# Principal Values as Mean Values with the Normal Distribution

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**Abstract**

For a random variable that has the normal distribution the mean value of its reciprocal does not exist. But under the condition that near zero its values appear with the same frequency to the left and right of zero (a condition that occurs in practical applications, some of which we present), the principal value of the pertinent integral can be used as mean value of the reciprocal variable. We also present some estimates and examine what happens with the mean values of the reciprocal of the square and the cube of any normal random variables.

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**Key words:** Normal distribution, reciprocal random variable, principal value, mean value, estimates.

We consider a random variable $X = x$ which has the normal distribution with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$,

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty < x < \infty,$$

and we are interested in finding the mean (average, expected) value of the reciprocal random variable $Y = y := \dfrac{1}{x}$.

The integral

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{x}\, dx,$$

in the mean value of $Y$, becomes $\infty - \infty$ near $x = 0$ and so it does not exist. (See also [2], page 171.) Depending on how we approach $x = 0$ from left and right, we may achieve any answer we wish.

[With other probability densities or distributions this question does not pose this problem. For example, certain Gamma or Beta (with $\alpha > 1$), Chi square (with $\nu \geq 3$), and others.]

But in special practical situations, we may consider as the mean value of $Y$ to be the principal value

$$E(Y) = \frac{1}{\sigma\sqrt{2\pi}}\ \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{x}\, dx :=$$

$$\frac{1}{\sigma\sqrt{2\pi}} \lim_{\substack{0<\epsilon\to 0\\ 0<R\to\infty}} \int_{-R}^{-\epsilon} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{x}\, dx + \frac{1}{\sigma\sqrt{2\pi}} \lim_{\substack{0<\epsilon\to 0\\ 0<R\to\infty}} \int_{\epsilon}^{R} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{x}\, dx. \qquad \mathbf{(1)}$$

This principal value always exists and with the help of complex integration, we can prove that is equal to

$$E(Y) =$$

$$\frac{e}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{x^2+2\sigma^2} \left\{ x\cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right] - \sqrt{2}\,\sigma\sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\} dx. \qquad \mathbf{(2)}$$

This integral be can easily evaluated numerically and thus, we can estimate its value to any satisfactory degree of accuracy.

A practical statistical case in which we can use this result may be described as follows. Suppose a large number of values of the normal random variable $X$ (coming from tests, experiments, observations, etc.) satisfy **the condition**:

"*The values of $X$ appear with equal frequency in the intervals $[-c, 0]$ and $[0, c]$, where $c > 0$ is a constant*", (✠),

or we have strong reasons and/or indications to suppose so. Then, it makes sense to consider as the mean value of $Y$ the principal value found above.

For all practical purposes, condition (✠) is automatically satisfied sufficiently, most of the times. This is due to the formula of the normal distribution whose derivative at $x = 0$ in absolute value is

$$\left|\frac{d}{dx} n(0;\mu,\sigma)\right| = \frac{1}{\sigma^2\sqrt{2\pi}} |\lambda| e^{-\frac{\lambda^2}{2}}, \quad \text{where} \quad \lambda = \frac{\mu}{\sigma}.$$

So, when the absolute value of this derivative is "almost" equal to zero, the bell curve is almost horizontal at $x = 0$ and the condition (✠) is valid adequately approximately. This happens most of the times and it is easy to check. E.g., if $|\lambda| \geq 5$ and $\sigma \geq 1$, then $\left|\frac{d}{dx} n(0;\mu,\sigma)\right| \leq 0.00000186$, or if $|\lambda| \geq 6$ and $\sigma \geq 1$, then $\left|\frac{d}{dx} n(0;\mu,\sigma)\right| \leq 0.00000000913$, and so on with all sorts of combinations of the values of $|\lambda|$ and $\sigma$, and especially when $|\lambda|$ is big. In this kind of situations, formula **(2)** gives the answer with great accuracy.

For example: We use *Mathematica* and with $\sigma = 1$ and $\mu = 6$, formula **(2)** gives the ideal value $E(Y) = 0.1717501$. We run a principal value estimation with

10,000, 1,000,000 and 10,000,000 randomly generated numbers that have the distribution $n(x; \mu = 6, \sigma = 1)$. The results given by *Mathematica* were 0.171884, 0.171799, 0.171755, respectively. We observe that in all three results the (first) 3 decimal digits are accurate. In the last one, 5 decimal digits were accurate, which is very satisfactory.

Also, because of the $\sigma^2$ in the denominator, $|\lambda|$ could be relatively small, $\leq 1$ let us say, but $\sigma$ big, $\geq 200$ let us say. Then $\left|\frac{d}{dx} n(0; \mu, \sigma)\right| \leq 0.000006$. If $\lambda = 0$, i.e., $\mu = 0$, the above derivative is zero. We also observe that when $\mu = 0$, the integrand in formula **(2)** is an odd function about the origin and so its principal value, and hence the mean value of $Y = y := \frac{1}{x}$, is zero, which is naturally expected. In this instance, the stipulated condition $(\maltese)$ is valid exactly and the answer is perfect.

Now, if, for instance, we let $\mu = 2$ and $\sigma = \frac{1}{\sqrt{2}} \simeq 0.70711\ldots$, then $\lambda = 2\sqrt{2}$ and $\left|\frac{d}{dx} n(0; \mu = 2, \sigma = \frac{1}{\sqrt{2}})\right| \simeq 0.0146137$. This result is not very satisfactory, i.e., close enough to zero, and so, we 'd better require the stipulated condition, $(\maltese)$, on the values of $X$ around $x = 0$. Under this condition, we can use a numerical method of integration to evaluate the mean value of the reciprocal values of $X$ as the principal value given by **(2)** and we find

$$E(Y) \simeq \frac{1}{\sigma\sqrt{2\pi}} \cdot 1.06822\ldots = \frac{1}{\sqrt{\pi}} \cdot 1.06822\ldots \simeq 0.602681\ldots .$$

We experiment again with *Mathematica* and we run twice a principal value estimation with 10,000, 1,000,000 and 10,000,000 randomly generated numbers that have the distribution $n\left(x\,; \mu = 2, \sigma = \frac{1}{\sqrt{2}}\right)$. The results given by *Mathematica* were
$\{0.521246,\ 0.335484\}$, $\{0.568177,\ 0.562185\}$, $\{0.630353,\ 0.604342\}$, respectively.
Here, besides the significant difference of these results from the actual value 0.602681, we observe instability of the results form one run to another. So, in this example, while keeping $\mu = 2$ and $\sigma = \sqrt{2}$ the same, estimating the principal value becomes unstable and the results, from one experiment to another, vary significantly. Here, as explained above, condition $(\maltese)$ was not sufficiently satisfied. To achieve a value satisfactorily close to the actual value, condition $(\maltese)$ must be somehow satisfied by the data.

A real situation in which we may need $E(Y)$ is the following. In an experiment in a physics laboratory, a ball of known mass $m$ is shot by a certain (mechanical, pressure, electrical) device from fixed point $A$ to fixed point $B$. The distance $d = |AB|$ is known. In each shot a chronometer measures the time that the ball takes to go from $A$ to $B$. This experiment is done many times. Instead of computing the average $t_a$ of the times measured and find as the average speed estimate the $s_1 = \frac{d}{t_a}$, we compute the average of all the ratios $\frac{d}{t}$, $s_2$ let us say. These two estimates are not the same, a fact proven easily mathematically. E.g., if $d = 1$, in the example with $\mu = 6$ above, we have $s_1 = \frac{1}{6} = 0.16666\ldots$, whereas $s_2 = E(Y) = 0.1717501$.

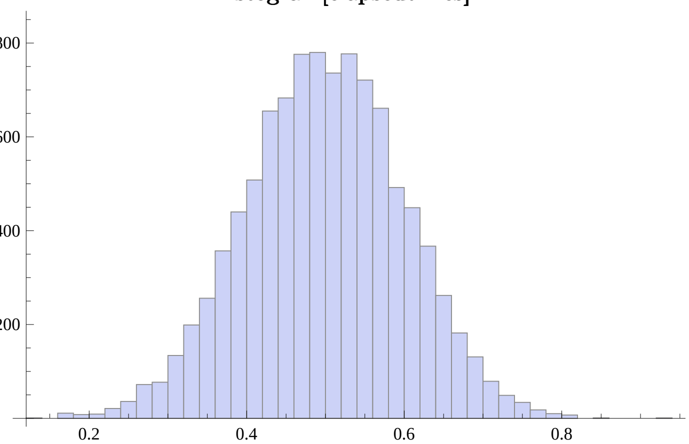
In this experiment the time indications, $t$, are positive and away from zero. So, the singularity problem, at $t = 0$, is not present. But, if in some experiment, two chronometers are needed, then the problem is present with the random variable $X$ defined to be the difference of the two time indications of the two chronometers.

This question was suggested to me by the physics professor Andy Rundquist of our physics laboratory and that is how this work was motivated and started. In fact he presented to me the following:

Suppose students are tasked with measuring the speed of passing cars. They set up two stations that are 10 meters apart. The synchronize two time pieces, and simply record the time at each station that the cars pass. Then they subtract the times from the two time pieces to get the elapsed time, which, in conjunction with the distance between the stations, can be used to calculate the speed of the cars.

Here is some sample data:

**Histogram[elapsedtimes]**




**elapsedtimes = RandomReal[NormalDistribution[0.5, 0.1], {10000}];**

In order to calculate the best estimate of the speed, we simply calculate the speed in each case and then find the mean and standard deviation:

**speeds** $= \frac{\mathbf{10}}{\mathbf{elapsedtimes}}$;

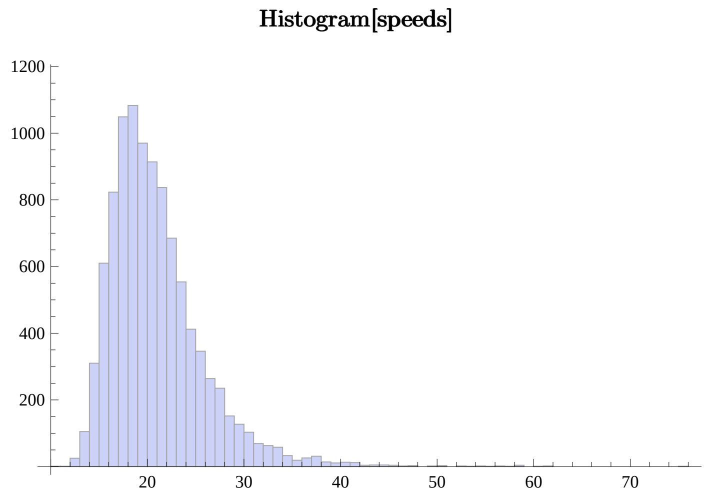
**Mean[speeds]**

20.9729

**StandardDeviation[speeds]**

4.98301

But there's a problem. Note how the histogram is not Gaussian in nature:




In fact, the tail on the right can go all the way to infinity because an elapsed time of zero seconds is possible (though improbable, being five standard deviations away from the mean).

**Proof of formula (2)**: We consider any $a > 0$, $b \in \mathbb{R} - \{0\}$, $R > 0$ and any $0 < \epsilon < \min\{R, |b|\}$ constants and for simplicity, we integrate the complex function

$$f(z) = \frac{e^{-\frac{(z-\mu)^2}{a^2}}}{z}$$

along the contour

$$C = [-R, -\epsilon] + S_\epsilon + [\epsilon, R] + [R, R + bi] + [R + bi, -R + bi] + [-R + bi, -R],$$

where a bracket $[z_1, z_2]$ indicates the straight segment between the complex numbers $z_1$ and $z_2$ and $S_\epsilon$ is the semicircle in the upper half-plane with center the origin, radius $\epsilon$. This contour, as written, is positively oriented if $b > 0$ and negatively oriented if $b < 0$. Then, the semicircle is travelled in the negative and positive direction, respectively.

Since this function is holomorphic on the contour and its interior, by the well-known Cauchy theorem of complex analysis (e.g., see, [4] Theorem 2.3.14, p. 152, or [5] Theorem 3.5.3, p. 332), its integral, along such a contour, is zero. I.e.,

$$\int_C f(z)dz = 0.$$

To achieve the above principal value, we let $\epsilon \to 0^+$ and $R \to \infty$. For the limit as $\epsilon \to 0^+$, we observe that $z = 0$ is a simple pole for $f(z)$ with residue (immediately evaluated)

$$\operatorname*{Res}_{z=0} f(z) = e^{-\frac{\mu^2}{a^2}}.$$

Also, the change of argument while travelling along $S_\epsilon^-$ is $0-\pi=-\pi$ and along $S_\epsilon^+$ is $\pi-0=\pi$ (depending on $b>0$ or $b<0$, respectively). Since the pole $z=0$ is simple (simple is important), we obtain the limit

$$\lim_{\epsilon\to 0}\int\limits_{S_\epsilon^-} f(z)dz = i\cdot(\text{change of argument})\cdot(\text{residue}) = -i\,\mathrm{sign}(b)\,\pi e^{-\frac{\mu^2}{a^2}}.$$

(See, [4] Lemma 4.3.13, p. 311, or [5] Lemma 3.7.3, p. 440.)

Now, as $R\to\infty$, the limits of the integral along the segments $[R, R+bi]$ and $[-R+bi, -R]$ of constant finite length $|b|>0$ are easily found to be zero. We also write the integral on the horizontal line $(-\infty+ib, \infty+ib)$ and we separate the real and the imaginary parts. From the equality of the real parts of the result, we obtain

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x}\,dx =$$

$$e^{\frac{b^2}{a^2}}\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2}\left\{x\,\cos\left[\frac{2b(x-\mu)}{a^2}\right]-b\,\sin\left[\frac{2b(x-\mu)}{a^2}\right]\right\}dx, \qquad (\mathbf{3})$$

(From the equality of the imaginary parts, we obtain

$$\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^2+b^2}\left\{b\,\cos\left[\frac{2b(x-\mu)}{a^2}\right]+x\,\sin\left[\frac{2b(x-\mu)}{a^2}\right]\right\}dx = \mathrm{sign}(b)\,\pi e^{\frac{-(\mu^2+b^2)}{a^2}}.$$

This equation is not needed in this work, but we write it as a interesting byproduct of this complex integration.)

Next, we notice that the answer in equation **(3)** is the same for any $b\neq 0$. Hence, if we let $a=b=\sqrt{2}\,\sigma\neq 0$ and multiply by the standard factor $\frac{1}{\sigma\sqrt{2\pi}}$, we get the claimed formula **(2)**.

We can evaluate **an almost sharp upper bound of the absolute value of** $E(Y)$. By Hölder's inequality we get

$$\int_{-\infty}^{\infty}\left|\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{x^2+2\sigma^2}\left\{x\cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]-\sqrt{2}\,\sigma\sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\}\right|dx \leq$$

$$\left\{\int_{-\infty}^{\infty}\left|e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right|^2 dx\right\}^{\frac{1}{2}}\times$$

$$\left\{\int_{-\infty}^{\infty}\left|\frac{1}{x^2+2\sigma^2}\left\{x\cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]-\sqrt{2}\,\sigma\sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\}\right|^2 dx\right\}^{\frac{1}{2}}.$$

Then, we have

$$\left\{\int_{-\infty}^{\infty}\left|e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right|^2 dx\right\}^{\frac{1}{2}}=\left\{\int_{-\infty}^{\infty}e^{-\frac{(x-\mu)^2}{\sigma^2}}dx\right\}^{\frac{1}{2}}=\sqrt{\sigma\sqrt{\pi}}. \qquad \mathbf{(4)}$$

By the Cauchy-Schwarz inequality (for numbers or vectors), we have

$$\left|x\cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]-\sqrt{2}\,\sigma\sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right|^2\leq x^2+2\sigma^2.$$

So,

$$\left\{\int_{-\infty}^{\infty}\left|\frac{1}{x^2+2\sigma^2}\left\{x\cos\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]-\sqrt{2}\,\sigma\sin\left[\frac{\sqrt{2}(x-\mu)}{\sigma}\right]\right\}\right|^2 dx\right\}^{\frac{1}{2}}\leq$$

$$\left(\int_{-\infty}^{\infty}\frac{1}{x^2+2\sigma^2}\,dx\right)^{\frac{1}{2}}=\sqrt{\frac{\pi}{\sqrt{2}\,\sigma}}, \qquad \mathbf{(5)}$$

Then, by **(2), (4)** and **(5)**, we get

$$|E(Y)|\leq\frac{e}{\sigma\sqrt{2\pi}}\sqrt{\sigma\sqrt{\pi}}\sqrt{\frac{\pi}{\sqrt{2}\,\sigma}}=\frac{1}{\sqrt{2}}\sqrt[4]{\frac{\pi}{2}}\,\frac{e}{\sigma}<\frac{2.1518375\ldots}{\sigma}.$$

This bound depends only on the standard deviation $\sigma$. It is rather counterintuitive that the mean $\mu$ is not included in this bound.

We conclude with the following two remarks:

**Remark 1:** It is easily seen that for the random variable $Z=z:=\dfrac{1}{x^2}$ the mean value, either as principal value or otherwise, is $+\infty$.

**Remark 2:** Using Remark **1**, the existence of the principal value **(1)**, that we have proved by proving **(2)**, and the integration by parts indicated below, we verify that

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{a^2}}}{x^3}\,dx=$$

$$\lim_{\substack{0<\epsilon\to 0\\ 0<R\to\infty}}\int_{-R}^{-\epsilon}e^{-\frac{(x-\mu)^2}{a^2}}d\left(\frac{x^{-2}}{-2}\right)+\lim_{\substack{0<\epsilon\to 0\\ 0<R\to\infty}}\int_{\epsilon}^{R}e^{-\frac{(x-\mu)^2}{a^2}}d\left(\frac{x^{-2}}{-2}\right)=$$

$$\ldots=\begin{cases}0, & \text{if }\ \mu=0\\ \\ \operatorname{sign}(\mu)\cdot\infty, & \text{if }\ \mu\neq 0.\end{cases}$$

So, unless $\mu = 0$, the mean value of $W = w := \frac{1}{x^3}$, even as principal value, is infinite with sign the sign of $\mu$. So, the side on which $\mu$ lies with respect to the origin always prevails in the case of $W$.

## References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