai, which is a simple pole for $e^{-iwx}\hat{f}(w) = \dfrac{e^{-iwx}}{a^2+w^2}$. (Make a figure of the contour.)

We have that

$$\operatorname*{Res}_{z=ai}\left[e^{-iwx}\hat{f}(w)\right] = \frac{e^{ax}}{2ai}.$$

Then, by analogous work as before, we find that

$$\text{for all } x < 0, \;\; f(x) = \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M+si}^{M+si}\frac{e^{-iwx}}{a^2+w^2}\,dw = \frac{e^{ax}}{2a}.$$

Subcase $x = 0$: If we plug $x = 0$ in both subcases above, we find the same result

$$f(0) = \frac{1}{2a},$$

which, with $-a < s < a$ and $w = t + is$, we can also compute directly by the integral

$$\frac{1}{2\pi}\int_{-\infty+si}^{\infty+si} \frac{1}{a^2+w^2}\,dw.$$

We see that all three subcases can be put in one formula as

$$f(x) = \frac{e^{-a|x|}}{2a}, \quad \text{for } x \in \mathbb{R},$$

which is, indeed, the function $f(x)$ we started with in the **previous Example**.

Case $s < -a$.
Subcase $x < 0$**:** As before, we consider the contour

$$C^+ = [-M + si, M + si] + A_R^+.$$

Notice that in this case the contour encloses both poles $-ai$ and ai. Then, working as in the previous case, we find

$$\text{for all } x < 0,\ \ f(x) = \frac{1}{2\pi}2\pi i\left(\frac{e^{ax}}{2ia} - \frac{e^{-ax}}{2ia}\right) = \frac{\sinh(ax)}{a}.$$

Subcase $x > 0$**:** Again, we consider the contour

$$C^- = [-M + si, M + si] + A_R^-.$$

Notice that in this case the contour does not enclose any singularity (pole). Then, for $x > 0$, we find $f(x) = 0$.

Both subcases agree at $x = 0$, and so we can eventually write

$$f(x) = \begin{cases} \dfrac{\sinh(ax)}{a}, & \text{if } x < 0, \\[2ex] 0, & \text{if } x \geq 0. \end{cases}$$

Case $s > a$.

This case is analogous to the **previous case**. We use the same contours as above for the subcases $x < 0$ and $x > 0$ but in reverse order. Eventually, we find

$$f(x) = \begin{cases} \dfrac{-\sinh(ax)}{a}, & \text{if } x > 0, \\[2ex] 0, & \text{if } x \leq 0. \end{cases}$$

We see that the answers to the inverse Fourier transforms in the three possible different cases of the horizontal strips of s are different.

▲

(6) We are going to prove a result that we tacitly used in the **previous Example**.

With $e^{-sx}f(x)$ absolutely integrable in $\mathbb{R}$, in the maximal open horizontal strip

$$\mathbb{S} = \{w = t + is \quad | \quad -\infty < t < \infty \text{ and } -\infty \leq p < s < q \leq \infty\} \subseteq \mathbb{C}$$

in which $\hat{f}(w) = \mathcal{F}[f(x)](w)$ exists and is holomorphic, the principal value in the **inversion formula (1.40)** is independent of s.

In the proof, we use the fact that the Fourier transform $\hat{f}(w) = \mathcal{F}[f(x)](w)$ has limit zero as $w \to \infty$ (where ∞ is the complex infinity within the strip), as we have already justified in **(4)** above.

We pick any $p < s_1 < s_2 < q$ and consider the contour

$$C^+ = [-M + is_1, M + is_1] + [M + is_1, M + is_2] +$$

$$[M + is_2, -M + is_2] + [-M + is_2, -M + is_1],$$

which is a positively oriented parallelogram.

Since on C^+ and in its interior the function $\hat{f}(w) = \mathcal{F}[f(x)](w)$ is complex analytic with no singularities, the **Cauchy-Goursat Theorem, 1.5.3**, gives

$$\oint_{C^+} \hat{f}(w)\, dw = 0.$$

Then, we let $M \longrightarrow \infty$ and eventually get

$$\text{P.V.} \int_{-\infty+is_1}^{+\infty+is_1} \hat{f}(w)\, dw = \text{P.V.} \int_{-\infty+is_2}^{+\infty+is_2} \hat{f}(w)\, dw,$$

since by the mentioned condition

$$\lim_{w\to\infty} \hat{f}(w) = 0,$$

the integrals along the constant length$(= s_2 - s_1)$, finite vertical segments $[M + is_1, M + is_2]$ and $[-M + is_2, -M + is_1]$ become zero, as $M \longrightarrow \infty$ (the complex infinity, and so the convergence is uniform). (Work out the details, as an easy exercise!)

Application:

In **Application 1** of **Subsection I 2.7.3**, we saw the Bessel function of order zero

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \tag{1.42}$$

as a solution of Bessel's differential equation of order zero

$$xy'' + y' + xy = 0. \tag{1.43}$$

We see that $J_0(x)$ is an even function in $\mathbb{R}$. We are going to use the method of Fourier transform with complex argument to find an integral representation of it up to a multiplicative constant.

Applying the Fourier transform to this differential equation, as done in **Application 2** of **Subsection 1.7.7**, with the help of the rules

$$\mathcal{F}[f'(x)](w) = -iw\mathcal{F}[f(x)](w)$$

and

$$\frac{d}{dw}\{\mathcal{F}[f(x)]\}(w) = \mathcal{F}[ixf(x)](w),$$

and assuming that all conditions are met to apply them, we find the following first-order linear homogeneous differential equation for the Fourier transform with complex argument of $y(x)$, which here we write by $\hat{y}(w)$:

$$(w^2 - 1)\frac{d[\hat{y}(w)]}{dw} + w\,\hat{y}(w) = 0.$$

The solution of this differential equation is easily found to be

$$\hat{y}(w) = c\left(w^2 - 1\right)^{\frac{-1}{2}}, \quad \text{where } c \text{ is an arbitrary complex constant.}$$

Now, to find $y(x)$ in integral form, we are going to use the **inversion formula(1.40)**, in **Item (4.)** above, and work in an analogous way as in **Example 1.7.46**. In the process, we may consider any multiplicative constant to be 1 for convenience.

First of all, we must notice that the function to be used in **inversion formula (1.40)** is

$$z = g(w) = e^{-ixw}\left(w^2 - 1\right)^{\frac{-1}{2}} = e^{-ixw}(w-1)^{\frac{-1}{2}} \cdot (w+1)^{\frac{-1}{2}}.$$

This has singular points at $w = 1$ and $w = -1$ and the fractional power $\frac{1}{2}$ dictates to use two branch cuts, one for each factor. (See also **Example 1.7.14**.)

As convenient branch cuts that shift $w-1$ and $w+1$ to the origin, we choose the following:

$$\mathbb{C} - [+1, +\infty) \text{ for } \sqrt{w-1} \quad \text{and} \quad \mathbb{C} - [-1, +\infty) \text{ for } \sqrt{w+1}.$$

The intersection of these branch cuts is $[+1, +\infty)$ and their union $[-1, +\infty)$, both intervals of the real axis.

The corresponding restrictions on the arguments are

$$0 < \arg(w-1) < 2\pi \quad \text{and} \quad 0 < \arg(w+1) < 2\pi.$$

Then, by the definition of the non-integer powers through complex logarithms, we finally have

$$z = g(w) = e^{-ixw} \frac{1}{\sqrt{|w^2-1|}} \cdot e^{\frac{-i[\arg(w-1)+\arg(w+1)]}{2}}.$$

Since the singular points and the branch cuts are on the x-axis, in **inversion formula (1.40)**, we must consider the two infinite horizontal strips:

$$\mathbb{S}_1 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad s > 0\}$$

and

$$\mathbb{S}_2 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad s < 0\}.$$

(I) First, we work in the strip $\mathbb{S}_1$. We pick any $s > 0$, and for any $M > 1$ we let $R = \sqrt{M^2 + s^2}$. We have to consider two cases $x < 0$ and $x > 0$. The work is analogous with **Example 1.7.46**.

Case (1): $x < 0$. We consider the contour

$$C^+ = [-M + si, M + si] + A_R^+,$$

where $[-M + si, M + si]$ is a positive horizontal segment and

$$A_R^+ = \left\{ w = u + vi = Re^{i\theta} \mid \arcsin\left(\frac{s}{M}\right) \le \theta \le \pi - \arcsin\left(\frac{s}{M}\right) \right\}$$

is a positive arc. This is depicted as the upper part in **Figure 1.18**.

Since $g(w)$ has no singularities on C^+ and inside of it, by the **Cauchy-Goursat Theorem, 1.5.3**, we have

$$\oint_{C^+} g(w)\, dw = 0.$$

As in **Example 1.7.46**, with $x < 0$, we find

$$\lim_{M \to \infty} \int_{A_R^+} g(w)\, dw = 0$$

and so we get

$$\text{P.V.} \int_{-\infty+is}^{\infty+is} g(w)\,dw = \lim_{M\to\infty} \int_{-M+is}^{M+is} g(w)\,dw = \int_{-\infty}^{\infty} g(w)\,dw = 0.$$

So, in $\mathbb{S}_1$, we get $y(x) = 0$ for all $x < 0$, which is a trivial solution for Bessel's equation. So, in this case, we do not obtain anything interesting.

Case (2): $x > 0$. We pick any $0 < r < 1$, M and R as before, and we consider the contour

$$C^- = [-M+si, M+si] + PA_R^- + [R, 1+r]^- + S_r^+$$
$$+[1-r, -1+r]^- + A_r^+ + [-1+r, 1-r]^+ + T_r^+ + [1+r, R]^+ + A_R^+.$$

depicted as the lower part in **Figure 1.18**.

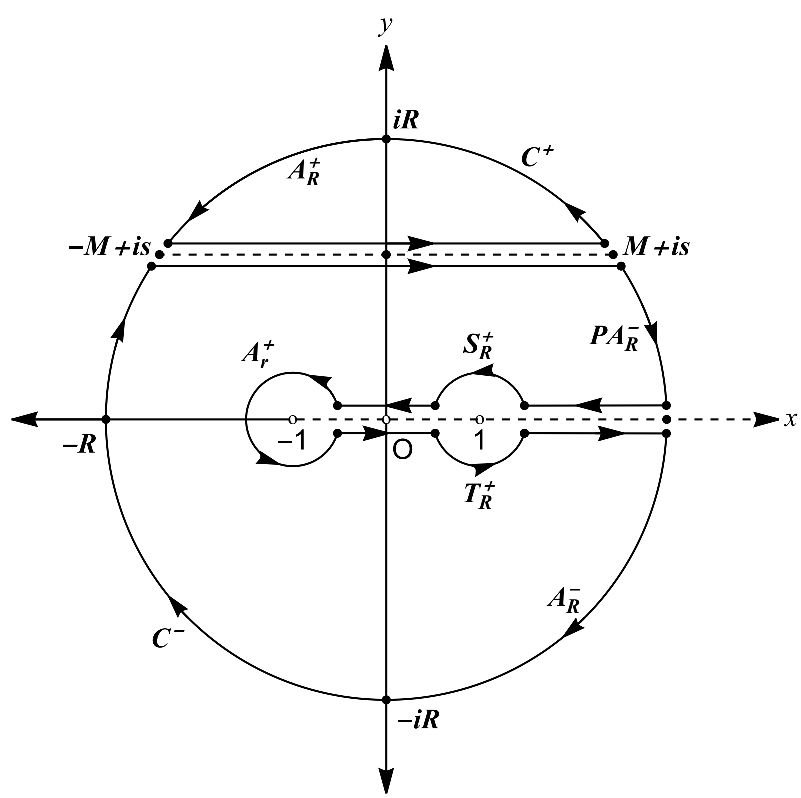

FIGURE 1.18: Contours for Application

So, $[-M+si, M+si]$ is a positive horizontal segment as before. The arcs are:

$$\begin{aligned}
PA_R^- &= \left\{w = u+vi = Re^{i\theta} \mid \arcsin\left(\frac{s}{M}\right) \geq \theta > 0\right\},\\
S_r^+ &= \left\{z = 1+re^{i\theta} \mid 0 < \theta < \pi\right\},\\
A_r^+ &= \left\{z = -1+re^{i\theta} \mid 0 < \theta < 2\pi\right\},\\
T_r^+ &= \left\{z = 1+re^{i\theta} \mid \pi < \theta < 2\pi\right\},\\
A_R^- &= \left\{z = Re^{i\theta} \mid 0 > \theta > -\pi - \arcsin\left(\frac{s}{M}\right)\right\}.
\end{aligned}$$

The negative segments $[R, 1+r]^-$ and $[1-r, -1+r]^-$ are approached by staying in the upper half plane, whereas the positive segments $[-1+r, 1-r]^+$ and $[1+r, R]^+$ are approached by staying in the lower half plane.

Then, working as in **Examples 1.7.14** and **1.7.46**, with $x > 0$, we find the following:

$$(1) \quad \int_{PA_R^-} g(w)\,dw \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty.$$

$$(2) \quad \int_{A_R^-} g(w)\,dw \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty.$$

Keeping track with the correct arguments, we find

$$(3) \quad \int_{[R,1+r]} g(w)\,dw + \int_{[1+r,R]} g(w)\,dw = 0$$

for any $0 < r < 1$ and $R > 1$. This is so because in the intersection of both branch cuts, $(1, \infty)$, the function $g(w)$ is continuous and therefore holomorphic. Since $[R, 1+r]$ and $[1+r, R]$ are opposite segments the two partial line integrals cancel each other.

We now prove:

$$(4) \quad \int_{A_r^+} g(w)\,dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

We have $0 < r < 1$ and

$$\int_{A_r^+} g(w)\,dw = \int_{A_r^+} \frac{e^{-ixw}}{\sqrt{w^2-1}}dw$$

with $w = re^{i\theta} + 1 \in A_r^+$ and $0 < \theta < 2\pi$. So,

$$\begin{aligned}
dw = re^{i\theta}d\theta &\implies |dw| = r\,d\theta.\\
e^{-ixw} = e^{-ix[1+r\cos(\theta)+i\sin(\theta)]} &\implies \left|e^{-ixw}\right| = e^{xr\sin(\theta)}.\\
w^2 - 1 = r^2e^{2i\theta} + 2re^{i\theta} &+ 1 - 1 = r\left(re^{2i\theta} + 2e^{i\theta}\right).
\end{aligned}$$

So,

$$\left|w^2-1\right| = r\left|re^{2i\theta}+2e^{i\theta}\right| > r\left|\left|2e^{i\theta}\right|-\left|re^{2i\theta}\right|\right| = r(2-r).$$

Putting these pieces together, we get

$$\left|\int_{A_r^+} g(w)\,dw\right| \le$$

$$\int_{A_r^+} |g(w)|\,dw < \int_0^{2\pi} \frac{e^{xr\sin(\theta)} r\,d\theta}{\sqrt{r(2-r)}} =$$

$$\int_0^{2\pi} \frac{e^{xr\sin(\theta)}\sqrt{r}\,d\theta}{\sqrt{2-r}} \longrightarrow \int_0^{2\pi} 0\,d\theta = 0, \text{ as } r \longrightarrow 0$$

(by the uniform or the bounded convergence), which implies what we wanted to prove.

In the same way, we prove the next two limits:

$$(5)\qquad \int_{S_r^+} g(w)\,dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

$$(6)\qquad \int_{T_r^+} g(w)\,dw \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0.$$

Now, we compute the integrals over the segments $[1-r,-1+r]$ and $[-1+r,1+r]$. We must consider the appropriate arguments in either one. So, $w=t+0i$, and we have

$$(7)\qquad \int_{[1-r,-1+r]} g(w)\,dw =$$

$$\int_{1-r}^{-1+r} e^{-ixt}\left|t^2-1\right|^{\frac{-1}{2}} e^{\frac{-i\left[\mathrm{Arg}(w-1)+\mathrm{Arg}(w+1)\right]}{2}}\,dt =$$

$$\int_{1-r}^{-1+r} [\cos(xt)-i\sin(xt)]\frac{1}{\sqrt{|t^2-1|}}e^{\frac{-i(\pi+\pi)}{2}}\,dt =$$

$$\int_{1-r}^{-1+r} \frac{\cos(xt)}{\sqrt{1-t^2}}(-1)\,dt = \int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt \longrightarrow$$

$$\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \text{ as } r \longrightarrow 0$$

(the sine integral is zero by the oddity of sine).

Similarly, on the interval $[-1+r, 1+r]$, we get

$$(8) \quad \int_{[-1+r,1-r]} g(w)\,dw =$$

$$\int_{-1+r}^{1-r} e^{-ixt}\left|t^2-1\right|^{\frac{-1}{2}} e^{\frac{-i\left[\mathrm{Arg}(w-1)+\mathrm{Arg}(w+1)\right]}{2}}\,dt =$$

$$\int_{-1+r}^{1-r} [\cos(xt) - i\sin(xt)]\frac{1}{\sqrt{|t^2-1|}}e^{\frac{-i(2\pi+2\pi)}{2}}\,dt =$$

$$\int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}(+1)\,dt = \int_{-1+r}^{1-r} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt \longrightarrow$$

$$\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \text{ as } r \longrightarrow 0$$

(the sine integral is zero by the oddity of sine).

Next, since

$$\oint_{C^-} g(w)\,dw = 0,$$

we find that for all $M > 1$,

$$\int_{-M+is}^{M+is} g(w)\,dw + 2\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt = 0.$$

So,

$$\text{P.V.}\int g(w)\,dw = \lim_{M\to\infty}\int_{-M+is}^{M+is} g(w)\,dw =$$

$$\int_{-\infty+is}^{\infty+is} g(w)\,dw = -2\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt.$$

Therefore, up to a multiplicative constant, the solution of the above Bessel's differential equation in integral form is

$$y(x) = \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\,dt, \quad \text{for all } x > 0.$$

Notice that $\lim\limits_{x\to 0} y(x) = [\arcsin(t)]_{-1}^{1} = \pi$, since limit and integral commute by, e.g., **Part I of Theorem I 2.2.1** or **Theorem I 2.3.11**.

(II) The work in the strip $\mathbb{S}_2$ is analogous. By choosing the correct contours and cases, we find

$$y(x) = 0, \quad \text{for all } x > 0,$$

and

$$y(x) = \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt, \quad \text{for all } \; x < 0.$$

Notice again that $\lim_{x \to 0} y(x) = \pi$. So, we can continuously set $y(0) = \pi$.

Regardless of how many conditions on taking the Fourier transform and applying the inversion formula were kept or violated, the solution found satisfies the differential equation **(1.43)**. We can verify this by plugging the integral found into the equation and then differentiating under the integral sign is legitimate, by **Part II of Theorem I 2.2.1**. Indeed,

$$x \frac{d^2}{dx^2} \left[\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt \right] + \frac{d}{dx} \left[\int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt \right] + x \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt =$$

$$\int_{-1}^{1} \left[-xt^2 \frac{\cos(xt)}{\sqrt{1-t^2}} - t \frac{\sin(xt)}{\sqrt{1-t^2}} + x \frac{\cos(xt)}{\sqrt{1-t^2}} \right] dt =$$

$$\ldots = \int_{-1}^{1} \frac{d}{dt} \left[\sqrt{1-t^2}\, \sin(xt) \right]\, dt = \left[\sqrt{1-t^2}\, \sin(xt) \right]_{-1}^{1} = 0.$$

Since $J_0(0) = 1$ and $y(0) = \pi$, by **(I)** and **(II)**, we finally have that the integral representation of $J_0(x)$ is

$$J_0(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt, \quad \text{for all } \; x \in \mathbb{R}.$$

Letting $t = \sin(\phi)$, we obtain the following integral representations of $J_0(x)$, for all $x \in \mathbb{R}$:

$$\begin{aligned} J_0(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\cos(xt)}{\sqrt{1-t^2}}\, dt = \\ \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos[x \sin(\phi)]\, d\phi = \\ \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \cos[x \sin(\phi)]\, d\phi = \\ \frac{1}{\pi} \int_{0}^{\pi} \cos[x \sin(\phi)]\, d\phi. \end{aligned}$$

(See also **Problem 1.7.129**.)

Problems

1.7.87 Using all results and examples of this section, find the values of the following four Fourier type integrals:

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{10x\cos(25x)}{x^2+100}\,dx, \qquad \text{(b)} \quad \int_{-\infty}^{\infty} \frac{10x\sin(25x)}{x^2+100}\,dx,$$

$$\text{(c)} \quad \int_{-\infty}^{\infty} \frac{10\cos(25x)}{x^2+100}\,dx, \qquad \text{(d)} \quad \int_{-\infty}^{\infty} \frac{10\sin(25x)}{x^2+100}\,dx.$$

1.7.88 Prove:

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{\sin(x)}{x\,(1+x^4)}\,dx = \pi\left[1 - e^{-\frac{\sqrt{2}}{2}}\cos\left(\frac{\sqrt{2}}{2}\right)\right],$$

$$\text{(b)} \qquad \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x\,(1+x^4)}\,dx = 0,$$

$$\text{(c)} \qquad \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 \pm x + 1}\,dx = \frac{2\pi\sqrt{3}}{3}\, e^{-\frac{\sqrt{3}}{2}}\cos\left(\frac{1}{2}\right),$$

$$\text{(d)} \qquad \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2 \pm x + 1}\,dx = \mp\frac{2\pi\sqrt{3}}{3}\, e^{-\frac{\sqrt{3}}{2}}\sin\left(\frac{1}{2}\right).$$

1.7.89 For any $a > 0$ and $b > 0$, show that

$$\int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2+a^2)(x^2+b^2)} = 2\int_{0}^{\infty} \frac{\cos(x)dx}{(x^2+a^2)(x^2+b^2)} =$$

$$2\int_{-\infty}^{0} \frac{\cos(x)dx}{(x^2+a^2)(x^2+b^2)} = \pi\frac{ae^{-b} - be^{-a}}{ab\,(a^2-b^2)}.$$

Write the result when $a = b$. (You may use L' Hôpital's rule.)

1.7.90 (a) For any $a > 0$ and $t \in \mathbb{R}$, show that

$$\int_{-\infty}^{\infty} \frac{\sin(tx)dx}{x\,(x^2+a^2)} = 2\int_{0}^{\infty} \frac{\sin(tx)dx}{x\,(x^2+a^2)} =$$

$$2\int_{-\infty}^{0} \frac{\sin(tx)dx}{x\,(x^2+a^2)} = \text{sign}(t)\cdot\frac{\pi\left(1-e^{-|t|a}\right)}{a^2}.$$

(b) For $t \in \mathbb{R}$, show that

$$\int_{-\infty}^{\infty} \frac{\cos(tx)dx}{(x^2+1)^2} = 2\int_{0}^{\infty} \frac{\cos(tx)dx}{(x^2+1)^2} = 2\int_{-\infty}^{0} \frac{\cos(tx)dx}{(x^2+1)^2} = \frac{\pi(|t|+1)}{2e^{|t|}}.$$

(c) For $t \in \mathbb{R}$, show that

$$\int_{-\infty}^{\infty} \frac{x\sin(tx)dx}{x^4+4} = 2\int_0^{\infty} \frac{x\sin(tx)dx}{x^4+4} = 2\int_{-\infty}^{0} \frac{x\sin(tx)dx}{x^4+4} = \frac{\pi\sin(t)}{e^{|t|}}.$$

(d) For $t \in \mathbb{R}$, show that

$$\int_{-\infty}^{\infty} \frac{\sin(tx)dx}{x^2+4x+5} = \frac{\pi\sin(2t)}{e^{|t|}}.$$

(e) For $t \in \mathbb{R}$ and $a \neq 0$, show that

$$\int_{-\infty}^{\infty} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = 2\int_0^{\infty} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = 2\int_{-\infty}^{0} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = \frac{\pi t}{2|a|e^{|at|}}.$$

(f) For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, show that

$$\int_{-\infty}^{\infty} \frac{1-\cos(tx)}{(2n\pi)^2-(tx)^2}\,dx = 0.$$

1.7.91 (a) For $R > 0$, consider the contour $C^+ = [0,R] + A_R^+ + [Ri, 0]$ where A_R^+ is the arc $A_R = \{z = Re^{i\theta} \mid 0 \leq \theta \leq \frac{\pi}{2}\}$ traveled in the positive direction. For any $\alpha \in \mathbb{R}$, explain why

$$\oint_{C^+} \frac{e^{\alpha iz}}{z+1}dz = 0.$$

(b) Expand the equality in **(a)**, and then for $\alpha > 0$ take the limit as $R \to \infty$ to prove that

$$\text{(a)} \qquad \int_0^{\infty} \frac{\sin(\alpha x)}{x+1}\,dx = \int_0^{\infty} \frac{\sin(x)}{x+\alpha}\,dx =$$
$$\int_0^{\infty} \frac{e^{-\alpha x}}{x^2+1}\,dx = \alpha\int_0^{\infty} e^{-\alpha x}\arctan(x)\,dx.$$

$$\text{(b)} \qquad \int_0^{\infty} \frac{\cos(\alpha x)}{x+1}\,dx = \int_0^{\infty} \frac{\cos(x)}{x+\alpha}\,dx =$$
$$\int_0^{\infty} \frac{xe^{-\alpha x}}{x^2+1}\,dx = \frac{\alpha}{2}\int_0^{\infty} e^{-\alpha x}\log(x^2+1)\,dx.$$

(See also **Problem I 2.2.33**.)

1.7.92 Prove that for any $a \in \mathbb{R}$, the limits of $\dfrac{\sin(z)}{z^a}$, $\dfrac{\cos(z)}{z^a}$, $\dfrac{\sinh(z)}{z^a}$, $\dfrac{\cosh(z)}{z^a}$, as $z \to \infty$, do not exist.

1.7.93 (1) For $a > 0$ and $0 < r < R$ integrate the function $f(z) = \dfrac{e^{iaz}}{z}$ along the contour $C = [r, R] + A_R^+ + [Ri, ri] + A_r^-$, where $A_R^+ = \left\{ Re^{i\theta} \mid 0 \leq \theta \leq \dfrac{\pi}{2} \right\}$ and $A_r^- = \left\{ Re^{i\theta} \mid \dfrac{\pi}{2} \geq \theta \geq 0 \right\}$ and then take limits as $r \to 0$ and $R \to \infty$ to prove

$$\text{(a)} \quad \int_0^\infty \frac{\cos(ax) - e^{-ax}}{x}\,dx = 0, \qquad \text{and} \qquad \text{(b)} \quad \int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{2}.$$

(You may need the **Lebesgue Dominated Convergence Theorem, I 2.3.11**, or **Example I 2.3.17**.)

(2) Prove that these integrals do not converge absolutely.

(See also **Examples I 1.3.18, I 2.2.8, I 2.3.11, 1.7.35** and **Problem I 1.3.16**.)

1.7.94 Use $f(z) = \dfrac{1 - e^{2iz}}{z^2}$, the **contour in Figure 1.16** and **Lemma 1.7.3** to find to find the following known results in a different way

$$\int_{-\infty}^\infty \frac{\sin^2(x)}{x^2}\,dx = \pi, \qquad \text{and} \qquad \text{P.V.} \int_{-\infty}^\infty \frac{\sin(2x)}{x^2}\,dx = 0.$$

1.7.95 If $a > 0$ constant, prove

$$\int_0^\infty \frac{x^2 - a^2}{x^2 + a^2} \cdot \frac{\sin(x)}{x}\,dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x^2 - a^2}{x^2 + a^2} \cdot \frac{\sin(x)}{x}\,dx = \pi \left(e^{-a} - \frac{1}{2} \right).$$

1.7.96 (a) For $a > 0$, $a = 0$ and $a < 0$ constant, find the two integrals

$$\int_{-\infty}^\infty \frac{e^{iax}}{x \pm i}\,dx.$$

(b) Separate real and imaginary parts to find the two corresponding real integrals in all cases.

If an integral does not exist, then find its principal value.

1.7.97 For $a > 0$, $a = 0$ and $a < 0$ constant, find the integral

$$\int_{-\infty}^\infty \frac{e^{iax}}{(x^2 + 1)(x - 2i)}\,dx.$$

Then separate real and imaginary parts to find the two corresponding real integrals in all cases.

If some integrals do not exist then find their principal values.

1.7.98 Use **Theorem 1.7.5** and imitate **Example 1.7.32** to find the

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{xe^{iax}}{x^2-b^2}\,dx=\text{P.V.}\int_{-\infty}^{\infty}\frac{x\cos(ax)+ix\sin(ax)}{x^2-b^2}\,dx,$$

for $a\in R$ and $b>0$.

Then find the two corresponding real principal values

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\cos(ax)}{x^2-b^2}\,dx=0,$$

and

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin(ax)}{x^2-b^2}\,dx=\begin{cases}\operatorname{sign}(a)\,\pi\cos(ab), & \text{if } a\neq 0,\\ \\ 0, \quad \text{if } a=0.\end{cases}$$

Now, in view of **Example 1.7.34** and its **Remarks** find

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin(ax)}{x^4-b^4}\,dx.$$

1.7.99 Prove that $\forall\ a\in\mathbb{R}$ and $\forall\ b\in\mathbb{R}$, we have

$$\int_{-\infty}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,dx=2\int_{0}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,dx=\pi(|b|-|a|).$$

(See also **Problem I 2.2.17** and the relative part of **Example 1.7.36**.)

1.7.100 Prove that the integral

$$\int_{-\infty}^{\infty}\frac{x\sin(x)}{x^4+2x^2+2}\,dx$$

exists and then evaluate it by computing its principal value.

1.7.101 Find the principal values

$$\text{(a)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{2ix}dx}{x-1},\quad \text{(b)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{3ix}dx}{x+1},$$

$$\text{(c)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{xe^{3ix}dx}{x^2-1},\quad \text{(d)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{e^{-2ix}dx}{x^3+1},$$

$$\text{(e)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{xe^{3ix}dx}{x^3+1},\quad \text{(f)}\quad \text{P.V.}\int_{-\infty}^{\infty}\frac{x^2e^{-5ix}dx}{x^4-1}.$$

1.7.102 Find the formula for the Fourier type integral

$$\int_{-\infty}^{\infty} \frac{x^m e^{iax}}{x^{2n}+b^2}\,dx$$

with $a \in \mathbb{R}$ and $b > 0$ constants and $m,\ n$ integers satisfying $2n \geq m+1$.

Then, separate real and imaginary parts to find two real integrals.

Check the hypotheses of the **inversion Theorem, 1.7.6**, and apply it appropriately.

1.7.103 If $a > 0$ and μ constants, compute the Fourier type integral

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x} e^{i\mu x}\,dx.$$

1.7.104 Use the techniques of this section, like in **Example 1.7.36** along with appropriate adjustments, and trigonometric identities such as:

$$4\sin^3(x) = 3\sin(x) - \sin(3x),$$

$$8\sin^4(x) = 3 - 4\cos(2x) + \cos(4x),$$

etc., to compute the following ten integrals:

$$(1) \qquad \int_{-\infty}^{\infty} \frac{\sin^3(x)}{x^k}\,dx \qquad \text{with} \qquad k = 1,\ 2,$$

$$(2) \qquad \int_{-\infty}^{\infty} \frac{\sin^4(x)}{x^l}\,dx \qquad \text{with} \qquad l = 1,\ 2,\ 3,$$

$$(3) \qquad \int_{0}^{\infty} \frac{\sin^3(x)}{x^k}\,dx \qquad \text{with} \qquad k = 1,\ 2,$$

$$(4) \qquad \int_{0}^{\infty} \frac{\sin^4(x)}{x^l}\,dx \qquad \text{with} \qquad l = 1,\ 2,\ 3.$$

(Compare with **Problems I 2.2.38** and **I 2.5.17** and the answers provided there.)

1.7.105 (a) Consider any $a \in \mathbb{R}$ and any $n \in \mathbb{N}$. Let $n = 2m$ when n is even, and $n = 2m+1$ when n is odd (so $m \in \mathbb{N}_0$). Prove that

$$\int_0^\infty \frac{\sin^n(ax)}{x^n}\,dx = \begin{cases} \dfrac{\pi a^{n-1}}{2^n(n-1)!}\displaystyle\sum_{k=0}^{m}(-1)^k\binom{n}{k}(n-2k)^{n-1}, & \text{if} \quad a>0, \\ 0, & \text{if} \quad a=0, \\ -\dfrac{\pi a^{n-1}}{2^n(n-1)!}\displaystyle\sum_{k=0}^{m}(-1)^k\binom{n}{k}(n-2k)^{n-1}, & \text{if} \quad a<0. \end{cases}$$

(E.g., for $a = 1$ and $n = 5,\ 6,\ 7$, the integral is $\dfrac{115\,\pi}{384},\ \dfrac{11\pi}{40},\ \dfrac{5887\pi}{23040}$, respectively, and so on.)

(b) If the integral is taken over $(-\infty,\infty)$, then the answer is two times the given answer because the integrand is an even function.

(c) Now compute

$$\int_0^\infty \frac{\sin^5(2x)}{x^5}\,dx, \qquad \text{and} \qquad \int_0^\infty \frac{\sin^6(5x)}{x^6}\,dx.$$

[Compare with **Problems I 2.2.15, I 2.2.16, I 2.2.28, (a), (f), I 2.2.37, I 2.2.38, I 2.5.17, I 2.5.19**, and especially **I 2.5.20**.

Also, $\forall\ a \neq 0$ in $\mathbb{R}$ and $n < m$ in $\mathbb{N}_0$, $\displaystyle\int_0^\infty \frac{\sin^n(ax)}{x^m}\,dx = \operatorname{sign}(a)\cdot\infty$, due to the degree of the unboundedness of the integrand at $x = 0$.]

[Hint: One way to prove this nice result is by completing the following steps. Without loss of generality consider $a = 1$. (Afterward use the substitution $u = ax$, with $a > 0$, etc.) Split the integral in two cases: $n = 2m+1$ odd and $n = 2m$ even. Notice that the integrand function is even over $\mathbb{R}$ and, by the **Binomial Theorem**, we get

$$\frac{\sin^n(z)}{z^n} = \frac{\left(e^{iz}-e^{-iz}\right)^n}{2^n i^n z^n} = \frac{1}{2^n i^n z^n}\sum_{l=0}^{n}(-1)^l\binom{n}{l}e^{i(n-2l)z}.$$

Pick $0 < \epsilon < R$ and integrate the last part of this equality over the following two simple contours that include $z = 0$ in their interiors:

$$C_1 = [-R,-\epsilon] + L_\epsilon^+ + [\epsilon, R] + S_R^+,$$

where the two straight segments are on the x-axis and we have the two half circle: $L_\epsilon^+ = \left\{z = \epsilon e^{i\theta} \ \mid\ \pi \le \theta \le 2\pi\right\}$ and
$S_R^+ = \left\{z = Re^{i\theta} \ \mid\ 0 \le \theta \le \pi\right\}$.

$$C_2 = [R,\epsilon] + U_\epsilon^+ + [-\epsilon,-R] + T_R^+,$$

where the two straight segments are on the x-axis and we have the two half circles: $U_\epsilon^+ = \{z = \epsilon e^{i\theta} \mid 0 \le \theta \le \pi\}$ and
$T_R^+ = \{z = Re^{i\theta} \mid \pi \le \theta \le 2\pi\}$.

Compute the residues at $z = 0$, using **Way 3** of **Subsection 1.6.2**. Use the **Residue Theorem, 1.7.1**, and add the two results of the two integrations sidewise. Use the symmetry of the binomial coefficients and keep track with the association of minus signs in either of the two cases and simplify the addition results in both sides of the equality obtained.

Now, take limits as $R \to \infty$ and use **Lemma 1.7.5** and limits as $\epsilon \to 0$. Then, divide both sides by 4. (With this method, four integrals $\displaystyle\int_0^\infty \frac{\sin^n(x)}{x^n}\,dx$ are produced.)]

1.7.106 (**Cauchy's discontinuous factor**.) In **Example I 2.2.11**, we have seen a discontinuous factor. Here we present another one due to Cauchy.

Consider any $a \in \mathbb{R}$, any $c > 0$, any $R > 0$, any $0 < \epsilon < R$, the function $f(z) = \dfrac{e^{iaz}}{z}$ and the contour

$$C^+ = [-Ri, c-Ri] + [c-Ri, c+Ri] + [c+Ri, Ri] + [Ri, \epsilon i] + A_\epsilon^- + [-\epsilon i, -Ri],$$

where $A_\epsilon^- = \left\{z = \epsilon\, e^{i\theta}, \mid \dfrac{\pi}{2} \ge \theta \ge -\dfrac{\pi}{2}\right\}$ is half a circle of radius $\epsilon > 0$, and all the other pieces are straight segments as indicated.

Consider the integral $\displaystyle\int_{C^+} f(z)\,dz$. Use appropriate results (such as the **Residue Theorem**, **1.7.1**, and **Lemma 1.7.3**) and then take limits as $\epsilon \longrightarrow 0$ and $R \longrightarrow \infty$, to prove that for any $c > 0$ constant

$$\mathrm{C}(a) := \frac{1}{\pi} \cdot \left[\mathrm{P.V.} \int_{-\infty}^{\infty} \frac{e^{a(c+ti)}}{c+ti}\,dt\right] = \begin{cases} 2, & \text{if} \quad a > 0, \\ 1, & \text{if} \quad a = 0, \\ 0, & \text{if} \quad a < 0. \end{cases}$$

[See also **Example I 2.2.11, Problem I 2.2.23, (b)**.]

1.7.107 Apply the **Parseval equation** separately and the **general Parseval equation** jointly to the Fourier transforms found in **Examples 1.7.33** and **1.7.34** to find three new integrals.

Find also these integrals by differentiating with respect to b the known integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+b^2}\, dx = \frac{\pi}{b}$$

and manipulating the result. Use the differentiation part of **Theorem I 2.2.1** to justify that this differentiation is legitimate.

1.7.108 Prove

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\, dx = 2\pi.$$

1.7.109 (a) Find the Fourier transforms of the following six functions in $\mathbb{R}$, or explain if they do not exist:

$$f_1(x) = \frac{1}{x^2+1}, \qquad f_2(x) = \frac{x}{x^2+1}, \qquad f_3(x) = \frac{x^3}{\left(x^2+1\right)^2},$$
$$f_4(x) = e^{-4x^2}, \qquad f_5(x) = e^{-x}, \qquad f_6(x) = xe^{-x}.$$

(b) Check and apply the **inversion Theorem, 1.7.6**, to $f_3(x)$ and $f_4(x)$.

[Hint: For the last three functions, you cannot use **Lemma 1.7.6**, since the condition $\lim\limits_{z\to\infty} f(z) = 0$ is not satisfied. Explain why!

In the last three questions, you must directly use

$$\int_{-\infty}^{\infty} f(x)e^{itx}dx = \int_{-\infty}^{\infty} f(x)\left[\cos(tx) + i\sin(tx)\right] dx =$$
$$\int_{-\infty}^{\infty} f(x)\cos(tx)\, dx + i\int_{-\infty}^{\infty} f(x)\sin(tx)\, dx$$

and compute the two integrals in any possible way, or use already computed integrals, or the evenness and/or the oddity of certain functions. See **Problem 1.7.32, (a)**, for the cosine integral of the even function $f_4(x)$, whereas its sine integral is automatically zero (by the oddity). Justify why the Fourier transforms of $f_5(x)$ and $f_6(x)$ do not exist.]

1.7.110 (a) Use **Problem 1.7.32, (a)**, to prove that for any $a > 0$ real constant

$$\mathcal{F}\left[e^{-a^2x^2}\right](t) = \frac{\sqrt{\pi}}{a}\, e^{-\frac{t^2}{4a^2}}.$$

(b) Derive the same result in the following way: Let

$$\phi(t) := \mathcal{F}\left[e^{-a^2x^2}\right](t).$$

Then, check that it is legitimate to take derivative under the integral sign

and use the correct integration by parts to show that $\dfrac{d[\phi(t)]}{dt} = \dfrac{-t}{2a^2}\phi(t)$.
Also, by **Problem I 2.1.11**, we have that $\phi(0) = \dfrac{\sqrt{\pi}}{a}$.
Now solve this initial value problem to get the result.

(c) Show that if $a = \dfrac{1}{\sqrt{2}}$, then $f(x) = e^{-a^2x^2}$ and
$\mathfrak{F}\left[e^{-a^2x^2}\right](t) = \dfrac{1}{\sqrt{2\pi}}\mathcal{F}\left[e^{-a^2x^2}\right](t)$ are given by the same formula.

1.7.111 Consider any $c > 0$ constant.

(a) Let $g_c(x) = e^{-cx^2}$, with $x \in \mathbb{R}$. Prove that

$$(g_a * g_b)(x) = \sqrt{\frac{\pi}{\sqrt{a+b}}}\, e^{\frac{-abx^2}{a+b}} = (g_b * g_a)(x) \longrightarrow 0, \quad \text{as} \quad x \longrightarrow \pm\infty.$$

You can prove this directly or by considering **convolution Property (7), Rule (g)**, of the Fourier transform. What is the answer if $a = b$?

$$\text{(b)} \quad \text{Let} \qquad f_c(x) = \begin{cases} e^{-cx}, & \text{if} \quad x \geq 0 \\ \\ 0, & \text{if} \quad x < 0 \end{cases}$$

Prove that

$$(f_a * f_b)(x) = (f_b * f_a)(x) = \begin{cases} \dfrac{e^{-bx} - e^{-ax}}{a-b}, & \text{if} \quad x \geq 0, \\ \\ 0, & \text{if} \quad x < 0. \end{cases}$$

What is the answer if $a = b$?

(c) Let $\quad h_c(x) = \left\{\begin{array}{ll} e^{cx}, & \text{if} \quad x \leq 0, \\ \\ 0, & \text{if} \quad x > 0. \end{array}\right\}. \qquad$ Find $\quad (h_a * h_b)(x)$.

(d) Find $(f_a * h_b)(x)$, where f_a and h_b are the functions defined in **(b)** and **(c)**.

(e) Let $k_c(x) = e^{-c|x|}$, $x \in \mathbb{R}$. Find $(k_a * k_b)(x)$.

(See also **Problem I 2.7.27** and compare.)

1.7.112 Consider $a < b$ real numbers, $n = 1, 2, 3, \ldots$, and a continuous function $f : [a, b] \longrightarrow \mathbb{R}$. Prove:

$$\text{(a)} \quad \lim_{n\to\infty} \int_a^b f(x)\cos(nx)dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_a^b f(x)\sin(nx)dx = 0.$$

$$\text{(b)} \qquad \lim_{n\to\infty}\int_a^b f(x)\sin^2(nx)dx = \frac{1}{2}\int_a^b f(x)dx.$$

[Hint: Use the **Riemann-Lebesgue Lemma, 1.7.7**.]

1.7.113 Let $f(x) = \chi_{[a,b]}(x)$, the characteristic function of an interval $[a, b]$ ($a \leq b$, reals), and similarly $g(x) = \chi_{[c,d]}(x)$ ($c \leq d$, reals).

(a) Prove: $\forall\ x \in \mathbb{R},\ \ (f * g)(x) =$

$$\int_{x-d}^{x-c} f(u)\,du = \max\{\,0,\ \ (\min\{b,\ x-c\} - \max\{a,\ x-d\})\,\} =$$

$$\int_{x-b}^{x-a} g(v)\,dv = \max\{\,0,\ \ (\min\{d,\ x-a\} - \max\{c,\ x-b\})\,\}\,.$$

(b) Graph the function $y = (f * g)(x)$.

1.7.114 Find explicitly $(f * f)(x)$ for $x \in \mathbb{R}$, if $f(x)$ is the function of **Example 1.7.39**

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}}\,, & \text{if } x \in [-1, 0) \cup (0, 1], \\ \\ 0\,, & \text{otherwise.} \end{cases}$$

[Hint: Consider the separate cases: $-\infty < x \leq -2$, $\ -2 < x \leq -1$, $-1 < x < 0$, $\ x = 0$, $\ 0 < x \leq 1$, $\ 1 < x \leq 2$, $\ 2 < x < \infty$. In the two extreme cases, the answer is 0. For $x = 0$, $(f * f)(0) = \infty$, etc.]

1.7.115 If $f(x) = \dfrac{x}{x^4 + 1}$ and $g(x) =$ constant, $\forall\ x \in \mathbb{R}$, prove that $f(x)$ is both absolutely and square integrable and $f * g \equiv 0$ in $\mathbb{R}$.

1.7.116 Consider the non-negative function (of infinitely many steep isosceles triangles) defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 2(x+1), & \text{if } \ -1 \leq x < 0, \\ \\ -2(x-1), & \text{if } \ 0 \leq x \leq 1, \\ \\ n^4\left(x - n + \dfrac{1}{|n|^3}\right), & \text{if}\, n \in \mathbb{Z} - \{0,\, \pm 1\} \,\text{and}\, n - \dfrac{1}{|n|^3} \leq x < n, \\ \\ -n^4\left(x - n - \dfrac{1}{|n|^3}\right), & \text{if}\, n \in \mathbb{Z} - \{0,\, \pm 1\} \,\text{and}\, n \leq x \leq n + \dfrac{1}{|n|^3}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Prove:

(a) $f(x)$ is even, unbounded, continuous, but not uniformly continuous.

(b) $\displaystyle\int_{\mathbb{R}} |f(x)|\,dx = \frac{\pi^2}{3}$ and $\displaystyle\int_{\mathbb{R}} |f(x)|^2\,dx = \infty$.

(c) $(f * f)(0) = \infty$. So, the convolution of two continuous, nonnegative and absolutely integrable functions may not be bounded and so may not be defined at some point(s), let alone be continuous. (Compare this with the **next Problem** below.)

(d) The Fourier transform $\hat{f}(t)$ exists but is not absolutely integrable. (E.g., see **Corollary 1.7.6**.)

(See also **Example 1.7.39**. Compare with **Problem 1.7.119**.)

1.7.117 (Compare with the **previous Problem**.) Consider two absolutely integrable functions f and $g : \mathbb{R} \to \mathbb{R}$. (I.e., $\|f\|_1 < \infty$ and $\|g\|_1 < \infty$.)

Prove that if one of them is both continuous and bounded, or one is continuous and the other bounded, then $f * g$ is also continuous and bounded.

In fact, for the boundedness part, if $|f| \le B$, where $B \ge 0$ constant, then $\|f * g\|_\infty \le B \cdot \|g\|_1$.

1.7.118 Let f and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be two absolutely integrable functions.

(a) Prove that if one of them is uniformly continuous, then $f * g$ is uniformly continuous.

(b) Prove that if g is differentiable and g' is absolutely integrable and either bounded or uniformly continuous, then $f * g$ is differentiable. In fact, $(f * g)' = f * g'$ and is either continuous or uniformly continuous, respectively. (Analogous result if the conditions on g transfer to f.)

(c) State the hypotheses for $f * g \in \mathfrak{C}^n$, where $n \in \mathbb{N} \cup \{\infty\}$, and also the corresponding results.

1.7.119 Consider two functions f and $g : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) Prove that if f is uniformly continuous, then so is $|f|$.
(b) Prove that if f is uniformly continuous and absolutely integrable, then $\displaystyle\lim_{x \to \pm\infty} f(x) = 0$.
(c) For any $s \in \mathbb{R}$, let $f_s(x) := f(x - s)$.
Prove that $\|f_s\|_p = \|f\|_p$, for any $p \ge 1$.
Next prove that, if $\|f\|_p < \infty$ for some $p \ge 1$, then
$\forall\ \epsilon > 0$, $\exists\ \delta = \delta(\epsilon) > 0$ such that $\forall\ s \in \mathbb{R}$ and $\forall\ t \in \mathbb{R}$,
$$|s - t| < \delta \implies \|f_s - f_t\|_p < \epsilon.$$
(d) If $p \ge 1$ and $q \ge 1$ are two conjugate exponents, (i.e., either one

of them is 1 and the other is ∞, or $\dfrac{1}{p}+\dfrac{1}{q}=1$), and $\|f\|_p<\infty$ and $\|g\|_q<\infty$, then $f*g$ is absolutely integrable and uniformly continuous and so, by **(b)**, $\lim_{x\to\pm\infty}(f*g)(x)=0$.

(Compare with **Problem 1.7.116**.)

1.7.120 We say that a function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is **Hölder continuous of order** $\alpha>0$ (uniformly) on (the whole) $\mathbb{R}$, if

$$\exists\ M\geq 0 \text{ constant } : \forall\ u\in\mathbb{R} \text{ and } \forall\ v\in\mathbb{R},\ |f(u)-f(v)|\leq M|u-v|^{\alpha}.$$

In particular, if $\alpha=1$, we say that f is **Lipschitz continuous**[41] (uniformly) on $\mathbb{R}$.

(a) Prove that if $\alpha>1$, then f is identically constant.

(b) Prove that if f is Hölder continuous then it is uniformly continuous.

(c) If $f(x)$ is defined by

$$f(x)=\begin{cases}0, & \text{if } x\leq 0,\\[2ex] \dfrac{1}{\ln(x)}, & \text{if } 0<x\leq\dfrac{1}{2},\\[2ex] \dfrac{-1}{\ln(2)}, & \text{if } x>\dfrac{1}{2},\end{cases}$$

prove that $f(x)$ is uniformly continuous over the whole $\mathbb{R}$, but not Hölder continuous of any order.

[Hint: For the non Hölder continuity consider any $\alpha>0$, $u=x$, $v=0$ and argue by contradiction on the inequality condition of the definition near $v=0$.]

(d) Prove that if f is differentiable and its derivative is (uniformly) bounded on $\mathbb{R}$, then f is Lipschitz continuous.

(e) Let f and $g:\mathbb{R}\longrightarrow\mathbb{R}$ be two absolutely integrable functions. Prove that if one of them is Hölder continuous of some order, then $f*g$ is Hölder continuous of the same order.

[41] There are several variations of these definitions of Hölder or Lipschitz continuities, which depend on the validity of the above definition at some points only, or locally in a domain, or globally in a whole domain, and on the fact that the domain is bounded or unbounded. There are many properties of these local or global continuities, which may vary drastically depending on considering the respected functions on bounded or unbounded domains. (In the above problem, for the sake of brevity, we have considered as domain the whole unbounded $\mathbb{R}$, omitting all the other cases.)

1.7.121 Prove that if $f(x)$ is a polynomial of degree n and $g(x)$ is a real function such that $(f * g)(x)$ exists for all $x \in \mathbb{R}$, then $(f * g)(x)$ is a polynomial of degree at most n.

1.7.122 (a) Let $f(x) = ax^2 + bx + c$ (neither absolutely, nor square integrable on $\mathbb{R}$, unless $a = b = c = 0$,) and $g(x) = \dfrac{\sin^4(x)}{x^4}$ on $\mathbb{R}$. Prove

$$(f * g)(x) = \left(ax^2 + bx + c\right)\frac{2\pi}{3} + a\frac{\pi}{2}, \quad \forall\ x \in \mathbb{R},$$

(polynomial of degree 2, if $a \neq 0$,).

(b) If $f(x) = ax + b$ and $g(x) = \dfrac{\sin^4(x)}{x^3}$ on $\mathbb{R}$, then

$$(f * g)(x) = -a\frac{\pi}{2}, \quad \forall\ x \in \mathbb{R}.$$

(c) Let $f(x) = e^{-x^2}$. Prove $(f * f)(x) = \sqrt{\dfrac{\pi}{2}}\, e^{\frac{-x^2}{2}}$, $\forall\ x \in \mathbb{R}$.

1.7.123 Consider f and $g : \mathbb{R} \longrightarrow \mathbb{R}$ functions such that $f * g$ exists. Consider the following three sets:
$A := \{x \mid x \in \mathbb{R} \ :\ f(x) \neq 0\}$,
$B := \{x \mid x \in \mathbb{R} \ :\ g(x) \neq 0\}$ and
$C := \{x \mid x \in \mathbb{R} \ :\ (f * g)(x) \neq 0\}$.
Prove that $C \subseteq A + B$ and equality may fail.

1.7.124 (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely integrable function, and let function $g : \mathbb{R} \longrightarrow \mathbb{R}$ be bounded. Prove that $f * g$ is bounded. In fact, using the norm notation of **Problem I 2.6.65**, prove

$$\|f * g\|_\infty \leq \|f\|_1 \cdot \|g\|_\infty.$$

(b) Can you find the conditions under which the inequality in **(a)** holds as equality?

[Hint: For the equality question, prove that g must be constant and f must not change sign.]

1.7.125 (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an absolutely integrable function, and let function $g : \mathbb{R} \longrightarrow \mathbb{R}$ be square integrable. Prove that $f * g$ is square integrable. In fact, using the norm notation of **Problem I 2.6.65**, prove

$$\|f * g\|_2 \leq \|f\|_1 \cdot \|g\|_2.$$

(b) For the three separate cases on p: (1) $p = 1$, (2) $1 < p < \infty$, and

(3) $p = \infty$, prove the general relation

$$\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p,$$

with f absolutely integrable and g such that $\|g\|_p < \infty$.[42]

(c) Prove that the inequality in **(b)** holds as equality:

(1) When $p = 1$: iff $\{f \geq 0$ and $g \geq 0\}$ or $\{f \leq 0$ and $g \leq 0\}$ or $\{f \geq 0$ and $g \leq 0\}$ or $\{f \leq 0$ and $g \geq 0\}$.

(2) When $1 < p < \infty$: iff $f \equiv 0$ or $g \equiv 0$. [This is the case for equality in **(a)** too.]

(3) When $p = \infty$: iff $f \equiv 0$ or $g =$ constant.

[Hint: (a) Go through the following computation, justify each step and draw the conclusion:

$$\int_{\mathbb{R}_x} |f * g|^2(x)\, dx = \int_{\mathbb{R}_x} \left| \int_{\mathbb{R}_u} f(x-u)g(u)\, du \right|^2 dx \leq$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|\, du \right]^2 dx =$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)|^{\frac{1}{2}} \cdot |f(x-u)|^{\frac{1}{2}} \cdot |g(u)|\, du \right]^2 dx \leq$$

$$\int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)|\, du \int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|^2 du \right] dx =$$

$$\|f\|_1 \int_{\mathbb{R}_x} \left[\int_{\mathbb{R}_u} |f(x-u)| \cdot |g(u)|^2 du \right] dx =$$

$$\|f\|_1 \int_{\mathbb{R}_u} \left[\int_{\mathbb{R}_x} |f(x-u)| \cdot |g(u)|^2\, dx \right] du =$$

$$\|f\|_1 \int_{\mathbb{R}_u} \left[|g(u)|^2 \int_{\mathbb{R}_x} |f(x-u)|\, dx \right] du =$$

$$\|f\|_1 \cdot \|f\|_1 \int_{\mathbb{R}_u} |g(u)|^2 du = \|f\|_1^2 \cdot \|g\|_2^2 .]$$

(b) For $p > 1$, consider the dual exponent $q > 1$, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and work analogously with the Hölder's inequality. (See **project Problem I 2.6.65, inequality I 2.12** (c) For the equality questions, the cases

[42] A **more general result** is the following: Let f and g be two real functions defined on $\mathbb{R}$ that satisfy $\|f\|_p < \infty$ and $\|g\|_q < \infty$ for some numbers $p \geq 1$ and $q \geq 1$ for which there are numbers $1 \leq p',\ q',\ r',\ r \leq \infty$ such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, $\dfrac{1}{r} + \dfrac{1}{r'} = 1$ and $\dfrac{1}{p'} + \dfrac{1}{q'} = \dfrac{1}{r'}$ (or $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$). Then $\|f * g\|_r \leq \|f\|_p \cdot \|g\|_q < \infty$.

$p = 1$ and $p = \infty$ are separate. For $1 < p < \infty$, see the condition under which the Hölder's inequality holds as equality and use it appropriately. (See **project Problem I 2.6.65, inequality I 2.12**.)]

1.7.126 (a) Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that its Fourier transform exists and $f * f = f$. Prove that $f(x) = \mathcal{F}^{-1}[\chi_A(t)](x)$, where χ_A is an (absolutely) integrable characteristic function of some $A \subset \mathbb{R}$.

(b) Check everything in **(a)** when A is the closed interval $[-a, a]$ or the open interval $(-a, a)$, where $a \geq 0$ constant.

(c) If we only assume $f * f = f$ and nothing about the Fourier transform, what can you say about f?

(Compare with **project Problem I 2.7.59**.)

1.7.127 Assume that for two functions f and $g : \mathbb{R} \longrightarrow \mathbb{R}$, $f * g = 0$, a.e., and their Fourier transforms exist and are analytic functions in the sense that their real and imaginary parts are power series locally. Then prove that either f or g (or both) is (are) zero, a.e.

[The Fourier transform of a function is not always analytic. For example, every square integrable function is the Fourier transform some square integrable function, but a square integrable function may not be analytic. This is a consequence of the Plancherel Theorem in the theory of the Fourier transform. For instance, see **Examples 1.7.34, 1.7.37**, the **previous Problem**, etc. If an absolutely or square integrable function is not zero in a bounded set, then its Fourier transform is analytic. (Analogous results on the real or complex Laplace transform.)]

1.7.128 Consider the function $\hat{f}(w) = \dfrac{1}{w^3 - 8}$, with $w = t + is$, to be the Fourier transform of some functions. Notice that its singularities are the simple poles $w_0 = 2 = 2+0i$, $w_1 = -1+\sqrt{3}i$ and $w_2 = -1-\sqrt{3}i$.

Consider the four horizontal strips in $\mathbb{C}$:

(1) $-\infty < s < -\sqrt{3}$, (2) $-\sqrt{3} < s < 0$,
(3) $0 < s < \sqrt{3}$, (4) $\sqrt{3} < s < \infty$,

and on each one find the corresponding inverse Fourier transform by applying the **inversion formula (1.40)**.

[Hint: Imitate **Example 1.7.46**. Use appropriate contours, as here we have three poles, one of which is on the real axis.]

1.7.129 Consider Bessel's equation **(1.43)** (in the last **Application** of this section). If we now choose as two branch cuts the
(1) $\mathbb{C} - [+1, +\infty)$ for $\sqrt{w-1}$, and (2) $\mathbb{C} - [-1, -\infty)$ for $\sqrt{w+1}$, and a contour similar to **Contour 8**, but not exactly the same, in **Figure**

1.12 of the practice **Problem 1.7.29**, show that with these choices we do not get a representation of the solutions of Bessel's equation. Can you give reasons why?

1.7.130 Consider Bessel's differential equation with imaginary argument of order 0, **(1.39)**,

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} - xu = 0.$$

Prove that the Fourier transform with complex argument $w = t + is$ of its solution, up to a multiplicative constant, is

$$\hat{u}(w) = \frac{1}{\sqrt{w^2+1}}.$$

[Hint: Notice that $w = -i = 0 + (-1)i$ and $w = i = 0 + 1i$ are singular points. Then, use appropriate branch cuts and contours in the three infinite horizontal strips

$$\mathbb{S}_1 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad -\infty < s < -1\},$$
$$\mathbb{S}_2 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad -1 < s < 1\},$$
$$\mathbb{S}_3 = \{w = t + si \mid \infty < t < \infty \quad \text{and} \quad 1 < s < \infty\},$$

to find integral representations of the solution of this differential equation (the Bessel function with imaginary argument of order 0), up to a (complex) multiplicative constant.]

Facts on Uniformly Continuous Functions.

In many instances and problems so far, we have encountered the importance of the **real functions** which are **uniformly continuous** on their domains. Here, we would like to make a compendium on the uniformly continuous real functions. So, we consider real functions

$$f : \mathcal{D} \longrightarrow \mathbb{R} \quad \text{with domain a set} \quad \mathcal{D} \subseteq \mathbb{R}$$

and we can provide proofs or counterexamples to the following facts:

1. If a function is uniformly continuous on $\mathcal{D}$, then so is its absolute value and any multiple of it by a constant.

2. If two functions are uniformly continuous on $\mathcal{D}$, then their sum and difference are uniformly continuous.

3. If two functions are uniformly continuous on $\mathcal{D}$, then their **maximum** and **minimum**[43] are uniformly continuous.

4. The product and the quotient of two uniformly continuous functions may not be uniformly continuous.

5. The composition of two uniformly continuous functions is uniformly continuous.

6. A continuous function with domain $\mathcal{D}$ a closed and bounded set is uniformly continuous.

7. A uniformly continuous function defined on an open interval (a, b), with $\infty < a < b < \infty$, can be extended continuously to the endpoints a and b.

8. The uniform limit of a sequence of uniformly continuous functions is uniformly continuous.

9. The point-wise limit of a sequence of uniformly continuous functions may or may not be uniformly continuous or even continuous.

10. A continuous function defined in (a, b), where $-\infty \le a < b \le \infty$ such that the limits $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ exist, is uniformly continuous.

11. A uniformly continuous function with domain $\mathcal{D}$ a bounded set is bounded.

12. A uniformly continuous function with domain $\mathcal{D}$ an unbounded set may or may not be bounded.

13. A Hölder continuous function is uniformly continuous, but not vice-versa.

14. An absolutely integrable and uniformly continuous function defined in $\mathbb{R}$ has limits $\lim\limits_{x \to \pm\infty} f(x) = 0$.

15. Now, verify or give counterexamples to the statements of the **following table** and hint:

[43] To prove this, we can use the preceding facts and the known formulae for the **maximum** and **minimum** of two functions f and g

$$\max(f, g)(x) := \max[f(x), g(x)] = \frac{1}{2}\left(f + g + |f - g|\right)(x)$$

and

$$\min(f, g)(x) := \min[f(x), g(x)] = \frac{1}{2}\left(f + g - |f - g|\right)(x).$$

Continuous Function $f : \mathcal{D} \to \mathbb{R}$ $\mathcal{D} \subseteq \mathbb{R}$	$\mathcal{D} = (a, b)$ with $-\infty < a < b < \infty$	$\mathcal{D} = [a, \infty)$ with $a \in \mathbb{R}$	$\mathcal{D} = \mathbb{R}$
f bounded	may/may not be unif. cont.	may/may not be unif. cont.	may/may not be unif. cont.
f unbounded	not uniformly continuous	may/may not be unif. cont.	may/may not be unif. cont.
f' bounded	uniformly continuous	uniformly continuous	uniformly continuous
f' unbounded	may/may not be unif. cont.	may/may not be unif. cont.	may/may not be unif. cont.
f bounded and f' unbounded	may/may not be uniformly continuous	may/may not be uniformly continuous	may/may not be uniformly continuous
f unbounded and f' unbounded	not uniformly continuous	may/may not be uniformly continuous	may/may not be uniformly continuous

{If the domain $\mathcal{D}$ is of the type $[a, b)$, $(a, b]$, $(-\infty, b]$, we can figure out the answers to the same questions with the help of this table.}

[Hint: Examine the **following table** and check the examples provide and give the proofs asked.

Continuous Function $f : \mathcal{D} \to \mathbb{R}$ $\mathcal{D} \subseteq \mathbb{R}$	$\mathcal{D} = (a, b)$ with $-\infty < a < b < \infty$	$\mathcal{D} = [a, \infty)$ with $a \in \mathbb{R}$	$\mathcal{D} = \mathbb{R}$
f bounded	$\mathcal{D} = (0, 1)$ $f(x) = x$ $f(x) = \sin\left(\frac{1}{x}\right)$	$\mathcal{D} = [0, \infty)$ $f(x) = e^{-x}$ $f(x) = \sin(e^x)$	$f(x) = e^{-\lvert x\rvert}$ $f(x) = \sin(e^x)$
f unbounded	Prove!	$\mathcal{D} = [0, \infty)$ $f(x) = x$ $f(x) = x^2$	$f(x) = x$ $f(x) = x^2$
f' bounded	Prove!	Prove!	Prove!
f' unbounded	$\mathcal{D} = (0, 1)$ $f(x) = \sqrt{x}$ $f(x) = \sin\left(\frac{1}{x}\right)$	$\mathcal{D} = [0, \infty)$ $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ $f(x) = x\sin(x)$	$f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ $f(x) = x\sin(x)$
f bounded and f' unbounded	$\mathcal{D} = (0, 1)$ $f(x) = \sqrt{x}$ $f(x) = \sin\left(\frac{1}{x}\right)$	$\mathcal{D} = [0, \infty)$ Construct example! $f(x) = \sin(e^x)$	Construct example! **(*)** $f(x) = \sin(e^x)$
f unbounded and f' unbounded	Prove!	$\mathcal{D} = [0, \infty)$ $f(x) = x+$ $x^2 \sin\left(\frac{1}{x^2}\right)$ $f(x) = x\sin(x)$	$f(x) = x+$ $x^2 \sin\left(\frac{1}{x^2}\right)$ $f(x) = x\sin(x)$

(*) In either construction asked in the table do, e.g., the following: For every integer $n \neq 0, \pm 1$, place a smooth ($\mathfrak{C}^\infty$), non-negative impulse over the interval $\left[n - \frac{1}{n^2}, n + \frac{1}{n^2}\right]$, symmetrical about the vertical line $x = n$, and of height $\frac{1}{n}$. Define the function to be 0, otherwise. Etc.]

1.7.9 Improper Integrals and Logarithms

We begin with the following:

Example 1.7.47 [Compare with **Problem I 2.1.23, (a)** and the generalization in **Problem 1.7.135**.]

Prove:

$$\int_0^\infty \frac{\ln(x)}{x^2+1}\,dx = 0.$$

We consider the complex function

$$f(z) = \frac{\log(z)}{z^2+1}.$$

This function has non-isolated singularities along a chosen branch cut of the complex $\log(z)$ and isolated singularities at the two roots $z = \pm i$ of the denominator.

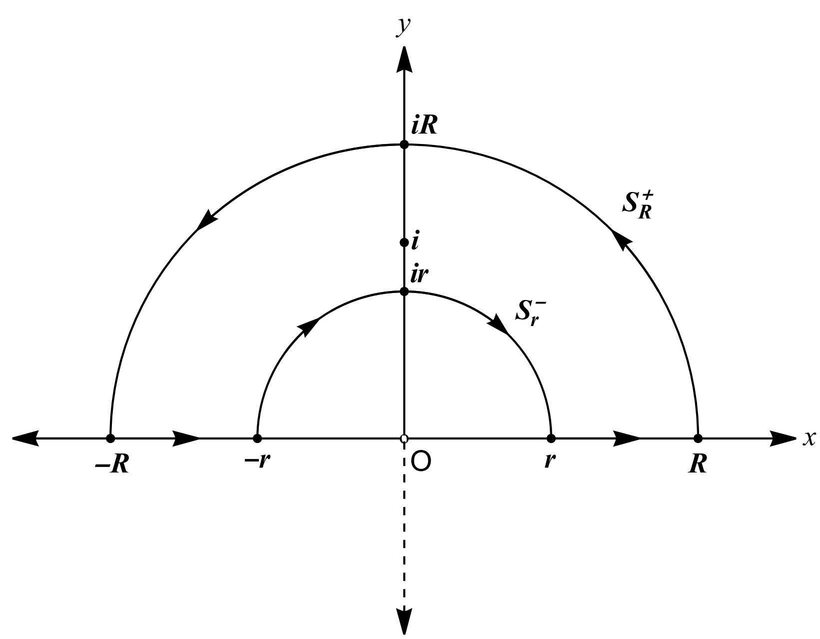

FIGURE 1.19: Contour 13 for Example 1.7.47

We choose as the branch cut the ray of the non-positive imaginary semi-axis $\left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \{-yi \mid y \geq 0\}$ and define the open domain

$$\mathcal{D} = \mathbb{C} - \left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \mathbb{C} - \{-yi \mid y \geq 0\} = \mathbb{C} - \{yi \mid y \leq 0\}.$$

Then, $f(z)$ is holomorphic in $\mathcal{D} - \{i\}$, and for any $z = re^{i\theta} \in \mathcal{D}$ we have

$$\log(z) = \log\left(re^{i\theta}\right) = \ln(r) + i\theta, \text{ with } -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

Next, we must choose a convenient simple closed piecewise differentiable contour in $\mathbb{C}$ that avoids

$$\left\{ye^{i\frac{3\pi}{2}} \mid y \geq 0\right\} = \{-yi \mid y \geq 0\} \quad \text{and} \quad z = i.$$

Then, we pick numbers r and R such that $0 < r < 1 < R < \infty$, and we let

$$C^+ = [r, R] + S_R^+ + [-R, -r] + S_r^-,$$

where S_R^+ is the upper half of $C(0, R)$, positively oriented, and S_r^- is the upper half of $C(0, r)$, negatively oriented. This contour lies in $\mathcal{D}$ and encloses the isolated singularity $z = i$ in its interior. See **Figure 1.19**.

Then,

$$\operatorname*{Res}_{z=i} f(z) = \frac{\log(i)}{2i} = \frac{\ln(1) + \frac{\pi}{2}i}{2i} = \frac{\pi}{4},$$

and by **the Residue Theorem, 1.7.1**, we get

$$\oint_{C^+} f(z)dz =$$

$$\int_{[r,R]} f(z)dz + \int_{S_R^+} f(z)dz + \int_{[-R,-r]} f(z)dz + \int_{S_r^-} f(z)dz =$$

$$2\pi i\frac{\pi}{4} = \frac{\pi^2}{2}\, i.$$

Since

$$\lim_{z\to\infty} [zf(z)] = \lim_{z\to\infty} \frac{z\log(z)}{z^2+1} = 0$$

(for the proof of this easy fact, see **Problem 1.7.131**), by **Lemma 1.7.1** we have that

$$\lim_{R\to\infty} \int_{S_R^+} f(z)\, dz = 0.$$

Also,

$$\lim_{r\to 0} \int_{S_r^-} f(z)\, dz = 0,$$

as it is seen by

$$\left| \int\limits_{S_r^-} f(z)\, dz \right| =$$

$$\left| \int_\pi^0 \frac{\log\left(re^{i\theta}\right)}{r^2 e^{2i\theta}+1} i\, r\, e^{i\theta}\, d\theta \right| \le \int_0^\pi \left| \frac{\ln(r)+i\theta}{r^2e^{2i\theta}+1} i\, r\, e^{i\theta} \right| d\theta =$$

$$\int_0^\pi \frac{\sqrt{\ln^2(r)+\theta^2}}{|r^2e^{2i\theta}+1|}\, r\, d\theta \le \int_0^\pi \frac{\sqrt{\ln^2(r)+\pi^2}}{1-r^2}\, r\, d\theta =$$

$$\frac{\sqrt{\ln^2(r)+\pi^2}}{1-r^2}\, r\, \pi \longrightarrow 0, \quad \text{as} \quad r \longrightarrow 0^+.$$

Therefore,

$$\lim_{\substack{r\to 0^+\\ R\to\infty}} \int_{-R}^{-r} f(z)\, dz + \lim_{\substack{r\to 0^+\\ R\to\infty}} \int_r^R f(z)\, dz = \frac{\pi^2}{2}\, i.$$

Since the complex argument along the negative real semi-axis is π and along the positive real semi-axis is 0, we find

$$\int_{-\infty}^0 \frac{\ln(|x|)+i\pi}{x^2+1}\, dx + \int_0^\infty \frac{\ln(x)}{x^2+1}\, dx = \frac{\pi^2}{2}\, i.$$

Then,

$$2\int_0^\infty \frac{\ln(x)}{x^2+1}\, dx + \int_{-\infty}^0 \frac{\pi\, i}{x^2+1}\, dx = \frac{\pi^2}{2}\, i.$$

This answer is purely imaginary. Therefore, its real part is zero. So, we obtain the **result**

$$\int_0^\infty \frac{\ln(x)}{x^2+1}\, dx = 0.$$

As for the equality of the imaginary parts, this yields

$$\int_{-\infty}^0 \frac{\pi\, i}{x^2+1}\, dx = \frac{\pi^2}{2}\, i, \quad \text{and so} \quad \int_{-\infty}^0 \frac{1}{x^2+1}\, dx = \frac{\pi}{2},$$

which is also obtained by a direct elementary computation and use of $\arctan(x)$.

▲

Remark 1: The given integral exists to begin with [see **Problems I 2.1.23, (a)**, and **1.1.3**]. Therefore, here we have evaluated its value by means of its principal value.

Along the same line but with lengthier computation, we prove the general formula

$$\forall\ n \in \mathbb{N}, \quad \int_0^\infty \frac{\ln(x)}{x^{2n}+1}\,dx = -\frac{\pi^2}{4n^2}\frac{\cos\left(\frac{\pi}{2n}\right)}{\sin^2\left(\frac{\pi}{2n}\right)}.$$

Remark 2: Using the above integral, for any constants $a > 0$ and $b > 0$, we find that

$$\int_0^\infty \frac{\ln(ax)}{x^2+b^2}\,dx = \frac{\pi\ln(ab)}{2b}.$$

[We first set $x = bu$ and so $dx = b\,du$, then we use $\ln(abu) = \ln(ab) + \ln(u)$, the result of **this example** and $\arctan(u)$.]

(See also **Problem 1.7.135**.)

Example 1.7.48 As we have seen in **Problem I 2.1.23, (b)** [see also **Problem 1.7.133, (a)**],

$$\frac{\pi}{4} - \frac{\ln(2)}{2} < \int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx < 1.$$

Here, we will evaluate this integral as a series of real numbers. We use integration by parts and get

$$\int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 -\ln(x)\,d\arctan(x) =$$

$$[-\ln(x)\arctan(x)]_0^1 + \int_0^1 \frac{\arctan(x)}{x}\,dx =$$

$$(-0+0) + \int_0^1 \frac{\arctan(x)}{x}\,dx = \int_0^1 \frac{\arctan(x)}{x}\,dx.$$

Since

$$\arctan(x) = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \forall\ -1 < x < 1,$$

after integrating term by term, we have

$$\int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{2n+1}\,dx = \sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^2}.$$

[Since the last series converges (absolutely), by **Abel's Lemma** (see **footnote of Example 1.7.24**) this equality is correct.]

Finally,

$$\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^2} =$$

(by the absolute convergence of this series)

$$\left(\frac{1}{1^2} - \frac{1}{3^2}\right) + \left(\frac{1}{5^2} - \frac{1}{7^2}\right) + \left(\frac{1}{9^2} - \frac{1}{11^2}\right) + \ldots =$$

$$8\sum_{n=0}^\infty \frac{2n+1}{(4n+1)^2(4n+3)^2} := G = 0.915965594\ldots,$$

where with the symbol G we denote the so-called **Catalan**[44] **constant**.[45] (See also **Problem 1.7.139**.)

▲

Example 1.7.49 As we work in **Examples 1.7.8** and **1.7.47**, especially **Case (b), of Example 1.7.8**, with $-1 < \alpha < 0$, using the contour of the latter and making the necessary adjustments of the arguments in the presence of the complex log, we find the important **result**:

$$\forall\ \alpha\ :\ -1 < \alpha < 0, \qquad \int_0^\infty \frac{x^\alpha \ln(x)}{x+1}\,dx = \pi^2 \cot(\alpha\pi)\csc(\alpha\pi).$$

[Look at **both examples** once more and make the necessary modifications to derive this result. Also, this result can be obtained by taking the derivative with respect to α of the integral computed in **Example 1.7.8, Case (b)**. Look at this fundamental integral again, justify that this differentiation is legitimate and obtain the result here in this way!]

Notice that this integral is $-\infty$ when $\alpha \le -1$ and $+\infty$ when $\alpha \ge 0$. (Prove this! See **Problem 1.7.142**.)

From this result, we obtain the following more general **result**:
For all real numbers q and $r \neq 0$ satisfying the inequality $0 < \frac{q+1}{r} < 1$, we have

$$\int_0^\infty \frac{x^q \ln(x)}{x^r+1}\,dx = -\mathrm{sign}(r)\,\frac{1}{r^2}\,\pi^2 \cot\left(\frac{q+1}{r}\pi\right)\csc\left(\frac{q+1}{r}\pi\right).$$

This follows by letting $u = x^r$ and working it out. (See also **Problems 1.7.142** and **1.7.145** for more variations.)

▲

[44] Eugène Charles Catalan, French-Belgian mathematician, 1814-1894.

[45] The **Catalan constant** appears in many applications, from combinatorics, to number theory, to integrals, etc. It can be found in several integral representations besides the ones developed in **the above example** and **Problem 1.7.139**.

Example 1.7.50 (Compare with **Problems I 2.1.19** and **I 2.2.46**. See also the end of **Subsection 1.5.4**.)

Prove that

$$\int_0^{2\pi} \ln\left|1-e^{i\theta}\right|\, d\theta = \int_0^{2\pi} \ln\left[2-2\cos(\theta)\right]\, d\theta = 0.$$

Then, conclude

$$\text{(a)} \quad \int_0^{2\pi} \ln\left[\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = -2\pi\ln(2),$$

and

$$\text{(b)} \quad \int_0^{\pi} \ln\left[\sin(u)\right] du = -\pi\ln(2).$$

We consider the holomorphic branch of the complex $\log(z)$

$$\mathrm{Log} \;:\; \mathbb{C} - \{x+0i \mid x \le 0\} \longrightarrow \mathbb{R} + i(-\pi,\pi).$$

So, if

$$z = re^{i\theta}, \quad \text{with} \quad \theta \ne \pi \pmod{2\pi},$$

we define

$$\mathrm{Log}(z) = \ln(r) + i\theta, \quad \text{with} \quad -\pi < \theta < \pi.$$

Therefore, the composite function

$$h(z) = \mathrm{Log}(1-z) = \mathrm{Log}(1-x-iy)$$

is holomorphic in the domain

$$\mathbb{C} - \{x \mid 1-x \le 0\} = \mathbb{C} - \{x \mid x \ge 1\}.$$

That is, for $h(z)$ the branch cut has been translated from $x=0$ to $x=1$ along the positive x-axis.

In this domain, we have

$$\mathrm{Re}\, h(z) = \ln|1-z| \quad \text{and} \quad -\pi < \mathrm{Im}\, h(z) < \pi.$$

Also,

$$h(0) = \ln(1) + 0\,i = 0 + 0\,i = 0.$$

So, for every $0<\epsilon<1$, we can write $h(z)$ as a power series $h(z) = a_1z + a_2z^2 + a_3z^3 + \dots$ in $D(0,\epsilon)$. Thus, $\dfrac{h(z)}{z} = a_1 + a_2z + a_3z^2 + \dots$. Therefore, the function $\dfrac{h(z)}{z}$ is holomorphic in $D(0,\epsilon)$. Hence, $z=0$ is not a singularity for $\dfrac{h(z)}{z}$. In fact, we can set $\dfrac{h(z)}{z}\Big|_{z=0} = a_1$.

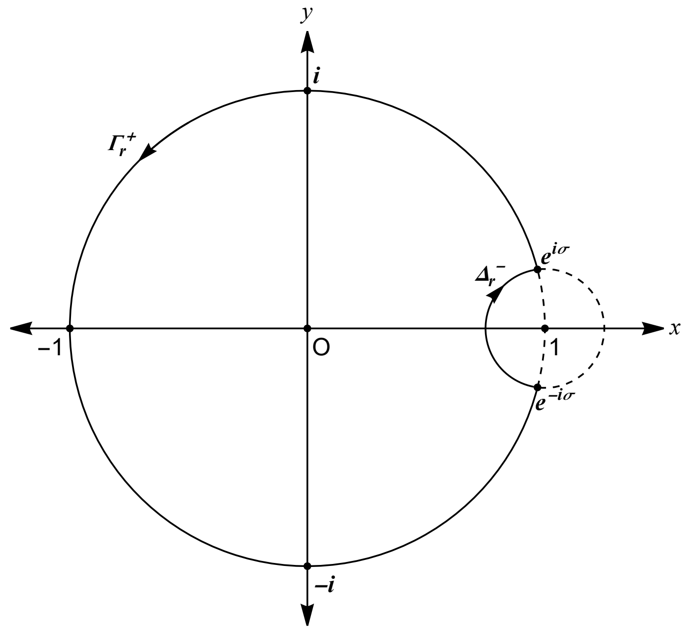

FIGURE 1.20: Contour 14 for Example 1.7.50

We now consider $C(0,1)$ and $C(1,r)$ for any $0 < r < \dfrac{1}{2}$. These two circles intersect at two points $e^{i\sigma}$ and $e^{-i\sigma}$ for some $0 < \sigma < \dfrac{\pi}{2}$.

We call Γ_r the part of $C(0,1)$ outside $C(1,r)$ and Δ_r the part of $C(1,r)$ inside $C(0,1)$. Then, we let our contour be $C^+ = \Gamma_r^+ + \Delta_r^-$ (see **Figure 1.20**), and we get

$$\oint_{C^+} \frac{h(z)}{z}\,dz = 0.$$

From this we obtain

$$\int_{\Gamma^+} \frac{h(z)}{z}\,dz = -\int_{\Delta^-} \frac{h(z)}{z}\,dz = \int_{\Delta^+} \frac{h(z)}{z}\,dz.$$

But,

$$\left|\int_{\Delta^+} \frac{h(z)}{z}\,dz\right| = \left|\int_{-\sigma}^{\sigma} \frac{\operatorname{Log}\left[1-\left(1-re^{i\theta}\right)\right]}{1-re^{i\theta}}\,r\,i\,e^{i\theta}\,d\theta\right| =$$

$$\left|\int_{-\sigma}^{\sigma} \frac{\operatorname{Log}\left(re^{i\theta}\right)}{1-re^{i\theta}}\, r\, i\, e^{i\theta}\, d\theta\right| = \left|\int_{-\sigma}^{\sigma} \frac{\ln(r)+i\theta}{1-re^{i\theta}}\, r\, i\, e^{i\theta}\, d\theta\right| \leq$$

$$\int_{-\sigma}^{\sigma} \left|\frac{\ln(r)+i\theta}{1-re^{i\theta}}\, r\, i\, e^{i\theta}\right| d\theta \leq \int_{-\sigma}^{\sigma} \frac{\sqrt{\ln^2(r)+\theta^2}}{\left|1-re^{i\theta}\right|}\, r\, d\theta \leq$$

$$\int_{-\sigma}^{\sigma} \frac{\sqrt{\ln^2(r)+\frac{\pi^2}{4}}}{1-r}\, r\, d\theta < \frac{\sqrt{\ln^2(r)+\frac{\pi^2}{4}}}{1-r}\, r\, \pi \to 0, \quad \text{as} \quad r \to 0^+.$$

Hence,

$$\lim_{r\to 0^+} \int\limits_{\Gamma^+} \frac{h(z)}{z}\, dz = 0.$$

Finally,

$$\int_0^{2\pi} \frac{\operatorname{Log}\left(1-e^{i\theta}\right)}{e^{i\theta}}\, i\, e^{i\theta}\, d\theta = 0, \quad \text{or} \quad \int_0^{2\pi} \operatorname{Log}\left(1-e^{i\theta}\right)\, d\theta = 0.$$

So,

$$\int_0^{2\pi} \ln\left|1-e^{i\theta}\right|\, d\theta + i\int_0^{2\pi} \arg\left(1-e^{i\theta}\right)\, d\theta = 0,$$

and therefore

$$\int_0^{2\pi} \ln\left|1-e^{i\theta}\right|\, d\theta = 0, \quad \text{and} \quad \int_0^{2\pi} \arg\left(1-e^{i\theta}\right)\, d\theta = 0.$$

But, $\forall \quad 0 \leq \theta \leq 2\pi$,

$$\left|1-e^{i\theta}\right| = \left|1-\cos(\theta)-i\sin(\theta)\right| = \left\{\left[1-\cos(\theta)\right]^2+\sin^2(\theta)\right\}^{\frac{1}{2}} =$$

$$\left\{\left[2-2\cos(\theta)\right]\right\}^{\frac{1}{2}} = \left|2\sin\left(\frac{\theta}{2}\right)\right| = 2\sin\left(\frac{\theta}{2}\right)$$

and so we find

$$\int_0^{2\pi} \ln\left[2\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = 0.$$

Then,

$$\int_0^{2\pi} \ln\left[\sin\left(\frac{\theta}{2}\right)\right]\, d\theta = -\int_0^{2\pi} \ln(2)\, d\theta = -2\pi\ln(2),$$

and by substituting $u = \dfrac{\theta}{2}$ we obtain the final **result**

$$\int_0^{\pi} \ln\left[\sin(u)\right] du = -\pi\ln(2).$$

Remarks: (1) Similarly we prove that

$$\int_0^{2\pi} \ln\left|1+e^{i\theta}\right| \, d\theta = \int_0^{2\pi} \ln\left[2+2\cos(\theta)\right] \, d\theta = 0.$$

Then, conclude

$$\text{(a)} \quad \int_0^{2\pi} \ln\left[\cos\left(\frac{\theta}{2}\right)\right] \, d\theta = -2\pi\ln(2),$$

and

$$\text{(b)} \quad \int_0^{\pi} \ln\left[\cos(u)\right] du = -\pi\ln(2).$$

(See also **Problem I 2.1.19**.)

(2) In this example, we have also proved that

$$\int_0^{2\pi} \arg\left(1 \pm e^{i\theta}\right) \, d\theta = 0.$$

(Prove this result directly as a **Problem**!)

(3) For the integrals

$$\int_0^{2\pi} \ln\left|1 \pm re^{i\theta}\right| \, d\theta = \frac{1}{2}\int_0^{2\pi} \ln\left[r^2 \pm 2r\cos(\theta)+1\right] \, d\theta,$$

with $r \in \mathbb{R}$, see **Problem I 2.2.46**.

▲

Problems

1.7.131 Consider $g(z) = \dfrac{\log^k(z)}{z^2+1}$, where $k > 0$ real constant. For every $R > 0$, we consider $C(0,R)$, the circumference of the circle with center the origin and radius R. Prove

$$\lim_{R\to\infty}\left[\max_{z\in C(0,R)} |zg(z)|\right] = 0 \quad \text{and} \quad \lim_{z\to\infty}\left[zg(z)\right] = 0.$$

1.7.132 (a) Use the appropriate result obtained in this **section** to prove

$$\int_0^{\frac{\pi}{2}} \ln[\sin(u)] \, du = \int_0^{\frac{\pi}{2}} \ln[\cos(u)] \, du = -\frac{\pi}{2}\ln(2).$$

(See also **Problem I 2.1.19**.)

(b) Let $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$. Use the substitution $x = \frac{1-u}{1+u}$ to prove that

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi \ln(2)}{8}.$$

1.7.133 (a) Prove that for any $n = 0,\ 1,\ 2, \ldots$ non-negative integer

$$\int_0^1 \frac{[\ln(x)]^n}{x^2+1}\,dx = \int_1^\infty \frac{(-1)^n[\ln(x)]^n}{x^2+1}\,dx.$$

(b) Prove that for any $n = 1,\ 3,\ 5, \ldots$ odd positive integer

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 0.$$

(c) Prove that for any $n = 0,\ 2,\ 4,\ \ldots$ even non-negative integer

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 2\int_0^1 \frac{[\ln(x)]^n}{x^2+1}\,dx = 2\int_1^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx.$$

1.7.134 Use contour integration to prove

$$\int_0^\infty \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{4}.$$

(This integral is also found by combining I_5 and I_4 of **Problems 1.7.62** and **1.7.63**.)

1.7.135 (a) Manipulate the result proven in **Example 1.7.47** to prove

$$\forall \quad a > 0,\ \ b > 0 \ \text{ and } \ c > 0,$$
$$\int_0^\infty \frac{\ln(cx)}{(ax)^2+b^2}\,dx = \frac{\pi}{2ab}\ln\left(\frac{bc}{a}\right).$$

(b) Prove the general formula

$$\forall \ b > 0 \text{ and } d \geq 0,$$
$$\int_0^\infty \frac{\ln(x)}{(x+d)^2+b^2}\,dx = \frac{\ln\left(b^2+d^2\right)}{2b} \cdot \arctan\left(\frac{b}{d}\right).$$

(c) Now, prove the general formula

$$\forall\ a>0,\ \ b>0,\ \ c>0 \text{ and } d\geq 0,$$
$$\int_0^\infty \frac{\ln(cx)}{(ax+d)^2+b^2}\,dx =$$
$$\frac{\ln\left(\frac{c}{a}\right)}{ab}\left[\frac{\pi}{2}-\arctan\left(\frac{d}{b}\right)\right]+\frac{\ln\left(b^2+d^2\right)}{2ab}\cdot\arctan\left(\frac{b}{d}\right).$$

(See also **Problems I 2.1.23** and **I 2.1.30**.)

1.7.136 (a) Prove

$$\int_0^\infty \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{8}.$$ [46]

Then, use the substitution $x=e^u$ to find

$$\int_{-\infty}^\infty \frac{u^2}{\cosh(u)}\,du = \int_{-\infty}^\infty u^2\operatorname{sech}(u)\,du = \frac{\pi^3}{4}.$$

Also, use $x=\dfrac{1}{u}$ to prove

$$\int_0^1 \frac{\ln^2(x)}{1+x^2}\,dx = \int_1^\infty \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{16}.$$

(b) Prove

$$\int_0^\infty \frac{\ln^4(x)}{1+x^2}\,dx = \frac{5\pi^5}{32}.$$

Then, use the substitution $x=e^u$ to find

$$\int_{-\infty}^\infty \frac{u^4}{\cosh(u)}\,du = \int_{-\infty}^\infty u^4\operatorname{sech}(u)\,du = \frac{5\pi^5}{16}.$$

Also, use $x=\dfrac{1}{u}$ to prove

$$\int_0^1 \frac{\ln^4(x)}{1+x^2}\,dx = \int_1^\infty \frac{\ln^4(x)}{1+x^2}\,dx = \frac{5\pi^5}{64}.$$

(c) Make the substitution $x=\tan(\theta)$ in the integrals in **(a)** and **(b)**

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[46] In general, we can prove:

$$\forall\ n\in\mathbb{N},\quad \int_0^\infty \frac{\ln^2(x)}{1+x^{2n}}\,dx = \frac{\pi^3}{8n^3}\,\frac{1+\cos^2\left(\frac{\pi}{2n}\right)}{\sin^3\left(\frac{\pi}{2n}\right)}.$$

that contain x to derive new forms of these integrals in logarithm and trigonometric functions.

1.7.137 Prove
$$\int_0^\infty \frac{\ln(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

(See also **Problem I 2.1.28**.)

1.7.138 Consider any $a \geq 0$ and notice that the function

$$f(x) = \frac{\ln[(ax)^2+1]}{x^2+1}$$

is even in $\mathbb{R}$. Also $(az)^2+1 = (az+i)(az-i)$.

Use the functions $f_1(z) = \dfrac{\log(az+i)}{z^2+1}$ and $f_2(z) = \dfrac{\log(az-i)}{z^2+1}$ and complex integration with appropriately chosen branch cuts (analogous to the branch cut in **Example 1.7.50**, but not the same) and contours, to find the integral in **Problem I 2.3.17**. I.e.,

$$\int_{-\infty}^\infty \frac{\ln\left[(ax)^2+1\right]}{x^2+1}\,dx = 2\int_0^\infty \frac{\ln\left[(ax)^2+1\right]}{x^2+1}\,dx = 2\pi\ln(1+a).$$

[Hint: Use complex integration to find $\displaystyle\int_{-\infty}^\infty f_1(x)dx$ and $\displaystyle\int_{-\infty}^\infty f_2(x)dx$ separately and add the two complex results found.]

1.7.139 (a) In **Example 1.7.48**, we have introduced the following three integral representations of the **Catalan constant**:

$$G := \int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{\arctan(x)}{x}\,dx.$$

Using appropriate changes of variables and manipulations, verify the following additional six integral representations of G:

$$g_1 = \int_0^1\int_0^1 \frac{1}{1+x^2y^2}\,dxdy, \qquad g_2 = \int_0^{\frac{\pi}{4}} \frac{t}{\sin(t)\cos(t)}\,dt,$$
$$g_3 = \frac{1}{2}\int_0^\infty \frac{x}{\cosh(x)}\,dx, \qquad g_4 = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t}{\sin(t)}\,dt,$$
$$g_5 = \int_0^{\frac{\pi}{4}} \ln[\cot(t)]\,dt, \qquad g_6 = \int_0^\infty \arctan\left(e^{-t}\right)\,dt.$$

[See also **Problem 1.7.39, (b)**.]

(b) See **Problem I 2.1.20** and prove

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln[\sin(\theta)]\,d\theta = \int_0^{\frac{\pi}{4}} \ln[\cos(\theta)]\,d\theta = \frac{G}{2} - \frac{\pi\ln(2)}{4}.$$

(c) Now see **Problems I 2.1.19**, and/or **1.7.132, (a)**, and prove

$$\int_0^{\frac{\pi}{4}} \ln[\sin(\theta)]\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln[\cos(\theta)]\,d\theta = -\frac{\pi\ln(2)}{4} - \frac{G}{2}.$$

(d) Prove that applying integration by parts to the integral

$$\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_1^\infty \ln(x)\,d\arctan(x),$$

we find $\infty - \infty$ and this indeterminate $\infty - \infty$ is equal to G.

1.7.140 (a) Prove that

$$\int_0^1 \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx - 2\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx - 2G.$$

where G is the **Catalan constant**.
[Hint: Use the substitution $x = \frac{1}{u}$ and **Problem 1.7.139, (a)**.]

(b) Now prove

$$\int_0^1 \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \frac{\pi}{2}\ln(2) - G$$

and

$$\int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \frac{\pi}{2}\ln(2) + G.$$

[Hint: Use **Problem I 2.1.24, (a)**, or **1.7.138**.]

1.7.141 Study the final result of **Example I 2.5.3**, make the appropriate transformations and use **Problem 1.7.68** to prove

$$\int_0^1 \frac{\arctan^2(x)}{x}\,dx = \frac{\pi}{2}G - \sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} = \frac{\pi}{2}G - \frac{7}{8}\zeta(3).$$

[Hint: You may begin with $u = \tan(x)$.]

1.7.142 (a) Prove that the integral $$\int_0^\infty \frac{x^\alpha \ln(x)}{x+1}\,dx,$$

in **Example 1.7.49**, is $-\infty$ when $\alpha \le -1$ and $+\infty$ when $\alpha \ge 0$.

(b) Prove that the result of **Example 1.7.49** can be rewritten as

$$\int_0^\infty \frac{x^{\beta-1}\ln(x)}{x+1}\,dx = -\pi^2 \cot(\beta\pi)\csc(\beta\pi), \quad \text{with } 0 < \beta < 1.$$

(c) Prove that this result can also be written as

$$\int_0^\infty \frac{\ln(x)}{x^\gamma+1}\,dx = -\left(\frac{\pi}{\gamma}\right)^2 \cot\left(\frac{\pi}{\gamma}\right)\csc\left(\frac{\pi}{\gamma}\right), \quad \text{with } \gamma > 1.$$

(See also **Problems I 2.2.48** and **I 2.6.63**.)

1.7.143 Prove the results of **Problem I 2.5.24, (a)-(b)**, using complex analysis methods with appropriate contour integration.

[Hint: Use the complex inverse tangent and **Problem I 2.1.25**.]

1.7.144 Compute the eight integrals:

$$I_1 = \int_0^\infty \frac{\ln(x)}{1+x^4}\,dx, \qquad I_2 = \int_0^\infty \frac{\sqrt{x}\ln(x)}{1+x^4}\,dx,$$
$$I_3 = \int_0^\infty \frac{x^{\frac{5}{2}}\ln(x)}{1+x^4}\,dx, \qquad I_4 = \int_0^\infty \frac{x^{-2}\ln(x)}{1+x^{-2}}\,dx,$$
$$I_5 = \int_0^\infty \frac{x^{-2}\ln(x)}{1+x^6}\,dx, \qquad I_6 = \int_0^\infty \frac{x^{\frac{-2}{3}}\ln(x)}{1+x}\,dx,$$
$$I_7 = \int_0^\infty \frac{\ln(x)}{1+x^5}\,dx, \qquad I_8 = \int_0^\infty \frac{x^3\ln(x)}{1+x^5}\,dx.$$

1.7.145 Compute the four integrals:

$$I_1 = \int_0^\infty \frac{(5x)^{-\frac{1}{4}}\ln(6x)}{7x+8}\,dx, \qquad I_2 = \int_0^\infty \frac{(5x)^4\ln(6x)}{(7x)^8+9}\,dx,$$
$$I_3 = \int_0^\infty \frac{\ln(10x)}{100+(5x)^2}\,dx, \qquad I_4 = \int_0^\infty \frac{x^3\ln(10x)}{64+(2x)^6}\,dx.$$

(See also **Example 1.7.49**.)

1.7.146 Consider the following six integrals:

$$I_1 = \int_{-\infty}^{\infty} \frac{dx}{x^2+x+1} \left(= \frac{2\pi\sqrt{3}}{3}\right),$$

$$I_2 = \int_{-\infty}^{0} \frac{dx}{x^2+x+1} \left(= \frac{4\pi\sqrt{3}}{9}\right), \quad I_3 = \int_{0}^{\infty} \frac{dx}{x^2+x+1} \left(= \frac{2\pi\sqrt{3}}{9}\right),$$

$$I_4 = \int_{-\infty}^{\infty} \frac{dx}{x^2-x+1}, \quad I_5 = \int_{-\infty}^{0} \frac{dx}{x^2-x+1}, \quad I_6 = \int_{0}^{\infty} \frac{dx}{x^2-x+1}.$$

(a) Compute all of them by using calculus.

[Hint: Observe that $x^2 \pm x + 1 = \left(x \pm \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$, and then for any $a \in \mathbb{R}$ and $b > 0$, we have

$$\int \frac{1}{(x+a)^2+b^2}\, dx = \frac{1}{b} \arctan\left(\frac{x+a}{b}\right) + C.]$$

(b) Now, compute all of them by using contour integration.

[Hint: For the first integral, use the complex function

$$f(z) = \frac{1}{z^2+z+1}.$$

For the second integral, use

$$g(z) = \frac{\log(z)}{z^2+z+1} \quad \text{or} \quad h(z) = \frac{\log(z)}{z^2-z+1},$$

whichever is more convenient, with branch cut $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ or another one that you may find more convenient. You may choose contour: $C^+ = [-R, -\delta] + S_\delta^- + [\delta, R] + S_R^+$ for $0 < \delta < R$ chosen appropriately and then let $\delta \to 0$ and $R \to \infty$, etc. Do similar work for the remaining integrals.]

1.7.147 Compute the six integrals

$$\int_{-\infty}^{\infty} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx, \qquad \int_{-\infty}^{0} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx, \qquad \int_{0}^{\infty} \frac{\sqrt[3]{x}}{x^2 \pm x + 1} dx,$$

by using contour integration. (Just calculus does not seem to work here!)
(See also **Example 1.7.16** for another method.)

[Hint: Consider the complex function

$$f(z) = \frac{e^{\frac{1}{3}\log(z)}}{z^2+z+1}, \qquad \text{or} \qquad g(z) = \frac{e^{\frac{1}{3}\log(z)}}{z^2-z+1}$$

with branch cut $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ or another one that you may find more convenient. You may choose contour:

$$C^+ = [-R, -\delta] + S_\delta^- + [\delta, R] + S_R^+$$

for $0 < \delta < R$ chosen appropriately and then let $\delta \to 0^+$ and $R \to \infty$. For example,

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx = \frac{4\pi}{3}\sin\left(\frac{\pi}{9}\right).]$$

1.7.148 Observe that $x^3+x^2+x+1 = (x^2+1)(x+1)$. Then:
(a) Prove

$$\int_{-\infty}^\infty \frac{dx}{x^3+x^2+x+1} \quad \text{does not exist.}$$

[Hint: Examine the singularity at $x = -1$, located on the x-axis, and show that the integral has the form $\infty - \infty$ around it.]

(b) Prove

$$\text{P.V.} \int_{-\infty}^\infty \frac{dx}{x^3+x^2+x+1} = \frac{\pi}{2}.$$

[Hint: Use **Theorem 1.7.4**.]

(c) Compute

$$\int_0^\infty \frac{dx}{x^3+x^2+x+1} \left(= \frac{\pi}{4}\right).$$

[Hint: As in **Problem 1.7.146**, consider the complex function

$$f(z) := \frac{\log(z)}{z^3+z^2+z+1} = \frac{\log(z)}{(z+i)(z-i)(z+1)},$$

or use partial fractions and calculus for

$$g(x) := \frac{1}{(x+1)(x^2+1)}.]$$

1.7.149 Notice $(x^5-1)=(x-1)(x^4+x^3+x^2+x+1)$ and compute the integrals:

$$\text{(a)} \qquad \int_{-\infty}^{\infty} \frac{dx}{x^4+x^3+x^2+x+1} \left[= \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right)\right],$$

$$\text{(b)} \qquad \int_{0}^{\infty} \frac{dx}{x^4+x^3+x^2+x+1},$$

$$\text{(c)} \qquad \int_{-\infty}^{0} \frac{dx}{x^4+x^3+x^2+x+1}.$$

[Hint: For **(b)** and **(c)**, work as in the **previous Problem** or **Problem 1.7.146**. Here, it is easier to work with the appropriate complex functions and the complex fifth roots of unity.]

1.7.150 For any a real constant, find the principal values or explain if they do not exist:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{iax}dx}{\ln|x|} \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{xe^{iax}dx}{(\ln|x|)^2}.$$

1.7.151 For $a>0$ constant prove:

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)\left[\ln^2(x)+\pi^2\right]} = \frac{\pi}{a\left[2\ln^2(a)+\frac{\pi^2}{2}\right]} - \frac{1}{1+a^2}.$$

[Hint: Consider the branch of $w=\log(z)$ for which $-\pi \le \arg(z) \le \pi$ and the function

$$f(z) = \frac{1}{(z^2+a^2)\log(z)}.$$

Take $0<\epsilon<a<r$, and the contour

$$C = [-r,-\epsilon] + A_{\epsilon}^{-} + [\epsilon, r] + A_r^{+},$$

where,

$$A_{\epsilon}^{-} = \{\epsilon e^{i\theta} \mid \pi > \theta > -\pi\}, \quad \text{and} \quad A_r^{+} = \{re^{i\theta} \mid -\pi < \theta < \pi\}.$$

Find the residues of $f(z)$ at $z=\pm ai$, compute $\oint_C f(z)dz$, take limits, etc.]

1.7.10 Application to Inverse Laplace Transform

As we have seen in **Section I 2.7**, on the classes of the absolutely[47] or square[48] integrable or of exponential order functions $y = f(x)$, with $x \in [0, \infty)$ or $x \in (0, \infty)$, we define the Laplace transform of $f(x)$ by the improper integral

$$F(s) := \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x) dx,$$

for all s for which it exists. [F(s) may also exist under other conditions. See **Problem I 2.7.6** and **Theorem I 2.7.1**.]

Under these conditions, there is a real constant c such that

$$F(s) = \mathcal{L}\{f(x)\}(s), \quad \text{exists for all } \ s > c,$$

and, as proven in **Theorem I 2.7.1** (and **Problems I 2.7.7** and **I 2.7.8**),

$$\lim_{s \to \infty} \mathcal{L}\{f(x)\}(s) = 0,$$

$F(s)$ is uniformly continuous on $[c+\epsilon, \infty)$, $\ \epsilon > 0$, **and the convergence of the improper integral is uniform**.

In the context here, we replace the real variable s with the complex variable $\zeta = s + i\tau$, and thus we consider the complex function $F(\zeta)$ for which we require the necessary condition

$$\lim_{\zeta \to \infty} \mathcal{L}\{f(x)\}(\zeta) = 0,$$

where ∞, in this context, is the complex infinity.

If we assume that $e^{-cx} f(x)$ is absolutely or square integrable for some real constant c, then as we have seen in **Subsection 1.7.8, properties (1)** and **(4)**, $\quad e^{-sx} f(x) \quad$ is absolutely integrable for all s such that $c < s < \infty$ and $F(\zeta)$ is holomorphic in the infinite strip $c < s < \infty$. Now, we state:

Theorem 1.7.7 (Mellin-Bromwich Inversion Theorem) *We consider a continuous function $y = f(x)$ where $x \in [0, \infty)$ or $x \in (0, \infty)$ such that $e^{-cx} f(x)$ is absolutely or square integrable for some real constant c, and so its Laplace transform*

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x) dx, \quad \textit{exists} \quad \forall \ s > c.$$

[47] $f(x)$ is **absolutely integrable** if $\displaystyle\int_{-\infty}^{\infty} |f(x)| dx < \infty$.

[48] $f(x)$ is **square integrable** if $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$.

[If $y = f(x)$ is absolutely integrable, then $s \geq c$.]

We consider the complex function $w = F(\zeta)$, where $\zeta = s + i\tau$, and we require that $F(\zeta)$ satisfies the necessary condition

$$\lim_{\zeta\to\infty} F(\zeta) = \lim_{\zeta\to\infty} \mathcal{L}\{f(x)\}(\zeta) = 0,$$

where ∞ is the complex infinity.

Then, for any $\gamma > c$,

$$P.V. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{x\zeta} F(\zeta) d\zeta \tag{1.44}$$

is independent of the constant $\gamma > c$, and if it exists for one choice of γ, then for any $\gamma > c$ we have

$$f(x) = \mathcal{L}^{-1}\left\{F(s)\right\}(x) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma-iR}^{\gamma+iR} e^{x\zeta} F(\zeta) d\zeta = \tag{1.45}$$

$$\frac{1}{2\pi i}\ P.V. \int_{\gamma-i\infty}^{\gamma+i\infty} e^{x\zeta} F(\zeta) d\zeta = \frac{1}{2\pi}\ P.V. \int_{-\infty}^{\infty} F(\gamma+iy) e^{(\gamma+iy)x}\, dy.$$

Before we give the proof, we make the following comments: In applications $w = F(\zeta)$ is given á-priori, i.e., before $y = f(x)$, in order to retrieve $y = f(x)$ from $w = F(\zeta)$. Under the above conditions, the limit in **(1.45)** exists and is independent of γ. This complex integral formula is called the **Mellin-Bromwich**[49] **inversion integral formula** for the **Laplace transform**. If the function $y = f(x)$ is not continuous at some points, the formula still works at all points at which $y = f(x)$ is continuous. For the existence of the principal value in equation **(1.44)**, see, e.g., **Condition (3.)** of the **next Theorem, 1.7.8**, and the **remark** that follows it.

Proof As we have seen, if $e^{-sx} f(x)$ is absolutely integrable for all s such that $c < s < \infty$, then $F(\zeta)$ is holomorphic in the infinite horizontal strip $c < s < \infty$. [See **Subsection 1.7.8, property (4)**.]

If for some $\gamma_1 > c$, the P.V. $\displaystyle\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} e^{x\zeta} F(\zeta) d\zeta$ exists, then it exists for all $\gamma > c$ and is independent of γ. Indeed, for any $\gamma_2 > \gamma_1 > c$ (or $\gamma_1 > \gamma_2 > c$) and any $R > 0$, we have $\displaystyle\oint_{C^+} e^{x\zeta} F(\zeta)\, d\zeta = 0$, where the path C^+ is the positively oriented parallelogram

$$C^+ = [\gamma_2 - iR, \gamma_2 + iR] + [\gamma_2 + iR, \gamma_1 + iR] + \\ [\gamma_1 + iR, \gamma_1 - iR] + [\gamma_1 - iR, \gamma_2 - iR].$$

[49] Thomas John l'Anson Bromwich, English mathematician, 1875-1929.

[Since on C^+ and in the interior of C^+ the function $e^{x\zeta}F(\zeta)$ is holomorphic, the **Cauchy-Goursat Theorem, 1.5.3**, applies.]

Then, we let $R \longrightarrow \infty$ and eventually get

$$\text{P.V.} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} e^{x\zeta} F(\zeta) d\zeta = \text{P.V.} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} e^{x\zeta} F(\zeta) d\zeta,$$

since by the condition

$$\lim_{\zeta \to \infty} \mathcal{L}\{f(x)\}(\zeta) = \lim_{\zeta \to \infty} F(\zeta) = 0$$

the limits of the integrals along the two finite horizontal segments $[\gamma_2 + iR, \gamma_1 + iR]$ and $[\gamma_1 - iR, \gamma_2 - iR]$ (of constant length $|\gamma_1 - \gamma_2|$) are zero, as $R \longrightarrow \infty$. (Work out the details, as an easy exercise!)

We have defined

$$F(\zeta) = \int_0^\infty e^{-\zeta x} f(x) dx = \int_0^\infty e^{-(s+i\tau)x} f(x) dx, \text{ where } \zeta = s + i\tau.$$

[So, $\tau = 0$ implies $F(s) = \mathcal{L}\{f(x)\}(s)$.] Then, with $f(x) = 0$ for $x < 0$, or $x \leq 0$, the **Fourier transform** of $y = f(x)$, defined by **(1.27)**, is

$$\hat{f}(\zeta) = \int_0^\infty e^{i\zeta x} f(x) dx = \int_0^\infty e^{-(-i\zeta)x} f(x) dx = F(-i\zeta).$$

Therefore, for $x > 0$ or $x \geq 0$ and for any $s = \gamma > c$, by the **inversion formula (1.40)** for the **Fourier transform with complex argument**, we obtain

$$f(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{-R+i\gamma}^{R+i\gamma} e^{-i\zeta x} F(-i\zeta) d\zeta,$$

where the integral takes place on the horizontal line $(-\infty + i\gamma, +\infty + i\gamma)$.

We make the change of variables $z = -i\zeta$ and so $d\zeta = \dfrac{1}{-i}\, dz = i dz$. Also, since multiplication by $-i$ amounts to a clockwise rotation of ζ by angle $\dfrac{\pi}{2}$, the new integral will take place on the vertical line $x = \gamma$ and y free. In fact, we get

$$f(x) = \frac{1}{2\pi} \lim_{R \to \infty} \int_{\gamma + iR}^{\gamma - iR} e^{zx} F(z) i\, dz =$$

$$\frac{-1}{2\pi i} \lim_{R \to \infty} \int_{\gamma + iR}^{\gamma - iR} e^{zx} F(z)\, dz = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{zx} F(z)\, dz,$$

which finishes the proof of the Theorem.

■

Example 1.7.51 We illustrate the use of the **inversion formula (1.45)** by verifying a known example. We can use this formula when the stated conditions are met, the principal value involved exists and, of course, if the formula is convenient enough to produce a result. Otherwise, we use the **Theorems** and the methods of the examples that follow in this subsection.

In the **table of Problem I 2.7.14**, **Rule (1.)**, we have that the Laplace transform of $f(x) = a$ (real constant), for $x > 0$ or $x \geq 0$, is

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} a dx = \frac{a}{s}, \quad \text{with} \quad s > 0.$$

By the **inversion formula (1.45)**, we should have

$$a = f(x) = \mathcal{L}^{-1}\left\{F(s)\right\}(x) = \frac{1}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy}\, e^{(\gamma+iy)x}\, dy$$

for any constant $\gamma > 0$, since when $a \neq 0$ the only singularity of $F(\zeta) = \dfrac{a}{\zeta}$ is $z_0 = 0$, and so the constant c in the **previous Theorem** is zero.

Indeed,

$$\frac{1}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy}\, e^{(\gamma+iy)x}\, dy = \frac{ae^{\gamma x}}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{\gamma - iy}{\gamma^2 + y^2}\, e^{ixy} dy.$$

Using the two Fourier type integrals computed in **Examples 1.7.33** and **1.7.34**, with $\gamma > 0$ and $x > 0$, we find

$$\frac{1}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{a}{\gamma + iy}\, e^{(\gamma+iy)x}\, dy =$$

$$\frac{a\gamma e^{\gamma x}}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{1}{\gamma^2 + y^2}\, e^{ixy} dy - i\frac{ae^{\gamma x}}{2\pi} \text{ P.V.} \int_{-\infty}^{\infty} \frac{y}{\gamma^2 + y^2}\, e^{ixy} dy =$$

$$\frac{a\gamma e^{\gamma x}}{2\pi} \frac{\pi}{\gamma}\, e^{-\gamma x} - i\frac{ae^{\gamma x}}{2\pi}\; i\pi e^{-x\gamma} = \frac{a}{2} + \frac{a}{2} = a,$$

which is correct (and independent of γ).

(As we see, this verification depends on the two Fourier transforms computed in **Examples 1.7.33** and **1.7.34**.)

▲

Next, we state and prove an **inversion Theorem** for the Laplace transform which is not as general as the previous one, but its proof is elementary because it does not depend on the Fourier transform with complex argument. Namely:

Theorem 1.7.8 *Given a complex function $w = F(\zeta)$, we define*

$$c = \inf\{r \mid r \in \mathbb{R} \text{ such that } F(\zeta) \text{ is analytic for all } \zeta \text{ with } Re(\zeta) \geq r\}.$$

Now, we assume

1. $c < +\infty$.
2. $\lim\limits_{\zeta \to \infty} F(\zeta) = 0$, *where ∞ is the complex infinity.*
3. *There is a constant $\gamma \in \mathbb{R}$ constant such that $\gamma > c$ and*

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |F(\zeta)| \cdot |d\zeta| = \int_{-\infty}^{\infty} |F(\gamma + iy)|\, dy < \infty.$$

Then, the integral

$$P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta$$

is independent of the constant $\gamma > c$, and

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta} F(\zeta) d\zeta \ = \tag{1.46}$$

$$\frac{1}{2\pi i} \ P.V. \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta = \frac{1}{2\pi} \ P.V. \int_{-\infty}^{\infty} F(\gamma + iy) e^{(\gamma + iy)x}\, dy,$$

i.e., the limit exists, is independent of γ and gives back the function $f(x)$, whose Laplace transform is $F(s)$.

Remark: The three conditions of this **Theorem** are satisfied in most cases of application. But, they are somewhat restrictive, and so this Theorem is not as general as **Theorem 1.7.7**.

For instance, **Condition (3.) of this Theorem** is not satisfied in the **previous Example**. Indeed, for $F(\zeta) = \dfrac{a}{\zeta}$, $c = 0$ and for any $\gamma > 0$ we have

$$\int_{-\infty}^{\infty} \left| \frac{1}{\gamma + iy} \right| dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\gamma^2 + y^2}}\, dy \overset{y = \gamma u}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1 + u^2}}\, du =$$

$$2 \int_0^{\infty} \frac{1}{\sqrt{1 + u^2}}\, du \overset{u = \tan(v)}{=} 2 \int_0^{\frac{\pi}{2}} \sec(v)\, dv =$$

$$\left[2 \ln |\sec(v) + \tan(v)| \right]_0^{\frac{\pi}{2}} = 2(\infty - 0) = \infty.$$

The same is true for $F(\zeta) = \dfrac{\zeta}{\zeta^2 + a^2}$, where $a > 0$ constant, even though for any $s > 0$, as we have seen in **Problem I 2.7.14**, **Rule (4.)**, $f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \cos(ax)$ with $x \geq 0$.

Now, we continue with:

Proof For the choice of $\gamma > c$ in **Condition (3.)**, the integral

$$\text{P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{x\zeta} F(\zeta) d\zeta$$

exists because it converges absolutely. Indeed,

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \left| e^{x\zeta} F(\zeta) \right| |d\zeta| =$$

$$\int_{-\infty}^{\infty} \left| e^{x(\gamma + iy)} F(\gamma + iy) \right| dy = e^{x\gamma} \int_{-\infty}^{\infty} |F(\gamma + iy)| \, dy < \infty.$$

Then, this integral exists for any $\gamma > c$ and is independent of the constant $\gamma > c$, by **conditions (1.)** and **(2.)** and the same argument we saw in the proof of **Theorem 1.7.7**.

Now, we consider any $z \in \mathbb{C}$ such that $\text{Re}(z) > c$. Then, take any γ such that $\text{Re}(z) > \gamma > c$, any real $R > |z - \gamma| > 0$ and consider the contour

$$C^+ = L^+_{\gamma R} + H^+_{\gamma R},$$

where

$$L^+_{\gamma R} = \{z = \gamma + iy \quad | \quad -R \leq y \leq R\} = [\gamma + iR, \gamma - iR]$$

is a vertical straight segment with mid-point $\gamma + i\,0$, and

$$H^+_{\gamma R} = \left\{ z = \gamma + Re^{i\theta} \quad | \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}$$

is the semicircle with diameter the segment $L^+_{\gamma R}$ and located to the right of $L^+_{\gamma R}$. (See **Figure 1.21**.)

By the **Cauchy integral formula** (**Theorem 1.5.7**), we have

$$F(z) = \frac{1}{2\pi i} \int_{C^+} \frac{F(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\gamma + iR}^{\gamma - iR} \frac{F(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{H^+_{\gamma R}} \frac{F(\zeta)}{\zeta - z} \, d\zeta.$$

But,

$$\left| \int_{H^+_{\gamma R}} \frac{F(\zeta)}{\zeta - z} \, d\zeta \right| \leq \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\left| F\left(\gamma + Re^{i\theta}\right) \right|}{|\gamma + Re^{i\theta} - z|} R \, d\theta \leq$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\left| F\left(\gamma + Re^{i\theta}\right) \right|}{R - |\gamma - z|} R \, d\theta \longrightarrow 0, \quad \text{as} \quad R \longrightarrow \infty,$$

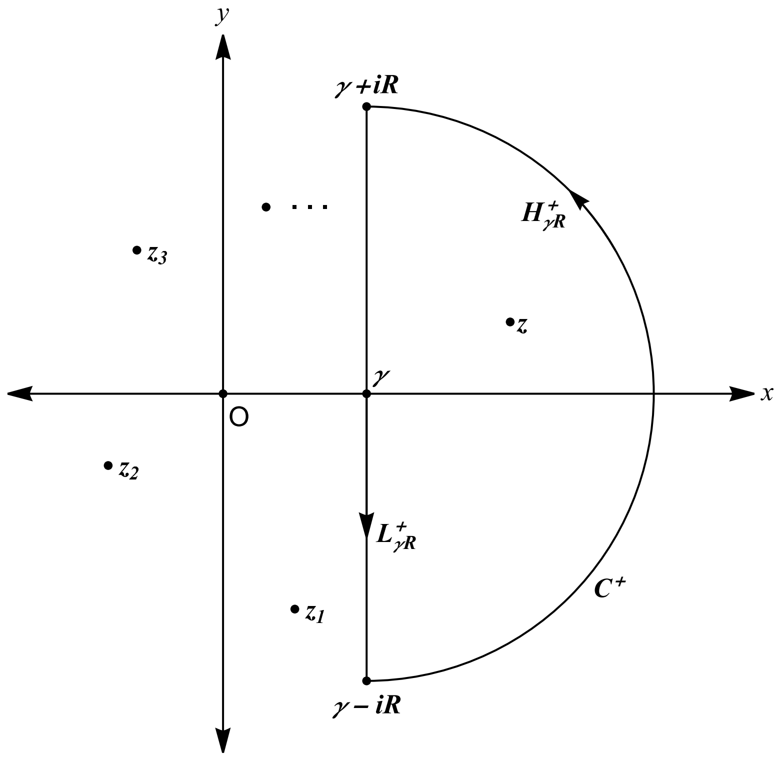

FIGURE 1.21: Contour 15 for Theorem 1.7.8

since $\lim_{R\to\infty}\dfrac{R}{R-|\gamma-z|}=1$ and $\lim_{\zeta\to\infty}F(\zeta)=0$ by **hypothesis (2.)**. (The argument is analogous to the proof of **Lemma 1.7.1**.)

Therefore, by letting $R\longrightarrow\infty$, we obtain

$$F(z)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma+iR}^{\gamma-iR}\frac{F(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\ \text{P.V.}\int_{\gamma+i\infty}^{\gamma-i\infty}\frac{F(\zeta)}{\zeta-z}\,d\zeta.$$

Since $\text{Re}(z-\zeta)=\text{Re}(z)-\text{Re}(\zeta)=\text{Re}(z)-\gamma>0$, by **Problem 1.7.58**, we have

$$-\frac{1}{\zeta-z}=\frac{1}{z-\zeta}=\int_0^\infty e^{-(z-\zeta)x}\,dx.$$

We switch the limits of integration to eliminate the minus $(-)$ sign and obtain

$$F(z) = \frac{1}{2\pi i} \text{ P.V. } \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) \left[\int_0^\infty e^{-(z-\zeta)x} \, dx \right] d\zeta.$$

By **hypothesis (3.)**, we get

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |F(\zeta)| \left[\int_0^\infty \left| e^{-(z-\zeta)x} \right| dx \right] |d\zeta| = \int_{-\infty}^{\infty} |F(\gamma + iy)| \frac{1}{\text{Re}(z) - \gamma} \, dy < \infty.$$

Thus, by the **Tonelli conditions** (see **Section I 2.4**), we are allowed to switch the order of integration to find

$$F(z) = \int_0^\infty e^{-zx} \left[\frac{1}{2\pi i} \text{ P.V. } \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta \right] dx.$$

Letting

$$f(x) = \frac{1}{2\pi i} \text{ P.V. } \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta,$$

we get

$$F(z) = \int_0^\infty e^{-zx} f(x) \, dx, \qquad \text{or} \qquad \mathcal{L}\{f(x)\}(z) = F(z).$$

Therefore,

$$f(x) = \mathcal{L}^{-1}\{F(z)\}(x) = \frac{1}{2\pi i} \text{ P.V. } \int_{\gamma - i\infty}^{\gamma + i\infty} F(\zeta) e^{\zeta x} d\zeta = \frac{1}{2\pi} \text{ P.V. } \int_{-\infty}^{\infty} F(\gamma + iy) e^{(\gamma + iy)x} \, dy,$$

and the proof is finished.

■

Remark: This Theorem can be proven with a slightly relaxed **Condition (2.)**. We can assume that there is a $\gamma > c$ and there are denumerably many semicircles

$$C^+_{\gamma R_k} = \left\{ \zeta \mid \zeta = \gamma + R_k e^{i\theta}, \ 0 \le \theta \le 2\pi \right\}, \quad k = 1, \ 2, \ 3, \ldots$$

such that $\lim_{k \to \infty} R_k = \infty$ and

$$\lim_{k \to \infty} F(\gamma + R_k e^{i\theta}) = 0, \quad \text{uniformly on } 0 \le \theta \le 2\pi.$$

So, under the conditions of this **Theorem** or, more generally, if the principal value of the above integral exists and is independent of γ, we obtain $f(x)$, the inverse Laplace transform of $F(s)$: I.e.,

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x).$$

We could use the **inversion formula (1.45)** or **(1.46)** to directly recover the function $y = f(x)$, if we are given its Laplace transform $F(s)$. But, the direct computation of the **complex integral** involved is not very convenient, in general. (Even in the simple case of **Example 1.7.51** the direct computation was rather involved.)

However, with the help of this integral formula, we obtain the following convenient computation of the inverse Laplace transform, which many times can apply to cases more general than **Theorem 1.7.8**. For instance, we can apply it to $F(\zeta) = \dfrac{a}{\zeta}$ and $F(\zeta) = \dfrac{\zeta}{\zeta^2 + a^2}$ (referred to in the **remark** before the **proof** of **Theorem 1.7.8**). So, we have the following applicable **Theorem**:

Theorem 1.7.9 *Suppose:*

1. *$F(\zeta)$ is analytic in $\mathbb{C} - \{z_1,\, z_2,\, z_3, \dots\}$, where $\{z_1,\, z_2,\, z_3,\, \dots\}$ are countably many isolated poles with real parts $Re\,(z_n) < \gamma$, $\forall \quad n = 1,\ 2,\ 3,\ \dots$, for some constant $\gamma \in \mathbb{R}$.*

2. *There are denumerably many semicircles*

$$H^+_{\gamma R_k} = \left\{ \zeta \,|\, \zeta = \gamma + R_k e^{i\theta},\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \right\}, \quad k = 1,\ 2,\ 3, \dots$$

   such that $\lim\limits_{k\to\infty} R_k = \infty$, and they contain no poles of $F(\zeta)$.

3. *$\lim\limits_{k\to\infty} F(\gamma + R_k e^{i\theta}) = 0$ uniformly in $\dfrac{\pi}{2} \le \theta \le \dfrac{3\pi}{2}$.*

4. *The Mellin-Bromwich inversion integral formula can apply. I.e., the principal value in it exists (and, as we have seen, is independent of γ).*

Then,

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \sum_{n=1}^{\infty} \underset{\zeta = z_n}{Res} \left[F(\zeta) e^{x\zeta}\right],$$

if this sum exists.

Remark: We note that if $\sum_{n=1}^{\infty} \operatorname*{Res}_{\zeta=z_n} \left[F(\zeta)e^{x\zeta}\right]$ exists, then it is independent of the constant γ, as long as all the poles of $F(\zeta)$ lie in the open half plane to the left of the vertical straight line $x = \gamma$. (Why?)

Proof For any $k \in \mathbb{N}$, we consider the simple closed contour

$$C_k^+ = L_{\gamma R_k}^+ + H_{\gamma R_k}^+$$

consisting of two pieces: the vertical straight segment

$$L_{\gamma R_k}^+\{z = \gamma + iy \quad | \quad -R_k \le y \le R_k\} = [\gamma - iR_k, \gamma + iR_k],$$

with mid-point $\gamma + i0$, and the semicircle

$$H_{\gamma R_k}^+ = \left\{\zeta \,|\, \zeta = \gamma + R_k e^{i\theta},\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\},$$

with diameter the straight segment $[\gamma - iR_k, \gamma + iR_k]$ and located to the left of it. (See **Figure 1.22**.)

Then, for any given $x > 0$ fixed, we evaluate the line integral of $F(\zeta)e^{x\zeta}$ along C_k^+ and take its limit as $k \to \infty$. Notice that the set of poles of $F(\zeta)$ is equal to the set of poles of $F(\zeta)e^{x\zeta}$. So, if we let A_k be the set of all poles of $F(\zeta)$ in the interior of the contour C_k^+, then by **the Residue Theorem, 1.7.1**, we have

$$\frac{1}{2\pi i}\int_{C_k^+} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi i}\int_{L_{\gamma R_k}^+} F(\zeta)e^{x\zeta}d\zeta + \frac{1}{2\pi i}\int_{H_{\gamma R_k}^+} F(\zeta)e^{x\zeta}d\zeta = \sum_{z\in A_k} \operatorname*{Res}_{\zeta=z}\left[F(\zeta)e^{x\zeta}\right].$$

In the limit, as $k \to \infty$, we get

$$\lim_{k\to\infty} \sum_{z\in A_k} \operatorname*{Res}_{\zeta=z}\left[F(\zeta)e^{x\zeta}\right] = \sum_{n=1}^{\infty} \operatorname*{Res}_{\zeta=z_n}\left[F(\zeta)e^{x\zeta}\right],$$

and by **Condition (4.)** and equation **(1.45)** or **(1.46)**, we obtain

$$\lim_{k\to\infty}\frac{1}{2\pi i}\int_{L_{\gamma R_k}^+} F(\zeta)e^{x\zeta}d\zeta = \frac{1}{2\pi i}\lim_{k\to\infty}\int_{\gamma-iR_k}^{\gamma+R_k} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi i}\lim_{R_k\to\infty}\int_{\gamma-iR_k}^{\gamma+R_k} F(\zeta)e^{x\zeta}d\zeta = \frac{1}{2\pi i}\ P.V.\int_{\gamma-i\infty}^{\gamma+i\infty} F(\zeta)e^{x\zeta}d\zeta =$$

$$\frac{1}{2\pi}\ P.V.\int_{-\infty}^{\infty} F(\gamma+iy)e^{x(\gamma+iy)}dy = \mathcal{L}^{-1}\{F(s)\}(x) = f(x).$$

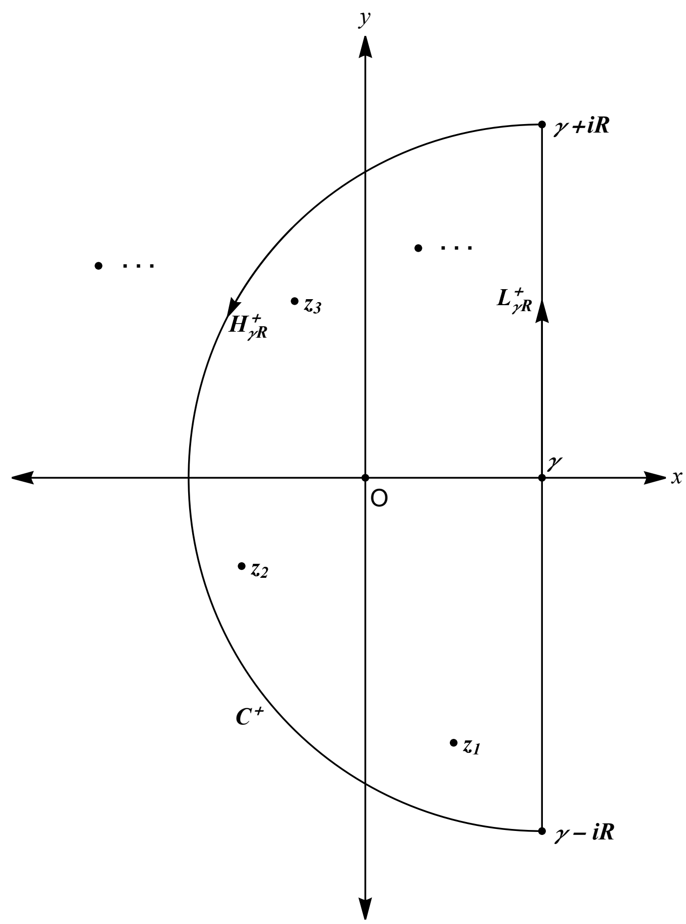

FIGURE 1.22: Contour 16 for Theorem 1.7.9

To finish the proof of the Theorem, we must show

$$\lim_{k\to\infty} \int\limits_{H^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = 0.$$

We have $\displaystyle\int\limits_{H^+_{\gamma R_k}} F(\zeta)e^{x\zeta}d\zeta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{x(\gamma+R_k e^{i\theta})} F\left(\gamma + R_k e^{i\theta}\right) R_k\, i\, e^{i\theta} d\theta,$

and we let $M_{R_k} = \underset{\zeta\in H^+_{\gamma R_k}}{\text{Maximum}}|F(\zeta)|$. By **hypothesis (3.)**, $\lim_{k\to\infty} M_{R_k} = 0$.

For $\zeta = \gamma + R_k e^{i\theta} \in H^+_{\gamma R_k}$, we have $|d\zeta| = |0 + R_k i e^{i\theta} d\theta| = R_k d\theta$ and $\left|e^{x\zeta}\right| = \left|e^{x(\gamma + R_k e^{i\theta})}\right| = e^{x\gamma} e^{xR_k \cos(\theta)}$. Hence, we obtain

$$\left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| \leq$$

$$\int\limits_{H^+_{\gamma R_k}} |F(\zeta)| \left|e^{x\zeta}\right| |d\zeta| \leq e^{x\gamma} M_{R_k} R_k \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{xR_k \cos(\theta)} d\theta.$$

By using the substitution $\theta = \phi + \dfrac{\pi}{2}$ and **Jordan's Lemma, 1.7.4,** we find that for any $x > 0$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{xR_k \cos(\theta)} d\theta = \int_0^{\pi} e^{-xR_k \sin(\phi)} d\phi < \frac{\pi}{R_k x}.$$

So, by the two previous relations, we obtain the inequality

$$\left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| < \frac{e^{x\gamma} M_{R_k} \pi}{x}.$$

Since $\lim\limits_{k\to\infty} M_{R_k} = 0$, we get that for any $x > 0$

$$\lim_{k\to\infty} \left| \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta \right| = 0 \quad \text{equivalent to} \quad \lim_{k\to\infty} \int\limits_{H^+_{\gamma R_k}} F(\zeta) e^{x\zeta} d\zeta = 0.$$

This proves what we wanted, and the Theorem follows.

■

Examples

Example 1.7.52 Let us apply the result of this Theorem to **Rule (5.)** in the **table of Problem I 2.7.14**, which generalizes **Example 1.7.51** in which $f(x)$ was constant.

We have the function $f(x) = x^n$, for $x > 0$ and $n = 0,\ 1,\ 2,\ \ldots$ (if $n = 0$, then $x^0 = 1$). Then,

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^{\infty} x^n e^{-sx} dx = \frac{n!}{s^{n+1}}, \quad \text{with} \quad s > 0,$$

and so

$$F(\zeta) = \frac{n!}{\zeta^{n+1}}, \quad \text{with} \quad \zeta \in \mathbb{C}.$$

This has only one pole of order $n+1$ at $\zeta = 0$.

Since

$$F(\zeta)e^{x\zeta} = \frac{n!}{\zeta^{n+1}} \sum_{k=0}^{\infty} \frac{x^k \zeta^k}{k!} = \sum_{k=0}^{\infty} \frac{n! x^k}{k!} \frac{1}{\zeta^{n+1-k}},$$

for $k = n$, we find the coefficient of $\dfrac{1}{\zeta}$, or

$$\operatorname*{Res}_{\zeta=0} \left[F(\zeta)e^{x\zeta}\right] = \frac{n! x^n}{n!} = x^n.$$

So, by the **previous Theorem**, for $x > 0$ and $n = 0,\ 1,\ 2,\ \ldots$, we have

$$\mathcal{L}^{-1}\left\{F(s)\right\}(x) = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}(x) = f(x) = x^n.$$

We observe that this method is much faster than directly computing the **integral (1.45)** or **(1.46)**, even when $n = 0$, as we did in **Example 1.7.51**.

▲

Example 1.7.53 Find the function $y = f(x)$ if we are given that

$$F(s) = \mathcal{L}\{f(x)\}(s) = \frac{18}{s^3 + 27}.$$

$F(s)$ obviously satisfies the three conditions of **Theorem 1.7.8**, so we can apply the **inversion integral formula (1.46)**, or more efficiently the result of **Theorem 1.7.9**.

The singularities of the function

$$e^{x\zeta} F(\zeta) = \frac{18 e^{x\zeta}}{\zeta^3 + 27}$$

are the third roots of -27, which are simple isolated poles. These are

$$z_k = 3e^{i\frac{(1+2k)\pi}{3}}, \quad \text{for} \quad k = 0,\ 1,\ 2$$

or

$$z_0 = \frac{3}{2}(1 + \sqrt{3}i), \qquad z_1 = -3, \qquad z_2 = \frac{3}{2}(1 - \sqrt{3}i).$$

Using

$$e^{x\zeta} F(\zeta) = \frac{18 e^{x\zeta}}{\zeta^3 + 27} = \frac{18 e^{x\zeta}}{(\zeta - z_0)(\zeta - z_1)(\zeta - z_2)},$$

we find that the residues of $e^{x\zeta}F(\zeta)$ at these poles are, respectively,

$$\frac{-(1+i\sqrt{3}\,)e^{\frac{3x}{2}(1+i\sqrt{3}\,)}}{3}, \qquad \frac{2e^{-3x}}{3}, \qquad \frac{-(1-i\sqrt{3}\,)e^{\frac{3x}{2}(1-i\sqrt{3}\,)}}{3}.$$

Since

$$\lim_{|\zeta|\to\infty} F(\zeta) = 0$$

and the constant γ can be taken $\gamma > \dfrac{3}{2}$, we can apply **Theorem 1.7.9**.

By adding the three residues and using **Euler's formulae** of exponential in terms of sines and cosines, we find

$$f(x) = \frac{2e^{-3x}}{3} + \frac{2e^{\frac{3x}{2}}}{3}\left[-\cos\left(\frac{3\sqrt{3}}{2}x\right) + \sqrt{3}\sin\left(\frac{3\sqrt{3}}{2}x\right)\right].$$

▲

The Example and the Remark that follow are very important. We study a more complicated situation in which $F(\zeta)$ has isolated and non-isolated singularities. This constitutes a variation or even generalization of **Theorem 1.7.9**.

As before, we assume:

1. $F(\zeta)$ is analytic for all $\zeta \in \mathbb{C}$ such that $\mathrm{Re}(\zeta) \geq \gamma$ for some constant γ, that is, all the singularities of $F(\zeta)$ have real parts less than γ.
2. $F(\zeta)$ **has isolated and non-isolated singularities**.
3. When applying the **Mellin-Bromwich inversion integral formula, (1.45)** [or **(1.46)**], by integrating along the vertical line segments $L^+_{\gamma R} = [\gamma - iR, \gamma + iR]$, for any $R > 0$, etc., we need to use appropriate contours that avoid not only the isolated but also the non-isolated singularities, and so we must also introduce appropriate branch cuts.
4. $\lim\limits_{R\to\infty} F(\gamma + Re^{i\theta}) = 0$ uniformly in θ.

We illustrate this situation with the **following example** in which all singularities are non-isolated. (If both isolated and non-isolated singularities are present, see the **remark** that follows.)

Example 1.7.54 For any $a > 0$, we would like to find

$$\mathcal{L}^{-1}\{F(s)\}(x) = f(x), \quad \text{if} \quad F(s) = \frac{e^{-a\sqrt{s}}}{\sqrt{s}}.$$

Thus, we must work with the function $F(\zeta) = \dfrac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}$, where $\zeta \in \mathbb{C}$.

Since the $\sqrt{\zeta}$ has non-isolated singularities, we choose the branch cut to be the closed non-positive real semi-axis $\{\zeta = x + 0i \mid x \leq 0\}$. In this domain $-\pi < \operatorname{Arg}(\zeta) < \pi$ and in its sub-domain $\operatorname{Re}(\zeta) > 0$, $F(\zeta)$ is complex analytic (holomorphic).

So, we can pick any $\gamma > 0$ constant, any $0 < r < \gamma$, any $R > 2\gamma$ and we consider the contour

$$C^+ = [\gamma - iR, \gamma + iR] + S^+_{\gamma R} + [\gamma - R, -r] + A^-_r + [-r, \gamma - R] + T^+_{\gamma R}$$

as in **Figure 1.23** where, apart from the obvious straight segments, we have set

$$\begin{aligned} S^+_{\gamma R} &= \left\{ \zeta = \gamma + Re^{i\theta} \mid \frac{\pi}{2} \leq \theta < \pi \right\}, \\ A^-_r &= \left\{ \zeta = Re^{i\theta} \mid \pi > \theta > -\pi \right\}, \\ T^-_{\gamma R} &= \left\{ \zeta = \gamma + Re^{i\theta} \mid \pi < \theta \leq \frac{3\pi}{2} \right\}. \end{aligned}$$

For any $x > 0$ fixed, we integrate the function $e^{x\zeta}F(\zeta) = e^{x\zeta}\dfrac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}$ along this simple closed contour. This line integral is equal to zero because this function has no singularity on C^+ and in its interior, and so the **Cauchy-Goursat Theorem, 1.5.3**, applies. Hence,

$$\begin{aligned} \int_{C^+} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{\gamma - iR}^{\gamma + iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \\ \int_{S^+_{\gamma R}} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int_{\gamma - R}^{-r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int_{A^-_r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \\ \int_{-r}^{\gamma - R} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta + \int_{T^-_{\gamma R}} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = 0. \end{aligned} \tag{1.47}$$

Next, we observe

$$\begin{aligned} \left| \int_{A^-_r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta \right| \leq \int_{A^-_r} \left| e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta \right| \leq \\ 2\pi r e^{(rx - a\sqrt{r})} r^{\frac{-1}{2}} = 2\pi\sqrt{r}e^{(rx - a\sqrt{r})} \longrightarrow 0, \text{ as } r \to 0. \end{aligned}$$

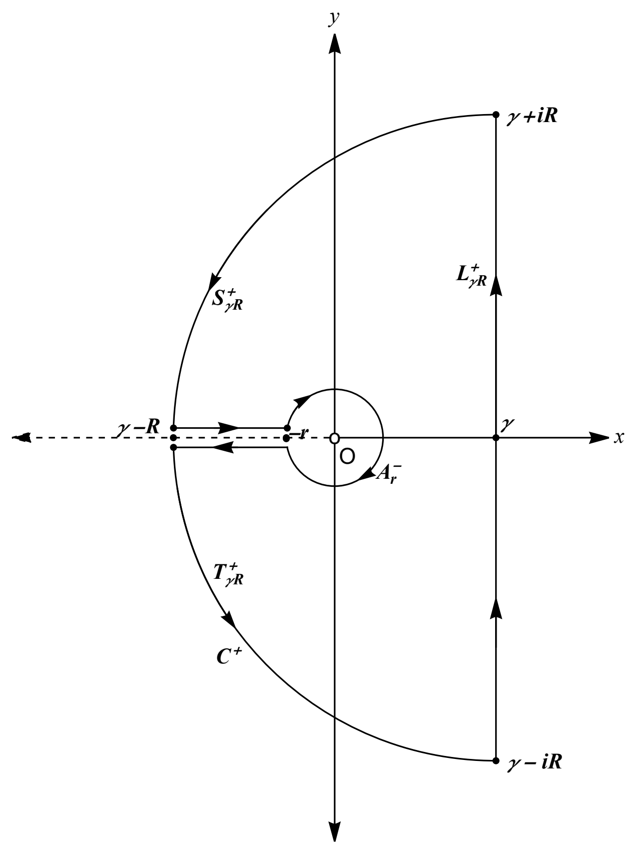

FIGURE 1.23: Contour 17 for Example 1.7.54

On $S^+_{\gamma R}$ and $T^-_{\gamma R}$, we have that $z = \gamma + Re^{i\theta}$. Then, by **Jordan's Lemma, 1.7.4,** we get (work out the details)

$$\left| \int\limits_{S^+_{\gamma R}} e^{x\zeta} \frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta \right| \longrightarrow 0, \text{ and } \left| \int\limits_{T^-_{\gamma R}} e^{x\zeta} \frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} d\zeta \right| \longrightarrow 0, \quad \text{as} \quad R \to \infty.$$

Along the segment $[\gamma - R, -r]$, we have that $\zeta = u + 0i = u < 0$ and $\text{Arg}(\zeta) = \pi$. So,

$$\sqrt{\zeta} = e^{\frac{1}{2}\log(\zeta)} = e^{\frac{1}{2}[\ln(|\zeta|)+i\pi]} = e^{\frac{i\pi}{2}} e^{(\frac{1}{2}\ln(|\zeta|)|} = i\sqrt{|\zeta|} = i\sqrt{|u|}.$$

Therefore, along the segment $[\gamma - R, -r]$ we have

$$e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}} = e^{xu}\frac{e^{-ia\sqrt{|u|}}}{i\sqrt{|u|}} = \frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}.$$

Then,

$$\int_{\gamma-R}^{-r} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{\gamma-R}^{-r}\frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}\,du.$$

Similarly, along the segment $[-r, \gamma - R]$, we have that $\zeta = u + 0i = u < 0$ and $\mathrm{Arg}(\zeta) = -\pi$. So,

$\sqrt{\zeta} = e^{\frac{1}{2}\log(\zeta)} = e^{\frac{1}{2}[\ln(|\zeta|)-i\pi]} = e^{-\frac{i\pi}{2}}e^{(\frac{1}{2}\ln(|\zeta|)|} = -i\sqrt{|\zeta|} = -i\sqrt{|u|}$ and

$$\int_{-r}^{\gamma-R} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}d\zeta = \int_{-r}^{\gamma-R}\frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) + i\sin\left(a\sqrt{|u|}\right)\right]}{-i\sqrt{|u|}}\,du.$$

So, by the relation **(1.47)** above, after letting $r \to 0$ and $R \to \infty$, we get

$$\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\,d\zeta =$$

$$\int_{-\infty}^{0}\frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) - i\sin\left(a\sqrt{|u|}\right)\right]}{i\sqrt{|u|}}\,du +$$

$$\int_{0}^{-\infty}\frac{e^{xu}\left[\cos\left(a\sqrt{|u|}\right) + i\sin\left(a\sqrt{|u|}\right)\right]}{-i\sqrt{|u|}}\,du =$$

$$-2i\int_{-\infty}^{0}\frac{e^{xu}\cos\left(a\sqrt{|u|}\right)}{\sqrt{|u|}}\,du.$$

We let $u = -v^2$ with $v > 0$, and so $du = -2vdv$, using the result of **Example I 2.2.14**, we find

$$\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\,d\zeta = 4i\int_{-\infty}^{0} e^{-xv^2}\cos(av)\,dv =$$

$$4i\int_{0}^{\infty} e^{-xv^2}\cos(av)\,dv = 4i\,\frac{1}{2}\sqrt{\frac{\pi}{x}}e^{\frac{-a^2}{4x}} = 2i\sqrt{\frac{\pi}{x}}e^{\frac{-a^2}{4x}}.$$

Hence, by the **Mellin inversion formula, (1.45)**, we obtain

$$f(x) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\sqrt{\zeta}}\,d\zeta = \frac{e^{\frac{-a^2}{4x}}}{\sqrt{\pi x}}, \quad \text{for } x > 0.$$

Finally, we have obtained the rule

$$\forall\ a > 0, \quad f(x) = \mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\}(x) = \frac{e^{\frac{-a^2}{4x}}}{\sqrt{\pi x}}, \quad \text{for } x > 0.$$

▲

Remark: Another **variation** of this method occurs when the function $F(\zeta)$ has both **isolated** and **non-isolated** singularities. Then, we must combine the methods of **this Example** with the method of the **previous Example** in order to find $f(x) = \mathcal{L}^{-1}\{F(s)\}(x)$. For the isolated singularities inside the contour C^+, we must use **the Residue Theorem, 1.7.1**.

For instance, $F(\zeta) = \dfrac{5\sqrt{\zeta}}{\zeta^2 + 4}$ has the non-isolated singularities of the $\sqrt{\zeta}$, and so we need a branch cut. It also has the two isolated singularities $\zeta = \pm 2i$, which can be easily included in the interior of the contour used.

If now, e.g., we consider the function $F(\zeta) = \dfrac{5\sqrt{\zeta}}{\zeta^2 - 4}$, we observe that this function has the two isolated singularities $\zeta = \pm 2$ and the non-isolated singularities of $\sqrt{\zeta}$. The isolated singularity $\zeta = +2$ can be easily included in the interior of the contour. But $\zeta = -2$ is on the negative x-semi-axis. So, with branch cut the non-positive x-semi-axis, we must put two equal circular arcs around $z = -2$, one in the upper half plane and the other in the lower half plane. Then, we take limits as the radius of these circular arcs approaches zero, etc.

Computing the inverse Laplace transforms of both of these $F(s)$ is left to the reader as a problem below.

Problems.

1.7.152 Check which of the three conditions of **Theorem 1.7.8** hold for the functions

$$f_1(\zeta) = e^{-\zeta^2}, \quad f_2(\zeta) = \sin\left(\frac{1}{\zeta}\right), \quad f_3(\zeta) = \frac{5\zeta}{\zeta^2 + 4}, \quad f_4(\zeta) = \frac{5}{\zeta^2 + 4}.$$

1.7.153 Use what we have learnt in **this section** and **Example 1.6.20** to prove that

$$\mathcal{L}^{-1}\left\{\sin\left(\frac{1}{s}\right)\right\}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)! \cdot (2n+1)!}.$$

In the following **Problems (1.7.154-1.7.158)**, justify why we can use the **Mellin inversion formula (1.45)** or **(1.46)** and use only the methods of this **Section**, to find the inverse Laplace transform $f(x)$ if the Laplace transform $F(s)$ is given by:

1.7.154

(a) $\dfrac{5s}{s^2+4}$, (b) $\dfrac{5}{s^2+4}$.

1.7.155

(a) $\dfrac{5s}{s^2-4}$, (b) $\dfrac{5}{s^2-4}$.

1.7.156

(a) $\dfrac{5\sqrt{s}}{s^2+4}$, (b) $\dfrac{5\sqrt{s}}{s^2-4}$.

1.7.157

(a) $\dfrac{5s^3}{s^4-4}$, (b) $\dfrac{5(s-2)}{(s+3)(s^2+4s+5)}$.

1.7.158

(a) $\dfrac{s^2-4}{(s^2+4)^2}$, (b) $\dfrac{s^2+4}{(s^2-4)^2}$.

1.7.159 Imitate the work done in **Example 1.7.54** to prove the following inverse Laplace transform rule: For any $a > 0$ constant,

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}(x) = \frac{a}{2\sqrt{\pi x^3}}\, e^{\frac{-a^2}{4x}} \text{ and so } \mathcal{L}\left\{\frac{a}{2\sqrt{\pi x^3}}\, e^{\frac{-a^2}{4x}}\right\}(s) = e^{-a\sqrt{s}}.$$

1.7.160 The **Laplace-Carson**[50] **transform** of $f\ :\ [0,\infty) \longrightarrow \mathbb{R}$ is

$$\mathcal{C}[f(x)](s) = s\int_0^\infty e^{-sx} f(x)dx = s\cdot\mathcal{L}[f(x)](s) = \mathcal{L}[f'(x)](s) + f(0).$$

Under what conditions we obtain the inverse transform

$$f(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{sx}\cdot\frac{\mathcal{C}[f(x)](s)}{s}\,ds\ ?$$

[50] John Renshaw Carson, American engineer, June 28, 1886 - October 31, 1940.

1.7.161 Imitate the work done in **Example 1.7.54** to prove the following inverse Laplace transform rule:

For any $a > 0$ constant,

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}(x) = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right),$$

$$\text{and so,}\quad \mathcal{L}\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \mathcal{L}\left\{1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \frac{e^{-a\sqrt{s}}}{s}.$$

[Hint: An important difference with **Example 1.7.54** is in

$$\lim_{r\to 0}\int\limits_{A_r^-} e^{x\zeta}\frac{e^{-a\sqrt{\zeta}}}{\zeta}d\zeta = \lim_{r\to 0}\int_{\pi}^{-\pi}\frac{e^{rxe^{i\theta}-a\sqrt{r}\,e^{i\frac{\theta}{2}}}}{re^{i\theta}}ire^{i\theta}d\theta =$$

$$i\cdot\lim_{r\to 0}\int_{\pi}^{-\pi} e^{rxe^{i\theta}-a\sqrt{r}\,e^{i\frac{\theta}{2}}}d\theta = \int_{\pi}^{-\pi} 1 d\theta = -2\pi i.$$

In the end, you need to use the integral representation of the error function as done in **Example I 2.2.15**.]

1.7.162 Prove that for any real $a \geq 0$, the Laplace transform of

$$f(x) = \frac{1}{\sqrt{x+a}},\quad \text{where}\quad x > -a,$$

$$\text{is}\quad F(s) = \frac{\sqrt{\pi}\,e^{as}\operatorname{erfc}\left(\sqrt{as}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}\operatorname{erfcx}\left(\sqrt{as}\right),\qquad \forall\ s > 0.$$

[Hint: Use the technique presented in this section or easier, the **Rule** in **Example I 2.7.8** and the definitions of the functions $\operatorname{erf}(x)$, $\operatorname{erfc}(x)$, and $\operatorname{erfcx}(x)$. Also see **Example I 2.1.1**.]

1.7.163 Find the inverse Laplace transforms of

$$F_1(s) = \frac{\sqrt{s}}{s^2+4},\qquad \text{and}\qquad F_2(s) = \frac{\sqrt{s}}{s^2-4}.$$

[Hint: Refer to the last **remark** of this section above.]

1.7.164 If $c > 0$, prove

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z^2}\,dz = \begin{cases}\ln(a), & \text{if}\quad a > 1\\[2ex] 0, & \text{if}\quad 0 < a \leq 1.\end{cases}$$

1.8 Definite Integrals with Sines and Cosines

In this section, we study some definite integrals of sines and cosines. There is the case of rational functions of sines and cosines on the whole interval $[0, 2\pi]$ and the case of rational or other functions of sines and cosines on some subintervals of $[0, 2\pi]$. In the first case, we have a more or less clear-cut answer by means of complex analysis and the Residue Theorem. In the other cases besides complex analysis, we can use and combine other techniques.

1.8.1 Rational Functions of Sines and Cosines

Calculus provides a general method for finding indefinite integrals of rational functions of $\sin(\theta)$ and $\cos(\theta)$. According to this method, we perform the so-called *tangent of half angle substitution*, then we use partial fractions decomposition and execute the ensuing computations. In the end, we perform an inverse substitution. (Review this method once more from a good calculus book and then provide the missing details in **Example I 1.1.2** and **Remark 2 of Example 1.7.14**.)

Complex functions provide some methods of computing certain definite integrals of rational functions of $\sin(\theta)$ and $\cos(\theta)$, under some mild hypotheses. One method that we have already encountered is the **Cauchy integral formula** as demonstrated in some examples and problems of **Section 1.5.8**. More general cases of such definite integrals are figured out by the following **Lemma**, which is very convenient and efficient when applicable. So, we have:

Lemma 1.8.1 *Let*

$$g(\theta) = R\left[\sin(\theta), \cos(\theta)\right], \quad \textit{with} \quad 0 \le \theta \le 2\pi,$$

be a rational function of $\sin(\theta)$ *and* $\cos(\theta)$. *We set*

$$f(z) = R\left(\frac{z - \frac{1}{z}}{2i}, \frac{z + \frac{1}{z}}{2}\right) \cdot \frac{1}{iz}$$

and suppose that $f(z)$ *has no singularities on the unit circle* $C(0,1)$ *and its singularities inside the open disc* $D(0,1)$ *are isolated and therefore finitely many.*

(a) If all the singularities of $f(z)$ *inside the open disc* $D(0,1)$ *are the*

complex numbers $z_1,\ z_2,\ z_3, \ldots, z_n$, $(n \geq 1)$, *then*

$$\int_0^{2\pi} g(\theta)\, d\theta = \int_0^{2\pi} R\left[\sin(\theta), \cos(\theta)\right] d\theta = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{Res}\, f(z).$$

(b) If $f(z)$ *has no singularities inside* $C(0,1)$, *then*

$$\int_0^{2\pi} g(\theta)\, d\theta = \int_0^{2\pi} R\left[\sin(\theta), \cos(\theta)\right] d\theta = 0.$$

[Singularities outside $C(0,1)$ *play no role in these two results.]*

Proof (a) If $C^+ = C^+(0,1)$, by the **Residue Theorem, 1.7.1**, we have

$$\oint_{C^+} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\operatorname{Res}}\, f(z).$$

But,

$$\oint_{C^+} f(z)\, dz = \int_0^{2\pi} f\left(e^{i\theta}\right)\, d\left(e^{i\theta}\right) =$$

$$\int_0^{2\pi} R\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}, \frac{e^{i\theta} + e^{-i\theta}}{2}\right) \frac{1}{ie^{i\theta}} ie^{i\theta}\, d\theta = \int_0^{2\pi} R[\sin(\theta), \cos(\theta)]\, d\theta.$$

Thus,

$$\int_0^{2\pi} R[\sin(\theta), \cos(\theta)]\, d\theta = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{\operatorname{Res}}\, f(z).$$

(b) This follows immediately from the **Cauchy-Goursat Theorem, 1.5.3**.

■

Examples

Example 1.8.1 We want to evaluate

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)}$$

with $a > 1$ real constant (so the denominator is never zero).

By **Lemma 1.8.1**, we consider the complex function

$$f(z) = \frac{1}{a + \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)} \cdot \frac{1}{iz} = \frac{2}{z^2 + 2aiz - 1}.$$

The roots of the denominator $z^2 + 2aiz - 1$, evaluated by the quadratic formula, are $z_1 = i(-a + \sqrt{a^2 - 1})$ and $z_2 = -i(a + \sqrt{a^2 - 1})$.

The first root is inside the $D(0, 1)$ and the second root is outside. Therefore, we need to compute the residue of $f(z)$ at $z = z_1$. So, we have

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{2}{i(-a + \sqrt{a^2-1}) + i(a + \sqrt{a^2-1})} = \frac{1}{i\sqrt{a^2-1}}.$$

Hence,

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)} = 2\pi i \frac{1}{i\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Remark 1: This result is equal to twice the result of the **Example 1.7.14**, since for $a > 1$

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a + \sin(\theta)}$$

(prove this equality!).

We see that this is an easier and more efficient treatment of **Example 1.7.14**.

Remark 2: This example can also be carried out with $a < -1$, real. In this case, we compute the residue at z_2, and in same way we find

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)} = \frac{-2\pi}{\sqrt{a^2-1}}.$$

[Another way is

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)} = -\int_{-2\pi}^{0} \frac{du}{-a + \sin(u)},$$

where $u = -\theta$. We use the periodicity of sine and, with $-a > 1$, the integral computed above.]

So, with $|a| > 1$ real, we get

$$\int_0^{2\pi} \frac{d\theta}{a + \sin(\theta)} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a + \sin(\theta)} = \frac{\operatorname{sign}(a) \cdot 2\pi}{\sqrt{a^2-1}}.$$

(See also **Problem 1.7.29**.)

Remark 3: Manipulating the above integral or directly, we also find that for any $|a| > 1$,

$$\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)} = 2\int_0^{\pi} \frac{d\theta}{a + \cos(\theta)} = \frac{\operatorname{sign}(a) \cdot 2\pi}{\sqrt{a^2-1}}.$$

Remark 4: By differentiation with respect to a we find that, if $|a| > 1$

$$\int_0^{2\pi} \frac{d\theta}{[a+\sin(\theta)]^2} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{[a+\sin(\theta)]^2} =$$

$$\int_0^{2\pi} \frac{d\theta}{[a+\cos(\theta)]^2} = 2\int_0^{\pi} \frac{d\theta}{[a+\cos(\theta)]^2} = \frac{2\pi|a|}{\sqrt{(a^2-1)^3}}.$$

▲

Example 1.8.2 If $a \in \mathbb{C}$ constant with $|a| < 1$, evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos(\theta) + a^2}.$$

(This integral is closely related to **Poisson kernel, Definition 1.5.3**, in the unit disc, if in that definition we put $a = 1$, $r = a$ and $\phi = 0$. Compare these two expressions carefully and compute the integral immediately from its series representation in that definition, as an exercise. See also **Problem 1.5.42**.)

We consider $0 < |a| < 1$ (so, $a \neq 0$) and again use **Lemma 1.8.1** with

$$f(z) = \frac{1}{1 - 2a\frac{1}{2}\left(z + \frac{1}{z}\right) + a^2} \cdot \frac{1}{iz} = \frac{-i}{-az^2 + (a^2+1)\,z - a} =$$

$$\frac{i}{az^2 - (a^2+1)\,z + a} = \frac{i}{(az-1)(z-a)} = \frac{i}{a} \cdot \frac{1}{\left(z - \frac{1}{a}\right)(z-a)}.$$

So, $f(z)$ has isolated singularities $z_1 = a$ inside the $D(0,1)$ and $z_2 = \frac{1}{a}$ outside it.

We compute

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{i}{a} \cdot \frac{1}{z - \frac{1}{a}}\Big|_{z=a} = \frac{i}{a^2-1}.$$

Hence, with $a \in \mathbb{C}$ such that $0 < |a| < 1$, we find

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos(\theta) + a^2} = 2\pi i \frac{i}{a^2-1} = \frac{2\pi}{1-a^2}, \quad \text{when} \quad 0 < |a| < 1.$$

We now observe that this formula is trivially true when $a = 0$, and so we can state that the formula is true for any $a \in \mathbb{C}$ such that $0 \leq |a| < 1$.

Remark: (a) When $|a| > 1$, we apply this result with $\frac{1}{a}$ in the place of a and we find that (check this)

$$\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos(\theta) + 1} = \frac{2\pi}{a^2 - 1}, \quad \text{when} \quad |a| > 1.$$

(b) Examine this integral when $a = 1$ as an **exercise!** Explain why the integral becomes improper at $\theta = 0$ and $\theta = 2\pi$. What value or principal value did you find eventually?

(c) In harmonic analysis and partial differential equations, for $R > r > 0$, we use the integral

$$\int_0^{2\pi} \frac{d\theta}{R^2 - 2Rr\cos(\theta) + r^2} = \frac{2\pi}{R^2 - r^2}.$$

[Verify this immediately from the result obtained in this example by making the appropriate transformation!

See also **Problem 1.5.46, (a)**.]

(d) The above integrals can be also found by using appropriately the indefinite integral

$$\int \frac{1 - r^2}{r^2 - 2r\cos(x) + 1}\, dx = 2\arctan\left[\frac{1 + r}{1 - r}\tan\left(\frac{x}{2}\right)\right] + C.$$

▲

Note: We must observe that both examples studied before can be figured out by means of the **Cauchy integral formula** (**Section 1.5.8**).

In both, we use $C = C^+(0, 1)$. In the first example, we apply the formula to the function

$$f(z) = \frac{2}{z^2 + 2aiz - 1} = \frac{\overline{\dfrac{2}{z - z_2}}}{z - z_1} = \frac{\overline{\dfrac{2}{z + i(a + \sqrt{a^2 - 1})}}}{z - i(-a + \sqrt{a^2 - 1})},$$

where $a > 1$ real. In the second example, we apply the formula to the function

$$f(z) = \frac{i}{a} \cdot \frac{1}{\left(z - \dfrac{1}{a}\right)(z - a)} = \frac{i}{a} \cdot \frac{\overline{\frac{1}{z - \frac{1}{a}}}}{z - a},$$

where $0 < |a| < 1$ complex. (Work out the application of the **Cauchy integral formula** in both examples.)

This way, using the **Cauchy integral formula**, is also very convenient here, because there is just one singularity inside the contour

$C = C^+(0,1)$. When there are two or more singularities inside $C = C^+(0,1)$, then **Lemma 1.8.1** is more convenient.

In the following example, **Lemma 1.8.1** is more convenient than the **Cauchy integral formula**.

Example 1.8.3 We prove the result in **Examples I 2.2.3, I 2.6.21** and **1.5.16** by means of **Lemma 1.8.1**. That is, if $a > 0$ and $b > 0$ constants, then

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x) + b^2\sin^2(x)} = \frac{2\pi}{ab}.$$

For $a = b > 0$, the result is immediate. So, we assume $a \neq b$ and consider

$$f(z) = \frac{1}{\left[a^2\left(\frac{z+\frac{1}{z}}{2}\right)^2 + b^2\left(\frac{z-\frac{1}{z}}{2i}\right)^2\right]iz} = \frac{1}{\left[a^2\left(\frac{z+\frac{1}{z}}{2}\right)^2 - b^2\left(\frac{z-\frac{1}{z}}{2}\right)^2\right]iz} =$$

$$\cdots = \frac{-i4z}{[(a-b)z^2 + (a+b)]\cdot[(a+b)z^2 + (a-b)]} =$$

$$\frac{-i4z}{[(a-b)z^2 + (a+b)]\cdot(a+b)\cdot\left(z - \sqrt{\frac{b-a}{a+b}}\right)\cdot\left(z + \sqrt{\frac{b-a}{a+b}}\right)}.$$

The four roots of the denominator are simple. With $a > 0$ and $b > 0$, two of these roots are outside the disc $\overline{D(0,1)}$, and the roots $z_1 = \sqrt{\dfrac{b-a}{a+b}}$ and $z_2 = -\sqrt{\dfrac{b-a}{a+b}}$ are inside. So, we compute

$$\operatorname*{Res}_{z=z_1} f(z) = \ldots = \frac{-i}{2ab} \qquad \text{and} \qquad \operatorname*{Res}_{z=z_2} f(z) = \ldots = \frac{-i}{2ab}.$$

Then, by **Lemma 1.8.1**, we have

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x) + b^2\sin^2(x)} = 2\pi i\left(\frac{-i}{2ab} + \frac{-i}{2ab}\right) = \frac{2\pi}{ab}.$$

Remark 1: This result could be found by using the **Cauchy integral formula** but after we break the function $f(z)$ above in complex partial fractions. So, **Lemma 1.8.1** is more convenient.

Remark 2: From this result, we obtain the following two byproducts:

(a) If $A > 0$ and $B > -A$ real constants, then

$$\int_0^{2\pi} \frac{dx}{A + B\sin^2(x)} = 2\int_0^{\pi} \frac{dx}{A + B\sin^2(x)} = 4\int_0^{\frac{\pi}{2}} \frac{dx}{A + B\sin^2(x)} =$$
$$\int_0^{2\pi} \frac{dx}{A\cos^2(x) + (A+B)\sin^2(x)} = \frac{2\pi}{\sqrt{A(A+B)}}.$$

(b) If $B > 0$ and $A > -B$ real constants, then

$$\int_0^{2\pi} \frac{dx}{A\cos^2(x) + B} = 2\int_0^{\pi} \frac{dx}{A\cos^2(x) + B} = 4\int_0^{\frac{\pi}{2}} \frac{dx}{A\cos^2(x) + B} =$$
$$\int_0^{2\pi} \frac{dx}{(A+B)\cos^2(x) + B\sin^2(x)} = \frac{2\pi}{\sqrt{(A+B)B}}\,.$$

(c) The integral of the previous example can be found using the above integral in **(a)** or **(b)**. (How?)

▲

Example 1.8.4 For any $n = 0,\ 1,\ 2,\ 3,\ \dots$, we obviously have

$$\int_0^{2\pi} \sin^{2n+1}(x)\,dx = \int_0^{2\pi} \cos^{2n+1}(x)\,dx = 0,$$

since sine and cosine are odd functions about π. But, we must prove the following integral

$$\int_0^{2\pi} \sin^{2n}(x)\,dx = \int_0^{2\pi} \cos^{2n}(x)\,dx = \frac{2\pi}{2^{2n}} \cdot \frac{(2n)!}{(n!)^2} =$$
$$\frac{2\pi}{2^{2n}} \cdot \binom{2n}{n} = 2\pi \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n}.$$

(See **Problems I 2.6.25, 1.8.12, 1.8.17** and **Examples I 2.6.19, 1.8.5, 1.8.6**, etc.)

We can also prove the last integral in the following way. We work with the cosine and similar work with the sine. We have

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \frac{e^{2ix} + 1}{2e^{ix}}.$$

Also, the transformation $z = e^{ix}$ maps the real interval $[0, 2\pi]$ onto the unit circle $C^+ = C^+(0,1)$. Then, $dx = \dfrac{1}{iz}\,dz = -i\dfrac{1}{z}\,dz$ and

$$\int_0^{2\pi} \cos^{2n}(x)\,dx = \frac{-i}{2^{2n}} \int_{C^+} \frac{\left(1 + z^2\right)^{2n}}{z^{2n+1}}\,dz.$$

The function $f(z)=\dfrac{\left(1+z^2\right)^{2n}}{z^{2n+1}}$ has one pole of order $2n+1$ at $z=0$. Its residue (by **Way 3**, **Subsection 1.6.2**) is

$$\operatorname*{Res}_{z=0} f(z)=\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\left(z^{2n+1}\right)f(z)\right]|_{z=0}=$$
$$\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\left(1+z^2\right)^{2n}\right]|_{z=0}=\ldots=\frac{(2n)!}{(n!)^2}=\binom{2n}{n}.$$

Therefore, by the **Residue Theorem, 1.7.1**, we get

$$\int_0^{2\pi}\cos^{2n}(x)\,dx=\frac{-i}{2^{2n}}\int\limits_{C^+}\frac{\left(1+z^2\right)^{2n}}{z^{2n+1}}\,dz=\frac{-i}{2^{2n}}\cdot 2\pi i\cdot\frac{(2n)!}{(n!)^2}=$$
$$\frac{2\pi}{2^{2n}}\cdot\frac{(2n)!}{(n!)^2}=\frac{2\pi}{2^{2n}}\cdot\binom{2n}{n}=2\pi\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}.$$

Remark: From this result, dividing by 4, we get

$$\int_0^{\frac{\pi}{2}}\sin^{2n}(x)\,dx=\int_0^{\frac{\pi}{2}}\cos^{2n}(x)\,dx=\frac{\pi}{2^{2n+1}}\cdot\frac{(2n)!}{(n!)^2}=$$
$$\frac{\pi}{2^{2n+1}}\cdot\binom{2n}{n}=\frac{\pi}{2}\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n},$$

as we have seen in **Problems I 2.6.25, 1.8.12, 1.8.17** and **Examples I 2.6.19, 1.8.5, 1.8.6**, etc.

▲

1.8.2 Other Techniques with Sines and Cosines

Next, we investigate some techniques of computing various integrals involving $\sin(x)$ and $\cos(x)$, based on various algebraic and trigonometric identities and substitutions (e.g., the tangent of the half angle substitution that we have seen in this text before, or in a calculus course, etc.).

For instance, we use the following **Euler's formulae**:

$$\begin{aligned}&(1)\;\; e^{ix}=\cos(x)+i\sin(x),\\ &(2)\;\; e^{-ix}=\cos(x)-i\sin(x),\\ &(3)\;\; \cos(x)=\frac{e^{ix}+e^{-ix}}{2},\\ &(4)\;\; \sin(x)=\frac{e^{ix}-e^{-ix}}{2i}.\end{aligned}$$

Along these lines, we also use the trigonometric formulae and identities that have been listed in **Problems I 2.2.28, 1.2.15, 1.2.27**, **1.2.28**, and others. (Have a look at them again.) Manipulating those and with the help of algebraic operations and other trigonometric identities, we can derive a few more, in the usual way.

We can use the fact stated in **Problem 1.7.58** that for any $-\infty < a,\ b < \infty$ real numbers and any non-zero complex number $\alpha + i\beta \neq 0$, the following integral formula

$$\int_a^b e^{(\alpha+i\beta)x}dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha+i\beta}$$

is valid. If $\alpha + i\beta = 0$, then the integral is equal to $b - a$.

From this fact, we immediately obtain the **result**:
For all integers $\boldsymbol{n}$ and $\boldsymbol{m}$, we have

$$\int_0^{2\pi} e^{-inx}e^{imx}dx = \begin{cases} 0, & \text{if} \quad m \neq n, \\ 2\pi, & \text{if} \quad m = n. \end{cases}$$

Combining this fact with Euler's formulae, we easily obtain the **result**:
For every integer $\boldsymbol{n \neq 0}$, we have

$$\int_0^{2\pi} \cos^2(nx)dx = \int_0^{2\pi} \sin^2(nx)dx = \pi.$$

Notice that for $\boldsymbol{n = 0}$ we trivially obtain that the first integral is equal to 2π, whereas the second integral is equal to 0.

We also obtain the following **results**:
For all integers $\boldsymbol{m \neq n}$, we have:

$$(1) \quad \int_0^{2\pi} \cos(nx)\cos(mx)\,dx = 0,$$

$$(2) \quad \int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0,$$

$$(3) \quad \int_0^{2\pi} \sin(nx)\sin(mx)\,dx = 0.$$

In the computations of integrals with sines and cosines, we may use the results in **Problems I 2.2.28, 1.2.16, 1.2.19, 1.2.27, 1.2.28**, etc. We can also use **algebraic identities**, such as:

1. **Binomial Theorem:** $\forall\ n = 0,\ 1,\ 2,\ \ldots$, and $\forall\ z,\ w \in \mathbb{C}$, we have

$$(z+w)^n = \sum_{k=0}^{n} z^{n-k} w^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   We remember that $0! = 1$ and so $\binom{n}{0} = 1$. For $k \in \mathbb{N}$, we simplify

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots(n-k+1)}{k(k-1)\ldots 3\cdot 2\cdot 1}.$$

2. $\forall\ n = 0,\ 1,\ 2,\ \ldots$, and $\forall\ z \neq w \in \mathbb{C}$, we have

$$\frac{z^n - w^n}{z-w} = z^{n-1} + z^{n-2}w + z^{n-3}w^2 + \ldots + zw^{n-2} + w^{n-1}.$$

   If $z = w$, then the limit of the first side, as $z \to w$, is equal to the second side if we replace z with w, that is, $nw^{n-1} = (z^n)'\,|_{z=w}$.

3. (Look at various sources of algebraic identities.)

We may also use **identities of combinatorial numbers** and factorials, such as:

1. $\forall\ n \geq k \geq 1$ integers, we have the **recursive identity of the combinatorial numbers**

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

2. $\forall\ n \geq k \geq 0$ integers, we have the **symmetry of the combinatorial numbers**

$$\binom{n}{n-k} = \binom{n}{k}.$$

3. $\forall\ n,\ m,\ k$, positive integers such that $n + m \geq k$, we have

$$\binom{n+m}{k} = \sum_{l=0}^{k} \binom{n}{l}\binom{m}{k-l}.$$

4. (Look also at **Problem I 2.2.28** and various sources of combinatorial identities.)

Examples

The methods and examples exposed here are rather straightforward. Sometimes they involve some lengthy, but not difficult, computations.

Example 1.8.5 In this example we consider integrals of powers of $\sin(x)$. [Similar work is done for powers of $\cos(x)$.]

For $n = 0,\ 1,\ 2,\ \ldots$, we have

$$\sin^n(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^n = \frac{1}{2^n i^n}\sum_{k=0}^{n}(-1)^k \binom{n}{k} e^{(n-2k)ix}.$$

(See also **Problem I 2.2.28**.)

We have already seen that

$$\int_a^b e^{(n-2k)ix}dx = \begin{cases} \dfrac{e^{(n-2k)ib} - e^{(n-2k)ia}}{(n-2k)i}, & \text{if} \quad n \neq 2k, \\ \\ b - a, & \text{if} \quad n = 2k. \end{cases}$$

So, for $0 \le k \le n$, we have the partial integrals

$$I_{n,k} := \int_0^{\frac{\pi}{2}} e^{(n-2k)ix}dx = \begin{cases} \dfrac{e^{(n-2k)i\frac{\pi}{2}} - 1}{(n-2k)i}, & \text{if} \quad n \neq 2k, \\ \\ \dfrac{\pi}{2}, & \text{if} \quad n = 2k. \end{cases}$$

(a) We consider the **case $\boldsymbol{n = 2m}$ is even**. Then,

$$I_{2m,k} = \begin{cases} \dfrac{[-(-1)^{m-k} + 1]i}{2(m-k)}, & \text{if} \quad m \neq k, \\ \\ \dfrac{\pi}{2}, & \text{if} \quad m = k. \end{cases}$$

So, this new method gives

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\, dx = \frac{1}{2^{2m}(-1)^m}\sum_{k=0}^{2m}(-1)^k \binom{2m}{k} I_{2m,k}.$$

Since this is a real number, by equating the real parts we finally have

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\, dx = \frac{1}{2^{2m}(-1)^m} \cdot (-1)^m \binom{2m}{m} I_{2m,m} =$$

$$\frac{\pi}{2^{2m+1}} \cdot \frac{(2m)!}{(m!)^2} = \frac{1 \cdot 3 \ldots (2m-1)}{2 \cdot 4 \ldots (2m)} \cdot \frac{\pi}{2}.$$

Remark 1: The imaginary parts must be equal to zero. Thus, we obtain the identity

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} (-1)^k \binom{2m}{k} \frac{1-(-1)^{m-k}}{m-k} = 0,$$

or

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{(-1)^k}{m-k} \binom{2m}{k} = (-1)^m \sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{1}{m-k} \binom{2m}{k}.$$

This identity is not very interesting because we can directly show, by the symmetry of the combinatorial numbers, that each of these two sums is equal to zero. I.e., for any integer $m \geq 1$, we have

$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{(-1)^k}{m-k} \binom{2m}{k} = 0, \qquad \text{and} \qquad \sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{1}{m-k} \binom{2m}{k} = 0.$$

[See also **Problem I 2.2.28, (e)**.]

Remark 2: This integral was also found in **Section I 2.6, Example I 2.6.18**, where we dealt with the **Beta and Gamma functions**. Again, the integral, with the help of **Problem I 2.6.14, (a)**, is

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\, dx = \frac{1}{2} \mathrm{B}\left(m + \frac{1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma(m+1)} =$$

$$\frac{\pi}{2^{2m+1}} \cdot \frac{(2m)!}{(m!)^2} = \frac{1 \cdot 3 \ldots (2m-1)}{2 \cdot 4 \ldots (2m)} \cdot \frac{\pi}{2}.$$

(b) Now, we consider the **case $\boldsymbol{n = 2m+1}$ is odd**. Then, we find

$$I_{2m+1,k} = \frac{(-1)^{m-k} + i}{2m+1-2k}.$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx = \int_0^{\frac{\pi}{2}} \sin^{2m+1}(x)\,dx =$$

$$\frac{1}{2^{2m+1}(-1)^m i} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \frac{(-1)^{m-k}+i}{2m+1-2k} =$$

$$\frac{(-1)^m}{2^{2m+1}} \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \frac{-i(-1)^{m-k}+1}{2m+1-2k}.$$

Separating real and imaginary parts, we find

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1}(x)\,dx = \frac{(-1)^m}{2^{2m+1}} \sum_{k=0}^{2m+1} \frac{(-1)^k}{2m+1-2k} \binom{2m+1}{k} (>0)$$

and the identity

$$\sum_{k=0}^{2m+1} \frac{1}{2m+1-2k} \binom{2m+1}{k} = 0.$$

[See **Problem I 2.2.28, (h)** and **Problem 1.8.25**.]

Remark 3: Again, this integral, in terms of the **Beta and Gamma functions, Section I 2.6, Example I 2.6.18** and the help of **Problem I 2.6.14, (a)**, is

$$\int_0^{\frac{\pi}{2}} \sin^{2m+1}(x)\,dx = \frac{1}{2}\mathrm{B}\left(m+1, \frac{1}{2}\right) = \sum_{l=0}^{m} \frac{(-1)^l}{2l+1} \binom{m}{l} =$$

$$\frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} = \frac{2^{2m}(m!)^2}{(2m+1)!} = \frac{2^{2m}}{2m+1} \cdot \frac{1}{\binom{2m}{m}} = \frac{2^{2m}}{2m+1} \cdot \frac{(m!)^2}{(2m)!} =$$

$$\frac{2 \cdot 4 \ldots (2m)}{1 \cdot 3 \ldots (2m+1)} = \frac{(-1)^m}{2^{2m+1}} \sum_{k=0}^{2m+1} \frac{(-1)^k}{2m+1-2k} \binom{2m+1}{k}.$$

(See **Problem 1.8.25**.)

Remark 4: We can do similar work with $\cos(x)$ in the place of $\sin(x)$ and with limits of integration $a = 0$ and $b = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi$.

(See also **Problems I 2.6.25, 1.8.12, 1.8.17**. **Examples I 2.6.19, 1.8.4, 1.8.6**.)

▲

Example 1.8.6 We can use **Euler's formulae** to prove

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!}.$$

We have

$$\sin^{2m}(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^{2m} = \frac{1}{2^{2m}(-1)^m}\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}e^{2(m-k)ix},$$

and

$$\cos^{2n}(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n} = \frac{1}{2^{2n}}\sum_{l=0}^{2n}\binom{2n}{l}e^{2(n-l)ix}.$$

(See also **Problems I 2.2.28** and **1.2.16**.) So,

$$\sin^{2m}(x)\cos^{2n}(x) = \frac{(-1)^m}{2^{2m+2n}}\sum_{k=0}^{2m}\sum_{l=0}^{2n}(-1)^k\binom{2m}{k}\binom{2n}{l}e^{2(m-k+n-l)ix}.$$

Therefore, the integral is

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{(-1)^m}{2^{2m+2n}}\sum_{k=0}^{2m}\sum_{l=0}^{2n}(-1)^k\binom{2m}{k}\binom{2n}{l}I_{mkln}\,,$$

where

$$I_{mkln} = \begin{cases} \dfrac{(-1)^{m-k+n-l}-1}{2(m-k+n-l)\,i} = \dfrac{[1-(-1)^{m-k+n-l}]\,i}{2(m-k+n-l)}, & \text{if} \quad m-k+n-l \neq 0 \\ & \iff \; l \neq m+n-k, \\ \dfrac{\pi}{2}, & \text{if} \quad m-k+n-l = 0 \\ & \iff \; l = m+n-k. \end{cases}$$

Since the integral is real, the imaginary part of this equation is zero. So, the summation over the indices such that $m-k+n-l \neq 0$ must be zero. That is, we obtain the identity

$$\sum_{\substack{k=0 \\ m+n\neq k+l}}^{2m}\sum_{l=0}^{2n}(-1)^k\binom{2m}{k}\binom{2n}{l}\frac{1-(-1)^{m-k+n-l}}{m-k+n-l} =$$

$$\sum_{\substack{k=0 \\ m+n\neq k+l}}^{2m}\sum_{l=0}^{2n}\binom{2m}{k}\binom{2n}{l}\frac{(-1)^k-(-1)^{m+n-l}}{m-k+n-l} = 0.$$

We observe that only when $m-k+n-l=0$ is there a contribution to the integral. Thus,

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{(-1)^m\pi}{2^{2m+2n+1}} \sum_{\substack{k=0\\k=m+n-l}}^{2m}\sum_{l=0}^{2n} (-1)^k \binom{2m}{k}\binom{2n}{l} =$$

$$\frac{(-1)^m\pi}{2^{2m+2n+1}} \sum_{l=0}^{2n}(-1)^{m+n-l}\binom{2m}{m+n-l}\binom{2n}{l} =$$

$$\frac{\pi}{2^{2m+2n+1}} \sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l}.$$

We could accept this as a final answer. However, someone experienced with **combinatorial identities** can prove that:

If $m \geq 0$ and $n \geq 0$ integers,

$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

[For the proof, see **Appendix 1.8.3, identity (1.48)**, after the end of **this subsection**.]

Finally, we obtain the answer

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!}.$$

Remark: We have seen that in terms of the **Beta and Gamma functions, Section I 2.6, Example I 2.6.19**, this integral is

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{1}{2}\mathrm{B}\left(m+\frac{1}{2}, n+\frac{1}{2}\right) =$$

$$\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(m+n+1)} = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!}.$$

(See also **Problems I 2.6.25, 1.8.12, 1.8.17** and **Examples I 2.6.19, 1.8.4, 1.8.5**.)

▲

Example 1.8.7 For any integer n, compute the integral

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx.$$

We notice

$$\frac{\sin(nx)}{\sin(x)} = \frac{\dfrac{e^{inx} - e^{-inx}}{2i}}{\dfrac{e^{ix} - e^{-ix}}{2i}} = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} =$$
$$e^{ix(n-1)} + e^{ix(n-2)}e^{-ix} + e^{ix(n-3)}e^{-2ix} + \ldots + e^{-ix(n-1)} =$$
$$e^{ix(n-1)} + e^{ix(n-3)} + e^{ix(n-5)} + \ldots + e^{ix(1-n)}.$$

If n is even, then every exponent of e^{ix}, in the above expression, is odd, and so the integral is

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\, dx = \frac{2i}{n-1} + \frac{2i}{n-3} + \frac{2i}{n-5} + \ldots + \frac{2i}{3-n} + \frac{2i}{1-n} = 0.$$

If n is odd, then exactly one exponent, the middle one, becomes zero. Since the sum of the imaginary numbers involved is zero, in this case we find

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\, dx = \int_0^{\pi} 1\, dx = \pi, \quad \text{for } n \text{ odd.}$$

Finally, we have rather easily obtained the useful **result**:

$$\int_0^{\pi} \frac{\sin(nx)}{\sin(x)}\, dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \\ \pi, & \text{if } n \text{ is odd.} \end{cases}$$

Since $\dfrac{\sin(nx)}{\sin(x)}$ is an even function, we also get

$$\int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin(x)}\, dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \\ 2\pi, & \text{if } n \text{ is odd.} \end{cases}$$

We observe that this result can also be obtained by integrating the formulae

$$\sum_{k=1}^{m} \cos[(2k-1)x] = \frac{\sin(2mx)}{2\sin(x)}, \quad \forall\ m \in \mathbb{N},$$
$$\frac{1}{2} + \sum_{k=1}^{m} \cos(2kx) = \frac{\sin\left[(2m+1)x\right]}{2\sin(x)}, \quad \forall\ m \in \mathbb{N}.$$

[See **Problem 1.2.27, (d)** and **(e)**.]

We also obtain the following two integrals. Integrating the second formula, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{\sin[(2m+1)x]}{\sin(x)}\,dx = \frac{\pi}{2}, \quad \forall\ m \in \mathbb{N}_0,$$

whereas integrating the first,

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2mx)}{\sin(x)}\,dx = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{(-1)^{m-1}}{2m-1}\right], \quad \forall\ m \in \mathbb{N}.$$

(Check the details. Try to establish this result by other techniques to see the difference!)

(See also **Problem 1.8.22**.)

▲

Problems

1.8.1 Consider any two real numbers $c > 1$ and $d > 1$. Use the integrals found in **Example 1.8.1** and integrate them with respect to the parameter a from c to d to prove

$$\int_0^{2\pi} \ln\left[\frac{\pm d + \sin(x)}{\pm c + \sin(x)}\right] dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\left[\frac{\pm d + \sin(x)}{\pm c + \sin(x)}\right] dx =$$
$$\int_0^{2\pi} \ln\left[\frac{\pm d + \cos(x)}{\pm c + \cos(x)}\right] dx = 2\int_0^{\pi} \ln\left[\frac{\pm d + \cos(x)}{\pm c + \cos(x)}\right] dx =$$
$$2\pi \cdot \ln\left[\frac{d + \sqrt{d^2-1}}{c + \sqrt{c^2-1}}\right],$$

where the correspondence of the signs is: $+$ with $+$ and $-$ with $-$.

[Hint: Prove this with the pluses first and then, using the periodicity of sine and cosine, conclude the result with the minuses too.]

1.8.2 Use the half angle substitution to prove that:

(a) If $-1 \le a \le 1$, then

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1 + a\cos(x)} = \frac{2}{\sqrt{1-a^2}} \arctan\left(\sqrt{\frac{1-a}{1+a}}\right) = \frac{\arccos(a)}{\sqrt{1-a^2}}.$$

(At $a = 1$ the answer is 1. For $a = -1$, we have the equality $\infty = \infty$.)

(b) If $a \geq 0$, then

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)dx}{\sin(x) + a\cos(x)} = \frac{a\pi}{2(a^2+1)} - \frac{\ln(a)}{a^2+1}.$$

(For $a = 0$, we have the equality $\infty = \infty$.)

1.8.3 For $R > r > 0$ and any θ constant, prove that

$$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi = 2\pi.$$

[See also **Problem 1.5.46 (a)**.]

1.8.4 (a) For $a \geq 1$, prove that

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\, dx = \pi \ln\left(1 + \frac{1}{a}\right).$$

[Hint: The integrand is even. Integrate the complex function

$$f(z) = \frac{z}{a - e^{-iz}}$$

along the rectangle with vertices $\pm\pi$ and $\pm\pi + iR$, etc.]

(b) Then, prove that for $0 < a \leq 1$

$$\int_0^{\pi} \frac{ax\sin(x)}{1 - 2a\cos(x) + a^2}\, dx = \pi \ln(1 + a).$$

[See also **Problem 1.5.46 (a)** and **(b)**.]

1.8.5 (a) If $|a| \neq 1$ and $n \in \mathbb{N}$, prove

$$\int_0^{\pi} \frac{\cos(nx)}{1 - 2a\cos(x) + a^2}\, dx = \begin{cases} \dfrac{\pi a^n}{1 - a^2}, & \text{if} \quad |a| < 1, \\[2ex] \dfrac{\pi}{a^n\left(a^2 - 1\right)}, & \text{if} \quad |a| > 1. \end{cases}$$

[Hint: Integrate along the unit circle the complex function

$$f(z) = \frac{z^n}{(z-a)\left(z - \frac{1}{a}\right)}.]$$

(b) Examine what happens when $|a| = 1$.

1.8.6 Compute the following integrals by whichever method is more efficient. (Either by using **Lemma 1.8.1** or by using any correct methods and integral formulae already used and proved previously anywhere in the text.)

$$I_1 = \int_0^{2\pi} \frac{d\theta}{1 + a\sin(\theta)} \quad -1 < a < 1, \quad \text{constant.}$$

$$I_2 = \int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}, \qquad a > 1, \quad \text{constant.}$$

(See also **Example 1.8.1, Remark 3**.)

$$I_3 = \int_0^{2\pi} \frac{d\theta}{1 + a\cos(\theta)}, \quad -1 < a < 1, \quad \text{constant.}$$

$$I_4 = \int_0^{2\pi} \frac{\cos(2\theta)}{8 + 3\cos(\theta)}\, d\theta.$$

$$I_5 = \int_0^{\pi} \frac{\cos(\theta)}{5 + 2\cos(\theta)}\, d\theta.$$

$$I_6 = \int_0^{2\pi} \frac{d\theta}{\left[9\cos^2(\theta) + 4\sin^2(\theta)\right]^2}.$$ (Compare with **Example I 2.2.3**.)

$$I_7 = \int_0^{2\pi} \frac{dx}{[2 + \cos(x)] \cdot [3 + \cos(x)]} \qquad \left[= 2\pi\left(\frac{1}{\sqrt{3}} - \frac{1}{2\sqrt{2}}\right)\right].$$

$$I_8 = \int_0^{2\pi} \frac{dx}{96\cos^2(x) + 25}.$$

$$I_9 = \int_0^{2\pi} \frac{dx}{100 + 69\sin^2(x)}.$$

$$I_{10} = \int_0^{2\pi} \frac{dx}{25\cos^2(x) + 100\sin^2(x)}.$$

$$I_{11} = \int_{-\pi}^{\pi} \frac{\cos(x)}{5 + 4\cos(x)}\, dx = -\frac{\pi}{3}.$$

$$I_{12} = \int_0^{2\pi} \frac{\cos^2(3x)}{5 - 4\cos(2x)}\, dx = \frac{3\pi}{8}.$$

$$I_{13} = \int_0^{\pi} \frac{1 + \cos(x)}{1 + \cos^2(x)}\, dx = \frac{\sqrt{\pi}}{2}.$$

1.8.7 Consider the positive improper integral

$$I_{\alpha\beta} := \int_0^{\infty} \frac{x^\beta}{1 + x^\alpha \sin^2(x)}\, dx \quad \text{with} \quad \alpha \geq 0 \quad \text{and} \quad \beta \geq 0 \quad \text{constants.}$$

(a) Prove that the integral converges (it is positive finite) iff $\alpha > 2(1+\beta)$ [and so it diverges (it is equal ∞) iff $\alpha \leq 2(1+\beta)$].
(See also **Problem I 1.3.13** and compare!)

[Hint: Notice that the integrand is positive and so

$$I_{\alpha\beta} = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{x^{\beta}}{1+x^{\alpha}\sin^2(x)}\, dx.$$

So,

$$\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{(n\pi)^{\beta}}{1+[(n+1)\pi]^{\alpha}\sin^2(x)}\, dx < I_{\alpha\beta} <$$
$$\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{[(n+1)\pi]^{\beta}}{1+(n\pi)^{\alpha}\sin^2(x)}\, dx.$$

Now use the result of **Example 1.8.3, Remark 2**, to obtain the result here.]

(b) Examine the same question if in the integral we replace $\sin^2(x)$ by $\cos^2(x)$.
(c) Examine what happens if we allow α and β to be any real constants.

1.8.8 Let $f(z)$ be holomorphic in $D(0,r)$, with $r > 1$. Evaluate

$$\oint_{C^+(0,1)} \left[2 \pm \left(z + \frac{1}{z}\right)\right] \frac{f(z)}{z}\, dz$$

in two different ways and find the formulae

$$\text{(a)} \qquad \int_0^{2\pi} f(e^{i\theta}) \cos^2\left(\frac{\theta}{2}\right) d\theta = \frac{\pi}{2}\left[2f(0) + f'(0)\right],$$
$$\text{(b)} \qquad \int_0^{2\pi} f(e^{i\theta}) \sin^2\left(\frac{\theta}{2}\right) d\theta = \frac{\pi}{2}\left[2f(0) - f'(0)\right].$$

1.8.9 Prove

$$\text{(a)} \qquad \int_0^{\frac{\pi}{2}} \sin(x)\sin(2x)\sin(3x)\, dx = \frac{1}{6},$$
$$\text{(b)} \qquad \int_0^{\pi} [x\sin(x)]^2\, dx = \frac{\pi^3}{6} - \frac{\pi}{4}.$$

1.8.10 Use **Lemma 1.8.1** to show that if $a > b > 0$, then:

(a) $$\int_0^{2\pi} \frac{dt}{[a+b\cos(t)]^2} = 2\int_0^{\pi} \frac{dt}{[a+b\cos(t)]^2} = \frac{2a\pi}{\sqrt{(a^2-b^2)^3}}.$$

(b) Prove this result by also taking the derivative of both sides of the result in **Example 1.8.1, Remark 3** (as we have seen in **Subsection I 2.2** when we use the **Leibniz Rule**), and then adjusting the given integral here to the result found after the differentiation.

(c) $$\int_0^{2\pi} \frac{\sin^4(t)dt}{a+b\cos(t)} = 2\int_0^{\pi} \frac{\sin^4(t)\,dt}{a+b\cos(t)} = \frac{2\pi}{b^4}\left[\sqrt{(a^2-b^2)^3} - a^3 + \frac{3}{2}\,ab^2\right].$$

1.8.11 Prove the four results:

(a) If $b > 1$ and $n = 0,\ 1,\ 2,\ \ldots,$

$$\int_0^{2\pi} \frac{\cos(nx)}{b+\cos(x)}\,dx = \frac{(-1)^n 2\pi}{\sqrt{b^2-1}\left(b+\sqrt{b^2-1}\,\right)^n},$$

and so, for any $a > 0$,

$$\int_0^{2\pi} \frac{\cos(nx)}{\cosh(a)+\cos(x)}\,dx = \frac{(-1)^n 2\pi}{\sinh(a)} e^{-na}.$$

(b) If $a > b > 0$,

$$\int_0^{2\pi} \frac{\sin^2(x)}{a+b\cos(x)}\,dx = \frac{2\pi\left(a-\sqrt{a^2-b^2}\,\right)}{b^2}.$$

(c) If $a > 0$,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2(x)} = \frac{\pi}{2\sqrt{a(a+1)}}.$$

(d) If $-1 < a < 1$,

$$\int_0^{\pi} \frac{\cos(2x)}{1-2a\cos(x)+a^2}\,dx = \frac{\pi a^2}{1-a^2}.$$

[See also **Problem 1.5.46, (b)**.]

1.8.12 (a) Use **Lemma 1.8.1** to prove the result of **Example I 2.6.19**. For $m \geq 0$ and $n \geq 0$ integers, we have

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi}{2^{2m+2n+1}} \cdot \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

(b) State the two byproducts that we get when $m = 0$ and when $n = 0$.

(c) Examine what happens if we use **Lemma 1.8.1** and one or both exponents of sine and/or cosine is/are odd.

1.8.13 We have proven that for $n \geq 0$ and $m \geq 0$ integers,

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\,dx = \frac{\pi\,(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!} =$$
$$\frac{1}{2}\mathrm{B}\left(m+\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{2\Gamma(m+n+1)}.$$

Find the integral formulae when the $2m$ and $2n$ are combined and/or replaced with $2m+1$ and $2n+1$. (There are three more cases. You may give the answers in summation forms if you cannot find them in terms of nice fractions with factorials, etc.) For practice and finding out the correct answer first, write the results in terms of the Beta and Gamma functions and then simplify.

1.8.14 Prove (and compare with the **next three Problems**) that for $n \in \mathbb{N}_0$,

$$\text{(a)} \qquad \int_0^{\pi} \cos^n(x)\cos(nx)\,dx = \frac{\pi}{2^n},$$
$$\text{(b)} \qquad \int_0^{\pi} \sin^n(x)\sin(nx)\,dx = \frac{\pi}{2^n}\,\sin\left(\frac{n\pi}{2}\right).$$

1.8.15 (a) If $0 \leq \alpha < 1$ constant, prove

$$\int_0^{\pi} \cos\left(\frac{\alpha x}{2}\right)\cos^{-\alpha}\left(\frac{x}{2}\right)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\left(\frac{\alpha x}{2}\right)\cos^{-\alpha}\left(\frac{x}{2}\right)\,dx =$$
$$2\int_0^{\frac{\pi}{2}} \cos(\alpha u)\cos^{-\alpha}(u)\,du = 2^{\alpha}\pi.$$

(b) Notice the discontinuity of the integral in **(a)** at $\alpha = 1$, (since $\pi < 2^1\pi = 2\pi$), and explain it.

1.8.16 (a) If $b > a > -1$, prove

$$\int_0^{\frac{\pi}{2}} \cos^a(x)\cos(bx)\,dx = \frac{\pi\Gamma(a+1)}{2^{a+1}\Gamma\left(1+\frac{a+b}{2}\right)\Gamma\left(1+\frac{a-b}{2}\right)}.$$

(b) Prove that this result is also true for any $a > -1$, regardless of b. (If the denominator of the fraction becomes $\pm\infty$, then the integral is 0.)

[Hint: Integrate the complex function $f(z) = \left(z+\frac{1}{z}\right)^a \cdot z^{b-1}$ along the contour derived from $[i,-i] + \left\{e^{ix} \mid \frac{-\pi}{2} \le x \le \frac{\pi}{2}\right\}$ after you indent it appropriately at the points i, 0, and $-i$. Also, remember **properties (B, 7)** and **(B, 8), (a)**, of the Beta function, **Subsection I 2.6.2**.]

1.8.17 For $n \ge 0$ and $m \ge 0$ integers and $s = \dfrac{\pi}{2}$, π, $\dfrac{3\pi}{2}$, 2π, using already known results and symmetries of sine and cosine around the unit circle when applicable, evaluate or give recursive formulae for the following four integrals:

$$\text{(a)} \quad \int_0^s \cos^n(x)\cos(mx)\,dx, \qquad \text{(b)} \quad \int_0^s \sin^n(x)\sin(mx)\,dx,$$

$$\text{(c)} \quad \int_0^s \cos^n(x)\sin(mx)\,dx, \qquad \text{(d)} \quad \int_0^s \sin^n(x)\cos(mx)\,dx.$$

[Hint: Begin with $m = 0$ first. Use appropriate integration by parts and derive recursive formulae. Also use already known results and symmetries of sine and cosine around the unit circle, when applicable. (See also **Example 1.8.5**. Also compare with the three **previous Problems**.)]

1.8.18 For any integer $m \ge 1$, prove directly

$$\text{(a)} \quad \sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{(-1)^k}{m-k}\binom{2m}{k} = 0 \quad \text{and} \quad \text{(b)} \quad \sum_{\substack{k=0 \\ k\neq m}}^{2m} \frac{1}{m-k}\binom{2m}{k} = 0.$$

See also **Problem I 2.2.28, (e)**.]

1.8.19 Establish the following Fejér[51] integral.

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2(nx)}{\sin^2(x)}\,dx = \frac{|n|\pi}{2}, \quad \forall\ n \in \mathbb{Z}.$$

[51] Lipót Fejér, Hungarian mathematician, 1880-1959.

Since $\dfrac{\sin^2(|n|x)}{\sin^2(x)}$ is even, we also get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(nx)}{\sin^2(x)}\, dx = |n|\pi, \quad \forall\ n \in \mathbb{Z}.$$

[Hint: Use the relation

$$\frac{\sin^2(nx)}{\sin^2(x)} = \left[e^{ix(n-1)} + e^{ix(n-3)} + e^{ix(n-5)} + \ldots + e^{ix(1-n)}\right]^2.$$

(See **Example 1.8.7**.)

Or easier, divide the identity

$$\sum_{k=1}^{n} \sin[(2k-1)x] = \frac{\sin^2(nx)}{\sin(x)},$$

(see **Problem 1.2.27**), by $\sin(x)$, then integrate and use the result

$$\int_0^{\frac{\pi}{2}} \frac{\sin\left[(2m+1)x\right]}{\sin(x)}\, dx = \frac{\pi}{2}, \quad \forall\ m \in \mathbb{N}_0,$$

(see **Example 1.8.7**), n times.]

1.8.20 If $0 < a < 1$ and $0 < b < 1$, prove

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{dz}{a^z \sin(\pi z)} = \frac{1}{\pi(1+a)}.$$

1.8.21 If $a > 0$ and $-\dfrac{\pi}{2} < a\lambda < \dfrac{\pi}{2}$, prove

$$\int_0^{\infty} e^{-r^a \cos(a\lambda)} \cos\left[r^a \sin(a\lambda)\right] dr = \frac{\cos(\lambda)}{a} \cdot \Gamma\left(\frac{1}{a}\right),$$

and

$$\int_0^{\infty} e^{-r^a \cos(a\lambda)} \sin\left[r^a \sin(a\lambda)\right] dr = \frac{\sin(\lambda)}{a} \cdot \Gamma\left(\frac{1}{a}\right).$$

1.8.22 (a) Let $0 < a \le \pi$ and $n \in \mathbb{N}_0$. Prove

$$\int_0^{a} \frac{\cos(nt)}{1-\cos(t)}\, dt = \infty \quad \text{and} \quad \int_{\pi-a}^{\pi} \frac{\cos(nt)}{1+\cos(t)}\, dt = \begin{cases} +\infty, & \text{if} \quad n \text{ is even} \\ -\infty, & \text{if} \quad n \text{ is odd.} \end{cases}$$

(b) Let $0 < s < \pi$ and $n \in \mathbb{N}_0$. Prove

$$\text{P.V.} \int_0^\pi \frac{\cos(nt)}{\cos(t) - \cos(s)}\, dt =$$

$$(\text{with} \quad 0 < \epsilon < \min\{s, \pi - s\})$$

$$\lim_{\epsilon \to 0} \left(\int_0^{s-\epsilon} \frac{\cos(nt)}{\cos(t) - \cos(s)}\, dt + \int_{s+\epsilon}^{\pi} \frac{\cos(nt)}{\cos(t) - \cos(s)}\, dt \right) =$$
$$\pi \cdot \frac{\sin(ns)}{\sin s}.$$

[Hint: $\cos(nt)$ and $\sin(nt)$ have been expressed by powers of $\cos(t)$ and $\sin(t)$ in **Problem 1.1.46, II, (a)** and **(c)**.

Observe that for any $m \in \mathbb{N}$ we have

$$\frac{u^m}{u-k} = u^{m-1} + ku^{m-2} + ku^{m-3} + \ldots + k^{m-2}u + k^{m-1} + \frac{k^m}{u-k}$$

(perform the division or multiply by both sides by $u - k$) and we let $u = \cos(t)$ and $k = \cos(s)(= \text{constant})$.

Next notice that

$$\frac{\cos^m(s)}{\cos(t) - \cos(s)} = \frac{\cos^m(s)}{\sin(s)} \cdot \frac{d}{dt} \left\{ \ln \left[\frac{\sin\left(\frac{s+t}{2}\right)}{\sin\left(\frac{s-t}{2}\right)} \right] \right\}.]$$

(c) When $s = \dfrac{\pi}{2}$ in **(b)**, we find

$$\text{P.V.} \int_0^\pi \frac{\cos(nt)}{\cos(t)}\, dt = \begin{cases} 0, & \text{if } n = 2k \text{ (even)} \\ \pi, & \text{if } n = 4k+1 \\ -\pi, & \text{if } n = 4k+3. \end{cases}$$

This result can also be found directly, without using **(b)**, working as in **Example 1.8.7**. Observe also that when $n = 4k+1$ and $n = 4k+3$ the corresponding integrals are proper (why?) and so the principal values π and $-\pi$ are the values of the integrals, respectively.

1.8.3 Appendix

In **Example 1.8.6**, we used the identity

$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}, \quad (1.48)$$

where $m \geq n \geq 1$ integers. We would like to give an **elementary proof** of it. Observe that this identity is symmetrical about m and n.[52]

[For simplifying expressions, we use $\binom{k}{r} = \dfrac{k!}{r!(k-r)!}$ for $0 \leq r \leq k$ integers. For any other integer r, the combination number $\binom{k}{r}$ is 0.]

We let

$$L := \sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l}$$
$$\left[= \sum_{l=0}^{2n}(-1)^{n-l}\frac{(2m)!(2n)!}{(m+n-l)!(m-n+l)!l!(2n-l)!}\right],$$

and so we want to prove

$$L = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}\left[= \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}\right].$$

So,

$$\frac{\binom{m+n}{n}^2}{\binom{2m}{m}\binom{2n}{n}}L = \frac{[(m+n)!]^2}{(2m)!(2n)!}L =$$
$$\sum_{l=0}^{2n}(-1)^{n-l}\frac{[(m+n)!]^2}{l!(2n-l)!(m+n-l)!(m-n+l)!} =$$
$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{m+n}{l}\binom{m+n}{2n-l}. \quad (1.49)$$

By the identity

$$(1-t^2)^{m+n} = (1+t)^{m+n}(1-t)^{m+n}$$

[52]These kinds of results have to do with the theory of **Hypergeometric Forms** and **Kummer's Summation Formulae**. These types of sums are very important in mathematics and applications, and so they have been standardized and tabulated. Also, some computer programs, such as ***Mathematica***, can evaluate them.

and the **Binomial Theorem**, we find

$$\sum_{k=0}^{m+n}\binom{m+n}{k}(-1)^k t^{2k} = \sum_{k_1=0}^{m+n}\binom{m+n}{k_1}t^{k_1}\sum_{k_2=0}^{m+n}(-1)^{k_2}\binom{m+n}{k_2}t^{k_2} =$$

$$\sum_{k_1=0}^{m+n}\sum_{k_2=0}^{m+n}\binom{m+n}{k_1}(-1)^{k_2}\binom{m+n}{k_2}t^{k_1+k_2} = \quad (\text{let } k_1+k_2=k)$$

$$\sum_{k=0}^{2(m+n)}\left[\sum_{k_1=0}^{k}(-1)^{k-k_1}\binom{m+n}{k_1}\binom{m+n}{k-k_1}\right]t^k.$$

By equating the coefficients of t^{2n}, we obtain

$$\binom{m+n}{n}(-1)^n = \sum_{k_1=0}^{2n}(-1)^{2n-k_1}\binom{m+n}{k_1}\binom{m+n}{2n-k_1}. \tag{1.50}$$

So, by simplifying the $(-1)^n$ and replacing k_1 with l in **(1.50)** and using **(1.49)** above, we find $\frac{\binom{m+n}{n}^2}{\binom{2m}{m}\binom{2n}{n}}L = \binom{m+n}{n}$. Therefore, $L = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}}$, which finishes the proof of **identity (1.48)**.

Next, for those who have sufficient knowledge on the hypergeometric functions and Kummer's formulae, but without wanting to go into a substantial exposition of this big chapter of the mathematical field of Special Functions, we would like to present a proof of the **identity (1.48)**, by using just what we need from the hypergeometric functions and Kummer's formulae.

With $m \geq n \geq 1$ and $l \geq 0$ integers, we let

$$c_l = (-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} =$$

$$(-1)^{n-l}\frac{(2m)!}{(m+n-l)!(m-n+l)!}\cdot\frac{(2n)!}{l!(2n-l)!}.$$

We compute and simplify the ratio

$$\frac{c_{l+1}}{c_l} = \ldots = -\frac{(l-m-n)(l-2n)}{(l+m-n+1)(l+1)}.$$

I.e., we have the **recursive formula**

$$c_{l+1} = -\frac{(l-m-n)(l-2n)}{(l+m-n+1)(l+1)}\cdot c_l$$

with starting term (for $l = 0$)

$$c_0 = (-1)^n \frac{(2m)!}{(m+n)!(m-n)!} \cdot \frac{(2n)!}{0!(2n)!} = (-1)^n \frac{(2m)!}{(m+n)!(m-n)!}.$$

Using this recursive formula, the expression of c_0, the **rising** or **ascending factorial of a real number** x **of order an integer** k [defined in **property (B, 8)** in **Subsection I 2.6.2**], and the **hypergeometric function**[53]

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{[a]_n [b]_n}{[c]_n} \cdot \frac{z^n}{n!}, \quad \text{for} \quad |z| < 1,$$

which also converges for $z = -1$ (e.g., by the **alternating series test**, see **Example I 1.7.12** and its **footnote**, and/or a book of calculus or mathematical analysis), we get:

$$\sum_{l=0}^{\infty} c_l = c_0 \sum_{l=0}^{\infty} \frac{[-m-n]_l [-2n]_l}{[m-n+1]_l} \cdot \frac{(-1)^l}{l!} =$$

$$(-1)^n \frac{(2m)!}{(m+n)!(m-n)!} \cdot {}_2F_1(-m-n, -2n; m-n+1; -1).$$

We see that $a = -m-n$, $b = -2n$ a negative integer, $c = m-n+1 \geq 1$ integer, $a - b + c = -m - n + 2n + m - n + 1 = 1$, and $z = -1$. Then, with these parameters, the following **Kummer's formula**

$${}_2F_1(a, b; c; -1) = 2\cos\left(\frac{b\pi}{2}\right) \cdot \frac{\Gamma\left(|b|\right)\Gamma(b-a+1)}{\Gamma\left(\frac{|b|}{2}\right)\Gamma\left(\frac{b}{2} - a + 1\right)}$$

applies. (Find and examine some related bibliography. E.g., Whittaker and Watson 1927-1996, Chapter XIV, Lebedev 1972, Chapter 9, etc.)

Since for $l > 2n$, $\binom{2n}{l} = 0$, from this formula, we finally find

[53]The **hypergeometric differential equation** is the differential equation : $z(1-z)u''(z) + [c - (a+b+1)z]u'(z) - ab\,u(z) = 0$, where z is a complex variable, and a, b, c are parameters with values complex numbers, in general. The **hypergeometric functions** are the power series solutions of this differential equations about the **regular singular points** 0, 1, and ∞. The **hypergeometric function** ${}_2F_1(a, b; c; z)$, above, is the power series solution about $z = 0$ and converges for all z such that $|z| < 1$, and also for $z = -1$.

$$\sum_{l=0}^{2n}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} = \sum_{l=0}^{\infty}(-1)^{n-l}\binom{2m}{m+n-l}\binom{2n}{l} =$$

$$(-1)^n\frac{(2m)!}{(m+n)!(m-n)!}\cdot {}_2F_1(-m-n,-2n;m-n+1;-1) =$$

$$(-1)^n\frac{(2m)!}{(m+n)!(m-n)!}\cdot 2\cos\left(\frac{-2n\pi}{2}\right)\cdot\frac{\Gamma(2n)\Gamma(m-n+1)}{\Gamma(n)\Gamma(m+1)} =$$

$$(-1)^n\frac{(2m)!}{(m+n)!(m-n)!}\cdot 2(-1)^n\cdot\frac{(2n-1)!(m-n)!}{(n-1)!m!} =$$

$$\frac{(2m)!}{(m+n)!}\cdot\frac{2n[(2n-1)!]}{n[(n-1)!m!]} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!},$$

which finishes the proof of **identity (1.48)**.

Problems

1.8.23 Can you prove the **identity (1.48)**, proven in **Appendix 1.8.3**, combinatorially?

1.8.24 Use any convenient test for power series convergence to prove that the radius of convergence of ${}_2F_1(a,b;c;z)$, defined above, is $R=1$.

1.8.25 Prove the series of equalities that appears in **Example 1.8.5, (b)** and **Remark 3**. Namely,

$$\frac{1}{2}\mathrm{B}\left(m+1,\frac{1}{2}\right) = \sum_{l=0}^{m}\frac{(-1)^l}{2l+1}\binom{m}{l} = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)} =$$

$$\frac{2^{2m}(m!)^2}{(2m+1)!} = \frac{2^{2m}}{2m+1}\cdot\frac{1}{\binom{2m}{m}} = \frac{2^{2m}}{2m+1}\cdot\frac{(m!)^2}{(2m)!} =$$

$$\frac{2\cdot 4\ldots(2m)}{1\cdot 3\ldots(2m+1)} = \frac{(-1)^m}{2^{2m+1}}\sum_{k=0}^{2m+1}\frac{(-1)^k}{2m+1-2k}\binom{2m+1}{k}.$$

Chapter 2

Lists of Non-elementary Integrals and Sums in Text

2.1 List of Non-elementary Integrals

Here, we cite a list of all major, non-elementary, general and important real or complex integrals that have been evaluated in the text or referred to in problems and footnotes. An addition to this list can be considered all the Laplace, Fourier and inverse transforms that have been evaluated in all relevant sections to which we direct your attention. More integrals, usually less general, can be found in various examples and problems presented in the text.

1. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x} \sin(\beta x)\, dx = \frac{\beta}{\alpha^2 + \beta^2}.$$

[**Problems I 1.2.13, 1.7.58**. **Example 1.7.25. Corollary 1.7.5 (C)**.]

2. If $\alpha > 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x} \cos(\beta x)\, dx = \frac{\alpha}{\alpha^2 + \beta^2}.$$

(**Problems I 1.2.15, 1.7.58**.)

3. If $\alpha > 0$ and $-\alpha < \beta < \alpha$ constants,

$$\int_0^\infty e^{-\alpha x} \sinh(\beta x)\, dx = \frac{\beta}{\alpha^2 - \beta^2}.$$

(**Problems I 1.2.17**.)

4. If $\alpha > 0$ and $-\alpha < \beta < \alpha$ constants,

$$\int_0^\infty e^{-\alpha x} \cosh(\beta x)\, dx = \frac{\alpha}{\alpha^2 - \beta^2}.$$

(**Problems I 1.2.18**.)

5. If $a \geq 0$, $b \geq -1$ and $c > 0$ real constants,

$$\int_0^\infty a^{-cx}\, dx = \begin{cases} +\infty, & \text{if} \quad 0 \leq a \leq 1, \\ \dfrac{1}{c \cdot \ln(a)}, & \text{if} \quad a > 1, \end{cases}$$

$$\int_0^\infty \frac{1}{a^{cx} + b}\, dx = \begin{cases} +\infty, & \text{if} \quad 0 \leq a \leq 1, \\ \dfrac{\ln(1+b)}{bc \cdot \ln(a)}, & \text{if} \quad a > 1. \end{cases}$$

(**Problem I 1.2.20**. **Example 1.7.26**.)

6. If $c > |b|$ real constants,

$$\int_{-\infty}^\infty \frac{dx}{x^2 + 2bx + c^2} = \frac{\pi}{\sqrt{c^2 - b^2}}.$$

(**Problem I 1.2.21**.)

7. If $a > 0$ and $b \geq 0$ real constants,

$$\int_0^1 a^{\left(x^b\right)} dx = \sum_{n=0}^\infty \frac{[\ln(a)]^n}{n!(bn+1)}.$$

(**Problem I 1.2.28**.)

8.
$$\int_0^\infty \frac{dx}{1 + x^3} = \frac{2\pi\sqrt{3}}{9}.$$

(**Problem I 1.3.2**.)

9.
$$\int_0^\infty \frac{dx}{1 + x^4} = \frac{\pi\sqrt{2}}{4} = \int_0^\infty \frac{x^2 dx}{1 + x^4} = \frac{1}{2}\int_0^\infty \frac{1 + x^2}{1 + x^4}\, dx.$$

[**Problems I 1.3.2, I 2.6.21.**. **Example I 2.2.6 (b)**.]

10.
$$\int_0^\infty \frac{dx}{\left(1+x^3\right)^2} = \frac{4\pi\sqrt{3}}{27}.$$

(**Problem I 1.3.2**.)

11.
$$\int_0^\infty \frac{dx}{\left(1+x^4\right)^2} = \frac{3\pi\sqrt{2}}{16}.$$

(**Problem I 1.3.2**.)

12. If $m,\ n \in \mathbb{N}$,
$$\int_0^\infty x^n (x+1)^{-m-n-1} dx = \frac{(m-1)!n!}{(m+n)!} = \frac{1}{m\binom{m+n}{m}} = \frac{1}{m\binom{m+n}{n}}.$$

If $0 \le m < n-1$,
$$\int_0^\infty \frac{x^m}{(1+x)^n}\, dx = \int_0^\infty \frac{x^{n-m-2}}{(1+x)^n}\, dx =$$
$$\int_1^\infty \frac{(u-1)^m}{u^n}\, du = \int_1^\infty \frac{(u-1)^{n-m-2}}{u^n}\, du = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{n-m+k-1}.$$

[**Problems I 1.3.20, I 1.3.25, (a)**.]

13.
$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = 2\int_{-\infty}^0 e^{-x^2}\, dx = \sqrt{\pi} = \Gamma\left(\frac{1}{2}\right).$$

[**Section I 2.1**. **Subsection I 2.6.1 ($\boldsymbol{\Gamma}$, 5)**.]

14. If $b \in \mathbb{R}$ and $s > 0$ constants,
$$\int_0^b e^{-su^2} du \overset{v=\sqrt{s}\,u}{=} \int_0^{\sqrt{s}b} e^{-v^2} \frac{dv}{\sqrt{s}} =$$
$$\frac{1}{2}\sqrt{\frac{\pi}{s}}\, \mathrm{erf}\left(\sqrt{s}b\right) = \frac{1}{2}\sqrt{\frac{\pi}{s}} \left[1 - \mathrm{erfc}\left(\sqrt{s}b\right)\right].$$

(**Example I 2.1.1.**)

15. If $a > 0$ constant,
$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}.$$

(**Problem I 2.1.4**.)

16. If $\alpha > 0$ constant,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = 2\int_0^{\infty} e^{-\alpha x^2}\,dx = 2\int_{-\infty}^{0} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}.$$

(**Problem I 2.1.11**.)

17. If $a > 0$ and $n \in \mathbb{N}$,

$$\int_0^{\infty} x^{2n} e^{-ax^2}\,dx = \frac{1 \cdot 3 \cdot 5\,\ldots\,(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}.$$

(**Problem I 2.1.14**.)

18. If $a \geq 0$ and $b \geq 0$ constants,

$$\int_0^{\infty} \left(e^{\frac{-a}{x^2}} - e^{\frac{-b}{x^2}}\right) dx = \sqrt{\pi b} - \sqrt{\pi a}.$$

[**Problems I 2.1.15 (c), I 2.5.6 (a)**.]

19. If m and n in $\mathbb{N}_0$,

$$\int_0^1 x^n \ln^m(u)\,du = \frac{(-1)^m m!}{(n+1)^{m+1}},$$

$$\int_0^1 x^n |\ln(u)|^m\,du = \frac{m!}{(n+1)^{m+1}}.$$

[**Problem I 2.1.17 (b)**.]

20. If $\alpha > 1$ real constant,

$$\int_1^{\infty} x^{-\alpha}\ln(x)\,dx = \int_1^{\infty} \frac{\ln(x)}{x^{\alpha}}\,dx = \frac{1}{(\alpha-1)^2}.$$

(**Problem I 2.1.18**.)

21.

$$\int_0^{\pi} \ln[\sin(x)]\,dx = 2\int_0^{\frac{\pi}{2}} \ln[\sin(x)]\,dx =$$
$$\int_0^{\pi} \ln[|\cos(x)|]\,dx = 2\int_0^{\frac{\pi}{2}} \ln[\cos(x)]\,dx =$$
$$\int_{-1}^{1} \frac{\ln(|u|)}{\sqrt{1-u^2}}\,du = 2\int_0^1 \frac{\ln(u)}{\sqrt{1-u^2}}\,du = -\pi\ln(2).$$

If $a > 0$ constant,

$$\int_0^{\frac{\pi}{2}} \ln[a\sin(x)]dx = \int_0^{\frac{\pi}{2}} \ln[a\cos(x)]dx = \frac{\pi}{2}\ln\left(\frac{a}{2}\right),$$

$$\int_0^{\pi} \ln[a\sin(x)]\,dx = \int_0^{\pi} \ln[a|\cos(x)|]\,dx = \pi\ln\left(\frac{a}{2}\right).$$

[**Problems I 2.1.19 (b)-(f), I 2.2.46, 1.2.34 (d), 1.7.132 (a)**. **Subsection 1.5.4**. **Example 1.7.50** and **Remarks**.]

22.

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\arctan(u)}{u\,(u^2+1)}\,du = \int_0^{\infty} \frac{\arctan(u)}{u\,(u^2+1)}\,du \overset{x=\arctan(u)}{=}$$
$$\int_0^{\frac{\pi}{2}} x\cot(x)\,dx = \frac{\pi}{2}\ln(2),$$

$$\int_0^{\frac{\pi}{2}} \frac{x^2}{\sin^2(x)}\,dx = \pi\ln(2).$$

[**Problem I 2.1.19 (g)**.]

23.

$$\int_0^{\pi} x\ln[\sin(x)]\,dx = -\frac{1}{2}\pi^2\ln(2).$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\ln[\cos(x)]\,dx = 0.$$

[**Problem I 2.1.21 (a)-(b)**.]

24.

$$\int_0^{\frac{\pi}{2}} \ln[\tan(x)]\,dx = 0 = \int_0^{\frac{\pi}{2}} \ln[\cot(x)]\,dx.$$

$$\int_0^{\pi} \ln[|\tan(x)|]\,dx = 0 = \int_0^{\pi} \ln[|\cot(x)|]\,dx.$$

(**Problem I 2.1.22**.)

25.

$$\int_0^{\infty} \frac{\ln(x)}{x^2+1}\,dx = 0.$$

(**Problems I 2.1.23**, **1.7.135**. **Example 1.7.47**.)

26.
$$\int_0^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \pi\ln(2).$$

If $a > 0$ constant,

$$\int_0^\infty \frac{\ln\left[(ax)^2+1\right]}{(ax)^2+1}\,dx = \frac{1}{a}\pi\ln(2).$$

(**Problems I 2.1.24, I 2.3.17, 1.7.138**.)

27.
$$\int_1^\infty \frac{\ln(1+y)}{y^2}\,dy = 2\ln(2).$$
$$\int_1^\infty \frac{\ln(y-1)}{y^2}\,dy = \int_0^\infty \frac{\ln(u)}{(u+1)^2}\,dy = 0.$$

[**Problem I 2.1.25 (a) (c)**.]

28.
$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx = \frac{\pi\ln(2)}{8}.$$

[**Problem I 2.1.26 (a)**.]

29.
$$\int_{-1}^\infty \frac{\ln(x+1)}{x^2+1}\,dx = \frac{3\pi\ln(2)}{8}.$$

(**Problem I 2.1.27**.)

30.
$$\int_0^\infty \frac{\ln(x)}{(x^2+1)^2}\,dx = -\frac{\pi}{4}.$$

(**Problems I 2.1.28, 1.7.137**.)

31. If $r \in \mathbb{R}$ constant,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^r(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\tan^r(u)}{1+\tan^r(u)}\,du =$$
$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^r(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cot^r(u)}{1+\cot^r(u)}\,du = \frac{\pi}{4}.$$

(**Problem I 2.1.29**.)

32. If $\beta > 0$ constant,

$$\int_0^\infty \frac{1}{\beta^2 + x^2}\, dx = \frac{\pi}{2\beta}.$$

(**Problem I 2.1.30**. **Example 1.7.5**.)

33. If $\alpha > 0$ and $\beta > 0$ constants,

$$\int_0^\infty \frac{\ln(\alpha x)}{\beta^2 + x^2}\, dx = \frac{\pi}{2\beta}\ln(\alpha\beta).$$

(**Problems I 2.1.30, 1.7.135**. **Example 1.7.47 Remark 2**.)

34. If $x \geq 0$,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^{\frac{\pi}{4}} e^{-x^2\csc^2(\theta)} d\theta\right]^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\left[\int_0^{\frac{\pi}{4}} e^{-x^2\sec^2(\theta)} d\theta\right]^{\frac{1}{2}}.$$

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-x^2\csc^2(\theta)} d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-x^2\sec^2(\theta)} d\theta.$$

[**Problem I 2.1.31 (a)-(b)**.]

35. If $a > 0$ and b constants,

$$\int_0^b \frac{dx}{(a^2 + x^2)^3} = \frac{b}{8a^4}\left[\frac{5a^2 + 3b^2}{(a^2 + b^2)^2} + \frac{3}{ab}\arctan\left(\frac{b}{a}\right)\right].$$

(**Example I 2.2.2.**.)

36. If $a > 0$ and $b > 0$ constants,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2(x) + b^2\sin^2(x)} = \frac{\pi}{2ab}.$$

$$\int_0^{2\pi} \frac{dx}{a^2\cos^2(x) + b^2\sin^2(x)} = \frac{2\pi}{ab}.$$

(**Examples I 2.2.3, I 2.6.21, 1.5.16**, **1.8.3**.)

37. If $a > 0$ and $b > 0$ constants,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\left[a^2\cos^2(x) + b^2\sin^2(x)\right]^2} = \frac{\pi\left(a^2 + b^2\right)}{4(ab)^3}.$$

(**Example I 2.2.3**.)

38. If $0 < p < 1$ and $0 < q < 1$ such that $p + q = 1$,

$$\int_0^\infty \frac{t^{p-1}}{1+t}\,dt = \int_0^\infty \frac{1}{t^{1-p}(1+t)}\,dt = \frac{\pi}{\sin(p\pi)} =$$
$$\frac{\pi}{\sin(q\pi)} = \int_0^\infty \frac{t^{q-1}}{1+t}\,dt = \int_0^\infty \frac{1}{t^{1-q}(1+t)}\,dt.$$

(**Examples I 2.2.5** and **Remark**, **1.7.7**, **1.7.8**.)

39. If a, b real constants, $b \neq 0$ such that $0 < \dfrac{a+1}{b} < 1$,

$$\int_0^\infty \frac{x^a}{1+x^b}\,dx = \frac{1}{|b|} \cdot \frac{\pi}{\sin\left(\frac{a+1}{b}\pi\right)} = \int_0^\infty \frac{x^{b-a-2}}{1+x^b}\,dx.$$

If $m \geq 0$ and $n \geq 2$ **integers** such that $0 \leq m < n - 1$,

$$\int_0^\infty \frac{x^m}{1+x^n}\,dx = \frac{1}{n} \cdot \frac{\pi}{\sin\left(\frac{m+1}{n}\pi\right)} =$$
$$\int_0^\infty \frac{x^{n-m-2}}{1+x^n}\,dx = \frac{1}{2}\int_0^\infty \frac{x^m + x^{n-m-2}}{1+x^n}\,dx.$$

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4} = \int_0^\infty \frac{x^2 dx}{1+x^4} = \frac{1}{2}\int_0^\infty \frac{1+x^2}{1+x^4}\,dx.$$

$$\int_0^\infty \frac{1+x^2}{1+x^4}\,dx = \frac{\pi\sqrt{2}}{2}.$$

$$\int_0^1 \frac{dx}{1+x^4} = \int_1^\infty \frac{x^2 dx}{1+x^4} = \frac{\sqrt{2}}{8}\left[\pi + \ln\left(3 + 2\sqrt{2}\right)\right].$$

$$\int_1^\infty \frac{dx}{1+x^4} = \int_0^1 \frac{x^2 dx}{1+x^4} = \frac{\sqrt{2}}{8}\left[\pi - \ln\left(3 + 2\sqrt{2}\right)\right].$$

[**Examples I 2.2.6 (a)-(b), I 2.4.2**, **1.7.6**, **1.7.7**, **1.7.8**.]

40. If β constant,

$$\int_0^\infty \frac{\sin(\beta x)}{x}\,dx = \begin{cases} \dfrac{\pi}{2}, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\dfrac{\pi}{2}, & \text{if } \beta < 0. \end{cases}$$

[**Examples I 1.3.18, I 2.2.8, I 2.3.11**, **1.7.35**. **Problems I 2.2.15, I 2.4.1, I 2.7.24 (d), 1.7.93**.]

41. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x} \frac{\sin(\beta x)}{x}\, dx = \arctan\left(\frac{\beta}{\alpha}\right).$$

(**Examples I 2.2.8, I 2.2.9, I 2.3.11**, and second half of **Example 1.7.41**. **Problems I 2.4.11, I 2.5.23**.)

42. If $-\infty < \beta < \infty$ constant,

$$\frac{2\beta}{\pi} \int_0^\infty \frac{\sin(\beta x)}{x}\, dx = |\beta|.$$

(**Example I 2.2.10**.)

43. If $\alpha \in \mathbb{R}$ constant,

$$\int_0^\infty \frac{\sin^2(\alpha x)}{x^2}\, dx = \int_{-\infty}^0 \frac{\sin^2(\alpha x)}{x^2}\, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin^2(\alpha x)}{x^2}\, dx =$$
$$\int_0^\infty \frac{1-\cos^2(\alpha x)}{x^2}\, dx = \int_{-\infty}^0 \frac{1-\cos^2(\alpha x)}{x^2}\, dx =$$
$$\frac{1}{2} \int_{-\infty}^\infty \frac{1-\cos^2(\alpha x)}{x^2}\, dx = |\alpha| \cdot \frac{\pi}{2}.$$

$$\int_0^\infty \frac{1-\cos(\alpha x)}{x^2}\, dx = \int_{-\infty}^0 \frac{1-\cos(\alpha x)}{x^2}\, dx =$$
$$\frac{1}{2} \int_{-\infty}^\infty \frac{1-\cos(\alpha x)}{x^2}\, dx = |\alpha| \cdot \frac{\pi}{2}.$$

(**Examples I 2.2.12, I 2.2.13, 1.7.36**.)

44. If $\alpha > 0$ and β real constants,

$$\int_{-\infty}^\infty e^{-\alpha x^2} \cos(\beta x)\, dx = 2 \int_0^\infty e^{-\alpha x^2} \cos(\beta x)\, dx = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-\beta^2}{4\alpha}}.$$

[**Examples I 2.2.14, 1.7.18**. **Problems I 2.3.16, 1.7.32 (a)**.]

45. If $a \geq 0$ and $k \geq 0$ constants not both zero,

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-kx}\sin(a\sqrt{x})}{x}\, dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ku^2}\sin(au)}{u}\, du = \operatorname{erf}\left(\frac{a}{2\sqrt{k}}\right).$$

If $k > 0$ constant and $t \geq 0$,

$$\operatorname{erf}(t) = \frac{1}{\pi}\int_0^\infty \frac{e^{-kx}\sin(2t\sqrt{kx})}{x}\, dx = \frac{2}{\pi}\int_0^\infty \frac{e^{-ku^2}\sin(2t\sqrt{k}\cdot u)}{u}\, du.$$

If $t > 0$,

$$\operatorname{erfc}(t) = \frac{2}{\pi}\int_0^\infty \left(1 - e^{-u^2}\right)\frac{\sin(2tu)}{u}\,du = \frac{1}{\pi}\int_0^\infty \left(1 - e^{-x}\right)\frac{\sin(2t\sqrt{x})}{x}\,dx.$$

(**Example I 2.2.15**.)

46. If $\mu \in \mathbb{R}$ and $\sigma > 0$ constants,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \sigma\sqrt{2\pi},$$

$$\int_{-\infty}^{\infty} xe^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \mu\sigma\sqrt{2\pi},$$

$$\int_{-\infty}^{\infty} x^2e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = (\mu^2+\sigma^2)\sigma\sqrt{2\pi},$$

$$\int_{-\infty}^{\infty} x^3e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \mu\left(\mu^2+3\sigma^2\right)\sigma\sqrt{2\pi},$$

$$\int_{-\infty}^{\infty} x^4e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \left(\mu^4+6\mu^2\sigma^2+3\sigma^4\right)\sigma\sqrt{2\pi}.$$

(**Examples I 2.2.16, I 2.3.24**.)

47. If $|t| < 1$ constant,

$$\int_0^\pi \ln[1 + t\cos(x)]\,dx = \pi\ln\left(\frac{1+\sqrt{1-t^2}}{2}\right).$$

If $a \geq b > 0$ constants,

$$\int_0^\pi \ln[a \pm b\cos(x)]\,dx = \pi\ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right).$$

[**Problem I 2.2.3 (b)**.]

48.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+x+1)^3}dx = \frac{4\pi\sqrt{3}}{9}.$$

(**Problem I 2.2.4**.)

49. If $0 < b < 1$ constant,

$$\int_0^\infty \frac{1}{t^b(1+t)}\,dt = \frac{\pi}{\sin(b\pi)}.$$

If $a \neq 0$ and $b \in \mathbb{R}$ constants such that $0 < \dfrac{1-b}{a} < 1$, and if $A > 0$ and $B > 0$ constants,

$$\int_0^\infty \frac{1}{t^b(A+Bt^a)}\,dt = \frac{1}{|a|A}\left(\frac{A}{B}\right)^{\frac{1-b}{a}} \frac{\pi}{\sin\left(\frac{1-b}{a}\pi\right)}.$$

(**Problem I 2.2.6**.)

50. If $0 < a < b$ constants,

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^{bx}}\,dx = \frac{1}{b}\cdot\frac{\pi}{\sin\left(\frac{a}{b}\pi\right)}.$$

[**Problem I 2.2.12 (a)**.]

51. If $1 < \alpha < 2$ constant,

$$\int_0^\infty \frac{\ln(x+1)}{x^\alpha}\,dx = \frac{\pi}{(1-\alpha)\sin(\alpha\pi)} \ (> 0).$$

If $\alpha \in \mathbb{R}$ and $\beta \neq 0$ constants such that $0 < \dfrac{\alpha-1}{\beta} < 1$, and $A > 0$ constant,

$$\int_0^\infty \frac{\ln(Ax^\beta+1)}{x^\alpha}\,dx = \operatorname{sign}(\beta)\cdot\frac{A^{\frac{\alpha-1}{\beta}}\pi}{(\alpha-1)\sin\left(\frac{\alpha-1}{\beta}\pi\right)}.$$

[**Problem I 2.2.14 (a)-(b)**.]

52. If $\alpha \in \mathbb{R}$ constant,

$$\int_0^\infty \frac{\sin^2(\alpha x)}{x^2}\,dx \left[= \int_0^\infty \frac{1-\cos^2(\alpha x)}{x^2}\,dx\right] = \int_0^\infty \frac{1-\cos(ax)}{x^2}\,dx = |\alpha|\frac{\pi}{2},$$

$$\int_0^\infty \frac{\cos(ax)[1-\cos(ax)]}{x^2}\,dx = 0.$$

(**Problems I 2.2.16, I 2.2.17**.)

53. If $a \in \mathbb{R}$ constant,

$$\int_0^\infty \frac{\sin(a\sqrt{x})}{x}\,dx = \begin{cases} \pi, & \text{if} \quad a > 0, \\ 0, & \text{if} \quad a = 0, \\ -\pi, & \text{if} \quad a < 0. \end{cases}$$

(**Problem I 2.2.18**.)

54. If a and b real constants,

$$\int_0^\infty \frac{\sin(ax)\cos(bx)}{x}\,dx = \begin{cases} \frac{\pi}{2}, & \text{if } 0 \le |b| < a, \\ \frac{\pi}{4}, & \text{if } |b| = a > 0, \\ 0, & \text{if } |b| > a \ne 0, \end{cases}$$

$$\frac{4}{\pi}\int_0^\infty \frac{\sin(ax)\cos(bx)}{x}\,dx = \begin{cases} 2, & \text{if } 0 \le b < a, \\ 1, & \text{if } b = a > 0, \\ 0, & \text{if } b > a > 0. \end{cases}$$

[**Problem I 2.2.23 (a)-(b)**.]

55. If $a > 0$, $b > 0$, and $c > 0$ constants such that $a > b + c$,

$$\int_0^\infty \frac{\sin(ax)\sin(bx)\sin(cx)}{x}\,dx = 0.$$

(**Problem I 2.2.24**.)

56. If $a \in \mathbb{R}$ constant, and $n \in \mathbb{N}_0$,

$$\int_0^\infty \frac{\sin^{2n+1}(ax)}{x}\,dx =$$

$$\begin{cases} \dfrac{\pi}{2^{2n+1}}\dfrac{(2n)!}{(n!)^2} = \dfrac{\pi}{2^{2n+1}}\dbinom{2n}{n}, & \text{if} \quad a > 0, \\ 0, & \text{if} \quad a = 0, \\ -\dfrac{\pi}{2^{2n+1}}\dfrac{(2n)!}{(n!)^2} = -\dfrac{\pi}{2^{2n+1}}\dbinom{2n}{n}, & \text{if} \quad a < 0. \end{cases}$$

[**Problem I 2.2.28 (a)**.]

57. If $n \in \mathbb{N}_0$

$$\int_0^\infty \frac{\sin^{2n}(x)}{x}\,dx = \infty, \qquad \text{and} \qquad \int_{\frac{-\pi}{2}}^\infty \frac{\cos^{2n}(x)}{x+\frac{\pi}{2}}\,dx = \infty.$$

[**Problem I 2.2.28 (c)**.]

58. If $n \in \mathbb{N}$,

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^2}\,dx = \frac{\pi}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^{n+k+1}\binom{2n}{k}(n-k).$$

[**Problem I 2.2.28 (f)**.]

59. If $\alpha \in \mathbb{R}$ constant,

$$\int_0^\infty e^{-\left(x-\frac{\alpha}{x}\right)^2}dx = \int_{-\infty}^0 e^{-\left(x-\frac{\alpha}{x}\right)^2}dx = \begin{cases} \frac{\sqrt{\pi}}{2}, & \text{if} \quad \alpha \geq 0, \\ \frac{\sqrt{\pi}}{2}\,e^{4\alpha}, & \text{if} \quad \alpha \leq 0. \end{cases}$$

$$\int_0^\infty e^{-\left(x^2+\alpha^2x^{-2}\right)}dx = \int_{-\infty}^0 e^{-\left(x^2+\alpha^2x^{-2}\right)}dx = \frac{\sqrt{\pi}}{2}\,e^{-2|\alpha|}.$$

If $\beta \in \mathbb{R} - \{0\}$ constant,

$$\int_0^\infty e^{-\left(\beta^2x^2+\alpha^2x^{-2}\right)}dx = \int_{-\infty}^0 e^{-\left(\beta^2x^2+\alpha^2x^{-2}\right)}dx = \frac{\sqrt{\pi}}{2|\beta|}\,e^{-2|\alpha\beta|}.$$

[**Problem I 2.2.31 (b), (f)-(i).**]

60. If $s \geq 0$ constant,

$$\int_0^\infty \frac{e^{-sx}}{x^2+1}\,dx = s\int_0^\infty e^{-sx}\arctan(x)\,dx =$$
$$\frac{\pi}{2}\cos(s) - \cos(s)\int_0^s \frac{\sin(t)}{t}\,dt - \sin(s)\int_s^\infty \frac{\cos(t)}{t}\,dt =$$
$$\cos(s)\int_s^\infty \frac{\sin(t)}{t}\,dt - \sin(s)\int_s^\infty \frac{\cos(t)}{t}\,dt =$$
$$\int_s^\infty \frac{\sin(t-s)}{t}\,dt = \int_0^\infty \frac{\sin(t)}{t+s}\,dt = \int_0^\infty \frac{\sin(st)}{t+1}\,dt.$$

[**Problem I 2.2.33 (d)-(e) (k)-(l)**.]

61. If $s > 0$ constant,

$$\int_0^\infty \frac{xe^{-sx}}{x^2+1}\,dx = \frac{s}{2}\int_0^\infty e^{-sx}\ln\left(x^2+1\right)\,dx =$$
$$\sin(s)\int_s^\infty \frac{\sin(t)}{t}\,dt + \cos(s)\int_s^\infty \frac{\cos(t)}{t}\,dt =$$
$$\int_s^\infty \frac{\cos(t-s)}{t}\,dt = \int_0^\infty \frac{\cos(t)}{t+s}\,dt = \int_0^\infty \frac{\cos(st)}{t+1}\,dt.$$

[**Problem I 2.2.33 (m)**.]

62.
$$\int_0^\infty \frac{\sin^2(x)}{x}\,dx = \infty.$$

[**Problems I 2.2.28 (c), I 2.2.37 (b) hint**. **Example I 2.5.10**.]

63.
$$\int_0^\infty \frac{\sin^3(x)}{x}\,dx = \frac{\pi}{4}.$$

(**Problems I 2.2.38 hint**, **1.7.104**.)

64.
$$\int_0^\infty \frac{\sin^3(x)}{x^3}\,dx = \frac{3\pi}{8}.$$

(**Problem I 2.2.38 hint**.)

65.
$$\int_0^\infty \frac{\sin^4(x)}{x}\,dx = \infty.$$

(**Problems I 2.2.38 hint**, **1.7.104**.)

66.
$$\int_0^\infty \frac{\sin^4(x)}{x^2}\,dx = \frac{\pi}{4}.$$
(**Problems I 2.2.38 hint**, **1.7.104**.)

67.
$$\int_0^\infty \frac{\sin^4(x)}{x^4}\,dx = \frac{\pi}{3}.$$
(**Problem I 2.2.38 hint**. **Example 1.7.42**.)

68.
$$\int_{-\infty}^\infty \frac{[1-\cos(x)]^2}{x^4}dx = 2\int_0^\infty \frac{[1-\cos(x)]^2}{x^4}dx =$$

$$2\int_{-\infty}^0 \frac{[1-\cos(x)]^2}{x^4}dx = \frac{\pi}{3}.$$
[**Problem I 2.2.38 (c)**.]

69. If $a \geq 0$ and $b \geq 0$ real constants,
$$\int_{-\infty}^\infty \frac{\sin(ax)\sin(bx)}{x^2}dx = \pi\min\{a,\ b\},$$

$$\int_{-\infty}^\infty \frac{\sin^2(ax)\sin^2(bx)}{x^4}dx = \frac{\pi}{2}\min\{a,\ b\}.$$
[**Problem I 2.2.39 (a)-(b)**. **Example 1.7.38**.]

70. If $\alpha > 0$ and $\beta \in \mathbb{R}$ constants,
$$\int_0^\infty e^{-\alpha x^2}\sin(\beta x)\,dx = \frac{1}{2\alpha}\,e^{-\frac{\beta^2}{4\alpha}}\int_0^\beta e^{\frac{\rho^2}{4\alpha}}\,d\rho.$$
(**Problem I 2.2.43**.)

71. If $r \in \mathbb{R}$,
$$I(r) = \int_0^\pi \ln[r^2 \pm 2r\cos(x) + 1]\,dx =$$
$$\frac{1}{2}\int_0^{2\pi} \ln[r^2 \pm 2r\cos(x) + 1]\,dx = \begin{cases} 0, & \text{if } |r| \leq 1, \\ 2\pi\ln(|r|), & \text{if } |r| > 1. \end{cases}$$
[**Problem I 2.2.46 (c)-(d)**.]

72. If $m \geq 0$ and $n \geq 2$ integers such that $0 \leq m < n-1$,

$$\int_0^\infty \frac{x^m}{(1+x^n)^2}\,dx = \frac{(n-m-1)\pi}{n^2 \sin[\frac{(m+1)\pi}{n}]}.$$

(**Problem I 2.2.47**.)

73. If $0 < p < 1$ constant,

$$\int_0^\infty \frac{t^{p-1}\ln(t)}{1+t}\,dt = -\pi^2 \cot(p\pi)\csc(p\pi).$$

(**Problems I 2.2.48, I 2.6.63, 1.7.142, 1.7.145**. **Examples 1.7.8**, **1.7.47**, **1.7.49**.)

74. If $a \geq 0$ and $b > 0$ constants,

$$\int_{-\infty}^\infty \frac{e^{-a^2x^2}}{b^2+x^2}\,dx = 2\int_0^\infty \frac{e^{-a^2x^2}}{b^2+x^2}\,dx = 2\int_{-\infty}^0 \frac{e^{-a^2x^2}}{b^2+x^2}\,dx = \frac{\pi}{b}\,e^{a^2b^2}[1-\operatorname{erf}(ab)] = \frac{\pi}{b}\,e^{a^2b^2}\operatorname{erfc}(ab).$$

(**Problem I 2.2.50**.)

75.

$$\int_{-1}^1 \frac{-\ln(1-x)}{x}\,dx = \int_0^2 \frac{-\ln(u)}{1-u}\,du = \frac{\pi^2}{4}.$$

$$\int_0^1 \frac{-\ln(1-x)}{x}\,dx = \int_0^1 \frac{-\ln(u)}{1-u}\,du = \sum_{m=0}^\infty \frac{1}{m^2} = \frac{\pi^2}{6}.$$

$$\int_{-1}^0 \frac{-\ln(1-x)}{x}\,dx = \int_1^2 \frac{-\ln(u)}{1-u}\,du = \frac{\pi^2}{12}.$$

$$\int_0^1 \frac{\ln^2(x)}{(1-x)^2}\,dx = \int_0^1 \frac{\ln^2(1-u)}{u^2}\,du = \frac{\pi^2}{3}.$$

$$\int_0^1 \frac{\ln^2(x)}{(1+x)^2}\,dx = \int_1^2 \frac{\ln^2(u-1)}{u^2}\,du = \frac{\pi^2}{6}.$$

(**Example I 2.3.23**.)

76. If $a \geq 0$ constant,

$$\int_{-\infty}^\infty \frac{\ln\left[(ax)^2+1\right]}{x^2+1}\,dx = 2\int_0^\infty \frac{\ln\left[(ax)^2+1\right]}{x^2+1}\,dx = 2\pi\ln(1+a).$$

(**Problems I 2.3.17, 1.7.138**.)

77. If $a \geq 0$ constant,

$$\int_0^\infty \frac{2ax^2}{(x^2+1)\left[(ax)^2+1\right]}\, dx = \frac{\pi}{1+a}.$$

(**Problem I 2.3.17 hint**.)

78.

$$\int_0^\infty e^{-x}\ln(x)\, dx = -\gamma, \qquad \text{or} \qquad \int_0^1 \ln[-\ln(u)]du = -\gamma,$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problems I 2.3.18, I 2.6.5 (a)**.]

79.

$$\int_0^1 \frac{1-e^{-t}-e^{\frac{-1}{t}}}{t}\, dt = \int_1^\infty \frac{1-e^{-s}-e^{\frac{-1}{s}}}{s}\, ds = \gamma.$$

$$\int_0^\infty \frac{1-e^{-v}-e^{\frac{-1}{v}}}{v}\, dv = 2\gamma. \qquad \int_0^\infty \left(\frac{1}{1+s} - e^{-s}\right)\frac{ds}{s} = \gamma.$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problem I 2.3.19 (a)-(b)**.]

80. If $a > 0$ and $b > 0$ constants,

$$\int_0^\infty \frac{e^{-u^a}-e^{-u^b}}{u}\, du = \int_{-\infty}^\infty \left(e^{-e^{au}} - e^{-e^{bu}}\right) du = \gamma \cdot \frac{a-b}{ab}.$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

(**Problems I 2.3.20, I 2.5.29**.)

81.

$$\gamma = \int_0^1 \left(1-e^{-y}\right)\frac{dy}{y} - \int_1^\infty e^{-y}\frac{dy}{y}.$$

$$\gamma = \int_0^1 [1-\cos(y)]\frac{dy}{y} - \int_1^\infty \cos(y)\frac{dy}{y}.$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

(**Problems I 2.3.21, 1.7.38**.)

82. If $a \in \mathbb{R}$, $b > 0$ and $-1 \le c \le 1$ constants,

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx = \int_0^\infty \left[\sum_{n=0}^{\infty}(-c)^n e^{-b(n+1)x}\sin(ax)\right] dx =$$
$$\sum_{n=0}^{\infty}\frac{a(-c)^n}{a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2},$$

$$\int_0^\infty \frac{x}{e^{bx}+c}\,dx = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2}.$$

[**Problem I 2.3.22 (a)-(b)**, **Example 1.7.25**, **Corollary 1.7.5**.]

83. If $b > 0$ and $-b < a < b$ constants,

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sinh(ax)}{\sinh(bx)}\,dx = \int_{-\infty}^{0}\frac{\sinh(ax)}{\sinh(bx)}\,dx = \int_0^\infty \frac{\sinh(ax)}{\sinh(bx)}\,dx =$$
$$\int_0^\infty 2\left[\sum_{n=0}^{\infty}\sinh(ax)\,e^{-b(2n+1)x}\right]dx = \sum_{n=0}^{\infty}\frac{2a}{b^2(2n+1)^2-a^2},$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{x}{\sinh(bx)}\,dx = \int_0^\infty \frac{x}{\sinh(bx)}\,dx = \int_{-\infty}^{0}\frac{x}{\sinh(bx)}\,dx =$$
$$\frac{1}{b^2}\sum_{n=1}^{\infty}\frac{2}{n^2} = \frac{\pi^2}{4b^2}.$$

[**Problems I 2.3.22 (c)-(d), 1.7.43 (a)-(b)**, **Example 1.7.25**, **Corollary 1.7.5**.]

84. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 \frac{x^{q-1}-x^{p-1}}{1-x}\,dx = \sum_{n=0}^{\infty}\left(\frac{1}{n+q} - \frac{1}{n+p}\right) = \sum_{n=0}^{\infty}\frac{p-q}{(n+q)(n+p)}.$$

[**Problem I 2.3.32 (a)**.]

85. If $k \in \mathbb{N}$,

$$\int_0^1 \frac{\ln^k(x)}{1-x}dx = \int_0^1 \frac{\ln^k(1-x)}{x}dx =$$
$$(-1)^k\, k! \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+1}} = (-1)^k\, k! \sum_{n=1}^{\infty} \frac{1}{n^{k+1}}.$$

[**Problem I 2.3.32 (d)**.]

86.

$$\int_{-\infty}^{\infty} \sin\left(x^2\right)\, dx = 2\int_0^{\infty} \sin\left(x^2\right)\, dx = \frac{\sqrt{2\pi}}{2}.$$

$$\int_{-\infty}^{\infty} \cos\left(x^2\right)\, dx = 2\int_0^{\infty} \cos\left(x^2\right)\, dx = \frac{\sqrt{2\pi}}{2}.$$

If $a \neq 0$ real constant,

$$\int_{-\infty}^{\infty} \sin\left(ax^2\right)\, dx = 2\int_0^{\infty} \sin\left(ax^2\right)\, dx = \operatorname{sign}(a)\sqrt{\frac{\pi}{2|a|}},$$

$$\int_{-\infty}^{\infty} \cos\left(ax^2\right)\, dx = 2\int_0^{\infty} \cos\left(ax^2\right)\, dx = \sqrt{\frac{\pi}{2|a|}}.$$

(**Fresnel Integrals**. **Examples I 2.4.2**, **1.7.17**. **Problems I 2.6.11, I 2.4.16**.)

87. If $a \geq 0$ constant and $k \in \mathbb{N}$ integer,

$$\int_0^1 \int_0^1 \frac{(xy)^a}{1-xy}\, dxdy = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2},$$

$$\int_0^1 \int_0^1 \frac{(xy)^a \ln^k(xy)}{1-xy}\, dxdy = (-1)^k (k+1)! \sum_{n=1}^{\infty} \frac{1}{(n+a)^{k+2}},$$

$$\int_0^1 \int_0^1 \frac{\ln^k(xy)}{1-xy}\, dxdy = (-1)^k (k+1)! \sum_{n=1}^{\infty} \frac{1}{n^{k+2}} =$$
$$(-1)^k (k+1)!\, \zeta(k+2).$$

(**Example I 2.4.3 Remark**.)

88.

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{-\ln(1-t)}{t}\,dt = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}.$$

$$\int_{\frac{1}{2}}^{1} \frac{-\ln(1-t)}{t}\,dt = \frac{\pi^2}{12} + \frac{\ln^2(2)}{2}.$$

(**Example I 2.4.5**.)

89. If $x > 0$ real,

$$\frac{1}{x} = \int_0^{\infty} e^{-xt}\,dt.$$

If $k \geq 1$ integer,

$$\frac{1}{x^k} = \frac{1}{(k-1)!}\int_0^{\infty} e^{-xt}t^{k-1}dt.$$

[**Problems I 2.4.1 (a) (d)** and **I 2.6.6 (b)**.]

90.

$$\int_0^1\int_0^1 \frac{1}{u+v}\,du\,dv = \int_0^{\infty}\left(\frac{1-e^{-t}}{t}\right)^2 dt = 2\int_0^{\infty}\frac{1}{(s+1)(s+2)}\,ds = 2\ln(2).$$

[**Problem I 2.4.1 (e)**.]

91. If $a \geq 0$ and $b \in \mathbb{R}$ constants,

$$\int_0^{\infty} e^{-ay}\,\frac{1-\cos(2by)}{2y}\,dy = \int_0^{\infty} e^{-ay}\,\frac{\sin^2(by)}{y}\,dy = \frac{1}{4}\ln\left(\frac{a^2+4b^2}{a^2}\right).$$

(**Problems I 2.4.10, I 2.5.23**.)

92. If $\alpha \geq 0$ and $\beta \in \mathbb{R}$ constants,

$$\int_0^{\infty} e^{-\alpha u}\,\frac{\sin^2(\beta u)}{u^2}\,du = \beta\arctan\left(\frac{2\beta}{\alpha}\right) - \frac{\alpha}{4}\ln\left(\frac{4\beta^2}{\alpha^2}+1\right).$$

(**Problem I 2.4.12**.)

93. If $a \neq 0$, b and c real constants,

$$\int_{-\infty}^{\infty} \sin\left(ax^2+bx+c\right)\,dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\operatorname{sign}(a)\cos\left(\frac{4ac-b^2}{4a}\right)+\sin\left(\frac{4ac-b^2}{4a}\right)\right],$$

$$\int_{-\infty}^{\infty} \cos\left(ax^2+bx+c\right)\,dx =$$
$$\sqrt{\frac{\pi}{2|a|}}\left[\cos\left(\frac{4ac-b^2}{4a}\right)-\operatorname{sign}(a)\sin\left(\frac{4ac-b^2}{4a}\right)\right].$$

[**Problem I 2.4.16 (a)**.]

94. If $k \geq 2$, m, n **integers** and $0 \leq m < n-1$,

$$\int_0^{\infty} \frac{x^m}{(x^n+1)^k}\,dx = \int_0^{\infty} \frac{x^{kn-m-2}}{(x^n+1)^k}\,dx =$$
$$\frac{\pi}{n\cdot\sin\left(\frac{m+1}{n}\pi\right)}\cdot\prod_{j=1}^{k-1}\left(1-\frac{m+1}{nj}\right),$$

(**Problem I 2.4.18**.)

95. If $0 < a, b < \infty$ constants,

$$\int_0^{\infty} \frac{e^{-bx}-e^{-ax}}{x}\,dx = -\ln\left(\frac{b}{a}\right) = \ln\left(\frac{a}{b}\right),$$

$$\int_0^{\infty} \frac{e^{-x}-e^{-ax}}{x}\,dx = \ln(a),$$

$$\int_0^1 \frac{t^{a-1}-t^{b-1}}{\ln(t)}\,dx = \ln\left(\frac{a}{b}\right), \qquad \int_0^1 \frac{t^{a-1}-1}{\ln(t)}\,dt = \ln(a).$$

[**Example I 2.5.1. Problem I 2.2.32. Application (d), after Example I 2.7.8**.]

96. If $0 < a, b < \infty$ constants,

$$\int_0^{\infty} \frac{\arctan(bx)-\arctan(ax)}{x}\,dx = \frac{\pi}{2}\ln\left(\frac{b}{a}\right).$$

If $0 < a, b < \infty$ and $r > 0$ constants,

$$\int_0^{\infty} \frac{\arctan^r(bx)-\arctan^r(ax)}{x}\,dx = \left(\frac{\pi}{2}\right)^r\ln\left(\frac{b}{a}\right),$$

$$\int_0^\infty \frac{\arctan^r(bx)}{x}\,dx = \left(\frac{\pi}{2}\right)^r \ln\left(\frac{b}{0^+}\right) = \infty, \quad \forall\ r \in \mathbb{R}.$$

If $0 < t$ constant,

$$\int_0^\infty \frac{\arctan(tx) - \arctan(x)}{x}\,dx = \frac{\pi}{2}\ln(t).$$

(**Examples I 2.5.2, I 2.2.17**.)

97.

$$\int_0^\infty \frac{\arctan^2(x)}{x\,(x^2+1)}\,dx = \int_0^{\frac{\pi}{2}} u^2\cot(u)\,du = -2\int_0^{\frac{\pi}{2}} u\ln[\sin(u)]\,du =$$
$$\frac{\pi^2}{4}\ln(2) - \frac{7}{8}\sum_{n=1}^\infty \frac{1}{n^3} = \frac{\pi^2}{4}\ln(2) - \sum_{k=1}^\infty \frac{1}{(2k-1)^3}.$$

$$\int_{-\infty}^\infty \frac{\arctan^2(x)}{x\,(x^2+1)}\,dx = 0.$$

(**Example I 2.5.3**.)

98. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\sin(bx) - \sin(ax)}{x}\,dx = 0.$$

(**Example I 2.5.7**. **Problem I 2.5.23**.)

99. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\cos(bx) - \cos(ax)}{x}\,dx = \ln\left(\frac{a}{b}\right),$$

$$\int_0^\infty \frac{\cos\left(\frac{1}{bx}\right) - \cos\left(\frac{1}{ax}\right)}{x}\,dx = \ln\left(\frac{b}{a}\right).$$

If $0 < t$ constant,

$$\int_0^\infty \frac{\cos(x) - \cos(tx)}{x}\,dx = \ln(t).$$

If $a \neq 0$ or $b \neq 0$ real constants,

$$\int_0^\infty \frac{\sin(ax)\sin(bx)}{x}\,dx = \frac{1}{2}\ln\left(\frac{|a+b|}{|a-b|}\right).$$

[**Example I 2.5.8 (a)-(d)**. **Problem I 2.5.23**.]

100. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \ln\left|\frac{\cos(bx)}{\cos(ax)}\right| \frac{1}{x}\,dx = \ln(2)\ln\left(\frac{a}{b}\right).$$

(**Example I 2.5.9**.)

101. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\dfrac{1}{1+(bx)^2} - \dfrac{1}{1+(ax)^2}}{x}\,dx = \ln\left(\frac{a}{b}\right).$$

(**Problem I 2.5.2**.)

102. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \left[b\,\sin\left(\frac{1}{bx}\right) - a\,\sin\left(\frac{1}{ax}\right)\right] dx = \ln\left(\frac{b}{a}\right).$$

[**Problem I 2.5.7 (a)**.]

103. If $0 < a,\ \ b < \infty$ constants,

$$\int_0^\infty \frac{|\sin(bx)| - |\sin(ax)|}{x}\,dx = \frac{2}{\pi}\ln\left(\frac{b}{a}\right).$$

(**Problem I 2.5.12**.)

104. If $0 < a,\ \ b < \infty$ constants,

$$\int_0^\infty \frac{|\cos(bx)| - |\cos(ax)|}{x}\,dx = \left(\frac{2}{\pi} - 1\right)\ln\left(\frac{b}{a}\right).$$

(**Problem I 2.5.12**.)

105. If $0 < \alpha < 1$ and $a,\ \ b > 0$ constants,

$$\int_0^\infty \frac{|\tan(bx)|^\alpha - |\tan(ax)|^\alpha}{x}\,dx = \frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)}\ln\left(\frac{b}{a}\right).$$

(**Problem I 2.5.13**.)

106. If $0 \le \alpha < 1$ and $a,\ \ b > 0$ constants,

$$\int_0^\infty \frac{|\sec(bx)|^\alpha - |\sec(ax)|^\alpha}{x}\,dx = \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)} - 1\right]\ln\left(\frac{b}{a}\right).$$

$$\int_0^\infty \frac{|\sec(bx)| - |\sec(ax)|}{x}\,dx = \operatorname{sign}\left[\ln\left(\frac{b}{a}\right)\right]\cdot\infty.$$

(**Problem I 2.5.14**.)

107. If $0 < a,\ b < \infty$ constants,

$$\int_0^\infty \frac{\frac{\sin^2(bx)}{bx} - \frac{\sin^2(ax)}{ax}}{x}\,dx = 0.$$

(**Problem I 2.5.16**.)

108.
$$\int_0^\infty \frac{\sin^3(x)}{x^2}\,dx = \frac{3}{4}\ln(3).$$

(**Problems I 2.5.17**, **1.7.104**.)

109.
$$\int_0^\infty \frac{\sin^4(x)}{x^3}\,dx = \ln(2).$$

(**Problems I 2.5.17**, **1.7.104**.)

110. If $\sum_{i=1}^{k} A_i = 0$ for real constants $A_i,\ \ i = 1,\ 2,\ 3,\ \ldots,\ k$ and $a_i > 0,\ \ i = 1,\ 2,\ 3,\ \ldots,\ k,$

$$\int_0^\infty \frac{A_1\cos(a_1x) + A_2\cos(a_2x) + \ldots + A_k\cos(a_kx)}{x}\,dx =$$
$$-A_1\ln(a_1) - A_2\ln(a_2) - A_3\ln(a_3) - \ldots - A_k\ln(a_k).$$

(**Problem I 2.5.18**.)

111. If $n \in \mathbb{N}$,

$$\int_0^\infty \frac{\sin^{2n+1}(x)}{x^2}\,dx =$$
$$\frac{(-1)^{n+1}}{2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}[2(n-k)+1]\cdot\ln[2(n-k)+1] =$$
$$\frac{(-1)^{n+1}}{2^{2n}}\sum_{k=0}^{n-1}(-1)^k\binom{2n+1}{k}[2(n-k)+1]\cdot\ln[2(n-k)+1].$$

(**Problem I 2.5.19**.)

112.
$$\int_0^\infty \frac{\sin^5(x)}{x^4}\,dx = \frac{125}{96}\ln(5) - \frac{45}{32}\ln(3).$$

$$\int_0^\infty \frac{\sin^6(x)}{x^5}\,dx = \frac{27}{16}\ln(3) - \ln(4).$$

$$\int_0^\infty \frac{\sin^6(x)}{x^4}\,dx = \frac{\pi}{8}.$$

[**Problem I 2.5.20 (a)-(c)**.]

113. General formulae for the integral $\int_0^\infty \frac{\sin^N(x)}{x^L}\,dx$, where N and L belong to $\mathbb{Z}$.

Case $N - L =$ **even** ≥ 0.

Subcases:

(1) $1 \leq L = 2l + 1 \leq N = 2n + 1$. (Both numbers are positive odd.)

(2) $2 \leq L = 2l \leq N = 2n$. (Both numbers are positive even ≥ 2.)

$$\int_0^\infty \frac{\sin^N(x)}{x^L}\,dx = \frac{(-1)^{n+l}\pi}{2^N(L-1)!}\sum_{k=0}^{n}(-1)^k\binom{N}{k}(N-2k)^{L-1}.$$

[In subcase **(2)** the maximum index in the summation may be obviously replaced by $n-1$, as the last summand obtained for $k = n$ is zero.]

Case $N - L =$ **odd** ≥ 1.

Subcase: **(3)** $2 \leq L = 2l \leq N = 2n + 1$.

$$\int_0^\infty \frac{\sin^N(x)}{x^L}\,dx =$$

$$\frac{(-1)^{n+l}}{2^{N-1}(L-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{N}{k}(N-2k)^{L-1}\ln(N-2k).$$

Subcase: **(4)** $3 \leq L = 2l + 1 \leq N = 2n$.

$$\int_0^\infty \frac{\sin^N(x)}{x^L}\,dx =$$

$$\frac{(-1)^{n+l+1}}{2^{N-1}(L-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{N}{k}(N-2k)^{L-1}\ln(N-2k).$$

In all other cases of combinations of N and L in $\mathbb{Z}$, the integral does not exist (is infinity or oscillates).

[**Problem I 2.5.20 (II)-(III)**.]

114. If $\beta \geq 0$ and $a,\ b$ real constants,

$$\int_0^\infty e^{-\beta x}\,\frac{\cos(ax)-\cos(bx)}{x}\,dx = \frac{1}{2}\ln\left(\frac{\beta^2+b^2}{\beta^2+a^2}\right).$$

[**Problems I 2.4.10, I 2.5.23 (a). Example I 2.5.4**.]

115. If $\beta \geq 0$ and $a,\ b$ real constants,

$$\int_0^\infty e^{-\beta x}\,\frac{\sin(ax)\pm\sin(bx)}{x}\,dx = \arctan\left(\frac{a}{\beta}\right) \pm \arctan\left(\frac{b}{\beta}\right).$$

[**Problems I 2.4.11, I 2.5.23 (b). Example I 2.2.9**.]

116.

$$\int_0^\infty \frac{\arctan^2(x)}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\arctan^2(x)}{x^2}\,dx = 2\int_0^\infty \frac{\arctan(x)}{x(x^2+1)}\,dx = \pi\ln(2).$$

$$\int_0^\infty \frac{\arctan^3(x)}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\arctan^3(x)}{x^3}\,dx = \frac{3}{2}\pi\ln(2) - \frac{\pi^3}{16}.$$

(**Problems I 2.5.24 (a)-(b), 1.7.143**.)

117. If $a,\ b$ real constants,

$$\int_{-\infty}^\infty \frac{\cos(bx)-\cos(ax)}{x^2}\,dx = 2\int_{-\infty}^0 \frac{\cos(bx)-\cos(ax)}{x^2}\,dx = 2\int_0^\infty \frac{\cos(bx)-\cos(ax)}{x^2}\,dx = (|a|-|b|)\pi.$$

$$\int_0^\infty \frac{\sin(bx)-\sin(ax)}{x^2}\,dx = -\int_{-\infty}^0 \frac{\sin(bx)-\sin(ax)}{x^2}\,dx = \begin{cases} +\infty, & \text{if } b > a, \\ 0, & \text{if } b = a, \\ -\infty, & \text{if } b < a. \end{cases}$$

$$\int_{-\infty}^{\infty} \frac{\cos^2(bx) - \cos^2(ax)}{x^2}\, dx = 2\int_{-\infty}^{0} \frac{\cos^2(bx) - \cos^2(ax)}{x^2}\, dx =$$
$$2\int_{0}^{\infty} \frac{\cos^2(bx) - \cos^2(ax)}{x^2}\, dx = (|a| - |b|)\pi.$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(bx) - \sin^2(ax)}{x^2}\, dx = 2\int_{-\infty}^{0} \frac{\sin^2(bx) - \sin^2(ax)}{x^2}\, dx =$$
$$2\int_{0}^{\infty} \frac{\sin^2(bx) - \sin^2(ax)}{x^2}\, dx = (|b| - |a|)\pi.$$

[**Problems I 2.5.27 (a)-(d), 1.7.99, I 2.2.17**. **Example 1.7.36**.]

118. If $a > 0$, constant,

$$\int_0^{\infty} \left(e^{-ae^x} + e^{-ae^{-x}} - 1\right) dx = \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{-ae^x} + e^{-ae^{-x}} - 1\right) dx =$$
$$-\ln(a) - \gamma.$$

(**Problem I 2.5.28**.)

119. If $p > 0$ constant,

$$\int_0^{\infty} x^{p-1} e^{-x}\, dx = \Gamma(p).$$

(**Section I 2.6. Subsection I 2.6.1**.)

120.
$$\int_0^{\infty} \ln(x)\, e^{-x}\, dx = \Gamma'(1) = -\gamma < 0,$$

$$\int_0^{\infty} x\ln(x)\, e^{-x}\, dx = \Gamma'(2) = 1 - \gamma > 0,$$

$$\int_0^{\infty} \ln^2(x)\, e^{-x}\, dx = \Gamma''(1) = \frac{\pi^2}{6} + \gamma^2,$$

where $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Subsection I 2.6.1 (Γ, 2). Problems I 2.1.32, I 2.3.18, I 2.3.19, I 2.6.67 (a)-(c), I 2.7.20 (a)-(c)**.]

121. If $p > 0$ positive constant,

$$\int_0^\infty x^p e^{-x}\,dx = \Gamma(p+1) = p\,\Gamma(p).$$

If $p \neq 0,\ -1,\ -2,\ \ldots$ real constant,

$$\Gamma(p) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{p+n} + \int_1^\infty x^{p-1}e^{-x}dx.$$

If $n \geq 0$ integer constant,

$$\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n!\,.$$

[**Subsection I 2.6.1 ($\boldsymbol{\Gamma}$, 8) ($\boldsymbol{\Gamma}$, 9)**.]

122. If $p > 0$ constant,

$$\int_0^\infty u^{p-1}e^{-xu}du\ = \frac{\Gamma(p)}{x^p}.$$

[**Subsection I 2.6.1 ($\boldsymbol{\Gamma}$, 11)**.]

123. If $0 < p < 2$ constant,

$$\int_0^\infty \frac{\sin(x)}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(\frac{p\pi}{2})}.$$

(**Example I 2.6.7**. **Problems I 2.2.25, I 2.2.26, I 2.6.56, 1.7.20**.)

124. If $0 < p < 1$ constant,

$$\int_0^\infty \frac{\cos(x)}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(\frac{p\pi}{2})}.$$

(**Example I 2.6.7**. **Problems I 2.2.25, I 2.2.26, I 2.6.9, I 2.6.56, 1.7.20**.)

125. If $\beta \neq 0$ and $-1 < \dfrac{\alpha+1}{\beta} < 1$ constants,

$$\int_0^\infty x^\alpha \sin(x^\beta)\,dx = \frac{1}{|\beta|}\cdot\frac{\pi}{2\cdot\Gamma\left(1-\dfrac{\alpha+1}{\beta}\right)\cdot\sin\left[\left(1-\dfrac{\alpha+1}{\beta}\right)\dfrac{\pi}{2}\right]}.$$

(**Example I 2.6.8**.)

126. If $\beta \neq 0$ and $0 < \dfrac{\alpha+1}{\beta} < 1$ constants,

$$\int_0^\infty x^\alpha \cos(x^\beta)\,dx = \frac{1}{|\beta|} \cdot \frac{\pi}{2 \cdot \Gamma\left(1 - \frac{\alpha+1}{\beta}\right) \cdot \cos\left[\left(1 - \frac{\alpha+1}{\beta}\right)\frac{\pi}{2}\right]}.$$

(**Example I 2.6.8**.)

127.

$$\int_0^\infty \frac{\sin^2(x)}{x^{\frac{3}{2}}}\,dx = \sqrt{\pi}.$$

$$\int_0^\infty \frac{\cos^3(x)}{\sqrt{x}}\,dx = \frac{\left(9+\sqrt{3}\right)\sqrt{2\pi}}{24}.$$

If $N = 2n \geq 2$ even and $R > 0$ real,

$$\int_0^\infty \frac{\cos^{2n}(x)}{x^R}\,dx = \infty.$$

If $N = 2n \geq 2$ even and $0 < R \leq 1$ or $2n+1 = N+1 \leq R$ real,

$$\int_0^\infty \frac{\sin^{2n}(x)}{x^R}\,dx = \infty.$$

$$\int_0^\infty \frac{\cos^5(x)}{x^{0.75}}\,dx = \left(\frac{3 \cdot 5^{0.75} + 25 \cdot 3^{0.75} + 150}{240}\right) \cdot \frac{\pi}{\Gamma\left(\frac{3}{4}\right)\sqrt{2+\sqrt{2}}}.$$

$$\int_0^\infty \frac{\sin^5(x)}{x^{5.25}}\,dx =$$

$$\frac{8}{1243.125} \cdot \left(625 \cdot 5^{0.25} - 405 \cdot 3^{0.25} + 10\right) \cdot \frac{\pi}{\Gamma\left(\frac{1}{4}\right)\sqrt{2+\sqrt{2}}}.$$

(**Example I 2.6.9**. Study also the general method in this **Example**.)

128. If $p > 0$ and $q > 0$ real constants,

$$\int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \mathrm{B}(p,q).$$

(**Subsection I 2.6.2**.)

129. If $p > 0$ and $q > 0$ constants,

$$\int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}\,du = \int_0^\infty \frac{u^{q-1}}{(1+u)^{p+q}}\,du = \frac{1}{2}\int_0^\infty \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = \mathrm{B}(p,q).$$

[**Subsection I 2.6.2 (B, 5)**.]

130. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = \int_1^\infty \frac{u^{p-1}+u^{q-1}}{(1+u)^{p+q}}\,du = \mathrm{B}(p,q).$$

[**Subsection I 2.6.2 (B, 5)**.]

131. If $p > 0$ and $q > 0$ constants,

$$\int_0^{\frac{\pi}{2}} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,d\theta = \frac{\mathrm{B}(p,q)}{2}.$$

[**Subsection I 2.6.2 (B, 6)**.]

132. If $p > 0$ and $q > 0$ constants,

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

[**Subsection I 2.6.2 (B, 7)**.]

133. If $m \geq 0$ and $n \geq 0$ integers,

$$\mathrm{B}(m+1,n+1) = \int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx = \frac{m!\,n!}{(m+n+1)!} = \frac{1}{(m+n+1)\binom{m+n}{m}} = \frac{1}{(m+n+1)\binom{m+n}{n}}.$$

[**Subsection I 2.6.2 (B, 7)**.]

134. If $0 < p < 1$ constant,

$$\mathrm{B}(p,1-p) = \int_0^\infty \frac{u^{p-1}}{1+u}du = \int_0^\infty \frac{u^{-p}}{1+u}du = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}.$$

Next:

If the **integer parts** of $a > 0$, $b > 0$ and $a + b > 0$, are

$$k = [\![a]\!] \geq 0, \qquad l = [\![b]\!] \geq 0, \qquad \text{and} \qquad n = [\![a + b]\!] \geq 0,$$

and the **fractional parts** of $a > 0$, $b > 0$ and $a + b > 0$, are

$$s = a - k \geq 0, \qquad t = b - l \geq 0, \quad \text{and} \quad r = a + b - n \geq 0,$$

then:

(a) If **both** $a > 0$ **and** $b > 0$ **are not integers, but** $a+b = n \geq 1$ **is integer**,

$$\mathrm{B}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\,du = \int_0^\infty \frac{u^{b-1}}{(1+u)^{a+b}}\,du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(n)} =$$
$$\frac{(a-1)(a-2)\ldots(1+s)s\Gamma(s)(b-1)(b-2)\ldots(1+t)t\Gamma(t)}{(n-1)!} =$$
$$\frac{(k-1+s)(k-2+s)\ldots(1+s)s(l-1+t)\ldots(1+t)t}{(n-1)!}\frac{\pi}{\sin(s\pi)} =$$
$$\frac{[s]_k \cdot [t]_l}{(n-1)!} \cdot \frac{\pi}{\sin(s\pi)}.$$

(Careful how to write this formula, when $k = 0$, or $l = 0$, or $k = l = 0$.)

(b) If **the three numbers** $a > 0$**,** $b > 0$ **and** $a + b$ **are not integers**,

$$\mathrm{B}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\,du = \int_0^\infty \frac{u^{b-1}}{(1+u)^{a+b}}\,du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} =$$
$$\frac{(k-1+s)\ldots(1+s)s\,(l-1+t)\ldots(1+t)t}{(n-1+r)(n-2+r)\ldots(1+r)r} \cdot \frac{\Gamma(s)\Gamma(t)}{\Gamma(r)} =$$
$$\frac{(a-1)(a-2)\ldots(1+s)s\,(b-1)(b-2)\ldots(1+t)t}{(a+b-1)(a+b-2)\ldots(1+r)r} \cdot \frac{\Gamma(s)\Gamma(t)}{\Gamma(r)} =$$
$$\frac{[s]_k \cdot [t]_l}{[r]_n} \cdot \frac{\Gamma(s)\Gamma(t)}{\Gamma(r)}.$$

(c) If $a > 0$ **is an integer, but** $b > 0$ **is not an integer (or vice-versa)**,

$$\mathrm{B}(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\,du = \int_0^\infty \frac{u^{b-1}}{(1+u)^{a+b}}\,du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} =$$
$$\frac{(a-1)!}{(n-1+t)(n-2+t)\ldots(l+t)} = \frac{(a-1)!}{(a+b-1)(a+b-2)\ldots b} =$$

$$= \frac{(a-1)!}{[b]_a} = \frac{(a-1)! \cdot [t]_l}{[t]_n}.$$

[**Subsection I 2.6.2 (B, 8) (B, 5)**. **Examples I 2.2.5** and **1.7.8 Case (b)**.]

135. If $p > 0$, $q > 0$, $r > 0$ constants,

$$\int_0^1 \frac{u^{p-1}(1-u)^{q-1}}{(r+u)^{p+q}}\, du = \frac{1}{(r+1)^p \cdot r^q}\, \mathrm{B}(p, q).$$

[**Subsection I 2.6.2 (B, 9)**.]

136. If $p > 0$, $s > 0$ constants,

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right) = \frac{2^{2p-\frac{1}{2}}}{\sqrt{2\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right),$$

$$\mathrm{B}(s, s) = 2^{1-2s}\mathrm{B}\left(s, \frac{1}{2}\right).$$

[**Subsection I 2.6.2 (B, 10)** and **Problem I 2.6.40**.]

137.

$$\int_0^\infty \frac{\sqrt[4]{x}}{(1+x)^2}\, dx = \frac{\pi\sqrt{2}}{4}.$$

(**Example I 2.6.15**.)

138. If $b > 0$ real constant,

$$\int_0^b \sqrt{b^4 - x^4}\, dx = \frac{b^3}{6\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right),$$

$$\int_{-b}^b \sqrt{b^4 - x^4}\, dx = \frac{b^3}{3\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right).$$

(**Example I 2.6.16**.)

139. If $p > 0$ constant,

$$\int_0^{\frac{\pi}{2}} \sin^{2p-1}(\theta)\, d\theta = \frac{1}{2}\mathrm{B}\left(p, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(p)}{\Gamma\left(p + \frac{1}{2}\right)}.$$

(**Example I 2.6.18**.)

140. If $q > 0$ constant,

$$\int_0^{\frac{\pi}{2}} \cos^{2q-1}(\theta)\, d\theta = \frac{1}{2}\mathrm{B}\left(q, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(q)}{\Gamma\left(q+\frac{1}{2}\right)}.$$

(**Example I 2.6.18**.)

141. If $m \geq 0$ and $n \geq 0$ integers,

$$\int_0^{\frac{\pi}{2}} \sin^{2m}(x)\cos^{2n}(x)\, dx = \frac{1}{2} \cdot \mathrm{B}\left(m+\frac{1}{2}, n+\frac{1}{2}\right) =$$
$$\frac{1}{2} \cdot \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(m+n+1)} = \frac{\pi}{2^{2m+2n+1}} \cdot \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

[**Examples I 2.6.19**, **1.8.5**, **1.8.6**. **Problems I 2.2.28 (g), 1.7.17 (c), 1.8.12, 1.8.17**.]

142. If $p > 0$ and $q > 0$ real constants,

$$\int_0^{\frac{\pi}{2}} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\, d\theta = \frac{1}{2}\mathrm{B}(p,q) = \frac{1}{2}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(**Example I 2.6.20**.)

143. If $a > 0$, $b > 0$, $p > 0$, $q > 0$ constants,

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2p-1}(\theta)\cos^{2q-1}(\theta)}{\left[a\,\sin^2(\theta) + b\,\cos^2(\theta)\right]^{p+q}}\, d\theta = \frac{\mathrm{B}(p,q)}{2\,a^p \cdot b^q} = \frac{1}{2\,a^p \cdot b^q} \cdot \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

(**Example I 2.6.21**.)

144. If a, b and c real constants, such that $a > 0$, $b > -1$ and $c = \dfrac{b+2}{a}$,

$$\int_0^1 \frac{x^b+1}{(x^a+1)^c}\, dx = \int_1^\infty \frac{u^b+1}{(u^a+1)^c}\, du = \frac{1}{2}\int_0^\infty \frac{v^b+1}{(v^a+1)^c}\, dv =$$
$$\frac{1}{a}\mathrm{B}\left(\frac{b+1}{a}, \frac{1}{a}\right) = \frac{1}{a}\frac{\Gamma\left(\frac{b+1}{a}\right)\Gamma\left(\frac{1}{a}\right)}{\Gamma(c)}.$$

$$\int_0^1 \frac{x^{10}+1}{(x^3+1)^4}\, dx = \int_1^\infty \frac{u^{10}+1}{(u^3+1)^4}\, du = \frac{1}{2}\int_0^\infty \frac{v^{10}+1}{(v^3+1)^4}\, dv =$$
$$\frac{80\pi\sqrt{3}}{729}.$$

$$\int_0^1 \frac{x^4+1}{x^6+1}\,dx = \int_1^\infty \frac{u^4+1}{u^6+1}\,du = \frac{1}{2}\int_0^\infty \frac{v^4+1}{v^6+1}\,dv = \frac{\pi}{3}.$$

(**Examples I 2.6.22, I 2.2.6,**, **Problem I 1.1.4**.)

145. If $0 < s < 1$ constant,

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \frac{2^{1-s}}{\sqrt{\pi}}\cos\left(\frac{\pi s}{2}\right)\Gamma(s).$$

(**Example I 2.6.23**.)

146. If $t \neq 0$ and $s \in \mathbb{R}$ constants such that $0 < \dfrac{s+1}{t} < 1$,

$$\int_0^\infty \frac{u^s}{1+u^t}\,du = \frac{1}{|t|}\frac{\pi}{\sin\left(\frac{s+1}{t}\pi\right)}.$$

If $t \neq 0$, $s \in \mathbb{R}$ and $r \in \mathbb{R}$ constants such that $0 < \dfrac{s+1}{t} < r$,

$$\int_0^\infty \frac{u^s}{(1+u^t)^r}\,du =$$

$$\frac{1}{|t|}\mathrm{B}\left(\frac{s+1}{t}, r-\frac{s+1}{t}\right) = \frac{1}{|t|}\frac{\Gamma\left(\dfrac{s+1}{t}\right)\Gamma\left(r-\dfrac{s+1}{t}\right)}{\Gamma(r)}.$$

(**Example I 2.6.24**.)

147.

$$K\left(\frac{1}{\sqrt{2}}\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2(\phi)}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}.$$

(**Subsection I 2.6.3, Application 1**.)

148.

$$\int_0^\pi \frac{1}{1-\cos(u)}\,du = \infty. \qquad \int_0^\pi\int_0^\pi \frac{du\,dv}{1-\cos(u)\cos(v)} = \infty.$$

$$\int_0^\pi\int_0^\pi\int_0^\pi \frac{du\,dv\,dw}{1-\cos(u)\cos(v)\cos(w)} =$$

$$\frac{\Gamma^4\left(\frac{1}{4}\right)}{4} = \pi^3\cdot 1.393203929\ldots = \frac{\pi^4}{\Gamma^4\left(\frac{3}{4}\right)}.$$

(**Subsection I 2.6.3, Application 7**.)

149. If $p > 0$ constant,

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx = \int_0^1 [-\ln(u)]^{p-1}du = \int_0^1 |\ln(u)|^{p-1}du.$$

[**Problems I 2.1.17, I 2.6.5 (a), I 2.6.8**.]

150.

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{-\frac{1}{2}} dx = \int_0^1 \frac{1}{\sqrt{-\ln(x)}}\,dx = \int_0^1 \frac{1}{\sqrt{|\ln(x)|}}\,dx = \sqrt{\pi}.$$

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{\frac{1}{2}} dx = \int_0^1 \sqrt{-\ln(x)}\,dx = \int_0^1 \sqrt{|\ln(x)|}\,dx = \frac{\sqrt{\pi}}{2}.$$

[**Problem I 2.6.5 (b), Properties (Γ, 5) (Γ, 9)**.]

151. If $\alpha > -1$, $\beta > 0$ and $c > 0$ constants,

$$\int_0^\infty x^\alpha e^{-\beta x^c}\,dx = \frac{1}{c\beta^{\frac{\alpha+1}{c}}}\Gamma\left(\frac{\alpha+1}{c}\right).$$

$$\int_0^\infty \ln(x)x^\alpha e^{-\beta x^c}\,dx = \frac{1}{c^2\beta^{\frac{\alpha+1}{c}}}\left[\Gamma'\left(\frac{\alpha+1}{c}\right) - \Gamma\left(\frac{\alpha+1}{c}\right)\ln(\beta)\right].$$

(**Problem I 2.6.6 (a) (d)**.)

152. If $p \geq 0$ constant

$$\int_0^\infty \frac{x^{p-1}}{e^x - 1}\,dx = \int_1^\infty \frac{\ln^{p-1}(u)}{(u-1)u}\,du = \int_0^1 \frac{[-\ln(v)]^{p-1}}{1-v}\,dv =$$

$$\int_0^1 \frac{[-\ln(1-t)]^{p-1}}{t}\,dt = \Gamma(p)\sum_{n=1}^\infty \frac{1}{n^p} = \Gamma(p)\zeta(p).$$

(**Problem I 2.6.7**.)

153. (a) If $n = 0, 1, 2, 3, \ldots$ integers and $\alpha > -1$ constant,

$$\int_0^1 x^\alpha[\ln(x)]^n\,dx = \frac{(-1)^n n!}{(\alpha+1)^{n+1}}.$$

[**Problems I 2.6.5, I 2.6.6, I 2.6.8 (a), I 2.1.17**.]

154. If $a \in \mathbb{R}$ and $b > 0$ constants,

$$\int_0^1 x^{\left(ax^b\right)}\,dx = \sum_{n=0}^{\infty} \frac{(-a)^n}{(b\,n+1)^{n+1}}.$$

[**Problems I 2.6.8 (b)**.]

155. If $n = 0,\ 1,\ 2,\ 3,\ \ldots$ integer,

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\,\sqrt{\pi}}{4^n n!} = \frac{1\cdot 3\cdot 5\cdots(2n-1)\,\sqrt{\pi}}{2^n}.$$

[**Problem I 2.6.14 (a)**.]

156. If $m = 1,\ 2,\ 3,\ 4,\ \ldots$ integer,

$$\Gamma\left(-m+\frac{1}{2}\right) = \frac{(-1)^m\,2^m\,\sqrt{\pi}}{1\cdot 3\cdot 5\cdots(2m-1)} = \frac{(-1)^m\,4^m\,m!\,\sqrt{\pi}}{(2m)!}.$$

[**Problem I 2.6.14 (b)**.]

157. If $-1 \le \alpha \le 1$ constant,

$$\int_0^{\pi} |\tan(x)|^{\alpha}\,dx = 2\int_0^{\frac{\pi}{2}} \tan^{\alpha}(x)\,dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)},$$

$$\int_0^{\pi} |\cot(x)|^{\alpha}\,dx = 2\int_0^{\frac{\pi}{2}} \cot^{\alpha}(x)\,dx = \frac{\pi}{\cos\left(\frac{\alpha\pi}{2}\right)}.$$

(**Problem I 2.6.19**.)

158. For all $\alpha \le 1$ constant,

$$\int_0^{\pi} |\sec(x)|^{\alpha}\,dx = 2\int_0^{\frac{\pi}{2}} \sec^{\alpha}(x)\,dx = \sqrt{\pi}\cdot\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)},$$

$$\int_0^{\pi} |\csc(x)|^{\alpha}\,dx = 2\int_0^{\frac{\pi}{2}} \csc^{\alpha}(x)\,dx = \sqrt{\pi}\cdot\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{2-\alpha}{2}\right)}.$$

(**Problem I 2.6.20**.)

159. If $b > 0$ constant,

$$\int_0^b \frac{dx}{\sqrt{b^4-x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{2\pi}} = \frac{1}{2}\int_{-b}^{b} \frac{dx}{\sqrt{b^4-x^4}}.$$

(**Problem I 2.6.23**.)

160. If $b > 0$ constant,

$$\int_0^\infty \frac{dx}{\sqrt{b^4+x^4}} = \frac{\left[\Gamma(\frac{1}{4})\right]^2}{4b\sqrt{\pi}} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\sqrt{b^4+x^4}}.$$

(**Problem I 2.6.24**.)

161. If $p \in \mathbb{N}_0$,

$$\int_0^{\frac{\pi}{2}} \sin^p(\phi)\,d\phi = \int_0^{\frac{\pi}{2}} \cos^p(\phi)\,d\phi =$$

$$\begin{cases} \dfrac{\pi}{2}, & \text{if } p = 0, \\[2ex] 1, & \text{if } p = 1, \\[2ex] \dfrac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots p}\cdot\dfrac{\pi}{2} = \dfrac{p\,!}{2^p\cdot\left[\left(\frac{p}{2}\right)!\right]^2}\cdot\dfrac{\pi}{2} = \dfrac{\binom{p}{\frac{p}{2}}}{2^p}\cdot\dfrac{\pi}{2}, & \text{if } p = \text{even} \geq 2, \\[2ex] \dfrac{2\cdot 4\cdot 6\cdots(p-1)}{1\cdot 3\cdot 5\cdots p} = \frac{2^{p-1}\cdot\left[\left(\frac{p-1}{2}\right)!\right]^2}{p\,!} = \frac{2^{p-1}}{p\cdot\binom{p-1}{\frac{p-1}{2}}}, & \text{if } p = \text{odd} \geq 3. \end{cases}$$

[**Problems I 2.2.28 (g), I 2.6.25, 1.8.12, 1.8.17**. **Examples I 2.6.19, 1.8.4, 1.8.5, 1.8.6**.]

162. If $p > 0$, $q > 0$, t real constants,

$$\int_0^1 e^{xt}x^{p-1}(1-x)^{q-1}dx = \sum_{n=0}^\infty \frac{\Gamma(n+p)\,\Gamma(q)}{n!\,\Gamma(n+p+q)}\,t^n.$$

(**Problem I 2.6.32**.)

163. If $a > -1$, $b > -1$, $c > 0$ constants,

$$\int_0^1 x^a\left(1-x^c\right)^b dx = \frac{1}{c}\,\mathrm{B}\left(\frac{a+1}{c}, b+1\right).$$

[**Problem I 2.6.33 (a)**.]

164. If $a > 0$, $p > 0$ and $q > 0$ constants,

$$\int_0^{\frac{1}{a}} x^{p-1}\left(1-ax\right)^{q-1} dx = \frac{1}{a^p}\,\mathrm{B}(p,q).$$

(**Problem I 2.6.35**.)

165. If $p > 0$ and $q > 0$ constants,

$$\int_0^\infty e^{-p\,t}\left(1-e^{-t}\right)^{q-1}\,dt = \mathrm{B}(p,q).$$

(**Problem I 2.6.38**.)

166. If $p > 0$ and $q > 0$ constants,

$$\int_0^1 u^{2p-1}\left(1-u^2\right)^{q-1}\,du = \frac{\mathrm{B}(p,q)}{2}.$$

(**Problem I 2.6.39**.)

167. If $b > a$, $p > -1$ and $q > -1$ constants,

$$\int_a^b (x-a)^p(b-x)^q dx = (b-a)^{p+q+1}\frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}.$$

(**Problem I 2.6.41**.)

168. If $a > b > 0$ constants,

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(x)]^{2a}}\,dx =$$
$$4^{a-1}\,\mathrm{B}(a+b,\,a-b) = 4^{a-1}\,\frac{\Gamma(a+b)\,\Gamma(a-b)}{\Gamma(2a)}.$$

[**Problem I 2.6.42 (a)**.]

169. If $a > 0$ constant,

$$\int_0^\infty \frac{1}{[\cosh(x)]^{2a}}\,dx = 4^{a-1}\,\frac{[\Gamma(a)]^2}{\Gamma(2a)}.$$

[**Problem I 2.6.42 (b)**.]

170. If $r > 0$, $b > 0$ and $a > \dfrac{b}{r}$ constants,

$$\int_0^\infty \frac{\cosh(2bx)}{[\cosh(rx)]^{2a}}\,dx =$$
$$\frac{4^{a-1}}{r}\,\mathrm{B}\left(a+\frac{b}{r},\,a-\frac{b}{r}\right) = \frac{4^{a-1}}{r}\,\frac{\Gamma\left(a+\frac{b}{r}\right)\,\Gamma\left(a-\frac{b}{r}\right)}{\Gamma(2a)}.$$

[**Problem I 2.6.42 (d)**.]

171. If $a > b > -1$ constants,

$$\int_0^\infty \frac{\sinh^b(x)}{\cosh^a(x)}\,dx = \frac{1}{2}\mathrm{B}\left(\frac{b+1}{2}, \frac{a-b}{2}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{a-b}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}.$$

(**Problem I 2.6.43**.)

172. If $p > 0$ constant,

$$\Gamma(3p) = \frac{3^{3p-\frac{1}{2}}}{2\pi}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{3}\right)\,\Gamma\left(p+\frac{2}{3}\right).$$

(**Problem I 2.6.45**.)

173. If $p > 0$ constant and $n \geq 1$ integer,

$$\Gamma(np) = \frac{n^{np-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\,\Gamma(p)\,\Gamma\left(p+\frac{1}{n}\right)\Gamma\left(p+\frac{2}{n}\right)\ldots\Gamma\left(p+\frac{n-1}{n}\right).$$

[**Problems I 2.6.45 footnote, I 2.6.65 Item (13.)**.]

174. If $z \in \mathbb{C}$,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}\right],$$

where $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problems I 2.6.45 footnote, I 2.6.65 Item (8.)**.]

175.

$$\int_0^1 \ln[\Gamma(x)]dx = \frac{1}{2}\ln(2\pi) = \ln\left(\sqrt{2\pi}\right).$$

$$\int_1^2 \ln[\Gamma(x)]\,dx = -1 + \ln\left(\sqrt{2\pi}\right) = -0.0810615\ldots.$$

If $s \geq 0$ constant,

$$\int_s^{s+1} \ln[\Gamma(x)]\,dx = s\cdot\ln(s) - s + \ln(\sqrt{2\pi}).$$

[**Problem I 2.6.55 (b) (d)-(e)**.]

176. If $0 < m < 1$ constant,

$$\int_0^\infty \frac{\cos(x)}{x^{1-m}} dx = \Gamma(m) \cos\left(\frac{m\pi}{2}\right).$$

$$\int_0^\infty \frac{\sin(x)}{x^{1-m}} dx = \Gamma(m) \sin\left(\frac{m\pi}{2}\right).$$

(**Problems I 2.6.56, 1.7.20**.)

177. If $0 < p < 1$,

$$\frac{d}{dp}\mathrm{B}(p, 1-p) = \frac{d}{dp}[\Gamma(p)\Gamma(1-p)] =$$

$$\int_0^1 x^{p-1}(1-x)^{-p} \ln\left(\frac{x}{1-x}\right) dx = \frac{-\pi^2 \cos(p\pi)}{\sin^2(p\pi)} =$$

$$\int_0^1 x^{-p}(1-x)^{p-1} \ln\left(\frac{1-x}{x}\right) dx = -\pi^2 \csc(p\pi)\cot(p\pi).$$

(**Problems I 2.6.63, I 2.2.48, 1.7.142, 1.7.145**. **Examples 1.7.8**, **1.7.47**, **1.7.49**.)

178.

$$\int_0^\infty \ln(x)\, e^{-x^2} dx = \frac{1}{4}\int_0^\infty \frac{\ln(u)\, e^{-u}}{\sqrt{u}}\, du = \frac{-\sqrt{\pi}}{4}\,[\gamma + 2\ln(2)].$$

[**Problem I 2.6.67 (g)**.]

179. If $\alpha > 0$, and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x} x \cos(\beta x)\, dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2},$$

$$\int_0^\infty e^{-\alpha x} x \sin(\beta x)\, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

[**Application (c)** after **Example I 2.7.8**.]

180. If $n \in \mathbb{N}_0$, and $b \geq 0$, $s > 0$ constants,

$$\int_0^\infty x^n \sin(bx)\, e^{-sx} dx = (-1)^n \frac{d^n}{ds^n}\left(\frac{b}{s^2 + b^2}\right).$$

$$\int_0^\infty x^n \cos(bx)\, e^{-sx} dx = (-1)^n \frac{d^n}{ds^n}\left(\frac{s}{s^2 + b^2}\right).$$

(See **Problem I 2.7.35**.)

181. If $s > 0$ constant,

$$\int_0^{\infty} e^{-sx}\ln(x)dx = \frac{-[\gamma + \ln(s)]}{s},$$

$$\int_0^{\infty} e^{-sx}\ln^2(x)dx = \frac{\pi^2}{6s} + \frac{[\gamma + \ln(s)]^2}{s},$$

$$\int_0^{\infty} e^{-sx}x\ln(x)dx = \frac{1-\gamma-\ln(s)}{s^2},$$

where $\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problem I 2.7.20 (a)-(c)**.]

182. If $0 \le r \le 1$ and a and b real constants,

$$\int_a^b \ln\left[1 + 2r\cos(\theta) + r^2\right]d\theta = 2\sum_{n=1}^{\infty}(-1)^{n-1}r^n\left[\frac{\sin(nb)-\sin(na)}{n^2}\right].$$

(**Subsection 1.5.4**.)

183.

$$\int_0^{\frac{\pi}{2}} \theta\ln[\cos(\theta)]d\theta = -\frac{\pi^2}{8}\ln(2) - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} =$$

$$-\frac{\pi^2}{8}\ln(2) - \frac{7}{16}\sum_{n=1}^{\infty}\frac{1}{n^3} = -\frac{\pi^2}{8}\ln(2) - \frac{7}{16}\zeta(3).$$

$$\int_0^{\frac{\pi}{2}} u\ln[\sin(u)]du = -\frac{\pi^2}{8}\ln(2) + \frac{7}{16}\zeta(3).$$

(**Example 1.5.5**.)

184. If $r > 0$ constant and $a \in \mathbb{C}$ such that $|a| \neq r$,

$$\int_{C^+(0,r)} \frac{|dz|}{|z-a|^4} = 2\pi r\frac{|a|^2 + r^2}{\left||a|^2 - r^2\right|^3}.$$

If $r > 0$ and $c,\ d \in \mathbb{R}$ constants,

$$\int_0^{2\pi} \frac{d\theta}{\{c^2 + d^2 + r^2 - 2r[c\cos(\theta) + d\sin(\theta)]\}^2} = 2\pi\frac{c^2 + d^2 + r^2}{\left|c^2 + d^2 - r^2\right|^3}.$$

(**Problem 1.5.33**.)

185. If $m \in \mathbb{N}_0$,

$$\int_{-\pi}^{\pi} P_{(a,\phi)}(r,\theta)\cos(m\theta)\,d\theta = \int_0^{2\pi} P_{(a,\phi)}(r,\theta)\cos(m\theta)\,d\theta = 2\pi\left(\frac{r}{a}\right)^m,$$

$$\text{where} \quad P_{(a,\phi)}(r,\theta) = \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} =$$

$$-1+2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)] = 1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)],$$

is the **Poisson kernel** and $0 \le r < a$ and $\theta,\ \phi$ are real constants. (**Problem 1.5.42**.)

186.

$$\int_0^{2\pi} \frac{d\theta}{a^2-2a\cos(\theta)+1} = \begin{cases} \dfrac{2\pi}{1-a^2}, & \text{when} \quad |a|<1, \\[2ex] \dfrac{2\pi}{a^2-1}, & \text{when} \quad |a|>1. \end{cases}$$

[**Problem 1.5.46 (a). Example 1.8.2**.]

187. If $R > r > 0$ constants,

$$\int_0^{2\pi} \frac{d\theta}{R^2-2Rr\cos(\theta)+r^2} = \frac{2\pi}{R^2-r^2}.$$

[**Problems 1.5.46 (a), 1.8.3**. **Example 1.8.2**.]

188. If $0 \le r < a$ and θ constants,

$$\int_0^{2\pi} \frac{\cos(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi = \frac{2\pi r\cos(\theta)}{a(a^2-r^2)},$$

$$\int_0^{2\pi} \frac{\sin(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\,d\phi = \frac{2\pi r\sin(\theta)}{a(a^2-r^2)}.$$

[**Problems 1.5.46 (b)**.]

189.

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\cos\left[\theta-3\pi\sin(\theta)\right]-\cos(\theta)}{1-2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right]+e^{6\pi\cos(\theta)}}\,d\theta = 2.$$

(**Example 1.7.2**.)

190.

$$\int_0^{2\pi} \frac{e^{3\pi\cos(\theta)}\sin\left[\theta-3\pi\sin(\theta)\right]-\sin(\theta)}{1-2e^{3\pi\cos(\theta)}\cos\left[3\pi\sin(\theta)\right]+e^{6\pi\cos(\theta)}}\,d\theta = 0.$$

(**Example 1.7.2**.)

191. If $z,\ w \in \mathbb{C}$, with $-1 < \mathrm{Re}(z) < 0$, and $0 < \mathrm{Re}(w) < 1$ constants,

$$\int_0^\infty \frac{x^z}{1+x}\,dx = \frac{\pi}{\sin[(z+1)\pi]}, \quad \text{and} \quad \int_0^\infty \frac{x^{w-1}}{1+x}\,dx = \frac{\pi}{\sin(w\pi)}.$$

[**Example 1.7.8 Case (b), footnote**.]

192. If $-1 < \alpha < 0$ and $b > 0$ constants,

$$\int_0^\infty \frac{x^\alpha}{b+x}\,dx = \frac{b^\alpha \pi}{\sin(-\alpha\pi)}.$$

[**Example 1.7.8 Remark (4)**.]

193. If $|a| > 1$ real constant,

$$\int_{-1}^{1} \frac{dx}{(x+a)\sqrt{1-x^2}} \overset{[x=\sin(\theta)]}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin(\theta)} = \frac{\mathrm{sign}(a)\cdot\pi}{\sqrt{a^2-1}}.$$

(**Examples 1.7.14**, **1.8.1 Remark 2**.)

194.

$$\int_0^\infty \frac{\sqrt[3]{x}}{x^2+x+1}dx = \frac{4}{3}\sin\left(\frac{\pi}{9}\right).$$

(**Example 1.7.16**. **Problem 1.7.147**.)

195.

$$2\int_0^\infty e^{-x^2}\cos(4x)\,dx = 2\int_{-\infty}^{0} e^{-x^2}\cos(4x)\,dx =$$
$$\int_{-\infty}^{\infty} e^{-x^2}\cos(4x)\,dx = e^{-4}\sqrt{\pi}.$$

[**Examples I 2.2.14**, **1.7.18**. **Problem 1.7.32 (a)**.]

196.

$$\int_0^\infty e^{-x^2}\sin(4x)\,dx = e^{-4}\int_0^2 e^{y^2}dy \simeq 0.3013403889237924\ldots.$$

(**Example 1.7.18 Remark 3**.)

197. If $k \in \mathbb{N}_0$,

$$\int_0^1 \frac{x^k+1}{x^{k+2}+1}\,dx = \int_1^\infty \frac{u^k+1}{u^{k+2}+1}\,du =$$
$$\int_0^\infty \frac{1}{x^{k+2}+1}\,dx = \int_0^\infty \frac{x^k}{x^{k+2}+1}\,dx = \frac{\pi}{(k+2)\cdot\sin\left(\frac{\pi}{k+2}\right)}.$$

[**Problems 1.7.14 (c)**.]

198. If $a > 0$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = 2\int_{-\infty}^{0} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

(**Problem 1.7.15**.)

199. If $a \in \mathbb{R}$ constant,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)^3} = 2\int_{0}^{\infty} \frac{x^2 dx}{(x^2+a^2)^3} = 2\int_{-\infty}^{0} \frac{x^2 dx}{(x^2+a^2)^3} = \frac{\pi}{8|a|^3}.$$

[**Problem 1.7.16 (a)**.]

200.

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^2+x^4} = 2\int_{0}^{\infty} \frac{x^2 dx}{1+x^2+x^4} = 2\int_{-\infty}^{0} \frac{x^2 dx}{1+x^2+x^4} = \frac{\pi\sqrt{3}}{3}.$$

[**Problem 1.7.16 (b)**.]

201.

$$\int_{-\infty}^{\infty} \frac{x^6 dx}{(1+x^4)^2} = 2\int_{0}^{\infty} \frac{x^6 dx}{(1+x^4)^2} = 2\int_{-\infty}^{0} \frac{x^6 dx}{(1+x^4)^2} = \frac{3\pi\sqrt{2}}{8}.$$

[**Problem 1.7.16 (c)**.]

202. If $a > 0$, $b > 0$, constants and $n = 0,\ 1,\ 2,\ \ldots$, integer,

$$\int_{-\infty}^{\infty} \frac{dx}{(ax^2+b)^{n+1}} = 2\int_{0}^{\infty} \frac{dx}{(ax^2+b)^{n+1}} = \frac{1}{b^n\sqrt{ab}}\frac{\pi(2n)!}{2^{2n}(n!)^2} = \frac{1}{b^n\sqrt{ab}}\frac{\pi}{2^{2n}}\binom{2n}{n}.$$

[**Problem 1.7.17 (a)-(b)**.]

203. If $\beta \in \mathbb{R}$ and $c > 0$ constants,

$$\int_{-\infty}^{\infty} e^{-cx^2} \sin(\beta x^2)\, dx =$$
$$2\int_0^{\infty} e^{-cx^2} \sin(\beta x^2)\, dx = \int_0^{\infty} e^{-cu} \frac{\sin(\beta u)}{\sqrt{u}}\, du =$$
$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\beta\, dv}{\beta^2 + \left(c+v^2\right)^2} = \operatorname{sign}(\beta) \sqrt{\frac{\pi}{2}} \sqrt{\frac{-c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

[**Problem 1.7.19 (a)**. **Example I 2.4.2 Remark 3**.]

204. If $\beta \in \mathbb{R}$ and $c > 0$ constants,

$$\int_{-\infty}^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx =$$
$$2\int_0^{\infty} e^{-cx^2} \cos(\beta x^2)\, dx = \int_0^{\infty} e^{-cu} \frac{\cos(\beta u)}{\sqrt{u}}\, du =$$
$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\left(c+v^2\right)\, dv}{\beta^2 + \left(c+v^2\right)^2} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{c+\sqrt{\beta^2+c^2}}{\beta^2+c^2}}.$$

[**Problem 1.7.19 (b)**.]

205.
$$\int_0^{\infty} \frac{\sqrt{x}}{x^3+x^2+x+1}\, dx = \frac{\pi}{2}\left(\sqrt{2}-1\right).$$

(**Problem 1.7.22**.)

206.
$$\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+4x+8}\, dx = \frac{\pi\sqrt{2}}{2} \cdot \frac{\sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{\pi\sqrt{6}}{6}\sqrt{2-\sqrt{3}}.$$

(**Problem 1.7.23**.)

207. If $-1 < a < 1$ and $b > 0$ constants,

$$\int_0^{\infty} \frac{x^a}{x^2+b^2}\, dx = \frac{\pi\, b^{a-1}}{2\cos\left(\dfrac{\pi a}{2}\right)}.$$

[**Problem 1.7.24 (a)**.]

208. If $-1 < a < 2$ constant

$$\int_0^{\infty} \frac{x^a}{x^3+1}\, dx = -\frac{\pi\left\{1+2\cos\left[\dfrac{2\pi(a+1)}{3}\right]\right\}}{3\sin(\pi a)}.$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \int_0^\infty \frac{x}{x^3+1}\,dx = \frac{2\pi\sqrt{3}}{9}.$$

[**Problem 1.7.24 (b)**.]

209. If $-1 < p < 1$ and $a > b > 0$ constants,

$$\int_0^\infty \frac{x^p}{(x+a)(x+b)}\,dx = \frac{\pi}{\sin(p\pi)} \cdot \frac{a^p - b^p}{a-b}.$$

[**Problem 1.7.25 (a)**.]

210. If $a > b > 0$ constants,

$$\int_0^\infty \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{a-b} \cdot \ln\left(\frac{a}{b}\right).$$

[**Problem 1.7.25 (b)**.]

211. If $-1 < p < 1$ and $b > 0$ constants,

$$\int_0^\infty \frac{x^p}{(x+b)^2}\,dx = \frac{p\,\pi b^{p-1}}{\sin(p\pi)}.$$

[**Problem 1.7.25 (c)**.]

212. If $-1 < a < 1$ and $0 < \theta < \pi$ constants,

$$\int_0^\infty \frac{x^a}{x^2 + 2x\cos(\theta) + 1}\,dx = \frac{\pi \sin(a\theta)}{\sin(a\pi)\sin\theta}.$$

[**Problem 1.7.26 (a)**.]

213. If $-1 < a < 1$ constant,

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,dx = \frac{a\pi}{\sin(a\pi)}.$$

[**Problem 1.7.26 (c)**.]

214. If $-1 < \alpha < 3$ constant,

$$\int_0^\infty \frac{x^\alpha}{\left(1+x^2\right)^2}\,dx = \frac{\pi}{4}\frac{1-\alpha}{\cos\left(\frac{\alpha\pi}{2}\right)} = \frac{1}{2}\mathrm{B}\left(\frac{\alpha+1}{2}, \frac{3-\alpha}{2}\right).$$

(**Problem 1.7.27**.)

215. If $0 < a \leq b$ and $0 < c < 2$ constants,

$$\int_a^b \left(\frac{b-x}{x-a}\right)^{c-1} \frac{dx}{x} = \frac{\pi}{\sin(c\pi)} \left[1 - \left(\frac{b}{a}\right)^{c-1}\right].$$

(**Problem 1.7.28**.)

216. If $a \neq 0$ and b real constants,

$$\int_{-\infty}^0 e^{-a^2x^2} \cos(bx)\, dx = \int_0^\infty e^{-a^2x^2} \cos(bx)\, dx =$$

$$\frac{1}{2} \int_{-\infty}^\infty e^{-a^2x^2} \cos(bx)\, dx = \frac{\sqrt{\pi}}{2|a|} e^{\frac{-b^2}{4a^2}}.$$

[**Problem 1.7.32 (a)**. **Examples I 2.2.14, 1.7.18**.]

217. If $c \in \mathbb{R}$ constant,

$$\int_{-\infty}^\infty e^{-(x+ic)^2}\, dx = \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$

[**Problem 1.7.32 (b)**.]

218. If $a \neq 0$ and b real constants,

$$\int_0^\infty e^{-a^2x^2} \sin(bx)\, dx = \frac{1}{2a^2}\, e^{-\frac{b^2}{4a^2}} \int_0^b e^{\frac{t^2}{4a^2}}\, dt.$$

(**Problem 1.7.33**.)

219. If $a \neq 0$ and τ real constants,

$$\int_{-\infty}^\infty e^{-a^2(\pm\sigma + i\tau)^2} d\sigma = \frac{\sqrt{\pi}}{a} e^{\tau^2\left(a^2 - \frac{1}{a^2}\right)}.$$

(**Problem 1.7.34**.)

220. If $-1 < \lambda < 1$ constant,

$$\int_{-\infty}^\infty e^{\lambda x} \operatorname{sech}(x)\, dx = \int_{-\infty}^\infty e^{\lambda x} \frac{1}{\cosh(x)}\, dx = \frac{\pi}{\cos\left(\frac{\lambda\pi}{2}\right)}.$$

[**Problems 1.7.35, 1.7.39, (a)**.]

221. If $s > 0$ and $\gamma > 0$ constants,

$$\int_{-\infty}^\infty \frac{e^{s(\gamma+it)}}{\sqrt{\gamma + 1 + it}}\, dt = 2\sqrt{\pi}\ \frac{e^{-s}}{\sqrt{s}},$$

(**Problem 1.7.36**.)

222. If $p > 0$, $q > 0$ and $n \in \mathbb{N}$ constants,

$$\int_0^\infty x^{n-1} e^{-px} \cos(qx)\, dx = \frac{(n-1)!\, \mathrm{Re}[(p+iq)^n]}{(p^2+q^2)^n},$$

$$\int_0^\infty x^{n-1} e^{-px} \sin(qx)\, dx = \frac{(n-1)!\, \mathrm{Im}[(p+iq)^n]}{(p^2+q^2)^n}.$$

(**Problem 1.7.37**.)

223.

$$\int_{-\infty}^0 \frac{dx}{(1+x^2)\cosh(x)} = \int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} =$$

$$\frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^\infty \frac{(-1)^n}{\left(n - \frac{1}{2}\right)^2 \pi^2 - 1}.$$

(**Example 1.7.19**. **Lemma 1.7.3**.)

224.

$$\int_{-\infty}^\infty \frac{x}{(1+x^4)\sinh(x)}\, dx = \int_{-\infty}^\infty \frac{x}{1+x^4}\, \operatorname{csch}(x)\, dx =$$

$$\frac{\pi \sinh\left(\frac{1}{\sqrt{2}}\right) \cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right) + \sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2 \sum_{n=1}^\infty \frac{(-1)^n n}{n^4\pi^4 + 1}.$$

(**Problem 1.7.40**.)

225. If $a > 0$ and $b \in \mathbb{R}$ constants,

$$\int_0^\infty \frac{\cos(bx)}{\cosh(x) + \cosh(a)}\, dx = \frac{\pi \sin(ab)}{\sinh(a)\sinh(\pi b)}.$$

(**Problem 1.7.42**.)

226. If $-\pi < a < \pi$ constant,

$$\int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\, dx = 2\int_0^\infty \frac{\sinh(ax)}{\sinh(\pi x)}\, dx = 2\int_{-\infty}^0 \frac{\sinh(ax)}{\sinh(\pi x)}\, dx =$$

$$\tan\left(\frac{a}{2}\right).$$

If $b > 0$ and $-b < a < b$ constants,

$$\int_{-\infty}^{\infty} \frac{\sinh(ax)}{\sinh(bx)}\,dx = 2\int_{0}^{\infty} \frac{\sinh(ax)}{\sinh(bx)}\,dx = 2\int_{-\infty}^{0} \frac{\sinh(ax)}{\sinh(bx)}\,dx = \frac{\pi}{b}\tan\left(\frac{a\pi}{2b}\right).$$

[**Problems 1.7.43,I 2.3.22 (c)-(d), 1.7.80**, **Example 1.7.25**, **Corollary 1.7.5**.]

227. If $a \neq 0$, b and c real constants,

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = 0 \quad \text{if} \quad b^2 > 4ac.$$

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \begin{cases} \operatorname{sign}(a)\infty, & \text{if} \quad b^2 = 4ac, \\ \dfrac{2\pi}{\sqrt{4ac-b^2}}, & \text{if} \quad b^2 < 4ac. \end{cases}$$

(**Problem 1.7.45**.)

228. If $a \in \mathbb{C}$ such that $|\mathrm{Im}(a)| < 2\pi$ constant,

$$\int_{0}^{\infty} \frac{\sin(ax)}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}.$$

[**Problem 1.7.49 (a)**.]

229. If α and β in $\mathbb{C}$ such that $\mathrm{Re}(\beta) > 0$ and $|\mathrm{Im}(\alpha)| < \mathrm{Re}(\beta)$ constants,

$$\int_{0}^{\infty} \frac{\sin(\alpha x)}{e^{\beta x}-1}\,dx = \frac{\pi}{2\beta}\coth\left(\frac{\alpha\pi}{\beta}\right) - \frac{1}{2a}.$$

[**Problem 1.7.49 (b)**. **Example 1.7.25. Corollary 1.7.5 (A)**.]

230. If α and β in $\mathbb{C}$ such that $\mathrm{Re}(\beta) > 0$ and $|\mathrm{Im}(\alpha)| < \mathrm{Re}(\beta)$ constants,

$$\int_{0}^{\infty} \frac{\sin(\alpha x)}{e^{\beta x}+1}\,dx = -\frac{\pi}{2}\operatorname{csch}\left(\frac{\alpha\pi}{\beta}\right) + \frac{1}{2\alpha}.$$

[**Problem 1.7.49 (d)**. **Example 1.7.25. Corollary 1.7.5 (B)**.]

231. If a and b in $\mathbb{C}$ such that $\mathrm{Re}(b) > 0$ and $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ constants,

$$\int_{0}^{\infty} \frac{\sinh(ax)}{e^{bx}-1}\,dx = -\frac{\pi}{2b}\cot\left(\frac{a\pi}{b}\right) + \frac{1}{2a}.$$

[**Problem 1.7.50 (a)**.]

232. If a and b in $\mathbb{C}$ such that $\operatorname{Re}(b) > 0$ and $|\operatorname{Re}(a)| < \operatorname{Re}(b)$ constants,

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx}+1}\,dx = \frac{\pi}{2b}\csc\left(\frac{a\pi}{b}\right) - \frac{1}{2a}.$$

[**Problem 1.7.50 (b)**.]

233. If $a > 0$, $\mu \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ constants,

$$\text{P.V.}\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\,dx =$$

$$e^{\frac{b^2}{a^2}}\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2}\left\{x\,\cos\left[\frac{2b(x-\mu)}{a^2}\right] - b\,\sin\left[\frac{2b(x-\mu)}{a^2}\right]\right\}dx =$$

$$e^{b^2}\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+(ab)^2}\left\{x\,\cos\left[\frac{2b(x-\mu)}{a}\right] - ab\,\sin\left[\frac{2b(x-\mu)}{a}\right]\right\}dx =$$

$$e\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+a^2}\left\{x\,\cos\left[\frac{2(x-\mu)}{a}\right] - a\,\sin\left[\frac{2(x-\mu)}{a}\right]\right\}dx.$$

(**Problem 1.7.51**.)

234. If $a > 0$, $\mu \in \mathbb{R}$ and $b \in \mathbb{R} - \{0\}$ constants,

$$\int_{-\infty}^\infty \frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2}\left\{b\,\cos\left[\frac{2b(x-\mu)}{a^2}\right] + x\,\sin\left[\frac{2b(x-\mu)}{a^2}\right]\right\}dx =$$

$$\operatorname{sign}(b)\,\pi e^{-\frac{\mu^2+b^2}{a^2}}.$$

[**Problem 1.7.51 (b)**.]

235. If a, b and c such that $\operatorname{Re}(b) > 0$, $|\operatorname{Im}(a)| < \operatorname{Re}(b)$ and $|c| \le 1$ complex constants,

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx = \sum_{n=0}^\infty \frac{a(-c)^n}{a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.25**. **Problem I 2.3.22**.)

236. If b and c such that $\operatorname{Re}(b) > 0$ and $|c| \le 1$ complex constants,

$$\int_0^\infty \frac{x}{e^{bx}+c}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^2},$$

$$\int_0^\infty \frac{x}{e^{bx}-1}\,dx = \frac{1}{b^2}\sum_{n=1}^\infty \frac{1}{n^2} = \frac{1}{b^2}\frac{\pi^2}{6},$$

$$\int_0^\infty \frac{x}{e^{bx}+1}\,dx = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{b^2}.\frac{\pi^2}{12}.$$

[**Example 1.7.25. Corollary 1.7.5 (E)**.]

237. If b and c such that $\mathrm{Re}(b) > 0$ and $|c| \le 1$ complex constants,

$$\int_1^\infty \frac{\ln(u)}{(u^b+c)u}\,du = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2},$$

$$\int_1^\infty \frac{\ln(u)}{(u^b-1)u}\,du = \frac{1}{b^2}\frac{\pi^2}{6},$$

$$\int_1^\infty \frac{\ln(u)}{(u^b+1)u}\,du = \frac{1}{b^2}\frac{\pi^2}{12}.$$

[**Example 1.7.25. Corollary 1.7.5 (F)**.]

238. If b and c such that $\mathrm{Re}(b) > 0$ and $|c| \le 1$ complex constants,

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{c+x^b}\,dx = \frac{1}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1-x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{6},$$

$$\int_0^1 \frac{-x^{b-1}\ln(x)}{1+x^b}\,dx = \frac{1}{b^2}\frac{\pi^2}{12}.$$

[**Example 1.7.25. Corollary 1.7.5 (G)**.]

239. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| \le 1$ complex constants,

$$\int_0^\infty \frac{\sinh(ax)}{e^{bx}+c}\,dx = \sum_{n=0}^{\infty}\frac{a(-c)^n}{-a^2+b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}}{n^2-\left(\frac{a}{b}\right)^2}.$$

[**Example 1.7.25. Corollary 1.7.5 (I)**. **Problem 1.7.78**.]

240. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.26**.)

241.

$$\int_0^\infty \frac{\cos(x)}{e^{x}+1}\,dx = \sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2+1}.$$

(**Example 1.7.26**.)

242. If b such that $\mathrm{Re}(b) > 0$ complex constant and $-1 < c \leq 1$ real constant,

$$\int_0^\infty \frac{1}{e^{bx}+1}\,dx = \int_1^\infty \frac{1}{u(u^b+1)}\,du = \frac{\ln(2)}{b},$$

$$\int_0^\infty \frac{1}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n} = \frac{\ln(1+c)}{bc}.$$

(**Example 1.7.26**. **Problem I 1.2.20**.)

243. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{x\sin(ax)}{e^{bx}+c}\,dx = \frac{2a}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}n}{\left[n^2+\left(\frac{a}{b}\right)^2\right]^2}.$$

(**Example 1.7.26**.)

244. If b and c such that $\mathrm{Re}(b) > 0$, and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{x^2}{e^{bx}+c}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-c)^{n-1}}{n^3},$$

$$\int_0^\infty \frac{x^2}{e^{bx}+1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_0^\infty \frac{x^2}{e^{bx}-1}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 1.7.26**.)

245. If $b > 0$ constant,

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b+1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_1^\infty \frac{\ln^2(u)}{u(u^b-1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 1.7.26**.)

246. If $b > 0$ constant,

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1+x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3},$$

$$\int_0^1 \frac{x^{b-1}\ln^2(x)}{1-x^b}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3}.$$

(**Example 1.7.26**.)

247. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}n}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.26**.)

248. If $a \in \mathbb{R}$, $b > 0$ and $c \geq 1$ real constants,

$$\int_0^\infty \frac{\sin(ax)}{e^{bx}+c}\,dx = \frac{1-\cos\left[\frac{a\ln(c)}{b}\right]}{ac} +$$
$$\frac{a}{c}\sum_{n=1}^\infty \frac{(-1)^n}{(a^2+b^2n^2)c^n} + \frac{2a}{c}\cos\left[\frac{a\ln(c)}{b}\right]\sum_{n=1}^\infty \frac{(-1)^{n-1}}{a^2+b^2n^2}.$$

(**Example 1.7.27**.)

249. If $b > 0$ and $c \geq 1$ real constants,

$$\int_0^\infty \frac{x}{e^{bx}+c}\,dx = \frac{1}{2c}\left[\frac{\ln(c)}{b}\right]^2 + \frac{1}{b^2c}\sum_{n=1}^\infty \frac{(-1)^n}{n^2c^n} + \frac{2}{b^2c}\frac{\pi^2}{12}.$$

(**Example 1.7.27**.)

250.

$$\int_1^\infty \frac{\ln(u)}{(u+1)u}\,du = \frac{\pi^2}{12}.$$

$$\int_1^\infty \frac{\ln(u)}{(u-1)u}\,du = \frac{\pi^2}{6}.$$

$$\int_1^\infty \frac{\ln(u)}{(u^2+1)u}\,du = \frac{\pi^2}{48}.$$

$$\int_1^\infty \frac{\ln(u)}{(u^2-1)u}\,du = \frac{\pi^2}{24}.$$

$$\int_1^\infty \frac{\ln(u)}{u^2-1}\,du = \frac{\pi^2}{8}.$$

(**Problem 1.7.62**.)

251.

$$\int_0^1 \frac{\ln(v)}{1+v}\,dv = \frac{-\pi^2}{12}.$$

$$\int_0^1 \frac{\ln(v)}{1-v}\,dv = \frac{-\pi^2}{6}.$$

$$\int_0^1 \frac{v\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{24}.$$

$$\int_0^1 \frac{\ln(v)}{1-v^2}\,dv = \frac{-\pi^2}{8}.$$

(**Problem 1.7.63**. **Example I 2.4.4**.)

252.

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$$

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = \frac{-\pi^2}{6}.$$

$$\int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = \int_1^\infty \ln\left(\frac{x+1}{x-1}\right)\frac{dx}{x} =$$
$$\int_0^\infty \ln\left(\frac{e^x+1}{e^x-1}\right)\,dx = \int_0^\infty \ln\left[\coth\left(\frac{x}{2}\right)\right]\,dx = \frac{\pi^2}{4}.$$

$$\int_0^1 \frac{x\ln(x)}{1-x}\,dx = 1-\frac{\pi^2}{6}.$$

$$\int_0^1 \ln(x)\ln(1-x)\,dx = 2-\frac{\pi^2}{6}.$$

$$\int_0^1 \ln(x)\ln(1+x)\,dx = 2-2\ln(2)-\frac{\pi^2}{12}.$$

(**Problem 1.7.64**. **Example I 2.4.4**.)

253.

$$\int_0^1 \frac{x^2\ln^2(x)}{1+x^2}\,dx = 2\left(1-\frac{\pi^3}{32}\right).$$

(**Problem 1.7.66**.)

254. If $a > 0$ and $b > 0$ constants,

$$\int_{-\infty}^\infty \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_0^\infty \frac{\cos(ax)}{x^2+b^2}\,dx =$$
$$2\int_{-\infty}^0 \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}.$$

If $b>0$ and $a \in \mathbb{R}$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx =$$
$$2\int_{-\infty}^{0} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-|a|b},$$

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}\,dx = 0.$$

(**Example 1.7.28 Remark**.)

255. $\forall\ n = 1,\ 2,\ 3,\ \ldots$

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^{2n}+1}\,dx = \frac{\pi}{n}\sum_{k=1}^{n} e^{-\sin(t_k)}\cdot \sin[t_k+\cos(t_k)],$$

where $t_k = \dfrac{\pi}{2n}(2k+1)$.

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^{2n}+1}\,dx = 0.$$

(**Example 1.7.28 Remark-footnote**.)

256. If $a \in \mathbb{R}$ and $b>0$ constants,

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = 2\int_0^{\infty} \frac{\sin(ax)}{x(x^2+b^2)}\,dx =$$
$$2\int_{-\infty}^{0} \frac{\sin(ax)}{x(x^2+b^2)}\,dx = \operatorname{sign}(a)\frac{\pi}{b^2}\cdot\left(1-e^{-|a|b}\right).$$

(**Example 1.7.29**.)

257. If $a \geq 0$ and $b>0$ constants,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}e^{-ab}(1+ab).$$

(**Example 1.7.30**.)

258.

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(x)}{x^3-1}\,dx = -\frac{\pi}{3}\,e^{\frac{-\sqrt{3}}{2}}\left[\sqrt{3}\cos\left(\frac{1}{2}\right)+\sin\left(\frac{1}{2}\right)-\sin(1)\right].$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(x)}{x^3-1}\,dx = -\frac{\pi}{3}\left[\cos\left(\frac{1}{2}\right)\sqrt{3}\sin\left(\frac{1}{2}\right)+\cos(1)\right].$$

(**Example 1.7.31**.)

259. If $a \in \mathbb{R}$ and $b > 0$ constants,

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 - b^2}\,dx = \begin{cases} -\operatorname{sign}(a)\,\pi\,\dfrac{\sin(ab)}{b}, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0, \end{cases}$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2 - b^2}\,dx = 0,$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4 - b^4}\,dx = \text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(x^2 - b^2)(x^2 + b^2)}\,dx =$$
$$\frac{1}{2b^2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 - b^2}\,dx - \frac{1}{2b^2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2 + b^2}\,dx =$$
$$\begin{cases} -\dfrac{\pi}{2b^3}\left[\operatorname{sign}(a)\dfrac{\sin(ab)}{b} + e^{-|a|b}\right], & \text{if } a \neq 0, \\ -\dfrac{\pi}{2b^3}, & \text{if } a = 0. \end{cases}$$

(**Example 1.7.32**.)

260. If $a > 0$ and $b > 0$ constants,

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2 + b^2}\,dx = 2\int_{0}^{\infty} \frac{x\sin(ax)}{x^2 + b^2}\,dx =$$
$$2\int_{-\infty}^{0} \frac{x\sin(ax)}{x^2 + b^2}\,dx = \frac{\pi}{e^{ab}}.$$

(**Example 1.7.34 Remark 1**.)

261.

$$\int_{-\infty}^{\infty} \left\{\frac{2[\cos(t) - 1]}{t^2}\right\}^2 dt =$$
$$2\pi\left[\int_{-1}^{0} (1 + x)^2\,dx + \int_{0}^{1} (1 - x)^2\,dx\right] = 2\pi\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{4\pi}{3}.$$

(**Example 1.7.42**.)

262.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos[x\sin(\phi)]\,d\phi =$$

$$\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\cos[x\sin(\phi)]\,d\phi = \frac{1}{\pi}\int_{0}^{\pi}\cos[x\sin(\phi)]\,d\phi =$$

$$J_0(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$

(**Subsection 1.7.8 Application**.)

263.

$$\int_{-\infty}^{\infty}\frac{\sin(x)}{x\left(1+x^4\right)}\,dx = \pi\left[1-e^{-\frac{\sqrt{2}}{2}}\cos\left(\frac{\sqrt{2}}{2}\right)\right].$$

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos(x)}{x\left(1+x^4\right)}\,dx = 0.$$

$$\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2\pm x+1}\,dx = \frac{2\pi\sqrt{3}}{3}\,e^{-\frac{\sqrt{3}}{2}}\cos\left(\frac{1}{2}\right).$$

$$\int_{-\infty}^{\infty}\frac{\sin(x)}{x^2\pm x+1}\,dx = \mp\frac{2\pi\sqrt{3}}{3}\,e^{-\frac{\sqrt{3}}{2}}\sin\left(\frac{1}{2}\right).$$

(**Problem 1.7.88**.)

264. If $a>0$ and $b>0$ constants,

$$\int_{-\infty}^{\infty}\frac{\cos(x)dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} = 2\int_{0}^{\infty}\frac{\cos(x)dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} =$$

$$2\int_{-\infty}^{0}\frac{\cos(x)dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)} = \pi\frac{ae^{-b}-be^{-a}}{ab\left(a^2-b^2\right)}.$$

(**Problem 1.7.89**.)

265. If $a>0$ and $t\in\mathbb{R}$ constants,

$$\int_{-\infty}^{\infty}\frac{\sin(tx)dx}{x\left(x^2+a^2\right)} = 2\int_{0}^{\infty}\frac{\sin(tx)dx}{x\left(x^2+a^2\right)} =$$

$$2\int_{-\infty}^{0}\frac{\sin(tx)dx}{x\left(x^2+a^2\right)} = \text{sign}(t)\cdot\frac{\pi\left(1-e^{-|t|a}\right)}{a^2}.$$

$$\int_{-\infty}^{\infty} \frac{\cos(tx)dx}{(x^2+1)^2} = 2\int_0^{\infty} \frac{\cos(tx)dx}{(x^2+1)^2} = 2\int_{-\infty}^{0} \frac{\cos(tx)dx}{(x^2+1)^2} = \frac{\pi(|t|+1)}{2e^{|t|}}.$$

$$\int_{-\infty}^{\infty} \frac{x\sin(tx)dx}{x^4+4} = 2\int_0^{\infty} \frac{x\sin(tx)dx}{x^4+4} = 2\int_{-\infty}^{0} \frac{x\sin(tx)dx}{x^4+4} = \frac{\pi\sin(t)}{e^{|t|}}.$$

$$\int_{-\infty}^{\infty} \frac{\sin(tx)dx}{x^2+4x+5} = \frac{\pi\sin(2t)}{e^{|t|}}.$$

If $t \in \mathbb{R}$ and $a \neq 0$ constants,

$$\int_{-\infty}^{\infty} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = 2\int_0^{\infty} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = 2\int_{-\infty}^{0} \frac{x\sin(tx)dx}{(x^2+a^2)^2} = \frac{\pi t}{2|a|e^{|at|}}.$$

If $t \in \mathbb{R}$ and $n \in \mathbb{N}$ constants,

$$\int_{-\infty}^{\infty} \frac{1-\cos(tx)}{(2n\pi)^2-(tx)^2}\,dx = 0.$$

[**Problem 1.7.90 (a)-(f)**.]

266. If $a > 0$ constant,

$$\int_0^{\infty} \frac{\cos(ax)-e^{-ax}}{x}\,dx = 0.$$

(**Problem 1.7.93**.)

267. If $a > 0$ constant,

$$\int_0^{\infty} \frac{x^2-a^2}{x^2+a^2}\cdot\frac{\sin(x)}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2-a^2}{x^2+a^2}\cdot\frac{\sin(x)}{x}\,dx = \pi\left(e^{-a}-\frac{1}{2}\right).$$

(**Problem 1.7.95**.)

268. If $a \in \mathbb{R}$ and $b > 0$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\cos(ax)}{x^2 - b^2}\, dx = 0,$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^2 - b^2}\, dx = \begin{cases} \operatorname{sign}(a)\,\pi\cos(ab), & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

(**Problem 1.7.98**.)

269. If $a \in \mathbb{R}$ constant, and $n \in \mathbb{N}$ with $n = 2k$ when n is even and $n = 2k+1$ when n is odd ($k \in \mathbb{N}_0$),

$$\int_0^{\infty} \frac{\sin^n(ax)}{x^n}\, dx =$$

$$= \begin{cases} \dfrac{\pi a^{n-1}}{2^n(n-1)!}\displaystyle\sum_{l=0}^{k}(-1)^l \binom{n}{l}(n-2l)^{n-1}, & \text{if} \quad a > 0, \\ 0, & \text{if} \quad a = 0, \\ -\dfrac{\pi a^{n-1}}{2^n(n-1)!}\displaystyle\sum_{l=0}^{k}(-1)^l \binom{n}{l}(n-2l)^{n-1}, & \text{if} \quad a < 0. \end{cases}$$

(E.g., for $a = 1$ and $n = 5,\ 6,\ 7$, the integral is $\dfrac{115\,\pi}{384}, \dfrac{11\pi}{40}, \dfrac{5887\pi}{23040}$, respectively, and so on.)

(**Problem 1.7.105**.)

270. If $a \in \mathbb{R}$ and $c > 0$ constants,

$$\frac{1}{\pi} \cdot \left[\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{a(c+ti)}}{c+ti}\, dt\right] = \begin{cases} 2, & \text{if} \quad a > 0, \\ 1, & \text{if} \quad a = 0, \\ 0, & \text{if} \quad a < 0. \end{cases}$$

(**Problem 1.7.106**.)

271.

$$\int_{-\infty}^{\infty} \frac{\sin(\pi x)}{x(1-x^2)}\, dx = 2\pi.$$

(**Problem 1.7.108**.)

272. If $n \in \mathbb{N}$,

$$\int_0^\infty \frac{\ln(x)}{x^{2n}+1}\,dx = -\frac{\pi^2}{4n^2}\frac{\cos\left(\frac{\pi}{2n}\right)}{\sin^2\left(\frac{\pi}{2n}\right)}.$$

(**Example 1.7.47 Remark 1**.)

273.

$$\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{\arctan(x)}{x}\,dx =$$
$$\sum_{n=0}^\infty (-1)^n \frac{1}{(2n+1)^2} = 8\sum_{n=0}^\infty \frac{2n+1}{(4n+1)^2(4n+3)^2} :=$$
$$\textbf{Catalan constant } G = 0.915965594\ldots.$$

[**Example 1.7.48**. **Problem 1.7.139 (a)**.]

274. If $-1 < \alpha < 0$,

$$\int_0^\infty \frac{x^\alpha \ln(x)}{x+1}\,dx = \pi^2 \cot(\alpha\pi)\csc(\alpha\pi).$$

(**Examples 1.7.8**, **1.7.47**, **1.7.49**. **Problems I 2.2.48, I 2.6.63, 1.7.142**.)

275. If q and $r \neq 0$ such that $0 < \dfrac{q+1}{r} < 1$ real constants,

$$\int_0^\infty \frac{x^q \ln(x)}{x^r+1}\,dx = -\operatorname{sign}(r)\,\frac{1}{r^2}\,\pi^2 \cot\left(\frac{q+1}{r}\pi\right)\csc\left(\frac{q+1}{r}\pi\right).$$

(**Examples 1.7.8**, **1.7.47**, **1.7.49**. **Problems I 2.2.48, I 2.6.63, 1.7.142, 1.7.145**.)

276.

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi\ln(2)}{8}.$$

[**Problem 1.7.132 (b)**.]

277. If $n = 1,\ 3,\ 5,\ \ldots$ odd positive integer,

$$\int_0^\infty \frac{[\ln(x)]^n}{x^2+1}\,dx = 0.$$

[**Problem 1.7.133 (b)**.]

278.

$$\int_0^{\infty} \frac{\ln(x)}{x^2-1}\,dx = \frac{\pi^2}{4}.$$

(**Problems 1.7.134, 1.7.62, 1.7.63**.)

279. If $a > 0$, $b > 0$, $c > 0$ and $d \geq 0$ constants,

$$\int_0^{\infty} \frac{\ln(cx)}{(ax)^2+b^2}\,dx = \frac{\pi}{2ab}\ln\left(\frac{bc}{a}\right),$$

$$\int_0^{\infty} \frac{\ln(x)}{(x+d)^2+b^2}\,dx = \frac{\ln\left(b^2+d^2\right)}{2b}\cdot\arctan\left(\frac{b}{d}\right),$$

$$\int_0^{\infty} \frac{\ln(cx)}{(ax+d)^2+b^2}\,dx =$$

$$\frac{\ln\left(\frac{c}{a}\right)}{ab}\left[\frac{\pi}{2}-\arctan\left(\frac{d}{b}\right)\right] + \frac{\ln\left(b^2+d^2\right)}{2ab}\cdot\arctan\left(\frac{b}{d}\right).$$

(**Problems 1.7.135, I 2.1.23, I 2.1.30**.)

280.

$$\int_0^{\infty} \frac{\ln^2(x)}{1+x^2}\,dx = 2\int_0^1 \frac{\ln^2(x)}{1+x^2}\,dx = 2\int_1^{\infty} \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{8}.$$

$$\int_{-\infty}^{\infty} \frac{u^2}{\cosh(u)}\,du = \int_{-\infty}^{\infty} u^2\,\mathrm{sech}(u)\,du = \frac{\pi^3}{4}.$$

$$\int_0^1 \frac{\ln^2(x)}{1+x^2}\,dx = \int_1^{\infty} \frac{\ln^2(x)}{1+x^2}\,dx = \frac{\pi^3}{16}.$$

$$\int_0^{\infty} \frac{\ln^4(x)}{1+x^2}\,dx = \frac{5\pi^5}{32}.$$

$$\int_{-\infty}^{\infty} \frac{u^4}{\cosh(u)}\,du = \int_{-\infty}^{\infty} u^4\,\mathrm{sech}(u)\,du = \frac{5\pi^5}{16}.$$

$$\int_0^1 \frac{\ln^4(x)}{1+x^2}\,dx = \int_1^{\infty} \frac{\ln^4(x)}{1+x^2}\,dx = \frac{5\pi^5}{64}.$$

Letting $x = \tan(\theta)$, we find another form of these integrals in logarithm and trigonometric functions.

[**Problem 1.7.136 (a)-(b)**.]

281.

$$\text{If} \quad n \in \mathbb{N}, \qquad \int_0^\infty \frac{\ln^2(x)}{1+x^{2n}}\,dx = \frac{\pi^3}{8n^3}\frac{1+\cos^2\left(\frac{\pi}{2n}\right)}{\sin^3\left(\frac{\pi}{2n}\right)}.$$

(**Problem 1.7.136 footnote**.)

282.

$$\int_0^1\int_0^1 \frac{1}{1+x^2y^2}\,dxdy = G,$$

$$\int_0^{\frac{\pi}{4}} \frac{t}{\sin(t)\cos(t)}\,dt = G,$$

$$\frac{1}{2}\int_0^\infty \frac{x}{\cosh(x)}\,dx = G,$$

$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t}{\sin(t)}\,dt = G,$$

$$\int_0^{\frac{\pi}{4}} \ln[\cot(t)]\,dt = G,$$

$$\int_0^\infty \arctan\left(e^{-t}\right)\,dt = G.$$

where, $\quad G := \int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_0^1 \frac{-\ln(x)}{x^2+1}\,dx =$ $\int_0^1 \frac{\arctan(x)}{x}\,dx \cong 0.915965594\ldots$ is the **Catalan constant**.

[**Problem 1.7.139 (a)**.]

283.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln[\sin(\theta)]\,d\theta = \int_0^{\frac{\pi}{4}} \ln[\cos(\theta)]\,d\theta = \frac{G}{2} - \frac{\pi\ln(2)}{4}.$$

$$\int_0^{\frac{\pi}{4}} \ln[\sin(\theta)]\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln[\cos(\theta)]\,d\theta = -\frac{\pi\ln(2)}{4} - \frac{G}{2}.$$

(G:= **Catalan constant** = 0.915965594... .)

[**Problem 1.7.139 (b)-(c)**.]

284.

$$\int_0^1 \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx - 2\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx = \int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx - 2G.$$

$$\int_0^1 \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \frac{\pi}{2}\ln(2) - G.$$

$$\int_1^\infty \frac{\ln\left(x^2+1\right)}{x^2+1}\,dx = \frac{\pi}{2}\ln(2) + G.$$

(G:= **Catalan constant** = 0.915965594... .)

[**Problem 1.7.140 (a)-(b)**.]

285.

$$\int_0^1 \frac{\arctan^2(x)}{x}\,dx = \frac{\pi}{2}\,G - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} = \frac{\pi}{2}\,G - \frac{7}{8}\zeta(3).$$

(G:= **Catalan constant** = 0.915965594... .)

(**Problem 1.7.141**.)

286.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^3+x^2+x+1} = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{dx}{x^3+x^2+x+1} = \frac{\pi}{4}.$$

[**Problem 1.7.148 (b)-(c)**.]

287.

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+x^3+x^2+x+1} = \frac{4\pi}{5}\sin\left(\frac{2\pi}{5}\right).$$

[**Problem 1.7.149 (a)**.]

288. If $a > 0$ constant,

$$\int_0^\infty \frac{dx}{(x^2+a^2)\left[\ln^2(x)+\pi^2\right]} = \frac{\pi}{a\left[2\ln^2(a)+\frac{\pi^2}{2}\right]} - \frac{1}{1+a^2}.$$

(**Problem 1.7.151**.)

289. If $c > 0$ constant,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z^2}\,dz = \begin{cases}\ln(a), & \text{if} \quad a > 1\\ 0, & \text{if} \quad 0 < a \leq 1.\end{cases}$$

(**Problem 1.7.164**.)

290. If $|a| > 1$ real constant,

$$\int_0^{2\pi}\frac{d\theta}{a+\sin(\theta)} = \int_0^{2\pi}\frac{d\theta}{a+\cos(\theta)} =$$
$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{a+\sin(\theta)} = 2\int_0^{\pi}\frac{d\theta}{a+\cos(\theta)} = \frac{\text{sign}(a)\cdot 2\pi}{\sqrt{a^2-1}}.$$

(**Examples 1.7.14**, **1.8.1 Remarks 2 and 3**.)

291. If $|a| > 1$ real constant,

$$\int_0^{2\pi}\frac{d\theta}{[a+\sin(\theta)]^2} = 2\int_0^{\pi}\frac{d\theta}{[a+\sin(\theta)]^2} =$$
$$\int_0^{2\pi}\frac{d\theta}{[a+\cos(\theta)]^2} = 2\int_0^{\pi}\frac{d\theta}{[a+\cos(\theta)]^2} = \frac{2\pi|a|}{\sqrt{(a^2-1)^3}}.$$

(**Example 1.8.1 Remark 4**.)

292. (a) If $A > 0$ and $B > -A$ real constants,

$$\int_0^{2\pi}\frac{dx}{A+B\sin^2(x)} = 2\int_0^{\pi}\frac{dx}{A+B\sin^2(x)} =$$
$$4\int_0^{\frac{\pi}{2}}\frac{dx}{A+B\sin^2(x)} = \frac{2\pi}{\sqrt{A(A+B)}}\,.$$

(b) If $B > 0$ and $A > -B$ real constants,

$$\int_0^{2\pi}\frac{dx}{A\cos^2(x)+B} = 2\int_0^{\pi}\frac{dx}{A\cos^2(x)+B} =$$
$$4\int_0^{\frac{\pi}{2}}\frac{dx}{A\cos^2(x)+B} = \frac{2\pi}{\sqrt{(A+B)B}}\,.$$

(**Example 1.8.3 Remark 2**.)

293.

$$\int_0^{2\pi} e^{-inx} e^{imx} dx = \begin{cases} 0, & \text{if} \quad m \neq n, \\ 2\pi, & \text{if} \quad m = n. \end{cases}$$

(**Subsection 1.8.2**.)

294. If n integer,

$$\int_0^{2\pi} \cos^2(nx) dx = \begin{cases} 2\pi, & \text{if} \quad n = 0, \\ \pi, & \text{if} \quad n \neq 0. \end{cases}$$

(**Subsection 1.8.2**.)

295. If n integer,

$$\int_0^{2\pi} \sin^2(nx) dx = \begin{cases} 0, & \text{if} \quad n = 0, \\ \pi, & \text{if} \quad n \neq 0. \end{cases}$$

(**Subsection 1.8.2**.)

296. If $m \neq n$ integers,

$$\int_0^{2\pi} \cos(nx) \cos(mx)\, dx = 0.$$

(**Subsection 1.8.2**.)

297. If $m \neq n$ integers,

$$\int_0^{2\pi} \sin(nx) \cos(mx)\, dx = 0.$$

(**Subsection 1.8.2**.)

298. If $m \neq n$ integers,

$$\int_0^{2\pi} \sin(nx) \sin(mx)\, dx = 0.$$

(**Subsection 1.8.2**.)

299.

$$\forall\ n \in \mathbb{Z}, \qquad \int_{-\pi}^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx =$$

$$2\int_{0}^{\pi} \frac{\sin(nx)}{\sin(x)}\,dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \\ 2\pi, & \text{if } n \text{ is odd.} \end{cases}$$

$$\forall\ m \in \mathbb{N}_0, \qquad \int_{0}^{\frac{\pi}{2}} \frac{\sin\left[(2m+1)x\right]}{\sin(x)}\,dx = \frac{\pi}{2}.$$

$$\forall\ m \in \mathbb{N}, \quad \int_{0}^{\frac{\pi}{2}} \frac{\sin(2mx)}{\sin(x)}\,dx = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{m-1}}{2m-1}\right].$$

(**Example 1.8.7**.)

300. If $c > 1$ and $d > 1$ real numbers,

$$\int_{0}^{2\pi} \ln\left[\frac{\pm d + \sin(x)}{\pm c + \sin(x)}\right] dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln\left[\frac{\pm d + \sin(x)}{\pm c + \sin(x)}\right] dx =$$

$$\int_{0}^{2\pi} \ln\left[\frac{\pm d + \cos(x)}{\pm c + \cos(x)}\right] dx = 2\int_{0}^{\pi} \ln\left[\frac{\pm d + \cos(x)}{\pm c + \cos(x)}\right] dx =$$

$$2\pi \cdot \ln\left[\frac{d + \sqrt{d^2 - 1}}{c + \sqrt{c^2 - 1}}\right],$$

where the correspondence of the signs is: $+$ with $+$ and $-$ with $-$.
(**Problem 1.8.1**.)

301. If $-1 \le a \le 1$,

$$\int_{0}^{\frac{\pi}{2}} \frac{dx}{1 + a\cos(x)} = \frac{2}{\sqrt{1 - a^2}} \arctan\left(\sqrt{\frac{1-a}{1+a}}\right) = \frac{\arccos(a)}{\sqrt{1 - a^2}}.$$

(At $a = 1$ the answer is 1. For $a = -1$, we have the equality $\infty = \infty$.)
[**Problem 1.8.2 (a)**.]

302. If $a \ge 0$,

$$\int_{0}^{\frac{\pi}{2}} \frac{\cos(x)dx}{\sin(x) + a\cos(x)} = \frac{a\pi}{2(a^2 + 1)} - \frac{\ln(a)}{a^2 + 1}.$$

(For $a = 0$, we have the equality $\infty = \infty$.)
[**Problem 1.8.2 (b)**.]

303. If $a \geq 1$ constant,

$$\int_0^{\pi} \frac{ax\sin(x)}{1-2a\cos(x)+a^2}\,dx = \pi\ln\left(1+\frac{1}{a}\right).$$

[**Problem 1.8.4 (a)**.]

304. If $0 < a < 1$ constant,

$$\int_0^{\pi} \frac{ax\sin(x)}{1-2a\cos(x)+a^2}\,dx = \pi\ln(1+a).$$

[**Problem 1.8.4 (b)**.]

305. If $|a| \neq 1$ and $n \in \mathbb{N}$ constants,

$$\int_0^{\pi} \frac{\cos(nx)}{1-2a\cos(x)+a^2}\,dx = \begin{cases} \dfrac{\pi a^n}{1-a^2}, & \text{if} \quad |a|<1, \\[2ex] \dfrac{\pi}{a^n\left(a^2-1\right)}, & \text{if} \quad |a|>1. \end{cases}$$

(**Problem 1.8.5**.)

306.

$$\int_0^{2\pi} \frac{dx}{[2+\cos(x)]\cdot[3+\cos(x)]} = 2\pi\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{2}}\right).$$

$$\int_{-\pi}^{\pi} \frac{\cos(x)}{5+4\cos(x)}\,dx = -\frac{\pi}{3}.$$

$$\int_0^{2\pi} \frac{\cos^2(3x)}{5-4\cos(2x)}\,dx = \frac{3\pi}{8}.$$

$$\int_0^{\pi} \frac{1+\cos(x)}{1+\cos^2(x)}\,dx = \frac{\sqrt{\pi}}{2}.$$

(**Problem 1.8.6**.)

307.

$$\int_0^{2\pi} f(e^{i\theta})\cos^2\left(\frac{\theta}{2}\right)\,d\theta = \frac{\pi}{2}\left[2f(0)+f'(0)\right],$$
$$\int_0^{2\pi} f(e^{i\theta})\sin^2\left(\frac{\theta}{2}\right)\,d\theta = \frac{\pi}{2}\left[2f(0)-f'(0)\right].$$

(**Problem 1.8.8**.)

308.

$$\int_0^{\frac{\pi}{2}} \sin(x)\sin(2x)\sin(3x)\,dx = \frac{1}{6}.$$

$$\int_0^{\pi} [x\sin(x)]^2\,dx = \frac{\pi^3}{6} - \frac{\pi}{4}.$$

(**Problem 1.8.9**.)

309. If $a > b > 0$ constants,

$$\int_0^{2\pi} \frac{dt}{[a+b\cos(t)]^2} = 2\int_0^{\pi} \frac{dt}{[a+b\cos(t)]^2} = \frac{2a\pi}{\sqrt{(a^2-b^2)^3}}.$$

[**Problem 1.8.10 (a)**.]

310. If $a > b > 0$ constants,

$$\int_0^{2\pi} \frac{\sin^4(t)dt}{a+b\cos(t)} = 2\int_0^{\pi} \frac{\sin^4(t)dt}{a+b\cos(t)} =$$
$$\frac{2\pi}{b^4}\left[\sqrt{(a^2-b^2)^3} - a^3 + \frac{3}{2}\,ab^2\right].$$

[**Problem 1.8.10 (c)**.]

311. If $a > 0$, $b > 1$ constants and $n = 0,\ 1,\ 2,\ \dots,$

$$\int_0^{2\pi} \frac{\cos(nx)}{b+\cos(x)}\,dx = \frac{(-1)^n 2\pi}{\sqrt{b^2-1}\left(b+\sqrt{b^2-1}\,\right)^n},$$

$$\int_0^{2\pi} \frac{\cos(nx)}{\cosh(a)+\cos(x)}\,dx = \frac{(-1)^n 2\pi}{\sinh(a)} e^{-na}.$$

[**Problem 1.8.11 (a)**.]

312. If $a > b > 0$ constants

$$\int_0^{2\pi} \frac{\sin^2(x)}{a+b\cos(x)}\,dx = \frac{2\pi\left(a-\sqrt{a^2-b^2}\,\right)}{b^2}.$$

[**Problem 1.8.11 (b)**.]

313. If $a > 0$ constant,

$$\int_0^{\frac{\pi}{2}} \frac{dx}{a+\sin^2(x)} = \frac{\pi}{2\sqrt{a(a+1)}}.$$

[**Problem 1.8.11 (c)**.]

314. If $-1 < a < 1$ constant,

$$\int_0^{\pi} \frac{\cos(2x)}{1 - 2a\cos(x) + a^2}\, dx = \frac{\pi a^2}{1 - a^2}.$$

[**Problem 1.8.11 (d)**.]

315. If n non-negative integer,

$$\int_0^{\pi} \cos^n(x)\cos(nx)\, dx \;= \frac{\pi}{2^n},$$

$$\int_0^{\pi} \sin^n(x)\sin(nx)\, dx \;= \frac{\pi}{2^n}\, \sin\left(\frac{n\pi}{2}\right).$$

(**Problem 1.8.14**.)

316. If $0 \le \alpha < 1$ constant,

$$\int_0^{\pi} \cos\left(\frac{\alpha x}{2}\right)\cos^{-\alpha}\left(\frac{x}{2}\right)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos\left(\frac{\alpha x}{2}\right)\cos^{-\alpha}\left(\frac{x}{2}\right)\, dx =$$

$$2\int_0^{\frac{\pi}{2}} \cos(\alpha u)\cos^{-\alpha}(u)\, du = 2^{\alpha}\pi.$$

(**Problem 1.8.15**.)

317. If $a > -1$ and b real constants,

$$\int_0^{\frac{\pi}{2}} \cos^a(x)\cos(bx)\, dx = \frac{\pi\Gamma(a+1)}{2^{a+1}\Gamma\left(1 + \frac{a+b}{2}\right)\Gamma\left(1 + \frac{a-b}{2}\right)}.$$

If the denominator of this fraction becomes $\pm\infty$, then the integral is 0.

(**Problem 1.8.16**.)

318. If $n \in \mathbb{Z}$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(nx)}{\sin^2(x)}\, dx = 2\int_0^{\frac{\pi}{2}} \frac{\sin^2(nx)}{\sin^2(x)}\, dx = |n|\pi.$$

(**Problem 1.8.19**.)

319. If $0 < a < 1$ and $0 < b < 1$ constants,

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{dz}{a^z \sin(\pi z)} = \frac{1}{\pi(1+a)}.$$

(**Problem 1.8.20**.)

320. If $a > 0$ and $-\dfrac{\pi}{2} < a\lambda < \dfrac{\pi}{2}$ constants,

$$\int_0^\infty e^{-r^a \cos(a\lambda)} \cos\left[r^a \sin(a\lambda)\right] dr = \frac{\cos(\lambda)}{a} \cdot \Gamma\left(\frac{1}{a}\right),$$

$$\int_0^\infty e^{-r^a \cos(a\lambda)} \sin\left[r^a \sin(a\lambda)\right] dr = \frac{\sin(\lambda)}{a} \cdot \Gamma\left(\frac{1}{a}\right).$$

(**Problem 1.8.21**.)

321. If $0 < a \leq \pi$, $0 < s < \pi$ constants and $n \in \mathbb{N}_0$,

$$\int_0^a \frac{\cos(nt)}{1-\cos(t)}\, dt = \infty,$$

$$\int_{\pi-a}^\pi \frac{\cos(nt)}{1+\cos(t)}\, dt = \begin{cases} +\infty, & \text{if} \quad n \text{ is even} \\ -\infty, & \text{if} \quad n \text{ is odd}, \end{cases}$$

$$\text{P.V.} \int_0^\pi \frac{\cos(nt)}{\cos(t)-\cos(s)}\, dt = \pi \cdot \frac{\sin(ns)}{\sin s},$$

$$\text{P.V.} \int_0^\pi \frac{\cos(nt)}{\cos(t)}\, dt = \begin{cases} 0, & \text{if } n = 2k \text{ (even)} \\ \pi, & \text{if } n = 4k+1 \\ -\pi, & \text{if } n = 4k+3. \end{cases}$$

(**Problem 1.8.22**.)

2.2 List of Non-elementary Sums and a few Products

Here, we cite a list of all major, non-elementary, general and important real or complex finite or infinite sums that have been evaluated in the text or referred to in problems and footnotes. More sums, usually less general, can be found in various examples and problems presented in the text.

1. **Finite geometric sum** for any $z \in \mathbb{C}$,

$$\sum_{n=0}^{k} z^n = 1 + z + z^2 + z^3 + \ldots + z^k =$$

$$\begin{cases} \dfrac{z^{k+1}-1}{z-1} = \dfrac{1-z^{k+1}}{1-z}, & \text{if} \quad z \neq 1, \\ k+1, & \text{if} \quad z = 1. \end{cases}$$

2.

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} =$$

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)} = \ln(2).$$

[**Example I 1.3.12**. **Problems I 2.3.7 (c), 1.7.74 (a)**. **Subsection 1.5.4**.]

3.

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \ldots =$$

$$4\left(\frac{1}{1 \cdot 3} - \frac{3}{5 \cdot 7} + \frac{5}{9 \cdot 11} - \frac{7}{13 \cdot 15} + \ldots\right) =$$

$$4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(4n+1)(4n+3)} = \frac{\pi\sqrt{2}}{4}.$$

(**Example I 2.2.6, Remark**.)

4. If $n \in \mathbb{N}_0$,

$$\sin^{2n+1}(x) = \frac{(-1)^n}{2^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \sin\{[2(n-k)+1]x\}.$$

[**Problem I 2.2.28 (a)**.]

5. If $n \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{2n+1}{k} = \frac{(2n)!}{(n!)^2} = \binom{2n}{n}.$$

[**Problem I 2.2.28 (a)**.]

6. If $n \in \mathbb{N}$,

$$\cos^{2n+1}(x) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos\{[2(n-k)+1]x\}.$$

[**Problems I 2.2.28 (b), 1.2.16**.]

7. If $n \in \mathbb{N}_0$ and $0 \le m \le n-1$,

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

[**Problem I 2.2.28 (d)**.]

8. If $n \in \mathbb{N}$,

$$\sin^{2n}(x) = \left\{ \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos[2(n-k)x] \right\} + \frac{\binom{2n}{n}}{2^{2n}}.$$

[**Problem I 2.2.28 (e)**.]

9. If $n \in \mathbb{N}$,

$$\cos^{2n}(x) = \left\{ \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[2(n-k)x] \right\} + \frac{\binom{2n}{n}}{2^{2n}}.$$

[**Problems I 2.2.28 (e), 1.2.16**.]

10. If $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} = (-1)^{n+1} \frac{\binom{2n}{n}}{2},$$

$$\text{or} \qquad \left[\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\right] + (-1)^n\frac{\binom{2n}{n}}{2} = 0.$$

[**Problem I 2.2.28 (e)**.]

11. If $n \in \mathbb{N}$,

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}[2(n-k)+1] = 0.$$

[**Problem I 2.2.28 (h)**.]

12. If $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}(n-k)\sin\left[(n-k)\frac{\pi}{2}\right] = (-1)^{n-1}n2^{n-1}.$$

[**Problem I 2.2.28 (i)**.]

13. If $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}(n-k)^2 = 0.$$

[**Problem I 2.2.28 (j)**.]

14. If $\alpha > 0$ constant,

$$\frac{1}{2} < \sum_{n=0}^{\infty}\frac{(-1)^n}{1+\alpha n} = \int_0^1 \frac{1}{1+x^\alpha}\,dx < 1.$$

[**Problem I 2.3.7 (b), 1.7.71 (d)**.]

15.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1+\frac{n}{3}} = 3\left[\ln(2) - \frac{1}{2}\right].$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1+\frac{n}{2}} = 2[1-\ln(2)].$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots =$$

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n)} = \ln(2).$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+2n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

$$= 2\left(\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \dots\right) = 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)(4n+3)} = \frac{\pi}{4}.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+3n} = \frac{1}{3}\left[\ln(2) + \frac{\pi\sqrt{3}}{3}\right].$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{1+4n} = \frac{\sqrt{2}}{8}\left[\ln(3+\sqrt{2}) + \pi\right].$$

[**Problem I 2.3.7 (e)**.]

16.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{6}.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(3n-2)(3n-1)} = \frac{2}{3}\ln(2).$$

[**Problem I 2.3.7 (f)**.]

17. If $\beta < 0$ constant,

$$0 < \sum_{n=0}^{\infty} \frac{(-1)^n}{1-\beta(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1-\beta n} = \int_0^1 \frac{1}{1+x^\beta}\, dx < \frac{1}{2}.$$

[**Problem I 2.3.7 (h)**.]

18.

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \frac{\pi^2}{6} - 1.$$

[**Problem I 2.3.10 (b)**.]

19. If $0 \le x < 1$,

$$\sum_{n=0}^{\infty} \ln\left(1 + x^{2^n}\right) = \ln\left(\frac{1}{1-x}\right).$$

$$\sum_{n=0}^{\infty} \frac{2^n x^{2^n-1}}{1+x^{2^n}} = \frac{1}{1-x}.$$

[**Problem I 2.3.15 (b)**.]

20. If $-1 < x < 1$,

$$\sum_{n=0}^{\infty} \frac{2^n |x|^{2^n-1}}{1+x^{2^n}} = \frac{1}{1-|x|} = \sum_{n=0}^{\infty} |x|^n.$$

[**Problem I 2.3.15 (c)**.]

21.
$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{12} + \frac{3\ln(3)}{4}.$$

[**Problems I 2.3.28, I 2.3.32 (a), 1.7.72 footnote**.]

22. If $p > 0$ and $q > 0$ constants,

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+q} - \frac{1}{n+p}\right) = \sum_{n=0}^{\infty} \frac{p-q}{(n+1)(n+p)} = \int_0^1 \frac{x^{q-1} - x^{p-1}}{1-x}\, dx.$$

[**Problem I 2.3.32 (a)**.]

23.
$$\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{2}\right)} = 4\,[1 - \ln(2)].$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(3n+1)} = \frac{1}{2}\left[\frac{\sqrt{3}\,\pi}{3} + \ln\left(\frac{27}{16}\right)\right].$$

[**Problem I 2.3.33 (4)-(5)**.]

24.
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(**Euler's sum**. **Examples I 2.4.3, 1.7.23. Corollary 1.7.3**. **Problem 1.7.52**.)

25. If $a \geq 0$ constant and $k \in \mathbb{N}$ integer,

$$\int_0^1 \int_0^1 \frac{(xy)^a}{1-xy}\, dxdy = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2}.$$

$$\int_0^1 \int_0^1 \frac{(xy)^a \ln^k(xy)}{1-xy}\, dxdy = (-1)^k (k+1) \sum_{n=1}^{\infty} \frac{1}{(n+a)^{k+2}}.$$

$$\int_0^1 \int_0^1 \frac{\ln^k(xy)}{1-xy}\, dxdy = (-1)^k (k+1) \sum_{n=1}^{\infty} \frac{1}{n^{k+2}} = (-1)^k (k+1)\, \zeta(k+2).$$

(**Example I 2.4.3, Remark**.)

26.

$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2} \quad \left[= \mathrm{Li}_2\left(\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{-\ln(1-t)}{t}\, dt\right].$$

(**Example I 2.4.5**.)

27. If $p > 0$ constant,

$$\Gamma(2p) = \frac{2^{2p-1}}{\sqrt{\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right) = \frac{2^{2p-\frac{1}{2}}}{\sqrt{2\pi}} \cdot \Gamma(p) \cdot \Gamma\left(p + \frac{1}{2}\right).$$

[**Subsection I 2.6.2 (B, 10)**.]

28.

$$3\left[\left(\frac{1}{1} + \frac{1}{5}\right) - \left(\frac{1}{7} + \frac{1}{11}\right) + \left(\frac{1}{13} + \frac{1}{17}\right) - \left(\frac{1}{19} + \frac{1}{23}\right) + \ldots\right] = 18 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(6n+1)(6n+5)} = 6 \sum_{n=0}^{\infty} \frac{(-1)^n (6n+3)}{(6n+1)(6n+5)} = \pi.$$

(**Example I 2.6.22**.)

29.

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^2}{(n+m)!} = \frac{5e}{6}.$$

$$\sum_{n=1}^{\infty} \frac{n}{(n-1)!} = 2e.$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n-1)!} = 0.$$

$$\sum_{k=1}^{\infty} \frac{2k}{(2k-1)!} = \sum_{l=1}^{\infty} \frac{2l-1}{(2l-2)!}.$$

$$\forall\ x \in \mathbb{R}, \qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n \cdot m}{(n+m)!} x^{n+m} = x^2 e^x \left(\frac{x}{6} + \frac{1}{2}\right).$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n \cdot m}{(n+m)!} = \frac{2e}{3}.$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n \cdot m}{(n+m)!}(-3)^{n+m} = 0.$$

(**Subsection I 2.6.3. Application 5**.)

30.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} = +\infty. \qquad \sum_{k=0}^{\infty} \left[\frac{\binom{2k}{k}}{2^{2k}}\right]^2 = \infty.$$

$$\sum_{k=0}^{\infty} \left[\frac{\binom{2k}{k}}{2^{2k}}\right]^3 = \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3} = \frac{\pi}{\Gamma^4\left(\frac{3}{4}\right)} = 1.393203929\ldots.$$

(**Subsection I 2.6.3. Application 7**.)

31. If $p > 1$ constant,

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{x^{p-1}}{e^x - 1}\, dx = \frac{1}{\Gamma(p)} \int_1^{\infty} \frac{\ln^{p-1}(u)}{(u-1)u}\, du =$$
$$\frac{1}{\Gamma(p)} \int_0^1 \frac{[-\ln(v)]^{p-1}}{1-v}\, dv = \frac{1}{\Gamma(p)} \int_0^1 \frac{[-\ln(1-t)]^{p-1}}{t}\, dt.$$

(**Problems I 2.6.7**, **1.7.86**.)

32.

$$\sum_{k=0}^{n} \frac{(-1)^k}{m+k+1}\, nCk = \sum_{l=0}^{m} \frac{(-1)^l}{n+l+1}\, mCl = \frac{m!\,n!}{(m+n+1)!} =$$
$$\frac{1}{(m+n+1)\cdot(m+n)Cm} = \frac{1}{(m+n+1)\cdot(m+n)Cn}.$$

(**Problem I 2.6.15**.)

33. If $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{2n+1}{k}}{2(n-k)+1} =$$

$$(-1)^n 2^{2n} \frac{2 \cdot 4 \cdot 6 \ldots 2n}{1 \cdot 3 \cdot 5 \ldots (2n+1)} = (-1)^n \frac{2^{4n}(n!)^2}{(2n+1)!}.$$

(**Problem I 2.6.26**.)

34. If $n \in \mathbb{N}$,

$$\sum_{n=0}^{\infty} \frac{(n+1)(n!)^2}{(2n)!} = \sum_{n=0}^{\infty} \frac{n+1}{\binom{2n}{n}} = 2 + \frac{4\sqrt{3}\pi}{27}.$$

$$\sum_{n=0}^{\infty} \frac{(n+1)2^n(n!)^2}{(2n)!} = \sum_{n=0}^{\infty} \frac{(n+1)2^n}{\binom{2n}{n}} = 5 + \frac{3\pi}{2}.$$

(**Problem I 2.6.30**.)

35. If $p > 0$ constant,

$$\Gamma(3p) = \frac{3^{3p-\frac{1}{2}}}{2\pi}\, \Gamma(p)\, \Gamma\left(p + \frac{1}{3}\right)\, \Gamma\left(p + \frac{2}{3}\right).$$

(**Problem I 2.6.45**.)

36. If $p > 0$ constant and $n \geq 1$ integer,

$$\Gamma(np) = \frac{n^{np-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\, \Gamma(p)\, \Gamma\left(p + \frac{1}{n}\right) \Gamma\left(p + \frac{2}{n}\right) \ldots \Gamma\left(p + \frac{n-1}{n}\right).$$

[**Problems I 2.6.45 footnote, I 2.6.65 Item (13.)**.]

37. If $z \in \mathbb{C}$,

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}}\right],$$

where $\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problems I 2.6.45 footnote, I 2.6.65 Item (8.)**.]

38. If a and b real constants,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a^n b^m}{(n+m)!} = \begin{cases} \dfrac{ae^b - be^a}{b-a} + 1, & \text{if} \quad a \neq b, \\ (a-1)e^a + 1, & \text{if} \quad a = b. \end{cases}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(n+m)!} = 1.$$

(**Problem I 2.6.60**.)

39. For any $p > 0$,

$$\Gamma(p) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+p)} + \sum_{n=0}^{\infty} c_n p^n,$$

where $c_n = \dfrac{1}{n!}\displaystyle\int_1^{\infty} x^{-1}e^{-x}\ln^n(x)\,dx$.
(**Problems I 2.6.62, 1.2.38**.)

40. If m and n integers such that $0 \leq m < n-1$,

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n-m+k-1} = \frac{m!(n-m-2)!}{(n-1)!} = \sum_{k=0}^{n-m-2}\binom{n-m-2}{k}\frac{(-1)^k}{m+k+1}.$$

[**Problems I 2.6.64, I 1.3.25 (a)**.]

41. If $x \in \mathbb{R}$ and $p \neq 0$ constants, let $\left[\left[\dfrac{x}{p}\right]\right]$ be the **integer part** or **floor function** of $\dfrac{x}{p}$ and H_b or H the **Heaviside functions** defined in **Problem I 2.7.19 (a)**,

$$\left[\left[\frac{x}{p}\right]\right] = \sum_{n=1}^{\infty} H_{pn}(x) = \sum_{n=1}^{\infty} H(x - pn).$$

[**Problem I 2.7.19 (g)**.]

42. If $n \geq 0$ and $m \geq 1$ integers and x real,

$$\cos(nx) = \sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k} \cos^{n-2k}(x) \sin^{2k}(x) =$$

$$\sin^n(x) \sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k} \cot^{n-2k}(x).$$

$$\sum_{k=0}^{m-1} \cot^2 \left[\frac{(1+2k)\pi}{4m}\right] = m(2m-1).$$

$$\sum_{k=0}^{m-1} \cot^4 \left[\frac{(1+2k)\pi}{4m}\right] = \frac{m(2m-1)(4m^2+2m-3)}{3}.$$

$$\prod_{k=0}^{m-1} \cot^2 \left[\frac{(1+2k)\pi}{4m}\right] = 1.$$

[**Problem 1.1.46 (II)**.]

43. If $n \geq 0$ and $m \geq 1$ integers and x real,

$$\sin(nx) = \sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(x) \sin^{2k+1}(x) =$$

$$\sin^n(x) \sum_{k=0}^{\llbracket \frac{n}{2} \rrbracket} (-1)^k \binom{n}{2k+1} \cot^{n-2k-1}(x).$$

$$\sum_{k=1}^{m} \cot^2 \left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

$$\sum_{k=1}^{m} \cot^4 \left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

$$\prod_{k=1}^{m} \cot^2 \left(\frac{k\pi}{2m+1}\right) = \frac{1}{2m+1}.$$

[**Problem 1.1.46 (II)**.]

44. If $n \geq 0$ integer and x real,

$$\tan(nx) = \frac{\sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k+1} \tan^{2k+1}(x)}{\sum_{k=0}^{[\![\frac{n}{2}]\!]} (-1)^k \binom{n}{2k} \tan^{2k}(x)}.$$

[**Problem 1.1.46 (II)**.]

45. **Infinite geometric sum**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots, \quad \forall\, z: \ |z| < 1.$$

(**Example 1.2.1**. **Subsection 3.5.1**.)

46.
$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad \forall\, z: \ |z| < 1.$$

(**Example 1.2.1**.)

47.
$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad \forall\, z: \ |z| < 1.$$

(**Example 1.2.1**.)

48.
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots, \quad \forall\, z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

49.
$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots, \forall\, z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

50.
$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots, \quad \forall\, z \in \mathbb{C}.$$

(**Section 3.2**. **Subsection 3.5.1**.)

51.

$$\arctan(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots,$$
$$\forall\, z : \; |z| < 1.$$

(**Section 3.2**. **Subsection 3.5.1**.)

52. If $r \in \mathbb{R}$ and all $\theta \in \mathbb{R}$,

$$e^{r[\cos(\theta)+i\sin(\theta)]} = \sum_{n=0}^{\infty} \frac{r^n[\cos(n\theta) + i\sin(n\theta)]}{n!},$$

$$e^{r\cos(\theta)}\{\cos[r\sin(\theta)] + i\sin[r\sin(\theta)]\} =$$
$$\sum_{n=0}^{\infty} \frac{r^n\cos(n\theta)}{n!} + i\sum_{n=0}^{\infty} \frac{r^n\sin(n\theta)}{n!}.$$

$$e^{r\cos(\theta)}\cos[r\sin(\theta)] = \sum_{n=0}^{\infty} \frac{r^n\cos(n\theta)}{n!},$$

$$e^{r\cos(\theta)}\sin[r\sin(\theta)] = \sum_{n=0}^{\infty} \frac{r^n\sin(n\theta)}{n!}.$$

(**Subsection 1.2.1**.)

53.

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^4 + 2n^3 + n^2}{4}.$$

If $x \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} x^n = x\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} + \frac{7}{2}x^2 \sum_{n=1}^{\infty} \frac{x^{n-2}}{(n-2)!} +$$
$$2x^3 \sum_{n=1}^{\infty} \frac{x^{n-3}}{(n-3)!} + \frac{1}{4}x^4 \sum_{n=1}^{\infty} \frac{x^{n-4}}{(n-4)!} = \left(x + \frac{7}{2}x^2 + 2x^3 + \frac{1}{4}x^4\right) e^x.$$

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot 2^{\frac{n}{2}} = (8 + 5\sqrt{2})e^{\sqrt{2}}.$$

If r, θ real constants,

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot r^n \cos(n\theta) =$$
$$e^{r\cos(\theta)} \left\{ r\cos[\theta + r\sin(\theta)] + \frac{7}{2} r^2 \cos[2\theta + r\sin(\theta)] + \right.$$
$$\left. 2r^3 \cos[3\theta + r\sin(\theta)] + \frac{1}{4} r^4 \cos[4\theta + r\sin(\theta)] \right\}$$

$$\sum_{n=1}^{\infty} \frac{1^3 + 2^3 + \ldots + n^3}{n!} \cdot r^n \sin(n\theta) =$$
$$e^{r\cos(\theta)} \left\{ r\sin[\theta + r\sin(\theta)] + \frac{7}{2} r^2 \sin[2\theta + r\sin(\theta)] + \right.$$
$$\left. 2r^3 \sin[3\theta + r\sin(\theta)] + \frac{1}{4} r^4 \sin[4\theta + r\sin(\theta)] \right\}$$

(**Example 1.2.2**.)

54.
$$\sum_{n=0}^{\infty} \frac{n^3}{n!} = 5\,e.$$

(**Problem 1.2.6**.)

55. If $-1 < x < 1$,

$$\arctan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots ,$$

$$\arctan^2(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \left(1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} \right) \frac{x^{2n}}{n} =$$
$$x^2 - \left(1 + \frac{1}{3} \right) \frac{x^4}{2} + \left(1 + \frac{1}{3} + \frac{1}{5} \right) \frac{x^6}{2} - \ldots .$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots .$$
$$\frac{\pi^2}{16} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} \right) \frac{1}{n}.$$

(**Problem 1.2.7**. **Subsection 1.5.4**. **Example 1.7.24**.)

56. If $|z| \neq 1$ complex number,

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \begin{cases} \dfrac{1}{(1-z)^2}, & \text{if } |z| < 1, \\ \\ \dfrac{1}{z(1-z)^2}, & \text{if } |z| > 1. \end{cases}$$

(**Problem 1.2.5**.)

57. If $n \geq 0$ integer and x real,

$$\sum_{k=0}^{n} \sin(kx) = \sum_{k=1}^{n} \sin(kx) =$$
$$\frac{\sin\left[\frac{(n+1)x}{2}\right] \sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{\cos\left(\frac{x}{2}\right) - \cos\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

[**Problems 1.2.15 (f), 1.2.27 (c)**.]

58. If $n \geq 0$ integer and x real,

$$\sum_{k=0}^{n} \cos(kx) = \frac{\sin\left[\frac{(n+1)x}{2}\right] \cos\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(nx + \frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)},$$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{\sin\left(nx + \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

[**Problems 1.2.15 (g), 1.2.27 (c)**.]

59. If $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$,

$$\cos^{2n+1}(z) = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos\{[2(n-k)+1]z\}.$$

If $n \in \mathbb{N}$ and $z \in \mathbb{C}$,

$$\cos^{2n}(z) = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[2(n-k)z] + \frac{\binom{2n}{n}}{2^{2n}}.$$

[**Problems 1.2.16, I 2.2.28 (b) (e)**.]

60. If $z \in \mathbb{C}$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + \frac{z^{2n+1}}{(2n+1)!} + \ldots ,$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \frac{z^{2n}}{(2n)!} + \ldots .$$

(**Problem 1.2.17**. **Subsection 3.5.1**.)

61. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin[a + (k-1)d] = \frac{\sin\left[a + \frac{(n-1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\cos\left(a - \frac{d}{2}\right) - \cos\left(a + nd - \frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

$$\sum_{k=1}^{n} \cos[a + (k-1)d] = \frac{\cos\left[a + \frac{(n-1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\sin\left(a + nd - \frac{d}{2}\right) - \sin\left(a - \frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

[**Problem 1.2.27 (a)**.]

62. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin(a + kd) = \frac{\sin\left[a + \frac{(n+1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\cos\left(a + \frac{d}{2}\right) - \cos\left(a + nd + \frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

$$\sum_{k=1}^{n} \cos(a + kd) = \frac{\cos\left[a + \frac{(n+1)d}{2}\right] \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} = \frac{\sin\left(a + nd + \frac{d}{2}\right) - \sin\left(a + \frac{d}{2}\right)}{2\sin\left(\frac{d}{2}\right)}.$$

[**Problem 1.2.27 (b)**.]

63. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin(ka) = \frac{\sin\left[\frac{(n+1)a}{2}\right] \sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)} = \frac{\cos\left(\frac{a}{2}\right) - \cos\left(na + \frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)}.$$

$$\sum_{k=1}^{n} \cos(ka) = \frac{\cos\left[\frac{(n+1)a}{2}\right] \sin\left(\frac{na}{2}\right)}{\sin\left(\frac{a}{2}\right)} = \frac{\sin\left(na + \frac{a}{2}\right) - \sin\left(\frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)}.$$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(ka) = \frac{\sin\left(na + \frac{a}{2}\right)}{2\sin\left(\frac{a}{2}\right)}.$$

$$\sum_{k=1}^{n} k\sin(kx) = \frac{(n+1)\sin(nx) - n\sin[(n+1)x]}{4\sin^2\left(\frac{x}{2}\right)}.$$

$$\sum_{k=1}^{n} k\cos(kx) = \frac{(n+1)\cos(nx) - n\cos[(n+1)x] - 1}{4\sin^2\left(\frac{x}{2}\right)}.$$

[**Problem 1.2.27 (c)**.]

64. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin[(2k-1)a] = \frac{\sin^2(na)}{\sin(a)}.$$

$$\sum_{k=1}^{n} \cos[(2k-1)a] = \frac{\sin(2na)}{2\sin(a)}.$$

[**Problem 1.2.27 (d)**.]

65. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin(2ka) = \frac{\sin[(n+1)a]\sin(na)}{\sin(a)} = \frac{\cos(a) - \cos[(2n+1)a]}{2\sin(a)}.$$

$$\sum_{k=1}^{n} \cos(2ka) = \frac{\cos[(n+1)a]\sin(na)}{\sin(a)} = \frac{\sin[(2n+1)a] - \sin(a)}{2\sin(a)}.$$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(2ka) = \frac{\sin\left[(2n+1)a\right]}{2\sin(a)}.$$

[**Problem 1.2.27 (e)**.]

66. If $a \in \mathbb{C}$, $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2[a+(k-1)d] = \frac{n}{2} - \frac{\cos\left[2a+(n-1)d\right]\sin(nd)}{2\sin(d)} =$$
$$\frac{n}{2} - \frac{\sin\left[2a+(2n-1)d\right]+\sin(2a-d)}{4\sin(d)}.$$

[**Problem 1.2.28 (a)**.]

67. If $a \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2(ka) =$$
$$\frac{n}{2} - \frac{\cos[(n+1)a]\sin(na)}{2\sin(a)} = \frac{n}{2} - \frac{\sin[(2n+1)a]+\sin(a)}{4\sin(a)}.$$

$$\sum_{k=1}^{n} \sin^2[(2k-1)a] = \frac{n}{2} - \frac{\cos(2na)\sin(2na)}{2\sin(2a)} = \frac{n}{2} - \frac{\sin(4na)}{4\sin(2a)}.$$

[**Problem 1.2.28 (b)**.]

68. If $d \in \mathbb{C}$ and $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \sin^2(2kd) =$$
$$\frac{n}{2} - \frac{\cos[2(n+1)d]\sin(2nd)}{2\sin(2d)} = \frac{n}{2} - \frac{\sin[2(2n+1)d]+\sin(d)}{4\sin(2d)}.$$

[**Problem 1.2.28 (c)**.]

69. If $-\frac{\pi}{6} < x < \frac{\pi}{6}$,

$$\sum_{n=1}^{\infty} 2^{n-1}\sin^n(x) = \frac{\sin(x)}{1-2\sin(x)}.$$

(**Problem 1.2.29**.)

70.
$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2^{n+2}}\right) \cos\left(\frac{3\pi}{2^{n+2}}\right) = \frac{1}{2}.$$

[**Problem 1.2.30 (a)**.]

71. If $z \in \mathbb{C}$,
$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan\left(\frac{z}{2^n}\right) = \frac{1}{z} - \cot(z).$$

[**Problem 1.2.30 (b)**.]

72. If $x \in \mathbb{R}$,
$$\sum_{n=1}^{\infty} \arctan\left[\frac{x}{1+n(n+1)x^2}\right] =$$

$$\begin{cases} \dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right), & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -\dfrac{\pi}{2} - \arctan(x) = \arctan\left(\dfrac{1}{x}\right), & \text{if} \quad x < 0. \end{cases}$$

[**Problem 1.2.31 (b)**.]

73.
$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{2n^2}\right) = \frac{\pi}{4}.$$

[**Problem 1.2.31 (c)**.]

74. If $n \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \ldots\ \omega_{n-1}$ are the n n^{th} roots of 1,

$$\omega_0^j + \omega_1^j + \ldots + \omega_{n-1}^j = \begin{cases} 0, & \text{if} \quad j \in \{1,\ 2,\ \ldots\ n-1\} + n\,\mathbb{Z}, \\ n, & \text{if} \quad j \in n\,\mathbb{Z}. \end{cases}$$

[**Problem 1.2.32 (b)**.]

75. If $n \in \mathbb{N}$ and $\omega_0,\ \omega_1,\ \dots\ \omega_{n-1}$ are the n n^{th} roots of 1,

$$\omega_0 + \omega_1 + \ldots + \omega_{n-1} = 0.$$

$$\omega_0\omega_1 + \omega_0\omega_2 + \ldots + \omega_{n-2}\omega_{n-1} = 0.$$

$$\omega_0\omega_1\omega_2 + \ldots + \omega_{n-3}\omega_{n-2}\omega_{n-1} = 0.$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\omega_0\omega_1 \ldots \omega_{n-2} + \ldots + \omega_1\omega_2 \ldots \omega_{n-1} = 0.$$

$$\omega_0\omega_1 \ldots \omega_{n-1} = (-1)^{n+1}.$$

[**Problem 1.2.33 (a)**.]

76. If $n \in \mathbb{N}$ and $m = \left[\!\left[\frac{n}{2}\right]\!\right]$ be the integer part of $\frac{n}{2}$,

$$\prod_{k=1}^{m} \sin\left(\frac{k\pi}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}.$$

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

[**Problem 1.2.34 (c)**.]

77.

$$\arccos(z) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \frac{z^{2n+1}}{2n+1} =$$

$$\frac{\pi}{2} - \frac{z}{1} - \frac{1}{2}\cdot\frac{z^3}{3} - \frac{1\cdot 3}{2\cdot 4}\cdot\frac{z^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{z^7}{7} - \ldots, \quad \forall\ z:\ |z| < 1.$$

(**Subsection 1.5.1**.)

78.

$$\arcsin(z) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}+n-1}{n} \frac{z^{2n+1}}{2n+1} =$$

$$\frac{z}{1} + \frac{1}{2}\cdot\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{z^7}{7} - \ldots, \quad \forall\ z:\ |z| < 1.$$

(**Subsection 1.5.1**.)

79. **Zero branch of complex logarithm**,

$$\log_{(0)}(1+z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots,$$
$$\forall\ z:\ |z| < 1.$$

(**Subsection 1.5.3**. **Section 1.2**.)

80. **Zero branch of complex binomial series**,

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty}\binom{\alpha}{n} z^n, \quad \alpha \in \mathbb{C}, \quad \forall\ z:\ |z| < 1,$$

where $\forall \quad n \in \mathbb{N}$,

$$\binom{\alpha}{0} = 1 \quad \text{and} \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}.$$

(**Subsection 1.5.5**. **Problem I 2.6.59**.)

81. If $r,\ \ \theta$ real constants,

$$\frac{1}{2}\ln\left[1 + 2r\cos(\theta) + r^2\right] = \sum_{n=1}^{\infty}(-1)^{n-1} r^n \frac{\cos(n\theta)}{n}.$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(n\theta)}{n} = \frac{1}{2}\ln\left[2 + 2\cos(\theta)\right] =$$
$$\ln\left[2\cos\left(\frac{\theta}{2}\right)\right] = \ln(2) + \ln\left[\cos\left(\frac{\theta}{2}\right)\right].$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(n\theta)}{n} = \frac{\theta}{2}.$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2n\theta)}{n} = \ln(2) + \ln[\cos(\theta)].$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin^2(n\theta)}{n} = -\frac{1}{2}\ln[\cos(\theta)].$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos^2(n\theta)}{n} = \ln(2) + \frac{1}{2}\ln[\cos(\theta)].$$

(**Subsection 1.5.4**.)

82. If $0 \le r < a$ and $\theta,\ \phi$ are real constants,

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)] = -1 + 2\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n \cos[n(\theta-\phi)] =$$

$$P_{(a,\phi)}(r,\theta) = \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta-\phi)}.$$

(**Definition 1.5.3 of Poisson kernel and Appendix 1.5.9**.)

83. If $k \in \mathbb{N}_0$,

$$\pi = (k+2)\cdot\sin\left(\frac{\pi}{k+2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n[2(k+2)n+k+2]}{[(k+2)n+1][(k+2)n+k+1]}.$$

[**Problem 1.7.14 (d)**.]

84.

$$\int_{-\infty}^{\infty}\frac{x}{(1+x^4)\sinh(x)}\,dx = \int_{-\infty}^{\infty}\frac{x}{1+x^4}\ \operatorname{csch}(x)\,dx =$$

$$\frac{\pi\,\sinh\left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{1}{\sqrt{2}}\right)}{\sinh^2\left(\frac{1}{\sqrt{2}}\right)+\sin^2\left(\frac{1}{\sqrt{2}}\right)} - 2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^4\pi^4+1}.$$

(**Problem 1.7.40**.)

85. If $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$,

$$\tan(x) = 8x\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\pi^2 - 4x^2}.$$

[**Problem 1.7.43 (c)**.]

86. For any $a \in \mathbb{C}$ such that $a \ne ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer) constants,

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a).$$

(**Example 1.7.23**.)

87. For any $a \in \mathbb{C}$ such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., $a\,i$ is not an integer) constants,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a} \left[\pi \coth(\pi a) - \frac{1}{a} \right].$$

For any $z \in \mathbb{C}$,

$$\coth(z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 + z^2}.$$

(**Example 1.7.23**.)

88.
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(**Example 1.7.23 footnote**.)

89.
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

(**Example 1.7.23 footnote**.)

90. If $a \notin \mathbb{Z}$ constant,

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} = \frac{1}{2a} \left[\frac{1}{a} - \pi \cot(\pi a) \right].$$

If $z \in \mathbb{C}$,

$$\cot(z) = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 - z^2}.$$

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

$$\left[\frac{\pi}{\sin(\pi z)} \right]^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}. \qquad \pi^2 = \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{1}{2} - n\right)^2}.$$

[**Examples 1.7.23, 1.7.24. Corollary 1.7.4**. **Problems 1.7.53**, **1.7.83 (a)**.]

91. If $a \notin \mathbb{Z}$ constant,

$$\pi\cot(\pi a) - \frac{1}{a} = \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a}{n(a-n)}.$$

$$\pi\cot(\pi a) - \frac{1}{a} = \lim_{m\to\infty} \sum_{\substack{n=-m \\ n\neq 0}}^{m} \frac{1}{a-n}.$$

$$\pi\cot(\pi a) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{a-n}.$$

$$\frac{\pi}{\sin(\pi a)} = \pi\csc(\pi a) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-n}.$$

$$\pi = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{1}{2} - n} = 2\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} + \sum_{n=-1}^{-\infty} \frac{(-1)^{n+1}}{2n-1}\right] = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right).$$

$$\pi = 2 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{1}{4} - n^2} = 2 + 2\left(\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots\right).$$

[**Example 1.7.24**. **Problem 1.7.83 (a)**.]

92. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| \leq 1$ complex constants,

$$\int_0^{\infty} \frac{\sin(ax)}{e^{bx} + c}\, dx = a\sum_{n=0}^{\infty} (-c)^n \frac{1}{a^2 + b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^2 + \left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.25**.)

93. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| \leq 1$ complex constants,

$$\int_0^{\infty} \frac{\sinh(ax)}{e^{bx} + c}\, dx = \sum_{n=0}^{\infty} \frac{a(-c)^n}{-a^2 + b^2(n+1)^2} = \frac{a}{b^2}\sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^2 - \left(\frac{a}{b}\right)^2}.$$

[**Example 1.7.25. Corollary 1.7.5 (I)**. **Problem 1.7.78**.]

94. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Im}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cos(ax)}{e^{bx}+c}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-c)^{n-1}n}{n^2+\left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.26**.)

95. If a, b and c such that $\mathrm{Re}(b) > 0$, $|\mathrm{Re}(a)| < \mathrm{Re}(b)$ and $|c| < 1$ complex constants,

$$\int_0^\infty \frac{\cosh(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2-\left(\frac{a}{b}\right)^2}.$$

(**Example 1.7.26**.)

96.

$$\sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{\pi^2}{24}. \qquad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

$$\sum_{n=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}. \qquad \sum_{k=1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} = \frac{\pi^2}{2}.$$

$$\sum_{k=0}^{\infty}\frac{2k+1}{k^2(k+1)} = \frac{\pi^2}{6}+1.$$

(**Problem 1.7.55**. **Example I 2.3.23**.)

97.

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}-\frac{1}{4k+3}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2-\left(\frac{1}{2}\right)^2} = \frac{\pi}{4}.$$

[**Problems 1.7.56, 1.7.71 (d)**.]

98. If $a \neq ni$ with $n \in \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+a^2} = \frac{\pi}{a}\,\mathrm{csch}(\pi a).$$

[**Problem 1.7.59 (a)**.]

99. If $a \neq ni$ with $n \in \mathbb{Z}$ complex constant,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{2a} \operatorname{csch}(\pi a) - \frac{1}{2a^2}.$$

$$\operatorname{csch}(z) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 + z^2}.$$

[**Problem 1.7.59 (a)**.]

100. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 - a^2} = \frac{-\pi}{a} \csc(\pi a).$$

[**Problem 1.7.59 (b)**.]

101. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi}{2a} \csc(\pi a).$$

$$\csc(z) = \frac{1}{z} - 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2 - z^2}.$$

[**Problem 1.7.59 (b)**.]

102.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

[**Problem 1.7.59 (c)**.]

103.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n + \frac{1}{2}\right)^3} = \frac{\pi^3}{32}.$$

(**Problem 1.7.65**.)

104.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n + \frac{1}{2}\right)^3} = \frac{\pi^3}{4}.$$

(**Problem 1.7.65**.)

105. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \pi^2 \csc^2(\pi a) = \frac{\pi^2}{\sin^2(\pi a)}.$$

[**Problem 1.7.69 (a)**.]

106. If $a \notin \mathbb{Z}$ complex constant,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \pi^2 \csc(\pi a)\cot(\pi a) = \frac{\pi^2 \cos(\pi a)}{\sin^2(\pi a)}.$$

[**Problem 1.7.69 (b)**.]

107. If $a \notin \mathbb{Z}$ and $b \notin \mathbb{Z}$ complex constants,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{\pi}{b-a}\left[\cot(\pi a) - \cot(\pi b)\right].$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2 + a^2} = \frac{\pi}{2ai}\left\{\cot[\pi(z-ai)] - \cot[\pi(z+ai)]\right\}.$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2 - a^2} = \frac{\pi}{2a}\left\{\cot[\pi(z-a)] - \cot[\pi(z+a)]\right\}.$$

[**Problem 1.7.70 (a)-(b)**.]

108. If $0 < a < 1$ constant,

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)(n+1-a)} = \frac{\pi}{1-2a}\cot(\pi a).$$

[**Problem 1.7.70 (d)**.]

109. If $a \in \mathbb{C} - \mathbb{Z}$ constant,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+a} = \pi \cot(\pi a).$$

[**Problem 1.7.70 (e)**.]

110. If $a \in \mathbb{C} - \mathbb{Z}$, $b \in \mathbb{C} - \mathbb{Z}$ and $c \in \mathbb{C} - \mathbb{Z}$ constants

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)(n+b)(n+c)} =$$
$$\pi \left[\frac{\cot(\pi a)}{(b-a)(c-a)} + \frac{\cot(\pi b)}{(a-b)(c-b)} + \frac{\cot(\pi c)}{(a-c)(b-c)} \right].$$

[**Problem 1.7.70 (f)**.]

111. If $a \notin \mathbb{Z}$ and $b \notin \mathbb{Z}$ complex constants,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)} = \frac{\pi}{b-a} \left[\csc(\pi a) - \csc(\pi b) \right].$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2 + a^2} = \frac{\pi}{2ai} \left\{ \csc[\pi(z-ai)] - \csc[\pi(z+ai)] \right\}.$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+z)^2 - a^2} = \frac{\pi}{2a} \left\{ \csc[\pi(z-a)] - \csc[\pi(z+a)] \right\}.$$

[**Problem 1.7.71 (a)-(b)**.]

112. If $a \notin \mathbb{Z}$ constant,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+a} = \pi \csc(\pi a).$$

[**Problems 1.7.71 (d), I 2.3.7 (b)**.]

113. If $a \in \mathbb{C} - \mathbb{Z}$, $b \in \mathbb{C} - \mathbb{Z}$ and $c \in \mathbb{C} - \mathbb{Z}$ constants,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)(n+b)(n+c)} =$$
$$\pi \left[\frac{\csc(\pi a)}{(b-a)(c-a)} + \frac{\csc(\pi b)}{(a-b)(c-b)} + \frac{\csc(\pi c)}{(a-c)(b-c)} \right].$$

[**Problem 1.7.71 (e)**.]

114.
$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)(3n+1)} = \frac{\sqrt{3}\,\pi}{3}.$$

(**Problem 1.7.72**.)

115.

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(3k+2)} = \frac{3\ln(3)}{2} - \frac{\sqrt{3}\,\pi}{6}.$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)(6k+1)} = \frac{\sqrt{3}\,\pi}{8} + \frac{3\ln(3)}{8}.$$

$$\sum_{k=1}^{\infty} \frac{1}{\left(k-\frac{1}{2}\right)\left(k-\frac{1}{6}\right)} = \frac{3\sqrt{3}\,\pi}{2} - \frac{9\ln(3)}{2}.$$

(**Problem 1.7.72, footnote**.)

116.

$$\sum_{n=1}^{\infty} \frac{1}{1+n^4} =$$

$$\frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)+\sinh\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right)+\sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right] =$$

$$\frac{\pi\sqrt{2}}{4}\cdot\frac{\sinh(\pi\sqrt{2})+\sin(\pi\sqrt{2})}{\cosh(\pi\sqrt{2})-\cos(\pi\sqrt{2})} - \frac{1}{2}.$$

If $a > 0$ constant,

$$\sum_{n=1}^{\infty} \frac{1}{a^4+n^4} = \frac{\pi\sqrt{2}}{4a^3}\cdot\frac{\sinh(\pi a\sqrt{2})+\sin(\pi a\sqrt{2})}{\cosh(\pi a\sqrt{2})-\cos(\pi a\sqrt{2})} - \frac{1}{2a^4}.$$

[**Problem 1.7.73 (a)**.]

117.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^4} =$$

$$\frac{1}{2}\left[\frac{\pi}{\sqrt{2}}\,\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)+\sinh\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)}{\sinh^2\left(\frac{\pi}{\sqrt{2}}\right)+\sin^2\left(\frac{\pi}{\sqrt{2}}\right)} - 1\right] =$$

$$\frac{\pi\sqrt{2}}{4}\cdot\frac{\sin\left(\frac{\pi}{\sqrt{2}}\right)\cosh\left(\frac{\pi}{\sqrt{2}}\right)+\sinh\left(\frac{\pi}{\sqrt{2}}\right)\cos\left(\frac{\pi}{\sqrt{2}}\right)}{\cosh(\pi\sqrt{2})-\cos(\pi\sqrt{2})} - \frac{1}{2}.$$

[**Problem 1.7.73 (b)**.]

118.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n^2}{n^3+1} = \frac{1}{3}\left[1 - \ln(2) + \pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right)\right].$$

[**Problem 1.7.74 (a)**.]

119.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}(2n-1)}{n^2-n+1} =$$

$$\pi \operatorname{sech}\left(\pi \frac{\sqrt{3}}{2}\right).$$

[**Problem 1.7.74 (b)**.]

120.

$$\sum_{n=1}^{\infty} \frac{1}{n^2-n+1} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2-n+1} = \frac{\pi\sqrt{3}}{3}\tanh\left(\pi \frac{\sqrt{3}}{2}\right).$$

[**Problem 1.7.74 (c)**.]

121. If $a \in \mathbb{C} - i\mathbb{Z}$ constant,

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{-2 + a\pi\coth(a\pi) + a^2\pi^2 \operatorname{csch}\left(a^2\pi^2\right)}{4a^4}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

[**Problem 1.7.75**.]

122.

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2+n^2} = \infty$$

[**Problem 1.7.76**.

123.

$$\operatorname{sech}(z) = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 + 4z^2}.$$

$$\sec(z) = 4\pi \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)}{(2n+1)^2\pi^2 - 4z^2}.$$

$$\tanh(z) = 8z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 + 4z^2}.$$

$$\tan(z) = 8z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\pi^2 - 4z^2}.$$

[**Problems 1.7.80**, **1.7.43 (c)**.]

124. If $-\pi < a < \pi$ and $t \notin \mathbb{Z}$ (not an integer) constants,

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos(na)}{t^2 - n^2} = \frac{\pi}{2t} \cdot \frac{\cos(at)}{\sin(\pi t)} - \frac{1}{2t^2}.$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n\sin(na)}{t^2 - n^2} = \frac{\pi}{2} \cdot \frac{\sin(at)}{\sin(\pi t)}.$$

(**Problem 1.7.81**.)

125. If $a > 0$ and $b > 0$ constants,

$$\sum_{n=1}^{\infty} \frac{1}{a + bn^2} = \frac{1}{2}\left[\frac{\pi}{\sqrt{ab}} \cdot \coth\left(\pi\sqrt{\frac{a}{b}}\right) - \frac{1}{a}\right].$$

$$\sum_{n=1}^{\infty} \frac{n^2}{a^4 + n^4} = \frac{\pi}{2\sqrt{2}a} \cdot \frac{\sinh\left(\pi a\sqrt{2}\right) - \sin\left(\pi a\sqrt{2}\right)}{\cosh\left(\pi a\sqrt{2}\right) - \cos\left(\pi a\sqrt{2}\right)}.$$

[**Problem 1.7.82 (a)**.]

126. For any $z \in \mathbb{C}$,

$$\operatorname{Log}\left[\frac{\sin(\pi z)}{z}\right] = \log(\pi) + \sum_{n=1}^{\infty} \operatorname{Log}\left(1 - \frac{z^2}{n^2}\right).$$

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right). \qquad \sinh(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right).$$

$$e^z - 1 = e^{\frac{z}{2}} z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

(**Problem 1.7.83**.)

127. For any $z \in \mathbb{C}$,

$$\cos(z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{\pi^2(2n-1)^2}\right].$$

$$\cosh(z) = \prod_{n=1}^{\infty} \left[1 + \frac{4z^2}{\pi^2(2n-1)^2}\right].$$

(**Problem 1.7.83 footnote**.)

128.

$$\pi = 2 \cdot \lim_{n\to\infty} \left[\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right].$$

$$\sqrt{\pi} = \lim_{n\to\infty} \left[\frac{2^{2n} \cdot (n!)^2}{(2n)! \cdot \sqrt{n}}\right].$$

[**Problem 1.7.84 (a) (c)**.]

129. If $k \in \mathbb{N}$, $0 < |a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k k$^{\text{th}}$ roots of unity$(= 1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{\omega_\rho a x}}{k a^{k-1}\,(e^x - 1)}\, dx.$$

(**Problem 1.7.86**.)

130. If $k \in \mathbb{N}$, $m \in \mathbb{N}$, $a \in \mathbb{C}$ such that $0 < |m - a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k k$^{\text{th}}$ roots of unity$(= 1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k - a^k} = \sum_{n=1}^{m} \frac{1}{n^k - a^k} + \int_0^\infty \frac{\sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\omega_\rho a)x}}{k a^{k-1}(e^x - 1)}\, dx.$$

(**Problem 1.7.86**.)

131. If $k \in \mathbb{N}$, ψ a k$^{\text{th}}$ root of -1, $0 < |a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k k$^{\text{th}}$ roots of unity$(= 1)$,

$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \int_0^\infty \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{\psi \omega_\rho a x}}{k a^{k-1}(e^x - 1)}\, dx$$

(**Problem 1.7.86**.)

132. If $k \in \mathbb{N}$, ψ a k$^{\text{th}}$ root of -1, $m \in \mathbb{N}$, $a \in \mathbb{C}$ such that $0 < |m-a| < 1$ and ω_ρ, $\rho = 0,\ 1,\ 2,\ \ldots,\ k-1$ the k k$^{\text{th}}$ roots of unity$(=1)$,
$$\sum_{n=1}^{\infty} \frac{1}{n^k + a^k} = \sum_{n=1}^{m} \frac{1}{n^k + a^k} + \int_0^{\infty} \frac{-\psi \sum_{\rho=0}^{k-1} \omega_\rho e^{-(m-\psi\omega_\rho a)x}}{k a^{k-1}(e^x - 1)}\, dx.$$
(**Problem 1.7.86**.)

133. If $m \geq 0$ integer,
$$\sum_{k=0}^{2m+1} \frac{1}{2m+1-2k} \binom{2m+1}{k} = 0.$$
(**Example 1.8.5**.)

134. If $m \geq 0$ and $n \geq 0$ integers,
$$\sum_{\substack{k=0 \\ m+n \neq k+l}}^{2m} \sum_{l=0}^{2n} (-1)^k \binom{2m}{k} \binom{2n}{l} \frac{1-(-1)^{m-k+n-l}}{m-k+n-l} =$$
$$\sum_{\substack{k=0 \\ m+n \neq k+l}}^{2m} \sum_{l=0}^{2n} \binom{2m}{k} \binom{2n}{l} \frac{(-1)^k - (-1)^{m+n-l}}{m-k+n-l} = 0.$$
(**Example 1.8.6**.)

135. If $m \geq 0$ and $n \geq 0$ integers,
$$\sum_{l=0}^{2n} (-1)^{n-l} \binom{2m}{m+n-l} \binom{2n}{l} = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{n}} = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$
(**Example 1.8.6**. **Appendix 1.8.3**.)

136. If $m \geq 1$ integer,
$$\sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{(-1)^k}{m-k} \binom{2m}{k} = 0. \qquad \sum_{\substack{k=0 \\ k \neq m}}^{2m} \frac{1}{m-k} \binom{2m}{k} = 0.$$
[**Example 1.8.6**. **Problem 1.8.18 (a)-(b)**.]

137. If a, b, and c real constants,
$${}_2F_1(a, b; c; -1) = 2 \cos\left(\frac{b\pi}{2}\right) \cdot \frac{\Gamma(|b|)\,\Gamma(b-a+1)}{\Gamma\left(\frac{|b|}{2}\right) \Gamma\left(\frac{b}{2} - a + 1\right)}$$
(A **Kummer formula**, **Subsection-Appendix 1.8.3**.)

2.3 List of Laplace and Inverse Laplace Transforms

Here, we cite a list of all major Laplace and Inverse Laplace Transforms that have been evaluated in the text or referred to in problems and footnotes.

Given a nice real function $y = f(x)$ defined on $[0, \infty)$, we define its **Laplace transform** to be the following improper integral with one parameter s:

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx.$$

Given $f(x)$ with Laplace transform $g(s) = \mathcal{L}\{f(x)\}(s)$, we call $f(x)$ the **inverse Laplace transform** of $g(s)$, and we write

$$\mathcal{L}^{-1}\{g(s)\}(x) = f(x).$$

General Rules:

1.
$$\mathcal{L}\{(f+h)(x)\}(s) = \mathcal{L}\{f(x)+h(x)\}(s) = \mathcal{L}\{f(x)\}(s)+\mathcal{L}\{h(x)\}(s).$$

2.
$$\mathcal{L}\{(cf)(x)\}(s) = \mathcal{L}\{cf(x)\}(s) = c\,\mathcal{L}\{f(x)\}(s).$$

3.
$$\mathcal{L}\{(f)(x)\}(s) = \mathcal{L}\{g(x)\}(s) \quad \Longleftrightarrow \quad f = g.$$

4.
$$\mathcal{L}^{-1}\{(f+h)(x)\}(s) = \mathcal{L}^{-1}\{f(x)+h(x)\}(s) = \mathcal{L}^{-1}\{f(x)\}(s) + \mathcal{L}^{-1}\{h(x)\}(s).$$

5.
$$\mathcal{L}^{-1}\{(cf)(x)\}(s) = \mathcal{L}^{-1}\{cf(x)\}(s) = c\,\mathcal{L}^{-1}\{f(x)\}(s).$$

6.
$$\mathcal{L}^{-1}\{(f)(x)\}(s) = \mathcal{L}^{-1}\{g(x)\}(s) \quad \Longleftrightarrow \quad f = g.$$

7. **Convolution Rule:** If the convolution, here, is defined by

$$(f*g)(x) := \int_0^x f(x-v)g(v)\,dv = \int_0^x f(v)g(x-v)\,dv = (g*f)(x),$$

$$\mathcal{L}\{(f*g)(x)\}(s) = \mathcal{L}\{f(x)\}(s) \cdot \mathcal{L}\{g(x)\}(s).$$

(**Example I 2.7.4.**)

8. If $a \geq 0$ constant,

$$\mathcal{L}\{f(x+a)\}(s) = e^{as}\mathcal{L}\{f(u)\}(s) - e^{as}\int_0^a e^{-su} f(u)\,du.$$

(**Example i 2.7.8.**)

9. If $y = f(x)$ defined on $[0, \infty)$ **periodic with period** p,

$$\mathcal{L}\{f(x)\}(s) = \frac{1}{1-e^{-sp}}\int_0^p e^{-sx} f(x)\,dx =$$

$$\frac{e^{sp}}{e^{sp}-1}\int_0^p e^{-sx} f(x)\,dx.$$

[**Problem I 2.7.18 (a).**]

10. If $a > 0$ and $b \geq 0$ constants,

$$\mathcal{L}\{H_{\frac{b}{a}}(x) \cdot f(ax-b)\}(s) = \frac{1}{a}\, e^{\frac{-bs}{a}}\ \mathcal{L}\{f(x)\}\left(\frac{s}{a}\right).$$

(**Problem I 2.7.22.**)

11. If $a > 0$ and $b \in \mathbb{R}$ constants,

$$\mathcal{L}\{f(ax-b)\}(s) = \frac{1}{a} e^{\frac{-bs}{a}}\left[\mathcal{L}\{f(x)\}\left(\frac{s}{a}\right) + \int_{-b}^0 e^{\frac{-su}{a}} f(u)\,du\right].$$

(**Problem I 2.7.23.**)

12. **Table I:**

	Function $h(x),\ x \in [0, \infty)$	Laplace transform $\mathcal{L}\{h(x)\}(s)$
1.	a (=constant)	$\frac{a}{s}, \quad s > 0$
2.	e^{ax}	$\frac{1}{s-a}, \quad s > a$
3.	$\sin(bx)$	$\frac{b}{s^2+b^2}, \quad s > 0$
4.	$\cos(bx)$	$\frac{s}{s^2+b^2}, \quad s > 0$
5.	$x^n, \quad n = 0, 1, 2, 3, \ldots.$	$\frac{n!}{s^{n+1}}, \quad s > 0, \quad [0! = \Gamma(1) = 1]$
6.	$x^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0, \quad [p! = \Gamma(p+1)]$
7.	$\sinh(bx)$	$\frac{b}{s^2-b^2}, \quad s > \lvert b \rvert$
8.	$\cosh(bx)$	$\frac{s}{s^2-b^2}, \quad s > \lvert b \rvert$
9.	$e^{ax}\sin(bx)$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{ax}\cos(bx)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$x^n e^{ax}, \quad n = 0, 1, 2, 3, \ldots.$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$

(**Problem I 2.7.14.**)

13. **Table II:**

	Function $h(x)$ $x \in [0,\infty)$ or $x \in (0,\infty)$	Laplace transform $\mathcal{L}\{h(x)\}(s)$, for $s > k \geq 0$, k constant
1.	$f'(x)$	$s\mathcal{L}\{f(x)\}(s) - f(0^+)$
2.	$f''(x)$	$s^2\mathcal{L}\{f(x)\}(s) - sf(0^+) - f'(0^+)$
3.	$f^{(n)}(x)$	$s^n\mathcal{L}\{f(x)\}(s) -$ $s^{n-1}f(0^+) - \ldots - f^{(n-1)}(0^+)$
4.	$e^{ax}f(x)$ with a constant	$\mathcal{L}\{f(x)\}(s-a)$ for $s > k + a$ (shift by a in the Laplace transform)
5.	$H_b(x) \cdot f(x-b)$ shift of $f(x)$ by constant $b \geq 0$ $[H_b(x) = \text{Heaviside function}]$	$e^{-bs}\mathcal{L}\{f(x)\}(s)$ for $s > k$
6.	$\displaystyle\int_0^x f(t)\,dt$	$\dfrac{1}{s}\,\mathcal{L}\{f(x)\}(s)$
7.	$\displaystyle\int_x^\infty f(t)\,dt$	$\dfrac{1}{s}\left[\displaystyle\int_0^\infty f(t)\,dt - \mathcal{L}\{f(x)\}(s)\right]$
8.	$x^n f(x), \quad n = 0, 1, 2, 3, \ldots$	$(-1)^n \dfrac{d^n}{ds^n}\mathcal{L}\{f(x)\}(s)$
9.	$\dfrac{f(x)}{x}$	$\displaystyle\int_s^\infty \mathcal{L}\{f(x)\}(u)\,du$
10.	$f(ax)$ $a > 0$ constant	$\dfrac{1}{a}\,\mathcal{L}\{f(x)\}\left(\dfrac{s}{a}\right)$
11.	$(-x)^n f(x), \quad n = 0, 1, 2, 3, \ldots$	$\dfrac{d^n}{ds^n}\mathcal{L}\{f(x)\}(s)$

(**Example I 2.7.4. Problem I 2.7.21**.)

Specific Laplace and Inverse Laplace transforms:

14. If $a > 0$ constant,

$$\mathcal{L}\{\mathrm{erf}(ax)\}(s) = \frac{e^{\frac{s^2}{4a^2}}}{s}\,\mathrm{erfc}\left(\frac{s}{2a}\right).$$

$$\mathcal{L}\{\mathrm{erfc}(ax)\}(s) = \frac{1}{s} - \frac{e^{\frac{s^2}{4a^2}}}{s}\,\mathrm{erfc}\left(\frac{s}{2a}\right) = \frac{1}{s}\left[1 - e^{\frac{s^2}{4a^2}}\,\mathrm{erfc}\left(\frac{s}{2a}\right)\right].$$

(**Example I 2.7.3.**)

15. Laplace transform of **Dirac Delta function:**

$$\mathcal{L}\{\delta(x)\}(s) = \int_0^\infty e^{-sx}\delta(x)\,dx = e^{-0s} = 1.$$

$$\mathcal{L}\{\delta(x-a)\}(s) = \int_0^\infty e^{-sx}\delta(x-a)\,dx = e^{-as}.$$

(**Example I 2.7.6**.)

16.
$$\mathcal{L}\{\sin(\sqrt{x})\}(s) = \frac{\sqrt{\pi}e^{\frac{-1}{4s}}}{2s^{\frac{3}{2}}}.$$

$$\mathcal{L}\{\cos(\sqrt{x})\}(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}\frac{n!}{s^{n+1}} =$$
$$\frac{1}{s}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)(n+2)\dots(2n)}\frac{1}{s^n}.$$

(**Example I 2.7.7**.)

17. If $\alpha > 0$, and $\beta \in \mathbb{R}$ constants,

$$\int_0^\infty e^{-\alpha x}x\cos(\beta x)\,dx = \frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^2},$$

$$\int_0^\infty e^{-\alpha x}x\sin(\beta x)\,dx = \frac{2\alpha\beta}{(\alpha^2+\beta^2)^2}.$$

[**Application (c)** after **Example I 2.7.8**.]

18. If $\beta \in \mathbb{R}$ real constant,

$$\mathcal{L}\left\{\frac{\sin(\beta x)}{x}\right\}(s) = \frac{\pi}{2} - \arctan\left(\frac{s}{\beta}\right) = \arctan\left(\frac{\beta}{s}\right).$$

[**Problems I 2.7.10 (c), I 2.7.24**.]

19. If $-\infty < \beta < \infty$ constant and $x > 0$,

$$\mathcal{L}\left\{\frac{\cos(\beta\sqrt{x})}{\sqrt{x}}\right\}(s) = \sqrt{\frac{\pi}{s}}\, e^{\frac{-\beta^2}{4s}}.$$

$$\mathcal{L}\left\{\frac{1}{\sqrt{x}}\right\}(s) = \sqrt{\frac{\pi}{s}}.$$

[**Problems I 2.7.11 (a)-(b), 1.7.32 (a).**]

20. If $a,\ b$ real constants,

$$\mathcal{L}\left\{\frac{\cos(ax) - \cos(bx)}{x}\right\}(s) = \frac{1}{2}\ln\left(\frac{s^2 + b^2}{s^2 + a^2}\right).$$

$$\mathcal{L}\left\{\frac{\sin(ax) + \sin(bx)}{x}\right\}(s)dx = \arctan\left(\frac{a}{s}\right) + \arctan\left(\frac{b}{s}\right).$$

$$\mathcal{L}\left\{\frac{\sin(ax) - \sin(bx)}{x}\right\}(s)dx = \arctan\left(\frac{a}{s}\right) - \arctan\left(\frac{b}{s}\right).$$

[**Problem I 2.7.12 (a)-(c).**]

21.
$$\mathcal{L}\left\{e^{-x^2}\right\}(s) = \frac{\sqrt{\pi}}{2} \cdot e^{\frac{s^2}{4}} \cdot \operatorname{erfc}\left(\frac{s}{2}\right).$$

[**Problem I 2.7.13**.]

22. If $a,\ b \in \mathbb{R}$ constants,

$$\mathcal{L}\left\{\sin(ax + b)\right\}(s) = \frac{a \cdot \cos(b) + s \cdot \sin(b)}{s^2 + a^2}.$$

$$\mathcal{L}\left\{\cos(ax + b)\right\}(s) = \frac{s \cdot \cos(b) - a \cdot \sin(b)}{s^2 + a^2}.$$

[**Problem I 2.7.16**.]

23. If $a > 0$ constant,

$$\mathcal{L}\left\{\frac{1}{x^2+a^2}\right\}(s) = \frac{1}{a}\left\{\cos(as)\left[\frac{\pi}{2} - \mathrm{Si}(as)\right] - \sin(as)\,\mathrm{Ci}(as)\right\}.$$

$$\mathcal{L}\left\{\frac{x}{x^2+a^2}\right\}(s) = \sin(as)\left[\frac{\pi}{2} - \mathrm{Si}(as)\right] + \cos(as)\,\mathrm{Ci}(as).$$

$$\mathcal{L}\{\arctan(ax)\}(s) = \frac{1}{s}\left\{\cos\left(\frac{s}{a}\right)\left[\frac{\pi}{2} - \mathrm{Si}\left(\frac{s}{a}\right)\right] - \sin\left(\frac{s}{a}\right)\,\mathrm{Ci}\left(\frac{s}{a}\right)\right\}.$$

$$\mathcal{L}\left\{\ln\left(x^2+a^2\right)\right\}(s) = \frac{2\ln(a)}{s} + \frac{2}{s}\left\{\sin(as)\left[\frac{\pi}{2} - \mathrm{Si}(as)\right] + \cos(as)\,\mathrm{Ci}(as)\right\}.$$

(**Problem I 2.7.17.**)

24. If $b \in \mathbb{R}$ constant,

$$H_b(x) := H(x-b) := \begin{cases} 0, & \text{if} \quad x < b, \\ 1, & \text{if} \quad x \geq b, \end{cases}$$

is the **Heaviside unit step function**.

$$\mathcal{L}\{H_b(x)\}(s) := \int_0^\infty e^{-sx} H_b(x)\,dx = \begin{cases} \dfrac{e^{-bs}}{s}, & \text{if} \quad b \geq 0, \\ \dfrac{1}{s}, & \text{if} \quad b \leq 0, \end{cases}$$

and $s > 0$.

[**Problem I 2.7.19 (a).**]

25.

$$\mathcal{L}\{\ln(x)\}(s) = \int_0^\infty e^{-sx}\ln(x)\,dx = \frac{-[\gamma + \ln(s)]}{s}.$$

$$\mathcal{L}\left\{\ln^2(x)\right\}(s) = \int_0^\infty e^{-sx}\ln^2(x)\,dx = \frac{\pi^2}{6s} + \frac{[\gamma + \ln(s)]^2}{s}.$$

$$\mathcal{L}\{x\ln(x)\}(s) = \int_0^\infty e^{-sx} x\ln(x)\,dx = \frac{1 - \gamma - \ln(s)}{s^2}.$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problem I 2.7.20 (a)-(c)**.]

26. $\forall\, n \in \mathbb{N}$ and $x \in \mathbb{R}$, the Laplace transform of the Cauchy n-tuple integral

$$\int_0^x \int_0^{u_1} \cdots \left\{\int_0^{u_{n-2}} \left[\int_0^{u_{n-1}} f(u_n)\,du_n\right] du_{n-1}\right\} \ldots du_2\,du_1 =$$

$$\int_0^x \frac{(x-\tau)^{n-1}}{(n-1)!} f(\tau)\,d\tau = \frac{1}{(n-1)!}\,g * f, \quad \text{where} \quad g(x) = x^{n-1},$$

is

$$\frac{\mathcal{L}\{f(x)\}(s)}{s^n}.$$

(**Problem I 2.7.26**.)

27.

$$\mathcal{L}\left\{\frac{1-\cos(x)}{x}\right\}(s) = \int_s^\infty \left(\frac{1}{t} - \frac{t}{1+t^2}\right) dt = \frac{1}{2}\ln\left(1 + \frac{1}{s^2}\right).$$

[**Problem I 2.7.29 (b)**.]

28. If $a > 0$ constant,

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x \geq a, \\ \\ 0 & \text{if} \quad 0 \leq x < a. \end{cases}$$

$$\mathcal{L}\{f(x)\}(s) = -\ln(s) - \sum_{n=1}^{\infty}\frac{(-1)^n a^n}{n!}\frac{s^n}{n} + c,$$

where

$$c = \int_a^\infty \frac{e^{-x}}{x}\,dx + \sum_{n=1}^{\infty}\frac{(-1)^n a^n}{n \cdot n!}.$$

(**Problem I 2.7.33**.)

29. If $n \in \mathbb{N}_0$, and $b \geq 0$, $s > 0$ constants,

$$\int_0^\infty x^n \sin(bx)\, e^{-sx} dx = (-1)^n \frac{d^n}{ds^n} \left(\frac{b}{s^2 + b^2} \right).$$

$$\int_0^\infty x^n \cos(bx)\, e^{-sx} dx = (-1)^n \frac{d^n}{ds^n} \left(\frac{s}{s^2 + b^2} \right).$$

(**Problem I 2.7.35**.)

30. If a and b real constants,

$$\mathcal{L}\left\{ \frac{e^{ax} - e^{bx}}{x} \right\}(s) = \int_s^\infty \left(\frac{1}{t-a} - \frac{1}{t-b} \right) dt = \ln \left| \frac{s-b}{s-a} \right|.$$

(**Problem I 2.7.36**.)

31. If $a > 0$ constant,

$$\mathcal{L}^{-1}\left\{ e^{-a\sqrt{s}} \right\}(x) = \frac{a}{2\sqrt{\pi}}\, x^{\frac{-3}{2}}\, e^{\frac{-a^2}{4x}}.$$

(**Example I 2.7.12**.)

32.

$$\mathcal{L}\{J_0(x)\}(s) := \mathcal{L}\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \right\}(s) = \frac{1}{\sqrt{1+s^2}}.$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{\sqrt{1+s^2}} \right\}(x) = J_0(x).$$

(**Subsection I 2.7.3. Application 1.**)

33.

$$\mathcal{L}^{-1}\left\{ \frac{e^{-x\sqrt{\frac{s}{k}}}}{s} \right\}(t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}} \right) = 1 - \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} \right).$$

(**Subsection I 2.7.3. Application 2.**)

34. If

$$Y(s) = \frac{Ce^{-t_0 s}}{s^2+4s+6} = \frac{C\,e^{-t_0 s}}{(s+2)^2+(\sqrt{2})^2},$$

$$\mathcal{L}^{-1}\{Y(s)\}(t) = H_{t_0}(t)\frac{C}{\sqrt{2}}\cdot e^{-2(t-t_0)}\sin\left[\sqrt{2}\,(t-t_0)\right] =$$

$$= \begin{cases} 0, & \text{if} \quad t < t_0, \\ \dfrac{C}{\sqrt{2}}\cdot e^{-2(t-t_0)}\sin\left[\sqrt{2}\,(t-t_0)\right], & \text{if} \quad t \geq t_0, \end{cases}$$

(**Subsection I 2.7.3. Application 3.**)

35.

$$\mathcal{L}^{-1}\left\{\frac{\ln(s)}{s}\right\}(x) = -[\ln(x)+\gamma].$$

$$\mathcal{L}^{-1}\left\{\frac{\ln^2(s)}{s}\right\}(x) = [\ln(x)+\gamma]^2 - \frac{\pi^2}{6}.$$

$$\mathcal{L}^{-1}\left\{\frac{\ln(s)}{s^2}\right\}(x) = x[1-\gamma-\ln(x)].$$

$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln(n)\right] \simeq 0.57721566\ldots > 0$ is the **Euler-Mascheroni constant**.

[**Problem I 2.7.43 (a)-(c).**]

36. The Laplace transform of **modified Bessel function of order zero**

$$I_0(x) = 1 + \frac{x^2}{2\cdot 2} + \frac{x^4}{2\cdot 4\cdot 2\cdot 4}\ldots = J_0(ix),$$

where $i = \sqrt{-1}$, is

$$\frac{1}{\sqrt{s^2-1}}.$$

(**Problem I 2.7.47.**)

37. If

$$\begin{cases} y'(x) + p\,y(x) = r(x), & p \text{ is a constant,} \\ y(0) = y_0, & (y_0 \text{ is constant),} \end{cases}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{r(x)\}(s) + y_0}{s+p}\right\}(x).$$

(**Problem I 2.7.48**.)

38. If

$$\begin{cases} y''(x) + p\,y'(x) + q\,y(x) = r(x), & p,\ q \text{ are constants,} \\ y(0) = y_0, & (y_0 \text{ is constant),} \\ y'(0) = y_0', & (y_0' \text{ is constant),} \end{cases}$$

$$y(x) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{r(x)\}(s) + (s+p)y_0 + y_0'}{s^2+ps+q}\right\}(x).$$

(**Problem I 2.7.50**.)

39. If β real constant, and $s > 0$,

$$\mathcal{L}\left\{\frac{\sin(\beta u)}{\sqrt{u}}\right\}(s) = \operatorname{sign}(\beta)\sqrt{\frac{\pi}{2}}\sqrt{\frac{-s+\sqrt{\beta^2+s^2}}{\beta^2+s^2}}.$$

$$\mathcal{L}\left\{\frac{\cos(\beta u)}{\sqrt{u}}\right\}(s) = \sqrt{\frac{\pi}{2}}\sqrt{\frac{s+\sqrt{\beta^2+s^2}}{\beta^2+s^2}}.$$

[**Problem 1.7.19 (a)-(b). Example I 2.4.2 Remark 3.**]

40. If $b \in \mathbb{R}$ constant, and $s > 0$,

$$\mathcal{L}\left\{\frac{\sin(b\sqrt{u})}{\sqrt{u}}\right\}(s) = \frac{1}{2s}\,e^{-\frac{b^2}{4s}}\int_0^b e^{\frac{t^2}{4s}}\,dt.$$

(**Problems 1.7.33, I 2.2.43**.)

41.
$$\mathcal{L}^{-1}\left\{\frac{18}{s^3+27}\right\}(x) =$$
$$\frac{2e^{-3x}}{3} + \frac{2e^{\frac{3x}{2}}}{3}\left[-\cos\left(\frac{3\sqrt{3}}{2}x\right) + \sqrt{3}\sin\left(\frac{3\sqrt{3}}{2}x\right)\right].$$

(**Example 1.7.53.**)

42. If $a > 0$ constant
$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\}(x) = \frac{e^{\frac{-a^2}{4x}}}{\sqrt{\pi x}}, \qquad \text{for} \quad x > 0.$$

(**Example 1.7.54.**)

43.
$$\mathcal{L}^{-1}\left\{\sin\left(\frac{1}{s}\right)\right\}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!\cdot(2n+1)!}.$$

(**Problem 1.7.153.**)

44. If $a > 0$ constant,
$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}(x) = \frac{a}{2\sqrt{\pi x^3}}\,e^{\frac{-a^2}{4x}}.$$
$$\mathcal{L}\left\{\frac{a}{2\sqrt{\pi x^3}}\,e^{\frac{-a^2}{4x}}\right\}(s) = e^{-a\sqrt{s}}.$$

(**Problem 1.7.159.**)

45. If $a > 0$ constant and $x > 0$,
$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}(x) = 1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right).$$
$$\mathcal{L}\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \mathcal{L}\left\{1 - \operatorname{erf}\left(\frac{a}{2\sqrt{x}}\right)\right\}(s) = \frac{e^{-a\sqrt{s}}}{s}.$$

(**Problem 1.7.161.**)

46. If $a \geq 0$ constant and $x \geq 0$,
$$\mathcal{L}\left\{\frac{1}{\sqrt{x+a}}\right\}(s) = \frac{\sqrt{\pi}\,e^{as}\operatorname{erfc}\left(\sqrt{as}\,\right)}{\sqrt{s}} =$$
$$\sqrt{\frac{\pi}{s}}\operatorname{erfcx}\left(\sqrt{as}\,\right), \qquad \forall\ s > 0.$$

(**Problem 1.7.162.**)

2.4 List of Fourier Transforms

Here, we cite a list of all major Fourier Transforms that have been evaluated in the text or referred to in problems and footnotes.

Given a nice real function $y = f(x)$ defined on $\mathbb{R}$, we define its **Fourier transform** to be the following complex improper integral with one parameter t:

$$\hat{f}(t) = \mathcal{F}[f(x)](t) := \int_{-\infty}^{\infty} e^{-itx} f(x)\,dx.$$

or

$$\hat{f}(t) = \mathcal{F}[f(x)](t) :=$$
$$\text{P.V.}\int_{-\infty}^{\infty} e^{-itx} f(x)\,dx = \lim_{0<M\to\infty} \int_{-M}^{M} e^{-itx} f(x)\,dx.$$

$\Big($or these integrals multiplied by $\dfrac{1}{\sqrt{2\pi}}\Big)$.

General Rules:

1. If $\mathcal{F}[f(x)](t)$ exists for all t and $\lim\limits_{x\to\pm\infty} f(x) = 0$,

$$\mathcal{F}[f'] = -it\mathcal{F}[f].$$

2. If $f(x)$ is **real**,

$$\hat{f}(-t) = \overline{\hat{f}(t)}.$$

$$\mathrm{Re}\{\mathcal{F}[f(x)](-t)\} = \mathrm{Re}\{\mathcal{F}[f(x)](t)\}.$$

$$\mathrm{Im}\{\mathcal{F}[f(x)](-t)\} = -\mathrm{Im}\{\mathcal{F}[f(x)](t)\}.$$

3. If $xf(x)$ is absolutely integrable,

$$\frac{d}{dt}\{\mathcal{F}[f(x)]\}(t) = \mathcal{F}[ixf(x)](t) = i\mathcal{F}[xf(x)](t).$$

4. If $a \neq 0$ and b real constants,

$$\mathcal{F}[f(ax-b)](t) = \frac{1}{|a|}\, e^{\frac{ibt}{a}}\, \mathcal{F}[f(x)]\left(\frac{t}{a}\right).$$

5. If c real constant,

$$\mathcal{F}\left[e^{icx}f(x)\right](t) = \mathcal{F}[f(x)](t+c).$$

6. If c real constant,

$$\mathcal{F}\left[\cos(cx)f(x)\right](t) = \frac{1}{2}\mathcal{F}\left[e^{icx}f(x)\right](t) + \frac{1}{2}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$

$$\frac{1}{2}\left\{\mathcal{F}[f(x)](t+c) + \mathcal{F}[f(x)](t-c)\right\},$$

and

$$\mathcal{F}\left[\sin(cx)f(x)\right](t) = \frac{1}{2i}\mathcal{F}\left[e^{icx}f(x)\right](t) - \frac{1}{2i}\mathcal{F}\left[e^{-icx}f(x)\right](t) =$$

$$\frac{-i}{2}\left\{\mathcal{F}[f(x)](t+c) - \mathcal{F}[f(x)](t-c)\right\}.$$

7. **Convolution Rule:** The convolution is defined by

$$(f*g)(x) := \int_{-\infty}^{\infty} f(x-v)g(v)\,dv = \int_{-\infty}^{\infty} f(v)g(x-v)\,dv = (g*f)(x).$$

$$\mathcal{F}\{(f*g)(x)\}(t) = \mathcal{F}\{f(x)\}(t)\cdot\mathcal{L}\{g(x)\}(t).$$

8. If $f(x) = \displaystyle\int_{-\infty}^{x} e^{-(x-u)}g(u)du$,

$$\mathcal{F}[f(x)](t) = \frac{1}{1-it}\,\mathcal{F}[g(x)](t).$$

(**Example 1.7.44.**)

Specific Fourier transforms:

9. If $f(x) = \dfrac{1}{x^2 + b^2}$, where $b > 0$ constant,

$$\hat{f}(t) = \frac{\pi}{b}\, e^{-b|t|}, \quad t \in \mathbb{R}\,.$$

(**Example 1.7.33.**)

10. If $f(x) = \dfrac{x}{x^2 + b^2}$, where $b > 0$ and $a \in \mathbb{R}$ constants,

$$\hat{f}(a) = \begin{cases} 0, & \text{if} \quad a = 0, \\ \operatorname{sign}(a)\, \pi i\, e^{-b|a|}, & \text{if} \quad a \neq 0. \end{cases}$$

(**Example 1.7.34.**)

11. If $f(x) = \dfrac{1}{x}$, $(x \neq 0)$,

$$\hat{f}(\alpha) = \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = \begin{cases} i\pi, & \text{if } \alpha > 0, \\ 0, & \text{if } \alpha = 0, \\ -i\pi, & \text{if } \alpha < 0. \end{cases}$$

(**Example 1.7.35 Remark 2.**)

12. If $f(x) = \dfrac{\sin(x)}{x}$,

$$\hat{f}(t) = \mathcal{F}[f(x)](t) = \begin{cases} 0, & \text{if} \quad |t| > 1, \\ \dfrac{\pi}{2}, & \text{if} \quad t = \pm 1, \\ \pi, & \text{if} \quad |t| < 1. \end{cases}$$

(**Example 1.7.37.**)

13. If $a > 0$ constant, and $f(x) = \begin{cases} 1, & \text{if} \quad |x| < a, \\ 0, & \text{if} \quad |x| \geq a, \end{cases}$

$$\hat{f}(\xi) = \mathcal{F}[f(x)](\xi) = 2\,\frac{\sin(a\xi)}{\xi}.$$

(**Example 1.7.38.**)

14.

$$\mathcal{F}\left[e^{-x^2}\right](t) = \sqrt{\pi}\,e^{-\frac{t^2}{4}}.$$

$$\mathcal{F}\left[e^{-a(x-b)^2}\right](t) = \mathcal{F}\left[e^{-[\sqrt{a}(x-b)]^2}\right](t) = \sqrt{\frac{\pi}{a}}\,e^{itb}e^{-\frac{t^2}{4a}}.$$

If c real constant,

$$\mathcal{F}\left[e^{-x^2}\sin(cx)\right](t) =$$

$$\frac{-i}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} - \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right] = i\sqrt{\pi}\,e^{-\frac{(t^2+c^2)}{4}}\sinh\left(\frac{ct}{2}\right),$$

$$\mathcal{F}\left[e^{-x^2}\cos(cx)\right](t) =$$

$$\frac{1}{2}\left[\sqrt{\pi}e^{-\frac{(t+c)^2}{4}} + \sqrt{\pi}e^{-\frac{(t-c)^2}{4}}\right] = \sqrt{\pi}\,e^{-\frac{(t^2+c^2)}{4}}\cosh\left(\frac{ct}{2}\right).$$

(**Example 1.7.40.**)

15.

$$\mathcal{F}\left[\frac{e^{-|x|}}{2}\right](t) = \frac{1}{1+t^2}.$$

$$\mathcal{F}^{-1}\left[\left(\frac{1}{1+t^2}\right)^2\right](x) = \frac{1}{4}(1+|x|)\,e^{-|x|}.$$

(**Example 1.7.41.**)

16.

$$\text{If} \qquad g(x) = \begin{cases} 1+x, & \text{if } -1 \le x \le 0, \\ 1-x, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } |x| \le 1, \end{cases}$$

$$g'(x) = \begin{cases} 1, & \text{if } -1 < x < 0, \\ -1, & \text{if } 0 < x < 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

$$\mathcal{F}\left[g'(x)\right](t) = \frac{-2i[1-\cos(t)]}{t},$$

$$\mathcal{F}[g(x)](t) = \frac{2[1-\cos(t)]}{t^2}.$$

(**Example 1.7.42.**)

17. If $f(x) = \dfrac{1}{x^2+6x+10} = \dfrac{1}{(x+3)^2+1}$,

$$\mathcal{F}[f(x)](t) = \pi e^{-|t|-3it}.$$

If $a \neq 0$, $b \in \mathbb{R}$ and $f(x) = \dfrac{1}{(ax+b)^2+1}$,

$$\mathcal{F}[f(x)](t) = \frac{\pi}{|a|}\, e^{-\left|\frac{x}{a}\right|-\frac{bit}{a}}.$$

(**Example 1.7.43.**)

18. If $a > 0$ constant, and

$$f(x) = \frac{1}{2a}\, e^{-a|x|} = \begin{cases} \dfrac{1}{2a}\, e^{-ax}, & \text{if } x > 0, \\[2ex] \dfrac{1}{2a}, & \text{if } x = 0, \\[2ex] \dfrac{1}{2a}\, e^{ax}, & \text{if } x < 0, \end{cases}$$

and $w = t + is$,

$$\hat{f}(w) = \mathcal{F}[f(x)](w) = \frac{1}{a^2+w^2}.$$

(**Example 1.7.45.**)

19. If $a \neq 0$ real constant

$$\mathcal{F}\left[e^{-a^2x^2}\right](t) = \frac{\sqrt{\pi}}{|a|}\, e^{-\frac{t^2}{4a^2}}.$$

(**Problem 1.7.110.**)

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